

## AVERAGE SUBMODULARITY OF MAXIMIZING ANTICOORDINATION IN NETWORK GAMES\*

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**Abstract.** We consider the control of decentralized learning dynamics for agents in an anticonoordination network game. In the anticonoordination network game, there is a preferred action in the absence of neighbors' actions, and the utility an agent receives from the preferred action decreases as more of its neighbors select the preferred action, potentially causing the agent to select a less desirable action. The decentralized dynamics that are based on the synchronous best-response dynamics converge for the considered payoffs. Given a convergent action profile, we measure anticonoordination by the number of edges in the underlying graph that have at least one agent in either end of the edge not taking the preferred action. A designer wants to find an optimal set of agents to control under a finite budget in order to achieve maximum anticonoordination (MAC) on game convergence as a result of the dynamics. We show that the MAC is submodular in expectation over all realizations of the payoff interaction constants in bipartite networks. The proof relies on characterizing well-behavedness of MAC instances for bipartite networks, and designing a coupling between the dynamics and another distribution preserving selection protocol, for which we can show the diminishing returns property. Utilizing this result, we obtain a performance guarantee for the greedy optimization of MAC. Finally, we provide a computational study to show the effectiveness of greedy node selection strategies to solve MAC on general bipartite networks.

**Key words.** game theory, submodular optimization, anticonoordination games, approximation algorithms, games on graphs

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**1. Introduction.** Anticonoordination games can be used to study competition among firms [5, 6], public goods scenarios [15], free-rider behavior during epidemics [2, 12], and network security [21]. In each of these scenarios, there is a desired action for each agent, e.g., not taking the costly preemptive measures during a disease outbreak, not investing in insurance/protection etc., in the absence of other agents. When other agents are around, they can affect the benefits of the desired action, providing incentives for agents to switch. Here we consider networked interactions, where the actions of an agent are only affected by its neighbors (a subset of the population)—see [13, 23] for more details on network games. Specifically, the action space of agents is the entire spectrum of real numbers in the range  $[0, 1]$  with 0 and 1 representing the most costly and most easy action polarities, respectively. Agents are engaged in an anticonoordination game where their payoffs from playing the costlier actions increase with more of their neighbors playing the easier action.

The selfish behavior can lead to the failure of anticonoordination in the population, when anticonoordination is desirable for the well-being and safety of the system as a whole. That is, some of the agents may continue to take the individually preferred action, endangering their peers and the rest of the population despite the peer influences.

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In such scenarios, we can envision the existence of a central coordinator with the goal to induce behavior that supports the well-being of the society. Here, we consider one such mechanism where the centralized coordinator intervenes by controlling a few agents in the network to incentivize antcoordination among agents that repeatedly take actions to maximize individual payoffs. In particular, the players follow a form of synchronous best-response dynamics, where the players best-respond considering the worst and best possible action profile of their neighbors (section 3). We showed in [11] that such dynamics will converge in finite time and eliminate all dominated strategies in the antcoordination network game considered here. Given such adaptive behavior of agents, the centralized player can steer the convergent action profile toward socially desirable outcomes by controlling the actions of a few players during the learning phase.

We define the goal of the central coordinator as to maximize antcoordination between connected pairs of agents upon convergence of the behavior (section 4) under limited intervention budget. The MAC problem is combinatorial, involving selection of a subset of the agents in the population. Indeed, we show that the MAC is NP-Hard to solve, in general, graphs (Theorem 4.2). This motivates us to consider a computationally tractable greedy selection protocol for solving MAC, where at each selection epoch the agent that yields the highest number of antcoordinating edges at the convergent action profile is added to the control set until the control budget is reached (section 4). We show that the MAC problem is monotone and submodular in expectation in bipartite networks (Theorems 5.6 and 5.5). Together, these results imply that the worst case performance of the greedy selection protocol is bounded by a fraction of the optimal solution (Corollary 5.7). Numerical experiments confirm that the greedy selection protocol provides near-optimal results on average for bipartite networks.

This work is most closely related to the following intervention mechanisms in games that aim to improve efficiency: nudging [14, 24, 28], influence maximization under limited control [16, 20], optimal targeting in supermodular games [9], and seeding in advertising [1]. All of these approaches aim to determine the emerging action profile resulting from adaptive learning dynamics under repeated game play by either providing incentives or suggestions of “good” behavior to agents or by directly controlling a set of agents, as we do in this paper. Similar themes have been investigated in the control and optimization in networks literature [8], flow of opinion dynamics in social networks [3, 4], and in the game theoretic control of multiagent networked systems [17]. Traditionally, a lot of work in this area is focused on consensus seeking or enhancing coordination among self interested agents with access to full or partial information. Here, we aim to maximize antcoordination instead of maximizing social welfare or reach consensus. That is, we ask if selfish agents can successfully use local interaction to reach maximum polarization in strategies, and if not, can we identify a set of agents whose actions when controlled would lead to the same global objective? Other forms of intervention mechanisms involve financial incentives in the form of taxations or rewards [7], and information design [25, 26]. These mechanisms do not consider repeated game play, and instead focus on improving the efficiency of Nash equilibria. Here our control selection policy is dependent on the adaptive learning behavior of agents.

We obtain the performance guarantee for the greedy selection by showing the monotonicity and submodularity of the MAC problem in well-behaved instances given the learning dynamics. A well-behaved instance is one where there are no extremely insensitive agents, i.e., agents that are stubborn and play their chosen strategies no

matter what their neighbors are playing. We capture this insensitivity to peer influence as having a very low payoff interaction constant. In such an instance, agents playing opposite strategies are concentrated on either side of the bipartite network. That is, if we control agents only on one side of the bipartite network to play the undesirable action, only agents on this side take the undesirable action as a result of the learning dynamics. This property makes anticonoordination akin to two parallel diffusion processes unfolding on both sides of the bipartite network. Each side reinforces the decision making of the other side, each side further polarizing the other side.

Given the characteristics of the dynamics in the well-behaved scenario, our approach to showing the submodularity of MAC follow ideas similar to the ones utilized in [16, 20]. In [16], Kempe, Kleinberg, and Tardos talk about an influence diffusion process (such as viral marketing), where the process is initiated from a set of active nodes and spreads further. They show that if the dynamics are the linear threshold type, then the expected value of the total influence (the set of all active nodes) in the social network on process convergence is submodular. In [20], Mossel and Roch tackle the more general problem. They establish that any influence diffusion process, under certain assumptions on the neighbor influence function (monotonicity and submodularity), can be shown to be submodular.

While our dynamics are different, we are still able to utilize similar ideas to show submodularity (Theorem 5.5). We begin by defining functions that measure one-step influence which encourages agents to get decided at each step of the dynamics. We establish that the one-step functions so defined are modular. Next, we describe the stochastic process of the set of agents getting decided when the dynamics are seeded by a set of agents who are controlled. We show that the distribution of agents getting decided on convergence of dynamics remains intact if instead of controlling the entire set from the get-go we break it into a set of smaller subsets (a partition) and control each set one by one in stages. This property allows us to provide an alternate equivalent description of the selection process based on a selection rule (Definition 5.18 and Lemma 5.19). The proof of the submodularity result entails designing such a *coupling* between the actual (greedy) selection method and another equivalent selection method for which we can show the diminishing returns property (submodularity).

The submodularity result establishes a similar result as in the case of contagion dynamics [16, 20] or as in best-response dynamics in supermodular games [9]. While these results aim to trigger contagion of a particular action (product), i.e., coordination, we aim to trigger anti-coordination among connected agents. Despite the fundamental difference, our submodularity result shows that a similar cascade of desirable behavior can be established via greedy selection of controlled agents.<sup>1</sup>

**2. Network anticonoordination game.** We consider a graph  $\mathcal{G}$  as the pair  $(V, E)$  where the set of vertices  $V = \{1, \dots, n\}$  represent the agents, and the set of edges  $E$  is a multiset of subsets of  $V$  of size at most two, that represents interactions between individual agents. We use the functions  $V(\cdot), E(\cdot)$  to represent the vertices and edges of a graph. Thus,  $V(\mathcal{G}) = V$  and  $E(\mathcal{G}) = E$ . We assume the edges are undirected. We denote the neighbors of agent  $i$  with  $n(i) := \{j : \{i, j\} \in E(\mathcal{G})\}$ . The degree of agent  $i$  is denoted with  $|n(i)|$ .

<sup>1</sup>In [10, 11], we considered a similar MAC problem for the same anti-coordination network game. In [10], we exploit the paradigm of approximate submodularity (bounded violation of submodularity) to provide guarantees for greedy optimization of MAC in line networks, which is a specific bipartite network. In [11], we numerically compare the greedy algorithm with other heuristic selection rules without any performance guarantees.

Each agent takes an action  $a_i \in [0, 1]$  to maximize its utility function

$$(2.1) \quad u_i(a_i, a_{n(i)}) = a_i \left( 1 - c_i \sum_{j \in n(i)} a_j \right)$$

where  $a_{n(i)} := \{a_j\}_{j \in n(i)}$  denotes the actions of agent  $i$ 's neighbors, and  $0 < c_i < 1$  is a payoff constant. The utility function above captures scenarios where agent  $i$  has a preferred action (action 1) but its incentive to choose this action decreases as more of its neighbors choose the preferred action. The decrease in incentive per neighbor is proportional to the constant  $c_i$ . The constant represents the sensitivity of agent  $i$ 's utility to its neighbor's actions. The network game is represented by the tuple  $\Gamma = \{V, A, \{u_i\}_{i \in V}\}$  where  $A = [0, 1]^n$  is the set of actions available to all players.

Below we provide scenarios that can be captured by the network game  $\Gamma$  with payoffs given by (2.1).

*Example 2.1* (disease spread on networks [12]). Consider a bipartite graph  $\mathcal{G}_B$  where agents on one side are sick and the other side healthy. The edges in  $\mathcal{G}_B$  represent the network of interactions. The disease spreads when agents on either end of an interaction link do not follow healthcare protocols, such as wearing masks, vaccinating, social distancing etc. We model this using action level 1 for the agent (the easier/preferred action). Action level 0 represents following epidemic mitigation protocols (the costlier action), and all the actions between 0 and 1 represent the relative importance given to disease prevention measures. When we have an agent playing 0 on the end of an interaction link, we have deactivated a disease transmission pathway. The utilities of agents in the epidemic game are the anticoordination type, i.e., its incentive to social distance increases with more of its neighbors flouting protocols, and hence can be captured using the utility function in (2.1). The payoff interaction constants  $c_i, i \in V$  represent the sensitivity of the agents to the neighbor influence.

*Example 2.2* (political polarization). The network represents the social interactions among players in opposing beliefs (agents on different sides of the bipartite network) that want to differentiate their actions from those with opposing beliefs [18]. Action 1 represents a monetary choice or support for a cause that is individually desirable in the absence of partisanship. A player's tendency to take the preferred action (action 1) reduces as it has more neighbors that take action 1. That is, a player can opt-out from individual benefits or societal impact to express partisan preferences. The payoff constants  $c_i$  capture the inclination of players to distinguish their actions from those in the opposite camp.

*Example 2.3* (hawk-dove network game). Two competing species ( $\mathcal{S}_0$  and  $\mathcal{S}_1$ ) face-off in an ecological environment. At each interaction players decide to be hawkish ( $a_i = 1$ ) or dovish ( $a_i = 0$ ). A hawk move gets the highest reward if its neighboring competitors play dove. If both interacting players play dove, they miss the opportunity to overcome their competitor. If both interacting players are hawkish, they challenge each other and face costs. The constants  $c_0$  and  $c_1$  represent the costs species 0 and 1 incur, respectively, when they act hawkish against a hawkish competitor.

