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Quantum K-theory Chevalley formulas in the parabolic case

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ABSTRACT

We derive cancellation-free Chevalley-type multiplication formulas for the T-equivariant quantum K-theory ring of Grassmannians of type A and C, and also those of two-step flag manifolds of type A. They are obtained based on the uniform Chevalley formula in the T-equivariant quantum K-theory ring of arbitrary flag manifolds G/B, which was derived earlier in terms of the quantum alcove model, by the last three authors.

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1. Introduction

Y.-P. Lee defined the (small) quantum K-theory of a smooth projective variety X, denoted by $QK(X)$ (see [13], and also [8]). This is a deformation of the ordinary K-ring of X, analogous to the relation between quantum cohomology and ordinary cohomology. The deformed product is defined in terms of certain generalizations of Gromov-Witten invariants (i.e., the structure constants in quantum cohomology), called quantum K-invariants of Gromov-Witten type.

Given a simply-connected simple algebraic group G over $\mathbb{C}$, with Borel subgroup B, and maximal torus $T \subset B$, we consider the corresponding flag manifold G/B, the T-equivariant K-theory $K_T(G/B)$, and the T-equivariant quantum K-ring $QK_T(G/B) := K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][\![Q]\!]$, where $\mathbb{Z}[\Lambda][\![Q]\!]$ is the ring of formal power series with coefficients in $\mathbb{Z}[\Lambda]$ in the (Novikov) variables $Q_i = Q^{\alpha_i^\vee}$, $i \in I$, with I the index set for the simple roots α_i of G; $QK_T(G/B)$ has a $\mathbb{Z}[\Lambda][\![Q]\!]$-basis given by the classes $[\mathcal{O}^w]$ of the structure sheaves of the (opposite) Schubert varieties $X^w \subset G/B$ indexed by the elements w of the Weyl group $W = \langle s_i := s_{\alpha_i} \mid i \in I \rangle$ of G. Also, given a (standard) parabolic subgroup $P_J \supset B$ corresponding to a subset J, we also consider the partial flag manifold G/P_J, the T-equivariant K-theory $K_T(G/P_J)$, and the T-equivariant quantum K-ring $QK_T(G/P_J) := K_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][\![Q_K]\!]$, where $\mathbb{Z}[\Lambda][\![Q_K]\!]$ is the ring of formal power series with coefficients in $\mathbb{Z}[\Lambda]$ in the (Novikov) variables Q_k, $k \in K := I \setminus J$; $QK_T(G/P_J)$ has a $\mathbb{Z}[\Lambda][\![Q_K]\!]$-basis given by the (opposite) Schubert classes $[\mathcal{O}_J^y]$, for $y \in W^J$, where W^J denotes the set of minimal-length coset representatives for the cosets in W/W_J, where $W_J := \langle s_j \mid j \in J \rangle \subset W$. A Chevalley formula (in cohomology, K-theory, or their quantum versions) expresses the Schubert basis expansion of the product between an arbitrary Schubert class and the class of a line bundle, or a Schubert class indexed by a simple reflection (i.e., a divisor class). Having an explicit Chevalley formula in the quantum K-ring of an arbitrary flag manifold is important because this algebra is uniquely determined by products with divisor classes [4], together with its $K_T(\mathrm{pt})$-module structure; here, $K_T(\mathrm{pt}) = R(T)$, the representation ring of T, is identified with the group algebra $\mathbb{Z}[\Lambda]$ of the weight lattice Λ of G.

A cancellation-free Chevalley formula in the T-equivariant quantum K-theory of G/B was recently given in [19] (see also [20]); cf. the related conjecture in [22]. This formula is expressed in terms of the so-called quantum alcove model, which was introduced in [17]. It generalizes the formula in the T-equivariant K-theory of G/B in [22], which can easily be restricted to the partial flag manifold G/P_J for $J \subset I$. However, such a restriction does not work in quantum K-theory, because of the lack of functoriality. In contrast, we know (see [10]) that for a subset $J \subset I$, the ($\mathbb{Z}[\Lambda]$-linear) push-forward $(\pi_J)_* : K_T(G/B) \to K_T(G/P_J)$, induced by the natural projection $\pi_J : G/B \to G/P_J$ with P_J the (standard) parabolic subgroup of G corresponding to J, yields a surjective $\mathbb{Z}[\Lambda]$-module homomorphism from $QK_T^{\mathrm{poly}}(G/B) := K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q] \subset QK_T(G/B)$ onto $QK_T^{\mathrm{poly}}(G/P_J) := K_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q_K]$ such that

$$\Phi_J([\mathcal{O}^w] \cdot [\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_J^{\lfloor w \rfloor}] \cdot [\mathcal{O}_{G/P_J}(-\varpi_k)]$$

for $w \in W$ and $k \in K := I \setminus J$, where $\mathcal{O}_{G/B}(-\varpi_k)$ (resp., $\mathcal{O}_{G/P_J}(-\varpi_k)$) denotes the G-equivariant line bundle $G \times^B \mathbb{C}_{\varpi_k}$ over G/B (resp., $G \times^{P_J} \mathbb{C}_{\varpi_k}$ over G/P_J) corresponding to the one-dimensional representation $\mathbb{C}_{\varpi_k}$ of B (resp., P_J) of weight ϖ_k, and $\lfloor w \rfloor$ denotes the minimal-length coset representative for the coset wW_J in W/W_J; here, $\mathbb{Z}[\Lambda][Q]$ (resp., $\mathbb{Z}[\Lambda][Q_K]$) is the polynomial ring with coefficients in $\mathbb{Z}[\Lambda]$ in the (Novikov) variables Q_i, $i \in I$, (resp., Q_k, $k \in K = I \setminus J$).

