Received: 6 September 2023 | Revised: 10 May 2024 | Accepted: 17 June 2024
DOI: 10.1002/rnc.7511

RESEARCH ARTICLE

WILEY

Anti-windup compensation for model reference adaptive control schemes

Jorge Sofrony[1] | **Matthew C. Turner**[2] | **Christopher M. Richards**[3]

[1]Faculty of Mechanical Engineering and Mechatronics, Universidad Nacional de Colombia, Bogota, Colombia
[2]School of Electronics and Computer Science, University of Southampton, Southampton, UK
[3]Department of Mechanical Engineering, University of Louisville, Louisville, Kentucky, USA

Correspondence
Jorge Sofrony, Faculty of Mechanical Engineering and Mechatronics, Universidad Nacional de Colombia, Carrera 30 No. 40-03, Oficina 209, Edificio 411, Bogota, Colombia.
Email: jsofronye@unal.edu.co

Funding information
National Science Foundation, Grant/Award Number: CMMI-2137030; UK Engineering and Physical Sciences Research Council, Grant/Award Number: EP/X012654/1; Colombian Ministry of Science–MINCIENCIAS, Grant/Award Number: 9812-913-93939

Abstract
Actuator constraints, particularly saturation limits, are an intrinsic and long-standing problem in the implementation of most control systems. Model reference adaptive control (MRAC) is no exception and it may suffer considerably when actuator saturation is encountered. With this in mind, this paper proposes an anti-windup strategy for model reference adaptive control schemes subject to actuator saturation. A prominent feature of the proposed compensator is that it has the same architecture as well-known nonadaptive schemes, namely model recovery anti-windup, which rely on the assumption that the system model is known accurately. Since, in the adaptive case, the model is largely unknown, the proposed approach uses an "estimate" of the system matrices for the anti-windup formulation and modifies the adaptation laws that update the controller gains; if the (unknown) ideal control gains are reached, the model recovery anti-windup formulation is recovered. The main results provide conditions under which, if the *ideal* control signal eventually lies within the control constraints, then the system states will converge to those of the reference model, that is, the tracking error will converge to zero asymptotically. The article deals with open-loop stable linear systems and highlights the main challenges involved in the design of anti-windup compensators for model-reference adaptive control systems, demonstrating its success via a flight control application.

KEYWORDS
anti-windup, model reference adaptive control, robust adaptive control

1 | INTRODUCTION

Model reference adaptive controller (MRAC) is one of the more popular adaptive control approaches, with much research being devoted to its development (see e.g., References 1 and 2). The main idea behind MRAC is to use a reference model (chosen by the designer) and a time-varying state-feedback controller, to generate a state-tracking error, which is then used to govern the adaptation of the control gains. MRAC has become one of the preferred adaptive control architectures and there is compelling evidence of successful deployment on real systems—see for example References 3–6. Unfortunately, MRAC is sensitive to the effects of unmatched uncertainty, disturbances, and unmodeled actuator dynamics, hence various modifications to the baseline algorithm have been proposed (see e.g., Reference 1).

Actuator saturation is another nonlinear uncertainty that is detrimental to adaptive control systems. In addition to the traditional wind-up effects that occur in many constrained systems,[7,8] the saturation nonlinearity corrupts the adaptation mechanisms by which the control gains are updated.[9] Crucially, in adaptive control systems, the control signal not only plays the role of regulation, it also probes the system for information and is often more active than one might expect of an LTI controller. Therefore, *in practice* adaptive control systems are naturally prone to saturation problems.

The impact of saturation in the adaptation process has been noted by many researchers and various papers have attempted to address the issue (e.g., References 6,10–15). Most of the work addresses input constraints using approaches that modify the reference model and/or the reference signal and diverge from the more traditional anti-windup philosophy used in linear control systems. The advantage of the "traditional" anti-windup approach is its two-step philosophy[16–18]: (i) when saturation is not active, the anti-windup compensator is inactive and the baseline (nominal) controller stabilizes the closed-loop and guarantees some performance criteria; (ii) when input saturation is present, an additional element (the anti-windup compensator) becomes active and improves performance and enhances stability properties. Most anti-windup schemes[7,8,17–19] have been developed for linear systems and require *knowledge of the plant model*. Since MRAC assumes that the model is *unknown*, applying model recovery AW to the adaptive control setting is not trivial.

Consequently, the anti-windup approach has not been investigated thoroughly in the field of adaptive control due to the complexity in demonstrating stability and correct adaptation of the controller gains. However some work exists, notably the work in Reference 20 where an indirect adaptive control is developed; the pseudo-hedging technique described in Reference 21; the sliding mode technique given in Reference 22 for systems with rate-limits; the output feedback adaptive controller with AW in Reference 23; the adaptive scalar AW gain for chaotic systems in Reference 24; and most recently the application of the approach of Reference 19 to systems with inertia variations.[25] Most of these schemes have drawbacks which include a lack of sufficient performance guarantees and a lack of practicality (most schemes do not apply the two-step philosophy).

Recent work on the *positive μ-modification*,[26,27] which relies on a modified reference model that includes information about the saturating input, has shown that it exhibits an AW-like behavior. In fact, under certain assumptions, the error between the ideal model state and the plant state will converge, provided that the ideal control signal is within the control bounds in the steady state.[28,29] However, the structure of this scheme is quite different from the standard anti-windup philosophy, which has shown itself so useful in practical applications such as aircraft flight control.[30,31] The main contribution of this paper is to formulate and solve a "model recovery anti-windup" (MRAW) for MRAC schemes. An initial version of this paper was presented in Reference 32, but lacked the generality presented in this article as the input distribution matrix was considered known. This article solves the anti-windup problem for MRAC schemes under no additional assumptions and demonstrates its effectiveness, and the practicalities associated with parameter choices, on a reasonably realistic flight control application: the longitudinal dynamics of JAXA's μ-pal experimental aircraft.

2 | MATERIALS AND METHODS

In this section, some notation used throughout the paper will be described and some facts and lemmas will be stated. A brief description of traditional MRAC will pave the way to formulate the AW compensator and provide a solution.

A positive (negative)-definite square matrix P is denoted as $P > 0$ $(P < 0)$. For a matrix G, $G^{\perp}$ denotes a full-rank matrix with rows that span the null space of G' such that $G^{\perp}G = 0$. The Hermitian of a square matrix is defined as

$$He\{A\} = A' + A.$$

A' denotes the transpose of a matrix A, and $\mathbf{tr}(A)$ its trace. An identity matrix of dimension n is denoted I_n. A signal $x(t)$ is said to belong to $\mathcal{L}_2$ if

$$\|x(t)\|_2 := \sqrt{\left(\int_0^{\infty} \|x(t)\|^2 dt\right)} < \infty$$

where $\|x\|$ denotes the Euclidean norm of the vector. A signal $x(t)$ is said to belong to $\mathcal{L}_\infty$ if

$$\|x(t)\|_\infty := \sup_{t\geq 0} \max_i |x_i(t)| < \infty.$$

The scalar saturation function, $\text{sat}_i(.) : \mathbb{R} \to [\underline{u}_i,\ \bar{u}_i]$, is defined as

$$\text{sat}_i(u_i) = \begin{cases} u_i & \text{if} \quad \underline{u}_i < u_i < \bar{u}_i, \\ \bar{u}_i & \text{if} \quad u_i \geq \bar{u}_i, \\ \underline{u}_i & \text{if} \quad u_i \leq \underline{u}_i. \end{cases}$$

The values $\bar{u}_i$ and $\underline{u}_i$ are the upper and lower limits respectively; if $\bar{u}_i = -\underline{u}_i$, the saturation is said to be symmetric; the values of $\bar{u}_i$ and $\underline{u}_i$ are of little importance to the technical results developed later. The vector saturation function $\text{sat}(.) : \mathbb{R}^m \to \mathbb{R}^m$ is simply

$$\text{sat}(u) = [\text{sat}_1(u_1) \ \ \ldots \ \ \text{sat}_m(u_m)]'.$$

Extensive use is made of the deadzone function, $\text{Dz}(u)$ which can be defined via the identity

$$\text{sat}(u) + \text{Dz}(u) = u.$$

Both saturation and dead-zone functions are globally Lipschitz with unit gain, such that the following property holds:

$$\|\psi_i(u_1 + u_2) - \psi(u_1)\| \leq \|u_2\| \quad \forall u_1,\ u_2 \in \mathbb{R}^m.$$

The main results will make use of the following facts and lemma.