**3. Synchronous best-response dynamics.** At each stage  $t = 1, 2, \dots$ , we assume agents observe the past actions of their neighbors  $a_{n(i)}^{t-1}$ , and determine their actions  $a_i^t$  according to the following rule:

$$\begin{aligned}
 (3.1) \quad a_i^t &= 1 \text{ if } \{1\} = BR_i(\lceil a_{n(i)}^{t-1} \rceil) = \operatorname{argmax}_{a_i \in [0,1]} u_i(a_i, \lceil a_{n(i)}^{t-1} \rceil), \\
 a_i^t &= 0 \text{ if } \{0\} = BR_i(\lfloor a_{n(i)}^{t-1} \rfloor) = \operatorname{argmax}_{a_i \in [0,1]} u_i(a_i, \lfloor a_{n(i)}^{t-1} \rfloor), \\
 a_i^t &= \epsilon \text{ otherwise,}
 \end{aligned}$$

where  $BR_i(a_{n(i)}) := \operatorname{argmax}_{a_i \in [0,1]} u_i(a_i, a_{n(i)})$  is the best response action profile, and  $\epsilon \in (0, 1)$  is an arbitrary action between 0 and 1. See that when an agent plays  $\epsilon$  (are undecided), it means that they can effectively play anything between 0 and 1. The decision dynamics proceeds with agents trying to figure out if they have a clear winning strategy, no matter what it's undecided neighbors may be playing. The best response action for the utility function in (2.1) is given by

$$(3.2) \quad BR_i(a_{n(i)}) = \mathbb{1} \left( 1 > c_i \sum_{j \in n(i)} a_j \right)$$

when  $1 \neq c_i \sum_{j \in n(i)} a_j$  (the best response is a singleton). When  $1 = c_i \sum_{j \in n(i)} a_j$ , then  $BR_i(a_{n(i)}) = [0, 1]$ . Given (3.2), the dynamics can also be equivalently represented as

$$\begin{aligned}
 (3.3) \quad a_i^t &= 1 \text{ if } 1 > c_i \sum_{j \in n(i)} \lceil a_j^{t-1} \rceil, \\
 a_i^t &= 0 \text{ if } 1 < c_i \sum_{j \in n(i)} \lfloor a_j^{t-1} \rfloor, \\
 a_i^t &= \epsilon \text{ otherwise.}
 \end{aligned}$$

*Remark 3.1.* The best response dynamics emulate the decentralized iterated elimination of dominated strategies [19] in the network game. Since  $\epsilon \in (0, 1)$ , we see that

$$(3.4) \quad \operatorname{argmax}_{a_i \in [0,1]} u_i(a_i, \lceil a_{n(i)}^{t-1} \rceil) = 1 \text{ implies that } \operatorname{argmax}_{a_i \in [0,1]} u_i(a_i, a_{n(i)}^{t-1}) = 1$$

that is, from (3.2), if  $1 > c_i \sum_{j \in n(i)} \lceil a_j \rceil$ , then  $1 > c_i \sum_{j \in n(i)} a_j$  holds for all  $a_j \in [0, 1]$   $j \in V$ . Thus, agents remove all dominated strategies and decide to play 1, when the first condition in (3.1) holds. The ceil operator implies a “worst-case” scenario where the agent assumes all of its “undecided” agents play 1. If action 1 is the best-response even under this worst-case scenario, then it would take action 1 regardless of the strategies of the other agents. Similarly, we see that

$$(3.5) \quad \operatorname{argmax}_{a_i \in [0,1]} u_i(a_i, \lfloor a_{n(i)}^{t-1} \rfloor) = 0 \text{ implies that } \operatorname{argmax}_{a_i \in [0,1]} u_i(a_i, a_{n(i)}^{t-1}) = 0$$

that is, from (3.2), if  $1 < c_i \sum_{j \in n(i)} \lfloor a_j \rfloor$ , then  $1 < c_i \sum_{j \in n(i)} a_j$  holds for all  $a_j \in [0, 1]$   $j \in V$ . Thus, the best response 0 is now the dominant strategy, and the dynamics in (3.1) cause agents to decide to play 0 whenever the second condition holds. The floor operator implies a “best-case” scenario where the agent assumes all of its “undecided” agents play 0. If action 0 is the best-response even under this best-case scenario, then it would take action 0 regardless of the strategies of the other agents. For all other situations, agents do not have a dominant strategy, and thus they stay undecided (play  $\epsilon \in (0, 1)$ ).

In [11], we provide a *finite time convergence guarantee* for the dynamics discussed. Specifically, we show that the dynamics in (3.1) converges in at most  $|V|$  iterations,

eliminating all strictly dominated actions for the network game  $\Gamma$  starting the dynamics with all agents undecided. We note that a strictly dominated action cannot be a rational action. These updates converge to a Nash equilibrium if the game is dominance solvable, i.e., if the game is such that a single action profile is left as a result of iterated elimination of dominated strategies. For instance, the anticonoordination game  $\Gamma$  with utility function in (2.1) is dominance solvable given constants  $c_i < \frac{1}{|n(i)|}$  for all  $i \in V$ . Indeed, all agents take action 1 after the first update in (3.1) because  $c_i \sum_{j \in n(i)} [a_j] < 1$  for all  $i \in V$ . For general payoff constants, the game  $\Gamma$  is not dominance solvable, i.e., some agents can stay undecided (play  $\epsilon$ ) at the end of  $|V|$  iterations.

**4. Maximum anticonoordination problem.** Assume we are given the network game  $\Gamma = \{V(\mathcal{G}), A, \{u_i\}_{i \in V(\mathcal{G})}\}$  where graph  $\mathcal{G}$  represents the interaction network between agents. Let  $c_i \in (0, 1)$  be the payoff interaction constants for all  $i \in V$ . Assume agents play according to the synchronous best response dynamics in (3.1). We define an undirected edge  $\{i, j\} \in E(\mathcal{G})$  between agents  $i, j \in V$  to be inactive (an anticonoordinating link) in action profile  $a \in A$  when at least one of the agents take action 0, i.e., when  $a_i = 0$  or  $a_j = 0$ . Our goal is to maximize the number of inactive edges by controlling a subset of the players (with set cardinality  $r \in \mathbb{Z}_+$ ) to play action 0 during the learning dynamics in (3.1). We state this goal to maximize anticonoordination (MAC) as follows:

$$(4.1) \quad \begin{aligned} \max_{X \subseteq V(\mathcal{G})} \mathcal{F}(X) &:= \sum_{\{i, j\} \in E(\mathcal{G})} \mathbb{1}(a_i^\infty a_j^\infty = 0) \\ \text{subject to} \quad &|X| = r, \\ &a_j^0 = \epsilon \quad \text{for all } j \in V, \\ &(a^0, a^1, \dots, a^\infty) = \Phi(a^0, c, X), \end{aligned}$$

where  $\Phi(a^0, c, X)$  represents the sequence of actions obtained when uncontrolled agents  $(V(\mathcal{G}) \setminus X)$  follow the learning dynamics in (3.1), and the actions of controlled agents are set to 0, i.e.,  $a_i^t = 0$  for all  $t \geq 0$  and  $i \in X$ .

*Remark 4.1.* The definition of MAC in (4.1) considers an edge  $\{u, v\}$  inactive if either of  $a_u^\infty$  or  $a_v^\infty$  is 0. That is, it intrinsically tolerates coordination on the undesirable action in the sense that an edge  $\{u, v\} \in E(\mathcal{G})$  is considered inactive and counted as anticonoordinating even if  $i$  and  $j$  both play action 0, i.e.,  $a_u^\infty = a_v^\infty = 0$ . Our motivation stems from the epidemic setting (Example 2.1), where an agent playing ‘costly’ action level 0 neutralizes (inactivates) all incident disease transmission pathways in the interaction graph; see section 4.1 for motivating examples. In the setting of well-behaved instances that we analyze in this paper (see following section), every deactivated edge  $\{u, v\}$  additionally satisfies  $a_u^\infty \neq a_v^\infty$ . That is, for the MAC instances we study in this paper  $a^\infty$  satisfies  $a_u^\infty \neq a_v^\infty$  whenever  $\{u, v\} \in E(\mathcal{G})$  and is deactivated.

By removing the agents that are controlled from the game, we can guarantee that the learning process converges in finite time as per the aforementioned convergence result in [11]. The control budget for the planner is restricted, i.e., the planner can only control a given  $r \in \mathbb{Z}_+$  number of agents as indicated by the first constraint in (4.1). Henceforth, we shall use the notation  $(\mathcal{G}, c)$  as the specification for an unique instance of the MAC problem, where  $\mathcal{G}$  is the interaction network and  $c$  is the vector of payoff interaction constants. Note that given the payoff interaction constants  $c$ ,

the interaction graph  $\mathcal{G}$  and the stationary initial action profile  $a_i^0 = \epsilon$  for all  $i \in V$ , every step of the local learning dynamics in (3.1) is fully determined. We can exclude  $\Phi(\cdot)$  from the specification since we assume (3.1) is fixed in the rest of this paper. We also drop  $r$  from the specification, and additionally specify the control budget whenever needed.

**4.1. Motivation for maximizing anticoordination.** Maximizing anticoordination is a valid objective for the system designer whenever there is gain achieved by maximizing differentiation among agents. The exact system designer's objective can change depending on the particular study. In the context of disease spread in a population (Example 2.1), which motivates our problem definition, maximizing anticoordination in the underlying relationship network by inactivating disease transmission links is desirable as an effective means of curbing spread of the disease between members of the society. The decentralized learning dynamics do not inactivate all edges on convergence and thereby a central planner would need to control/enforce certain agents to coordinate with policy guidelines (playing action level 0) so that the maximum number of transmission pathways shall be dismissed.

Another motivating example follows. The hawk-dove game (Example 2.3) can serve as a general framework for understanding conflicts, where agents playing against each other incur high overall costs if both players choose to escalate (play "hawk", the preferred strategy), but if at least one of them de-escalates (plays "dove", the less preferred strategy) the conflict can be resolved. De-escalation comes with its own cost for the agent choosing that strategy, and selfish agents may not be able to avoid conflict by themselves following their individual model for rational behavior. The system designer interested in conflict resolution may want to maximize anticoordination by enforcing de-escalation on one or both agents.

**4.2. MAC is NP-hard.** Now, let us first characterize the hardness of MAC formalized in (4.1).

**THEOREM 4.2.** *The MAC problem in (4.1) is NP-Hard for general graphs.*

*Proof.* Consider the NP-Complete problem of whether a vertex cover  $C$  of a general graph  $\mathcal{G}$  with cardinality  $|C| = k$  exists or not.<sup>2</sup> Given a vertex cover problem on graph  $\mathcal{G}$  and cardinality constant  $k$ , construct a MAC instance for  $\mathcal{G}$  where  $c_i < |n(i)|^{-1}$  where  $n(i)$  is the set of neighbors of node  $i \in V$ , and set  $r = k$ . Given the payoff constants  $c_i$ , all agents play 1 given the uncontrolled dynamics, i.e.,  $a^t = \mathbf{1}$  for all  $t \geq 1$ . If there exists a solution  $X$  to MAC where the objective value  $g(X) = |E(\mathcal{G})|$ , then  $C = X$  is a vertex cover with cardinality  $k$ . If there does not exist a solution to MAC where the objective value  $g(X) = |E(\mathcal{G})|$ , then we can conclude that there does not exist a solution to the vertex cover problem for the graph  $\mathcal{G}$  with  $k$  vertices. Thus if we can solve MAC efficiently, we can solve every vertex cover problem efficiently, which is a contradiction.  $\square$

Given that MAC is NP-Hard, we provide performance guarantees for tractable greedy approaches to solving MAC defined over bipartite networks.

**5. Submodularity for bipartite MAC instances.** In a greedy approach, we obtain a solution to a cardinality constrained maximization problem

<sup>2</sup>A vertex cover  $C$  of a graph  $\mathcal{G}$  is such that every edge has at least one endpoint incident (vertex) in  $C$ ; see [27] for a formal definition.

$$(5.1) \quad \max_{X \subseteq V, |X| \leq r} f(X)$$

by selecting one element at a time, i.e.,

$$(5.2) \quad \begin{aligned} u &= \operatorname{argmax}_{w \in V} f(G_j \cup \{w\}), \\ G_j &= G_{j-1} \cup \{u\}, \text{ for } 1 \leq j \leq r, \end{aligned}$$

where  $G_0 = \emptyset$ . The greedy approach is a computationally tractable way to build a set of maximum cardinality  $r$  for solving MAC in (4.1) when, in addition, we consider the learning dynamics. If the MAC is submodular, then the greedy approach can obtain a solution comparable to the optimal set  $X^*$ . In the following, we provide preliminary definitions of submodularity and monotonicity, and then characterize the optimality loss when we implement a greedy selection (5.2).

**DEFINITION 5.1** (submodularity). *A set function  $f(\cdot) : 2^V \rightarrow \mathbb{R}$  is submodular if for any subsets  $X \subseteq Y \subseteq V$ , where  $V$  is the ground set, and any  $u \in V \setminus Y$  we have*

$$(5.3) \quad f(X \cup \{u\}) - f(X) \geq f(Y \cup \{u\}) - f(Y),$$

where  $V$  is finite. Alternatively, for any two sets  $S, T \subseteq V$ ,  $f(\cdot)$  is submodular if

$$(5.4) \quad f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

Equations (5.3) and (5.4) are equivalent representations of the property, and we shall use both representations in our analyses going forward. The representation in (5.3) is often referred as the diminishing returns description, and it provides a useful intuition. Say  $X \subset V$  is a control that generates a benefit  $f(X)$ . Then, while already controlling  $X$ , the marginal benefit of adding  $u \in V \setminus X$  to the control set does not increase whenever  $f(\cdot)$  is submodular.