Originally, in [10], the fact above was proved by using the relationship between the T-equivariant K-group of a (full or partial) semi-infinite flag manifold and the T-equivariant quantum K-theory of a (full or partial) flag manifold. Here we should mention that the existence of the surjective $\mathbb{Z}[\Lambda]$-algebra homomorphism Φ_J can also be verified by using the K-theoretic analog, conjectured in [18], of the Peterson homomorphism (K-Peterson homomorphism for short), which is a homomorphism of $\mathbb{Z}[\Lambda]$-algebras from the K-homology of the affine Grassmannian to (the localization, with respect to the positive part $Q^{\vee,+}$ of the coroot lattice $Q^\vee$, of) the quantum K-ring of G/P_J; a (new) proof of the existence of the K-Peterson homomorphism has been given recently by [6]. Indeed, as stated in the proof of [6, Lemma 2.12], under the K-Peterson homomorphism (which is a $\mathbb{Z}[\Lambda]$-algebra homomorphism) in the case of the Borel subgroup B, the classes of the structure sheaves of Schubert varieties in the affine Grassmannian indexed by the minimal-length coset representatives for W_{af}/W, with W_{af} the affine Weyl group and W the finite Weyl group, are sent injectively to the corresponding (opposite) Schubert classes in $QK_T(G/B)$ multiplied by explicit monomials in the Novikov variables corresponding to anti-dominant coroots in $-Q^{\vee,+}$. Hence, by composing the inverse of the K-Peterson homomorphism in the case of B with the K-Peterson homomorphism (which is also a $\mathbb{Z}[\Lambda]$-algebra homomorphism) in the case of $P_J \supset B$, we obtain the desired surjective $\mathbb{Z}[\Lambda]$-algebra homomorphism Φ_J; here we use the fact that all the (opposite) Schubert classes will lie in the image of the K-Peterson homomorphism in the case of B if we multiply them by a monomial in the Novikov variables corresponding to a (fixed) regular anti-dominant coroot in $-Q^{\vee,+}$. The details of these arguments are explained in Appendix A.

In this paper, on the basis of the fact above, we derive cancellation-free Chevalley formulas in the T-equivariant quantum K-ring $QK_T(G/P_J)$ of the partial flag manifold G/P_J, where $P_J \supset B$ is the (standard) parabolic subgroup of G corresponding to $J \subset I$ in the following two cases: (i) G is of type A or C and $J = I \setminus \{k\}$ for $k \in I$; (ii) G is of type A and $J = I \setminus \{k_1, k_2\}$ for $k_1, k_2 \in I$ with $k_1 \neq k_2$. More precisely, the mentioned Chevalley formulas express the quantum multiplication in $QK_T(G/P_J)$ with the class of the line bundle associated to the anti-dominant fundamental weight $-\varpi_k$ for $k \in I \setminus J$. Our strategy is the following: start with the Chevalley formula for $QK_T^{\mathrm{poly}}(G/B) \subset QK_T(G/B)$ in [19]; apply the $\mathbb{Z}[\Lambda]$-module surjection $\Phi_J : QK_T^{\mathrm{poly}}(G/B) \to QK_T^{\mathrm{poly}}(G/P_J)$ (which respects quantum multiplications) above; perform all cancellations, which arise via a sign-reversing involution. In addition, as an application of our Chevalley formulas, we prove the positivity property of certain structure constants of the quantum K-ring of a Grassmannian of type C and that of a two-step flag manifold of type A, as well as that for an arbitrary full flag manifold.

The resulting Chevalley formulas for Grassmannians of types A and C and also those for two-step flag manifolds of type A are no longer uniform, and they might also involve several cases. This fact validates our approach of deriving them from the uniform formula for G/B. Note that, in many cases, the opposite approach works better, namely the formulas for Grassmannians are obtained first, because they are easier.

We now compare our work with two related papers. In [12], a quantum K-theory Chevalley formula is given in $QK_T(G/P_J)$, where $J = I \setminus \{k\}$, for the line bundle associated to $-\varpi_k$, assuming that ϖ_k is a minuscule fundamental weight in type A, D, E, or B. The formulas are expressed in terms of the quantum Bruhat graph (on which the quantum alcove model is based). The approach in the present paper is simpler, and has the advantage of being easier to be extended to other partial flag manifolds; in fact, we also obtain a quantum K-theory Chevalley formula for two-step flag manifolds of type A. On another hand, the Chevalley formulas in [4] for cominuscule varieties are of a different nature than the corresponding cases of the formulas in this paper. Indeed, the role of the quantum Bruhat graph is not transparent in [4].

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2. Background

Consider a simply-connected simple algebraic group G over $\mathbb{C}$, with Borel subgroup B, and maximal torus T. Let $\mathfrak{g}$ be the corresponding finite-dimensional simple Lie algebra over $\mathbb{C}$, and W its Weyl group, with length function denoted by $\ell(\cdot)$. Let Φ, Φ^+, and Φ^- be the set of roots, positive roots, and negative roots of $\mathfrak{g}$, respectively, and let Λ be the corresponding weight lattice. Let α_i, $i \in I$, be the simple roots, $\Delta := \{\alpha_i \mid i \in I\}$ the set of all simple roots, θ the highest root, and $\alpha^\vee$ the coroot associated to the root α. The reflection corresponding to α is denoted, as usual, by s_α, and we let $s_i := s_{\alpha_i}$, $i \in I$, be the simple reflections. Set $\rho := (1/2) \sum_{\alpha \in \Phi^+} \alpha$.

Let J be a subset of I. We denote by $W_J := \langle s_i \mid i \in J \rangle$ the parabolic subgroup of W corresponding to J, and we identify W/W_J with the corresponding set of minimal coset representatives, denoted by W^J; note that if $J = \emptyset$, then $W^J = W^\emptyset$ is identical to W. For $w \in W$, we denote by $\lfloor w \rfloor = \lfloor w \rfloor^J \in W^J$ the minimal coset representative for the coset wW_J in W/W_J.

