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Geography of surface bundles over surfaces

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Abstract

We construct symplectic surface bundles over surfaces with positive signatures for all but 19 possible pairs of fiber and base genera. Meanwhile, we determine the commutator lengths of a few new mapping classes.

1 Introduction

Surface bundles over surfaces constitute an interesting family of 4-manifolds, which is amenable to techniques from different areas of mathematics, such as algebraic geometry, symplectic topology and geometic group theory. Let Σ_g denote a closed orientable surface of genus g and let σ denote the signature of a 4-manifold. Surface bundles with $\sigma = 0$ are certainly easy to generate for any fiber and base genera; in fact, a Σ_g-bundle over Σ_h has $\sigma = 0$ in many situations, such as when $\pi_1(\Sigma_h)$ acts trivially on $H^*(\Sigma_g)$, when the fibration is hyperelliptic (in particular when $g \leq 2$), or when the base genus $h \leq 1$ [16, 18, 39]. Further, $\sigma \equiv 0 \pmod 4$ for any surface bundle [39].

Our goal in this article is to provide a comprehensive answer to the following outstanding geography problem:

For which pairs of $(g, h) \in \mathbb{N}^2$ *are there* Σ_g*–bundles over* Σ_h *with signature* $\sigma > 0$?

This problem on surface bundles has a long and rich history going back to the pioneering works of Kodaira, Atiyah and Hirzebruch in the late 1960s [1, 27, 30], who produced the first examples of surface bundles with $\sigma > 0$ via branched coverings of products of complex curves, albeit for fairly large fiber or base genera. Endo's inno-

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vative work on signatures of surface bundles in the late 1990s [17, 18], via Meyer's cocycle and relations in the mapping class group, made it possible to approach this geography problem more systematically. Over the past two decades, these methods have led to a myriad of examples of surface bundles with positive signatures; see e.g. [2, 7, 10, 11, 15, 19, 22, 36, 37, 41, 43, 46].

Our main result is an extensive advancement in this line of research:

Theorem 1 *There exists a symplectic Σ_g–bundle over Σ_h with positive signature for every $g \geq 15$, $h = 2$; $g \geq 9$, $h = 3$; $g \geq 4$, $h = 4$; and $g \geq 3$, $h \geq 5$.*

Since $\sigma = 0$ when $g \leq 2$ or $h \leq 1$, our theorem leaves out 19 possible (g, h) pairs. While constructing surface bundles with positive signatures for all but 19 possible pairs of (g, h) is the best we could achieve at the time of writing, it seems plausible that variations of our techniques, which we will discuss shortly, may succeed in shrinking the gap even further. Note that Hamenstädt claims in [25] that the Euler characteristic e and the signature σ of a surface bundle always satisfies the inequality $|\mathrm{e}| \geq 3\,|\sigma|$, ruling out the existence of a surface bundle with positive signature for one (and just one) more possible pair: $(g, h) = (3, 2)$.

All surface bundles we built in the theorem have signature $\sigma = 4$. Therefore, for $\mathcal{M}_g$ the moduli space of genus–g curves and m_g denoting the minimal genus among the genera of all surfaces representing the generator of the infinite cyclic group $H_2(\mathcal{M}_g; \mathbb{Z})/\mathrm{Tor}$, with $g \geq 3$, as observed in [10], we can conclude from our results that $m_g = 2$ for any $g \geq 15$, and we have the estimates $m_g = 2$ or 3 when $9 \leq g \leq 14$, $m_g = 2, 3$ or 4 when $4 \leq g \leq 8$, and $m_g = 3, 4$ or 5 when $g = 3$.

We describe all but one of our surface bundles in Theorem 1 via explicit monodromy factorizations in the mapping class group $\mathrm{Mod}(\Sigma_g)$. (The remaining example uses a semi-stable holomorphic fibration due to Catanese–Corvaja–Zannier in [14] as an ingredient.) The breakthrough in our understanding of relations that generate these small surface bundles with positive signatures is due to shorter commutator expressions we are able to obtain for both products of commutators themselves and multi-twists in the mapping class group. In particular, Theorem 4, the proof of which adapts an ingenious argument of Tsuboi in [44] and Burago, Ivanov and Polterovich in [12], allows one to derive examples with base genus $h = 2$ and 3 from those over higher genera surfaces. Leveraging these ingredients, we calculate in Corollaries 11 and 14 the commutator and stable commutator lengths of a few new mapping classes, in particular providing new answers to the Kirby Problem 2.13(b) [29].

One of the motivations for our work is to better understand how the geography of surface bundles compare to that of symplectic 4–manifolds and compact complex surfaces. While all surface bundles with positive signatures admit symplectic forms a lá Thurston, their total spaces do not necessarily admit complex structures. In fact, by the first author's work in [2], the surface bundles we construct in this article yield infinitely many such examples for all possible fiber and base genera except for less than two dozen pairs; see Remark 16.

A particularly interesting comparative geography problem is for the border case of Bogomolov–Miyaoka–Yau inequality [8, 40, 45]. By Yau's celebrated solution of the Calabi conjecture [45], any compact complex surface of general type with $\mathrm{e} = 3\,\sigma$ is a complex ball quotient. These constitute a rather small but very interesting class of complex surfaces, which can not contain any surface bundles [28, 38]. In contrast, it is still not known whether there are symplectic 4–manifolds of general type with $\mathrm{e} = 3\,\sigma$ that are not complex ball quotients, leading to the more specific question:

Question 2 *Is there a symplectic surface bundle over a surface with* $\mathrm{e} = 3\,\sigma$*?*

This amounts to asking in particular whether there is a Σ_g–bundle over Σ_h with positive signature for $(g, h) = (4, 2)$—where our example with $(g, h) = (4, 4)$ gets provokingly close. And more generally, it is part of the bigger question on the existence of any further constraints on the geography of surface bundles with positive signatures, while obviously the very examples in this article limit much wilder constraints to be expected.

Basic conventions: All manifolds and maps we consider in this article are smooth. We denote a compact orientable surface of genus g with b boundary components by Σ_g^b, whereas we omit b when there is no boundary. We denote by $\mathrm{Diff}^+(\Sigma_g^b)$ the group of orientation–preserving diffeomorphisms $\Sigma_g^b \to \Sigma_g^b$ that restrict to the identity in a collar neighborhood of the boundary. The *mapping class group* of Σ_g^b is defined as $\mathrm{Mod}(\Sigma_g^b) := \pi_0(\mathrm{Diff}^+(\Sigma_g^b))$. Our products of mapping classes act on curves starting with the rightmost factor. Whenever we study relations in the mapping class group of Σ_g^b, we consider the curves on Σ_g^b and the elements in $\mathrm{Diff}^+(\Sigma_g^b)$ only up to isotopy and we denote their isotopy classes by the same symbols. We denote by t_c the right-handed, or the *positive Dehn twist*, along a simple closed curve c on a surface Σ_g^b. For any A and B in $\mathrm{Mod}(\Sigma_g^b)$, we let $[A, B] := ABA^{-1}B^{-1}$ denote their commutator, and $A^B := BAB^{-1}$ denote the conjugate of A by B. We denote by $\lfloor r \rfloor$ the largest integer less than or equal to the real number r.

Further conventions: By a *genus–g surface bundle* (X, f) *over a genus–h surface* we mean a smooth locally trivial Σ_g–bundle $f\colon X \to \Sigma_h$, where X is an oriented 4–manifold. A *monodromy factorization* for (X, f) with b disjoint sections $\{S_j\}$ of self-intersections $S_j \cdot S_j = -k_j$ is a relation of the form

$$[A_1, B_1] \cdots [A_h, B_h] = t_{\delta_1}^{k_1} \cdots t_{\delta_b}^{k_b} \quad \text{in } \mathrm{Mod}(\Sigma_g^b),$$

where A_i, B_i are general elements in $\mathrm{Mod}(\Sigma_g^b)$ and $\{\delta_j\}$ are boundary parallel curves along distinct boundary components of Σ_g^b. Finally, for any relator $W = 1$ in $\mathrm{Mod}(\Sigma_g^b)$ we define the signature $\sigma(W)$ as the algebraic sum of the signatures of the relators that are used to derive it from the trivial word with respect to Dehn twist generators [17, 20]. We refer the reader to [5, 6, 19–21, 41] for the general background on surface bundles, monodromy factorizations, mapping class group relations, and signatures.

2 Shorter expressions for products of commutators

There are many situations when a given product of commutators in a group can be re-expressed as a product of less number of commutators. For example, the famous Hall-Witt identity for arbitrary a, b, c in a group G can be arranged to read

$$[[a, b], c^b]\,[[b, c], a^c] = [b^a, [c, a]].$$

Here are a couple of other instances that might be less well–known:

Lemma 3 *For a, b, c, d any elements in a group G, the following hold:*

(i) $[a, b]\,[b, c]\,[c, d]\,[d, a] = [a^{-1}c, b^{-1}d]^{ab}$*, and*
(ii) $\prod_{i=1}^{k}[a_i, b_i] = [\prod_{i=1}^{k} a_i, \prod_{i=1}^{k} b_i]$, *if* $[a_i, a_j] = [a_i, b_j] = [b_i, b_j] = 1$ *for all* $i \neq j$.