Our results in this section focus on the bipartite topology for the interaction graph in the network game  $\Gamma$ . A graph  $\mathcal{G}_B$  is bipartite if  $V(\mathcal{G}_B)$  can be partitioned into two (possibly empty) sets  $S_0, S_1$  such that every edge of  $\mathcal{G}_B$  has one end in  $S_0$  and the other in  $S_1$ . We call such a partition  $\{S_0, S_1\}$  as the *bipartition* of  $\mathcal{G}_B$ . We say that a MAC instance  $(\mathcal{G}, c)$  is bipartite (equivalently, bipartite MAC) if  $\mathcal{G}$  is a bipartite graph. The bipartiteness of the MAC instance arises as a natural modelling assumption in the disease spread on networks example (2.1), where each partition of the vertex set can represent the agents who are sick and healthy, respectively. Since we are only concerned with valid disease transmission links, any interactions between two sick or two healthy agents can be ignored.

**5.1. Main result.** First, we proceed with a definition that limits agent insensitivity to peer influence in the network game, based on the payoff interaction constants  $c$ .

**DEFINITION 5.2** (stubbornness). *For a MAC instance  $(\mathcal{G}, c)$ , we define an agent  $i \in V(\mathcal{G})$  as stubborn if  $c_i < |n(i)|^{-1}$ .*

**LEMMA 5.3.** *A stubborn agent will always play 1, i.e.,  $a_i^t = 1$  for all  $t > 1$ .*

*Proof.* If  $c_i < \frac{1}{|n(i)|}$ , then  $1 > c_i |n(i)| > c_i \sum_{j \in n(i)} [a_j^t]$  for any time step  $t$  in the learning dynamics (3.1) ( $BR_i(a_{n(i)}) = 1$ ). Thus, starting from  $\epsilon$ , agent  $i$  decides to play 1 in the next time step, and stays decided on playing 1 no matter what.  $\square$

In this paper, we concern ourselves with MAC instances which do not have any stubborn agents. See the following definition.



DEFINITION 5.4 (well-behaved MAC). A MAC-instance  $(\mathcal{G}, c)$  is well-behaved if there are no stubborn agents.

Under some minor constraints on the payoff interaction parameters and the minimum degree of the graph, we can guarantee that there are no stubborn agents. Specifically, the constraints  $c_i \geq |n(i)|^{-1}$  and  $|n(i)| \geq 2$  for all  $i \in V(\mathcal{G})$  are sufficient to ensure that there are no stubborn agents in graph  $\mathcal{G}$ . For an agent  $i$  with only one neighbor,  $c_i < |n(i)|^{-1}$  is trivially satisfied for any  $c_i \in (0, 1)$ . Thus, for this study, we require the underlying graph to additionally have a minimum degree of 2. The line network that we study in the previous paper [10] in the context of MAC does not satisfy this restriction, since the end nodes of the line graph will always have one neighbor, and therefore always be stubborn.

Now, we state the main result in this paper, which directly produces a performance guarantee for the greedy optimization of MAC (4.1).

THEOREM 5.5. Given a random, well-behaved instance of bipartite MAC  $(\mathcal{G}_B, c)$  where  $c_i \sim \text{uniform}(\frac{1}{|n(i)|}, 1)$  for all  $i \in V(\mathcal{G}_B)$  is drawn uniformly at random,  $\mathbb{E}_c \mathcal{F}(\cdot)$  is submodular when we restrict the control input to  $\mathcal{F}(\cdot)$  defined in (4.1) to one side of the bipartite graph  $\mathcal{G}_B$  and  $\mathbb{E}_c$  is the expectation with respect to the random payoff interaction constants  $\{c_i\}_{i \in V(\mathcal{G}_B)}$ .

That is, bipartite MAC is submodular in expectation for well-behaved instances when we assume a uniform distribution on the payoff interaction constants. To leverage the result, we need to condition the control set for MAC (4.1) to belong to one side of the bipartition of the underlying interaction graph  $\mathcal{G}_B$ . We also show that MAC is monotone increasing for any well-behaved instance.

THEOREM 5.6. The bipartite MAC objective  $\mathcal{F}(\cdot)$  (4.1) is monotone increasing for well-behaved instances  $(\mathcal{G}_B, c)$ , with bipartition  $\{S_0, S_1\}$  when the control input to  $\mathcal{F}(\cdot)$  is restricted to one-side of the graph. Given selections of control sets  $X \subseteq Y \subseteq S_0$ , we have  $\mathcal{F}(X) \leq \mathcal{F}(Y)$ .

The theorems stated above lead to the proceeding result.

COROLLARY 5.7 (greedy performance guarantee). Let  $(\mathcal{G}_B, c)$  be a random well-behaved bipartite MAC instance with the bipartition  $\{S_0, S_1\}$ , and  $c_i \sim \text{uniform}(\frac{1}{|n(i)|}, 1)$  for all  $i \in V(\mathcal{G}_B)$  is drawn uniformly at random. Then the solution obtained by the greedy algorithm satisfies  $\mathbb{E}_c \mathcal{F}(G_r) \geq (1 - e^{-1}) \mathbb{E}_c \mathcal{F}(X^*)$  where  $r$  is the control budget and  $G_r \subseteq S_0$  is the control set obtained after  $r$  steps of the algorithm.

The result follows from Theorems 5.5 and 5.6, and the standard  $1 - 1/e$  performance guarantee for submodular function maximization. See [22] for details. For the proof of Theorem 5.5, we track the set of people who get decided as a result of the initial control via the dynamics in (3.1). The proof utilizes the fact that the one-step influence on an undecided agent is modular (see section 5.3). We then design an alternate equivalent description of the dynamics in (3.1) following which the distribution of the random set of decided agents on convergence of the learning dynamics stays unperturbed. Thereby, we design a coupling argument (see sections 5.4 and 5.5) which allows us to have the properties we desire for the coupled stochastic process, from which we recover the results we need for our dynamics via expectation over the random payoff interaction parameters. Throughout, we leverage our understanding of how the dynamics in (3.1) unfold on a bipartite graph if we additionally assume well-behavedness.

*Remark 5.8.* The restriction for control to be confined to one side of the bipartite graph can be motivated from the epidemic setting (Example 2.1), where the two sides of the bipartite graph refer to sick and healthy agents, respectively, and enforcing strict health protocols for healthy agents is not practical. From a technical standpoint, however, controlling agents on opposite sides of the bipartite graph can potentially cause conflicting decision cascades (see the discussion in the next section), where the potential positive benefits of control cancel out.

**5.2. Well-behaved dynamics of MAC.** Let us look at some consequences of the well-behavedness assumption for bipartite MAC instances. First, we define a scenario where every agent remains undecided according to the learning dynamics (3.1).

**LEMMA 5.9.** *The action profile  $a_i = \epsilon$  for all  $i \in V(\mathcal{G})$  is a fixed point for the dynamics in (3.1) for well-behaved MAC instances.*

*Proof.* Let  $(\mathcal{G}, c)$  be a well-behaved MAC instance (not necessarily bipartite). Then, we converge in one step when there is no control with dynamics in (3.1). Agents start undecided (play  $\epsilon$  initially). For all  $i \in V(\mathcal{G})$ ,  $1 \leq c_i \sum_{j \in n(i)} [a_j] = c_i |n(i)|$  is satisfied, whereas  $c_i \sum_{j \in n(i)} [a_j] = 0$  as everyone plays  $\epsilon$  in the zeroth step. Thus agents stay undecided.  $\square$

This implies that without any external intervention, in a well-behaved instance of MAC, elimination of dominated strategies corresponds to agents staying undecided. Now, consider a well-behaved bipartite instance of MAC  $(\mathcal{G}_B, c)$  with the bipartition  $\{S_0, S_1\}$ . At  $t = 0$ , all agents are undecided in the absence of control and the dynamics have converged (Lemma 5.9). Let  $X \subset S_0$  be the set of agents we control initially at time  $t = 1$  to play action level 0. We call this set  $X$  the *initiating set*. The notation  $X_t$  and  $\bar{X}_t$  represents the set of agents playing action level 0 and 1 at time  $t$ , respectively. We have  $X_1 = X$  and  $\bar{X}_1 = \emptyset$  in this setting. The following lemma specifies conditions on the evolution of  $X_t$  and  $\bar{X}_t$  using the dynamics (3.1).

**LEMMA 5.10 (concentration lemma).** *Let  $(\mathcal{G}_B, c)$  be a well-behaved bipartite MAC instance with the bipartition  $\{S_0, S_1\}$ , and let  $X \subseteq S_0$  be the initiating set. Then  $X_t \subseteq X_{t+1} \subseteq S_0$  and  $\bar{X}_t \subseteq \bar{X}_{t+1} \subseteq S_1$  is satisfied for all  $t$ .*

*Proof.* Initially,  $X_1 = X \subseteq S_0$  and  $\bar{X}_1 = \emptyset$ . For all  $v \in S_1$ ,  $\sum_{j \in n(v)} [a_j^1] \leq \sum_{j \in n(v)} [a_j^0]$ . Thus, following dynamics, additional agents in  $S_1$  are encouraged to play 1 at  $t = 2$ . But, for all  $v \in S_0$ ,  $\sum_{j \in n(v)} [a_j^1] = |n(v)|$ ,  $\sum_{j \in n(v)} [a_j^0] = 0$  as  $a_j^1 = \epsilon$  for all  $j \in S_1$ . This ensures that, following dynamics in (3.1),  $X_1 = X_2 \subseteq S_0$ , and  $\bar{X}_2 \cap S_0 = \emptyset$  as there are no stubborn agents. Thus  $\bar{X}_1 \subseteq \bar{X}_2 \subseteq S_1$ , and the property holds at  $t = 2$ .

Let the property hold at an arbitrary time step  $t - 1$ , i.e.,  $X_{t-1} \subseteq X_t \subseteq S_0$  and  $\bar{X}_{t-1} \subseteq \bar{X}_t \subseteq S_1$ . If  $a^t$  is a fixed point of the dynamics,  $X_{t+1} = X_t$ ,  $\bar{X}_{t+1} = \bar{X}_t$ , and we are done. Assume that  $a^t$  is not a fixed point. Let  $X_{t+1} \subset X_t$ . Then,  $X_t \subset S_0$ . Moreover, there exists  $v \in X_t \setminus X_{t+1}$  such that  $1 < \sum_{j \in n(v)} [a_j^{t-1}]$  but  $1 \geq \sum_{j \in n(v)} [a_j^t]$ , or  $\sum_{j \in n(v)} [a_j^t] < \sum_{j \in n(v)} [a_j^{t-1}]$ . Now,  $n(v) \subseteq S_1$  since  $v \in S_0$ . Also  $\bar{X}_t \cap S_1 = \emptyset$  from our assumption, that is agents in  $S_1$  only play  $\epsilon, 1$  at time  $t, t - 1$ . This implies that  $\bar{X}_t \subset \bar{X}_{t-1}$ , which is a contradiction. Thus  $X_t \subseteq X_{t+1}$ . Similarly, when  $\bar{X}_{t+1} \subset \bar{X}_t$ , there exists  $v \in \bar{X}_t \setminus \bar{X}_{t+1}$  such that  $1 > \sum_{j \in n(v)} [a_j^{t-1}]$  but  $1 \leq \sum_{j \in n(v)} [a_j^t]$ , or  $\sum_{j \in n(v)} [a_j^{t-1}] < \sum_{j \in n(v)} [a_j^t]$ , which implies  $X_{t-1} \supset X_t$ , which is a contradiction as  $X_{t-1} \subseteq X_t$  and  $\bar{X}_t \cap S_0 = \emptyset$ . Thus  $\bar{X}_t \subseteq \bar{X}_{t+1}$ .

Now we need to show that  $X_{t+1} \subseteq S_0$ . Assume, for contradiction, that there exists  $v \in X_{t+1} \cap S_1$ . Therefore, since  $X_t \subseteq S_0, v \notin X_t$ . Now,  $v$  satisfies  $1 \geq \sum_{j \in n(v)} [a_j^{t-1}] (\cdot \cdot v \notin X_t)$  but  $1 < \sum_{j \in n(v)} [a_j^t] (\cdot \cdot v \in X_{t+1})$ . This implies that  $\sum_{j \in n(v)} [a_j^{t-1}] < \sum_{j \in n(v)} [a_j^t]$ . But,  $v \in S_1$ , which means  $n(v) \in S_0$ . Since  $\bar{X}_{t-1} \subseteq \bar{X}_t \subseteq S_1$ , all agents in  $S_0$  are playing  $\epsilon$  or 0 at times  $t-1, t$ . This means that  $\sum_{j \in n(v)} [a_j^{t-1}] = \sum_{j \in n(v)} [a_j^t] = 0$  which is a contradiction. Therefore,  $X_{t+1} \cap S_1 = \emptyset$ , or  $X_{t+1} \subseteq S_0$ . Following similar arguments, we are also able to show that  $\bar{X}_{t+1} \subseteq S_1$ . Now, using the principle of mathematical induction, our result holds for all  $t$ .  $\square$

**COROLLARY 5.11.** *When  $X \subseteq S_0$  is the initiating set for a well-behaved bipartite MAC instance, we have*

$$(5.5) \quad \begin{aligned} X_t &\subseteq X_{t+1} \subseteq \cdots \subseteq X_\infty \subseteq S_0, \\ \bar{X}_t &\subseteq \bar{X}_{t+1} \subseteq \cdots \subseteq \bar{X}_\infty \subseteq S_1, \end{aligned}$$

where  $S_0 \cap S_1 = \emptyset$ .