2.1. The quantum Bruhat graph

We start with the definition of this graph, which plays a fundamental role in our combinatorial model.

Definition 1. The quantum Bruhat graph $\mathrm{QB}(W)$ is the Φ^+-labeled directed graph whose vertices are the elements of W, and whose directed edges are of the form: $w \xrightarrow{\beta} v$ for $w, v \in W$ and $\beta \in \Phi^+$ such that $v = ws_\beta$, and such that either of the following holds: (i) $\ell(v) = \ell(w) + 1$; (ii) $\ell(v) = \ell(w) + 1 - 2\langle\rho,\, \beta^\vee\rangle$. An edge satisfying (i) (resp., (ii)) is called a Bruhat (resp., quantum) edge.

In [5], it is proved that the quantum Bruhat graph $\mathrm{QB}(W)$ has the following property (called the *shellability*): for all $x, y \in W$, there exists a unique directed path from in $\mathrm{QB}(W)$ from x to y whose edge labels are increasing with respect to an arbitrarily fixed reflection order on Φ^+.

We recall an explicit description of the edges of the quantum Bruhat graphs of types A and C. These results generalize the well-known criteria for covers of the Bruhat order in these cases [3].

In type A_{n-1}, the Weyl group elements (i.e., permutations) $w \in W(A_{n-1}) = S_n$ are written in one-line notation $w = [w(1), \dots, w(n)]$. For simplicity, we use the same notation (i, j) with $1 \le i < j \le n$ for the root α_{ij} and the reflection $s_{\alpha_{ij}}$, which is the transposition t_{ij} of i and j. We have $\theta = (1, n)$. We recall a criterion for the edges of the type A_{n-1} quantum Bruhat graph. We need the circular order $\prec_i$ on $[n]$ starting at i, namely $i \prec_i i+1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i-1$. It is convenient to think of this order in terms of the numbers $1, \dots, n$ arranged clockwise on a circle, in this order. We make the convention that, whenever we write $a \prec b \prec c \prec \cdots$; i.e., the leftmost of the chain $a \prec b \prec c \prec \cdots$ we are writing is a, we refer to the circular order $\prec = \prec_a$.

Proposition 2 *([15]). For $w \in S_n$ and $1 \le i < j \le n$, we have an edge $w \xrightarrow{(i,j)} w(i, j)$ if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$.*

If there is a position k as above, we say that the transposition of $w(i)$ and $w(j)$ straddles $w(k)$. We also let $w[i, j] := [w(i), w(i+1), \cdots, w(j)]$. We continue to use this terminology and notation for the other classical types.

The Weyl group of type C_n is the group of signed permutations. These are bijections w from $[\overline{n}] := \{1 < 2 < \cdots < n < \overline{n} < \overline{n-1} < \cdots < \overline{1}\}$ to $[\overline{n}]$ satisfying $w(\overline{\imath}) = \overline{w(i)}$. Here $\overline{\imath}$ is viewed as $-i$, so $\overline{\overline{\imath}} = i$, $|\overline{\imath}| = i$, and $\mathrm{sign}(\overline{\imath}) = -1$. We use both the window notation $w = [w(1), \dots, w(n)]$ and the full one-line notation $w = [w(1), \dots, w(n), w(\overline{n}), \dots, w(\overline{1})]$ for signed permutations. For simplicity, given $1 \le i < j \le n$, we denote by (i, j) the root $\varepsilon_i - \varepsilon_j$ and the corresponding reflection, which is identified with the composition of transpositions $t_{ij} t_{\overline{\jmath}\overline{\imath}}$. Similarly, for $1 \le i < j \le n$, we denote by $(i, \overline{\jmath}) = (j, \overline{\imath})$ the root $\varepsilon_i + \varepsilon_j$ and the corresponding reflection, which is identified with the composition of transpositions $t_{i\overline{j}}t_{j\overline{i}}$. Finally, we denote by $(i,\overline{\imath})$ the root $2\varepsilon_i$ and the corresponding reflection, which is identified with the transposition $t_{i\overline{\imath}}$. We have $\theta = (1,\overline{1})$.

We now recall the criterion for the edges of the type C_n quantum Bruhat graph. We need the circular order $\prec_i$ on $[\overline{n}]$ starting at i, which is defined similarly to the circular order on $[n]$, by thinking of the numbers $1, 2, \dots, n, \overline{n}, \overline{n-1}, \dots, \overline{1}$ arranged clockwise on a circle, in this order. We make the same convention as above that, whenever we write $a \prec b \prec c \prec \cdots$, we refer to the circular order $\prec = \prec_a$.

Proposition 3 *([15]). Let $w \in W(C_n)$ be a signed permutation.*

(1) *Given $1 \le i < j \le n$, we have an edge $w \xrightarrow{(i,j)} w(i,j)$ if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$.*
(2) *Given $1 \le i < j \le n$, we have an edge $w \xrightarrow{(i,\overline{\jmath})}$ if and only if $w(i) < w(\overline{\jmath})$, $\mathrm{sign}(w(i)) = \mathrm{sign}(w(\overline{\jmath}))$, and there is no k such that $i < k < \overline{\jmath}$ and $w(i) < w(k) < w(\overline{\jmath})$.*
(3) *Given $1 \le i \le n$, we have an edge $w \xrightarrow{(i,\overline{\imath})} w(i,\overline{\imath})$ if and only if there is no k such that $i < k < \overline{\imath}$ (or, equivalently, $i < k \le n$) and $w(i) \prec w(k) \prec w(\overline{\imath})$.*

2.2. The quantum alcove model

We need basic notions related to the combinatorial model known as the *alcove model*, which was defined in [22]. In particular, we need the notion of a λ-*chain of roots*, where λ is a weight. In this section, we recall definitions of these notions from [22].