Fact 1. The saturation and dead-zone functions are slope-restricted, namely,

$$0 \leq \frac{\sigma_i(u_1) - \sigma_i(u_2)}{u_1 - u_2} \leq 1 \quad \forall u_1, u_2 \in \mathbb{R}$$

$\sigma_i(\cdot)$ is the i'th component of either the saturation or the deadzone function.

Fact 1 implies that the saturation and dead-zone functions both satisfy the following incremental sector condition

$$(\sigma(u_1) - \sigma(u_2))' W (u_1 - u_2 - \sigma(u_1) + \sigma(u_2)) \geq 0 \tag{1}$$

for all diagonal matrices $W > 0$ and all $u_1, u_2 \in \mathbb{R}^m$. The following lemma from Reference 28 is also required.

Lemma 1. *Consider the dynamics*

$$\dot{x} = Ax + B\Lambda \text{sat}(u), \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_u}$ and Λ is a positive-definite constant matrix. If A is Hurwitz, then the state $x(t)$ is bounded for all $u(t) \in \mathbb{R}^{n_u}$.

2.1 | Model reference adaptive control

Consider the linear-time-invariant (LTI) plant

$$G \sim \left\{ \dot{x} \ = \ Ax + B\Lambda u, \right. \tag{3}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$. Matrix $\Lambda \in \mathbb{R}^{n_u \times n_u}$ is constant and positive-definite; it captures the unknown components in the input distribution matrix, for example, actuator efficiency loss; full measurement of the system states is assumed.

Assumption 1. Matrix A is unknown but Hurwitz; matrix B is completely known; matrix Λ is an unknown, positive-definite matrix.

Remark 1. In addition to standard MRAC assumptions, matrix A is assumed to be Hurwitz. This condition is necessary to prove boundedness under saturation, which requires the application of Lemma 1.

The plant is controlled via an MRAC where the reference model is:

$$G_m \sim \{\dot{x}_m = A_m x_m + B_m r. \tag{4}$$

The following *Model Matching Conditions* are assumed to hold.

Assumption 2 (Model matching conditions). There exist matrices $K_x^* \in \mathbb{R}^{n_u \times n}$ and $K_r^* \in \mathbb{R}^{n_u \times n_r}$ such that

$$A_m = A + B\Lambda K_x^* \qquad B_m = B\Lambda K_r^*. \tag{5}$$

The controller gains K_x^* and K_r^* are not known and are the "ideal" feedback gains. Note that the selection process of the reference model must guarantee the existence of such gains; hence limitations in the applicability of MRAC strategies appear in practical problems.

The standard MRAC controller, (6)–(7) below, ensures that the tracking error $e(t) = x_m(t) - x(t)$ converges to zero and all controller gains are bounded[1] when no saturation is present in the plant (3).

$$u = \hat{K}_x x + \hat{K}_r r, \tag{6}$$

$$\mathcal{A} \sim \begin{cases} \dot{\hat{K}}_x' & = & \Gamma_x x(e'\mathbf{P}B), \\ \dot{\hat{K}}_r' & = & \Gamma_r r(e'\mathbf{P}B). \end{cases} \tag{7}$$

The symmetric positive-definite matrix $\mathbf{P}$ is obtained from the solution, for some $Q > 0$, of the Lyapunov equation

$$A_m'\mathbf{P} + \mathbf{P}A_m + Q = 0. \tag{8}$$

Remark 2. Observe that the unknown input distribution matrix Λ is not estimated for purposes of "standard" MRAC controller synthesis. In contrast, if actuator saturation is present, the estimation of the system's (unknown) input gain, Λ, *is* required for the synthesis of the AW compensator, as shown below.

Although MRAC schemes have proven to be successful in a wide variety of settings, control signal saturation has shown to be highly detrimental to such control applications. Besides stability concerns, the performance of the time-varying closed-loop system must be guaranteed, hence adding difficulty to the design and analysis of saturating MRAC systems.

3 | MAIN RESULTS

Consider the plant with input saturation:

$$\dot{x} = Ax + B\Lambda \mathrm{sat}(u). \tag{9}$$

An AW compensator is retro-fitted to modify the nominal adaptive control algorithms (6)–(7) during periods of saturation (and the recovery from it) so that convergence of the tracking error is guaranteed. Following the ethos of traditional anti-windup schemes, the nominal controller dynamics are left unchanged during periods of no saturation. The scheme presented in this paper is based on the scheme proposed by Reference 18, and is similar to the so-called model-recovery AW scheme (MRAW) proposed by Reference 17 and advocated in Reference 7 and elsewhere. A key feature of these

AW schemes is that the plant matrices (A, B) are an intrinsic part of the AW design. In contrast, here, the system matrix A is *unknown*, hence the MRAW structure, which relies on a coprime factorization of the plant,[19] *cannot* be directly implemented. Therefore an estimate of matrix A, that is, $A_m - B\hat{\Lambda}\hat{K}_x(t)$, is required for implementation, which in turn requires the estimation of Λ; hence the structure and application of the AW compensator is more complex.

The following AW compensator structure is proposed:

$$\Sigma \sim \begin{cases} \dot{x}_{aw} = (A_m - B\hat{\Lambda}\hat{K}_x + B\hat{\Lambda}\mathbf{F})x_{aw} + B\hat{\Lambda}\mathrm{Dz(u)}, \\ v_1 = \mathbf{F}x_{aw}, \\ v_2 = x_{aw}. \end{cases} \tag{10}$$

Since it is assumed that A is unknown, an "estimate" is used, based on the model matching conditions in Assumption 2. The compensator has two outputs, v_1 and v_2. These are the compensation signals that modify the output and input of the controller, respectively. The (compensated) control signal is defined as the difference between an adaptive control law and some linear combination of the states that emanate from the anti-windup compensator dynamics.

$$u = \hat{K}_x(x + v_2) + \hat{K}_r r - v_1 = \hat{K}_x(x + x_{aw}) + \hat{K}_r r - \mathbf{F}x_{aw}. \tag{11}$$

The anti-windup "gain" $\mathbf{F}$ from (10) is a parameter to be designed and used to obtain stability. In contrast to the μ-modification presented in References 26,28,29, the adaptive update laws for the feedback gains (i.e., $\hat{K}_x$ and $\hat{K}_r$) are constructed in much the same way as described in (7).

Define new states that capture the system's nominal (i.e., no saturation present) dynamics as $x_l = x + x_{aw}$; hence the state tracking error is defined as:

$$e = \underbrace{x + x_{aw}}_{x_l} - x_m. \tag{12}$$

The fact that the system matrices A and Λ are unknown has two main effects on the proposed model recovery AW and MRAC configurations: (i) the usual decoupling of the closed-loop system into nominal closed-loop dynamics and non-linear dynamics (as presented by Reference 18) is not achievable, hence performance is more difficult to define; and (ii) the AW states feed back into the model reference error dynamics, hence proving boundedness of the states and controller gain estimates is more involved. These two issues add complexity to the proof of stability and asymptotic convergence to the origin of the tracking error. Define the modified controller as

$$C \sim \begin{cases} u_l = \hat{K}_x x_l + \hat{K}_r r, \\ u = u_l - \mathbf{F}x_{aw}, \end{cases} \tag{13}$$

$$\mathcal{A} \sim \begin{cases} \dot{\hat{K}}_x' = \Gamma_x x(e'\mathbf{P}B), \\ \dot{\hat{K}}_r' = \Gamma_r r(e'\mathbf{P}B). \end{cases} \tag{14}$$

The controller has an LQR-like structure, where the compensated states x_l are used for feedback instead of the real system states x. Another characteristic of the proposed MRAC is that the adaptation law for $\hat{K}_x$ is not driven by the feedback state that generates the error as in traditional formulations (i.e., x_l); instead, it is driven by x.

The main results are presented as two propositions. The first proposition ensures that the error $e(t) = x_l(t) - x_m(t)$ decays asymptotically to zero and that the adaptive gains are bounded; the second provides conditions under which $x_{aw}(t)$ will also decay asymptotically to zero and hence ensure that $x(t)$ approaches the ideal reference model states, $x_m(t)$, as $t \to \infty$. Finally, in Lemma 2, a synthesis procedure for the anti-windup gain $\mathbf{F}$ is proposed based on the Elimination Lemma (Finsler's Lemma).[33]

Proposition 1. *Let Assumptions 1 and 2 be satisfied and consider the interconnection of the plant (9), the reference model (4), the control and adaptive law (13)–(14),*

$$\dot{\hat{\Lambda}} = \Gamma_\lambda(\hat{K}_x x_{aw} - \mathbf{F}x_{aw} - \mathrm{Dz}(u))(e'\mathbf{P}B) \tag{15}$$

and the anti-windup compensator (10). Additionally, assume $r \in \mathcal{L}_\infty$. Then the error defined in equation (12) is such that $\lim_{t\to\infty} e(t) = 0$ and the adaptive gains $\hat{K}_x(t)$ and $\hat{K}_r(t)$, and input gain estimate $\hat{\Lambda}$, are bounded.