This implies that for well-behaved instances of bipartite MAC, agents playing 0 and 1 are concentrated on either side of the bipartite graph, with decided agent  $v \in S_0$  playing 0 since the initiating set  $X \subseteq S_0$ .

**COROLLARY 5.12.** *Undecided agents can only change their strategies in alternate time steps.*

The second corollary follows from the first. When all agents playing 0 and 1 are aligned on different sides of the bipartition  $\{S_0, S_1\}$ , we need undecided agents on  $S_0$  to wait for undecided agents on  $S_1$  to change their strategies, before they can update their strategies again. This unique nature of the anticoordination effect in well-behaved instances of MAC arises as a response to the asymmetric control that we exert, i.e., enforce  $X \subset S_0$  to play 0 at  $t = 1$ . In the context of the epidemic game, controlling one type of agent can mean that we can only choose to isolate individuals who are sick. The agents playing zero on one side of the bipartite graph (set  $S_0$ ) reinforce the decisions of agents playing 1 on the other side of the bipartite network (set  $S_1$ ), and vice versa, till there is convergence. We call this a *decision cascade*. Having payoff interaction constants in the range  $c_i \in (\frac{1}{|n(i)|}, 1)$  for all  $i \in V(\mathcal{G}_B)$  allows for the removal of *extremely insensitive* agents ( $c_i < \frac{1}{|n(i)|}$ ) who choose to play 1 no matter what their neighbors are doing. In the absence of these agents, the only *decision cascade* is triggered by the asymmetric control we exert at  $t = 1$ . This is the foundation of the well-behaved scenario, where agents playing 0 and 1 are concentrated on either side of the bipartite network.

For an illustration, see Figure 1 where the dynamics unfold on a well-behaved instance for MAC and the control set is confined to one side of the network. See that agents playing strategy levels 0 and 1 are concentrated on either side of the network. Initially, at  $t = 1$  all agents are undecided, which is a convergent action profile by itself in the absence of control. We introduce control in the next time step which triggers the dynamics and leads to agents updating their strategies to reach a new equilibrium profile for the dynamics.

**5.2.1. Monotonicity of well-behaved MAC instances.** We provide the proof of Theorem 5.6.

*Proof.* Given  $X \subseteq Y$ , we claim that any  $X_t \subseteq Y_t$ , where  $X_t$  refers to the set of agents in  $S_0$  that choose strategy 0 at time  $t$ . Initially, the property is true ( $t = 0$ ),

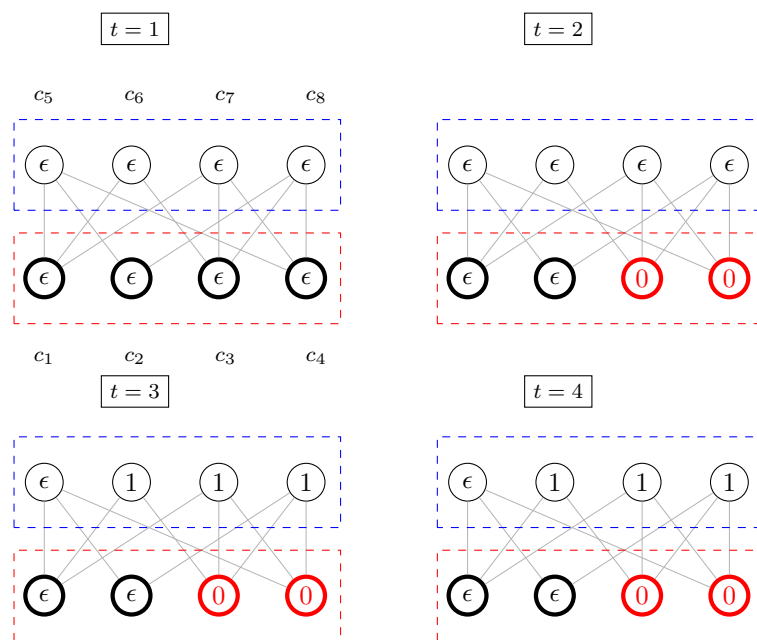


FIG. 1. We have a  $4 \times 4$  bipartite well-behaved instance for MAC, with payoff interaction constants for agents given as  $c_1 = 0.41, c_2 = 0.55, c_3 = 0.57, c_4 = 0.86, c_5 = 0.92, c_6 = 0.60, c_7 = 0.34, c_8 = 0.39$ . See that all  $c_i, i \in V$  satisfy  $c_i > 1/|n(i)|$ . Sets  $S_0$  and  $S_1$  are marked in red and blue boxes. At time  $t = 1$ , all agents are undecided. This is a fixed point of (3.1). We initiate dynamics by applying control to agents in set  $S_0$  (shown in red) at  $t = 2$ . The dynamics in (3.1) converge at  $t = 3$ . No agent following the dynamics would change their strategy beyond  $t = 3$ . (Figure in color online.)

i.e.,  $X_0 = X \subseteq Y = Y_0$ . All agents in  $S_1$  are undecided, whereas agent  $i \in S_0$  plays 0 if  $i \in Y$  or  $i \in X$ . Assume now that the property is true for some  $t$ , and let  $\bar{X}_t$  and  $\bar{Y}_t$  represent the corresponding sets of agents playing 1 in  $S_1$ , respectively. Following the dynamics, agents in set  $S_1$  update at  $t+1$  according to the actions of agents in  $S_0$  at time  $t$ . Since  $X_t \subseteq Y_t$ , then  $\bar{X}_{t+1} \subseteq \bar{Y}_{t+1}$  as any agent  $i \in S_1$  which plays 1 under the influence of zero-set  $X_t$  will have to choose 1 under the influence of zero-set  $Y_t$ . This again implies that  $X_{t+2} \subseteq Y_{t+2}$ . Since  $X_{t+1} = X_t$  and  $Y_{t+1} = Y_t$  (as the sets  $S_0$  and  $S_1$  update alternately, Corollary 5.12) the property holds for all  $t$ . Eventually, we have  $X_\infty \subseteq Y_\infty$  on convergence. Since all agents playing strategy 0 are in set  $S_0$  and agents in  $S_1$  are either 1 or undecided, the only edges deactivated are the ones incident to agents in set  $X_\infty$  and  $Y_\infty$ . Thus  $\mathcal{F}(X) \leq \mathcal{F}(Y)$ .  $\square$

**COROLLARY 5.13.** *The set of zeros  $X_t$  and  $Y_t$  for the processes initiated with zero sets  $X \subseteq Y \subseteq S_0$  satisfy  $X_t \subseteq Y_t \subseteq S_0$ .*

**COROLLARY 5.14.** *The set of decided agents  $X_t \cup \bar{X}_t$  and  $Y_t \cup \bar{Y}_t$  for the processes initiated with zero sets  $X$  and  $Y$ , respectively,  $X \subseteq Y \subseteq S_0$ , satisfy  $X_t \cup \bar{X}_t \subseteq Y_t \cup \bar{Y}_t \subseteq V(\mathcal{G}_B)$ .*

**5.3. Modularity of one-step influence.** We begin by defining the one-step influence function for an undecided agent (playing  $\epsilon$ ) following the local learning dynamics. Given the set of decided agents at a time step, the one-step influence quantifies the net neighbor influence on an undecided agent that persuades it to get

decided in the next time step. In the following, let  $(\mathcal{G}_B, c)$  be a well-behaved bipartite MAC instance with bipartition  $\{S_0, S_1\}$ , and let  $T_k$  be the set of decided agents (agents playing 0 or 1) in  $V(\mathcal{G}_B)$  at time step  $k$  following dynamics (3.1) initiated by zero-set  $T \subseteq S_0$ .

**DEFINITION 5.15.** *The one step influence experienced by an undecided agent  $v \in S_0 \setminus T_k$  at the next time step  $k+1$  of the dynamics is  $f_v^0 : 2^{V(\mathcal{G}) \setminus \{v\}} \rightarrow \mathbb{R}$  which is given by  $f_v^0(T_k) = \sum_{j \in n(v)} \lfloor a_j(k) \rfloor$ . Similarly, the one-step influence experienced by an undecided agent  $v \in S_1 \setminus T_k$  at the next time step  $k+1$  of the dynamics in (3.1) is  $f_v^1 : 2^{V(\mathcal{G}) \setminus \{v\}} \rightarrow \mathbb{R}$  given by  $f_v^1(T_k) = \sum_{j \in n(v)} \lceil a_j(k) \rceil$ .*

Following the concentration lemma, Lemma 5.10, we know that agents playing 0 and 1 will be concentrated on sides  $S_0$  and  $S_1$  of the graph, respectively. Thus, for an undecided agent in  $S_0$ , which cannot decide to play 1, the decision rule that decides whether it will select to play 0 is if  $1 < c_v \sum_{j \in n(v)} \lfloor a_j(k) \rfloor$  is satisfied. Thus, following our definition, if the net influence  $c_v f_v^0(T_k)$  on agent  $v \in S_0 \setminus T_k$ , which is the one-step influence scaled by the individual agent's payoff interaction constant, exceeds threshold 1, then agent  $v$  switches to playing 0, i.e.,  $a_v(k+1) = 0$  following the rules in dynamics (3.1). Similarly, following our Definition 5.15, for undecided agent  $v \in S_1 \setminus T_k$  at time  $k$ , following the update rules in (3.1), if the net influence  $c_v f_v^1(T_k)$  is less than the threshold 1, the agent gets decided and switches to playing 1. For well-behaved bipartite instances of MAC, it is easy to see that the influence functions have the closed-form representation given by

$$(5.6) \quad \begin{aligned} f_v^0(T_k) &= |n(v) \cap T_k|, \\ f_v^1(T_k) &= |n(v) \setminus T_k| \end{aligned}$$

for  $v \in S_0 \setminus T_k$  and  $v \in S_1 \setminus T_k$  respectively. Notice that in the bipartite graph  $\mathcal{G}_B$ , agent  $v \in S_0$  implies  $n(v) \subseteq S_1$ , and agents in  $S_1$  cannot choose to play 0 from Lemma 5.10. Thus  $\sum_{j \in n(v)} \lfloor a_j \rfloor$  is equal to the number of agents in the neighborhood of  $v$  playing 1 (as  $\lfloor \epsilon \rfloor = 0$  for  $\epsilon \in (0, 1)$ ), which is given by cardinality of set  $n(v) \cap T_k$ . Similarly,  $\sum_{j \in n(v)} \lceil a_j \rceil$  is equal to the number of undecided agents in the neighborhood of  $v \in S_1$  (the set  $n(v) \setminus T_k$ ), since  $n(v) \subseteq S_0$  and decided agents in  $S_0$  play 0.

We now state two important results.

**LEMMA 5.16.** *The set functions  $f_v^0$  and  $f_v^1$  are modular, i.e., the relation in (5.3) holds with equality. The set functions  $f_v^0$  and  $f_v^1$  are monotone increasing and monotone decreasing, respectively.*

We provide the proofs in the appendix. In the next part of this paper, we show how modularity of the one-step influence function leads to submodularity of the function  $\mathcal{F}(\cdot)$  described in (4.1).

**5.4. The coupling argument.** We develop the mathematical machinery needed for the coupling argument. We are given a random, well-behaved instance of bipartite MAC  $(\mathcal{G}_B, c)$  with the bipartition  $\{S_0, S_1\}$ . Following Lemma 5.9, all agents are undecided initially. When we initiate a *decision cascade* by controlling certain agents (let's say  $S \subset S_0$ ), the randomness of the payoff interaction constants imparts stochasticity to the set of agents that get decided (choose 0 or 1) over time. Here  $S$  is referred to as the initiating set for the decision cascade. Also recall that all controlled agents play 0 at all times. Since the dynamics converge in  $n$  time steps, it is enough to consider the time interval  $0 \leq k \leq n-1$ . If we now run the dynamics in (3.1) given an initiating set  $S \subset S_0$ , we represent the distribution of the stochastic process  $\mathbb{T} = (T_k)_{k=0}^{n-1}$  where  $T_k \subseteq V(\mathcal{G}_B)$  represents the set of decided agents at time  $k$ ,  $T_0 = S$  with  $\mathcal{Q}(S)$ .

In the following, we prove equivalence of the process when we consider  $R^{(1)}, \dots, R^{(K)}$  a partition of  $S$ , i.e.,  $\bigcup_{i=1}^K R^{(i)} = S$ , and sequentially apply control to the sets  $R^{(j)}$ , starting from  $R^{(1)}$  by letting the process run with control set till convergence for each  $j = 2, \dots, K$  before adding  $R^{(j)}$  to the control set. Let  $\mathbb{V} = (V_k)_{k=0}^{Kn-1}$  represent this process where  $V_k \subset V(\mathcal{G}_B)$  represents the set of decided agents at time  $k$ . Then, for  $1 \leq j \leq K$ , we set  $(V_k)_{k=(j-1)n}^{jn-1} \sim \mathcal{Q}(V_{(j-1)n-1} \cup R^{(j)})$ , where  $V_{-1} = \emptyset$  and  $\mathcal{Q}(R^{(1)}, \dots, R^{(K)})$  denotes the distribution of  $\mathbb{V}$ . Our first step is to show that processes  $\mathbb{T}$  and  $\mathbb{V}$  lead to the same end result on convergence.