Let Λ be the weight lattice of G and set $\mathfrak{h}^*_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we define a hyperplane $H_{\alpha,k}$ by $H_{\alpha,k} := \{\xi \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \xi, \alpha^\vee \rangle = k\}$. We denote by $s_{\beta,k}$, $\beta \in \Phi$ and $k \in \mathbb{Z}$, the reflection with respect to $H_{\alpha,k}$. Then, an *alcove* is defined to be a connected component of the space

$$\mathfrak{h}^*_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Phi,\ k \in \mathbb{Z}} H_{\alpha,k}.$$

If two alcoves A and B have a common wall, then A and B are said to be *adjacent*. Let us take adjacent alcoves A and B. If the common wall of A and B is contained in a hyperplane $H_{\alpha,k}$ for some $\alpha \in \Phi$ and $k \in \mathbb{Z}$, and the vector α points a direction from A to B, then we write $A \xrightarrow{\alpha} B$. We define a specific alcove $A_\circ$, called *the fundamental alcove*, by

$$A_\circ := \{\xi \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \xi, \alpha^\vee \rangle \ge 0 \text{ for all } \alpha \in \Phi^+\}.$$

In addition, for $\lambda \in \Lambda$, we define an alcove A_λ by $A_\lambda := A_\circ + \lambda = \{\xi + \lambda \mid \xi \in A_\circ\}$.

Definition 4 *([22, Definitions 5.2 and 5.4]).*

(1) An *alcove path* is a sequence $(A_0, A_1, \ldots, A_m)$ of alcoves such that for each $0 \le k \le m-1$, A_k and A_{k+1} are adjacent. If an alcove path $\Pi = (A_0, \ldots, A_m)$ is shortest among all alcove paths from A_0 to A_m, we say that Π is *reduced*.

(2) Let $\lambda \in \Lambda$. A λ-*chain of roots* is a sequence $\Gamma = (\beta_1, \ldots, \beta_m)$ of roots such that there exists an alcove path $\Pi = (A_\circ = A_0, \ldots, A_m = A_{-\lambda})$ such that

$$A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m.$$

If Π is reduced, then we also say that Γ is *reduced*.

Let $\lambda \in \Lambda$. Take a λ-chain $\Gamma = (\beta_1, \ldots, \beta_m)$ and corresponding alcove path $(A_0, \ldots, A_m)$. Set $r_i := s_{\beta_i}$, $i = 1, \ldots, m$. Below, we present an explicit description of the chains of roots corresponding to the anti-dominant fundamental weights in the classical types, i.e., $\lambda = -\varpi_k$.

We also need to recall the more general *quantum alcove model* [17]. We refer to [19, Section 3.2] for more details. In the next definition, we use the following notation: for $\beta \in \Phi$,

$$|\beta| := \begin{cases} \beta & \text{if } \beta \in \Phi^+, \\ -\beta & \text{if } \beta \in \Phi^-. \end{cases}$$

Definition 5 *([17]).* A subset $A = \{j_1 < j_2 < \cdots < j_s\}$ of $[m] := \{1, \ldots, m\}$ (possibly empty) is a w-*admissible subset* if we have the following directed path in the quantum Bruhat graph $\mathrm{QB}(W)$:

$$\Pi(w, A): \quad w \xrightarrow{|\beta_{j_1}|} wr_{j_1} \xrightarrow{|\beta_{j_2}|} wr_{j_1}r_{j_2} \xrightarrow{|\beta_{j_3}|} \cdots \xrightarrow{|\beta_{j_s}|} wr_{j_1}r_{j_2}\cdots r_{j_s} =: \mathrm{end}(w, A)\,.$$

We denote by A^- the subset of A corresponding to quantum steps in $\Pi(w, A)$. Let $\mathcal{A}(w, \Gamma)$ be the collection of all w-admissible subsets corresponding to the λ-chain Γ, and $\mathcal{A}_\lessdot(w, \Gamma)$ its subset consisting of all those A with $A^- = \emptyset$ (i.e., $\Pi(w, A)$ is a saturated chain in Bruhat order). For convenience, we identify an admissible subset $J = \{j_1 < \cdots < j_s\}$ with the corresponding sequence of roots $\{\beta_{j_1}, \ldots, \beta_{j_s}\}$ in the λ-chain Γ (in case of multiple occurrences of a root in Γ, we specify which one is considered). Also, we define statistics $\mathrm{down}(w, A)$ for $A \in \mathcal{A}(w, \Gamma)$ as follows:

$$\mathrm{down}(w, A) := \sum_{j \in A^-} |\beta_j|^\vee.$$

In addition, let $H_{\beta_j, -l_j}$, $j = 1 \ldots m$, be the hyperplane containing the common wall of A_{j-1} and A_j. Then we define $\mathrm{wt}(w, A)$ by

$$\mathrm{wt}(w, A) := -ws_{\beta_{j_1}, -l_{j_1}} \cdots s_{j_s, -l_{j_s}}(-\lambda).$$

We use the same notation as in Section 2.1, and we start with type A_{n-1}. It is proved in [22, Corollary 15.4] that, for any $k = 1, \dots, n-1$, we have the following reduced $(-\varpi_k)$-chain of roots, denoted by $\Gamma(k)$ (note that all the roots in this $(-\varpi_k)$-chain are negated for simplicity of notation, and hence they are all positive roots):

$$\begin{array}{llll} (\,(1,n), & (1,n-1), & \dots, & (1,k+1)\,, \\ (2,n), & (2,n-1), & \dots, & (2,k+1)\,, \\ & & \cdots & \\ (k,n), & (k,n-1), & \dots, & (k,k+1)\,)\,. \end{array} \tag{1}$$