Proof Both identities can be checked in a straightforward fashion:

$$\begin{aligned} [a, b]\,[b, c]\,[c, d]\,[d, a] &= aba^{-1}\underline{b^{-1}\,b}cb^{-1}\underline{c^{-1}\,c}dc^{-1}\underline{d^{-1}\,d}ad^{-1}a^{-1} \\ &= aba^{-1}\,cb^{-1}\,dc^{-1}\,ad^{-1}a^{-1} \\ &= aba^{-1}cb^{-1}dc^{-1}ad^{-1}\,\underline{bb^{-1}}\,a^{-1} \\ &= (ab)\,a^{-1}c\,b^{-1}d\,(a^{-1}c)^{-1}\,(b^{-1}\,d)^{-1}\,(ab)^{-1} \\ &= [a^{-1}c, b^{-1}d]^{ab}, \end{aligned}$$

where we have underlined the canceling pairs. Likewise,

$$\begin{aligned} \prod_{i=1}^{k}[a_i, b_i] &= a_1b_1a_1^{-1}b_1^{-1}\,a_2b_2a_2^{-1}b_2^{-1}\cdots a_kb_ka_k^{-1}b_k^{-1} \\ &= a_1a_2\cdots a_k\,b_1a_1^{-1}b_1^{-1}\,b_2a_2^{-1}b_2^{-1}\cdots b_ka_k^{-1}b_k^{-1} \\ &= a_1a_2\cdots a_k\,b_1b_2\cdots b_k\,a_1^{-1}b_1^{-1}\,a_2^{-1}b_2^{-1}\cdots a_k^{-1}b_k^{-1} \\ &= a_1a_2\cdots a_k\,b_1b_2\cdots b_k\,a_1^{-1}a_2^{-1}\cdots a_k^{-1}\,b_1^{-1}b_2^{-1}\cdots b_k^{-1} \\ &= a_1a_2\cdots a_k\,b_1b_2\cdots b_k\,a_k^{-1}\cdots a_2^{-1}a_1^{-1}\,b_k^{-1}\cdots b_2^{-1}b_1^{-1} \\ &= \left[\prod_{i=1}^{k} a_i, \prod_{i=1}^{k} b_i\right], \end{aligned}$$

where in each one of the intermediate steps we have repeatedly used only the given commutativity relations. □

Recall that a conjugate of a commutator is again a commutator, so that all the commutator identities we have listed so far, in fact, describe a product of commutators as a single commutator. The first identity in the lemma appears in the literature as early as in [42] and contains two special cases which appear more frequently:

$$[a, b]\,[b, c]\,[c, a] = [a^{-1}c, b^{-1}a]^{ab}$$

which one derives by taking $d = a$ in Lemma 3(i), whereas taking $d = 1$ we get

$$[a, b]\,[b, c] = [a^{-1}c, b^{-1}]^{ab}. \tag{1}$$

The second identity in the lemma is perhaps more contemporary but was clearly already known to experts [12, 44] (more on this below).

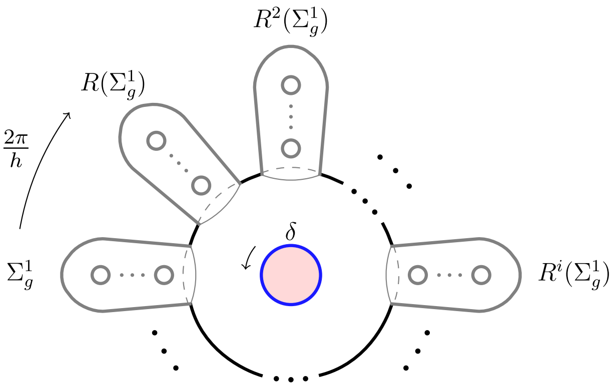

Fig. 1 The rotation R

For monodromy factorizations of surface bundles, commutator identities as above allow one to derive new surface bundles over surfaces of smaller genera. Our next theorem, the proof of which is leveraging a beautiful argument of Tsuboi in [44] and Burago, Ivanov and Polterovich in [12], shows that one can moreover lower the base genus dramatically at the expense of increasing the fiber genus:

Theorem 4 *Let (X, f) be a genus–g surface bundle over a genus–h surface with a section of self-intersection zero. Then there also exist surface bundles*

(i) (X', f') of fiber genus $g' = gh$ and base genus 2, for $h \geq 2$, and
(ii) (X'', f'') of fiber genus $g'' = g\lfloor \frac{h+1}{2} \rfloor$ and base genus 3, for $h \geq 3$,

also with sections of self-intersection zero and signatures $\sigma(X') = \sigma(X'') = \sigma(X)$. Further, given an explicit monodromy factorization for (X, f) with a self-intersection zero section S, we can explicitly describe the monodromy factorization of (X', f') and (X'', f'') with self-intersection zero sections S' and S'', respectively.

Proof Let a_i, b_i, for $i = 1, \ldots, h$, be elements of $\mathrm{Diff}^+(\Sigma_g^1)$ which restrict to the identity in some collar neighborhood of $\partial\Sigma_g^1$. Let $\phi = \prod_{i=1}^h [a_i, b_i]$. That is to say, we have the following relation:

$$\phi = [a_1, b_1] \cdots [a_h, b_h] \ \text{ in } \mathrm{Mod}(\Sigma_g^1)\,, \tag{2}$$

where we simply denote the corresponding mapping classes by the same letters.

Now let R be the clockwise $\frac{2\pi}{h}$–rotation of $\Sigma_{g'}^1$, with $g' = gh$ as illustrated in Fig. 1, followed by a counter-clockwise $\frac{2\pi}{h}$–rotation of $\partial\Sigma_{g'}^1$ supported in a small collar neighborhood of its boundary. We take an embedding $\Sigma_g^1 \hookrightarrow \Sigma_{g'}^1$ with image as shown in Fig. 1, away from the support of the above boundary rotation. With this identification of Σ_g^1 with a subsurface of $\Sigma_{g'}^1$, we can then define $\tilde{\phi}, \tilde{a}_i, \tilde{b}_i \in \mathrm{Diff}^+(\Sigma_{g'}^1)$ by extending each ϕ, a_i, b_i as the identity on $\Sigma_{g'}^1 \setminus \Sigma_g^1$. We thus have the relation

$$\tilde{\phi} = [\tilde{a}_1, \tilde{b}_1] \cdots [\tilde{a}_h, \tilde{b}_h] \ \text{ in } \mathrm{Mod}(\Sigma_{g'}^1)\,. \tag{3}$$

A two-commutator identity: Reviewing the details of Tsuboi's algebraic argument in [44] (cf. [12]) will be essential for the remaining part of the proof of our theorem. Let $C_i := [\tilde{a}_i, \tilde{b}_i]$, for all $i = 1, \ldots, h$, so $\tilde{\phi} = \prod_{i=1}^{h} C_i$. Following [44], we set

$$P := \prod_{i=1}^{h} (C_1^{R^{h-i}} \cdots C_i^{R^{h-i}}) .$$

There are quite a few commutativity relations which are important to note here. First of all, $[\tilde{a}_i^{R^p}, \tilde{b}_j^{R^q}] = [\tilde{a}_i^{R^p}, \tilde{a}_j^{R^q}] = [\tilde{b}_i^{R^p}, \tilde{b}_j^{R^q}] = 1$ for any $p \neq q$ since each pair of diffeomorphisms in these commutators have disjoint supports in $\Sigma^1_{g'}$. It follows that $[C_i^{R^p}, C_j^{R^q}] = 1$ for any i, j and $p \neq q$. In turn, $[C_1^{R^{h-i}} \cdots C_i^{R^{h-i}},\ C_1^{R^{h-j}} \cdots C_j^{R^{h-j}}] = 1$ whenever $i \neq j$, so one can spell out the parenthetical factors in the above product expression of P in any—and in particular, in reversed—order. Last but not least, since R^h is identity on the compact support of any $\tilde{a}_i^{R^p}$ and $\tilde{b}_i^{R^p}$ (even though it is isotopic to a boundary parallel Dehn twist on $\Sigma^1_{g'}$), we have $[R^h, \tilde{a}_i^{R^p}] = [R^h, \tilde{b}_i^{R^p}] = 1$, and thus $[R^h, C_i^{R^p}] = 1$ for any i and p, and in turn, $(C_i^{R^p})^{R^h} = C_i^{R^p}$.

We have the product expressions

$$P^{-1} = \prod_{i=1}^{h} (C_1^{R^{h-i}} \cdots C_i^{R^{h-i}})^{-1} = \tilde{\phi}^{-1} \prod_{i=1}^{h-1} (C_1^{R^{h-i}} \cdots C_i^{R^{h-i}})^{-1}$$

and

$$P^R = \prod_{i=1}^{h} (C_1^{R^{h-i+1}} \cdots C_i^{R^{h-i+1}}) = \prod_{i=0}^{h-1} (C_1^{R^{h-i}} \cdots C_{i+1}^{R^{h-i}}) = C_1 \prod_{i=1}^{h-1} (C_1^{R^{h-i}} \cdots C_{i+1}^{R^{h-i}}).$$

Therefore,

$$\begin{aligned}
[P^{-1}, R] &= P^{-1} P^R \\
&= \tilde{\phi}^{-1} \prod_{i=1}^{h-1} (C_1^{R^{h-i}} \cdots C_i^{R^{h-i}})^{-1} \cdot C_1 \prod_{i=1}^{h-1} (C_1^{R^{h-i}} \cdots C_{i+1}^{R^{h-i}}) \\
&= \tilde{\phi}^{-1}\, C_1 \prod_{i=1}^{h-1} ((C_1^{R^{h-i}} \cdots C_i^{R^{h-i}})^{-1} (C_1^{R^{h-i}} \cdots C_{i+1}^{R^{h-i}})) \\
&= \tilde{\phi}^{-1}\, C_1 \prod_{i=1}^{h-1} C_{i+1}^{R^{h-i}} \\
&= \tilde{\phi}^{-1} \prod_{i=1}^{h} C_i^{R^{h-i+1}}
\end{aligned}$$

$$= \tilde{\phi}^{-1} \, [\prod_{i=1}^{h} \tilde{a}_i^{R^{h-i+1}}, \prod_{i=1}^{h} \tilde{b}_i^{R^{h-i+1}}]$$

where we repeatedly used the commutativity relations mentioned above and invoked Lemma 3(ii) at the final step.