LEMMA 5.17.  $V_{Kn-1}$  has the same distribution as  $T_{n-1}$  for any partition of the initiating set  $\bigcup_{i=1}^K R^{(i)} = S \subset S_0$ .

*Proof.* We have  $\mathbb{T} = (T_k)_{k=0}^{Kn-1} \sim \mathcal{Q}(S)$  and  $\mathbb{V} = (V_k)_{k=0}^{Kn-1} \sim \mathcal{Q}(R^{(1)}, \dots, R^{(K)})$ . Let  $\mathbb{V}' = (V'_k)_{k=0}^{Kn-1} \sim \mathcal{Q}(S, \emptyset, \dots, \emptyset)$ . Let  $\mathbb{V}'' = (V''_k)_{k=0}^{Kn-1} \sim \mathcal{Q}(\emptyset, \dots, \emptyset, S)$ . Since the process is monotonic (see Theorem 5.6 and Corollary 5.14), by induction on the  $K$  stages we have

$$(5.7) \quad V''_{Kn-1} \subseteq V_{Kn-1} \subseteq V'_{Kn-1}.$$

However, we have  $V''_{Kn-1} = V'_{Kn-1} = T_{n-1}$ . Thus we have  $T_{n-1} = V_{Kn-1}$ .  $\square$

According to the above result, it does not matter whether we control all the agents in our control set right from the start, or we apply control progressively in stages by working through partitions, as eventually either process will lead to the same distribution of the decided agents on convergence.

Next, we consider  $S \cup T \subset S_0$  as the initiating set. As per the above result, we can consider a partition of the set  $S$ ,  $R^{(1)}, \dots, R^{(K)}$ , and apply control progressively for all stages until all of  $S$  is controlled, i.e., starting from  $R^{(1)}$ , let the dynamics run with control sets  $\bigcup_{i=1}^{j-1} R^{(i)}$  before adding  $R^{(j)}$  to the control set, and do this till we reach  $S = \bigcup_{i=1}^K R^{(i)}$  as the control set. Next, we can add nodes from  $T$  to obtain the same process as initially controlling  $S \cup T$  again by the above result. In the following definition, we consider an alternative *selection rule* instead of adding  $T$  to the control set.

DEFINITION 5.18. Given a random well-behaved instance of bipartite MAC  $(\mathcal{G}_B, c)$ , with bipartition  $\{S_0, S_1\}$  and  $c_i \sim \text{uniform}(\frac{1}{|n(i)|}, 1)$  for all  $i \in V(\mathcal{G}_B)$ , let  $R^{(1)}, \dots, R^{(K)}$  be a partition of  $S \subseteq S_0$ , and let  $T \subseteq S_0 \setminus S$ . We consider the initiating set  $S \cup T \subseteq S_0$  and controlled agents play 0. We define the process that tracks the set of decided agents over time as  $\mathbb{T}^- = (T_k^-)_{k=0}^{(K+1)n-1} \sim \mathcal{Q}_-(R^{(1)}, \dots, R^{(K)}; T)$  where,

1. Let  $(T_k^-)_{k=0}^{Kn-1} \sim \mathcal{Q}(R^{(1)}, \dots, R^{(K)})$ .
2. Set  $T_{Kn}^- = T_{Kn-1}^- \cup T$ .
3. At time  $(Kn+1) \leq k \leq (K+1)n-1$  initialize  $T_k^- = T_{k-1}^-$  and add to  $T_k^-$  the set of nodes in  $v \in V(\mathcal{G}_B) \setminus T_{k-1}^-$  such that

$$(5.8) \quad \begin{aligned} |n(v)|^{-1} + f_v^0(T_{Kn-1}^-)^{-1} - f_v^0(T_{k-1}^-)^{-1} &> c_v \text{ for } v \in S_0, \\ 1 + f_v^1(T_{Kn-1}^-)^{-1} - f_v^1(T_{k-1}^-)^{-1} &< c_v \text{ for } v \in S_1 \end{aligned}$$

until we run out of nodes to add.

We refer to the left-hand side of rule 3. as the selection quotient.

We show next that the *selection rule* we propose is distribution preserving, which means that following the rule we would reach the same distribution of the random set of agents getting decided on convergence as when we add  $T$  to the pre-existing control set and run the dynamics.

LEMMA 5.19. *The stochastic process  $\mathbb{T}^- \sim \mathcal{Q}_-(R^{(1)}, \dots, R^{(K)}; T)$  as defined in 5.18 has the same distribution as the process  $\mathbb{T} = (T_k)_{k=0}^{(K+1)n-1} \sim \mathcal{Q}(R^{(1)}, \dots, R^{(K)}, T)$ .*

*Proof.* We are given the payoff interaction constant  $c_v \sim \text{uniform}(|n(v)|^{-1}, 1)$  for each  $v \in V(\mathcal{G}_B)$ . We use the same payoff interaction constants for both processes. Both  $\mathbb{T}^-$  and  $\mathbb{T}$  progress exactly the same way till time  $k = Kn$ . That is for all  $0 \leq k \leq Kn$  we have  $T_k = T_k^-$ . We know from the concentration lemma, Lemma 5.10 that agents playing 0 and 1 will be positioned on sides  $S_0$  and  $S_1$  of the bipartition, whenever the initiating set is on side  $S_0$ . Since both processes  $T$  and  $T^-$  run the same way for the first  $K$  stages, we get  $T_{Kn-1} = T_{Kn-1}^-$ . Moreover, in either case we initialize  $T_{Kn} = T_{Kn}^- = T_{Kn-1} \cup T$ .

*Case 1.* For an agent  $v \in S_0, v \notin T_{Kn}$ , but  $v \in T_k$  for some time  $Kn+1 \leq k \leq (K+1)n-1$ , we need

$$(5.9) \quad f_v^0(T_{k-1})^{-1} \leq c_v < f_v^0(T_{Kn-1})^{-1}.$$

This is from application of the rules in the dynamics (3.1) while recognizing that an agent  $v \in S_0$  is only eligible to get decided by choosing to play 0. Equation (5.9) says that the net influence  $c_v f_v^0(T_{Kn-1})$  did not exceed threshold 1, while the net influence  $c_v f_v^0(T_{k-1})$  exceeds threshold 1 for some  $k \geq Kn+1$ . Conditioning only on  $v \notin T_{Kn}$ , we have  $c_v \sim \text{uniform}(|n(v)|^{-1}, f_v^0(T_{Kn-1})^{-1})$ . Define the shadow learner as

$$(5.10) \quad \tilde{c}_v^0 := |n(v)|^{-1} + f_v^0(T_{Kn-1})^{-1} - c_v.$$

Then,  $\tilde{c}_v^0$  is also distributed uniformly in  $(|n(v)|^{-1}, f_v^0(T_{Kn-1})^{-1})$ . Now, since  $v \in T_k$ , then  $c_v f_v^0(T_{k-1}) > 1$  must be satisfied. In constructing the set  $T_k$ , if we select agents  $v \notin T_{k-1}$  such that  $\tilde{c}_v^0 f_v^0(T_{k-1}) > 1$ , then the distribution of the set of decided agents  $T_k$  will not be altered as both  $c_v$  and  $\tilde{c}_v^0$  are identically distributed and, therefore, can serve as proxies for each other. This means that if we pick agent  $v$  using the shadow learner  $\tilde{c}_v^0$  rather than its original payoff interaction constant  $c_v$ , we preserve the distribution of  $T_k$ . Plugging in the value of  $\tilde{c}_v^0$  in  $\tilde{c}_v^0 f_v^0(T_{k-1}) > 1$ , we get

$$(5.11) \quad |n(v)|^{-1} + f_v^0(T_{Kn-1})^{-1} - f_v^0(T_{k-1})^{-1} > c_v,$$

the equivalent of (5.8) for process  $\mathbb{T}$  for agents on side  $S_0$ , which is guaranteed to not alter the distribution of  $T_k$  if we would just follow the dynamics (3.1).

*Case 2.* Again, for an agent  $u \in S_1, u \notin T_{Kn}, c_u f_u^1(T_{Kn-1}) \geq 1$  must hold, following updates (3.1). Conditioning on  $u \notin T_{Kn}$  we must have  $c_u \sim \text{uniform}(f_u^1(T_{Kn-1})^{-1}, 1)$ . Define shadow learner  $\tilde{c}_u^1$  as follows:

$$(5.12) \quad \tilde{c}_u^1 := 1 + f_u^1(T_{Kn-1})^{-1} - c_u.$$

Then  $\tilde{c}_u^1$  is also uniformly distributed in  $(f_u^1(T_{Kn-1})^{-1}, 1)$ . If  $u \in T_k$  (agent  $u$  gets decided to play 1), for some  $k$  in  $Kn+1 \leq k \leq (K+1)n-1$ , then following dynamics (3.1)  $c_u f_u^1(T_{k-1}) < 1$  must hold. Selecting agent  $u$  to be added to set  $T_k$  by alternatively checking for the condition  $\tilde{c}_u^1 f_u^1(T_{k-1}) < 1$  cannot alter the distribution of the set  $T_k$  as  $c_u$  and  $\tilde{c}_u^1$  are identically distributed. Now, plugging in the value of  $\tilde{c}_u^1$  in  $\tilde{c}_u^1 f_u^1(T_{k-1}) < 1$  gives us

$$(5.13) \quad 1 + f_u^1(T_{Kn-1})^{-1} - f_u^1(T_{k-1})^{-1} < c_u.$$

the equivalent of the selection rule for process  $\mathbb{T}$ , for agents on side  $S_1$ . This completes the proof.  $\square$

Next, we provide two technical results related to monotone submodularity and coupling which we use in the proof of Theorem 5.5.

LEMMA 5.20. Let  $h: 2^V \rightarrow \mathbb{R}_+$  be monotone and submodular. Then if  $I \subseteq I' \subseteq V$  and  $J \subseteq J' \subseteq V$  are given, we have

$$(5.14) \quad h(I \cup J') - h(I) \geq h(I' \cup J) - h(I').$$

*Proof.*

$$(5.15) \quad \begin{aligned} h(I \cup J') - h(I) &\geq h(I \cup J) - h(I) = h(I \cup (J \setminus I)) - h(I) \\ &\geq h(I \cup (J \setminus I) \cup (I' \setminus (I \cup J))) - h(I \cup (I' \setminus (I \cup J))) \\ &= h(I' \cup J) - h(I'). \end{aligned}$$

Here the first inequality is due to monotonicity, and the second inequality is due to submodularity of  $h$ .  $\square$

LEMMA 5.21. Let  $(\mathcal{G}_B, c)$  be a random, well-behaved instance of bipartite MAC, with the bipartition  $\{S_0, S_1\}$ ,  $c_i \sim \text{uniform}(|n(i)|^{-1}, 1)$ . Consider initiating sets  $A, B \subseteq S_0$ ,  $Z = A \cap B$ , and  $D = A \cup B$ . Let  $A_t \subseteq V(\mathcal{G}_B)$  represent the decided agents at time  $t$  when initiation set in (3.1) is  $A$ . We let the dynamics in (3.1) unfold, and then couple the following processes:

$$\begin{aligned} \mathbb{A} &= (A_t)_{t=0}^\infty \sim \mathcal{Q}(A), & \mathbb{B} &= (B_t)_{t=0}^\infty \sim \mathcal{Q}(B), \\ \mathbb{Z} &= (Z_t)_{t=0}^\infty \sim \mathcal{Q}(Z), & \mathbb{D} &= (D_t)_{t=0}^\infty \sim \mathcal{Q}(D) \end{aligned}$$

in such a way that  $Z_\infty \subseteq A_\infty \cap B_\infty$  and  $D_\infty \subseteq A_\infty \cup B_\infty$ . If such a coupling exists for any selections of  $A, B$ , then we obtain submodularity (5.4) of  $\mathbb{E}_c \mathcal{F}(\cdot)$  defined in (4.1).

The proof is in the appendix. We now have all the elements in place for us to prove Theorem 5.5.