In type A_{n-1}, we have the (Dynkin) diagram automorphism

$$\omega : [n-1] \to [n-1], \quad l \mapsto n-l.$$

By applying the diagram automorphism ω to $\Gamma(n-k)$, we obtain another reduced $(-\varpi_k)$-chain (with all the roots negated), denoted by $\Gamma^*(k)$:

$$\begin{array}{llll} (\,(1,n), & (2,n), & \dots, & (k,n)\,, \\ (1,n-1), & (2,n-1), & \dots, & (k,n-1)\,, \\ & & \cdots & \\ (1,k+1), & (2,k+1), & \dots, & (k,k+1)\,)\,. \end{array}$$

In type C_n, let

$$\Gamma(k) := \Gamma'_2 \cdots \Gamma'_k \Gamma_1(k) \cdots \Gamma_k(k)\,, \tag{2}$$

where

$$\begin{array}{lllll} \Gamma'_j := ((1,\overline{\jmath}), (2,\overline{\jmath}), \dots, (j-1,\overline{\jmath}))\,, & & & & \\ \Gamma_j(k) := (\,(1,\overline{\jmath}), & (2,\overline{\jmath}), & \dots, & (j-1,\overline{\jmath}), & \\ (j,\overline{k+1}), & (j,\overline{k+2}), & \dots, & (j,\overline{n}), & \\ (j,\overline{\jmath}), & & & & \\ (j,n), & (j,n-1), & \dots, & (j,k+1)\,)\,. & \end{array} \tag{3}$$

It is proved in [14, Lemma 4.1] that $\Gamma(k)$ is a reduced $(-\varpi_k)$-chain (with all the roots negated), for $1 \le k \le n$.

2.3. *The quantum K-theory of flag manifolds*

In order to describe the (small) T-equivariant quantum K-ring $QK_T(G/B)$, for the finite-dimensional flag manifold G/B, we associate a variable Q_i to each simple coroot $\alpha_i^\vee$, and set $\mathbb{Z}[Q] := \mathbb{Z}[Q_i \mid i \in I]$, $\mathbb{Z}[\![Q]\!] := \mathbb{Z}[\![Q_i \mid i \in I]\!]$; for each $\xi = \sum_{i \in I} d_i \alpha_i^\vee$ in $Q^{\vee,+}$, we set $Q^\xi := \prod_{i \in I} Q_i^{d_i}$. Also, we set $\mathbb{Z}[\Lambda][Q] := \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}} \mathbb{Z}[Q]$, $\mathbb{Z}[\Lambda][\![Q]\!] := \mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}} \mathbb{Z}[\![Q]\!]$, where $\mathbb{Z}[\Lambda]$ is the group algebra of the weight lattice Λ of G, and is identified with the representation ring $R(T) = K_T(\mathrm{pt})$. Following [13] (and also [8]), we define the quantum K-ring $QK_T(G/B)$ to be the $\mathbb{Z}[\Lambda][\![Q]\!]$-module $K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][\![Q]\!]$, equipped with the quantum product $\star$ given in terms of quantum K-invariants of Gromov-Witten type. The quantum K-ring $QK_T(G/B)$ has a $\mathbb{Z}[\Lambda][\![Q]\!]$-basis given by the classes $[\mathcal{O}^w]$ of the structure sheaves of the (opposite) Schubert varieties $X^w \subset G/B$ of codimension $\ell(w)$, for $w \in W$.

We consider the maximal (standard) parabolic subgroup $P_J \supset B$ of G corresponding to the subset $J := I \setminus \{k\}$, for some $k \in I$. The T-equivariant quantum K-ring $QK_T(G/P_J)$ of the partial flag manifold G/P_J is defined as $K_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][\![Q_k]\!]$, where $K_T(G/P_J)$ is the T-equivariant K-theory of G/P_J, and $\mathbb{Z}[\Lambda][\![Q_k]\!]$ is the ring of formal power series with coefficients in $\mathbb{Z}[\Lambda]$ in the single (Novikov) variable $Q_k = Q^{\alpha_k^\vee}$ corresponding to the simple coroot $\alpha_k^\vee$. The (opposite) Schubert classes $[\mathcal{O}_J^y]$, for $y \in W^J$, form a $\mathbb{Z}[\Lambda][\![Q_k]\!]$-basis.

We also consider the (standard) parabolic subgroup $P_J \supset B$ of G corresponding to the subset $J := I \setminus \{k_1, k_2\}$, for some $k_1, k_2 \in I$ with $k_1 \neq k_2$. In this case, the T-equivariant quantum K-ring $QK_T(G/P_J)$ is defined as $K_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][\![Q_{k_1}, Q_{k_2}]\!]$, where $\mathbb{Z}[\Lambda][\![Q_{k_1}, Q_{k_2}]\!]$ is the ring of formal power series with coefficients in $\mathbb{Z}[\Lambda]$ in the two (Novikov) variables Q_{k_1}, Q_{k_2}. As in the maximal parabolic case, the (opposite) Schubert classes $[\mathcal{O}_J^y]$, for $y \in W^J$, form a $\mathbb{Z}[\Lambda][\![Q_{k_1}, Q_{k_2}]\!]$-basis.

