Viscous shock solutions to the stochastic Burgers equation

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Abstract

We define a notion of a viscous shock solution of the stochastic Burgers equation that connects "top" and "bottom" spatially stationary solutions of the same equation. Such shocks generally travel in space, but we show that they admit time-invariant measures when viewed in their own reference frames. Under such a measure, the viscous shock is a deterministic function of the bottom and top solutions and the shock location. However, the measure of the bottom and top solutions must be tilted to account for the change of reference frame. We also show a convergence result to these stationary shock solutions from solutions initially connecting two constants, as time goes to infinity.

1 Introduction

We consider the one-dimensional stochastic Burgers equation, forced by the gradient of a Gaussian noise that is smooth in space and white in time:

$$\mathrm{d}u(t,x) = \frac{1}{2}[\partial_x^2 u(t,x) - \partial_x(u^2)(t,x)]\mathrm{d}t + \mathrm{d}(\partial_x V)(t,x), \qquad t,x \in \mathbb{R}. \tag{1.1}$$

Here, $V = \rho * W$, where W is a cylindrical Wiener process on $L^2(\mathbb{R})$ whose covariance kernel is the identity, so the Itô time differential $\mathrm{d}W$ is a white noise on $\mathbb{R}\times\mathbb{R}$, and $\rho \in \mathcal{C}^\infty(\mathbb{R}) \cap H^1(\mathbb{R})$. We use $*$ to denote spatial convolution. A detailed construction of the solutions to (1.1) in a weighted space $\mathcal{X}$ of continuous functions that grow at most as $|x|^{1/2+}$ at infinity an be found in [13]. We recall the precise result and the definition of this space in Section 2.

Spacetime-stationary solutions to the stochastic Burgers equation on the whole real line have been studied extensively in the recent years. With apologies for the clumsiness, we will refer to the single-time laws of such spacetime-stationary solutions as "space-translation-invariant invariant measures." Kick-type random forcing in (1.1) was considered in [1, 2, 4], and the white in time setting, as in the present paper, was treated in [13]. We also refer to these papers for references to the extensive literature on the torus case $x \in \mathbb{R}/\mathbb{Z}$.

For the unforced Burgers equation ($V \equiv 0$ in (1.1)), spacetime-stationary solutions are simply constants. In addition, the unforced problem also admits traveling wave solutions, known as viscous shocks, that are perhaps of a more direct interest in applications than constant solutions. They have the explicit form

$$u(t,x) = -a\tanh(a(x-bt-c)) + b = \frac{b-a}{1+\mathrm{e}^{-2a(x-bt-c)}} + \frac{b+a}{1+\mathrm{e}^{2a(x-bt-c)}} \tag{1.2}$$

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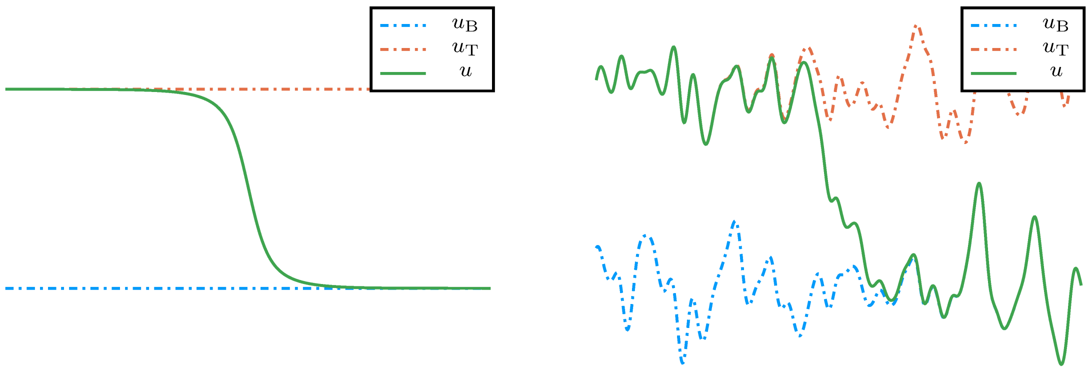

(a) Viscous shock solution u between lower and upper solutions u_B, u_T of the deterministic Burgers equation.

(b) Viscous shock solution u between lower and upper solutions u_B, u_T of the stochastic Burgers equation (1.3).

Figure 1.1: Viscous shock solutions to the deterministic and stochastic Burgers equations.

for constants $a, b, c \in \mathbb{R}$, as can be checked directly. Such solutions "connect" the two constant solutions $b \pm |a|$, by which we mean that

$$\lim_{x \to \mp\infty} u(t, x) = b \pm |a|,$$

as depicted in Figure 1.1a.

Existence and classification of the random shock measures

In this paper, we discuss analogues of these viscous shocks in the stochastically forced case. We first must have analogues of the constant solutions that are connected by the shocks. We recall that [13] considered spacetime-stationary *families* of solutions to (1.1), by which we mean jointly spacetime-stationary solutions to the system of equations

$$\mathrm{d}u_i(t, x) = \frac{1}{2}[\partial_x^2 u_i(t, x) - \partial_x(u_i^2)(t, x)]\mathrm{d}t + \mathrm{d}(\partial_x V)(t, x), \qquad t, x \in \mathbb{R}, \tag{1.3}$$

which are coupled through the noise V. As shown in [13, Theorem 1.1], such coupled spacetime-stationary solutions are almost-surely ordered according to their mean. One can construct them as long-time limits of solutions starting with constant initial conditions, and the limits preserve the order of the constants. It was also shown in [13, Theorem 1.1] that for any $a_k \in \mathbb{R}$, $k = 1, \ldots, N$, there exists a unique extremal space-translation-invariant invariant measure $\nu_{a_1,\ldots,a_N}$ for (1.3) such that if $(v_1, \ldots, v_N) \sim \nu_{a_1,\ldots,a_N}$, then $\mathbb{E}v_k(x) = a_k$ and $\mathbb{E}v_k(x)^2 < \infty$ for each $x \in \mathbb{R}$ and $k = 1, \ldots, N$. The extremal invariant measures serve as attractors for the solutions to the Cauchy problem for a large class of "not far from periodic" initial conditions.

A stochastic shock, rather than connecting two constants as in the deterministic case, connects two ordered components u_B and u_T ("bottom" and "top") of a (space-stationary, say) solution to (1.3) with $N = 2$, as illustrated in Figure 1.1b. We define the set of bottom and top solutions

$$\mathcal{X}_{BT} = \left\{(u_B, u_T) \in \mathcal{X}^2 \ : \ u_B < u_T \text{ and } \lim_{x \to \pm\infty} \int_0^x [u_T - u_B](y)\,\mathrm{d}y = \pm\infty\right\}. \tag{1.4}$$

The space of viscous shocks is then

$$\mathcal{X}_{\mathrm{Sh}} = \left\{(u_{\mathrm{B}}, u_{\mathrm{T}}, u) \in \mathcal{X}_{\mathrm{BT}} \times \mathcal{X} \ : \ \int_{-\infty}^{0} |u_{\mathrm{T}} - u| + \int_{0}^{\infty} |u - u_{\mathrm{B}}| < \infty\right\}. \tag{1.5}$$

If $(u_{\mathrm{B}}, u_{\mathrm{T}}, u) \in \mathcal{X}_{\mathrm{Sh}}$, then we say that u is a shock connecting u_{T} on the left to u_{B} on the right. We note that (1.4)–(1.5) give an "L^1" notion of a shock, which is convenient because of the nice L^1 properties of the stochastic Burgers dynamics (described in [13, Section 3]).

Given a pair $(u_{\mathrm{B}}, u_{\mathrm{T}}) \in \mathcal{X}_{\mathrm{BT}}$ of bottom and top solutions to (1.1), one can construct a semi-explicit shock solution to this equation in terms of u_{B} and u_{T}, generalizing (1.2), so that the triple $(u_{\mathrm{B}}, u_{\mathrm{T}}, u)$ lies in $\mathcal{X}_{\mathrm{Sh}}$, as follows. If $v_{\mathrm{B}}(x) < v_{\mathrm{T}}(x)$ for all $x \in \mathbb{R}$, and $b, \gamma \in \mathbb{R}$, define

$$\mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}](x) = \frac{v_{\mathrm{B}}(x)}{1 + \exp\{\gamma - \int_b^x [v_{\mathrm{T}} - v_{\mathrm{B}}](y)\,\mathrm{d}y\}} + \frac{v_{\mathrm{T}}(x)}{1 + \exp\{-\gamma + \int_b^x [v_{\mathrm{T}} - v_{\mathrm{B}}](y)\,\mathrm{d}y\}}. \tag{1.6}$$

Let $(u_{\mathrm{B}}, u_{\mathrm{T}})$ be a solution to (1.3) with $N = 2$ such that $u_{\mathrm{B}}(t,x) < u_{\mathrm{T}}(t,x)$ for all t and x, and b_t be the solution to the non-autonomous ordinary differential equation

$$\partial_t b_t = \frac{1}{2}(-\partial_x(\log(u_{\mathrm{T}} - u_{\mathrm{B}})) + u_{\mathrm{B}} + u_{\mathrm{T}})(t, b_t). \tag{1.7}$$

If we set

$$u(t,x) = \mathscr{S}_{b_t,\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)] \tag{1.8}$$

for some fixed $\gamma \in \mathbb{R}$, then it turns out that $(u_{\mathrm{B}}, u_{\mathrm{T}}, u)$ solves (1.3) with $N = 3$. This is a general fact true for any pair of ordered solutions u_{B} and u_{T} of (1.3). We will refer to b_t as the "shock position." A more useful interpretation of b_t, in terms of the KPZ equation, is presented in Lemma 3.1 in Section 3. We postpone it until then as it requires some additional notions.

If $(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot) \in \mathcal{X}_{\mathrm{BT}}$ (for which it suffices that this holds at $t = 0$, as shown in Lemma 2.2 below), and $u(t,x)$ is given by (1.8), then for $x - b_t \gg 1$ we have $u \approx u_{\mathrm{B}}$, while for $x - b_t \ll -1$ we have $u \approx u_{\mathrm{T}}$. This is a direct way to see that (1.8) defines a connection between u_{T} on the left and u_{B} on the right. The width of the transition region around b_t depends on the size of $u_{\mathrm{T}} - u_{\mathrm{B}}$ near b_t: the closer u_{T} and u_{B} get near b_t, the wider the shock region. We will see this reflected in the tilt of the invariant measure in Theorem 1.1 below.

The system (1.7)–(1.8) involves the random noise V only through u_{B} and u_{T}: conditional on the top and bottom solutions, the shock position and profile are completely determined by (1.7) and (1.8), respectively. The expression (1.8) is a direct generalization of (1.2). Indeed, if $u_{\mathrm{B}} \equiv b - a$ and $u_{\mathrm{T}} = b + a$, with some $b \in \mathbb{R}$ and $a > 0$, then for any $b_0 \in \mathbb{R}$, $b_t = bt + b_0$ solves (1.7). Then (1.8) reduces to (1.2), with $c = b_0 - \gamma/2$.

Motivated by (1.8) and continuing to assume the ordering of u_{B} and u_{T}, we can make a change of variables

$$\zeta = \frac{1}{2}\int_{b_t}^{x} [u_{\mathrm{T}} - u_{\mathrm{B}}](t,x)\,\mathrm{d}x, \qquad U = \frac{2u - u_{\mathrm{B}} - u_{\mathrm{T}}}{u_{\mathrm{T}} - u_{\mathrm{B}}}. \tag{1.9}$$

Under this change of variables, (1.8) becomes the deterministic and time-independent profile

$$U(t,\zeta) = -\tanh\zeta, \tag{1.10}$$

which is the same as (1.2) in the deterministic case. As we will see, under the same change of variables, the stochastic Burgers equation (1.1) takes the strikingly simple form

$$\partial_t U(t,\zeta) = \frac{1}{8}\partial_\zeta\left((u_{\mathrm{T}} - u_{\mathrm{B}})^2 \cdot (\partial_\zeta U - U^2 + 1)\right)(t,\zeta), \tag{1.11}$$

to which (1.10) is a solution.

The above computations did not use any statistical properties of u_{B} and u_{T}. Of particular interest to us is the case when $(u_{\mathrm{B}}, u_{\mathrm{T}})$ is a spacetime-stationary solution to (1.3) as constructed in [13, Theorem 1.2]. Assume that $u_{\mathrm{B}}(t,x) < u_{\mathrm{T}}(t,x)$ for all t and x almost surely. In the deterministic case, the viscous shock profile is stationary in the reference frame that moves with the constant speed b of the shock. In the random case, the triple $(u_{\mathrm{B}}, u_{\mathrm{T}}, u)$ is not expected to be stationary in time, despite the time-stationarity of the difference $u_{\mathrm{T}} - u_{\mathrm{B}}$ driving (1.11), because the shock location b_t need not be stationary. It is natural to expect that $(u_{\mathrm{B}}, u_{\mathrm{T}}, u)$ would at least be time-stationary in a reference frame moving along with b_t: that is, that the randomly shifted triple $\tau_{b_0-b_t}(u_{\mathrm{B}}, u_{\mathrm{T}}, u)$ would be time-stationary. Here, τ is the spatial translation defined by

$$\tau_x f(y) = f(y-x). \tag{1.12}$$

This is not quite right either, because b_t is not independent of $(u_{\mathrm{B}}, u_{\mathrm{T}})$. We need to tilt the invariant measure to account for this dependence, as described in the following theorem.

Theorem 1.1. *Let ν be a space-translation-invariant invariant measure for the dynamics (1.3) with $N=2$, such that if $(v_{\mathrm{B}}, v_{\mathrm{T}}) \sim \nu$, then $\mathbb{E}v_{\mathrm{B}}(x)^2, \mathbb{E}v_{\mathrm{T}}(x)^2 < \infty$ and $v_{\mathrm{B}}(x) < v_{\mathrm{T}}(x)$ for all $x \in \mathbb{R}$ almost surely. Fix $b \in \mathbb{R}$ and define the measure $\hat{\nu}^{[b]}$, absolutely continuous with respect to ν, with Radon–Nikodym derivative*

$$\frac{\mathrm{d}\hat{\nu}^{[b]}}{\mathrm{d}\nu}(v_{\mathrm{B}}, v_{\mathrm{T}}) = \frac{(v_{\mathrm{T}} - v_{\mathrm{B}})(b)}{\lim\limits_{L\to\infty} \frac{1}{L}\int_0^L [v_{\mathrm{T}} - v_{\mathrm{B}}](x)\,\mathrm{d}x}. \tag{1.13}$$

Fix $\gamma \in \mathbb{R}$ and let $(u_{\mathrm{B}}, u_{\mathrm{T}}, u)$ solve (1.3) with initial condition $(u_{\mathrm{B}}, u_{\mathrm{T}}) \sim \hat{\nu}^{[b]}$, independent of the noise, and

$$u(0,x) = \mathscr{S}_{b,\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(0,\cdot)].$$

Let b_t solve (1.7) with $b_0 = b$. Then for all $t \geq 0$ we have

$$\mathrm{Law}(\tau_{b-b_t}(u_{\mathrm{B}}, u_{\mathrm{T}}, u)(t,\cdot)) = \mathrm{Law}((u_{\mathrm{B}}, u_{\mathrm{T}}, u)(0,\cdot)). \tag{1.14}$$

Note that the limit in the denominator in (1.13) exists ν-almost surely by the Birkhoff–Khinchin theorem.