Setting $A := \prod_{i=1}^{h} \tilde{a}_i^{R^{h-i+1}}$ and $B := \prod_{i=1}^{h} \tilde{b}_i^{R^{h-i+1}}$ we arrive at the two-commutator identity

$$\tilde{\phi} = [A, B][R, P^{-1}] \text{ in } \mathrm{Mod}(\Sigma_{g'}^1)\,. \tag{4}$$

A variation: Let $\tilde{\phi} = \prod_{i=1}^{h} C_i$ and R be as above. This time set

$$Q := \prod_{i=0}^{h-1} (C_1^{R^i} \cdots C_{h-i}^{R^i}) = \tilde{\phi} \prod_{i=1}^{h-1} (C_1^{R^i} \cdots C_{h-i}^{R^i}).$$

Note that by the commutativity relations mentioned above, the parenthetical terms in this product can also be spelled out in any order. So by using the equality $(C_1^{-1})^{R^h} = C_1^{-1}$, we have

$$\begin{aligned} (Q^{-1})^R &= \prod_{i=0}^{h-1} (C_1^{R^{i+1}} \cdots C_{h-i}^{R^{i+1}})^{-1} = \prod_{i=1}^{h} (C_1^{R^i} \cdots C_{h-i+1}^{R^i})^{-1} \\ &= C_1^{-1} \prod_{i=1}^{h-1} (C_1^{R^i} \cdots C_{h-i+1}^{R^i})^{-1}. \end{aligned}$$

It follows that for $[Q, R] = QRQ^{-1}R^{-1} = Q(Q^{-1})^R$, applying the same arguments as earlier, we have

$$\begin{aligned} [Q, R] &= \tilde{\phi} \prod_{i=1}^{h-1} (C_1^{R^i} \cdots C_{h-i}^{R^i}) \;\; \cdot C_1^{-1} \prod_{i=1}^{h-1} (C_1^{R^i} \cdots C_{h-i+1}^{R^i})^{-1} \\ &= \tilde{\phi}\, C_1^{-1} \prod_{i=1}^{h-1} (C_1^{R^i} \cdots C_{h-i}^{R^i})\, (C_1^{R^i} \cdots C_{h-i+1}^{R^i})^{-1} \\ &= \tilde{\phi}\, C_1^{-1} \prod_{i=1}^{h-1} (C_1^{R^i} \cdots C_{h-i}^{R^i})\, (C_{h-i+1}^{R^i})^{-1} (C_1^{R^i} \cdots C_{h-i}^{R^i})^{-1} \\ &= \tilde{\phi}\, C_1^{-1} \prod_{i=1}^{h-1} (C_{h-i+1}^{C_1^{R^i} \cdots C_{h-i}^{R^i} R^i})^{-1} \\ &= \tilde{\phi} \prod_{i=1}^{h} (C_{h-i+1}^{C_1^{R^i} \cdots C_{h-i}^{R^i} R^i})^{-1} \end{aligned}$$

$$= \tilde{\phi} \prod_{i=1}^{h} (C_i^{C_1^{R^{h-i+1}} \cdots C_{i-1}^{R^{h-i+1}} R^{h-i+1}})^{-1}$$
$$= \tilde{\phi} \left[\prod_{i=1}^{h} \tilde{b}_i^{C_1^{R^{h-i+1}} \cdots C_{i-1}^{R^{h-i+1}} R^{h-i+1}}, \prod_{i=1}^{h} \tilde{a}_i^{C_1^{R^{h-i+1}} \cdots C_{i-1}^{R^{h-i+1}} R^{h-i+1}} \right].$$

(Here we take $C_1^{R^{h-i+1}} \cdots C_{i-1}^{R^{h-i+1}} = 1$ if $i = 1$, and so on.) Setting

$$A' := \prod_{i=1}^{h} \tilde{a}_i^{C_1^{R^{h-i+1}} \cdots C_{i-1}^{R^{h-i+1}} R^{h-i+1}} \text{ and } B' := \prod_{i=1}^{h} \tilde{b}_i^{C_1^{R^{h-i+1}} \cdots C_{i-1}^{R^{h-i+1}} R^{h-i+1}}$$

give us the equality

$$\tilde{\phi} = [Q, R][A', B'] \text{ in } \mathrm{Mod}(\Sigma_{g'}^1)\,. \tag{5}$$

A three-commutator identity: Assume that for $j = 1, 2$ we have

$$\phi_j := \prod_{i=1}^{h_j} [a_i(j), b_i(j)] \text{ in } \mathrm{Mod}(\Sigma_g^1)$$

and let h be the maximum of h_1 and h_2. Possibly after adding trivial commutators, we can express both as $\phi_j := \prod_{i=1}^{h} [a_i(j), b_i(j)]$. Running Tsuboi's trick for $j = 1$ and its above variation for $j = 2$, respectively, we obtain two identities

$$\tilde{\phi}_1 = [A, B][R, P^{-1}] = [R, P^{-1}][A, B]^{[P^{-1}, R]}$$
$$\tilde{\phi}_2 = [Q, R][A', B'] = [A', B']^{[Q, R]}[Q, R]$$

in $\mathrm{Mod}(\Sigma_{g'}^1)$. Here, the triples A, B, P and A', B', Q are determined by diffeomorphisms coming from the entries of the commutators in ϕ_1 and ϕ_2, respectively, but R is the same diffeomorphism of $\Sigma_{g'}^1$. By the special case of Lemma 3(i), we have

$$\tilde{\phi}_2 \tilde{\phi}_1 = [A', B']^{[Q, R]}[Q, R][R, P^{-1}][A, B]^{[P^{-1}, R]}$$
$$= [A', B']^{[Q, R]}[Q^{-1} P^{-1}, R^{-1}]^{QR}[A, B]^{[P^{-1}, R]}.$$

Relabeling the conjugated commutator entries, we get a new three commutator expression

$$\tilde{\phi}_2 \tilde{\phi}_1 = [A_1, B_1][A_2, B_2][A_3, B_3] \text{ in } \mathrm{Mod}(\Sigma_{g'}^1). \tag{6}$$

We note that here $g' = gh$ for $h = \max\{h_1, h_2\}$ and not $h_1 + h_2$.

Constructions of (X', f') *and* (X'', f''): There is a monodromy factorization for (X, f) with a section S of self-intersection zero of the following form:

$$[a_1, b_1] \cdots [a_h, b_h] = 1 \text{ in } \mathrm{Mod}(\Sigma_g^1). \tag{7}$$

Taking $\phi = 1$ in (2), and in turn getting $\tilde{\phi} = 1$ in (3), the equality (4) becomes:

$$[A, B][R, P^{-1}] = 1 \text{ in } \mathrm{Mod}(\Sigma^1_{g'}). \tag{8}$$

Prescribed by this relation is the surface bundle (X', f') with fiber genus $g' = gh$ and base genus 2, along with a section S' of self-intersection zero.

On the other hand, if we take $\phi_2 = [a_1, b_1] \cdots [a_k, b_k]$ and $\phi_1 = [a_{k+1}, b_{k+1}] \cdots [a_h, b_h]$ for $k := \lfloor \frac{h+1}{2} \rfloor$ in our construction of the three-commutator identity, the equality (6) becomes

$$[A_1, B_1][A_2, B_2][A_3, B_3] = 1 \text{ in } \mathrm{Mod}(\Sigma^1_{g''}), \tag{9}$$

where $g'' = gk$. Prescribed by this relation is the surface bundle (X'', f'') with fiber genus $g'' = g\lfloor \frac{h+1}{2} \rfloor$ and base genus 3, along with a section S'' of self-intersection zero.

The monodromy factorization of a surface bundle is derived from the trivial word in the mapping class group of a fiber using some sequence of basic relators between Dehn twists. By [20], the signature of this surface bundle can be expressed as an algebraic sum of the signature contributions of these basic relators, and the result is independent of the sequence. Moreover the signature contribution of a relator and its conjugate are the same, so it suffices to look at a few basic relators.

Post factum, after embedding a positive factorization in Σ^1_g into the mapping class group of another surface as we did via the embedding $\Sigma_g \hookrightarrow \Sigma_{g'}$, we can in fact regard it to be obtained from the trivial word in $\mathrm{Mod}(\Sigma^1_{g'})$ using the same basic relators in $\mathrm{Mod}(\Sigma^1_{g'})$. (See next section for more on this.) Therefore, the initial positive factorization in our monodromy constructions, namely (7) has the same signature as the positive factorization (2) with $\phi = 1$. Pivotal to our construction is that after that point, in our derivation of the positive factorizations (8) and (9) all the relations we have used can be easily seen to be only commutativity and conjugation relators, along with insertion/removal of canceling pairs, all of which have zero signature contribution. Hence, we have $\sigma(X) = \sigma(X') = \sigma(X'')$, as promised. □

3 Shorter expressions for products of Dehn twists

We now turn to expressing products of positive Dehn twists as products of small numbers of commutators, while using only the relators (between Dehn twist generators) in the mapping class group with non-negative signature contributions.[1] Despite this resriction, several of our commutator expressions will make it possible for us to calculate the precise (and positive) commutator and stable commutator lengths of a few new mapping classes. These are discussed at the very end of this section.

More explicitly, our focus here is on expressing a product P of positive Dehn twists as a product C of a few commutators, so that the signature of the relator $P^{-1}C = 1$ is

[1] This seemingly unnatural restriction is due to our aspirations to build surface bundles with positive signatures via monodromy factorizations we will obtain in the final section by combining the relators we get here with other known relators that have negative signatures.

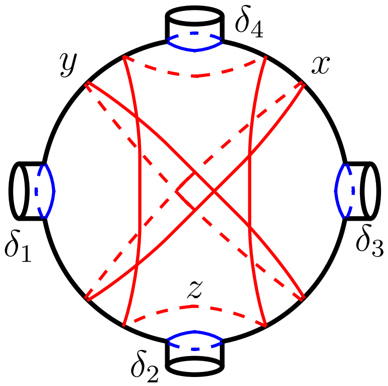

Fig. 2 The lantern curves on Σ_0^4

positive. Following [17, 20], we will take the infinite presentation of $\mathrm{Mod}(\Sigma_g^b)$ with Dehn twists along all curves as the generators, and the relators between them. We review some of these relators and their signatures first.

3.1 Basic relators and signatures in the mapping class group

We have the obvious relator $t_a t_a^{-1} = 1$ for any a. If two curves a and b are disjoint, then we have the *commutativity* relator $t_a t_b t_a^{-1} t_b^{-1} = 1$. Similarly, for simple closed curves a and b intersecting transversely at one point, there is the *braid* relator $t_a t_b t_a t_b^{-1} t_a^{-1} t_b^{-1} = 1$. All these basic relators have signature $\sigma = 0$. It then follows that for any $A, B \in \mathrm{Mod}(\Sigma_g^b)$, the *conjugation* relator $A^{-1} B^{-1} A^B B = 1$ has $\sigma = 0$ as well. This has several implications. For one, if one induces a relator $W' = 1$ in $\mathrm{Mod}(\Sigma_{g'}^{b'})$ from a given relator $W = 1$ in $\mathrm{Mod}(\Sigma_g^b)$ through some embedding $\Sigma_g^b \hookrightarrow \Sigma_{g'}^{b'}$, then $\sigma(W') = \sigma(W)$, and $\sigma(W')$ is in fact independent of the embedding.

There are two basic relators with positive signatures that are of importance to us. First, for a null-homotopic curve a in Σ_g^b, the well-known relator $t_a^{-1} = 1$ has $\sigma = +1$. One implication of this is the following: Say we have a monodromy factorization for a surface bundle with sections expressed by a relator $W := C\, t_{\delta_1}^{-k_1} t_{\delta_2}^{-k_2} \cdots t_{\delta_b}^{-k_b} = 1$ in $\mathrm{Mod}(\Sigma_g^b)$, where C is some product of commutators. Suppose that a relator $W' = 1$ in $\mathrm{Mod}(\Sigma_g^{b-1})$ is derived from W by capping off the boundary component δ_i with a disk. Then $\sigma(W') = \sigma(W) + k_i$ by the above reasoning.