### 5.5. Proof of Theorem 5.5.

*Proof.* Our endeavor is to construct the exact coupling outlined in Lemma 5.21 which leads to submodularity in expectation of  $\mathcal{F}$  defined in (4.1). Take initiation sets  $A, B \subseteq S_0$ . Let  $Z = A \cap B$  and  $D = A \cup B$ . Let  $A_t \subseteq S_0$  represent the zero-set at time  $t$  when we initiate the dynamics in (3.1) with the control set  $A$ . We let the dynamics in (3.1) unfold on a bipartite MAC instance  $(\mathcal{G}_B, c)$  with these four control sets, i.e.,  $A, B, Z$ , and  $D$ , respectively, as initiations, which lead to the following four stochastic processes recording the set of agents playing action level 0 over time:

$$(5.16) \quad \begin{aligned} \mathbb{A} &= (A_t)_{t=0}^\infty \sim \mathcal{Q}(A), & \mathbb{B} &= (B_t)_{t=0}^\infty \sim \mathcal{Q}(B), \\ \mathbb{Z} &= (Z_t)_{t=0}^\infty \sim \mathcal{Q}(Z), & \mathbb{D} &= (D_t)_{t=0}^\infty \sim \mathcal{Q}(D). \end{aligned}$$

The four processes above can be equivalently represented as

$$(5.17) \quad \begin{aligned} \mathbb{A} &= (A_k)_{k=0}^{3n-1} \sim \mathcal{Q}(A \cap B, A \setminus B, \emptyset), \\ \mathbb{B} &= (B_k)_{k=0}^{3n-1} \sim \mathcal{Q}_-(A \cap B, \emptyset; B \setminus A), \\ \mathbb{Z} &= (Z_k)_{k=0}^{3n-1} \sim \mathcal{Q}(A \cap B, \emptyset, \emptyset), \\ \mathbb{D} &= (D_k)_{k=0}^{3n-1} \sim \mathcal{Q}_-(A \cap B, A \setminus B; B \setminus A), \end{aligned}$$

using Lemmas 5.17 and 5.19 and the finite time convergence guarantee for the dynamics. We will now show that our proposed coupling in (5.17) will have  $Z_k \subseteq A_k \cap B_k$  and  $D_k \subseteq A_k \cup B_k$  for all  $0 \leq k \leq 3n-1$ . We use the same payoff interaction constants for all the four processes. By construction, for all  $0 \leq k \leq 2n-1$ ,  $B_k = Z_k \subseteq A_k$ ,



and  $Z_k = A_k \cap B_k$ . Similarly, for all  $0 \leq k \leq 2n-1$ , we have  $D_k = A_k$  which implies  $D_k \subseteq A_k \cup B_k$ . Thus for all *update times*  $0 \leq k \leq 2n-1$  our result holds. We need to additionally show that the inequalities hold for all times  $2n \leq k \leq 3n-1$ . Consider the following statements.

*Statement 1:*

$$(5.18) \quad D_k \setminus D_{2n-1} \subseteq B_k \setminus B_{2n-1}.$$

*Statement 2a:*

$$(5.19) \quad f_v^0(B_{2n-1})^{-1} - f_v^0(B_k)^{-1} \geq f_v^0(D_{2n-1})^{-1} - f_v^0(D_k)^{-1}, \text{ when } v \in S_0.$$

*Statement 2b:*

$$(5.20) \quad f_v^1(B_{2n-1})^{-1} - f_v^1(B_k)^{-1} \leq f_v^1(D_{2n-1})^{-1} - f_v^1(D_k)^{-1}, \text{ when } v \in S_1.$$

First, we shall use the principle of mathematical induction to show that these three statements are valid for all time  $2n \leq k \leq 3n-1$ .

Consider  $k = 2n$ . For the first statement, we have  $D_{2n} = D_{2n-1} \cup (B \setminus A)$ ,  $B_{2n} = B_{2n-1} \cup (B \setminus A)$ . That means we have

$$\begin{aligned} D_{2n} \setminus D_{2n-1} &= (D_{2n-1} \cup (B \setminus A)) \setminus D_{2n-1} \\ &= \{(B \setminus A) \setminus D_{2n-1}\} \cup \{(D_{2n-1}) \setminus D_{2n-1}\} = (B \setminus A) \setminus D_{2n-1}. \end{aligned}$$

Now using  $B_{2n-1} \subseteq D_{2n-1}$ , we have

$$\begin{aligned} D_{2n} \setminus D_{2n-1} &\subseteq (B \setminus A) \setminus B_{2n-1} \\ &= \{(B \setminus A) \setminus B_{2n-1}\} \cup \{B_{2n-1} \setminus B_{2n-1}\} \\ (5.21) \quad &= (B_{2n-1} \cup (B \setminus A)) \setminus B_{2n-1} = B_{2n} \setminus B_{2n-1}. \end{aligned}$$

Next, we prove Statement 2a for  $k = 2n$ . Consider  $v \in S_0$ . Now in (5.14), plug in  $I = B_{2n-1}$ ,  $I' = D_{2n-1}$ ,  $J = J' = B \setminus A$ . Noting that  $f_v^0$  is monotone increasing and modular (by Lemma 5.16), we have

$$(5.22) \quad f_v^0(B_{2n}) - f_v^0(B_{2n-1}) \geq f_v^0(D_{2n}) - f_v^0(D_{2n-1}).$$

Also, as a consequence of the monotonicity of  $f_v^0$  (see Lemma 5.16,  $f_v^0$  is monotone increasing) we have

$$(5.23) \quad f_v^0(B_{2n})f_v^0(B_{2n-1}) \leq f_v^0(D_{2n})f_v^0(D_{2n-1}).$$

Dividing the left (right) hand side of (5.22) with the left- (right-) hand side of (5.23), we get (5.20) for  $k = 2n$ .

Now we prove Statement 2b for  $k = 2n$  in a similar way. Consider  $v \in S_1$ . We know from Lemma 5.16 that  $f_v^1$  is modular and monotone decreasing. In (5.14), plug in  $I = B_{2n-1}$ ,  $I' = D_{2n-1}$ ,  $J = J' = B \setminus A$  like before. Then we get

$$(5.24) \quad f_v^1(B_{2n}) - f_v^1(B_{2n-1}) \leq f_v^1(D_{2n}) - f_v^1(D_{2n-1}).$$

Moreover, using Lemma 5.16, we have

$$(5.25) \quad f_v^1(B_{2n})f_v^1(B_{2n-1}) \geq f_v^1(D_{2n})f_v^1(D_{2n-1}).$$

Dividing the left- (right-) hand side of (5.22) with the left- (right-) hand side of (5.23), we get (5.20) for  $k = 2n$ .

Assume now that the statements (5.18), (5.19), and (5.20) hold true at time  $k$ . We can write the following inequalities by adding  $|n(v)|^{-1}$  and 1, respectively, to both sides of Statements 2a and 2b at time  $k$  (for  $v \in S_0$  and  $v \in S_1$ , respectively),

$$(5.26) \quad |n(v)|^{-1} + f_v^0(B_{2n-1})^{-1} - f_v^0(B_k)^{-1} \geq |n(v)|^{-1} + f_v^0(D_{2n-1})^{-1} - f_v^0(D_k)^{-1}$$

$$(5.27) \quad 1 + f_v^1(B_{2n-1})^{-1} - f_v^1(B_k)^{-1} \leq 1 + f_v^1(D_{2n-1})^{-1} - f_v^1(D_k)^{-1}.$$

For  $v \in S_0$ , inequality (5.26) says that the selection quotient for the process  $\mathbb{B}$  is larger than that for process  $\mathbb{D}$ . Recall that the selection quotient was the left-hand side of rule 3. in definition 5.18. See that for both processes  $\mathbb{B}$  and  $\mathbb{D}$  we select elements using selection rule 3. in definition 5.18 in the last stage (for  $2n \leq k \leq 3n-1$ ). Since process  $\mathbb{B}$  features a larger selection quotient for agents in  $S_0$ , any  $v \in S_0 \setminus D_k$  that is prescribed to be added to set  $D_{k+1}$ , i.e.,  $|n(v)|^{-1} + f_v^0(D_{2n-1})^{-1} - f_v^0(D_k)^{-1} > c_v$  is also prescribed to be added to set  $B_{k+1}$ , i.e.,  $|n(v)|^{-1} + f_v^0(B_{2n-1})^{-1} - f_v^0(B_k)^{-1} > c_v$ , unless it is already in set  $B_k$ . For  $v \in S_1$ , inequality (5.27) says that the selection quotient for the process  $\mathbb{B}$  is smaller than that for process  $\mathbb{D}$ . As a result, any  $v \in S_1 \setminus D_k$  that is prescribed to be added to set  $D_{k+1}$ , i.e.,  $1 + f_v^1(D_{2n-1})^{-1} - f_v^1(D_k)^{-1} < c_v$  is also prescribed to be added to set  $B_{k+1}$ , i.e.,  $1 + f_v^1(B_{2n-1})^{-1} - f_v^1(B_k)^{-1} < c_v$ .

Since  $D_k \setminus D_{2n-1} \subseteq B_k \setminus B_{2n-1}$  by assumption, this further implies that

$$(5.28) \quad D_{k+1} \setminus D_{2n-1} \subseteq B_{k+1} \setminus B_{2n-1}.$$

This is exactly Statement 1 holding at time  $k+1$ . We now plug in  $I = B_{2n-1} \subseteq I' = D_{2n-1}$  and  $J = D_{k+1} \setminus D_{2n-1} \subseteq J' = B_{k+1} \setminus B_{2n-1}$  in (5.14) to get that

$$(5.29) \quad f_v^0(B_{k+1}) - f_v^0(B_{2n-1}) \geq f_v^0(D_{k+1}) - f_v^0(D_{2n-1})$$

using that  $f_v^0$  is modular and monotone increasing (Lemma 5.16). Also, since

$$(5.30) \quad f_v^0(B_{k+1})f_v(B_{2n-1}) \leq f_v(D_{k+1})f_v(D_{2n-1})$$

by monotonicity of  $f_v^0$ , we get

$$(5.31) \quad f_v^0(B_{2n-1})^{-1} - f_v^0(B_{k+1})^{-1} \geq f_v^0(D_{2n-1})^{-1} - f_v^0(D_{k+1})^{-1}$$

by dividing the left- (right-) hand side of inequation (5.29) with the inequation (5.30). Adding  $n(v)^{-1}$  on both sides, we recover Statement 2a at time  $k+1$ . Similarly, plugging in  $I = B_{2n-1} \subseteq I' = D_{2n-1}$  and  $J = D_{k+1} \setminus D_{2n-1} \subseteq J' = B_{k+1} \setminus B_{2n-1}$  in (5.14), and using the fact that  $f_v^1$  is modular and monotone decreasing, we get

$$(5.32) \quad f_v^1(B_{k+1}) - f_v^1(B_{2n-1}) \leq f_v^1(D_{k+1}) - f_v^1(D_{2n-1}).$$

Moreover,

$$(5.33) \quad f_v^1(B_{k+1})f_v^1(B_{2n-1}) \geq f_v^1(D_{k+1})f_v^1(D_{2n-1})$$

since  $f_v^1$  is monotone decreasing. Combining inequalities (5.32) and (5.33), we get

$$(5.34) \quad f_v^1(B_{2n-1})^{-1} - f_v^1(B_{k+1})^{-1} \leq f_v^1(D_{2n-1})^{-1} - f_v^1(D_{k+1})^{-1}.$$

Adding 1 to either side reveals Statement 2b holding true at time  $k+1$ .

Thus Statements 1, 2a, and 2b are true for all update times, and all intermediate time steps between two update time steps. Now, for  $2n \leq k \leq 3n - 1$ , we have  $A_k = D_{2n-1}$ . Also  $D_k \setminus D_{2n-1} \subseteq B_k \setminus B_{2n-1} \subseteq B_k$  which implies that

$$(5.35) \quad \begin{aligned} (D_k \setminus D_{2n-1}) \cup D_{2n-1} &\subseteq A_k \cup B_k, \\ D_k &\subseteq A_k \cup B_k. \end{aligned}$$

Also,  $Z_k \subseteq A_k \cap B_k$  holds from the construction for this time range. As the dynamics converge by  $k = 3n - 1$  by the finite time convergence guarantee established in [11], we have  $Z_\infty \subseteq A_\infty \cap B_\infty$  and  $D_\infty \subseteq A_\infty \cup B_\infty$ . We have successfully constructed the coupling we need. Now, the main result in 5.5 follows from Lemma 5.21.  $\square$

**6. Submodularity violation: An example.** We shall now construct an example that shows that MAC is not submodular for every realization of the payoff interaction constants within a selected range.

*Example 6.1.* Consider a complete bipartite graph  $\mathcal{G} = K_{N,N}$  with the bipartition  $\{V_0, V_1\}$  such that  $E(\mathcal{G})$  is the set of all unordered pairs  $\{u, v\}$  such that  $u \in V_0$  and  $v \in V_1$ . Let  $|V_0| = |V_1| = N \geq 4$ . We refer to this MAC instance as  $(K_{N,N}, c)$ . See Figure 2 for a visual on the counterexample. Additionally, let the payoff interaction constants satisfy  $c_v \in (\frac{1}{N}, \frac{1}{N-1})$  for all  $v \in V_0$  and  $c_v \in (\frac{1}{N-1}, \frac{1}{N-2})$  for  $v \in V_1$ . Observe that any MAC instance  $(K_{N,N}, c)$  so constructed, with the  $c$ 's in the above range is well-behaved by construction, i.e.,  $c_v > 1/|n(i)|$  for all vertices  $v \in V(\mathcal{G})$ .