According to [13, Theorem 1.2], any space-translation-invariant invariant measure for (1.3) having bounded second moment can be decomposed into a mixture of extremal such measures, which are classified by their means. As in [13], we use the notation $\nu_{a_{\mathrm{B}},a_{\mathrm{T}}}$ for the extremal measure with mean $(a_{\mathrm{B}}, a_{\mathrm{T}})$, and we write $\hat{\nu}^{[b]}_{a_{\mathrm{B}},a_{\mathrm{T}}}$ for the tilt of this measure defined by (1.13). If $(v_{\mathrm{B}}, v_{\mathrm{T}}) \sim \nu_{a_{\mathrm{B}},a_{\mathrm{T}}}$, then [13, Theorem 1.2, property (P5)] and the Birkhoff-Khinchin theorem imply that

$$\lim_{L\to\infty} \frac{1}{L}\int_0^L [v_{\mathrm{T}} - v_{\mathrm{B}}](x)\,\mathrm{d}x = a_{\mathrm{T}} - a_{\mathrm{B}},$$

so in that case the change of measure formula (1.13) has the simpler form

$$\frac{\mathrm{d}\hat{\nu}^{[b]}_{a_{\mathrm{B}},a_{\mathrm{T}}}}{\mathrm{d}\nu_{a_{\mathrm{B}},a_{\mathrm{T}}}}(v_{\mathrm{B}}, v_{\mathrm{T}}) = \frac{(v_{\mathrm{T}} - v_{\mathrm{B}})(b)}{a_{\mathrm{T}} - a_{\mathrm{B}}}. \tag{1.15}$$

Note that (1.14) includes the statement that $\mathrm{Law}(\tau_{b-b_t}(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)) = \mathrm{Law}((u_{\mathrm{B}}, u_{\mathrm{T}})(0,\cdot))$. In fact, this statement contains most of the content of (1.14) once the semi-explicit nature of the shock profiles is understood as in the discussion following (1.6). Nonetheless, the change of measure (1.13) can be more easily understood in the context of the shocks. (A more direct computational reason for the tilt can be found in Lemma 3.1 below, and in particular, in expression (3.12).) The tilt (1.13) is a type of size-biasing (or "mass-biasing"), arising from the mass conservation of both the Burgers dynamics and the change of variables (1.9). The Burgers dynamics (1.3) has the form of a conservation law and so preserves the integrals of differences between solutions (as is proven formally in [13, Proposition 3.3]). As we show in Proposition 3.3, we have

$$\int_{\mathbb{R}} (\mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}] - \mathscr{S}_{b,\gamma'}[v_{\mathrm{B}}, v_{\mathrm{T}}]) = \gamma - \gamma' \tag{1.16}$$

for any $b, \gamma, \gamma' \in \mathbb{R}$ and $(v_{\mathrm{B}}, v_{\mathrm{T}})$ in an appropriate function space. This is why γ remains fixed in the evolution (1.8) and is thus a convenient way to parametrize the shocks. Consider the entire ensemble of shocks $(\mathscr{S}_{b_t,\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)])_{\gamma\in\mathbb{R}}$ evolving together between upper and lower solutions u_{B} and u_{T}. Now consider $\gamma, \gamma' \in \mathbb{R}$ with $|\gamma-\gamma'| \ll 1$, and let $b_t^{(\gamma')}$ be a solution to (1.7) (with a different initial condition than b_t) such that

$$\mathscr{S}_{b_t,\gamma'}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)] = \mathscr{S}_{b_t^{(\gamma')},\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)]. \tag{1.17}$$

It follows from (1.16) and (1.17) that

$$\gamma - \gamma' = \int (\mathscr{S}_{b_t,\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)] - \mathscr{S}_{b_t^{(\gamma')},\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)]) \sim (b_t - b_t^{(\gamma')}) \cdot (u_{\mathrm{T}} - u_{\mathrm{B}})(t, b_t)$$

is independent of t, and we must have

$$b_t - b_t^{(\gamma')} \sim \frac{\gamma - \gamma'}{(u_{\mathrm{T}} - u_{\mathrm{B}})(t, b_t)}.$$

This means that in an interval of size $\varepsilon \ll 1$ around b_t, we may expect to find $b_t^{(\gamma')}$ for

$$|\gamma' - \gamma| \lesssim \varepsilon (u_{\mathrm{T}} - u_{\mathrm{B}})(t, b_t),$$

hence the change of measure (1.13).

One may ask about the uniqueness of the stationary shock profile measures given by (1.13). This question is not entirely well-posed because one must specify the reference frame in which we require stationarity. We give a uniqueness statement for the shock in Proposition 5.5 if the reference frame is assumed to be given by a shock location $\{b_t\}_{t\geq 0}$ satisfying (1.7). The more intrinsic definition of b_t in Section 3 indicates that this choice of the reference frame is quite natural but further work is needed to understand uniqueness without fixing a particular reference frame.

Stationary shock behavior has been extensively studied for asymmetric simple exclusion processes, which are discrete microscopic models for Burgers-type dynamics. Similar phenomenology occurs there: a shock moves randomly through space, but in the reference frame of the shock itself, there is a stationary measure for the particle system [12, 14, 15]. We refer to the book [25] for more discussion and references.

Stability of the random shocks

We now turn to the stability of the shocks (1.8). The study of the stability of the shocks (1.2) in the deterministic case has a long history. Without any attempt at completeness, we mention in particular the works [16, 17, 18, 19, 20, 21, 22, 24, 28, 29, 30, 31, 32] and the books [11, 33]. In a similar spirit to our problem is [35], which shows convergence to shock waves when the equation is deterministic but the initial condition is random. As the Burgers equation is nonlinear, these issues are closely related.

In the present stochastic setting, we show that if the initial condition is sandwiched between two hyperbolic tangent functions (translated and scaled appropriately), with the same limits at infinity, then an intermediate solution, shifted appropriately, converges to a shock of the form (1.8). Actually, we show a somewhat stronger statement, that if we consider a finite collection of such solutions, then they converge jointly to a family of such shocks. In the following theorem, as above, $\mathcal{X}$ denotes the Fréchet space of continuous functions on $\mathbb{R}$ growing more slowly at infinity than $(1+|x|)^\ell$ for all $\ell > 1/2$, equipped with the corresponding family of weighted seminorms specified in Section 2.

Theorem 1.2. *Fix real constants $a_{\mathrm{B}} < a_{\mathrm{T}}$ and $\gamma_{\mathrm{L}} < \gamma_{\mathrm{R}}$. Let $(u_{\mathrm{B}}, u_{\mathrm{T}}, u_1, \ldots, u_{\mathrm{N}}) \in \mathcal{C}([0,\infty); \mathcal{X}^{2+N})$ solve (1.3) with initial conditions $u_{\mathrm{B}}(0,\cdot) \equiv a_{\mathrm{B}}$, $u_{\mathrm{T}}(0,\cdot) \equiv a_{\mathrm{T}}$, and for all $x \in \mathbb{R}$ and $i = 1, \ldots, N$,*

$$\mathscr{S}_{0,\gamma_{\mathrm{L}}}[a_{\mathrm{B}}, a_{\mathrm{T}}](x) \le u_i(0,x) \le \mathscr{S}_{0,\gamma_{\mathrm{R}}}[a_{\mathrm{B}}, a_{\mathrm{T}}](x).$$

For each $i = 1, \ldots, N$, let $b^{(i)}$ be the unique b so that

$$\int_{-\infty}^{b} [u_{\mathrm{T}} - u_i](0,x)\,\mathrm{d}x = \int_{b}^{\infty} [u_i - u_{\mathrm{B}}](0,x)\,\mathrm{d}x, \tag{1.18}$$

and let $b_t^{(i)}$ solve (1.7) with initial condition $b_0^{(i)} = b^{(i)}$. Let $(v_{\mathrm{B}}, v_{\mathrm{T}}) \sim \hat{\nu}_{a_{\mathrm{B}},a_{\mathrm{T}}}^{[b^{(1)}]}$ (defined after Theorem 1.1) and for $i = 1, \ldots, N$, put

$$v_i = \mathscr{S}_{b^{(1)}, b^{(i)} - b^{(1)}}[v_{\mathrm{B}}, v_{\mathrm{T}}]$$

and $\mathbf{v} = (v_{\mathrm{B}}, v_{\mathrm{T}}, v_1, \ldots, v_{\mathrm{N}})$. Then we have

$$\mathrm{Law}(\tau_{b^{(1)} - b_t^{(1)}}(u_{\mathrm{B}}, u_{\mathrm{T}}, u_1, \ldots, u_N)(t,\cdot)) \to \mathrm{Law}\,\mathbf{v} \tag{1.19}$$

weakly with respect to the topology of $\mathcal{X}^{2+N}$. Also, with probability 1 we have

$$\lim_{t\to\infty} \|u_i(t,\cdot) - \mathscr{S}_{b_t^{(i)},0}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)]\|_{L^1(\mathbb{R})} = 0. \tag{1.20}$$

We recall that, as far as the stability of u_{B} and u_{T} themselves is concerned, it was shown in [13, Theorem 1.3] that if $\mathbf{u} \in \mathcal{C}([0,\infty); \mathcal{X}^N)$ solves (1.3) with initial condition that is a decaying perturbation of a spatially-periodic state $\mathbf{u}(0,\cdot)$, then $\mathrm{Law}(\mathbf{u}(t,\cdot))$ converges to $\nu_{a_1,\ldots,a_N}$ weakly with respect to the topology of $\mathcal{X}^N$ as $t \to \infty$. Even stronger results are available for the stability of the spacetime-stationary solutions for the kick forcing of the Burgers equation considered in [4]. Theorem 1.2, however, only considers the case when the top and bottom solutions are initially constant in space.

An interpretation of the shocks in terms of the Cole–Hopf transform

The Burgers viscous shocks can be interpreted in terms of the Cole–Hopf transform [6, 8, 18]. Recall that if ϕ solves the multiplicative stochastic heat equation (SHE)

$$\mathrm{d}\phi = \frac{1}{2}\partial_x^2 \phi - \phi \mathrm{d}V, \tag{1.21}$$

then $h = -\log\phi$ solves the KPZ equation [23]

$$\mathrm{d}h = \frac{1}{2}[\partial_x^2 h - (\partial_x h)^2 + \|\rho\|_{L^2(\mathbb{R})}^2]\mathrm{d}t + \mathrm{d}V, \tag{1.22}$$

and $u = \partial_x h = -(\partial_x\phi)/\phi$ solves the stochastic Burgers equation (1.1). Of course, this transform can be extended to the system of equations (1.3). The multiplicative SHE (1.21) has the obvious advantage of being linear, but for our purposes both (1.21) and (1.22) have the disadvantage that they do not admit spacetime-stationary solutions. Spacetime-stationary solutions only arise when the derivative is taken to form u, which destroys the growing zero-frequency mode of h.

Nonetheless, the stable viscous shock solutions (1.8) have a simple interpretation in terms of solutions to the SHE (1.21). Indeed, if for $\mathrm{X} \in \{\mathrm{B}, \mathrm{T}\}$, we have $u_\mathrm{X} = -(\partial_x\phi_\mathrm{X})/\phi_\mathrm{X}$, and ϕ_X solves (1.21), then by linearity $\phi_\mathrm{B} + \phi_\mathrm{T}$ solves (1.21) as well, so that

$$u = -\frac{\partial_x(\phi_\mathrm{B} + \phi_\mathrm{T})}{\phi_\mathrm{B} + \phi_\mathrm{T}} = \frac{u_\mathrm{B}}{1 + \phi_\mathrm{T}/\phi_\mathrm{B}} + \frac{u_\mathrm{T}}{1 + \phi_\mathrm{B}/\phi_\mathrm{T}}$$

solves (1.1). Noting that

$$(\phi_\mathrm{T}/\phi_\mathrm{B})(t, x) = (\phi_\mathrm{T}/\phi_\mathrm{B})(t, 0)\exp\left\{-\int_0^x [u_\mathrm{T} - u_\mathrm{B}](t, y)\,\mathrm{d}y\right\},$$

we recover an expression of the form (1.8) by appropriate choices of γ and b_t.

Another, even more explicit, perspective considers the KPZ equation in relation to the change of variables (1.9). As we show in Lemma 3.1, solutions to (1.7) are given by inverting (as a function from $\mathbb{R} \to \mathbb{R}$) half the difference between two solutions to (1.22), started at the corresponding integrals of the initial conditions for u_B and u_T. Therefore, the integral appearing in the change of variables (1.9) is exactly half the difference of two solutions to (1.22). In addition, as shown in Lemma 3.1, the definition of the shock location b_t is more naturally given in terms of the solution to (1.22) than directly in terms of the Burgers equation itself.

Estimating the scale of fluctuations of b_t is thus a question about the growth of the difference between two solutions to (1.22). Long-time statistics for solutions to (1.22) are in general difficult to estimate, especially in non-integrable cases such as ours where exact calculations are not available. See [5, 7, 9, 27, 26] and their references for some results for integrable models, and [3] for more background and conjectures in this direction. We do not address the question of estimating b_t in the present paper, reserving it for future work.

Organization of the paper We begin by introducing the relevant function spaces and recalling the necessary setup and results from [13] in Section 2. We discuss the change of variables (1.9), the resulting PDE (1.11), and the explicit shock solutions (1.8) in Section 3. We derive the change of measure (1.13) and prove Theorem 1.1 in Section 4. In Section 5, we discuss more general shock profiles and give a partial characterization of a certain notion of stationary shock profile (assuming some nontrivial integrability conditions). Finally, we prove our stability result Theorem 1.2 in Section 6. A technical lemma is relegated to Appendix A.

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2 Function spaces and spacetime-stationary solutions

Because the viscous shock solutions to (1.1) are so intimately tied to the spacetime-stationary solutions they connect, we rely on the framework and many ingredients from [13]. Here, we review the setup and quote some of the results we will use.

First, we recall some definitions and set the notation. For a positive weight $w = w(x)$, we denote by $\mathcal{C}_w$ the Banach space of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that the norm

$$\|f\|_{\mathcal{C}_w} = \sup_{x\in\mathbb{R}} \frac{|f(x)|}{w(x)}$$

is finite. Given $\ell \in \mathbb{R}$, we set $\mathrm{p}_\ell = \langle x\rangle^\ell$, where $\langle x\rangle = \sqrt{4+x^2}$, and let

$$\mathcal{X} = \bigcap_{\ell>1/2} \mathcal{C}_{\mathrm{p}_\ell},$$

equipped with the Fréchet space topology induced by the family of norms $\{\|\cdot\|_{\mathcal{C}_{\mathrm{p}_{1/2+1/k}}}\}_{k\in\mathbb{N}}$. This space is denoted by $\mathcal{X}_{1/2}$ in [13]. The space $\mathcal{X}$ is separable and hence a Polish space.

The equation (1.3) is well-posed in $\mathcal{X}^N$, as was proved in [13, Theorem 1.1]. In particular, there is a random solution map $\Psi : \mathcal{X}^N \to \mathcal{C}([0,\infty); \mathcal{X}^N)$ for the equation (1.3). The map Ψ is almost surely continuous with respect to the locally uniform topology on $\mathcal{C}([0,\infty); \mathcal{X}^N)$. It was also shown in [13] that (1.3) has a comparison principle ([13, Proposition 3.1]), and if the difference of two components of a solution to (1.3) is in $L^1(\mathbb{R})$ at $t = 0$, then its $L^1(\mathbb{R})$ norm is non-increasing in time ([13, Proposition 3.2]).

As we have mentioned, it is shown in [13] that for any given set of means $a_1, \ldots, a_N$, there is a unique extremal space-translation-invariant and (1.3)-invariant measure $\nu_{a_1,\ldots,a_N}$ on $\mathcal{X}^N$ such that if $\mathbf{v} = (v_1, \ldots, v_N) \sim \nu_{a_1,\ldots,a_N}$, then $\mathbb{E}v_i(x) = a_i$ and $\mathbb{E}v_i(x)^2 < \infty$ for all $x \in \mathbb{R}$. Here, "extremal" means that the measure cannot be written as a nontrivial convex combination of measures with the same properties.

In deriving properties of the shock solutions, it will be convenient to state some necessary properties of the "bottom" and "top" spatially-stationary solutions in a nonprobabilistic way. We encode these properties in the function space

$$\mathcal{X}_{\mathrm{BT}} = \left\{(v_{\mathrm{B}}, v_{\mathrm{T}}) \in \mathcal{X}^2 \ : \ v_{\mathrm{B}} < v_{\mathrm{T}} \text{ and } \lim_{x\to\pm\infty} \int_0^x [v_{\mathrm{T}} - v_{\mathrm{B}}](y)\,\mathrm{d}y = \pm\infty\right\}, \tag{2.1}$$

as previously defined in (1.4). The conditions in (2.1) are necessary so that the change of variables (1.9) is invertible.

The next two lemmas in this section are technical in nature. Lemma 2.1 shows that $\mathcal{X}_{\mathrm{BT}}$ is a Polish space, so we can apply standard probabilistic tools such as Prokhorov's theorem and the Skorokhod representation theorem. Lemma 2.2 shows that the dynamics (1.3) preserves $\mathcal{X}_{\mathrm{BT}}$, so we can think of solutions to (1.3) as Markov processes on $\mathcal{X}_{\mathrm{BT}}$.