The point guard in our game is the *lantern relator* in $\mathrm{Mod}(\Sigma_0^4)$

$$t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} t_x t_y t_z = 1,$$

where the curves x, y, z, δ_i are as shown in Fig. 2. This relator also has $\sigma = +1$, which now follows from the facts we laid out above, once you embed $\Sigma_0^4 \hookrightarrow \Sigma_g^b$.

3.2 Multitwists as products of a few commutators

In the next several lemmas, we are going to express various *multitwists*, i.e. products of positive Dehn twists about disjoint curves, as small number of commutators using only the relators (with non-negative signatures) we have listed earlier. Our relations are supported in Σ_g^b, for $g = 2$, $g = 3$ and $g = 5$ (and varying b), respectively, where, importantly, we always have $b > 0$, so the same relations can be embedded into any $\mathrm{Mod}(\Sigma_{g'}^{b'})$ for $g' \geq g$.

We are going to make repeated use of the following simple but highly useful observation, which has been well-exploited in several prior works; e.g. [4, 5, 19, 23, 26, 32, 33, 41].

Lemma 5 *Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ be curves on Σ_g^b such that there is a diffeomorphism $F \in \mathit{Diff}^+(\Sigma_g^b)$ with $F(a_i) = b_i$ for all $i = 1, \ldots, m$. We then have the one-commutator expression*

$$t_{a_1}^{k_1} \cdots t_{a_m}^{k_m} t_{b_m}^{-k_m} \cdots t_{b_1}^{-k_1} = [t_{a_1}^{k_1} \cdots t_{a_m}^{k_n}, F] \quad \textit{in } \mathrm{Mod}(\Sigma_g^b) \tag{10}$$

and the corresponding relator has signature zero.

Proof Given the hypotheses, the following equalities hold in $\mathrm{Mod}(\Sigma_g^b)$:

$$\begin{aligned} t_{a_1}^{k_1} \cdots t_{a_m}^{k_m} t_{b_m}^{-k_m} \cdots t_{b_1}^{-k_1} &= (t_{a_1}^{k_1} \cdots t_{a_m}^{k_m})\, (t_{F(a_m)}^{-k_m} \cdots t_{F(a_1)}^{-k_1}) \\ &= (t_{a_1}^{k_1} \cdots t_{a_m}^{k_m})\, F\, (t_{a_m}^{-k_m} \cdots t_{a_1}^{-k_1}) F^{-1} \\ &= (t_{a_1}^{k_1} \cdots t_{a_m}^{k_m})\, F\, (t_{a_1}^{k_1} \cdots t_{a_m}^{k_m})^{-1} F^{-1} \\ &= [t_{a_1}^{k_1} \cdots t_{a_m}^{k_m}, F]. \end{aligned}$$

Since we have not used any relators with non-trivial signature contributions above, the relator $(t_{a_1}^{k_1} \cdots t_{a_m}^{k_m})^{-1} [t_{a_1}^{k_1} \cdots t_{a_m}^{k_n}, F] = 1$ in $\mathrm{Mod}(\Sigma_g^b)$ has signature zero. □

Remark 6 When applying Lemma 5, we are going to typically take $\{a_i\}$ and $\{b_j\}$ each as collection of disjoint curves on Σ_g^b, so we can argue the existence of a diffeomorphism F simply by looking at the homeomorphism types of the cut surfaces $\Sigma_g^b \setminus \{a_i\}$ and $\Sigma_g^b \setminus \{b_j\}$, while making sure that the boundary component match in the fashion we want to map a_i to b_j.

In the proof of the next lemma, our arguments work for any power of the same (multi)twist that is of particular interest to us, so we will derive the commutator expressions for all of them at once.

Lemma 7 *For δ_1, δ_2 any two boundary components of Σ_2^3, and n any positive integer, there is a relator*

$$t_{\delta_1}^{-n} t_{\delta_2}^{-n} \prod_{i=1}^{N} C_i(n) = 1 \quad \textit{in } \mathrm{Mod}(\Sigma_2^3)$$

with $N = \lfloor (n+3)/2 \rfloor$ *commutators* $\{C_i(n)\}$ *and signature* $2n$.

Proof Consider Σ_2^3 in Fig. 3. By the lantern relation, we have

$$t_{a_1} t_{a_2} t_{a_3} t_{\delta_1} = t_{x_3} t_{x_1} t_{x_2},$$
$$t_b t_{a_2} t_{a_3} t_{\delta_2} = t_{y_1} t_{y_2} t_{y_3}$$

so that

$$t_{a_1} t_{a_2} t_{a_3} t_{\delta_1} t_{x_2}^{-1} = t_{x_3} t_{x_1},$$
$$t_b t_{a_2} t_{a_3} t_{\delta_2} t_{y_3}^{-1} = t_{y_1} t_{y_2}.$$

Hence,

$$t_{a_1}^n t_{a_2}^n t_{a_3}^n t_{\delta_1}^n t_{x_2}^{-n} = (t_{x_3} t_{x_1})^n = \left(\prod_{i=1}^{n} t_{x_1}^{i-1} t_{x_3} t_{x_1}^{-i+1} \right) t_{x_1}^n = \left(\prod_{i=1}^{n} (t_{x_3})^{t_{x_1}^{i-1}} \right) t_{x_1}^n,$$
$$t_b^n t_{a_2}^n t_{a_3}^n t_{\delta_2}^n t_{y_3}^{-n} = (t_{y_1} t_{y_2})^n = \left(\prod_{i=1}^{n} t_{y_2}^{i-1} t_{y_1} t_{y_2}^{-i+1} \right) t_{y_2}^n = \left(\prod_{i=1}^{n} (t_{y_1})^{t_{y_2}^{i-1}} \right) t_{y_2}^n.$$

By multiplying the last two equalities, we get

$$t_b^n t_{a_1}^n t_{a_2}^{2n} t_{a_3}^{2n} t_{\delta_1}^n t_{\delta_2}^n = \left(\prod_{i=1}^{n} (t_{x_3} t_{y_1})^{V^{i-1}} \right) t_{x_1}^n t_{y_2}^n t_{x_2}^n t_{y_3}^n,$$

where $V = t_{x_1} t_{y_2}$. Thus

$$t_{\delta_1}^n t_{\delta_2}^n = \left(\prod_{i=1}^{n} \left(t_{x_3} t_{y_1} t_b^{-1} t_{a_3}^{-1} \right)^{V^{i-1}} \right) (t_{x_1}^n t_{y_2}^n t_{a_1}^{-n} t_{a_2}^{-n} t_{x_2}^n t_{y_3}^n t_{a_2}^{-n} t_{a_3}^{-n}).$$

Since the complements of $x_1 \cup y_2 \cup a_1 \cup a_2$ and $a_2 \cup a_3 \cup x_2 \cup y_3$ in Σ_2^3 are diffeomorphic, by the classification of surfaces, there is a diffeomorphism of Σ_2^3 mapping (x_1, y_2, a_1, a_2) to (a_2, a_3, x_2, y_3). Such a diffeomorhism may be taken as $t_{c_1} t_{a_2} t_{x_1}^{-1} t_{c_1}^{-1} \cdot t_{c_2}^{-1} t_{y_3}^{-1} t_{a_2} t_{c_2}$. By Lemma 5, the product

$$t_{x_1}^n t_{y_2}^n t_{a_1}^{-n} t_{a_2}^{-n} t_{x_2}^n t_{y_3}^n t_{a_2}^{-n} t_{a_3}^{-n}$$

is a commutator. Likewise, the diffeomorphism $t_{c_1} t_b t_{y_1} t_{c_1} \cdot t_{c_2} t_{a_3} t_{x_3} t_{c_2}$ of Σ_2^3 maps (x_3, y_1, b, a_3) to (a_3, b, y_1, x_3), so that the product

$$t_{x_3} t_{y_1} t_b^{-1} t_{a_3}^{-1} (t_{x_3} t_{y_1} t_b^{-1} t_{a_3}^{-1})^V$$

is a commutator.

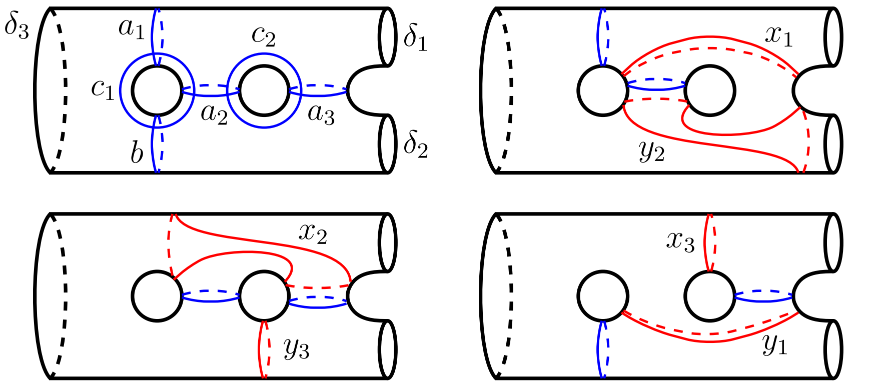

Fig. 3 The curves on Σ_2^3 used in the proof of Lemma 7

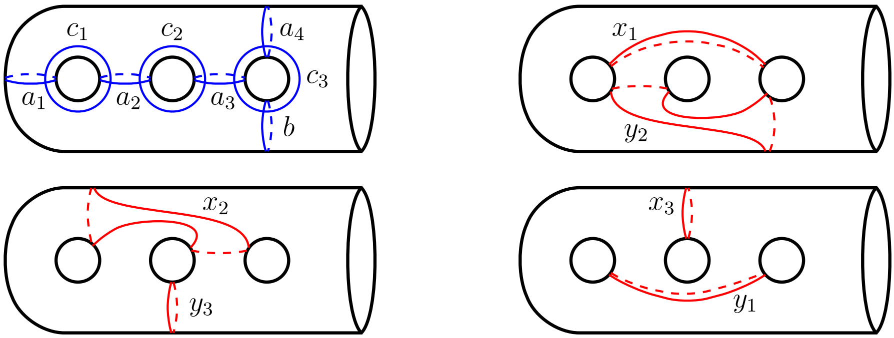

Fig. 4 The curves on Σ_3^1 used in Lemma 8 and its proof

It follows that if $n = 2k$ is even, then

$$\prod_{i=1}^{n} \left(t_{x_3} t_{y_1} t_b^{-1} t_{a_3}^{-1}\right)^{V^{i-1}} = \prod_{i=1}^{k} \left(t_{x_3} t_{y_1} t_b^{-1} t_{a_3}^{-1} (t_{x_3} t_{y_1} t_b^{-1} t_{a_3}^{-1})^{V}\right)^{V^{2i-2}}$$

is a product of k commutators, and if $n = 2k + 1$, then

$$\prod_{i=1}^{n} \left(t_{x_3} t_{y_1} t_b^{-1} t_{a_3}^{-1}\right)^{V^{i-1}} = \left(\prod_{i=1}^{n-1} \left(t_{x_3} t_{y_1} t_b^{-1} t_{a_3}^{-1}\right)^{V^{i-1}}\right) (t_{x_3} t_{y_1} t_b^{-1} t_{a_3}^{-1})^{V^{2k}}$$

is a product of $k + 1$ commutators.