For an arbitrary subset $J \subset I$, let $\pi_J : G/B \to G/P_J$ be the natural projection, and let $(\pi_J)_* : K_T(G/B) \to K_T(G/P_J)$ denote the induced push-forward, which is $\mathbb{Z}[\Lambda]$-linear. Also, it is well-known that $\pi_J([\mathcal{O}^w]) = [\mathcal{O}_J^{\lfloor w \rfloor}]$ for each $w \in W$, where $\lfloor w \rfloor$ denotes the minimal-length coset representative for the coset wW_J in W/W_J, and that $\pi_J([\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_{G/P_J}(-\varpi_k)]$ for $k \in K = I \setminus J$ (see, for example, [24, Section 9.2]); recall that $\mathcal{O}_{G/B}(-\varpi_k)$ (resp., $\mathcal{O}_{G/P_J}(-\varpi_k)$) denotes the G-equivariant line bundle $G \times^B \mathbb{C}_{\varpi_k}$ over G/B (resp., $G \times^{P_J} \mathbb{C}_{\varpi_k}$ over G/P_J) corresponding to the one-dimensional representation $\mathbb{C}_{\varpi_k}$ of B (resp., P_J) of weight ϖ_k. Now, we set $QK_T^{\mathrm{poly}}(G/B) := K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q] \subset QK_T(G/B)$, and $QK_T^{\mathrm{poly}}(G/P_J) := K_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q_K]$, where $\mathbb{Z}[\Lambda][Q_K]$ is the ring of polynomials with coefficients in $\mathbb{Z}[\Lambda]$ in the (Novikov) variables $Q_k = Q^{\alpha_k^\vee}$, $k \in K := I \setminus J$. Based on the finiteness result on the quantum multiplication in $QK_T(G/P_J)$ with the line bundle classes $[\mathcal{O}_{G/P_J}(-\varpi_k)]$ for $k \in K = I \setminus J$ (see also [1]), Kato proved (see [10]) that the ($\mathbb{Z}[\Lambda]$-linear) push-forward $(\pi_J)_* : K_T(G/B) \to K_T(G/P_J)$ induces a surjective $\mathbb{Z}[\Lambda]$-module homomorphism $\Phi_J : QK_T^{\mathrm{poly}}(G/B) \to QK_T^{\mathrm{poly}}(G/P_J)$ such that for $w \in W$ and $k \in K = I \setminus J$, the following equality holds:

$$\Phi_J([\mathcal{O}^w] \cdot [\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_J^{\lfloor w \rfloor}] \cdot [\mathcal{O}_{G/P_J}(-\varpi_k)],$$

by defining $\Phi_J(Q^\xi) := Q^{[\xi]^J}$ for each $\xi \in Q^{\vee,+}$, where $[\xi]^J := \sum_{k \in I \setminus J} c_k \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$. Namely, Kato proved the following.

Theorem 6 *([10]). Let J be an arbitrary subset of I. Then, the surjective $\mathbb{Z}[\Lambda]$-module homomorphism*

$$\Phi_J : QK_T^{\mathrm{poly}}(G/B) \to QK_T^{\mathrm{poly}}(G/P_J)$$

defined by $\Phi_J(Q^{\xi}[\mathcal{O}^w]) = Q^{[\xi]^J}[\mathcal{O}_J^{\lfloor w \rfloor}]$ for $w \in W$ and $\xi \in Q^{\vee,+}$, where $[\xi]^J = \sum_{k\in I\setminus J} c_k \alpha_k^{\vee}$ for $\xi = \sum_{i\in I} c_i \alpha_i^{\vee} \in Q^{\vee,+}$, has the following multiplicativity:

$$\Phi_J([\mathcal{O}^w]\cdot[\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_J^{\lfloor w \rfloor}]\cdot[\mathcal{O}_{G/P_J}(-\varpi_k)]$$

for $w \in W$ and $k \in K = I \setminus J$.

In Appendix A, we give another proof of the existence of the multiplicative $\mathbb{Z}[\Lambda]$-module surjection Φ_J above by using the K-Peterson homomorphism, which is a homomorphism of $\mathbb{Z}[\Lambda]$-algebras from the K-homology of the affine Grassmannian associated to G to (the localization, with respect to $Q^{\vee,+}$, of) the quantum K-ring $QK_T(G/P_J)$; a (new) proof of the existence of the K-Peterson homomorphism has been given by [6].

We now recall the (cancellation-free) quantum K-theory Chevalley formula in [19, Theorem 47] (see also [20, Theorem 12]) for G/B, which is based on the quantum alcove model; in fact, we use the slight modification corresponding to the multiplication by the class $[\mathcal{O}(-\varpi_k)] := [\mathcal{O}_{G/B}(-\varpi_k)]$ of the line bundle associated to $-\varpi_k$. Throughout this paper, we denote by $|S|$ for a set S the cardinality of S. This formula is expressed in terms of a $(-\varpi_k)$-chain of roots, cf. Section 2.2.

Theorem 7. *Let $k \in I$, and fix a reduced $(-\varpi_k)$-chain $\Gamma(k)$. Then, in $QK_T^{\mathrm{poly}}(G/B) \subset QK_T(G/B)$, we have for $w \in W$,*

$$[\mathcal{O}(-\varpi_k)]\cdot[\mathcal{O}^w] = \sum_{A\in\mathcal{A}(w,\Gamma(k))} (-1)^{|A|}\, Q^{\mathrm{down}(w,A)} \mathbf{e}^{-\,\mathrm{wt}(w,A)} [\mathcal{O}^{\mathrm{end}(w,A)}]. \tag{4}$$

Remark 8. The right-hand side of equation (4) is cancellation-free. Indeed, suppose, for a contradiction, that there exist two admissible subsets $A, A' \in \mathcal{A}(w,\Gamma(k))$ satisfying $\mathrm{end}(w,A) = \mathrm{end}(w,A')$ and $(-1)^{|A|} = -(-1)^{|A'|}$ (together with $\mathrm{down}(w,A) = \mathrm{down}(w,A')$ and $\mathrm{wt}(w,A) = \mathrm{wt}(w,A')$). Here we know (see [5] and also [25]) that for directed paths $\mathbf{p}_1$, $\mathbf{p}_2$ in $\mathrm{QB}(W)$ starting from the same element $v \in W$ and ending at the same element $u \in W$, the equality $(-1)^{\ell(\mathbf{p}_1)} = (-1)^{\ell(\mathbf{p}_2)}$ holds, where $\ell(\cdot)$ denotes the length of a directed path. This contradicts the equality $(-1)^{|A|} = -(-1)^{|A'|}$, as desired.