Lemma 2.1. *The space $\mathcal{X}_{\mathrm{BT}}$ is a Polish space.*

Proof. We can write

$$\{(v_{\mathrm{B}}, v_{\mathrm{T}}) \in \mathcal{X}^2 \ : \ v_{\mathrm{B}} < v_{\mathrm{T}}\} = \bigcap_{L\in\mathbb{N}} \{(v_{\mathrm{B}}, v_{\mathrm{T}}) \in \mathcal{X}^2 \ : \ v_{\mathrm{B}}(x) < v_{\mathrm{T}}(x) \text{ for all } x \in [-L, L]\}$$

and

$$\left\{(v_{\mathrm{B}},v_{\mathrm{T}})\in\mathcal{X}^2\ :\ \lim_{x\to\pm\infty}\int_0^x[v_{\mathrm{T}}-v_{\mathrm{B}}](y)\,\mathrm{d}y=\pm\infty\right\}$$
$$=\bigcap_{M\in\mathbb{N}}\left(\bigcup_{L\in(0,\infty)}\left\{(v_{\mathrm{B}},v_{\mathrm{T}})\in\mathcal{X}^2\ :\ \pm\int_0^{\pm L}[v_{\mathrm{T}}-v_{\mathrm{B}}](y)\,\mathrm{d}y>M\right\}\right).$$

Therefore, $\mathcal{X}_{\mathrm{BT}}$ is a countable intersection of open subsets of $\mathcal{X}^2$, or in other words a G_δ subset of the Polish space $\mathcal{X}^2$. By Alexandrov's theorem (see e.g. [34, Theorem 2.2.1]) a G_δ subset of a Polish space is again a Polish space. ☐

Lemma 2.2. *If $\mathbf{u}\in\mathcal{C}([0,\infty);\mathcal{X}^{2+N})$ solves (1.3) with initial condition $\mathbf{u}(0,\cdot)\in\mathcal{X}_{\mathrm{BT}}\times\mathcal{X}^N$, then with probability 1 we have, for all $t\geq 0$, that $\mathbf{u}(t,\cdot)\in\mathcal{X}_{\mathrm{BT}}\times\mathcal{X}^N$.*

Proof. Let us write $\mathbf{u}=(\mathbf{u}_{\mathrm{BT}},\tilde{\mathbf{u}})=((u_{\mathrm{B}},u_{\mathrm{T}}),\tilde{\mathbf{u}})$. The comparison principle ([13, Theorem 3.1]) implies that, with probability 1, we have $u_{\mathrm{B}}(t,x)<u_{\mathrm{T}}(t,x)$ for all $t\geq 0$ and all $x\in\mathbb{R}$. Thus it remains to prove that, with probability 1, we have for all $t\geq 0$ that

$$\lim_{x\to\pm\infty}\int_0^x[u_{\mathrm{T}}-u_{\mathrm{B}}](t,y)\,\mathrm{d}y=\pm\infty. \tag{2.2}$$

We will prove that $+$ case of (2.2); the $-$ case is analogous. The proof proceeds in a similar manner to that of [13, Proposition 3.3]. Fix $\ell>1/2$ and define

$$\zeta(x)=\mathrm{e}^{2^{1-\ell}-\langle x\rangle^{1-\ell}},\qquad x\in\mathbb{R},$$

and let χ be a smooth positive function so that $\chi|_{(-\infty,-1]}\equiv 0$ and $\chi|_{[0,\infty)}\equiv 1$. For $\delta>0$ we set

$$\zeta_\delta(x)=\zeta(\delta x)\qquad\text{and}\qquad\omega_\delta(x)=\chi(x)\zeta_\delta(x).$$

Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}[u_{\mathrm{T}}-u_{\mathrm{B}}](t,x)\omega_\delta(x)\,\mathrm{d}x&=\frac{1}{2}\int_{\mathbb{R}}[\partial_x^2(u_{\mathrm{T}}-u_{\mathrm{B}})-\partial_x(u_{\mathrm{T}}^2-u_{\mathrm{B}}^2)](t,x)\omega_\delta(x)\,\mathrm{d}x\\&=\frac{1}{2}\int_{\mathbb{R}}(u_{\mathrm{T}}-u_{\mathrm{B}})(t,x)[\omega_\delta''(x)+(u_T+u_{\mathrm{B}})(t,x)\omega_\delta'(x)]\,\mathrm{d}x\end{aligned}\tag{2.3}$$

using integration by parts. The boundary terms at infinity vanish due to the at most polynomial growth of $u_{\mathrm{B}}(t,x)$ and $u_{\mathrm{T}}(t,x)$ as $|x|\to\infty$ ([13, Theorem 1.2, property (P4)]) and the superpolynomial decay of ζ_δ at infinity. Now, as in [13, (3.12)–(3.13)], there is a constant $C<\infty$ (independent of δ) so that

$$|\zeta_\delta''(x)|\leq C\delta^2\zeta_\delta(x)\qquad\text{and}\qquad\mathrm{p}_\ell(x)|\zeta_\delta'(x)|\leq C\delta^{1-\ell}\zeta_\delta(x)\qquad\text{for all }x\in\mathbb{R}.$$

Therefore, we have

$$|\omega_\delta''(x)|\leq C\delta^2\omega_\delta(x)\qquad\text{and}\qquad\mathrm{p}_\ell(x)|\omega_\delta'(x)|\leq C\delta^{1-\ell}\omega_\delta(x)\qquad\text{for all }x\geq 0,$$

and moreover (making C larger if necessary)

$$|\omega_\delta''(x)|,\mathrm{p}_\ell(x)|\omega_\delta'(x)|\leq C\qquad\text{for all }x\in[-1,0].$$

Using these bounds in (2.3), we have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}[u_{\mathrm{T}}-u_{\mathrm{B}}](t,x)\omega_\delta(x)\,\mathrm{d}x\right|&\\ \le \frac{1}{2}\int_0^\infty (u_{\mathrm{T}}-u_{\mathrm{B}})(t,x)[|\omega_\delta''(x)|+2\|\mathbf{u}_{\mathrm{BT}}(t,\cdot)\|_{\mathcal{C}_{\mathrm{p}_\ell}}\mathrm{p}_\ell(x)|\omega_\delta'(x)|]\,\mathrm{d}x&\\ +C\int_{-1}^0 (u_{\mathrm{T}}-u_{\mathrm{B}})(t,x)[1+\|\mathbf{u}_{\mathrm{BT}}(t,\cdot)\|_{\mathcal{C}_{\mathrm{p}_\ell}}]\,\mathrm{d}x&\\ \le C(\delta^2+\|\mathbf{u}_{\mathrm{BT}}(t,\cdot)\|_{\mathcal{C}_{\mathrm{p}_\ell}}\delta^{1-\ell})\int_0^\infty (u_{\mathrm{T}}-u_{\mathrm{B}})(t,x)\omega_\delta(x)\,\mathrm{d}x+C\langle\|\mathbf{u}_{\mathrm{BT}}(t,\cdot)\|_{\mathcal{C}_{\mathrm{p}_\ell}}\rangle^2,&\end{aligned}\tag{2.4}$$

where we have allowed the constant C to change from line to line. Now by the well-posedness proved in [13, Theorem 1.1], for any $T \ge 0$ we have

$$\sup_{t\in[0,T]}\|\mathbf{u}_{\mathrm{BT}}(t,\cdot)\|_{\mathcal{C}_{\mathrm{p}_\ell}}<\infty \tag{2.5}$$

almost surely. By the assumption that $\mathbf{u}_{\mathrm{BT}}(0,\cdot)\in\mathcal{X}_{\mathrm{BT}}$, we have

$$\lim_{\delta\downarrow 0}\int_{\mathbb{R}}[u_{\mathrm{T}}-u_{\mathrm{B}}](0,x)\omega_\delta(x)\,\mathrm{d}x\to\infty. \tag{2.6}$$

Combining (2.4), (2.5), (2.6), and Grönwall's inequality, we see that

$$\lim_{\delta\downarrow 0}\int_{\mathbb{R}}[u_{\mathrm{T}}-u_{\mathrm{B}}](t,x)\omega_\delta(x)\,\mathrm{d}x=\infty,$$

which implies that

$$\lim_{x\to\infty}\int_0^x[u_{\mathrm{T}}-u_{\mathrm{B}}](t,y)\,\mathrm{d}y=\infty. \qquad \square$$

3 The change of variables and the explicit shock profiles

In this section we describe the change of variables (1.9) leading to the equation (1.11), and show how this leads to the explcit shock profiles (1.8).

3.1 The change of variables

To understand the change of variables (1.9), the first step is to understand the shock position b_t in terms of the solution to the KPZ equation (1.22). Given a triple $\mathbf{v}=(v_{\mathrm{B}},v_{\mathrm{T}},v)\in\mathcal{X}_{\mathrm{BT}}\times\mathcal{X}$, let $\mathbf{u}=(u_{\mathrm{B}},u_{\mathrm{T}},u)\in\mathcal{C}([0,\infty);\mathcal{X}^3)$ solve (1.3) with initial condition $\mathbf{u}(0,\cdot)=\mathbf{v}$. By Lemma 2.2, we have with probability 1 that

$$\mathbf{u}(t,\cdot)\in\mathcal{X}_{\mathrm{BT}}\times\mathcal{X} \tag{3.1}$$

for all $t\ge 0$. In addition, for a given $b\in\mathbb{R}$, let $\mathbf{h}^{[b]}=(h_{\mathrm{B}}^{[b]},h_{\mathrm{T}}^{[b]},h^{[b]})$ solve the KPZ equation (1.22) with initial condition

$$\mathbf{h}^{[b]}(0,x)=\int_b^x\mathbf{u}(0,y)\,\mathrm{d}y.$$

We emphasize that $\mathbf{h}^{[b]}(t,x)$ is *not* equal to $\int_b^x\mathbf{u}(t,y)\,\mathrm{d}y$, even though

$$\partial_x\mathbf{h}^{[b]}(t,x)=\mathbf{u}(t,x)\qquad\text{for all } t\ge 0 \text{ and } x\in\mathbb{R}. \tag{3.2}$$

Indeed, we have

$$\mathbf{h}^{[b]}(t,x) = \mathbf{h}^{[b]}(t,b) + \int_b^x \mathbf{u}(t,y)\,\mathrm{d}y, \tag{3.3}$$

and $\mathbf{h}^{[b]}(t,b)$ is not zero in general. Now we define

$$\overline{Z}_b[\mathbf{v}](x) = \frac{1}{2}\int_b^x [v_\mathrm{T} - v_\mathrm{B}](y)\,\mathrm{d}y = \frac{1}{2}[h_\mathrm{T}^{[b]} - h_\mathrm{B}^{[b]}](0,x), \tag{3.4}$$

and

$$Z_{b,t}[\mathbf{v}](x) = \frac{1}{2}[h_\mathrm{T}^{[b]} - h_\mathrm{B}^{[b]}](t,x). \tag{3.5}$$

Later on, we will use the notation $\overline{Z}_b[\mathbf{v}]$ and $Z_{b,t}[\mathbf{v}]$ when $\mathbf{v} \in \mathcal{X}_\mathrm{BT} \times \mathcal{X}^N$ for some $N \geq 0$; the extension is obvious because these quantities only depend on the first two components of $\mathbf{v}$. Observe that $Z_{b,0}[\mathbf{v}](x) = \overline{Z}_b[\mathbf{v}](x)$ and that for any $b' \in \mathbb{R}$ we have using (3.3) that

$$Z_{b,t}[\mathbf{v}](x) = Z_{b,t}[\mathbf{v}](b') + \frac{1}{2}\int_{b'}^x [u_\mathrm{T} - u_\mathrm{B}](t,y)\,\mathrm{d}y. \tag{3.6}$$

By (3.1) and (2.1), $Z_{b,t}[\mathbf{v}](x)$ is an invertible function of x for each fixed b and t. For the rest of this section we will fix $\mathbf{v}$ and $\mathbf{u}$ as above, and write $Z_{b,t} = Z_{b,t}[\mathbf{v}]$.

Lemma 3.1. *Fix $b, \zeta \in \mathbb{R}$, and for $t \geq 0$, define $b_t = Z_{b,t}^{-1}(\zeta)$. Then $(b_t)_{t\geq 0}$ is the unique solution to the ordinary differential equation (1.7) with initial condition $b_0 = \overline{Z}_b[\mathbf{v}]^{-1}(\zeta)$.*

Proof. We compute

$$0 = \partial_t \left(Z_{b,t}(Z_{b,t}^{-1}(\zeta))\right) = (\partial_t Z_{b,t})(Z_{b,t}^{-1}(\zeta)) + (\partial_x Z_{b,t})(Z_{b,t}^{-1}(\zeta)) \cdot \partial_t(Z_{b,t}^{-1}(\zeta).$$

The fact that $(u_\mathrm{B}, u_\mathrm{T})(t,\cdot) \in \mathcal{X}_\mathrm{BT}$ means that $\partial_x Z_{b,t}(x) \neq 0$ for all $x \in \mathbb{R}$, so we can compute

$$\begin{aligned}\partial_t(Z_{b,t}^{-1})(\zeta) &= -\left(\frac{\partial_t Z_{b,t}}{\partial_x Z_{b,t}}\right)(Z_{b,t}^{-1}(\zeta)) = -\frac{1}{2}\left(\frac{\partial_x(u_\mathrm{T} - u_\mathrm{B}) - u_\mathrm{T}^2 + u_\mathrm{B}^2}{u_\mathrm{T} - u_\mathrm{B}}\right)(t, Z_{b,t}^{-1}(\zeta)) \\ &= \frac{1}{2}\left(-\partial_x(\log(u_\mathrm{T} - u_\mathrm{B})) + u_\mathrm{B} + u_\mathrm{T}\right)(t, Z_{b,t}^{-1}(\zeta)),\end{aligned} \tag{3.7}$$

so $t \mapsto Z_{b,t}^{-1}(\zeta)$ satisfies (1.7) for any fixed b and ζ. The vector field on the right side of (3.7) is locally Lipschitz, so the uniqueness comes from the basic theory of ordinary differential equations. ☐

Note that the initial condition b_0 does not determine b and ζ uniquely. However, if we fix some $\zeta \in \mathbb{R}$, then b is determined uniquely by b_0. In particular, if $\zeta = 0$ then $b = b_0$. Alternatively, if we fix b, which determines $h_\mathrm{B}^{[b]}$ and $h_\mathrm{T}^{[b]}$, then the choice of b_0 is equivalent to the choice of ζ, as

$$\zeta = \frac{1}{2}[h_\mathrm{T}^{[b]} - h_\mathrm{B}^{[b]}](0, b_0),$$

and then the solution $(b_t)_{t\geq 0}$ to (1.7) with initial condition b_0 is determined by the condition that

$$\zeta = \frac{1}{2}[h_\mathrm{T}^{[b]} - h_\mathrm{B}^{[b]}](t, b_t).$$

This gives a very simple geometric interpretation of b_t in terms of the graphs of $h_\mathrm{B}^{[b]}(t,\cdot)$ and $h_\mathrm{T}^{[b]}(t,\cdot)$.

With this notation introduced, we see that the change of variables (1.9) becomes

$$\zeta = Z_{b_0,t}(x), \qquad U = \frac{2u - u_{\mathrm{T}} - u_{\mathrm{B}}}{u_{\mathrm{T}} - u_{\mathrm{B}}}. \tag{3.8}$$

The inverse change of variables is

$$x = Z_{b_0,t}^{-1}(\zeta), \qquad u = \frac{1}{2}[(u_{\mathrm{T}} - u_{\mathrm{B}})U + u_{\mathrm{T}} + u_{\mathrm{B}}]. \tag{3.9}$$

A convenient way to carry out this change of variables is to first define the corresponding KPZ object

$$Q^{[b_0]}(t,\zeta) = \left(h^{[b_0]} - \frac{1}{2}h_{\mathrm{T}}^{[b_0]} - \frac{1}{2}h_{\mathrm{B}}^{[b_0]}\right)(t, Z_{b_0,t}^{-1}(\zeta)), \tag{3.10}$$

and then put

$$U^{[b_0]}(t,\zeta) = \partial_\zeta Q^{[b_0]}(t,\zeta) = \left(\frac{2u - u_{\mathrm{T}} - u_{\mathrm{B}}}{u_{\mathrm{T}} - u_{\mathrm{B}}}\right)(t, Z_{b_0,t}^{-1}(\zeta)). \tag{3.11}$$

In (3.11) we used the fact that

$$\partial_\zeta Z_{b_0,t}^{-1}(\zeta) = \frac{1}{(\partial_x Z_{b_0,t})(Z_{b_0,t}^{-1}(\zeta))} = \frac{2}{(u_{\mathrm{T}} - u_{\mathrm{B}})(t, Z_{b_0,t}^{-1}(\zeta))} \tag{3.12}$$

by (3.2).

Having carried out the change of variables, we now show that $U^{[b_0]}$ solves the PDE (1.11).