Therefore, $t_{\delta_1}^n t_{\delta_2}^n = \prod_{i=1}^{N} C_i(n)$ for $N = \lfloor (n+3)/2 \rfloor$, where $C_i(n)$ are commutators, varying with n. As we have only employed signature zero relators and $2n$ lantern relators, the relator $t_{\delta_1}^{-n} t_{\delta_2}^{-n} \prod_{i=1}^{N} C_i(n) = 1$ has $\sigma = 2n$. □

Next is a three-commutator expression we derive on the genus–3 surface with one boundary component.

Lemma 8 *For* a_1, a_2, a_3, a_4 *the curves shown in* Σ_3^1 *in* Fig. 4, there is a relator

$$t_{a_1}^{-2}t_{a_2}^{-2}t_{a_3}^{-2}t_{a_4}^{-2}C_1C_2C_3 = 1 \quad \text{in } \mathrm{Mod}(\Sigma_3^1)$$

of signature 8, where C_1, C_2 and C_3 are commutators.

Proof By the lantern relation, we have

$$t_{a_1}t_{a_2}t_{a_3}t_{a_4} = t_{x_2}t_{x_3}t_{x_1}$$

or

$$t_{x_2}^{-1}t_{a_1}t_{a_2}t_{a_3}t_{a_4} = t_{x_3}t_{x_1}.$$

This yields the equality

$$\begin{aligned} t_{x_2}^{-2}t_{a_1}^2t_{a_2}^2t_{a_3}^2t_{a_4}^2 &= (t_{x_3}t_{x_1})^2 \\ &= t_{x_3}(t_{x_3})^{t_{x_1}}t_{x_1}^2 \end{aligned}$$

so that

$$t_{a_1}^2t_{a_2}^2t_{a_3}^2t_{a_4}^2 = t_{x_1}^2t_{x_2}^2t_{x_3}(t_{x_3})^{t_{x_1}}.$$

Hence,

$$\begin{aligned} t_{a_1}^4t_{a_2}^4t_{a_3}^4t_{a_4}^4 &= t_{x_1}^2t_{x_2}^2t_{x_3}\left(t_{a_1}^2t_{a_2}^2t_{a_3}^2t_{a_4}^2\right)(t_{x_3})^{t_{x_1}} \\ &= t_{x_1}^2t_{x_2}^2t_{x_3}\left(t_{x_1}^2t_{x_2}^2t_{x_3}(t_{x_3})^{t_{x_1}}\right)(t_{x_3})^{t_{x_1}} \\ &= t_{x_1}^2t_{x_2}^2\left(t_{x_1}^2t_{x_2}^2\right)^{t_{x_3}}t_{x_3}^2(t_{x_3}^2)^{t_{x_1}}. \end{aligned} \tag{11}$$

By a similar computation, the lantern relation

$$t_{a_1}t_{a_2}t_{a_3}t_b = t_{y_1}t_{y_2}t_{y_3}$$

yields

$$t_{a_1}^2t_{a_2}^2t_{a_3}^2t_b^2 = t_{y_2}^2t_{y_3}^2t_{y_1}(t_{y_1})^{t_{y_2}}$$

and

$$t_{a_1}^4t_{a_2}^4t_{a_3}^4t_b^4 = t_{y_2}^2t_{y_3}^2\left(t_{y_2}^2t_{y_3}^2\right)^{t_{y_1}}t_{y_1}^2(t_{y_1}^2)^{t_{y_2}}. \tag{12}$$

From the equalities (11) and (12) we get

$$t_{a_1}^8t_{a_2}^8t_{a_3}^8t_{a_4}^4t_b^4 = t_{x_1}^2t_{x_2}^2\left(t_{x_1}^2t_{x_2}^2\right)^{t_{x_3}}t_{x_3}^2(t_{x_3}^2)^{t_{x_1}}\cdot t_{y_2}^2t_{y_3}^2\left(t_{y_2}^2t_{y_3}^2\right)^{t_{y_1}}t_{y_1}^2(t_{y_1}^2)^{t_{y_2}}$$

$$= \left(t_{x_1}^2 t_{y_2}^2 t_{x_2}^2 t_{y_3}^2\right) \left(t_{x_1}^2 t_{y_2}^2 t_{x_2}^2 t_{y_3}^2\right)^X \left(t_{x_3}^2 t_{y_1}^2 (t_{x_3}^2 t_{y_1}^2)^Y\right),$$

where $X = t_{x_3} t_{y_1}$, $Y = t_{x_1} t_{y_2}$. We then write

$$t_{a_1}^2 t_{a_2}^2 t_{a_3}^2 t_{a_4}^2 = \left(t_{x_1}^2 t_{y_2}^2 t_{a_1}^{-2} t_{a_2}^{-2} \cdot t_{x_2}^2 t_{y_3}^2 t_{a_2}^{-2} t_{a_3}^{-2}\right) \left(t_{x_1}^2 t_{y_2}^2 t_{a_3}^{-2} t_{a_4}^{-2} \cdot t_{x_2}^2 t_{y_3}^2 t_{a_1}^{-2} t_b^{-2}\right)^X \left(t_{x_3}^2 t_{y_1}^2 t_{a_1}^{-2} t_{a_3}^{-2} \cdot (t_{x_3}^2 t_{y_1}^2 t_{a_2}^{-2} t_b^{-2})^Y\right).$$

We now show that each of the three factors on the right-hand side is a commutator, by invoking Lemma 5 repeatedly. It is easy to see (e.g. by cutting the surface along the given quadruples of curves) that there are $F_i \in \mathrm{Diff}^+(\Sigma_3^1)$, for $i = 1, 2, 3$, such that

- $F_1(x_1, y_2, a_1, a_2) = (a_2, a_3, x_2, y_3)$
- $F_2(x_1, y_2, a_3, a_4) = (a_1, b, x_2, y_3)$
- $F_3(x_3, y_1, a_1, a_3) = (a_2, b, x_3, y_1)$.

The diffeomorphisms F_1, F_2 and F_3 can be chosen as follows:

- $F_1 = t_{c_1} t_{a_2} t_{x_1}^{-1} t_{c_1}^{-1} \cdot t_{c_2}^{-1} t_{y_3}^{-1} t_{a_2} t_{c_2}$,
- $F_2 = t_w t_{y_3} t_b t_w \cdot t_{c_2} t_{x_2} t_{y_3} t_{c_2} \cdot t_z t_{a_1} t_{y_1}^{-1} t_z^{-1} \cdot h$, and
- $F_3 = t_{c_2} t_{a_2} t_{x_3} t_{c_2} \cdot t_{c_3} t_{y_1} t_b t_{c_3} \cdot t_{c_1}^{-1} t_{a_2}^{-1} t_{a_1}^{-1} t_{c_1}^{-1} \cdot t_{c_3} t_b t_{a_3} t_{c_3}$,

where $z = t_{x_3} t_{c_2} t_{a_2} t_{c_1}(a_1)$, $w = t_{a_4} t_{c_3} t_{a_3}(c_2)$ and $h = (t_{a_1} t_{c_1} t_{a_2} t_{c_2} t_{a_3} t_{c_3})^7$. Note that h projects to a hyperelliptic involution of the closed surface of genus 3 obtained by gluing a disc along the boundary of Σ_3^1.

We now have

$$\begin{aligned}
\left(t_{x_1}^2 t_{y_2}^2 t_{a_1}^{-2} t_{a_2}^{-2}\right) \cdot \left(t_{x_2}^2 t_{y_3}^2 t_{a_2}^{-2} t_{a_3}^{-2}\right) &= \left(t_{x_1}^2 t_{y_2}^2 t_{a_1}^{-2} t_{a_2}^{-2}\right) \left(t_{a_1}^2 t_{a_2}^2 t_{x_1}^{-2} t_{y_2}^{-2}\right)^{F_1} \\
&= [t_{x_1}^2 t_{y_2}^2 t_{a_1}^{-2} t_{a_2}^{-2}, F_1], \\
\left(t_{x_1}^2 t_{y_2}^2 t_{a_3}^{-2} t_{a_4}^{-2}\right) \cdot \left(t_{x_2}^2 t_{y_3}^2 t_{a_1}^{-2} t_b^{-2}\right) &= \left(t_{x_1}^2 t_{y_2}^2 t_{a_3}^{-2} t_{a_4}^{-2}\right) \left(t_{a_3}^2 t_{a_4}^2 t_{x_1}^{-2} t_{y_2}^{-2}\right)^{F_2} \\
&= [t_{x_1}^2 t_{y_2}^2 t_{a_3}^{-2} t_{a_4}^{-2}, F_2],
\end{aligned}$$

and

$$\begin{aligned}
\left(t_{x_3}^2 t_{y_1}^2 t_{a_1}^{-2} t_{a_3}^{-2}\right) \left(t_{x_3}^2 t_{y_1}^2 t_{a_2}^{-2} t_b^{-2}\right)^Y &= \left(t_{x_3}^2 t_{y_1}^2 t_{a_1}^{-2} t_{a_3}^{-2}\right) \left(t_{a_1}^2 t_{a_3}^2 t_{x_3}^{-2} t_{y_1}^{-2}\right)^{YF_3} \\
&= [t_{x_3}^2 t_{y_1}^2 t_{a_1}^{-2} t_{a_3}^{-2}, YF_3].
\end{aligned}$$

Since a conjugate of a commutator is a commutator, we have written $t_{a_1}^2 t_{a_2}^2 t_{a_3}^2 t_{a_4}^2$ as a product $C_1 C_2 C_3$ of three commutators.