Proof. Using the dynamics (4), (5), (9), and (10) it follows that

$$\begin{aligned}
\dot{e} &= Ax + B\Lambda\mathrm{sat}(u) - A_m x_m - B_m r + (A_m - B\hat{\Lambda}\hat{K}_x + B\hat{\Lambda}\mathbf{F})x_{aw} + B\hat{\Lambda}\mathrm{Dz}(u)\\
&= Ax + B\Lambda\left(\hat{K}_x(x + x_{aw}) + \hat{K}_r r - \mathbf{F}x_{aw}\right) - B\Lambda\mathrm{Dz}(u) - A_m x_m - B_m r + (A_m - B\hat{\Lambda}\hat{K}_x + B\hat{\Lambda}\mathbf{F})x_{aw} + B\hat{\Lambda}\mathrm{Dz}(u)\\
&= Ax + B\Lambda\hat{K}_x x + B\Lambda\hat{K}_r r - B\tilde{\Lambda}(\hat{K}_x - \mathbf{F})x_{aw} + B\tilde{\Lambda}\mathrm{Dz}(u) - A_m x_m - B_m r + A_m x_{aw}\\
&= A_m e + B\Lambda(\hat{K}_x - K_x^*)x + B\Lambda(\hat{K}_r - K_r^*)r - B\tilde{\Lambda}(\hat{K}_x - \mathbf{F})x_{aw} + B\tilde{\Lambda}\mathrm{Dz}(u),
\end{aligned}$$

where the matching condition in Assumption 2 has been used, along with the control law (11). Matrix $\hat{\Lambda}$ is an estimate of the unknown input matrix and $\tilde{\Lambda} = \hat{\Lambda} - \Lambda$. Defining $\Delta K_x(t) = \hat{K}_x(t) - K_x^*$ and $\Delta K_r(t) = \hat{K}_r(t) - K_r^*$ then yields

$$\dot{e} = A_m e + B\Lambda\Delta K_x x + B\Lambda\Delta K_r r - B\tilde{\Lambda}[(\hat{K}_x - \mathbf{F})x_{aw} - \mathrm{Dz}(u)]. \tag{16}$$

Forming the Lyapunov function

$$V(t) = e'\mathbf{P}e + \mathbf{tr}[\Lambda\Delta K_x\Gamma_x^{-1}\Delta K_x'] + \mathbf{tr}[\Lambda\Delta K_r\Gamma_r^{-1}\Delta K_r'] + \mathbf{tr}[\tilde{\Lambda}\Gamma_\lambda^{-1}\tilde{\Lambda}'] \tag{17}$$

it follows in the same way as in standard MRAC, using the adaptive laws (7), that

$$\dot{V} = -e'Qe - 2e'\mathbf{P}B\tilde{\Lambda}[(\hat{K}_x - \mathbf{F})x_{aw} + \mathrm{Dz}(u)] + 2\mathbf{tr}[\tilde{\Lambda}\Gamma_\lambda^{-1}\dot{\hat{\Lambda}}']. \tag{18}$$

By using adaptive law (15), the last two terms of the previous equation cancel out and render $\dot{V} = -e'Qe$ and one can conclude that $e(t)$, $\Delta K_x(t)$, $\Delta K_r(t)$, and $\tilde{\Lambda}(t)$ are bounded, hence $\hat{K}_x(t)$, $\hat{K}_r(t)$, and $\hat{\Lambda}(t)$ are also bounded.

Now, from Barbalat's lemma, it follows that if $\dot{V}(t)$ is uniformly continuous, then $\lim_{t\to\infty}\dot{V}(t) = 0$, hence, since $Q > 0$, $\lim_{t\to\infty} e(t) = 0$. Note that $\dot{V}(t)$ is uniformly continuous if $\ddot{V}(t)$ is bounded, where

$$\ddot{V}(t) = -2e'Q\left(A_m e + B\Lambda\Delta K_x x + B\Lambda\Delta K_r r - B\tilde{\Lambda}\left[(\hat{K}_x - \mathbf{F})x_{aw} + \mathrm{Dz}(u)\right]\right). \tag{19}$$

From Proposition 1, $e(t)$, ΔK_x, ΔK_r, $\tilde{\Lambda}$, control signal u (hence $\mathrm{Dz}(u)$) and r are all bounded. Now note that there are two cases: (i) no saturation is present (i.e., $\mathrm{Dz}(u) = 0$) and (ii) saturation is present (hence $\mathrm{Dz}(u) \neq 0$). For the first case, assuming that the AW compensator dynamics are stable, $x_{aw} = 0$ and x are bounded. In the latter case, from Lemma 1, $x(t)$ is bounded, hence x_{aw} is also bounded (since e and x_m are bounded). As a result, uniform continuity is established, hence convergence (to the origin) of $e(t)$ may be inferred. ■

The above proposition guarantees convergence of the error $e(t) = x_l(t) - x_m(t)$ to the origin. Note that the error depends on x_{aw}, hence $x(t) \to x_m(t)$ is only achieved if the AW states converge to zero—see Equation (12). If it can be proven that, under certain conditions, $x_{aw}(t)$ converges to zero, then $x_l(t) \to x(t)$ and $x(t)$ will converge to $x_m(t)$. Proposition 2 below gives conditions under which this will be achieved.

Proposition 2. *Let Assumptions 1 and 2 be satisfied and suppose there exists a scalar $\eta > 0$, a positive-definite matrix $\Lambda_0 \in \mathbb{R}^{n_u \times n_u}$, a positive-definite matrix $\mathbf{P}_1 > 0$, a diagonal positive-definite matrix $\mathbf{W} > 0$ and a matrix $\mathbf{F}$ of suitable dimensions, such that the following conditions are satisfied:*

1. $\hat{K}_x'(t)\hat{K}_x(t) \leq \eta I_{n_u} \quad \forall t \geq 0,$
2. $\|\hat{\Lambda}(t)\| \leq \|\Lambda_0\| \quad \forall t \geq 0.$
3. *The matrix* Ψ_0 *is negative-definite where*

$$\Psi_0 = \begin{bmatrix} He\{\mathbf{P}_1(A_m + B\Lambda_0\mathbf{F})\} + \frac{1}{\epsilon}\mathbf{P}_1 B\Lambda_0\Lambda_0' B'\mathbf{P}_1 + \eta\epsilon I & \mathbf{P}_1 B\Lambda_0 - \mathbf{F}'\mathbf{W} \\ \star & -2\mathbf{W} \end{bmatrix}. \tag{20}$$

Then, with $\mathbf{F}$ *calculated from inequality (20), the anti-windup compensator (10) ensures that* $\lim_{t\to\infty} x(t) = x_m(t)$ *for all* $x(0), x_{aw}(0) \in \mathbb{R}^n$ *and all* x_m *and* r *such that*

$$\mathrm{Dz}(K_x^{ss} x_m + K_r^{ss} r) \in \mathcal{L}_2,$$

where K_x^{ss} *and* K_r^{ss} *are the steady state values of the adaptive gains* $\hat{K}_x$ *and* $\hat{K}_r$ *respectively, that is,* $\lim_{t\to\infty} \hat{K}_{(x,r)} = K_{(x,r)}^{ss}$.