Proposition 3.2. *We have*

$$\partial_t U^{[b_0]}(t,\zeta) = \frac{1}{8}\partial_\zeta\left(J^{[b_0]}(\zeta)\left(\partial_\zeta U^{[b_0]}(t,\zeta) - (U^{[b_0]}(t,\zeta))^2 + 1\right)\right), \tag{3.13}$$

where

$$J^{[b_0]}(\zeta) = (u_{\mathrm{T}} - u_{\mathrm{B}})^2(t, Z_{b_0,t}^{-1}(\zeta)).$$

Proof. We start by computing a PDE for $Q^{[b_0]}$. Using (3.2) and (3.7), we can differentiate (3.10) to obtain

$$\begin{aligned}\partial_t Q^{[b_0]}(t,\zeta) &= \frac{1}{2}\left(\partial_x\left(u - \frac{1}{2}u_{\mathrm{T}} - \frac{1}{2}u_{\mathrm{B}}\right) - u^2 + \frac{1}{2}u_{\mathrm{T}}^2 + \frac{1}{2}u_{\mathrm{B}}^2\right)(t, Z_{b_0,t}^{-1}(\zeta)) \\ &\quad - \frac{1}{2}\left(\left(u - \frac{1}{2}u_{\mathrm{T}} - \frac{1}{2}u_{\mathrm{B}}\right)\cdot((\partial_x(\log(u_{\mathrm{T}} - u_{\mathrm{B}})) - (u_{\mathrm{B}} + u_{\mathrm{T}}))\right)(t, Z_{b_0,t}^{-1}(\zeta)).\end{aligned} \tag{3.14}$$

On the other hand, we can differentiate the second equality in (3.11) (using (3.12) again) to get

$$\begin{aligned}&\partial_\zeta^2 Q^{[b_0]}(t,\zeta) \\ &\quad = \frac{4}{(u_{\mathrm{T}} - u_{\mathrm{B}})^2}\left(\partial_x\left(u - \frac{1}{2}u_{\mathrm{T}} - \frac{1}{2}u_{\mathrm{B}}\right) - \left(u - \frac{1}{2}u_{\mathrm{T}} - \frac{1}{2}u_{\mathrm{B}}\right)\partial_x(\log(u_{\mathrm{T}} - u_{\mathrm{B}}))\right)(t, Z_{b_0,t}^{-1}(\zeta)).\end{aligned} \tag{3.15}$$

Recognizing the two terms in brackets in (3.15) in (3.14), we see that

$$\begin{aligned}\partial_t Q^{[b_0]}(t,\zeta) &= \frac{1}{8}(u_{\mathrm{T}} - u_{\mathrm{B}})^2(t, Z_{b_0,t}^{-1}(\zeta))\cdot\partial_\zeta^2 Q^{[b_0]}(t,\zeta) \\ &\qquad + \frac{1}{2}\left(\frac{1}{2}u_{\mathrm{T}}^2 + \frac{1}{2}u_{\mathrm{B}}^2 - u^2 + \left(u - \frac{1}{2}u_{\mathrm{T}} - \frac{1}{2}u_{\mathrm{B}}\right)(u_{\mathrm{B}} + u_{\mathrm{T}})\right)(t, Z_{b_0,t}^{-1}(\zeta)) \\ &= \frac{1}{8}(u_{\mathrm{T}} - u_{\mathrm{B}})^2(t, Z_{b_0,t}^{-1}(\zeta))\cdot\partial_\zeta^2 Q^{[b_0]}(t,\zeta) + \frac{1}{2}((u_{\mathrm{T}} - u)(u - u_{\mathrm{B}}))(t, Z_{b_0,t}^{-1}(\zeta)) \\ &= \frac{1}{8}(u_{\mathrm{T}} - u_{\mathrm{B}})^2(t, Z_{b_0,t}^{-1}(\zeta))\cdot\left[\partial_\zeta^2 Q^{[b_0]}(t,\zeta) - \left(\partial_\zeta Q^{[b_0]}(t,\zeta)\right)^2 + 1\right].\end{aligned} \tag{3.16}$$

Differentiating (3.16) in ζ and recalling (3.11), we get

$$\partial_t U^{[b_0]}(t,\zeta) = \frac{1}{8}\partial_\zeta \left((u_{\mathrm{T}} - u_{\mathrm{B}})^2(t, Z_{b_0,t}^{-1}(\zeta)) \cdot \left[\partial_\zeta U^{[b_0]}(t,\zeta) - \left(U^{[b_0]}(t,\zeta) \right)^2 + 1 \right] \right),$$

which is (3.13). □

3.2 The shock profiles

We now describe the explicit shock profiles introduced in (1.8). It is clear from (3.13) that, for any $\gamma \in \mathbb{R}$, the deterministic profile

$$U_\gamma(t,\zeta) = -\tanh(\zeta - \gamma/2)$$

is a solution to (3.13). Applying the change of variables (3.9), we see that if we define

$$\begin{aligned} u^{[b_0,\gamma]}(t,x) &= \frac{1}{2}[-(u_{\mathrm{T}} - u_{\mathrm{B}})(t,x)\tanh(Z_{b_0,t}(x) - \gamma/2) + (u_{\mathrm{B}} + u_{\mathrm{T}})(t,x)] \\ &= \frac{\mathrm{e}^{Z_{b_0,t}(x)-\gamma/2}}{\mathrm{e}^{Z_{b_0,t}(x)-\gamma/2} + \mathrm{e}^{-Z_{b_0,t}(x)+\gamma/2}} u_{\mathrm{B}}(t,x) + \frac{\mathrm{e}^{-Z_{b_0,t}(x)+\gamma/2}}{\mathrm{e}^{Z_{b_0,t}(x)-\gamma/2} + \mathrm{e}^{-Z_{b_0,t}(x)+\gamma/2}} u_{\mathrm{T}}(t,x) \\ &= \frac{1}{1 + \mathrm{e}^{\gamma - 2Z_{b_0,t}(x)}} u_{\mathrm{B}}(t,x) + \frac{1}{1 + \mathrm{e}^{2Z_{b_0,t}(x)-\gamma}} u_{\mathrm{T}}(t,x), \end{aligned} \tag{3.17}$$

then $(u_{\mathrm{B}}, u_{\mathrm{T}}, u^{[b_0,\gamma]})$ solves (1.3).

We note (recalling the definitions (1.6) and (3.4)) that

$$\mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}](x) = \frac{v_{\mathrm{B}}(x)}{1 + \mathrm{e}^{\gamma - 2\overline{Z}_b[v_{\mathrm{B}},v_{\mathrm{T}}](x)}} + \frac{v_{\mathrm{T}}(x)}{1 + \mathrm{e}^{2\overline{Z}_b[v_{\mathrm{B}},v_{\mathrm{T}}](x)-\gamma}}. \tag{3.18}$$

Using (3.6) with $b = b_0$ and $b' = b_t$, and noting by Lemma 3.1 (with $\zeta = 0$) that $Z_{b_0,t}(b_t) = 0$, we have

$$\begin{aligned} Z_{b_0,t}(x) &= Z_{b_0,t}(b_t) + \frac{1}{2}\int_{b_t}^x [u_{\mathrm{T}} - u_{\mathrm{B}}](t,y)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{b_t}^x [u_{\mathrm{T}} - u_{\mathrm{B}}](t,y)\,\mathrm{d}y = \overline{Z}_{b_t}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)](x). \end{aligned}$$

Substituting this into (3.17) and using (3.18), we get

$$u^{[b_0,\gamma]}(t,x) = \frac{u_{\mathrm{B}}(t,x)}{1 + \mathrm{e}^{\gamma - 2\overline{Z}_{b_t}[(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot)](x)}} + \frac{u_{\mathrm{T}}(t,x)}{1 + \mathrm{e}^{2\overline{Z}_{b_t}[(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot)](x)-\gamma}} = \mathscr{S}_{b_t,\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)](x). \tag{3.19}$$

Let us record the $L^1(\mathbb{R})$ distances between two of these explicit shock profiles.

Proposition 3.3. *If* $\mathbf{v}_{\mathrm{BT}} = (v_{\mathrm{B}}, v_{\mathrm{T}}) \in \mathcal{X}_{\mathrm{BT}}$, *then*

$$\|\mathscr{S}_{b_0,\gamma}[\mathbf{v}_{\mathrm{BT}}] - \mathscr{S}_{b_0,\gamma'}[\mathbf{v}_{\mathrm{BT}}]\|_{L^1(\mathbb{R})} = \left| \int_{\mathbb{R}} (\mathscr{S}_{b_0,\gamma}[\mathbf{v}_{\mathrm{BT}}] - \mathscr{S}_{b_0,\gamma'}[\mathbf{v}_{\mathrm{BT}}]) \right| = |\gamma - \gamma'|. \tag{3.20}$$

Proof. It is clear from the definition (1.6) that $\mathscr{S}_{b_0,\gamma}[\mathbf{v}_{\mathrm{BT}}]$ and $\mathscr{S}_{b_0,\gamma'}[\mathbf{v}_{\mathrm{BT}}]$ are ordered, hence the first equality. For the second equality, we note that the change of variables (3.8) (with $t = 0$) can be written as

$$U(t,\zeta) = u(t, \overline{Z}_b[\mathbf{v}_{\mathrm{BT}}]^{-1}(\zeta))\partial_\zeta \overline{Z}_b[\mathbf{v}_{\mathrm{BT}}]^{-1}(\zeta) - \frac{u_{\mathrm{B}} + u_{\mathrm{T}}}{u_{\mathrm{T}} - u_{\mathrm{B}}}(t, \overline{Z}_b[\mathbf{v}_{\mathrm{BT}}]^{-1}(\zeta)),$$

hence the integral in (3.20) becomes

$$\int_{\mathbb{R}} |-\tanh(\zeta-\gamma/2)-(-\tanh(\zeta-\gamma'/2))|\,\mathrm{d}\zeta=\gamma-\gamma'. \qquad \square$$

Proposition 3.4. *The map* $\mathbb{R}^2\times\mathcal{X}_{\mathrm{BT}}\ni((b,\gamma),(v_{\mathrm{B}},v_{\mathrm{T}}))\mapsto\mathscr{S}_{b,\gamma}[v_{\mathrm{B}},v_{\mathrm{T}}]\in\mathcal{X}$ *is continuous.*

Proof. Suppose that $(b^{(n)},\gamma^{(n)},v_{\mathrm{B}}^{(n)},v_{\mathrm{T}}^{(n)})\to(b,\gamma,v_{\mathrm{B}},v_{\mathrm{T}})$ in $\mathbb{R}^2\times\mathcal{X}_{\mathrm{BT}}$. It is clear that

$$\mathscr{S}_{b^{(n)},\gamma^{(n)}}[v_{\mathrm{B}}^{(n)},v_{\mathrm{T}}^{(n)}]\to\mathscr{S}_{b,\gamma}[v_{\mathrm{B}},v_{\mathrm{T}}]$$

uniformly on compact subsets of $\mathbb{R}$. We note that, for each $\ell>1/2$, we have

$$\|\mathscr{S}_{b^{(n)},\gamma^{(n)}}[v_{\mathrm{B}}^{(n)},v_{\mathrm{T}}^{(n)}]\|_{\mathcal{C}_{\mathrm{p}_\ell}}\le\max\{\|v_{\mathrm{B}}^{(n)}\|_{\mathcal{C}_{\mathrm{p}_\ell}},\|v_{\mathrm{T}}^{(n)}\|_{\mathcal{C}_{\mathrm{p}_\ell}}\},$$

so $(\mathscr{S}_{b^{(n)},\gamma^{(n)}}[v_{\mathrm{B}}^{(n)},v_{\mathrm{T}}^{(n)}])_n$ is bounded in each $\mathcal{C}_{\mathrm{p}_\ell}$, $\ell>1/2$. Therefore,

$$\mathscr{S}_{b^{(n)},\gamma^{(n)}}[v_{\mathrm{B}}^{(n)},v_{\mathrm{T}}^{(n)}]\to\mathscr{S}_{b,\gamma}[v_{\mathrm{B}},v_{\mathrm{T}}]$$

in the topology of each $\mathcal{C}_{\mathrm{p}_\ell}$, $\ell>1/2$, and hence in the topology of $\mathcal{X}$. $\square$

4 Bottom and top solutions in the shock location reference frame

In this section we consider what happens when we look at the bottom and top solutions u_{B} and u_{T} in the reference frame of the shock location b_t. We first compute the translation formula

$$\tau_y\mathscr{S}_{b,\gamma}[v_{\mathrm{B}},v_{\mathrm{T}}]=\mathscr{S}_{b+y,\gamma}[\tau_y(v_{\mathrm{B}},v_{\mathrm{T}})],$$

which is easily checked from the definition (1.6). Therefore, we can translate (3.19) in space to see that (with notation as in that expression)

$$\tau_{b_0-b_t}(u_{\mathrm{B}},u_{\mathrm{T}},u^{[b_0,\gamma]})(t,\cdot)=(\tau_{b_0-b_t}(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot),\mathscr{S}_{b_0,\gamma}[\tau_{b_0-b_t}(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot)]).$$

We note that the right side depends only on b_0,γ, and $\tau_{b_0-b_t}(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot)$. In other words, the shock $u^{[b_0,\gamma]}$ is a deterministic and time-independent functional of the top and bottom solutions in the reference frame of the shock location. Thus, in this section we study just the translated top and botom solutions, i.e. $\tau_{b_0-b_t}(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot)$. The main results of this section concern the invariant measure in this reference frame and its stability.

First we must define the evolution semigroup in the reference frame of the shock. Given an initial condition $\mathbf{v}=(\mathbf{v}_{\mathrm{BT}},\tilde{\mathbf{v}})\in\mathcal{X}_{\mathrm{BT}}\times\mathcal{X}^N$, with some $N\ge0$, let

$$\mathbf{u}=(\mathbf{u}_{\mathrm{BT}},\tilde{\mathbf{u}})=\Psi(\mathbf{v})\in\mathcal{C}([0,\infty);\mathcal{X}_{\mathrm{BT}}\times\mathcal{X}^N)$$

solve (1.3) with $\mathbf{u}(0,\cdot)=\mathbf{v}$. As in [13], we define, for $F\in\mathcal{C}_{\mathrm{b}}(\mathcal{X}_{\mathrm{BT}}\times\mathcal{X}^N)$ (a bounded continuous function on $\mathcal{X}_{\mathrm{BT}}\times\mathcal{X}^N$),

$$P_tF(\mathbf{v})=\mathbb{E}F(\mathbf{u}(t,\cdot)),$$

so that $\{P_t\}_{t\ge0}$ is the Markov semigroup for the dynamics (1.3) in the original reference frame. Next let $\{b_t\}_{t\ge0}$ solve (1.7) with initial condition $b_0=b$, set

$$\Phi^{[b]}(\mathbf{v})(t,\cdot)=\tau_{b-b_t}\mathbf{u}(t,\cdot)\in\mathcal{C}([0,\infty);\mathcal{X}_{\mathrm{BT}}\times\mathcal{X}^N),$$

and put, again for $F \in \mathcal{C}_{\mathrm{b}}(\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$,

$$\hat{P}_t^{[b]} F(\mathbf{v}) = \mathbb{E} F(\Phi^{[b]}(\mathbf{v})(t, \cdot)).$$

It is easily checked that $\{\hat{P}_t^{[b]}\}_{t \geq 0}$ has the semigroup property. It is the evolution semigroup in the reference frame of the shock. Moreover, $\{\hat{P}_t^{[b]}\}_{t \geq 0}$ has the Feller property (which was checked for $\{P_t\}_{t \geq 0}$ in [13, Theorem 1.1]). We endow the space $\mathcal{C}([0, \infty); \mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$ with the topology of uniform convergence (in the $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$ norm) on compact subsets of $[0, \infty)$.

Proposition 4.1. *The map $\Phi^{[b]} : \mathcal{X}_{\mathrm{BT}} \times \mathcal{X} \to \mathcal{C}([0, \infty); \mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$ is continuous (with respect to the just-defined topology on the target). Moreover, the semigroup $\{\hat{P}_t^{[b]}\}_{t \geq 0}$ has the Feller property: if $F \in \mathcal{C}_{\mathrm{b}}(\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$, then $\hat{P}_t^{[b]} F \in \mathcal{C}_{\mathrm{b}}(\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$ as well.*

We will prove Proposition 4.1 at the end of this section.

The first main result of this section concerns the invariance of the tilted measures introduced in the statement of Theorem 1.1.

Proposition 4.2. *Let ν and $\hat{\nu}^{[b]}$ be as in the statement of Theorem 1.1. Then*

$$\hat{\nu}^{[b]}(\mathcal{X}_{\mathrm{BT}}) = 1 \tag{4.1}$$

and

$$(\hat{P}_t^{[b]})^* \hat{\nu}^{[b]} = \hat{\nu}^{[b]}. \tag{4.2}$$

The second main result concerns the stability of the tilted measures $\hat{\nu}_{a_{\mathrm{B}}, a_{\mathrm{T}}}^{[b]}$ defined after the statement of Theorem 1.1.

Proposition 4.3. *Let $a_{\mathrm{B}} < a_{\mathrm{T}}$. Let $\delta_{a_{\mathrm{B}}, a_{\mathrm{T}}}$ be the measure on $\mathcal{X}_{\mathrm{BT}}$ with a single atom at the constant function $(a_{\mathrm{B}}, a_{\mathrm{T}})$. Then for any $b \in \mathbb{R}$, we have*

$$\lim_{t \to \infty} (\hat{P}_t^{[b]})^* \delta_{a_{\mathrm{B}}, a_{\mathrm{T}}} = \hat{\nu}_{a_{\mathrm{B}}, a_{\mathrm{T}}}^{[b]}$$

weakly with respect to the topology of $\mathcal{X}_{\mathrm{BT}}$.

The key ingredient in the proofs of Proposition 4.2 and 4.3 is Proposition 4.4 below, which describes how a translation-invariant measure evolves under $\hat{P}_t^{[b]}$. This will allow us to tilt the invariant measures constructed in [13] to obtain invariant measures in the reference frame of the shocks. We use the notation from [13] that $\mathscr{P}_{\mathbb{R}}(\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$ is the space of translation-invariant probability measures on $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$. (The subscript $\mathbb{R}$ denotes invariance under the action of $\mathbb{R}$ on the line by translations.) If $\mu \in \mathscr{P}_{\mathbb{R}}(\mathcal{X}_{\mathrm{BT}})$ and $(w_{\mathrm{B}}, w_{\mathrm{T}}) \sim \mu$, then (as noted in the statement of Theorem 1.1) the quantity

$$B[w_{\mathrm{B}}, w_{\mathrm{T}}] \coloneqq \lim_{L \to \infty} \frac{1}{L} \int_0^L [w_{\mathrm{T}} - w_{\mathrm{B}}](x) \, \mathrm{d}x$$

exists almost surely by the Birkhoff–Khinchin theorem.