A simple book keeping of the relators we have used now shows that the relator $t_{a_1}^{-2} t_{a_2}^{-2} t_{a_3}^{-2} t_{a_4}^{-2} \cdot C_1 C_2 C_3 = 1$ has signature 8, as a result of the eight lantern relators we employed in its derivation. □

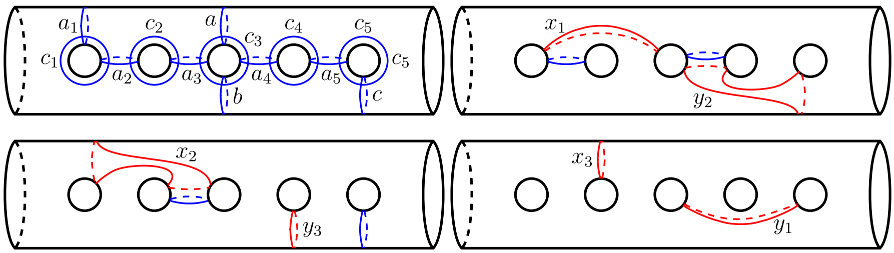

Fig. 5 The curves on Σ_5^2 used in the proof of Lemma 9

Lastly, we obtain a two-commutator expression on the genus–5 surface Σ_5^2:

Lemma 9 *Let a and b be pairwise disjoint nonseparating curves on Σ_5^2 such that $a \cup b$ separates the surface Σ_5^2 into two genus–2 components, and let x be another curve nonseparating in the complement of $a \cup b$. Then there is a relator*

$$t_a^{-1} t_b^{-1} t_x^{-2} C_1 C_2 = 1 \quad \textit{in } \mathrm{Mod}(\Sigma_5^2)$$

with signature 4.

Proof Consider the curves on Σ_5^2 as illustrated in Fig. 5. By the lantern relation, we have

$$\begin{aligned} t_{a_1} t_{a_2} t_{a_3} t_a &= t_{x_2} t_{x_3} t_{x_1}, \\ t_b t_{a_4} t_{a_5} t_c &= t_{y_1} t_{y_2} t_{y_3}. \end{aligned}$$

It follows now as in the proof of Lemma 8 that

$$\begin{aligned} t_{a_1}^2 t_{a_2}^2 t_{a_3}^2 t_a^2 &= t_{x_1}^2 t_{x_2}^2 t_{x_3} t_{x_3}^{t_{x_1}}, \\ t_b^2 t_{a_4}^2 t_{a_5}^2 t_c^2 &= t_{y_2}^2 t_{y_3}^2 t_{y_1} t_{y_1}^{t_{y_2}}, \end{aligned}$$

so that we have

$$t_{a_1}^2 t_{a_2}^2 t_{a_3}^2 t_{a_4}^2 t_{a_5}^2 t_a^2 t_b^2 t_c^2 = t_{x_1}^2 t_{y_2}^2 \cdot t_{x_2}^2 t_{y_3}^2 \cdot t_{x_3} t_{y_1} \cdot (t_{x_3} t_{y_1})^{t_{x_1} t_{y_2}},$$

or equivalently

$$t_a t_b t_{a_1}^2 = (t_{x_1}^2 t_{y_2}^2 t_{a_2}^{-2} t_{a_4}^{-2} \cdot t_{x_2}^2 t_{y_3}^2 t_{a_3}^{-2} t_c^{-2}) \cdot (t_{x_3} t_{y_1} t_a^{-1} t_{a_5}^{-1} \cdot (t_{x_3} t_{y_1} t_b^{-1} t_{a_5}^{-1})^{t_{x_1} t_{y_2}}).$$

One can easily find $F_1, F_2 \in \mathrm{Diff}^+(\Sigma_5^2)$ such that

- $F_1(x_1, y_2, a_2, a_4) = (a_3, c, x_2, y_3)$,
- $F_2(x_3, y_1, a, a_5) = (b, a_5, x_3, y_1)$.

The diffeomorphisms F_1 and F_2 can be chosen as

- $F_1 = t_{c_1} t_{x_2} t_{a_2} t_{c_1} \cdot t_{c_3} t_{a_3} t_{x_1} t_{c_3} \cdot t_{c_4} t_{y_3} t_{a_4} t_{c_4} \cdot t_{c_5} t_c t_{y_2} t_{c_5}$ and
- $F_2 = t_{c_5} t_{a_5} t_{y_1} t_{c_5} \cdot t_{z_2} t_{x_3} t_b t_{z_2} \cdot t_{z_1} t_b t_a t_{z_1}$,

where

- z_1 is any simple closed curve intersecting a and b transversely once and is disjoint from x_3, y_1 and a_5, and
- z_2 is any simple closed curve intersecting b and x_3 transversely once and is disjoint from y_1 and a_5.

It follows that, by using Lemma 5 once again, we can express $t_{x_1}^2 t_{y_2}^2 t_{a_2}^{-2} t_{a_4}^{-2} \cdot t_{x_2}^2 t_{y_3}^2 t_{a_3}^{-2} t_c^{-2}$, as well as $t_{x_3} t_{y_1} t_a^{-1} t_{a_5}^{-1} \cdot (t_{x_3} t_{y_1} t_b^{-1} t_{a_5}^{-1})^{t_{x_1} t_{y_2}}$, as a single commutator. We thus get a relator $t_a^{-1} t_b^{-1} t_{a_1}^{-2} C_1 C_2 = 1$, with signature 4. ◻

3.3 Commutator lengths of some mapping classes

For an element x in the commutator subgroup of a group G, let $\mathrm{cl}(x)$ denote its *commutator length*, the minimum number of commutators needed to express x as a product of commutators, and let $\mathrm{scl}(x) := \lim_{n\to\infty} \frac{\mathrm{cl}(x^n)}{n}$ be its *stable commutator length* [13].

Let c be a nonseparating curve on Σ_g^b. Recall that the mapping class group $\mathrm{Mod}(\Sigma_g^b)$ is perfect for $g \geq 3$, whereas $H_1(\mathrm{Mod}(\Sigma_2^b)) = \mathbb{Z}_{10}$, generated by the class of t_c; see e.g. [31]. It should be clear from our rendition of Tsuboi's trick in Sect. 2 that $\mathrm{cl}(t_c^n) = 2$ in $\mathrm{Mod}(\Sigma_g^b)$ when g is large enough (c.f. [35]). On the other hand, $\mathrm{cl}(t_c^n) \geq 2$ for any $g, n \in \mathbb{Z}^+$ [9]. Ensuing these facts is the question below, which is a refinement of Mess' question on $\mathrm{cl}(t_c^n)$ in [29]. To ease the upcoming discussion, let us define $g_2(n)$ to be the minimum genus $g \geq 2$ such that $\mathrm{cl}(t_c^n) = 2$ in $\mathrm{Mod}(\Sigma_g^1)$ for a nonseparating curve c, where it makes sense only to consider $n \equiv 0 \pmod{10}$ when $g = 2$.

Question 10 *For a given positive integer n, what is $g_2(n)$?*

The minimal genus in $\mathrm{Mod}(\Sigma_g^b)$ with $b > 1$ is the same as the one in $\mathrm{Mod}(\Sigma_g^1)$ since two surfaces of the same genus with boundaries embed into each other, and thus, we can cater commutator expressions from one another. By the same reasoning, any commutator expression for t_c^n in $\mathrm{Mod}(\Sigma_g^1)$ provides an upper bound for the minimal genus in $\mathrm{Mod}(\Sigma_g)$, too. However, it is not immediately clear to us whether the qualitative difference here would translate to a quantitative one; in fact, when $g = 1$, we can show that $\mathrm{cl}(t_c^{12}) = 2$ in $\mathrm{Mod}(\Sigma_1)$, but certainly not in $\mathrm{Mod}(\Sigma_1^1)$, where no nontrivial power of t_c is a product of commutators. Lastly, let us add that there is no reason—other than us not wanting to digress here any further—for not considering this question for separating curves.

Without sharper lower bounds in hand, it is challenging to answer the above question in full generality. Nonetheless we are able to determine the minimal genus $g_2(n)$ for a few values of n. We record them here:

Corollary 11 *We have $g_2(1) = g_2(2) = g_2(4) = 3$, and $g_2(10) = 2$.*

Proof It was shown by Ozbagci and the second author in [33] that $\mathrm{cl}(t_c) = 2$ when $g = 3$, and by the second author in [32] that $\mathrm{cl}(t_c^{10}) = 2$ when $g = 2$.

For t_c^2 and t_c^4 we proceed as follows. Let $\{\delta_1, \delta_2, \delta_3\}$ denote the three components of $\partial \Sigma_2^3$. By Lemma 7, we have $t_{\delta_1} t_{\delta_2} = C_1 C_2$ as well as $t_{\delta_1}^2 t_{\delta_2}^2 = D_1 D_2$, for some commutators C_j, D_j in $\mathrm{Mod}(\Sigma_2^3)$. Consider an embedding $\Sigma_2^3 \hookrightarrow \Sigma_3^1$ obtained by attaching a cylinder to the two boundary components δ_1 and δ_2 of Σ_2^3. The image of δ_1 and δ_2 are isotopic to the same non-separating curve c in Σ_3^1, whereas the remaining boundary component maps to the unique boundary component of Σ_3^1. Using the homomorphism $\mathrm{Mod}(\Sigma_2^3) \to \mathrm{Mod}(\Sigma_3^1)$ induced by this embedding, we thus derive two new expressions of the form $t_c^2 = C_1' C_2'$ and $t_c^4 = D_1' D_2'$ in $\mathrm{Mod}(\Sigma_3^1)$, where C_j', D_j' are commutators.

None of the t_c^n considered above are in the commutator subgroup of $\mathrm{Mod}(\Sigma_g^1)$ for smaller g, so we get the claimed values for $g_2(n)$. □

Remark 12 Corollary 11 provides a complete (meaning, for all $g \geq 3$) answer to Problem 2.13(b) in Kirby's List [29] for $n = 1, 2$ and 4. Using similar arguments, we can also conclude that $\mathrm{cl}(t_c^3) \leq 3$ for all $g \geq 3$ and is equal to 2 for $g \geq 5$.

Remark 13 If t_c^n is expressed as a product of two commutators in $\mathrm{Mod}(\Sigma_g^1)$, we get a genus–g Lefschetz fibration over Σ_2 with n nodes clustered all in one fiber. Then [9, Theorem 8] dictates that $g \geq \frac{n+6}{18}$. Thus, we have $g_2(n) \geq \frac{n+6}{18}$ for every n.

Lastly, we look at the stable commutator length of the boundary multitwist $\Delta = t_{\delta_1} t_{\delta_2} \cdots t_{\delta_b}$ in $\mathrm{Mod}(\Sigma_g^b)$, for $g \geq 2$, $b > 0$. Note that Δ is in the commutator subgroup of $\mathrm{Mod}(\Sigma_g^b)$, even when $g = 2$.