Proof. Since, by Proposition 1, $\lim_{t\to\infty} e(t) = 0$, then $\lim_{t\to\infty} x(t) = x_m(t)$, if $\lim_{t\to\infty} x_{aw}(t) = 0$. Consider the AW compensator dynamics

$$\dot{x}_{aw} = (A_m - B\hat{\Lambda}\hat{K}_x + B\hat{\Lambda}\mathbf{F})x_{aw} + B\hat{\Lambda}\mathrm{Dz}(u)$$

and note that u can be re-written as

$$u = \underbrace{\hat{K}_x(x + x_{aw}) + \hat{K}_r r}_{u_0} - \mathbf{F}x_{aw}.$$

Defining $\phi(u_0, x_{aw}) = \mathrm{Dz}(u_0 - \mathbf{F}x_{aw}) - \mathrm{Dz}(u_0)$ and adding and subtracting $B\hat{\Lambda}\mathrm{Dz}(u_0)$, yields

$$\dot{x}_{aw} = (A_m - B\hat{\Lambda}\hat{K}_x + B\hat{\Lambda}\mathbf{F})x_{aw} + B\hat{\Lambda}\phi(u_0, x_{aw}) + B\hat{\Lambda}\mathrm{Dz}(u_0).$$

Since $\mathrm{Dz}(\cdot)$ is a slope-restricted nonlinearity (Fact 1), it follows from inequality (1) that, for all diagonal matrices $W > 0$,

$$\phi(u_0, x_{aw})'\mathbf{W}(-\mathbf{F}x_{aw} - \phi(u_0, x_{aw})) \geq 0. \tag{21}$$

Next, choosing a Lyapunov function $V_1(x_{aw}) = x_{aw}'\mathbf{P}_1 x_{aw}$, its derivative (including the sector condition (21)) is bounded by

$$\begin{aligned}\dot{V}_1(x_{aw}) &\leq 2x_{aw}'\mathbf{P}_1[(A_m + B\hat{\Lambda}\mathbf{F})x_{aw} - B\hat{\Lambda}\hat{K}_x x_{aw} + B\hat{\Lambda}\phi + B\hat{\Lambda}\mathrm{Dz}(u_0)] + 2\phi'\mathbf{W}(-\mathbf{F}x_{aw} - \phi) \\ &= \begin{bmatrix} x_{aw} \\ \phi \end{bmatrix}' \begin{bmatrix} He\{\mathbf{P}_1(A_m + B\hat{\Lambda}\mathbf{F})\} & \mathbf{P}_1 B\hat{\Lambda} - \mathbf{F}'\mathbf{W} \\ \star & -2\mathbf{W} \end{bmatrix} \begin{bmatrix} x_{aw} \\ \phi \end{bmatrix} + 2x_{aw}'\mathbf{P}_1 B\hat{\Lambda}\hat{K}_x x_{aw} + 2x_{aw}'\mathbf{P}_1 B\hat{\Lambda}\mathrm{Dz}(u_0).\end{aligned}$$

If, for some $\epsilon > 0$, Condition 1 of the proposition holds, then

$$2x_{aw}'\mathbf{P}_1 B\hat{\Lambda}\hat{K}_x x_{aw} \leq x_{aw}'\left(\frac{1}{\epsilon}\mathbf{P_1}B\hat{\Lambda}\hat{\Lambda}'B'\mathbf{P}_1 + \epsilon\eta I\right)x_{aw}$$

which implies that

$$\dot{V}_1(x_{aw}) \leq \begin{bmatrix} x_{aw} \\ \phi \end{bmatrix}' \Psi \begin{bmatrix} x_{aw} \\ \phi \end{bmatrix} + 2x_{aw}'\mathbf{P}_1 B\mathrm{Dz}(u_0), \tag{22}$$

where

$$\Psi = \begin{bmatrix} He\{\mathbf{P}_1(A_m + B\Lambda\mathbf{F})\} + \frac{1}{\epsilon}\mathbf{P}_1 B\Lambda\Lambda' B'\mathbf{P}_1 + \eta\epsilon I & \mathbf{P}_1 B\Lambda - \mathbf{F}'\mathbf{W} \\ \star & -2\mathbf{W} \end{bmatrix}. \tag{23}$$

Condition 2 of the proposition implies that $\Psi < 0$ if $\Psi_0 < 0$ for all bounded $\Lambda > 0$. If Condition 3) holds, it may be inferred that there exist positive scalars c_1 and c_2 such that (22) can be rewritten as:

$$\dot{V}_1(x_{aw}) \leq -c_1\|x_{aw}\|^2 + c_2\|x_{aw}\|\|\mathrm{Dz}(u_0)\|. \tag{24}$$

Now observe that by using the Lipschitz property of the dead-zone, then $\|\mathrm{Dz}(u_0)\|$ may be rewritten as

$$\|\mathrm{Dz}(u_0)\| = \|\mathrm{Dz}(\hat{K}_x(x_m + e) + \hat{K}_r r)\| \leq \|\mathrm{Dz}(K_x^{ss}x_m + K_r^{ss}r)\| + \|\hat{K}_x e\| + \|(\hat{K}_x - K_x^{ss})x_m + (\hat{K}_r - K_r^{ss})r\|.$$

Thus, since $\lim_{t\to\infty} e(t) = 0$, $\lim_{t\to\infty} \hat{K}_{(x,r)} = K_{(x,r)}^{ss}$ and x_m, r and $\hat{K}_x$ are bounded, it follows that $\mathrm{Dz}(u_0) \in \mathcal{L}_2$ if $\mathrm{Dz}(K_x^{ss}x_m + K_r^{ss}r) \in \mathcal{L}_2$. Applying the Comparison Lemma to Equation (24), it therefore follows that $x_{aw} \to 0$ as $t \to \infty$ if $\mathrm{Dz}(u_0) \in \mathcal{L}_2$, which is exactly as stipulated in the proposition. ∎

Remark 3. For cases where $\mathrm{Dz}(K_x^{ss}x_m + K_r^{ss}r)$ is not in $\mathcal{L}_2$, but is rather bounded, it can be observed from condition (24) that there exist (large enough) states x_{aw} such that the system is stable (i.e., bounded). The (estimated) boundedness set of the AW states can be calculated as

$$\mathcal{S} = \left\{ x_{aw} \,:\, \|x_{aw}\| \leq \frac{c_2}{c_1}\|\mathrm{Dz}(K_x^{ss}x_m + K_r^{ss}r)\| \right\} \tag{25}$$

This means that during cases of saturation, the AW states are bounded, that is, $x_{aw} \in \mathcal{S}$.

As a special case of Propositions 1 and 2, consider that there is no uncertainty on the input distribution matrix; it is possible to assume, without any loss of generality, that $\Lambda = I_{n_u}$. The great advantage of considering this special case is that it leads to an emphatic simplification of the anti-windup compensator. These propositions were proved in a conference version of this paper,[32] but are easily obtained as special cases of the more general results proved here.

Corollary 1. *Let Assumptions 1 and 2 be satisfied and consider the interconnection of the plant (9) with known $\Lambda = I_{n_u}$, the reference model (4), the control and adaptive laws (14), and the anti-windup compensator (10) with $\hat{\Lambda} = \Lambda = I_{n_u}$. Additionally, assume $r \in \mathcal{L}_\infty$. Then the error defined in equation (12) is such that $\lim_{t\to\infty} e(t) = 0$ and the adaptive gains $\hat{K}_x(t)$ and $\hat{K}_r(t)$, are bounded.*

Proof. This proof follows similarly to the proof of Proposition 1, except that the adaptive law (15) is not needed since $\Lambda = \hat{\Lambda} = I_{n_u}$. ∎

Corollary 2. *Let Assumptions 1 and 2 be satisfied, and assume that $\Lambda = \hat{\Lambda} = I_{n_u}$. Suppose there exists a scalar $\eta > 0$, a positive-definite matrix $\mathbf{P}_1 > 0$, a diagonal positive-definite matrix $\mathbf{W} > 0$ and a matrix $\mathbf{F}$ of suitable dimensions, such that the following conditions are satisfied*:

1. $\hat{K}_x'(t)\hat{K}_x(t) \leq \eta \quad \forall t \geq 0$.
2. *The matrix Ψ_2 is negative-definite where*

$$\Psi_2 = \begin{bmatrix} He\{\mathbf{P}_1(A_m + B\mathbf{F})\} + \eta\epsilon I & \mathbf{P}_1 B - \mathbf{F}'\mathbf{W} & \mathbf{P}_1 B \\ \star & -2\mathbf{W} & 0 \\ \star & \star & -\epsilon I \end{bmatrix}. \tag{26}$$

Then, with $\mathbf{F}$ calculated from inequality (26), the anti-windup compensator (10) ensures that $\lim_{t\to\infty} x(t) = x_m(t)$ for all $x(0), x_{aw}(0) \in \mathbb{R}^n$ and all x_m and r such that

$$\mathrm{Dz}(K_x^{ss}x_m + K_r^{ss}r) \in \mathcal{L}_2,$$

where K_x^{ss} and K_r^{ss} are the steady state values of the adaptive gains $\hat{K}_x$ and $\hat{K}_r$ respectively, that is, $\lim_{t\to\infty} \hat{K}_{(x,r)} = K_{(x,r)}^{ss}$.