Proposition 4.4. *Let $N \geq 0$. Let $\mu_0 \in \mathscr{P}_{\mathbb{R}}(\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$. For each $t \geq 0$, define another measure $\hat{\mu}_t^{[b]}$ on $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$, absolutely continuous with respect to $\mu_t \coloneqq P_t^* \mu_0$, by*

$$\frac{\mathrm{d}\hat{\mu}_t^{[b]}}{\mathrm{d}\mu_t}((w_{\mathrm{B}}, w_{\mathrm{T}}), \tilde{\mathbf{w}}) = \frac{w_{\mathrm{T}}(b) - w_{\mathrm{B}}(b)}{B[w_{\mathrm{B}}, w_{\mathrm{T}}]}. \tag{4.3}$$

Then $\hat{\mu}_t^{[b]}$ *is a probability measure and*

$$(\hat{P}_t^{[b]})^* \hat{\mu}_0^{[b]} = \hat{\mu}_t^{[b]}. \tag{4.4}$$

Moreover, for any $t \geq 0$*, if* $\hat{\mathbf{v}} \sim \hat{\mu}_t^{[b]}$*, then for any deterministic* $\zeta \in \mathbb{R}$*, we have*

$$\tau_{b-\overline{Z}_b[\hat{\mathbf{v}}]^{-1}(\zeta)} \hat{\mathbf{v}} \overset{\text{law}}{=} \hat{\mathbf{v}}. \tag{4.5}$$

Proof. First we check that $\hat{\mu}_t^{[b]}$ is a probability measure. Let $\mathcal{I}$ be the translation-invariant sub-σ-algebra of the Borel σ-algebra on $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$. Then by the Birkhoff-Khinchin theorem, $B[w_{\mathrm{B}}, w_{\mathrm{T}}]$ is $\mathcal{I}$-measurable and in fact

$$B[w_{\mathrm{B}}, w_{\mathrm{T}}] = \mathbb{E}[w_{\mathrm{T}}(b) - w_{\mathrm{B}}(b) \mid \mathcal{I}] > 0.$$

It follows that

$$\mathbb{E}\left[\frac{w_{\mathrm{T}}(b) - w_{\mathrm{B}}(b)}{B[w_{\mathrm{B}}, w_{\mathrm{T}}]}\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{w_{\mathrm{T}}(b) - w_{\mathrm{B}}(b)}{B[w_{\mathrm{B}}, w_{\mathrm{T}}]} \;\middle|\; \mathcal{I}\right]\right] = \mathbb{E}\left[\frac{\mathbb{E}[w_{\mathrm{T}}(b) - w_{\mathrm{B}}(b) \mid \mathcal{I}]}{B[w_{\mathrm{B}}, w_{\mathrm{T}}]}\right] = 1,$$

so $\hat{\mu}_t^{[b]}$ is a probability measure as claimed.

Let $\hat{\mathbf{v}}(t, \cdot) \sim \hat{\mu}_t^{[b]}$ for all $t \geq 0$; we will not use any coupling between $\hat{\mathbf{v}}(t, \cdot)$ and $\hat{\mathbf{v}}(s, \cdot)$ for $t \neq s$. Consider a function $F \in L^\infty(\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$. To prove (4.4), we need to show that

$$\mathbb{E}F(\hat{\mathbf{v}}(t, \cdot)) = \mathbb{E}\hat{P}_t^{[b]} F(\hat{\mathbf{v}}^{[b]}(0, \cdot)). \tag{4.6}$$

Let $\mathbf{u} \in \mathcal{C}([0, \infty); \mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$ solve (1.3) with initial condition $\mathbf{u}(0, \cdot) \sim \mu_0$ (independent of the noise). We abbreviate $Z_{b,t} = Z_{b,t}[\mathbf{u}(0, \cdot)]$. We will show that both the left and right sides of (4.6) are equal to

$$\lim_{M \to \infty} \frac{1}{M} \int_0^M \mathbb{E}[F(\tau_{b - Z_{b,t}^{-1}(\zeta)} \mathbf{u}(t, \cdot)] \, \mathrm{d}\zeta.$$

We first show that

$$\mathbb{E}F(\hat{\mathbf{v}}(t, \cdot)) = \lim_{M \to \infty} \frac{1}{M} \int_0^M \mathbb{E}[F(\tau_{b - Z_{b,t}^{-1}(\zeta)} \mathbf{u}(t, \cdot))] \, \mathrm{d}\zeta. \tag{4.7}$$

The crux of the argument is the simple identity

$$\frac{1}{L} \int_0^L F(\tau_{-x} \mathbf{u}(t, \cdot))[u_{\mathrm{T}} - u_{\mathrm{B}}](t, b + x) \, \mathrm{d}x = \frac{2}{L} \int_{Z_{b,t}(b)}^{Z_{b,t}(L+b)} F(\tau_{b - Z_{b,t}^{-1}(\zeta)} \mathbf{u}(t, \cdot)) \, \mathrm{d}\zeta, \tag{4.8}$$

which comes from making the change of variables

$$x = Z_{b,t}^{-1}(\zeta) - b, \qquad \mathrm{d}x = \frac{2}{[u_{\mathrm{T}} - u_{\mathrm{B}}](t, b + x)} \mathrm{d}\zeta. \tag{4.9}$$

By the Birkhoff–Khinchin theorem, we have the limit

$$\lim_{L \to \infty} \frac{1}{L} \int_0^L F(\tau_{-x} \mathbf{u}(t, \cdot))[u_{\mathrm{T}} - u_{\mathrm{B}}](t, b + x) \, \mathrm{d}x = \mathbb{E}[F(\mathbf{u}(t, \cdot))[u_{\mathrm{T}} - u_{\mathrm{B}}](t, b) \mid \mathcal{I}] \tag{4.10}$$

almost surely. Also by the Birkhoff-Khinchin theorem (recalling (3.6)), we have

$$\lim_{L \to \infty} \frac{Z_{b,t}(L + b) - Z_{b,t}(b)}{L} = \frac{1}{2} B[(u_{\mathrm{B}}, u_{\mathrm{T}})(t, \cdot)] \tag{4.11}$$

almost surely. Combining (4.8)–(4.11), we have

$$\lim_{L\to\infty}\frac{1}{Z_{b,t}(L+b)-Z_{b,t}(b)}\int_{Z_{b,t}(b)}^{Z_{b,t}(L+b)}F(\tau_{b-Z_{b,t}^{-1}(\zeta)}\mathbf{u}(t,\cdot))\,\mathrm{d}\zeta=\frac{\mathbb{E}[F(\mathbf{u}(t,\cdot))[u_{\mathrm{T}}-u_{\mathrm{B}}](t,b)\mid\mathcal{I}]}{B[(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot)]}$$

almost surely. Since $\lim\limits_{L\to\infty}Z_{b,t}(L+b)=\infty$ almost surely by (3.6) and (3.1), and F is bounded, this means that

$$\lim_{M\to\infty}\frac{1}{M}\int_0^M F(\tau_{b-Z_{b,t}^{-1}(\zeta)}\mathbf{u}(t,\cdot))\,\mathrm{d}\zeta=\frac{\mathbb{E}[F(\mathbf{u}(t,\cdot))[u_{\mathrm{T}}-u_{\mathrm{B}}](t,b)\mid\mathcal{I}]}{B[(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot)]}\tag{4.12}$$

almost surely. Since F is bounded and $B[(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot)]$ is $\mathcal{I}$-measurable, taking the expectation in (4.12) and using the bounded convergence theorem we deduce that

$$\lim_{M\to\infty}\frac{1}{M}\int_0^M \mathbb{E}[F(\tau_{b-Z_{b,t}^{-1}(\zeta)}\mathbf{u}(t,\cdot))]\,\mathrm{d}\zeta=\mathbb{E}\left[\frac{F(\mathbf{u}(t,\cdot))[u_{\mathrm{T}}-u_{\mathrm{B}}](t,b)}{B[(u_{\mathrm{B}},u_{\mathrm{T}})(t,\cdot)]}\right],$$

which implies (4.7) because $\mathbf{u}(t,\cdot)\sim\mu_t$.

The next step is to show that, for any $s\geq 0$, we have

$$\mathbb{E}\hat{P}_s^{[b]}F(\hat{\mathbf{v}}^{[b]}(0,\cdot))=\lim_{M\to\infty}\frac{1}{M}\int_0^M \mathbb{E}[F(\tau_{b-Z_{b,s}^{-1}(\zeta)}\mathbf{u}(s,\cdot))]\,\mathrm{d}\zeta.\tag{4.13}$$

For a random variable $y\in\mathbb{R}$, measurable with respect to $\mathbf{u}(0,\cdot)$, let $\mathbf{u}^{[y]}$ solve (1.3) with initial condition $\mathbf{u}^{[y]}(0,\cdot)=\tau_y\mathbf{u}(0,\cdot)$. We can compute

$$\begin{aligned}\hat{P}_s^{[b]}F(\tau_y\mathbf{u}(0,\cdot))&=\mathbb{E}[F(\tau_{b-Z_{b,s}[\tau_y\mathbf{u}(0,\cdot)]^{-1}(0)}\mathbf{u}^{[y]}(s,\cdot))\mid\mathbf{u}(0,\cdot)]\\&=\mathbb{E}[F(\tau_{b-(Z_{b,s}[\tau_y\mathbf{u}(0,\cdot)]^{-1}(0)-y)}\tau_{-y}\mathbf{u}^{[y]}(s,\cdot))\mid\mathbf{u}(0,\cdot)]\\&=\mathbb{E}[F(\tau_{b-Z_{b-y,s}^{-1}(0)}\mathbf{u}(s,\cdot))\mid\mathbf{u}(0,\cdot)].\end{aligned}\tag{4.14}$$

The first equality above is by the definition of $\hat{P}_s^{[b]}$ and the second is a tautology. The third holds because by the translation-invariance of the noise, $(Z_{b,s}[\tau_y\mathbf{u}(0,\cdot)]^{-1}(0)-y,\tau_y\mathbf{u}^{[y]}(s,\cdot))$ and $(Z_{b-y,s}^{-1}(0),\mathbf{u}(s,\cdot))$ have the same conditional law given $\mathbf{u}(0,\cdot)$.

Now apply (4.7) with $t=0$ and F in that equation taken to be $\hat{P}_s^{[b]}F$. This gives

$$\begin{aligned}\mathbb{E}[\hat{P}_s^{[b]}F(\hat{\mathbf{v}}(0,\cdot))]&=\lim_{M\to\infty}\frac{1}{M}\int_0^M\mathbb{E}\hat{P}_s^{[b]}F(\tau_{b-Z_{b,0}^{-1}(\zeta)}\mathbf{u}(0,\cdot))\,\mathrm{d}\zeta\\&=\lim_{M\to\infty}\frac{1}{M}\int_0^M\mathbb{E}F(\tau_{b-Z^{-1}_{Z_{b,0}^{-1}(\zeta),s}(0)}\mathbf{u}(s,\cdot))\,\mathrm{d}\zeta,\end{aligned}\tag{4.15}$$

where in the second equality we used (4.14) with $y=b-Z_{b,0}^{-1}(\zeta)$. Now we can compute

$$\begin{aligned}Z_{Z_{b,0}^{-1}(\zeta),s}(x)&=\frac{1}{2}\left(h_{\mathrm{T}}^{[Z_{b,0}^{-1}(\zeta)]}-h_{\mathrm{B}}^{[Z_{b,0}^{-1}(\zeta)]}\right)(s,x)=\frac{1}{2}\left(h_{\mathrm{T}}^{[b]}-h_{\mathrm{B}}^{[b]}\right)(s,x)-\frac{1}{2}\int_b^{Z_{b,0}^{-1}(\zeta)}[u_{\mathrm{T}}-u_{\mathrm{B}}](0,y)\,\mathrm{d}y\\&=Z_{b,s}(x)-\zeta=Z_{b,s}(b)+\overline{Z}_b[\mathbf{u}(s,\cdot)](x)-\zeta.\end{aligned}\tag{4.16}$$

In the second equality we used the fact that the identity

$$\frac{1}{2}\left(h_{\mathrm{T}}^{[Z_{b,0}^{-1}(\zeta)]}-h_{\mathrm{B}}^{[Z_{b,0}^{-1}(\zeta)]}\right)(s,x)=\frac{1}{2}\left(h_{\mathrm{T}}^{[b]}-h_{\mathrm{B}}^{[b]}\right)(s,x)-\frac{1}{2}\int_b^{Z_{b,0}^{-1}(\zeta)}[u_{\mathrm{T}}-u_{\mathrm{B}}](0,y)\,\mathrm{d}y$$

for all $x \in \mathbb{R}$ holds at $s = 0$ and thus for all $s \geq 0$ as well. In the third equality of (4.16) we used (3.6). It follows from (4.16) that

$$Z^{-1}_{Z^{-1}_{b,0}(\zeta),s}(\kappa) = \overline{Z}_b[\mathbf{u}(s,\cdot)]^{-1}(\kappa + \zeta - Z_{b,s}(b)). \tag{4.17}$$

Substituting (4.17) (with $\kappa = 0$) into (4.15), we get

$$\begin{aligned}\mathbb{E}[\hat{P}^{[b]}_s F(\hat{\mathbf{v}}(0,\cdot))] &= \lim_{M\to\infty} \frac{1}{M} \int_0^M \mathbb{E}F(\tau_{b-\overline{Z}_b[\mathbf{u}(s,\cdot)](x)(\zeta - Z_{b,s}(b))}\mathbf{u}(s,\cdot))\,\mathrm{d}\zeta \\ &= \lim_{M\to\infty} \frac{1}{M} \int_{-Z_{b,s}(b)}^{M-Z_{b,s}(b)} \mathbb{E}F(\tau_{b-\overline{Z}_b[\mathbf{u}(s,\cdot)]^{-1}(\zeta)}\mathbf{u}(s,\cdot)) \\ &= \mathbb{E}F(\hat{\mathbf{v}}(s,\cdot)),\end{aligned}$$

where the last equality is again by (4.7), this time with $t = s$. This completes the proof of (4.13). As indicated above, (4.7) and (4.13) together imply (4.6).

The proof of (4.5) is similar but easier. Without loss of generality, assume that $t = 0$. Let $\mathbf{w} = (w_{\mathrm{B}}, w_{\mathrm{T}}, \tilde{\mathbf{w}}) \sim \mu_0$ and $\hat{\mathbf{w}} \sim \hat{\mu}^{[b]}_0$. For $\mathbf{y} \in \mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$ put $F_{b,\zeta}(\mathbf{y}) = F(\tau_{b-\overline{Z}_b[\mathbf{y}]^{-1}(\zeta)}\mathbf{y})$. By (4.7), applied at $t = 0$, we have

$$\begin{aligned}\mathbb{E}F(\tau_{b-\overline{Z}_b[\hat{\mathbf{v}}]^{-1}(\zeta)}\hat{\mathbf{w}}) &= \mathbb{E}F_{b,\zeta}(\hat{\mathbf{w}}) \\ &= \lim_{M\to\infty} \frac{1}{M} \int_0^M \mathbb{E}[F_{b,\zeta}(\tau_{b-\overline{Z}_b[\mathbf{w}]^{-1}(\zeta')}\mathbf{w})]\,\mathrm{d}\zeta' \\ &= \lim_{M\to\infty} \frac{1}{M} \int_0^M \mathbb{E}[F(\tau_{b-\overline{Z}_b[\tau_{b-\overline{Z}_b[\mathbf{w}]^{-1}(\zeta')}\mathbf{w}]^{-1}(\zeta)}\tau_{b-\overline{Z}_b[\mathbf{w}]^{-1}(\zeta')}\mathbf{w})]\,\mathrm{d}\zeta'.\end{aligned} \tag{4.18}$$

For any $y \in \mathbb{R}$, we have

$$\overline{Z}_b[\tau_y \mathbf{w}]^{-1}(\zeta) = \overline{Z}_{b-y}[\mathbf{w}]^{-1}(\zeta) + y$$

almost surely, which means that (taking $y = b - \overline{Z}_b^{-1}[\mathbf{w}](\zeta')$)

$$\overline{Z}_b[\tau_{b-\overline{Z}_b^{-1}[\mathbf{w}](\zeta')}\mathbf{w}]^{-1}(\zeta) = \overline{Z}_{\overline{Z}_b^{-1}[\mathbf{w}](\zeta')}[\mathbf{w}]^{-1}(\zeta) + b - \overline{Z}_b^{-1}[\mathbf{w}](\zeta'),$$

so

$$b - \overline{Z}_b[\tau_{b-\overline{Z}_b[\mathbf{w}]^{-1}(\zeta')}\mathbf{w}]^{-1}(\zeta) = \overline{Z}_b^{-1}[\mathbf{w}](\zeta') - \overline{Z}_{\overline{Z}_b^{-1}[\mathbf{w}](\zeta')}[\mathbf{w}]^{-1}(\zeta) = \overline{Z}_b^{-1}[\mathbf{w}](\zeta') - \overline{Z}_b[\mathbf{w}]^{-1}(\zeta + \zeta'),$$

where the second equality can be seen either by (4.17) with $s = 0$, $\zeta = \zeta'$, $\mathbf{u}(0,\cdot) = \mathbf{w}$, and $\kappa = \zeta$ or by a simple direct argument. Substituting this into (4.18) we obtain

$$\begin{aligned}\mathbb{E}F(\tau_{b-\overline{Z}_b[\hat{\mathbf{w}}]^{-1}(\zeta)}\hat{\mathbf{w}}) &= \lim_{M\to\infty} \frac{1}{M} \int_0^M \mathbb{E}[F(\tau_{b-\overline{Z}_b[\mathbf{w}]^{-1}(\zeta+\zeta')}\mathbf{w})]\,\mathrm{d}\zeta' \\ &= \lim_{M\to\infty} \frac{1}{M} \int_\zeta^{M+\zeta} \mathbb{E}[F(\tau_{b-\overline{Z}_b[\mathbf{w}]^{-1}(\zeta')}\mathbf{w})]\,\mathrm{d}\zeta' = \mathbb{E}F(\hat{\mathbf{w}}),\end{aligned}$$

with the last equality again by (4.7). ☐

Now we can prove Propositions 4.2 and 4.3.