In [5], it was shown by Monden and the authors of this article that when $b = 1$, $\mathrm{cl}(\Delta^n) = \lfloor (n+3)/2 \rfloor$ for any positive n, so that $\mathrm{scl}(\Delta) = 1/2$. The main gain in the case of a boundary multitwist is the sharp lower bounds we get, which can be interpreted as a manifestation of the Milnor-Wood inequality [3, 24]. As the same lower bound carries over to $\mathrm{Mod}(\Sigma_g^b)$ for any $b > 1$, precise calculations of $\mathrm{cl}(\Delta^n)$ and $\mathrm{scl}(\Delta)$ are possible any time we are able to realize the lower bound. And when we cap the extra boundary component, Lemma 7 does precisely this for $b = 2$, as we can now express $\Delta^n = t_{\delta_1}^n t_{\delta_2}^n$ as a product of $\lfloor (n+3)/2 \rfloor$ commutators in $\mathrm{Mod}(\Sigma_2^2)$. In summary, when $b = 2$, we also have $\mathrm{cl}(\Delta^n) = \lfloor (n+3)/2 \rfloor$ and $\mathrm{scl}(\Delta) = 1/2$ in $\mathrm{Mod}(\Sigma_g^2)$. We record these calculations as well:

Corollary 14 *Let Δ be the boundary multitwist in* $\mathrm{Mod}(\Sigma_g^b)$. *For any $g \geq 2$ and $b = 1, 2$, we have* $\mathrm{scl}(\Delta) = 1/2$.

There is a little more we can say here: Let us say that a sequence $(c_1, c_2, \ldots, c_k)$ of curves on a surface is a *chain* if c_i and c_j intersect transversely at one point for $j = i \pm 1$ and are disjoint otherwise. For a chain $(c_1, c_2, \ldots, c_{2g})$ on Σ_g^1, and a chain $(c_1, c_2, \ldots, c_{2g+1})$ on Σ_g^2, consider the elements

$$S := t_{c_1} t_{c_2} \cdots t_{c_{2g}} \in \mathrm{Mod}(\Sigma_g^1) \quad \text{and} \quad T := t_{c_1} t_{c_2} \cdots t_{c_{2g+1}} \in \mathrm{Mod}(\Sigma_g^2).$$

It is well-known that $S^{4g+2} = \Delta$ and $T^{2g+2} = \Delta$; see e.g. [21]. Since scl is homogeneous and since $\mathrm{scl}(\Delta) = 1/2$, we get $\mathrm{scl}(S) = 1/(4(2g+1))$ in $\mathrm{Mod}(\Sigma_g^1)$ and $\mathrm{scl}(T) = 1/(4(g+1))$ in $\mathrm{Mod}(\Sigma_g^2)$, $g \geq 2$.

4 New surface bundles with positive signatures

This final section is dedicated to the proof of our main theorem.

Proof of Theorem 1 We begin with an elementary observation (cf. [19]): Say we have a monodromy factorization

$$C_1 \cdots C_h = 1 \quad \text{in } \mathrm{Mod}(\Sigma_g^1)$$

for a Σ_g–bundle over Σ_h with a section of self-intersection zero, where $\{C_i\}$ are commutators. Call this surface bundle (X, f). Then for any given $g' \geq g$ and $h' \geq h$, we can derive another monodromy factorization

$$C_1' \cdots C_h' C_{h+1}' \cdots C_{h'} = 1 \quad \text{in } \mathrm{Mod}(\Sigma_{g'}^1)$$

for a $\Sigma_{g'}$–bundle over $\Sigma_{h'}$ with a section of self-intersection zero, by using any embedding $\Sigma_g^1 \hookrightarrow \Sigma_{g'}^1$ and concatenating the trivial commutators $C_{h+1}', \ldots, C_h'$ to the product of the commutators C_i' which are the images of the original commutators C_i. Since embedding a relation or adding trivial commutators do not change the signature, for the new surface bundle (X', f') we obtained, we have $\sigma(X') = \sigma(X)$.[2]

It should be noted that in order to increase the fiber genus here, we needed the initial surface bundle (X, f) to have a section of self-intersection zero, or equivalently, a commutator identity supported on Σ_g^1 as opposed to Σ_g. Recall that this is also a necessary condition to invoke Theorem 4.

With the above in mind, we will generate the promised surface bundles with positive signatures from a few base examples. □

$h \geq 5$ and $g \geq 3$: For the curves a_i, δ_j on Σ_1^4 as shown on left in Fig. 6, the *four-holed torus relation* in [34] gives us a relator

$$t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} (t_{c_0} t_{c_1} t_{c_3} t_{c_0} t_{c_2} t_{c_4})^2 = 1 \quad \text{in } \mathrm{Mod}(\Sigma_1^4)$$

with $\sigma = -4$. We embed $\Sigma_1^4 \hookrightarrow \Sigma_3^1$ so that the images of the boundary components δ_i of Σ_1^4 are as shown on the right hand side of Fig. 6, denoted by the same letters. As δ_2 and δ_4 become isotopic in Σ_3^1, we represent them by the same curve after the embedding. The image of each c_i is labeled as a_i.

Thus, we have the following relator in $\mathrm{Mod}(\Sigma_3^1)$ with $\sigma = -4$:

$$1 = t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} (t_{a_0} t_{a_1} t_{a_3} t_{a_0} t_{a_2} t_{a_4})^2 = t_{\delta_1}^{-1} t_{\delta_2}^{-1} t_{\delta_3}^{-1} t_{\delta_4}^{-1} \, t_{a_0} t_{z_1} t_{z_2} t_{z_3} (t_{a_1} t_{a_2} t_{a_3} t_{a_4})^2 \,,$$

[2] This construction is good enough to address the mere existence of positive signature surface bundles with prescribed fiber and base genera. Otherwise, to generate surface bundles with larger signatures relative to their topology, it is certainly better to use extensions with nontrivial surface bundles with positive signatures.

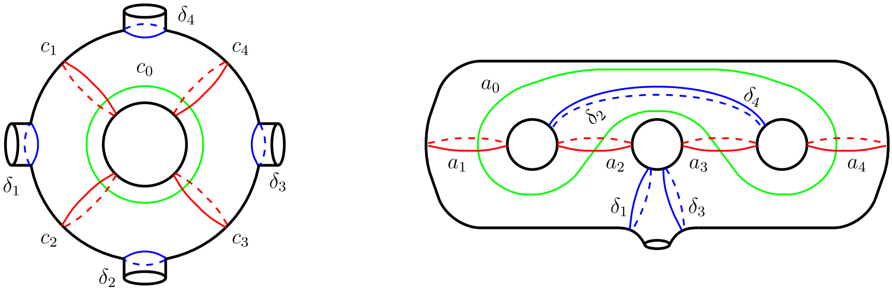

Fig. 6 4–holed torus curves and the embedding $\Sigma_1^4 \hookrightarrow \Sigma_3^1$

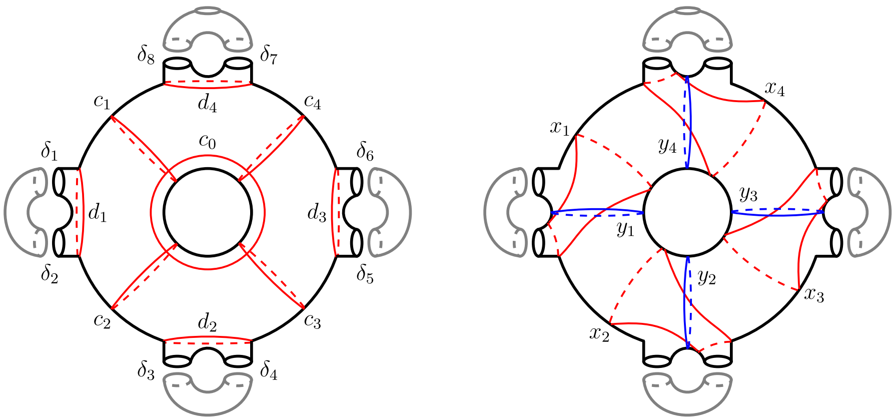

Fig. 7 The curves on Σ_5^1

where $z_1 = t_{a_1} t_{a_3}(a_0)$, $z_2 = t_{a_1} t_{a_2} t_{a_3} t_{a_4}(a_0)$ and $z_3 = t_{a_1} t_{a_1} t_{a_2} t_{a_3} t_{a_3} t_{a_4}(a_0)$. Using Lemmas 5 and 8, we can change this into

$$\begin{aligned} 1 &= (t_{\delta_1}^{-1} t_{a_0}\, t_{z_1} t_{\delta_2}^{-1})(t_{\delta_3}^{-1} t_{z_2}\, t_{z_3} t_{\delta_4}^{-1})(t_{a_1} t_{a_2} t_{a_3} t_{a_4})^2 \\ &= C_4 C_5 (t_{a_1} t_{a_2} t_{a_3} t_{a_4})^2 \cdot (t_{a_1} t_{a_2} t_{a_3} t_{a_4})^{-2} C_1 C_2 C_3 \\ &= C_4 C_5 \cdot C_1 C_2 C_3, \end{aligned}$$

or equivalently, into the relator

$$C_1 C_2 C_3 C_4 C_5 = 1 \quad \text{in} \quad \mathrm{Mod}(\Sigma_3^1)\,,$$

with signature $\sigma = -4 + 8 = 4$. This prescribes a Σ_3–bundle over Σ_5 with $\sigma = 4$ and a section of self-intersection zero. In turn, we get Σ_g–bundle over Σ_h with $\sigma = 4$ (and a section of self-intersection zero) for any $g \geq 3$ and $h \geq 5$.

$h = 4$ and $g \geq 5$: Take an embedding $\Sigma_1^4 \hookrightarrow \Sigma_5^1$ so that the c_i curves of the 4–holed torus relation (c.f. Fig. 6) are as shown on the left hand-side of Fig. 7, whereas the boundary curves δ_j are mapped to the separating curves d_j.