Proof. The proof is similar to that of Proposition 2 except that $\Lambda = \hat{\Lambda} = I_{n_u}$. The main aspect of the proof that changes is Condition 2, which is no longer required. From the proof of Proposition 2, note that

$$\dot{V}_1(x_{aw}) < -c_1\|x_{aw}\|^2 + c_2\|x_{aw}\|\|\mathrm{D}z(u_0)\|. \tag{27}$$

If the matrix $\Psi_0 < 0$ where Ψ_0 is given in Equation (20) but with $\Lambda_0 = I_m$. Using Schur complements and a change of variables, $\Psi_0 < 0$ is then reduced to inequality (26). ■

3.1 | Solving the matrix inequality

Condition 3 of Proposition 2 contains a nonlinear matrix inequality which contains products of the matrix variables **F** and Λ_0, and thus is difficult to solve. To circumvent this difficulty, the Elimination Lemma can be used to remove **F** from the inequality, leaving an inequality linear in the matrix variables. In particular, the matrix Λ_0 is retained, allowing one to maximize the size of Λ_0, and thus provide an estimate of the anti-windup compensator's robustness against uncertainty in the input distribution matrix. To this end, note that, by the Schur complement, $\Psi_0 < 0$ is equivalent to $\Psi_1 < 0$, where

$$\Psi_1 = \begin{bmatrix} He\{\mathbf{P}_1(A_m + B\Lambda_0\mathbf{F})\} + \eta\epsilon I & \mathbf{P}_1 B\Lambda_0 - \mathbf{F}'\mathbf{W} & \mathbf{P}_1 B\Lambda_0 \\ \star & -2\mathbf{W} & 0 \\ \star & \star & -\epsilon \end{bmatrix}. \tag{28}$$

This inequality can be re-written as

$$\Psi_1 = \Theta + H\mathbf{F}G' + G\mathbf{F}'H' < 0, \tag{29}$$

$$\Theta = \begin{bmatrix} He\{\mathbf{P}_1 A_m\} + \eta\epsilon I & \mathbf{P}_1 B\Lambda_0 & \mathbf{P}_1 B\Lambda_0 \\ \star & -2\mathbf{W} & 0 \\ \star & \star & -\epsilon I \end{bmatrix}, \tag{30}$$

$$H = \begin{bmatrix} \mathbf{P}_1 B\Lambda_0 \\ -\mathbf{W} \\ 0 \end{bmatrix} \quad G = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}. \tag{31}$$

From the Projection Lemma,[33] if $H^{\perp}\Theta(H^{\perp})' < 0$ and $G^{\perp}\Theta(G^{\perp})' < 0$, then $\Psi_1 < 0$. Possible choices for $H^{\perp} \in \mathbb{R}^{(n+m)\times(n+2m)}$ and $G^{\perp} \in (2m \times (n+2m))$ are

$$H^{\perp} = \begin{bmatrix} I & B\Lambda_0 & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} \mathbf{P}_1^{-1} & 0 & 0 \\ 0 & \mathbf{W}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \qquad G^{\perp} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{32}$$

This then leads to the following result.

Lemma 2. *For a given scalar η, there exist matrices* **F**, $\Lambda_0 > 0$, $\mathbf{P_1} > 0$, $\mathbf{W} > 0$ *(and diagonal) and scalar ϵ solving inequality (28) if there exist matrices $\mathbf{Q}_1 > 0$, $\Lambda_\epsilon > 0$ and a positive scalar $\tilde{\epsilon}$ satisfying the matrix inequality*

$$\begin{bmatrix} He\{A_m\mathbf{Q}_1\} + B\Lambda_\epsilon B' & \mathbf{Q}_1 \\ \star & -\eta\tilde{\epsilon}I \end{bmatrix} < 0. \tag{33}$$

Furthermore, matrix **F** *can be computed by solving inequality (28) for unknowns* **F** *and* $\mathbf{W} > 0$, *using the matrices* $\mathbf{P}_1 = \mathbf{Q}_1^{-1}$, $\Lambda_0 = \sqrt{\Lambda_\epsilon}/\tilde{\epsilon}$ *and* $\tilde{\epsilon} = 1/\epsilon$.

Proof. First, observe that the condition $G^{\perp}\Theta(G^{\perp})' < 0$ reduces to

$$\begin{bmatrix} -2\mathbf{W} & 0 \\ \star & -\epsilon I \end{bmatrix} < 0$$

which holds unconditionally due to the assumptions $\mathbf{W} > 0$ and $\epsilon > 0$.

Using (32), condition $H^{\perp}\Theta(H^{\perp})' < 0$ may be calculated as

$$\begin{bmatrix} He\{A_m\mathbf{Q}_1\} & B\Lambda_0 & \mathbf{Q}_1 \\ \star & \epsilon & 0 \\ \star & \star & -\eta\tilde{\epsilon}I \end{bmatrix} < 0. \tag{34}$$

The Schur Complement is equivalent to

$$\begin{bmatrix} He\{A_m\mathbf{Q}_1\} + B\Lambda_0\Lambda_0'B'\tilde{\epsilon} & \mathbf{Q}_1 \\ \star & -\eta\tilde{\epsilon}I \end{bmatrix} < 0 \tag{35}$$

from which the LMI in the lemma follows by defining $\Lambda_{\epsilon} = \Lambda_0^2\tilde{\epsilon}$. Therefore, $\Psi_0 < 0$ will be ensured if inequality (33) holds. Moreover, inequality (33) can be solved to obtain Λ_0, $\tilde{\epsilon}$ and $\mathbf{Q}_1$ and then these values can be used to find a solution to the linear inequality (28) in the unknowns $\mathbf{F}$ and $\mathbf{W}$. ■

Remark 4. The utility of Lemma 2 is that it can be incorporated into a convex optimisation problem to maximize the "size" of Λ_0. Since maximizing Λ_0 is roughly equivalent to maximizing Λ_0^2, then since $\Lambda_0^2 = \Lambda_{\epsilon}\epsilon$, one could therefore seek to ensure

$$\epsilon > 1 \Leftrightarrow \tilde{\epsilon} < 1. \tag{36}$$

Additionally, one might seek $\Lambda_{\epsilon} > \mu$ for some $\mu > 0$. This last inequality is equivalent to

$$\Lambda_{\epsilon}^{-1} < \tilde{\mu} \Leftrightarrow \begin{bmatrix} -\tilde{\mu}I & I \\ \star & -\Lambda_{\epsilon} \end{bmatrix} < 0 \tag{37}$$

for some $\tilde{\mu} = 1/\mu$. Therefore, maximizing the size of Λ_0 can, indirectly, be achieved by solving the following convex optimisation problem

$$\min \tilde{\mu} \quad \text{subject to (33), (36), (37)} \tag{38}$$

for a given η. Note that this optimisation problem is inherently conservative since the optimisation of the size of Λ_0 is achieved indirectly via constraining $\tilde{\epsilon}$ and Λ_{ϵ} and then optimizing the dummy variable $\tilde{\mu}$.

3.2 | Estimating η

A key element of Proposition 2 is the estimation of η, since this is used in the matrix inequality from which the anti-windup gain $\mathbf{F}$ is computed. Furthermore, η must be chosen correctly since if it is under-estimated, the stability results will be local rather than global.

From the proof of Proposition 1 it is known that $\dot{V}_1(e, \Delta K_x, \Delta K_r, \tilde{\Lambda}) \leq 0$ which implies that

$$\mathbf{tr}(\Delta K_x(t)\Gamma_x^{-1}\Delta Kx(t)) \leq e(0)'\mathbf{P}e(0) + \mathbf{tr}(\tilde{\Lambda}(0)\Gamma_{\lambda}^{-1}\tilde{\Lambda}(0)') + \mathbf{tr}(\Delta K_x(0)\Gamma_x^{-1}\Delta K_x(0)') + \mathbf{tr}(\Delta K_r(0)\Gamma_r^{-1}\Delta K_r(0)'). \tag{39}$$

Assuming that the error $e(t)$ is initially zero and that $\hat{K}_x(0) = 0$, $\hat{K}_r(0) = 0$ and $\hat{\Lambda}_r(0) = 0$ also, the above inequality simplifies to

$$\mathbf{tr}(\hat{K}_x\Gamma_x^{-1}\hat{K}_x') \leq \mathbf{tr}(K_r^*\Gamma_r^{-1}K_r^{*'}) + \mathbf{tr}(\Lambda\Gamma_\lambda^{-1}\Lambda') + \delta\mathbf{tr}(\hat{K}_x\Gamma_x^{-1}\hat{K}_x') + \frac{1}{\delta}\mathbf{tr}(K_x^*\Gamma_x^{-1}K_x^{*'}) \tag{40}$$

for some non-negative $\delta < 1$. Thus,

$$\mathbf{tr}(\hat{K}_x'\hat{K}_x) \leq \frac{\mathbf{tr}(\Gamma_x^{-1})\|K_r^*\|^2 + \mathbf{tr}(\Gamma_\lambda^{-1})\|\Lambda\|^2 + \frac{1}{\delta}\mathbf{tr}(\Gamma_r^{-1})\|K_x^*\|^2}{(1-\delta)\mathbf{tr}(\Gamma_x^{-1})}. \tag{41}$$

The left-hand side is simply $\|\hat{K}_x\|^2$ and thus the right-hand side can be used to obtain η, under the conditions assumed on the initial values of $e(0)$, $\hat{K}_x(0)$, $\hat{K}_r(0)$ and $\hat{\Lambda}(0)$. This implies that η can be calculated provided the bounds on $\|K_x^*\|$, $\|K_r^*\|$ and $\|\Lambda\|$ can themselves be estimated adequately.