Proof of Proposition 4.2. The Birkhoff–Khinchin theorem, along with the assumed ordering of the components of a function distributed according to ν, implies that $\nu(\mathcal{X}_{\mathrm{BT}}) = 1$. Then, since $\hat{\nu}^{[b]}$ is absolutely continuous with respect to ν, we have (4.1). Then (4.2) holds by Proposition 4.4 applied with $\mu_t = \nu$ for all t, since ν is an invariant measure for (1.3). ☐

Proof of Proposition 4.3. We note that $(\hat{P}_t^{[b]})^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}}$ is ergodic with respect to the group $\mathbb{R}$ of spatial translations due to the spatial ergodicity of the driving noise V. Therefore, by Proposition 4.4 and the Birkhoff–Khinchin theorem, $(\hat{P}_t^{[b]})^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}}$ is absolutely continuous with respect to $P_t^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}}$ with Radon–Nikodym derivative

$$\frac{\mathrm{d}\left((\hat{P}_t^{[b]})^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}}\right)}{\mathrm{d}(P_t^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}})}(w_{\mathrm{B}}, w_{\mathrm{T}}) = \frac{w_{\mathrm{T}}(b) - w_{\mathrm{B}}(b)}{a_{\mathrm{T}} - a_{\mathrm{B}}}.$$

For any $F \in L^\infty(\mathcal{X}^2)$, if $\hat{\mathbf{u}}^{[b]}(t,\cdot) \sim (\hat{P}_t^{[b]})^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}}$ and $\mathbf{u}(t,\cdot) = (u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot) \sim P_t^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}}$, then we have by the definitions that

$$\mathbb{E}F(\hat{\mathbf{u}}^{[b]}(t,\cdot)) = \mathbb{E}\left[F(\mathbf{u}(t,\cdot))\left(\frac{u_{\mathrm{T}}(t,b) - u_{\mathrm{B}}(t,b)}{a_{\mathrm{T}} - a_{\mathrm{B}}}\right)\right]. \tag{4.19}$$

By the L^2 bound proved as [13, Lemma 5.3], there is a constant $C < \infty$ so that, for all $t \geq 0$, we have

$$\mathbb{E}\left(\frac{u_{\mathrm{T}}(t,b) - u_{\mathrm{B}}(t,b)}{a_{\mathrm{T}} - a_{\mathrm{B}}}\right)^2 \leq C.$$

This means that the term inside the expectation on the right side of (4.19) is uniformly integrable. Since $P_t^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}}$ converges to $\nu_{a_{\mathrm{B}},a_{\mathrm{T}}}$ (weakly with respect to the topology of $\mathcal{X}^2$) by the stability result [13, Theorem 1.3], we have

$$\begin{aligned}\lim_{t\to\infty} F(\hat{\mathbf{u}}^{[b]}(t,\cdot)) = \lim_{t\to\infty}\mathbb{E}\left[F(\mathbf{u}(t,\cdot))\left(\frac{u_{\mathrm{T}}(t,b) - u_{\mathrm{B}}(t,b)}{a_{\mathrm{T}} - a_{\mathrm{B}}}\right)\right] &= \mathbb{E}\left[F(\mathbf{v})\left(\frac{v_{\mathrm{T}}(b) - v_{\mathrm{B}}(b)}{a_{\mathrm{T}} - a_{\mathrm{B}}}\right)\right] \\ &= \mathbb{E}F(\hat{\mathbf{v}}^{[b]}),\end{aligned}$$

where $\mathbf{v} = (v_{\mathrm{B}}, v_{\mathrm{T}}) \sim \nu_{a_{\mathrm{B}},a_{\mathrm{T}}}$ and $\hat{\mathbf{v}}^{[b]} \sim \hat{\nu}^{[b]}_{a_{\mathrm{B}},a_{\mathrm{T}}}$. Hence, $(\hat{P}_t^{[b]})^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}}$ converges weakly to $\hat{\nu}^{[b]}_{a_{\mathrm{B}},a_{\mathrm{T}}}$ with respect to the topology of $\mathcal{X}^2$.

It remains to show that in fact $(\hat{P}_t^{[b]})^*\delta_{a_{\mathrm{B}},a_{\mathrm{T}}}$ converges weakly to $\hat{\nu}^{[b]}_{a_{\mathrm{B}},a_{\mathrm{T}}}$ with respect to the topology of $\mathcal{X}_{\mathrm{BT}}$. If F is a bounded Lipschitz function on $\mathcal{X}_{\mathrm{BT}}$, then F is in particular uniformly continuous, so it can be extended to a bounded continuous function on the closure of $\mathcal{X}_{\mathrm{BT}}$ in $\mathcal{X}^2$, and hence by the Tietze extension theorem to a bounded continuous function on $\mathcal{X}^2$. Then the argument of the previous paragraph applies, and by the portmanteau lemma this completes the proof. ☐

Proof of the Feller property

Now we prove the Feller property Proposition 4.1.

Proof of Proposition 4.1. We recall that by [13, Theorem 1.1], the solution map $\Psi : \mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N \to \mathcal{C}([0,\infty); \mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$ is continuous with probability 1, if the target space is given the topology of uniform convergence on compact subsets of $[0,\infty)$. Thus, to show that Φ is continuous it suffices to show that the map $\mathcal{X}_{\mathrm{BT}} \ni \mathbf{v} \mapsto (Z_{b,t}[\mathbf{v}]^{-1}(0))_{t\geq 0} \in \mathcal{C}([0,\infty))$ is continuous with probability 1, where $\mathcal{C}([0,\infty))$ is similarly given the topology of uniform convergence on compact sets. This could be proved using the ODE (1.7), but we will instead argue using the KPZ equation and the formulas of the previous section.

Let $\mathbf{v} \in \mathcal{X}_{\mathrm{BT}}$ and let $\mathbf{u} \in \mathcal{C}([0,\infty);\mathcal{X}_{\mathrm{BT}})$ solve (1.3) with initial condition $\mathbf{u}(0,\cdot) = \mathbf{v}$. Observe that, if $(h_{\mathrm{B}}, h_{\mathrm{T}})$ solves (1.22) with initial condition

$$(h_{\mathrm{B}}, h_{\mathrm{T}})(0,x) = \int_b^x (v_{\mathrm{B}}, v_{\mathrm{T}})(y)\,\mathrm{d}y,$$

so that

$$Z_{b,t}[\mathbf{v}](x) = \frac{1}{2}[h_{\mathrm{T}} - h_{\mathrm{B}}](t,x) = Z_{b,t}[\mathbf{v}](b) + \frac{1}{2}\int_b^x [u_{\mathrm{T}} - u_{\mathrm{B}}](t,y)\,\mathrm{d}y, \tag{4.20}$$

then $Z_{b,t}[\mathbf{v}]$ satisfies the ODE

$$\begin{aligned}\partial_t Z_{b,t}[\mathbf{v}](b) &= \frac{1}{2}\partial_x^2[h_{\mathrm{T}} - h_{\mathrm{B}}](t,b) - \frac{1}{2}\partial_x[h_{\mathrm{T}} - h_{\mathrm{B}}](t,b)\cdot\partial_x[h_{\mathrm{T}} + h_{\mathrm{B}}](t,b) \\ &= \frac{1}{2}\partial_x(u_{\mathrm{T}} - u_{\mathrm{B}})(t,b) - \frac{1}{2}(u_{\mathrm{T}} - u_{\mathrm{B}})(t,b)\cdot(u_{\mathrm{T}} + u_{\mathrm{B}})(t,b).\end{aligned} \tag{4.21}$$

Fix a smooth, compactly supported function φ on $\mathbb{R}$ such that $\int \varphi = 1$ and define

$$Q_t[\mathbf{v}] = \int_{\mathbb{R}} Z_{b,t}[\mathbf{v}](x)\varphi(x-b)\,\mathrm{d}x. \tag{4.22}$$

Integrating (4.21) in time and against $\varphi(\cdot - b)$ in space, and integrating by parts, we obtain

$$Q_t[\mathbf{v}] = \frac{1}{2}\int_0^t \int_{\mathbb{R}} (u_{\mathrm{T}} - u_{\mathrm{B}})(s,x)[-\varphi'(x-b) - (u_{\mathrm{T}} + u_{\mathrm{B}})(s,x)\varphi(x-b)]\,\mathrm{d}x\,\mathrm{d}s. \tag{4.23}$$

On the other hand, using (4.20) in (4.22) we can also write

$$\begin{aligned}Q_t[\mathbf{v}] &= \int_{\mathbb{R}} \left(Z_{b,t}[\mathbf{v}](b) + \frac{1}{2}\int_b^x [u_{\mathrm{T}} - u_{\mathrm{B}}](t,y)\,\mathrm{d}y\right)\varphi(x-b)\,\mathrm{d}x \\ &= Z_{b,t}[\mathbf{v}](b) + \frac{1}{2}\int_{\mathbb{R}}\int_b^x [u_{\mathrm{T}} - u_{\mathrm{B}}](t,y)\varphi(x-b)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{4.24}$$

Combining (4.23) and (4.24) gives

$$\begin{aligned}Z_{b,t}[\mathbf{v}](b) = \frac{1}{2}\int_0^t \int_{\mathbb{R}} &(u_{\mathrm{T}} - u_{\mathrm{B}})(s,x)[-\varphi'(x-b) - (u_{\mathrm{T}} + u_{\mathrm{B}})(s,x)\varphi(x-b)]\,\mathrm{d}x\,\mathrm{d}s \\ &- \frac{1}{2}\int_{\mathbb{R}}\int_b^x [u_{\mathrm{T}} - u_{\mathrm{B}}](t,y)\varphi(x-b)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

By this and [13, Theorem 1.1], the map $\mathbf{v} \mapsto (Z_{b,t}[\mathbf{v}](b))_{t\geq 0}$ is almost-surely continuous.

Now we can write, by (4.20), that

$$Z_{b,t}[\mathbf{v}]^{-1}(0) = \overline{Z}_b[\mathbf{u}(t,\cdot)]^{-1}(-Z_{b,t}[\mathbf{v}](b)).$$

By Lemma A.1 and [13, Theorem 1.1], this implies that $\mathbf{v} \mapsto (Z_{b,t}[\mathbf{v}]^{-1}(0))_{t\geq 0}$ is almost-surely continuous. Therefore, Φ is almost-surely continuous. Now if $F \in \mathcal{C}_{\mathrm{b}}(\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N)$ and $\mathbf{v}_n \to \mathbf{v}$ in $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$ then

$$\hat{P}_t^{[b]}F(\mathbf{v}_n) = \mathbb{E}F(\Phi(\mathbf{v}_n)(t,\cdot)) = \mathbb{E}F(\Phi(\mathbf{v})(t,\cdot)) = \hat{P}_t^{[b]}F(\mathbf{v})$$

by the bounded convergence theorem. This proves that $\hat{P}_t^{[b]}$ is Feller. □

5 Uniqueness of the stationary shock profiles

In this section we show that shock profiles of the form (1.8) are the only "stationary" shock profiles that satisfy a certain integrability condition. We define this integrability condition through the space

$$\mathcal{X}_{\mathrm{Sh}} = \left\{(v_{\mathrm{B}}, v_{\mathrm{T}}, v) \in \mathcal{X}_{\mathrm{BT}} \times \mathcal{X} \ : \ \int_{-\infty}^{0} |v_{\mathrm{T}} - v| + \int_{0}^{\infty} |v - v_{\mathrm{B}}| < \infty\right\}, \tag{5.1}$$

as previously given in (1.5). This is a space of viscous shock fronts. As in the previous sections, v_{B} and v_{T} are the "bottom" and "top" solutions, respectively, while v is a viscous shock. Note that for any $(v_{\mathrm{B}}, v_{\mathrm{T}}) \in \mathcal{X}_{\mathrm{BT}}$ and $b, \gamma \in \mathbb{R}$, we have $\mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}] \in \mathcal{X}_{\mathrm{Sh}}$.

Next, we need a way to track the location of a moving shock. We define

$$\mathcal{X}_{\mathrm{Sh},b,\gamma} = \left\{(v_{\mathrm{B}}, v_{\mathrm{T}}, v) \in \mathcal{X}_{\mathrm{Sh}} \ : \ \int_{-\infty}^{b} [v - v_{\mathrm{T}}] + \int_{b}^{\infty} [v - v_{\mathrm{B}}] = \gamma\right\}.$$

Observe that (recalling the definition (2.1)), for any $(v_{\mathrm{B}}, v_{\mathrm{T}}, v) \in \mathcal{X}_{\mathrm{Sh}}$, the map

$$I(c) = \int_{-\infty}^{c} [v - v_{\mathrm{T}}] + \int_{c}^{\infty} [v - v_{\mathrm{B}}]$$

is decreasing and, moreover,

$$\lim_{c \to \pm\infty} I(c) = \mp\infty.$$

Therefore, for each fixed $b \in \mathbb{R}$, we have

$$\mathcal{X}_{\mathrm{Sh}} = \bigsqcup_{\gamma \in \mathbb{R}} \mathcal{X}_{\mathrm{Sh},b,\gamma},$$

and for each fixed $\gamma \in \mathbb{R}$, we have

$$\mathcal{X}_{\mathrm{Sh}} = \bigsqcup_{b \in \mathbb{R}} \mathcal{X}_{\mathrm{Sh},b,\gamma},$$

where $\bigsqcup$ denotes disjoint union.

We now show that the shocks (1.8) lie in the corresponding $\mathcal{X}_{\mathrm{Sh},b,\gamma}$.

Lemma 5.1. *We have $(v_{\mathrm{B}}, v_{\mathrm{T}}, \mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}]) \in \mathcal{X}_{\mathrm{Sh},b,\gamma}$ for any $(v_{\mathrm{B}}, v_{\mathrm{T}}) \in \mathcal{X}_{\mathrm{BT}}$ and any $b, \gamma \in \mathbb{R}$.*

Proof. By (3.18) and the change of variables

$$\zeta = \overline{Z}_b[v_{\mathrm{B}}, v_{\mathrm{T}}](x), \qquad \mathrm{d}\zeta = \frac{1}{2}[v_{\mathrm{T}} - v_{\mathrm{B}}](x)\mathrm{d}x,$$

(similar to (1.9)), we have

$$\begin{aligned}\int_{-\infty}^{b} (\mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}] - v_{\mathrm{T}})(x)\, \mathrm{d}x &= \int_{-\infty}^{b} \frac{v_{\mathrm{B}}(x) - v_{\mathrm{T}}(x)}{1 + \mathrm{e}^{\gamma - 2\overline{Z}_b[v_{\mathrm{B}}, v_{\mathrm{T}}](x)}}\, \mathrm{d}x \\ &= -2\int_{-\infty}^{0} \frac{1}{1 + \mathrm{e}^{\gamma - 2\zeta}}\, \mathrm{d}\zeta = -\log(1 + \mathrm{e}^{-\gamma}).\end{aligned} \tag{5.2}$$

Similarly, we have

$$\int_{b}^{\infty} (\mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}] - v_{\mathrm{B}})(x)\, \mathrm{d}x = \int_{b}^{\infty} \frac{v_{\mathrm{T}}(x) - v_{\mathrm{B}}(x)}{1 + \mathrm{e}^{2\overline{Z}_b[v_{\mathrm{B}}, v_{\mathrm{T}}](x) - \gamma}}\, \mathrm{d}x = 2\int_{0}^{\infty} \frac{1}{1 + \mathrm{e}^{2\zeta - \gamma}\, \mathrm{d}\zeta} = \log(1 + \mathrm{e}^{\gamma}). \tag{5.3}$$

Adding (5.2) and (5.3) yields

$$\int_{-\infty}^{b} (\mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}] - v_{\mathrm{T}})(x) \, \mathrm{d}x + \int_{b}^{\infty} (\mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}] - v_{\mathrm{B}})(x) \, \mathrm{d}x = \gamma,$$

completing the proof. □

The following simple lemma gives an alternative characterization of $\mathcal{X}_{\mathrm{Sh},b,\gamma}$.