Let $N_{u,v}^{w,\xi} \in \mathbb{Z}[P]$, with $v, w, u \in W^J$, $\xi \in Q_{I\setminus J}^{\vee,+} := \sum_{i\in I\setminus J} \mathbb{Z}_{\geq 0}\alpha_i^{\vee}$, denote the *structure constants* of $QK_T(G/P_J)$ defined by:

$$[\mathcal{O}^v]\cdot[\mathcal{O}^w] = \sum_{u\in W^J,\ \xi\in Q_{I\setminus J}^{\vee,+}} N_{v,w}^{u,\xi} Q^{\xi}[\mathcal{O}^u].$$

Let ρ_J be a half of the sum of all positive roots of P_J, and set $\deg(Q^\xi) := 2\langle \rho - \rho_J, \xi\rangle$ for $\xi \in Q^{\vee,+}_{I\setminus J}$. It is expected that the structure constants of $QK_T(G/P_J)$ have the following *positivity property.*

Conjecture 9 *([4, Conjecture 2.2]). For $v, w, u \in W^J$ and $\xi \in Q^{\vee,+}_{I\setminus J}$, we have*

$$(-1)^{\ell(v)+\ell(w)+\ell(u)+\deg(Q^\xi)} N^{u,\xi}_{v,w} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^\gamma - 1 \mid \gamma \in -\Delta].$$

The positivity property of the structure constants $N^{u,\xi}_{s_k,w}$, with $k \in K = I \setminus J$, is proved for cominuscule varieties G/P_J, which include Grassmannians of type A, by Buch-Chaput-Mihalcea-Perrin in [4] by writing explicitly the structure constants. Also, the positivity property of the structure constants $N^{u,\xi}_{v,w}$, with $\xi = 0$, is proved by Anderson-Griffeth-Miller [2] since these are the structure constants of the ordinary T-equivariant K-theory $K_T(G/P_J)$. In this paper, we prove the positivity property of the structure constants $N^{u,\xi}_{s_k,w}$, with $k \in K = I \setminus J$, for full flag manifolds of arbitrary types, two-step flag manifolds of type A, and Grassmannians of type C.

Let us define $C^{u,\xi}_w \in \mathbb{Z}[P]$, with $w, u \in W$, $\xi \in Q^{\vee,+}_{I\setminus J}$, by:

$$[\mathcal{O}(-\varpi_k)] \cdot [\mathcal{O}^w] = \sum_{u\in W^J,\ \xi\in Q^{\vee,+}_{I\setminus J}} C^{u,\xi}_w Q^\xi [\mathcal{O}^u].$$

Since it is well-known that $[\mathcal{O}^{s_k}] = 1 - \mathbf{e}^{-\varpi_k}[\mathcal{O}(-\varpi_k)]$ for $k \in I \setminus J$, we see that

$$\begin{aligned}
[\mathcal{O}^{s_k}] \cdot [\mathcal{O}^w] &= (1 - \mathbf{e}^{-\varpi_k}[\mathcal{O}(-\varpi_k)]) \cdot [\mathcal{O}^w] \\
&= [\mathcal{O}^w] - \mathbf{e}^{-\varpi_k}[\mathcal{O}(-\varpi_k)] \cdot [\mathcal{O}^w] \\
&= (1 - \mathbf{e}^{-\varpi_k} C^{w,0}_w)[\mathcal{O}^w] + \sum_{\xi\in Q^{\vee,+}_{I\setminus J}\setminus\{0\}} (-\mathbf{e}^{-\varpi_k} C^{w,\xi}_w) Q^\xi [\mathcal{O}^w] \\
&\quad + \sum_{u\in W\setminus\{w\},\ \xi\in Q^{\vee,+}_{I\setminus J}} (-\mathbf{e}^{-\varpi_k} C^{u,\xi}_w) Q^\xi [\mathcal{O}^u].
\end{aligned}$$

Hence it follows that for $w, u \in W^J$ and $\xi \in Q^{\vee,+}_{I\setminus J}$,

$$N^{u,\xi}_{s_k,w} = \begin{cases} 1 - \mathbf{e}^{-\varpi_k} C^{w,0}_w & \text{if } u = w \text{ and } \xi = 0, \\ -\mathbf{e}^{-\varpi_k} C^{u,\xi}_w & \text{otherwise.} \end{cases}$$

For the proof of the positivity property, we need the following lemma.

Lemma 10. *Let $w \in W$. Let $\lambda \in \Lambda$ be a dominant weight, and take a reduced $(-\lambda)$-chain Γ. For $A \in \mathcal{A}(w, \Gamma)$, we have $\mathrm{wt}(w, A) \in -\lambda + Q^+$.*

Proof. Let $A \in \mathcal{A}(w, \Gamma)$. We denote by Λ_{af}^0 the set of all level-zero weights of the (untwisted) affine Lie algebra $\mathfrak{g}_{\mathrm{af}} = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$ associated to $\mathfrak{g}$; in the following, we regard λ as an element of Λ_{af}^0.