Remark 5. Inequality (41) reveals an interesting balance that needs to be struck in adaptive anti-windup: choosing the adaptive gains, $\Gamma_x, \Gamma_r, \Gamma_\Lambda$ small implies the bound on the gains, η, will be large and thus the system will be more prone to saturation. On the other hand, choosing the adaptive gains large implies the bound, η, will be small, and hence one would expect fewer saturation problems and more robustness to input uncertainty Λ—but at the cost of rapid adaptation, which would potentially lead to performance loss and implementation problems.

3.3 | An adaptive internal model anti-windup compensator

In anti-windup for linear control schemes, a particular choice of compensator is the so-called IMC scheme, in which the anti-windup gain matrix $\mathbf{F}$ is set to zero. This has the desirable property of being optimally robust against input additive uncertainties,[19] although its time-domain performance is often disappointing.

It is instructive to consider the adaptive version of the IMC anti-windup compensator which is obtained, as in the linear control case, that is, setting $\mathbf{F} = 0$. This results in considerable simplification of the anti-windup control scheme. The anti-windup dynamics (10) become

$$\Sigma_{IMC} \sim \begin{cases} \dot{x}_{aw} &= (A_m - B\hat{\Lambda}\hat{K}_x)x_{aw} + B\hat{\Lambda}Dz(u), \\ v_1 &= 0, \\ v_2 &= x_{aw}, \end{cases} \tag{42}$$

and the update of the $\hat{\Lambda}$ becomes

$$\dot{\hat{\Lambda}} = -\Gamma_\lambda(\hat{K}_x x_{aw} - Dz(u))(e'\mathbf{P}B). \tag{43}$$

Furthermore, Conditions (1) and (2) of Proposition 2 can be replaced by a more convenient condition. Similar to the proof of Proposition 2 and since $\mathbf{F} = 0$, Condition (2) reduces to

$$\begin{bmatrix} x_{aw} \\ \phi \end{bmatrix}' \begin{bmatrix} \mathbf{P}_1 A_m + A_m \mathbf{P}_1 & \mathbf{P}_1 B\hat{\Lambda} \\ \star & -2\mathbf{W} \end{bmatrix} \begin{bmatrix} x_{aw} \\ \phi \end{bmatrix} < 0. \tag{44}$$

Since $\mathbf{W} > 0$, this expression can be enforced to be negative-definite for any $\hat{\Lambda}$, provided there exists a $\mathbf{P}_1$ such that

$$\mathbf{P}_1 A_m + A_m \mathbf{P}_1 < 0.$$

This is guaranteed by construction since A_m is Hurwitz by assumption. Therefore, Condition (2) can be replaced by a condition which is satisfied *for any bounded* $\hat{\Lambda}$. Furthermore, a bound on $\hat{K}_x$ is not required to solve the IMC compensator problem, so Condition (1) vanishes. One can therefore conclude that the IMC compensator provides, in some sense, an optimally robust adaptive anti-windup compensator because it is entirely insensitive to variations in the input distribution matrix, that is, $\mathbf{\Lambda_0}$ can be arbitrarily large. The main drawback of the IMC anti-windup compensator is that performance can be dire, since the anti-windup compensator inherits the plant dynamics and $\hat{\Lambda}$ can attain very large magnitudes.

4 | SIMULATION EXAMPLE–JAXA EXPERIMENTAL AIRCRAFT

4.1 | Aircraft dynamics

Consider a flight control application where the plant is the longitudinal attitude dynamics of the JAXA μ-Pal experimental aircraft.[34] The aircraft is trimmed at straight-level flight condition, at of an altitude 1524 m, velocity $VTAS = 66.5$ m/s and angle of attack $\alpha = 4.98$ deg. The linearized plant has state-space matrices:

$$A = \begin{bmatrix} -0.0175 & 0.173 & -9.77 & -5.63 \\ -0.192 & -1.09 & -0.846 & 64.6 \\ 0 & 0 & 0 & 1 \\ 0.0081 & -0.0738 & 0.0062 & -1.9 \end{bmatrix} \quad B = \begin{bmatrix} -0.428 \\ 4.91 \\ 0 \\ 4.22 \end{bmatrix}.$$

The states are given by $x = [u_x, u_z, \theta, q]^T$, where the first two states are the translational velocity in the x and z directions respectively, and the last two are pitch angle θ, and pitch rate q. The control signal is the deflection angle of the elevator, which is assumed to be (symmetrically) saturated with $u_{\max} = 50$ *deg*. In this example, it is assumed that B is known, but A and Λ are *unknown*. Access to all the states is assumed.

The reference model matrices A_m and B_m are defined as:

$$A_m = \begin{bmatrix} -0.03 & 0.166 & 12.56 & 37.29 \\ -0.052 & -1.02 & -1554.7 & -427.82 \\ 0 & 0 & 0 & 1 \\ 0.128 & -0.0142 & -1335.49 & -425.12 \end{bmatrix} B_m = \begin{bmatrix} -138.1 \\ 1584.2 \\ 0 \\ 1361.6 \end{bmatrix}.$$

It is clear that (A_m, B_m) and (A, B) satisfy the matching conditions of Assumption 2.

4.2 | Adaptive control and anti-windup design

The nominal MRAC parameters were chosen as $\Gamma_x = \text{diag}\{1, 1, 1, 10\}$, $\Gamma_r = 0.1$ and the matrix Q for the solution of the Lyapunov equation (8) was chosen as $Q = 0.4I_4$, enabling $\mathbf{P}$ to be calculated for use in the nominal adaptive control law. Such a choice of parameters led to an excellent reference model following in the absence of input saturation but did lead to large peaks in control activity, even for relatively small reference demands, which would lead to saturation of the control signal.

Since Λ is assumed unknown, the anti-windup strategy of Proposition 2 was adopted, using the optimisation approach of Remark 4. The following adaptive anti-windup parameter was chosen as $\Gamma_\lambda = 0.1$.

To obtain the anti-windup compensator matrix $\mathbf{F}$, first inequality (33) was solved with $\eta = 100$. This then provided $\mathbf{Q_1} > 0$ and Λ_ϵ from which $\mathbf{P}_1$ and Λ_0 could be found. They were then used in inequality (28) to find $\mathbf{F}$. It must be noted that it is possible to solve LMI (29) directly instead of solving the full LMI (28). To do this and use predefined toolboxes for the solution of the LMI, it was necessary to choose the diagonal matrix $\mathbf{W} > 0$ and this was chosen as $\mathbf{W} = 0.01I_4$.

Generally speaking, since a"small" **W** leads to a "large" **F**, some iteration was required. With these parameters, the AW gain was calculated as

$$\mathbf{F} = \begin{bmatrix} 995.19, & -10.29, & -12368.75 & -335.43 \end{bmatrix}$$

and $\Lambda_0 = 0.15$. This value of Λ_0 was smaller than ideal, but one must note the conservatism present in the optimisation proposed in Remark 4. For the simulations, $\Lambda = 1.5$ was used.

4.3 | Small reference demand

The system is subject to a reference pulse train with a period of 100 seconds, the duty cycle of 50%, and an amplitude of 10 degrees. Figure 1 shows the response of the adaptive control system with no saturation/anti-windup applied. Excellent tracking is achieved, with the model response and plant response almost indistinguishable. Control activity is quite restrained, but upon the change in reference amplitude, large spikes in control activity may be observed; the control signal reaches peaks of up to 10^4 degrees, which would obviously cause saturation.