Next, we show that the MAC objective  $\mathcal{F}$  as defined in (4.1) is not submodular whenever payoff constants satisfy  $c_v \in (\frac{1}{N}, \frac{1}{N-1})$  for all  $v \in V_0$  and  $c_v \in (\frac{1}{N-1}, \frac{1}{N-2})$  for  $v \in V_1$ . Since the instance satisfies the assumptions of well-behavedness,  $\epsilon$  is a fixed point of the dynamics, without any external control. Consider the empty control set  $X = \emptyset$ . When  $X$  is controlled,  $\epsilon$  continues to be the fixed point. Thus  $\mathcal{F}(X) = 0$ . Let  $u \in V_0 \setminus X$ . When  $X \cup \{u\}$  is controlled, for all agents  $v \in V_1$ ,  $c_v(\lceil a_{n(v)} \rceil) = c_v(N-1) > 1$  since  $c_v > \frac{1}{N-1}$  and  $c_v(\lfloor a_{n(v)} \rfloor) = c_v(0) = 0 < 1$  is satisfied. Thus all such agents  $v \in V_1$  stay undecided following dynamics. Thus, for all  $v \in V_0$ , in the next time step, nothing changes in terms of neighbor influence, and  $v \in V_0 \setminus \{u\}$  stays undecided. Thus,  $\mathcal{F}(X \cup \{u\}) = N$ .

Let  $y \in V_0, y \neq u$ . Define  $Y = \{y\}$ . Then  $X \subset Y$  by construction. When set  $Y$  is the control set, we get  $\mathcal{F}(Y) = N$  as this scenario is the same as the previous scenario (control of  $X \cup \{u\}$ ). Now, when the control set is  $Y \cup \{u\}$ , for all  $v \in V_1$ ,  $c_v(\lceil a_{n(v)} \rceil) = c_v(N-2) < 1$  as  $c_v < \frac{1}{N-2}$ , thus all agents in  $V_1$  play 1. In the following time step, for all agents  $v \in V_0 \setminus (Y \cup \{u\})$ ,  $c_v(\lfloor a_{n(v)} \rfloor) = c_v(N) > 1$  as  $c_v > \frac{1}{N}$ , and thus such agents choose 0. Thus, we get  $\mathcal{F}(Y \cup \{u\}) = N^2$ .

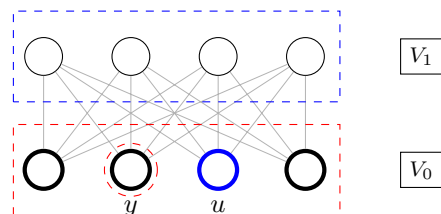


FIG. 2. We have a  $N \times N$  complete bipartite network, with the bipartition  $\{V_0, V_1\}$ , with selection for agent  $u$  shown in blue. Sets  $V_0$  and  $V_1$  are marked in red and blue boxes, respectively. (Figure in color online.)

Then the magnitude of violation of submodularity in this scenario is given by  $(\mathcal{F}(Y \cup \{u\}) - \mathcal{F}(Y)) - (\mathcal{F}(X \cup \{u\}) - \mathcal{F}(X)) = (N^2 - N) - (N - 0) = N^2$  which is equal to the number of edges  $|E(\mathcal{G})|$ . Thus  $\mathcal{F}$  is not submodular for any  $c_i$  value satisfying the given ranges.

When we constrain the payoff interaction constants to be within the above range, we see that for any instance, submodularity of MAC does not hold. Now, if we assume a uniform distribution on the payoff constants, that is  $c_v \sim \text{uniform}(\frac{1}{N}, \frac{1}{N-1})$  for all  $v \in V_0$ ,  $c_v \sim \text{uniform}(\frac{1}{N-1}, \frac{1}{N-2})$  for all  $v \in V_1$ , then for the sets  $X \subset Y \subset V_0$ ,  $u \in V_0 \setminus Y$  that we constructed above, the inequality  $\mathbb{E}_c(\mathcal{F}(X \cup \{u\}) - \mathcal{F}(X)) \geq \mathbb{E}_c(\mathcal{F}(Y \cup \{u\}) - \mathcal{F}(Y))$  does not hold. In fact, the magnitude of violation of the inequality is exactly  $N^2 = |E(K_{N,N})|$ . Thus submodularity in expectation does not arise with stronger conditions on the payoff constants, even if the instance is well-behaved. However, as we have established, when we allow the payoff interaction constants to belong to the entire spectrum, submodularity of MAC is established in expectation. The next section of this paper conveys the same intuition, where the graphs  $\mathcal{G}$  of limited size are generated using the Erdos-Renyi probabilistic model, with payoff interaction constants uniformly distributed over the entire range.

**7. Simulation.** MAC is concerned with the deactivation of as many edges as possible on convergence of learning dynamics, perturbed by controlling a select few agents. We define the *Inactivation Ratio* as the ratio of the number of edges inactivated on convergence to the number of active edges in the network before the dynamics progress. *Inactivation Ratio*, therefore, is a measure of how successful MAC is on the particular graph instance, given the control.

For our simulations we consider random Erdos-Renyi bipartite graph instances of the form  $K_{\frac{N}{2}, \frac{N}{2}}$  with edge formation probability equal to 0.3 and 0.8. Every realization of a network for given network sizes ( $N \in \{4, 8, 12, \dots, 40\}$ ) has a random topology with random payoff interaction constants for the agents. We sample the payoff interaction constants  $c$  for every agent from a uniform distribution between the limits  $(0, 1)$  (thus the instances are not necessarily well-behaved). The control budget is fixed at  $\lceil \frac{N}{10} \rceil$ , where  $N$  is the number of nodes in the graph. Given the budget, we select the control profile using a greedy cascade based algorithm (5.2). We compare its anticoordination performance with a control set generated using brute force search. In the brute force approach, we go over all the possible control sets for the budget specified and find the one that maximizes the number of edges deactivated. For every network instance, we calculate the *Inactivation Ratio* for both the control sets found using the greedy algorithm (5.2) and brute force search. For a given network size, we sample 40 instances of random bipartite graphs and evaluate the performance of the greedy algorithm.

We plot the average *Inactivation Ratio* against the size of network in Figure 3. The *Inactivation Ratio*, on average, for the greedy algorithm is close to the optimal inactivation at the current control budget for every network size. The maximum inactivation ratio gap for our simulations stands at 0.106 for networks with low edge formation probability and 0.095 for the networks with high edge formation probability, further highlighting the good performance of (5.2) in selecting control agents to induce anticoordination. In Figure 4, we show how long it takes to build the greedy control set using (5.2) for different control budgets and network sizes. All our simulations have been performed on Apple M1 CPU (Arm based, 8-core) with 16 GB of RAM.

*Remark 7.1* (computational complexity). The greedy algorithm has a time complexity of  $O(n^3)$ . This is because at each step we potentially scan  $O(n)$  agents for

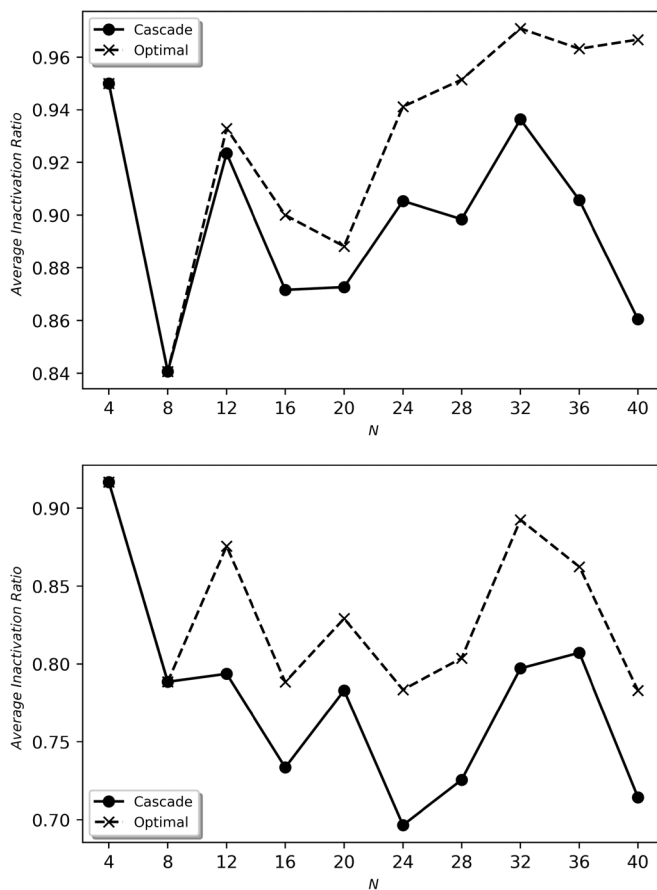


FIG. 3. Average Inactivation Ratio versus Graph Cardinality ( $N$ ) for Erdos-Renyi bipartite graphs with connection probability equal to 0.3 and 0.8, respectively. Control budget set at  $\lceil \frac{N}{10} \rceil$  for every network realization.

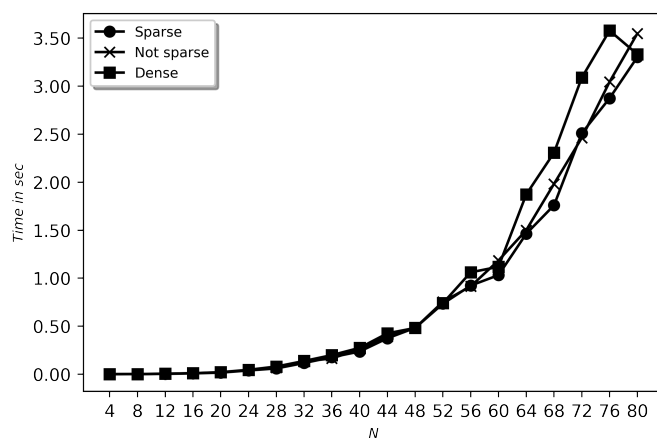


FIG. 4. Average time needed to construct greedy set for different network sizes and Erdos-Renyi edge formation probabilities 0.3, 0.5, 0.8 referred to as sparse, not sparse and locally large, respectively. Control budget fixed at  $\lceil \frac{N}{10} \rceil$  where  $N$  is the network size.

finding the one which causes the highest cascade. For each such scan, we run learning dynamics which takes at most  $O(n)$  time steps to converge [11] and each step of the learning dynamics involves  $O(n)$  updates.

**8. Conclusion.** We defined the combinatorial problem of selecting agents to control to maximize antcoordination among rational agents in a network game. Antcoordination is measured as the number of edges deactivated from the network on convergence of decentralized learning dynamics. We first show that MAC is NP-Hard to solve, in general, graphs. Our paper establishes that whenever MAC instances are bipartite, the objective in (4.1) is submodular in expectation. We discuss the stochastic nature of the dynamics, given random payoff interaction constants. We establish that the decision process has a unique property that instead of applying control in a consolidated fashion, we can apply it in a distributed manner which opens up possibilities for alternate equivalent descriptions of the dynamics (Lemmas 5.17 and 5.19). We then derive a selection rule which when substituted for the dynamics in the ultimate stage of control does not alter the distribution of decided agents on convergence. Using all these results in conjunction allows us to derive the coupling argument which guarantees that the four processes we couple have the properties we desire in Lemma 5.21. This leads directly to the proof of submodularity (Theorem 5.5). We also show that MAC is monotone increasing in the set of control agents. Using these results together, we provide the approximation guarantee for greedy node selection for MAC (Corollary 5.7). Our computational results confirm that greedy selection strategies are effective in producing near-optimal control sets for MAC in bipartite network scenarios.

## 9. Appendix.

### 9.1. Proof of Lemma 5.21.

*Proof.* Let  $e(A)_t$  and  $A_t^0$  be the set of edges deactivated and the set of decided agents playing 0, respectively, at time  $t$  when we initiate the dynamics in (3.1) with control set  $A$ . Let  $\delta(S)$  be the set of edges of  $\mathcal{G}_B$  which are incident to  $S$ , for some  $S \subseteq V(\mathcal{G}_B)$ . From Lemma 5.10, for a well-behaved instance at any time  $t$ , we have decided agents in  $S_0$  playing 0 and decided agents in  $S_1$  playing 1 whenever initiating sets are on side  $S_0$ , that is  $A_t^0 \subseteq S_0$  whenever  $A \subseteq S_0$ . All edges deactivated at time  $t$  are incident with the zero-set at time  $t$ , by definition of MAC (4.1). Thus  $e(A)_t = \delta(A_t^0)$ .