Lemma 5.2. *We have the equivalence*

$$(v_{\mathrm{B}}, v_{\mathrm{T}}, v) \in \mathcal{X}_{\mathrm{Sh},b,\gamma} \iff v - \mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}] \in L^1(\mathbb{R}) \text{ and } \int_{\mathbb{R}} (v - \mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}]) = 0. \tag{5.4}$$

Proof. If $(v_{\mathrm{B}}, v_{\mathrm{T}}, v) \in \mathcal{X}_{\mathrm{Sh},b,\gamma}$ and $(v_{\mathrm{B}}, v_{\mathrm{T}}, \tilde{v}) \in \mathcal{X}_{\mathrm{Sh},b,\tilde{\gamma}}$, then

$$\int_{-\infty}^{\infty} [v - \tilde{v}](y) \, \mathrm{d}y = \int_{-\infty}^{b} [v - v_{\mathrm{T}}] + \int_{-\infty}^{b} [v_{\mathrm{T}} - \tilde{v}] + \int_{b}^{\infty} [v - v_{\mathrm{B}}] + \int_{b}^{\infty} [v_{\mathrm{B}} - \tilde{v}] = \gamma - \tilde{\gamma}. \tag{5.5}$$

Combining (5.5) and Lemma 5.1 yields the " $\Longrightarrow$ " direction of (5.4).

On the other hand, if $v - \mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}] \in L^1(\mathbb{R})$, then $(v_{\mathrm{B}}, v_{\mathrm{T}}, v) \in \mathcal{X}_{\mathrm{Sh}}$ since $\mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}] \in \mathcal{X}_{\mathrm{Sh}}$. Thus the " $\Longleftarrow$ " direction of (5.4) follows immediately from the second equality in (5.5). □

The next lemma shows that for an arbitrary shock, the shock location follows the location b_t.

Lemma 5.3. *Suppose that* $\mathbf{u} = (u_{\mathrm{B}}, u_{\mathrm{T}}, u) \in \mathcal{C}([0, \infty); \mathcal{X}_{\mathrm{BT}} \times \mathcal{X})$ *is a solution to (1.3) such that* $\mathbf{u}(0, \cdot) \in \mathcal{X}_{\mathrm{Sh},b,\gamma}$, *and let* $(b_t)_{t \geq 0}$ *solve with (1.7) with initial condition* $b_0 = b$. *Then, with probability* 1, *for all* $t \geq 0$, *we have* $\mathbf{u}(t, \cdot) \in \mathcal{X}_{\mathrm{Sh},b_t,\gamma}$.

Proof. Define

$$u_{\mathrm{explicit}}(t, x) = \mathscr{S}_{b_t,\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t, \cdot)](x),$$

so $(u_{\mathrm{B}}, u_{\mathrm{T}}, u, u_{\mathrm{explicit}})$ solves (1.3) by (3.19). Now by the mass conservation of the Burgers dynamics ([13, Proposition 3.3]) we have

$$\begin{aligned} 0 = \int_{\mathbb{R}} (u_{\mathrm{expicit}}(0, x) - u(0, x)) \, \mathrm{d}x &= \int_{\mathbb{R}} (u_{\mathrm{explicit}}(t, x) - u(t, x)) \, \mathrm{d}x \\ &= \int_{\mathbb{R}} (\mathscr{S}_{b_t,\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t, \cdot)](x) - u(t, x)) \, \mathrm{d}x. \end{aligned}$$

Thus Lemma 5.2 implies that $\mathbf{u}(t, \cdot) \in \mathcal{X}_{\mathrm{Sh},b_t,\gamma}$ for all $t \geq 0$. □

Definition 5.4. Let μ be a probability measure on $\mathcal{X}_{\mathrm{Sh}}$ and $b \in \mathbb{R}$. We say that μ is the law of a *stationary shock profile* with respect to b if $(\hat{P}_t^{[b]})^* \mu = \mu$.

An immediate consequence of (3.19) and Proposition 4.2 is that if $a_{\mathrm{B}} < a_{\mathrm{T}}$, $b \in \mathbb{R}$, and $(v_{\mathrm{B}}, v_{\mathrm{T}}) \sim \hat{\nu}_{a_{\mathrm{B}},a_{\mathrm{T}}}^{[b]}$ (as defined in (1.13)), then for any $\gamma \in \mathbb{R}$, $(v_{\mathrm{B}}, v_{\mathrm{T}}, \mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}])$ has the law of a stationary shock profile with respect to b. We can also prove a partial converse of this property.

Proposition 5.5. *If* $\mathbf{v} = (v_{\mathrm{B}}, v_{\mathrm{T}}, v)$ *has the law of a stationary shock profile with respect to* b, *then there is a random* $\gamma \in \mathbb{R}$ *so that* $v = \mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}]$ *almost surely.*

Proof. Let γ be such that $\mathbf{v} \in \mathcal{X}_{\mathrm{Sh},b,\gamma}$ and $\mathbf{u} = (u_{\mathrm{B}}, u_{\mathrm{T}}, u, u_{\mathrm{explicit}})$ solve (1.3) with initial condition

$$\mathbf{u}(0,\cdot) = (\mathbf{v}, \mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}]).$$

Then, in particular, if $(b_t)_{t\geq 0}$ solves (1.7) with initial condition $b_0 = b$, then

$$u_{\mathrm{explicit}}(t,x) = \mathscr{S}_{b_t,\gamma}[(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)](x)$$

by (3.19). By the mass conservation of the Burgers equation ([13, Proposition 3.3]) we also know that

$$\int_{\mathbb{R}} (u - u_{\mathrm{explicit}})(t,x)\,\mathrm{d}x = 0 \tag{5.6}$$

for all $t \geq 0$. Since $(v_{\mathrm{B}}, v_{\mathrm{T}}, v)$ is a stationary shock profile, it follows that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |\tau_{-b_t} u(t,\cdot) - \tau_{-b_t} u_{\mathrm{explicit}}(t,\cdot)| = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} |u(t,\cdot) - u_{\mathrm{explicit}}(t,\cdot)|.$$

This allows us to use the ordering result proved in [13, Proposition 3.9] (using hypothesis (H2') there) which then implies that u and u_{explicit} must be ordered almost surely. In light of (5.6), this means that $v = \mathscr{S}_{b,\gamma}[v_{\mathrm{B}}, v_{\mathrm{T}}]$ almost surely, as claimed. ☐

6 Stability of the viscous shocks

In this section we study the stability of the viscous shocks (1.8) and prove Theorem 1.2. The proof follows a strategy, based on ordering and L^1 contraction, similar to [13]. We begin with a time-averaged result.

Proposition 6.1. *Fix real numbers $a_{\mathrm{B}} < a_{\mathrm{T}}$ and $\gamma_{\mathrm{L}} < \gamma_{\mathrm{R}}$. Let $(u_{\mathrm{B}}, u_{\mathrm{T}}, u) \in \mathcal{C}([0,\infty); \mathcal{X}_{\mathrm{BT}} \times \mathcal{X})$ solve (1.3) with initial conditions satisfying $u_{\mathrm{Y}}(0,\cdot) \equiv a_{\mathrm{Y}}$ for $\mathrm{Y} \in \{\mathrm{B}, \mathrm{T}\}$. Let $b, \gamma \in \mathbb{R}$ be such that $(a_{\mathrm{B}}, a_{\mathrm{T}}, u(0,\cdot)) \in \mathcal{X}_{\mathrm{Sh},b,\gamma}$, and let $(b_t)_{t\geq 0}$ solve (1.7) with initial condition $b_0 = b$. Further asume that for all $x \in \mathbb{R}$, we have*

$$\mathscr{S}_{b,\gamma_{\mathrm{L}}}[a_{\mathrm{B}}, a_{\mathrm{T}}](x) \leq u(0,x) \leq \mathscr{S}_{b,\gamma_{\mathrm{R}}}[a_{\mathrm{B}}, a_{\mathrm{T}}](x). \tag{6.1}$$

If $(w_{\mathrm{B}}, w_{\mathrm{T}}) \sim \hat{\nu}^{[b]}_{a_{\mathrm{B}},a_{\mathrm{T}}}$, then we have

$$\lim_{T\to\infty} \int_1^{T+1} \mathrm{Law}(\tau_{b-b_t}(u_{\mathrm{B}}, u_{\mathrm{T}}, u)(t,\cdot)) = \mathrm{Law}(w_{\mathrm{B}}, w_{\mathrm{T}}, \mathscr{S}_{b,\gamma}[w_{\mathrm{B}}, w_{\mathrm{T}}])$$

weakly with respect to the topology of $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}$.

Proof. Let us define $\gamma_{\mathrm{C}} = \gamma$ for simplicity of notation later on. Consider the joint families

$$\tilde{\mathbf{u}} = (u_{\mathrm{B}}, u_{\mathrm{T}}, u) \in \mathcal{C}([0,\infty); \mathcal{X}_{\mathrm{BT}} \times \mathcal{X})$$

and

$$\mathbf{u} = (\mathbf{u}_{\mathrm{explicit}}, u) = ((u_{\mathrm{B}}, u_{\mathrm{T}}, u_{\mathrm{L}}, u_{\mathrm{C}}, u_{\mathrm{R}}), u) \in \mathcal{C}([0,\infty); \mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^4)$$

solving (1.3) with initial conditions

$$u_{\mathrm{X}}(0,\cdot) = \mathscr{S}_{b,\gamma_{\mathrm{X}}}[v_{\mathrm{B}}, v_{\mathrm{T}}]$$

for $\mathrm{X} \in \{\mathrm{L}, \mathrm{C}, \mathrm{R}\}$. We note that

$$\tau_{b-b_t} u_{\mathrm{X}}(t,\cdot) = \mathscr{S}_{b,\gamma_{\mathrm{X}}}[\tau_{b-b_t}(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)] \tag{6.2}$$

for $\mathrm{X} \in \{\mathrm{L}, \mathrm{C}, \mathrm{R}\}$. Also, the comparison principle and (6.1) imply that

$$u_{\mathrm{L}}(t,x) \le u(t,x) \le u_{\mathrm{R}}(t,x), \qquad \text{for all } t \ge 0 \text{ and } x \in \mathbb{R}$$

and

$$u_{\mathrm{L}}(t,x) \le u_{\mathrm{C}}(t,x) \le u_{\mathrm{R}}(t,x), \qquad \text{for all } t \ge 0 \text{ and } x \in \mathbb{R}.$$

In addition, by Lemmas 5.2 and 5.3, we have

$$\int_{\mathbb{R}} [u - u_{\mathrm{C}}](t,x)\,\mathrm{d}x = 0. \tag{6.3}$$

Therefore, we have, for $\mathrm{X} \in \{\mathrm{L}, \mathrm{R}\}$, that

$$\|\tau_{b-b_t}[u - u_{\mathrm{X}}](t,\cdot)\|_{L^1(\mathbb{R})} = \left|\int_{\mathbb{R}} [u - u_{\mathrm{X}}](t,x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}} [u_{\mathrm{C}} - u_{\mathrm{X}}](t,\cdot)\right| = |\gamma - \gamma_{\mathrm{X}}|,$$

with the second equality by (6.3) and the third by Proposition 3.3.

We claim that the family $\{\tau_{b-b_t}\mathbf{u}(t,\cdot)\}_{t\ge 1}$ is tight in $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}$. Indeed, by Proposition 4.3, the family $\{\tau_{b-b_t}(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)\}_{t\ge 0}$ converges in law with respect to the topology of $\mathcal{X}_{\mathrm{BT}}$ as $t \to \infty$, so in particular by Prokhorov's theorem (which applies since $\mathcal{X}_{\mathrm{BT}}$ is a Polish space as proved in Lemma 2.1) this family is tight in $\mathcal{X}_{\mathrm{BT}}$. By the comparison principle ([13, Proposition 3.1] we have

$$u_{\mathrm{B}}(t,x) \le u(t,x) \le u_{\mathrm{T}}(t,x)$$

for all $t \ge 0$ and $x \in \mathbb{R}$, so the family $\{\tau_{b-b_t} u(t,\cdot)\}_{t\ge 0}$ is uniformly bounded in probability in $\mathcal{X}$. Then [13, Proposition 2.2] this implies that $\{\tau_{b-b_t} u(t,\cdot)\}_{t\ge 1}$ is tight in $\mathcal{X}$. Therefore, $\{\tau_{b-b_t}\mathbf{u}(t,\cdot)\}_{t\ge 1}$ is tight in the topology of $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^4$.

Now let $T_k \uparrow \infty$ be a sequence so that

$$\mu = \lim_{k\to\infty} \frac{1}{T_k} \int_1^{T_{k+1}} \mathrm{Law}(\tau_{b-b_t}\mathbf{u}(t,\cdot))\,\mathrm{d}t$$

exists in the sense of weak convergence of probability measures on $\mathcal{X}_{\mathrm{BT}}$. Consider

$$\mathbf{w} = (w_{\mathrm{B}}, w_{\mathrm{T}}, w_{\mathrm{L}}, w_{\mathrm{C}}, w_{\mathrm{R}}) \sim \mu,$$

and $\tilde{\mathbf{w}} = (w_{\mathrm{B}}, w_{\mathrm{T}}, w)$. By (6.2) and Proposition 3.4, we have $w_{\mathrm{X}} = \mathscr{S}_{b,\gamma_X}[w_{\mathrm{B}}, w_{\mathrm{T}}]$ for $\mathrm{X} \in \{\mathrm{L}, \mathrm{C}, \mathrm{R}\}$ almost surely. By the Skorokhod representation theorem, Fatou's lemma, and the $L^1(\mathbb{R})$ contraction property of the Burgers equation as stated in [13, Proposition 3.2], we therefore have

$$\|w - w_{\mathrm{C}}\|_{L^1(\mathbb{R})} \le \|(u - u_{\mathrm{C}})(0,\cdot)\|_{L^1(\mathbb{R})} < \infty \tag{6.4}$$

almost surely. Similarly, for $\mathrm{X} \in \{\mathrm{L}, \mathrm{R}\}$, we have

$$\|w - w_{\mathrm{X}}\|_{L^1(\mathbb{R})} \le \|(u - u_{\mathrm{X}})(0,\cdot)\|_{L^1(\mathbb{R})} = |\gamma_{\mathrm{X}} - \gamma| \tag{6.5}$$

almost surely. We see from (6.4) that $(w_{\mathrm{B}}, w_{\mathrm{T}}, w) \in \mathcal{X}_{\mathrm{Sh}}$ almost surely. Moreover, the Krylov–Bogolyubov theorem (see e.g. [10, Theorem 3.1.1]) tells us that

$$(\hat{P}_t^{[b]})^* \,\mathrm{Law}(\tilde{\mathbf{w}}) = \mathrm{Law}(\tilde{\mathbf{w}}) \qquad \text{for any } t \ge 0.$$

Therefore, $\tilde{\mathbf{w}}$ is a stationary shock profile in the sense of Definition 5.4. By Proposition 5.5, there is a random $\tilde{\gamma} \in \mathbb{R}$ so that $w \in \mathscr{S}_{b,\tilde{\gamma}}[w_{\mathrm{B}}, w_{\mathrm{T}}]$ with probability 1. This means that $\|w - w_{\mathrm{X}}\|_{L^1(\mathbb{R})} = |\gamma_{\mathrm{X}} - \tilde{\gamma}|$ for $\mathrm{X} \in \{\mathrm{L}, \mathrm{R}\}$. Combined with (6.5), this means that $\tilde{\gamma} = \gamma$ almost surely. This uniquely identifies μ. Since the topology of weak convergence of probability measures is metrizable, we therefore have

$$\lim_{T\to\infty} \frac{1}{T} \int_1^{T+1} \mathrm{Law}(\tau_{b-b_t}\tilde{\mathbf{u}}(t,\cdot))\,\mathrm{d}t = \mu$$

weakly with respect to the topology of $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}$, as claimed. ☐

The next proposition shows the almost sure $L^1(\mathbb{R})$ convergence of the solution to an initial value problem to a viscous shock arising from a corresponding shift of $(u_{\mathrm{B}}, u_{\mathrm{T}})$.

Proposition 6.2. *With the same notation and assumptions as Proposition 6.1, we have*

$$\lim_{t\to\infty} \|\tau_{b-b_t} u(t,\cdot) - \mathscr{S}_{b,\gamma}[\tau_{b-b_t}(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)]\|_{L^1(\mathbb{R})} = 0$$

almost surely.

To prove Proposition 6.2, we first prove the following lemma.