Figure 2 shows the system's response to the same reference demand, but now with saturation imposed. The output response has noticeable degradation. Interestingly after an initial period where the response begins to settle down, the system then begins a period of oscillation, having the appearance of the *bursting* phenomenon. Further investigation reveals that by setting $t = 200$, one finds that

$$\mathbf{spec}(A_p + B_p\Lambda\hat{K}(t)|_{t=200}) = \left\{-1.672 \times 10^5, -84.496, -0.165, 0.167\right\}.$$

Note that there is an unstable eigenvalue consistent with the bursting phenomena. Another by-product of saturation is that it drives the adaptive gains to very large values, as illustrated in the bottom plot of Figure 2.

Finally, the system is simulated with saturation and anti-windup. The tracking performance is shown in Figure 3. There is little performance degradation over that seen when no saturation is present, with the plant and model pitch attitude responses very close. The control signal saturates, but its behavior is reasonable. The evolution of the adaptive gain $\hat{K}_x$ is shown in the bottom plot in Figure 3: the presence of the anti-windup compensator prevents it from reaching

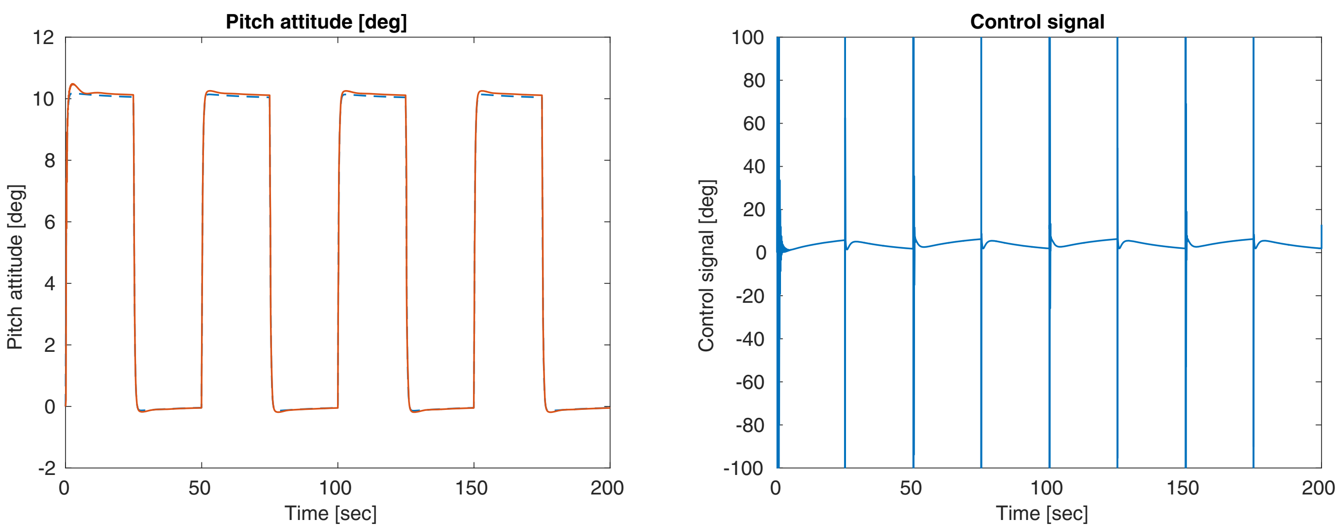

FIGURE 1 Response of JAXA aircraft with MRAC and no saturation/anti-windup. The first plot shows the pitch attitude response of the aircraft (solid line) and reference model (dashed line); the second plot shows the control signal response, plot constrained to ±100 degrees (peak control response is around 10^4 degrees).

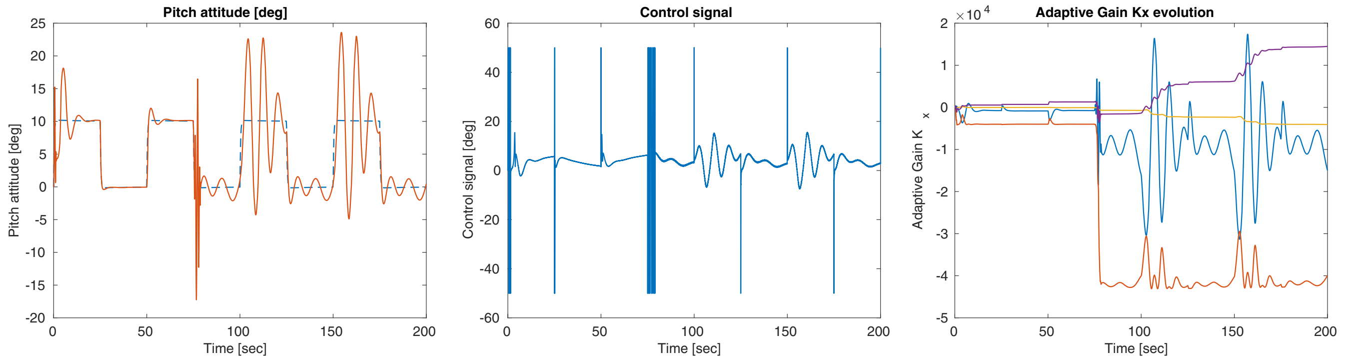

FIGURE 2 Response of JAXA aircraft with MRAC and saturation, but no anti-windup. The first plot shows the pitch attitude response of aircraft (solid line) and reference model (dashed line); the second plot shows saturated control signal response; the third plot shows the evolution of the adaptive gains $\hat{K}_x(t)$.

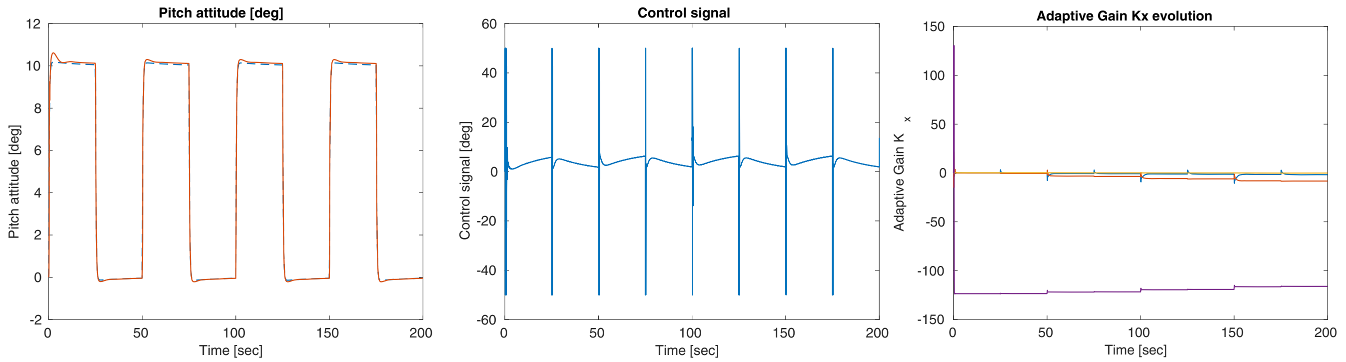

FIGURE 3 Response of JAXA aircraft with MRAC and saturation and anti-windup. The first plot shows the pitch attitude response of aircraft (solid line) and reference model (dashed line); the second plot shows saturated control signal response; the third plot shows the evolution of the adaptive gains $\hat{K}_x(t)$.

large values. Interestingly, no sign of bursting occurs and calculating $K_{ss} = \hat{K}(t)|_{t=200}$ in the same way as before it transpires that

$$\mathbf{spec}(A_p + B_p \Lambda K_{ss}) = \{-686.69, -5.284, -0.583, 0.038\}.$$

Although one eigenvalue is unstable, it is one order of magnitude smaller than the one when no anti-windup is present. Simulation over a longer period (500 s) confirms no bursting occurs.

4.4 | Large reference demand

The system is subject to a reference pulse train with a period of 100 s, the duty cycle of 50%, but this time with an amplitude 50 degrees. Although the system is nonlinear, the adaptive controller being nonlinear, when saturation is not present, the reference model tracking remains qualitatively the same as for the 10-degree demand case, hence the system dynamics of the unsaturated system are not presented.

The system behaves poorly when saturation is present but no anti-windup compensator is implemented. Not only is the transient response severely degraded, but the adaptive gains $\hat{K}_x(t)$ acquire immense values very quickly, making it difficult to simulate the system; it suffices to say that the behavior of the adaptive control system with saturation is unacceptable.