In the following, let us denote t_{c_i} by t_i for simplicity. From the 4-holed torus relation, we have the following in $\mathrm{Mod}(\Sigma_5^1)$:

$$\begin{aligned} t_{d_1}t_{d_2}t_{d_3}t_{d_4} &= t_0t_1t_3t_0t_2t_4t_0t_1t_3t_0t_2t_4 \\ &= t_0t_0^{t_1t_3t_0t_2t_4}t_1t_3t_0t_2t_4t_1t_3t_0t_2t_4 \\ &= t_0t_0^{t_1t_3t_0t_2t_4}t_0^{t_1t_3}t_1^2t_3^2t_2t_4t_0t_2t_4 \\ &= t_0t_0^{t_1t_3t_0t_2t_4}t_0^{t_1t_3}t_0^{t_1^2t_3^2t_2t_4}t_1^2t_2^2t_3^2t_4^2 \\ &= t_0t_{v_0}t_{v_1}t_{v_2}t_1^2t_2^2t_3^2t_4^2, \end{aligned}$$

where $v_0 = t_1t_3t_0t_2t_4(c_0)$, $v_1 = t_1t_3(c_0)$ and $v_2 = t_1^2t_3^2t_2t_4(c_0)$. Note that c_0 and v_0 cobound a subsurface Σ_2^2 in Σ_5^1. We multiply both sides of this equality by $t_{\delta_1}t_{\delta_2}\cdots t_{\delta_8}$ and, for $i = 1, 2, 3, 4$, apply the four lantern relations of the form

$$t_it_{i+1}t_{\delta_{2i-1}}t_{\delta_{2i}} = t_{y_i}t_{x_i}t_{d_i}\ ,$$

with the agreement that $t_5 = t_1$ to get

$$\begin{aligned} &t_{d_1}t_{d_2}t_{d_3}t_{d_4}t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4}t_{\delta_5}t_{\delta_6}t_{\delta_7}t_{\delta_8} \\ &\quad = t_0t_{v_0}t_{v_1}t_{v_2}\cdot t_1t_2t_{\delta_1}t_{\delta_2}\cdot t_2t_3t_{\delta_3}t_{\delta_4}\cdot t_3t_4t_{\delta_5}t_{\delta_6}\cdot t_4t_1t_{\delta_7}t_{\delta_8} \\ &\quad = t_0t_{v_0}t_{v_1}t_{v_2}\cdot t_{y_1}t_{x_1}t_{d_1}\cdot t_{y_2}t_{x_2}t_{d_2}\cdot t_{y_3}t_{x_3}t_{d_3}\cdot t_{y_4}t_{x_4}t_{d_4} \\ &\quad = t_0t_{v_0}t_{v_1}t_{v_2}\cdot t_{y_1}t_{x_1}\cdot t_{y_2}t_{x_2}\cdot t_{y_3}t_{x_3}\cdot t_{y_4}t_{x_4}\cdot t_{d_1}t_{d_2}t_{d_3}t_{d_4}. \end{aligned}$$

Now, cancelling all t_{d_i} gives

$$\begin{aligned} t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4}t_{\delta_5}t_{\delta_6}t_{\delta_7}t_{\delta_8} &= t_0t_{v_0}t_{v_1}t_{v_2}\cdot t_{y_1}t_{x_1}\cdot t_{y_2}t_{x_2}\cdot t_{y_3}t_{x_3}\cdot t_{y_4}t_{x_4} \\ &= t_0t_{v_0}t_{v_1}t_{v_2}\cdot t_{y_1}t_{y_2}t_{y_3}t_{y_4}\cdot t_{x_1}t_{x_2}t_{x_3}t_{x_4}\ . \end{aligned}$$

Supported in Σ_1^8, this is *an* 8*-holed torus relation* (cf. [23, 34]).

Hence, using Lemmas 5 and 9, we have

$$\begin{aligned} 1 &= t_0t_{v_0}t_{v_1}t_{v_2}\cdot t_{y_1}t_{y_2}t_{y_3}t_{y_4}t_{\delta_1}^{-1}t_{\delta_3}^{-1}t_{\delta_5}^{-1}t_{\delta_7}^{-1}\cdot t_{x_1}t_{x_2}t_{x_3}t_{x_4}t_{\delta_2}^{-1}t_{\delta_4}^{-1}t_{\delta_6}^{-1}t_{\delta_8}^{-1} \\ &= (t_0t_{v_0}t_{\delta_1}^2)\cdot(t_{v_1}t_{\delta_1}^{-1}t_{\delta_1}^{-1}t_{v_2}) \\ &\qquad\cdot(t_{y_1}t_{y_2}t_{y_3}t_{y_4}t_{\delta_1}^{-1}t_{\delta_3}^{-1}t_{\delta_3}^{-1}t_{\delta_7}^{-1}\cdot t_{x_1}t_{x_2}t_{x_3}t_{x_4}t_{\delta_2}^{-1}t_{\delta_4}^{-1}t_{\delta_6}^{-1}t_{\delta_8}^{-1}) \\ &= C_1C_2\cdot C_3\cdot C_4\ , \end{aligned}$$

in $\mathrm{Mod}(\Sigma_5^1)$, where all C_i are commutators. Here we have used the fact that there are two diffeomorphisms G_1 and G_2 of Σ_5^1 mapping (v_1, δ_1) to (δ_1, v_2) and $(y_1, y_2, y_3, y_4, \delta_1, \delta_3, \delta_5, \delta_7)$ to $(\delta_2, \delta_4, \delta_6, \delta_8, x_1, x_2, x_3, x_4)$, respectively. (We note that $\delta_i = \delta_{i+1}$ for $i = 1, 3, 5$.) We may take, for example,

- $G_1 = t_{w_3}t_{v_2}t_{\delta_3}t_{w_3}\cdot t_{w_2}t_{\delta_1}t_{v_1}t_{w_2}\cdot t_{w_1}t_{\delta_3}t_{\delta_1}t_{w_1}$, and
- $G_2 = \prod_{i=1}^4 t_{e_i}^{-1}t_{\delta_{2i}}^{-1}t_{y_i}t_{e_i}$,

where w_j and e_i are any curves with the property

- w_1 is disjoint from v_1 and intersects each of δ_1 and δ_3 transversely once,
- w_2 is disjoint from δ_3 and intersects each of v_1 and δ_1 transversely once,
- w_3 is disjoint from δ_1 and intersects each of v_2 and δ_3 transversely once, and
- e_i is the obvious longitude of the torus bounded by d_i.

Keeping track of the involved basic relators, one can easily see that the 8-holed torus relator has $\sigma = -4 + 4(+1) = 0$. Since the relator $t_0^{-1} t_{v_0}^{-1} t_{\delta_1}^{-2} \cdot C_1 C_2 = 1$ from Lemma 9 has $\sigma = 4$, we then conclude that the relator $C_1C_2C_3C_4 = 1$ in $\mathrm{Mod}(\Sigma_5^1)$ has $\sigma = 0 + 4 = 4$. This prescribes a Σ_5– bundle over Σ_4 with $\sigma = 4$ and a section of self-intersection zero. In turn, we get Σ_g–bundle over Σ_4 with $\sigma = 4$ (and a section of self-intersection zero) for any $g \geq 5$.
$g = 4$ *and* $h = 4$: We next construct a Σ_4–bundle over Σ_4 *without a section of self-intersection zero*. By [14, Theorem 1.3], there exists a genus–4 semi-simple holomorphic fibration over T^2, with exactly two singular fibers, each consisting of a genus–2 curve and an elliptic curve meeting transversally in two points. The total space of this fibration is smoothly a product $\Sigma_2 \times \Sigma_2$, so it has $\sigma = 0$. Thus, we have a relator

$$t_{a_1} t_{a_2} t_{b_1} t_{b_2} C_4 = 1 \quad \text{in } \mathrm{Mod}(\Sigma_4)$$

with $\sigma = 0$, where C_4 is a commutator. Since there exists an $F \in \mathrm{Diff}^+(\Sigma_4)$ with $F(a_i) = b_i$, we can rewrite the above relator as

$$\begin{aligned} 1 &= t_{a_1} t_{a_2} (t_{a_1} t_{a_2})^F C_4 \\ &= (t_{a_1} t_{a_2})^2 [(t_{a_1} t_{a_2})^{-1}, F]\, C_4 \\ &= (t_{a_1}^2 t_{a_2}^2) \cdot C_3 C_4\,, \end{aligned}$$

where we have set $C_3 := [(t_{a_1} t_{a_2})^{-1}, F]$. We can now invoke Lemma 7 to replace the first factor and get the relator

$$C_1C_2C_3C_4 = 1 \quad \text{in } \mathrm{Mod}(\Sigma_4)$$

which has $\sigma = 0 + 4 = 4$.
$h = 3, g \geq 9$ *and* $h = 2, g \geq 15$: Consider the Σ_3–bundle over Σ_5 we constructed above, which has $\sigma = 4$ and a section of self-intersection zero. By Theorem 4, we can derive two more bundles from it: a Σ_{15}–bundle over Σ_2 and a Σ_9–bundle over Σ_3, both also with $\sigma = 4$ and sections of self-intersection zero. In turn, we get a Σ_g–bundle over Σ_2 with $\sigma = 4$ for any $g \geq 15$, and a Σ_g–bundle over Σ_3 with $\sigma = 4$ for any $g \geq 9$.

All the surface bundles we constructed can be equipped with a Thurston symplectic form. This completes the proof of Theorem 1. □

Remark 15 In [19], Endo, Kotschick, Ozbagci, Stipsicz and the second author constructed surface bundles with positive signatures for all $g \geq 3$ and $h \geq 9$. On the other hand, Bryan and Donagi established in [10] that the base genus of a positive signature surface bundle could be as small as 2. These results were partially improved by Lee in [36], and most successfully by Monden in [41], who in particular produced positive signature surface bundles for all $g \geq 39$ and $h = 2$.

Remark 16 Following the recipe of [2], our surface bundles yield further examples of *non-holomorphic* surface bundles over surfaces with positive signatures at least for every $g \geq 16, h = 2$; $g \geq 10, h = 3$; $g \geq 6, h = 4$; $g \geq 4, h = 5$ and $g \geq 3, h \geq 6$. In particular, we answer the question of existence of non-holomorphic Σ_g–bundles over Σ_h with $\sigma \neq 0$ for all but finitely many pairs of (g, h). The signatures for all of these non-holomorphic examples can be chosen to be 4. In fact, to the best of our knowledge, there are no examples of holomorphic surface bundles with $\sigma = 4$. Is there an obstruction?

Remark 17 While the techniques of our paper can possibly be employed to generate surface bundles with high *Chern slope* c_1^2/c_2 (equivalently, high σ/e ratio) we do not currently have any examples with slopes higher than the ones obtained by Catanese and Rollenske in [15], making it all the more curious whether the Chern numbers of any (symplectic) surface bundle over a surface always satisfy $c_1^2/c_2 \leq 2 + 2/3$.

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