Lemma 6.3. *Suppose that $(v_{\mathrm{B}}, v_{\mathrm{T}}, v), (v_{\mathrm{B}}, v_{\mathrm{T}}, \tilde{v}) \in \mathcal{X}_{\mathrm{Sh}}$ and, for some $b \in \mathbb{R}$ and $\gamma_{\mathrm{L}} < \gamma_{\mathrm{R}}$ we have*

$$\mathscr{S}_{b,\gamma_{\mathrm{L}}}[v_{\mathrm{B}}, v_{\mathrm{T}}](x) \le v(x) \le \mathscr{S}_{b,\gamma_{\mathrm{R}}}[v_{\mathrm{B}}, v_{\mathrm{T}}](x), \tag{6.6}$$

$$\mathscr{S}_{b,\gamma_{\mathrm{L}}}[v_{\mathrm{B}}, v_{\mathrm{T}}](x) \le \tilde{v}(x) \le \mathscr{S}_{b,\gamma_{\mathrm{R}}}[v_{\mathrm{B}}, v_{\mathrm{T}}](x) \tag{6.7}$$

for all $x \in \mathbb{R}$. Then there is a constant $C < \infty$, depending only on γ_{L} and γ_{R}, so that for all $L > 0$ and all $\ell > 1/2$ we have

$$\|v - \tilde{v}\|_{L^1(\mathbb{R})} \le 2\langle |b| + L\rangle^{1+\ell} \|v - \tilde{v}\|_{\mathcal{C}_{\mathrm{p}_\ell}} + C\left(\mathrm{e}^{2\overline{Z}_b[v_{\mathrm{B}}, v_{\mathrm{T}}](b-L)} + \mathrm{e}^{-2\overline{Z}_b[v_{\mathrm{B}}, v_{\mathrm{T}}](b+L)}\right). \tag{6.8}$$

Proof. For each $L > 0$, we have

$$\|v - \tilde{v}\|_{L^1(\mathbb{R})} = \|v - \tilde{v}\|_{L^1([b-L,b+L])} + \|v - \tilde{v}\|_{L^1(\mathbb{R}\setminus[b-L,b+L])}, \tag{6.9}$$

and

$$\|v - \tilde{v}\|_{L^1([b-L,b+L])} \le 2\langle |b| + L\rangle^{1+\ell} \|v - \tilde{v}\|_{\mathcal{C}_{\mathrm{p}_\ell}}. \tag{6.10}$$

Using (6.6)–(6.7) and arguing as in Proposition 3.3, we have

$$\begin{aligned} \int_{-\infty}^{b-L} |[v - \tilde{v}](x)|\,\mathrm{d}x &\le \int_{-\infty}^{b-L} [\mathscr{S}_{b,\gamma_{\mathrm{R}}}[v_{\mathrm{B}}, v_{\mathrm{T}}](x) - \mathscr{S}_{b,\gamma_{\mathrm{L}}}[v_{\mathrm{B}}, v_{\mathrm{T}}](x)|\,\mathrm{d}x \\ &= \int_{-\infty}^{\overline{Z}_b[v_{\mathrm{B}}, v_{\mathrm{T}}](b-L)} [-\tanh(\zeta - \gamma_{\mathrm{R}}) + \tanh(\zeta - \gamma_{\mathrm{R}})]\,\mathrm{d}\zeta \\ &\le C\mathrm{e}^{2\overline{Z}_b[v_{\mathrm{B}}, v_{\mathrm{T}}](b-L)}, \end{aligned} \tag{6.11}$$

with a constant C depending only on γ_{L} and γ_{R}. Similarly,

$$\int_{b+L}^{\infty} |[v - \tilde{v}](x)|\,\mathrm{d}x \le C\mathrm{e}^{-2\overline{Z}_b[v_{\mathrm{B}}, v_{\mathrm{T}}](b+L)}. \tag{6.12}$$

Using (6.10)–(6.12) in (6.9) yields (6.8). ☐

Now we can prove Proposition 6.2.

Proof of Proposition 6.2. We set $Z_{b,t} = Z_{b,t}[(u_{\mathrm{B}}, u_{\mathrm{T}})(0,\cdot)]$ and consider

$$\mathbf{w} = (w_{\mathrm{B}}, w_{\mathrm{T}}) \sim \hat{\nu}^{[b]}_{a_{\mathrm{B}},a_{\mathrm{T}}}.$$

By the Birkhoff ergodic theorem, [13, Theorem 1.2, property (P5)], and the fact that $\hat{\nu}^{[b]}_{a_{\mathrm{B}},a_{\mathrm{T}}}$ is absolutely continuous with respect to $\nu_{a_{\mathrm{B}},a_{\mathrm{T}}}$, we have

$$\lim_{L\to\pm\infty} \frac{1}{L}\int_b^{b+L} [w_{\mathrm{T}} - w_{\mathrm{B}}](x)\,\mathrm{d}x = \pm(a_{\mathrm{T}} - a_{\mathrm{B}})$$

almost surely, and in particular in probability. Hence, given $\varepsilon > 0$, there is an $L_\varepsilon < \infty$ so that $L \ge L_\varepsilon$ then

$$\mathbb{P}\left(\overline{Z}_b[\mathbf{w}](b-L) \ge -\frac{1}{2}(a_{\mathrm{T}} - a_{\mathrm{B}})L \text{ or } \overline{Z}_b[\mathbf{w}](b+L) \le \frac{1}{2}(a_{\mathrm{T}} - a_{\mathrm{B}})L\right) < \frac{\varepsilon}{4}. \tag{6.13}$$

In addition, we can choose L_ε so large that for all $L \ge L_\varepsilon$ we have

$$2C\mathrm{e}^{-(a_{\mathrm{T}}-a_{\mathrm{B}})L} < \varepsilon/2, \tag{6.14}$$

with C as in Lemma 6.3. By Proposition 6.2, we can find a $T_\varepsilon < \infty$ so large that if $T \ge T_\varepsilon$ and $S_T \sim \mathrm{Uniform}([1, T+1])$ is independent of everything else, then (using in addition (6.13))

$$\mathbb{P}\begin{pmatrix} \overline{Z}_b[\tau_{b-b_{S_T}}(u_{\mathrm{B}}, u_{\mathrm{T}})(S_T,\cdot)](b-L) \ge -\dfrac{L}{2}(a_{\mathrm{T}} - a_{\mathrm{B}}) \\ \text{or } \overline{Z}_b[\tau_{b-b_{S_T}}(u_{\mathrm{B}}, u_{\mathrm{T}})(S_{\mathrm{T}},\cdot)](b+L) \le \dfrac{L}{2}(a_{\mathrm{T}} - a_{\mathrm{B}}) \end{pmatrix} < \frac{\varepsilon}{2}. \tag{6.15}$$

and (using in addition Proposition 3.4)

$$\mathbb{P}\left(\left\|\tau_{b-b_{S_T}}u(S_T,\cdot) - \mathscr{S}_{b,\gamma}[\tau_{b-b_{S_T}}(u_{\mathrm{B}}, u_{\mathrm{T}})(S_T,\cdot)]\right\|_{\mathcal{C}_{\mathrm{p}_\ell}} \ge \frac{\varepsilon}{4L^{1+\ell}}\right) < \frac{\varepsilon}{2}. \tag{6.16}$$

Then we can compute, using (6.8),

$$\begin{aligned}
&\left\|\tau_{b-b_{S_T}}u(S_T,\cdot) - \mathscr{S}_{b,\gamma}[\tau_{b-b_{S_T}}(u_{\mathrm{B}}, u_{\mathrm{T}})(S_T,\cdot)]\right\|_{L^1(\mathbb{R})} \\
&\quad\le 2\langle |b| + L\rangle^{1+\ell}\left\|\tau_{b-b_{S_T}}u(S_T,\cdot) - \mathscr{S}_{b,\gamma}[\tau_{b-b_{S_T}}(u_{\mathrm{B}}, u_{\mathrm{T}})(S_T,\cdot)]\right\|_{\mathcal{C}_{\mathrm{p}_\ell}} \\
&\qquad + C\exp\left\{2\overline{Z}_b[\tau_{b-b_{S_T}}(u_{\mathrm{B}}, u_{\mathrm{T}})(S_T,\cdot)](b-L)\right\} \\
&\qquad + C\exp\left\{-2\overline{Z}_b[\tau_{b-b_{S_T}}(u_{\mathrm{B}}, u_{\mathrm{T}})(S_T,\cdot)](b+L)\right\}.
\end{aligned} \tag{6.17}$$

Using (6.14)–(6.16) in (6.17), we get

$$\mathbb{P}\left(\left\|\tau_{b-b_{S_T}}u(S_T,\cdot) - \mathscr{S}_{b,\gamma}[\tau_{b-b_{S_T}}(u_{\mathrm{B}}, u_{\mathrm{T}})(S_T,\cdot)]\right\|_{L^1(\mathbb{R})} \ge \varepsilon\right) < \varepsilon,$$

so

$$\|\tau_{b-b_{S_T}}u(S_T,\cdot) - \mathscr{S}_{b,\gamma}[\tau_{b-b_{S_T}}(u_{\mathrm{B}}, u_{\mathrm{T}})(S_T,\cdot)]\|_{L^1(\mathbb{R})} \to 0 \text{ in probability as } T\to\infty. \tag{6.18}$$

On the other hand, by the L^1 contractivity property (proved as [13, Proposition 3.2]), with probability 1 the norm

$$\|\tau_{b-b_t}u(t,\cdot) - \mathscr{S}_{b,\gamma}[\tau_{b-b_t}(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)]\|_{L^1(\mathbb{R})}$$

is decreasing in t. Together with (6.18) this means in fact this norm goes to zero almost surely. □

Now we can remove the random time S_T in the statement of Proposition 6.1, proving Theorem 1.2.

Proof of Theorem 1.2. First we note that by Proposition 6.2 for each $i \in \{1, \ldots, N\}$ we have (setting $b = b^{(i)}$, $b_t = b_t^{(i)}$, and $\gamma = 0$)

$$\begin{aligned} 0 &= \lim_{t\to\infty} \|\tau_{b^{(i)}-b_t^{(i)}} u(t,\cdot) - \mathscr{S}_{b^{(i)},0}[\tau_{b^{(i)}-b_t^{(i)}}(u_{\mathrm{B}}, u_T)(t,\cdot)\|_{L^1(\mathbb{R})} \\ &= \lim_{t\to\infty} \|u(t,\cdot) - \tau_{-b^{(i)}+b_t^{(i)}} \mathscr{S}_{b^{(i)},0}[\tau_{b^{(i)}-b_t^{(i)}}(u_{\mathrm{B}}, u_T)(t,\cdot)\|_{L^1(\mathbb{R})} \\ &= \lim_{t\to\infty} \|u(t,\cdot) - \mathscr{S}_{b_t^{(i)},0}[(u_{\mathrm{B}}, u_T)(t,\cdot)\|_{L^1(\mathbb{R})}, \end{aligned}$$

which is (1.20).

Let $\tilde{\mathbf{u}} = (u_1, \ldots, u_N)$, $\mathbf{u} = (u_{\mathrm{B}}, u_{\mathrm{T}}, \tilde{\mathbf{u}})$, with notation as in the statement of the theorem. The assumption (1.18) means that $(a_{\mathrm{B}}, a_{\mathrm{T}}, u_i(0,\cdot)) \in \mathcal{X}_{\mathrm{Sh},b^{(i)},0}$. We set $Z_t = Z_{b^{(1)},t}[a_{\mathrm{B}}, a_{\mathrm{T}}]$ and $b = b^{(1)}$, $b_t = b_t^{(1)}$. By the same argument as in the proof of tightness in Proposition 6.1, we see that $\{\tau_{b-b_t}\mathbf{u}(t,\cdot)\}_{t\geq 1}$ is tight in the topology of $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$. Suppose that we have a sequence $t_k \uparrow \infty$ and a limiting random variable $\mathbf{w} = (w_{\mathrm{B}}, w_{\mathrm{T}}, w_1, \ldots, w_N) \in \mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$ so that

$$\tau_{b-b_{t_k}}\mathbf{u}(t_k,\cdot) \to \mathbf{w}$$

in law in the topology of $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$. By Proposition 4.3, we have

$$\mathrm{Law}(w_{\mathrm{B}}, w_{\mathrm{T}}) = \hat{\nu}^{[b]}_{a_{\mathrm{B}},a_{\mathrm{T}}}. \tag{6.19}$$

Therefore, using Proposition 3.4, we have

$$\tau_{b-b_{t_k}} u_i(t_k,\cdot) - \mathscr{S}_{b^{(i)},0}[\tau_{b-b_{t_k}}(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)] \xrightarrow[k\to\infty]{\text{law}} w_i - \mathscr{S}_{b^{(1)},b^{(i)}-b^{(1)}}[w_{\mathrm{B}}, w_{\mathrm{T}}]$$

with respect to the topology of $\mathcal{X}$. On the other hand, Proposition 6.2 implies that, with probability 1, for each $1 \leq i \leq N$

$$\lim_{t\to\infty} \|\tau_{b-b_t} u_i(t,\cdot) - \mathscr{S}_{b^{(i)},0}[\tau_{b-b_t}(u_{\mathrm{B}}, u_{\mathrm{T}})(t,\cdot)]\|_{L^1(\mathbb{R})} = 0.$$

Combined, the last two displays show that $w_i = \mathscr{S}_{b^{(1)},b^{(i)}-b^{(1)}}[w_{\mathrm{B}}, w_{\mathrm{T}}]$ almost surely. Since the topology of weak convergence of probability measures with respect to the topology of $\mathcal{X}_{\mathrm{BT}} \times \mathcal{X}^N$ is metrizable, this, (6.19), and Proposition 3.4 imply (1.19). ☐

A A technical lemma

Lemma A.1. *Let $\mathcal{Y}$ be a metric space and let $(q \mapsto F_q) : \mathcal{Y} \to \mathcal{C}^1_{\mathrm{loc}}(\mathbb{R})$ be continuous and such that $\partial_x[F_q(x)] > 0$ for all $q \in \mathcal{Y}$ and all $x \in \mathbb{R}$. Let $G : \mathcal{Y} \to \mathbb{R}$ be continuous. Then the map $\mathcal{Y} \ni q \mapsto F_q^{-1}(G(q)) \in \mathbb{R}$ is continuous.*

Proof. Let $q \in \mathcal{Y}$ and let $\varepsilon > 0$. There is a $\kappa > 0$ so that

$$\inf_{x\,:\,|x-F_q^{-1}(G(q))|<2\varepsilon} F_q'(x) \geq \kappa. \tag{A.1}$$

Since $F_q^{-1} \circ G : \mathcal{Y} \to \mathbb{R}$ is continuous, there is a $\delta > 0$ so that if $d_{\mathcal{Y}}(q,\tilde{q}) < \delta$, then

$$\left|F_q^{-1}(G(q)) - F_q^{-1}(G(\tilde{q}))\right| < \varepsilon \tag{A.2}$$

and

$$\sup_{x\,:\,|x-F_q^{-1}(G(q))|<2\varepsilon} |F_{\tilde{q}}(x) - F_q(x)| < \kappa\varepsilon/2. \tag{A.3}$$

Now if $d_{\mathcal{Y}}(q,\tilde{q}) < \delta$ then $|F_q^{-1}(G(\tilde{q})) + \varepsilon - F_q^{-1}(G(q))| < 2\varepsilon$, so

$$\begin{aligned}
&F_{\tilde{q}}(F_q^{-1}(G(\tilde{q})) + \varepsilon) - G(\tilde{q}) \\
&\quad = F_{\tilde{q}}(F_q^{-1}(G(\tilde{q})) + \varepsilon) - F_q(F_q^{-1}(G(\tilde{q})) + \varepsilon) + F_q(F_q^{-1}(G(\tilde{q})) + \varepsilon) - F_q(F_q^{-1}(G(\tilde{q}))) \\
&\quad > -\kappa\varepsilon/2 + \kappa\varepsilon = \kappa\varepsilon/2
\end{aligned}$$

by (A.1) and (A.3). This means that

$$F_q^{-1}(G(\tilde{q})) + \varepsilon > F_{\tilde{q}}^{-1}(G(\tilde{q}) + \kappa\varepsilon/2) \geq F_{\tilde{q}}^{-1}(G(\tilde{q})).$$

Similarly, we have

$$F_q^{-1}(G(\tilde{q})) - \varepsilon < F_{\tilde{q}}^{-1}(G(\tilde{q})),$$

so in fact we have

$$|F_q^{-1}(G(\tilde{q})) - F_{\tilde{q}}^{-1}(G(\tilde{q}))| < \varepsilon. \tag{A.4}$$

Combining (A.2) and (A.4), we obtain

$$|F_q^{-1}(G(q)) - F_{\tilde{q}}^{-1}(G(\tilde{q}))| \leq |F_q^{-1}(G(q)) - F_q^{-1}(G(\tilde{q}))| + |F_q^{-1}(G(\tilde{q})) - F_{\tilde{q}}^{-1}(G(\tilde{q}))| < 2\varepsilon.$$

This completes the proof. □

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