When saturation is present and anti-windup is implemented, Figure 4 shows the pitch attitude response of the system. The plant response has a mild deviation from that of the model response, with some overshoot and "steady state"

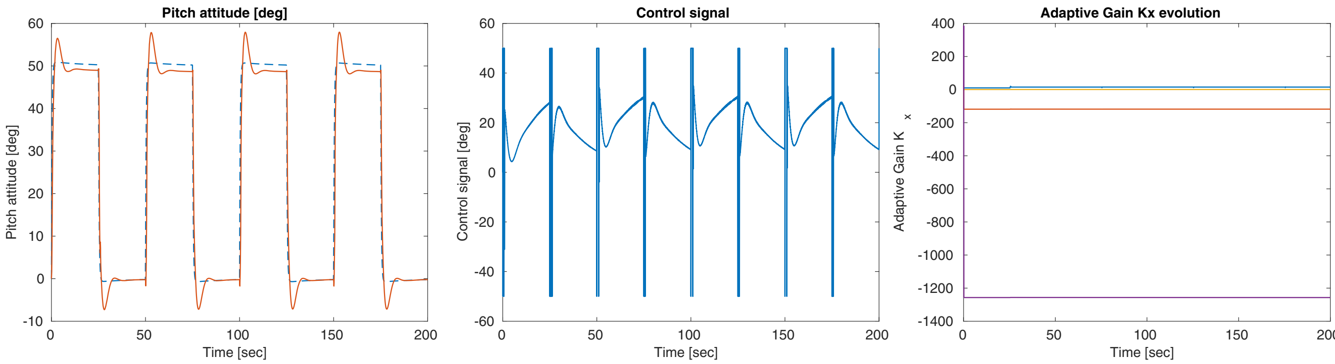

FIGURE 4 Response of JAXA aircraft with MRAC and saturation and anti-windup. The first plot shows the pitch attitude response of the aircraft (solid line) and reference model (dashed line); the second plot shows saturated control signal response; the third plot shows the evolution of the adaptive gains $\hat{K}_x(t)$.

error. The response of the compensated system is far superior to that without anti-windup. The adaptive gains converge faster when large amplitude references are applied, but their steady-state values are also larger, certainly much greater than the $\eta = 100$ used in the computation of the AW compensator; nevertheless the response appears satisfactory. Once again, notice that there is no "bursting" present in the anti-windup response and once again with $K_{ss}|_{t=200}$, one finds that

$$\mathbf{spec}(A_p + B_p \Lambda K_{ss}) = \{-7654.6, -6.617, -0.013 \pm 0.203j\}.$$

In this case all eigenvalues are stable and thus consistent with no bursting phenomena.

Remark 6. It is possible to improve the pitch attitude response of the system during saturation by increasing the rate of adaptation, that is, increase values of Γ_x and Γ_r. This leads to closer tracking of the model response but comes at the cost of something akin to control signal "chattering", which is not acceptable in aircraft applications.

4.5 | Preliminary comments on σ-modification

In practice, adaptive control laws are often implemented with so-called σ-modification to: (a) provide improved robustness and (b) prevent the adaptive gains from acquiring too great a magnitude (i.e., prevent parameter drift). The σ-modification seems of use when saturation is present and provides improved behavior. Although further studies need to be made to provide stability and performance guarantees, the following amended adaptive control laws were implemented:

$$\dot{\hat{K}}_x = -\sigma \hat{K}_x - \Gamma_x x(e'\mathbf{P}B), \tag{45}$$

$$\dot{\hat{K}}_r = -\sigma \hat{K}_r - \Gamma_r r(e'\mathbf{P}B), \tag{46}$$

$$\dot{\hat{\Lambda}} = -\sigma \hat{\Lambda} + \Gamma_\lambda \big((\hat{K}_x - \mathbf{F})x_{aw} - \mathrm{Dz}(u)\big)(e'\mathbf{P}B). \tag{47}$$

With little iteration, the parameter $\sigma = 0.2$ was chosen. The addition of σ-modification had a negligible effect on the tracking of system when no saturation is present: the pitch attitude still closely followed the model, although the σ term made the control gains gradually decay, leading to more control activity (simulations not included).

When saturation is present, but without anti-windup compensation, the presence of σ-modification has a somewhat positive effect, preventing control gains from acquiring such large values and providing damping to the pitch attitude response (shown in Figure 5). It is clear, however, that the response is far from ideal. Also, the σ-modification led to

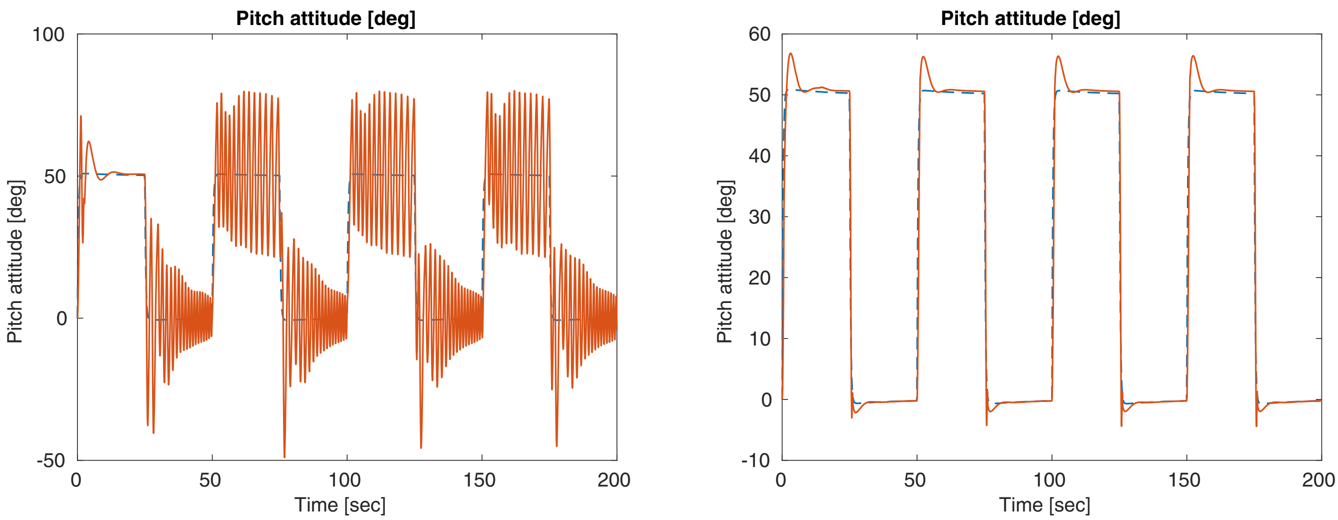

FIGURE 5 Pitch attitude response of JAXA aircraft with MRAC, saturation, and sigma modification; no AW (left) compensation and AW compensator active (right). Pitch Attitude response of aircraft (solid line) and reference model (dashed line).

an increase in control activity (not shown). When saturation is present and anti-windup compensation is deployed, σ-modification has a neutral effect: the pitch attitude tracking performance is improved (Figure 5), but control is activity dramatically increased (not shown). This comparison was somewhat crude and further investigation is required before concrete conclusions can be made.

5 | CONCLUSIONS

This article has proposed a full-order, model recovery anti-windup compensator for MRAC schemes, assuming that the plant matrix A is Hurwitz and that both A and Λ, which provide uncertainty on the input distribution matrix, are unknown. The compensator has an architecture similar to that found in conventional model recovery anti-windup schemes but requires an additional adaptive element to estimate the uncertainty on the input distribution matrix. The main results give practical conditions under which the system with saturation and anti-windup will provide asymptotic tracking of the reference model state: some of these conditions are similar to what is known for linear anti-windup schemes,[17] but additional ones are present due to the lack of information about the plant. With a flight control example, we have illustrated the potential of the scheme where it can suppress undesirable behavior due to saturation. The effects of the disturbances, measurement noise, and unmodelled dynamics have not been accounted for in this paper. Hence future work must address the addition of the so-called robust adaptive control modification and their implications for the anti-windup scheme under consideration.

ACKNOWLEDGMENTS

The work of C. M. Richards was supported by the National Science Foundation (Award CMMI-2137030). The work of M. C. Turner was supported by the UK Engineering and Physical Sciences Research Council, Grant number EP/X012654/1. The work of J. Sofrony was supported by the Colombian Ministry of Science–MINCIENCIAS, grant Res. 2488 of 2022 (Cod. 9812-913-93939).

CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

ORCID

Jorge Sofrony https://orcid.org/0000-0003-3159-1280
Matthew C. Turner https://orcid.org/0000-0003-2161-7635

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How to cite this article: Sofrony J, Turner MC, Richards CM. Anti-windup compensation for model reference adaptive control schemes. *Int J Robust Nonlinear Control*. 2024;1-18. doi: 10.1002/rnc.7511