Assume now the coupling in Lemma 5.21 exists. As a consequence of the aforementioned coupling, we have  $Z_\infty \subseteq A_\infty \cap B_\infty$ , the relationship between the set of decided agents obtained via the initiating sets  $Z = (A \cap B)$ ,  $A$ , and  $B$ . Now since all decided agents playing 1 are on side  $S_1$  via the concentration lemma, Lemma 5.10, we get the following relationship between the set of agents playing 0:

$$\begin{aligned} Z_\infty^0 &= Z_\infty \setminus S_1 \subseteq (A_\infty \cap B_\infty) \setminus S_1 \\ (9.1) \quad &= (A_\infty \setminus S_1) \cap (B_\infty \setminus S_1) = A_\infty^0 \cap B_\infty^0, \end{aligned}$$

Similarly, we get

$$(9.2) \quad D_\infty^0 \subseteq A_\infty^0 \cup B_\infty^0$$

from the coupling assumption  $D_\infty \subseteq A_\infty \cup B_\infty$  and by subtracting set  $S_1$  from both sides. We now make the following claims.

*Claim 1.* We claim that  $Z_\infty^0 \subseteq A_\infty^0 \cap B_\infty^0$  (9.1) implies  $\delta(Z_\infty^0) \subseteq \delta(A_\infty^0) \cap \delta(B_\infty^0)$ .

(Proof of Claim 1). Because of Lemma 5.10, since all initiating sets are on side  $S_0$ , the set of edges deactivated are exactly the set of edges incident to the zero-set on convergence of dynamics. Thus  $\delta(Z_\infty^0) \subseteq \delta(A_\infty^0 \cap B_\infty^0)$ . Now we argue that  $\delta(A_\infty^0 \cap B_\infty^0) \subseteq \delta(A_\infty^0) \cap \delta(B_\infty^0)$ . To see why, assume for contradiction that there exists edge  $\tilde{e} \in \delta(A_\infty^0 \cap B_\infty^0)$ ,  $\tilde{e} \notin \delta(A_\infty^0) \cap \delta(B_\infty^0)$ . But  $\tilde{e}$  is incident to some node  $u \in A_\infty^0 \cap B_\infty^0$ , which implies that  $u \in A_\infty^0$  and  $u \in B_\infty^0$ . Thus  $\tilde{e}$  is incident to both  $A_\infty^0$  and  $B_\infty^0$ , or  $\tilde{e} \in \delta(A_\infty^0)$  and  $\tilde{e} \in \delta(B_\infty^0)$ , which is a contradiction.

*Claim 2.* The inequality  $D_\infty^0 \subseteq A_\infty^0 \cup B_\infty^0$  (9.2) implies  $\delta(D_\infty^0) \subseteq \delta(A_\infty^0) \cup \delta(B_\infty^0)$ .

(Proof of Claim 2). As in the previous claim,  $\delta(D_\infty^0) \subseteq \delta(A_\infty^0 \cup B_\infty^0)$  holds. We argue now that  $\delta(A_\infty^0 \cup B_\infty^0) \subseteq \delta(A_\infty^0) \cup \delta(B_\infty^0)$ . For contradiction, assume that indeed there exists  $\tilde{e} \in \delta(A_\infty^0 \cup B_\infty^0)$ ,  $\tilde{e} \notin \delta(A_\infty^0) \cup \delta(B_\infty^0)$ . But  $\tilde{e}$  is incident to some node  $u \in A_\infty^0 \cup B_\infty^0$ , which means either  $u \in A_\infty^0$  or  $u \in B_\infty^0$  (or both), which directly implies that either  $\tilde{e} \in \delta(A_\infty^0)$  or  $\tilde{e} \in \delta(B_\infty^0)$ , or both, which is a contradiction.

From Claim 1 we get that  $e(Z)_\infty \subseteq e(A)_\infty \cap e(B)_\infty$  (since  $e(Z)_\infty = \delta(Z_\infty^0)$ , and so on). Moreover, from Claim 2, we get that  $e(D)_\infty \subseteq e(A)_\infty \cup e(B)_\infty$ . Recall we can represent the objective in MAC (4.1) as  $\mathcal{F}(X) = |e(X)_\infty|$ . We can now write

$$\begin{aligned} \mathcal{F}(A) + \mathcal{F}(B) &= |e(A)_\infty| + |e(B)_\infty| = |e(A)_\infty \cap e(B)_\infty| + |e(A)_\infty \cup e(B)_\infty| \\ &\geq |e(Z)_\infty| + |e(D)_\infty| \\ (9.3) \quad &= \mathcal{F}(Z) + \mathcal{F}(D) = \mathcal{F}(A \cap B) + \mathcal{F}(A \cup B). \end{aligned}$$

The second equality is due to the fact that the cardinality function is modular, i.e., both submodular and supermodular, which means that for any two sets  $I_1, I_2$  subset of a ground set  $I_0$  we have  $|I_1| + |I_2| = |I_1 \cup I_2| + |I_1 \cap I_2|$ . The inequality is a result of monotonicity of the cardinality function. Taking expectation on either side, we get the desired result.  $\square$

## 9.2. Proof of Lemma 5.16.

*Proof.* Following definitions of  $f_v^0$  and  $f_v^1$  we define the marginal gains  $\Delta_u f_v^0(S) = f_v^0(S \cup \{u\}) - f_v^0(S)$  and  $\Delta_u f_v^1(S) = f_v^1(S \cup \{u\}) - f_v^1(S)$  for some set  $S$  which is a valid input to  $f_v^0$  and  $f_v^1$  and some vertex  $u \notin S$ . Now we expand  $\Delta_u f_v^0(S)$  as follows:

$$\begin{aligned} \Delta_u f_v^0(S) &= |n(v) \cap (S \cup \{u\})| - |n(v) \cap S| \\ (9.4) \quad &= |(n(v) \cap S) \cup (n(v) \cap \{u\})| - |n(v) \cap S|. \end{aligned}$$

Since  $u \notin S$ ,  $n(v) \cap S$  is disjoint from  $n(v) \cap \{u\}$ , which implies

$$\begin{aligned} \Delta_u f_v^0(S) &= |n(v) \cap S| + |n(v) \cap \{u\}| - |n(v) \cap S| \\ (9.5) \quad &= |n(v) \cap \{u\}|. \end{aligned}$$

Thus  $\Delta_u f_v^0(S)$  is independent of input  $S$ . Thus for any  $S \subseteq T$ ,  $\Delta_u f_v^0(S) = \Delta_u f_v^0(T)$ . Thus  $f_v^0$  is modular.

Again,

$$\begin{aligned} \Delta_u f_v^1(S) &= |n(v) \setminus (S \cup \{u\})| - |n(v) \setminus S| \\ (9.6) \quad &= |(n(v) \setminus S) \setminus \{u\}| - |n(v) \setminus S|. \end{aligned}$$

Since  $u \notin S$ , we can break the first term on the right-hand side as

$$\begin{aligned} \Delta_u f_v^1(S) &= |n(v) \setminus S| - |n(v) \cap \{u\}| - |n(v) \setminus S| \\ (9.7) \quad &= -|n(v) \cap \{u\}|. \end{aligned}$$

Thus  $\Delta_u f_v^1(S)$  is independent of input  $S$ , which means that  $f_v^1$  is modular. Finally, from the closed form representation (5.6) of  $f_v^0$  and  $f_v^1$  it is trivial to see that the two functions are monotone increasing and decreasing, respectively, since  $n(v)$  is a constant.  $\square$

## REFERENCES

- [1] M.-F. BALKAN, S. KREHBIEL, G. PILIOURAS, AND J. SHIN, *Minimally invasive mechanism design: Distributed covering with carefully chosen advice*, in Proceedings of the 51st IEEE Conference on Decision and Control (CDC), IEEE, 2012, pp. 2690–2695.
- [2] C. T. BAUCH AND A. P. GALVANI, *Social factors in epidemiology*, Science, 342 (2013), pp. 47–49.
- [3] D. BAUSO, H. TEMBINE, AND T. BASAR, *Opinion dynamics in social networks through mean-field games*, SIAM J. Control Optim., 54 (2016), pp. 3225–3257, <https://doi.org/10.1137/140985676>.
- [4] P. BOLZERN, P. COLANERI, AND G. DE NICOLAO, *Opinion influence and evolution in social networks: A Markovian agents model*, Automatica J. IFAC, 100 (2019), pp. 219–230.
- [5] Y. BRAMOULLÉ, *Anti-coordination and social interactions*, Games Econom. Behav., 58 (2007), pp. 30–49.
- [6] Y. BRAMOULLÉ AND R. KRANTON, *Public goods in networks*, J. Econom. Theory, 135 (2007), pp. 478–494.
- [7] P. N. BROWN AND J. R. MARDEN, *Studies on robust social influence mechanisms: Incentives for efficient network routing in uncertain settings*, IEEE Control Syst. Mag., 37 (2017), pp. 98–115.
- [8] F. BULLO, J. CORTÉS, AND B. PICCOLI, *Special issue on control and optimization in cooperative networks*, SIAM J. Control Optim., 48 (2009), pp. vii.
- [9] G. COMO, S. DURAND, AND F. FAGNANI, *Optimal targeting in super-modular games*, IEEE Trans. Automat. Control, 67 (2021), pp. 6366–6380.
- [10] S. DAS AND C. EKSIN, *Approximate submodularity of maximizing anticoordination in network games*, in Proceedings of the 61st IEEE Conference on Decision and Control (CDC), IEEE, 2022, pp. 3151–3157.
- [11] C. EKSIN AND K. PAARPORN, *Control of learning in anticoordination network games*, IEEE Trans. Control Netw. Syst., 7 (2020), pp. 1823–1835.
- [12] C. EKSIN, J. S. SHAMMA, AND J. S. WEITZ, *Disease dynamics in a stochastic network game: A little empathy goes a long way in averting outbreaks*, Sci. Rep., 7 (2017), 44122.
- [13] A. GALEOTTI, S. GOYAL, M. O. JACKSON, F. VEGA-REDONDO, AND L. YARIV, *Network games*, Rev. Econ. Stud., 77 (2010), pp. 218–244.
- [14] R. GUERS, C. LANGBORT, AND D. WORK, *On informational nudging and control of payoff-based learning*, IFAC Proceedings Volumes, 46 (2013), pp. 69–74.
- [15] J. HIRSHLEIFER, *From weakest-link to best-shot: The voluntary provision of public goods*, Public Choice, 41 (1983), pp. 371–386.
- [16] D. KEMPE, J. KLEINBERG, AND É. TARDOS, *Maximizing the spread of influence through a social network*, in Proceedings of the Ninth ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, 2003, pp. 137–146.
- [17] T. LIU, J. WANG, X. ZHANG, AND D. CHENG, *Game theoretic control of multiagent systems*, SIAM J. Control Optim., 57 (2019), pp. 1691–1709, <https://doi.org/10.1137/18M1177615>.
- [18] C. MCCONNELL, Y. MARGALIT, N. MALHOTRA, AND M. LEVENDUSKY, *The economic consequences of partisanship in a polarized era*, Amer. J. Polit. Sci., 62 (2018), pp. 5–18.
- [19] I. MENACHE AND A. OZDAGLAR, *Network Games: Theory, Models, and Dynamics*, Synth. Lect. Commun. Netw. 4, Springer Cham, 2011.
- [20] E. MOSSEL AND S. ROCH, *Submodularity of influence in social networks: From local to global*, SIAM J. Comput., 39 (2010), pp. 2176–2188.
- [21] P. NAGHIZADEH AND M. LIU, *Exit equilibrium: Towards understanding voluntary participation in security games*, in Proceedings of the 35th Annual IEEE International Conference on Computer Communications, IEEE, 2016, pp. 1–9.
- [22] G. L. NEMHAUSER, L. A. WOLSEY, AND M. L. FISHER, *An analysis of approximations for maximizing submodular set functions—I*, Math. Program., 14 (1978), pp. 265–294.
- [23] F. PARISE AND A. OZDAGLAR, *Analysis and interventions in large network games*, Annu. Rev. Control Robot. Auton. Syst., 4 (2020), pp. 455–486.



- [24] J. RIEHL, P. RAMAZI, AND M. CAO, *Incentive-based control of asynchronous best-response dynamics on binary decision networks*, IEEE Trans. Control Netw. Syst., 6 (2018), pp. 727–736.
- [25] F. SEZER, H. KHAZAEI, AND C. EKSIN, *Maximizing social welfare and agreement via information design in linear-quadratic-gaussian games*, IEEE Trans. Automat. Control, 69 (2023), pp. 463–470.
- [26] H. TAVAFOGHI AND D. TENEKETZIS, *Informational incentives for congestion games*, in 2017 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton), IEEE, 2017, pp. 1285–1292.
- [27] D. P. WILLIAMSON AND D. B. SHMOYS, *The Design of Approximation Algorithms*, Cambridge University Press, 2011.
- [28] Y. XIAO, J. PARK, AND M. VAN DER SCHAAR, *Intervention in power control games with selfish users*, IEEE J. Sel. Top. Signal Process., 6 (2011), pp. 165–179.