We use *quantum Lakshmibai-Seshadri* (QLS) *paths* of shape λ, which are defined in [21, Definition 3.1]. We first assume that Γ is the *lex* $(-\lambda)$-chain, defined in [20, Section 4.2]. In this case, we know from [19, Proposition 31] that there exists a QLS path η of shape λ such that $\mathrm{wt}(w, A) = -\mathrm{wt}(\eta)$, where $\mathrm{wt}(\eta) := \eta(1)$. Let us write η in the form $\eta = (\nu_1, \dots, \nu_s; 0 = a_0 < a_1 < \cdots < a_s = 1)$, with $\nu_1, \dots, \nu_s \in W\lambda$ and $a_0, \dots, a_s \in \mathbb{Q}$. Then we see that $\nu_k \in \lambda - Q^+$, $k = 1, \dots, s$, since $\lambda \in \Lambda$ is dominant and W is the finite Weyl group. Hence we have

$$\mathrm{wt}(\eta) = \eta(1) = \sum_{k=1}^{s} (a_k - a_{k-1}) \nu_k \in \lambda - \sum_{j \in I} \mathbb{Q}_{\geq 0} \alpha_j.$$

Also, we have

$$\mathrm{wt}(\eta) = \eta(1) = \nu_s + \sum_{k=1}^{s-1} a_k (\nu_k - \nu_{k+1}).$$

Since (ν_k, ν_{k+1}) is an a_k-chain (see [16, Section 4]), it follows that $a_k(\nu_k - \nu_{k+1}) \in Q$ for $k = 1, \dots, s-1$. In addition, we have that $\nu_s \in \lambda - Q^+$. Hence we see that $\mathrm{wt}(\eta) \in \lambda + Q$. Therefore, we deduce that $\mathrm{wt}(\eta) \in \lambda - Q^+$, as desired.

We next assume that Γ is an arbitrary reduced $(-\lambda)$-chain. Then we know that Γ can be deformed to the lex $(-\lambda)$-chain Γ' by repeated application of *Yang-Baxter transformations* in [11, Section 3.1] (see also [20, Remark 40]). In this situation, [11, Theorems 3.2 and 3.4] implies that there exists a bijection $Y : \mathcal{A}(w, \Gamma) \to \mathcal{A}(w, \Gamma')$, given by *quantum Yang-Baxter moves*, such that $\mathrm{wt}(w, Y(A)) = \mathrm{wt}(w, A)$ for all $A \in \mathcal{A}(w, \Gamma)$. Here we note that [11, Theorem 3.2] states that Y is a *sijection* ([7, Section 2]), i.e., a "signed bijection", where $\mathcal{A}(w, \Gamma)$ and $\mathcal{A}(w, \Gamma')$ are regarded as signed sets equipped with sign functions. However, since $-\lambda$ is anti-dominant, we have no sign-reversing involution on any non-empty subset of $\mathcal{A}(w, \Gamma)$ or $\mathcal{A}(w, \Gamma')$. Therefore, Y is, in fact, a bijection. Since Γ' is the lex $(-\lambda)$-chain and $Y(A) \in \mathcal{A}(w, \Gamma')$, we deduce that $\mathrm{wt}(w, A) = \mathrm{wt}(w, Y(A)) \in -\lambda + Q^+$. This proves the lemma. $\square$

Note that if there exists an edge $x \to y$ in $\mathrm{QB}(W)$ for $x, y \in W$, then we have $\ell(y) \equiv \ell(x) + 1 \mod 2$ by the definition of $\mathrm{QB}(W)$. This implies that for $w \in W$ and $A \in \mathcal{A}(w, \Gamma(k))$, we have $(-1)^{|A|} = (-1)^{\ell(\mathrm{end}(w,A)) - \ell(w)}$.

In this section, we prove the positivity property of structure constants for full flag manifolds as a corollary of the Chevalley formula (Theorem 7). We will give a proof of the positivity property for Grassmannians of type C (resp., two-step flag manifolds of type A) in Section 3.2 (resp., Section 4.2).

Corollary 11. *Let G be of an arbitrary type, $J = \emptyset$ (hence $P_J = B$), and $k \in I$. Then, for $w, u \in W^J$ and $\xi \in Q^{\vee,+}$, we have*

$$(-1)^{1+\ell(w)+\ell(u)+\deg(Q^{\xi})} N_{s_k,w}^{u,\xi} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^{\gamma} - 1 \mid \gamma \in -\Delta].$$

Proof. Let $w \in W$. Take $A \in \mathcal{A}(w, \Gamma(k))$ such that $A^- = \emptyset$. If $A = \emptyset$, then we have

$$(-1)^{|A|} Q^{\mathrm{down}(w,A)} \mathbf{e}^{-\mathrm{wt}(w,A)} [\mathcal{O}^{\mathrm{end}(w,A)}] = \mathbf{e}^{w\varpi_k} [\mathcal{O}^w].$$

Since there exists no $A \in \mathcal{A}(w, \Gamma(k))$ such that $\mathrm{end}(w, A) = w$ and $\mathrm{down}(w, A) = 0$ except for $A = \emptyset$, we have $C_w^{w,0} = \mathbf{e}^{w\varpi_k}$. In addition, we have $\deg(Q^0) = 0$. Hence it follows that

$$N_{s_k,w}^{w,0} = 1 - \mathbf{e}^{w\varpi_k - \varpi_k} = (-1)^{1+\ell(w)+\ell(w)+\deg(Q^0)} (\mathbf{e}^{w\varpi_k - \varpi_k} - 1);$$

note that $w\varpi_k - \varpi_k \in -Q^+$. Since

$$\begin{aligned} \mathbf{e}^{-\mu} &= \prod_{i \in I} (\mathbf{e}^{-\alpha_i})^{c_i} \\ &= \prod_{i \in I} ((\mathbf{e}^{-\alpha_i} - 1) + 1)^{c_i} \\ &= \prod_{i \in I} \left(\sum_{k=0}^{c_i} \binom{c_i}{k} (\mathbf{e}^{-\alpha_i} - 1)^k \right) \in \mathbb{Z}_{\geq 0}[\mathbf{e}^{\gamma} - 1 \mid \gamma \in -\Delta] \end{aligned}$$

for $\mu = \sum_{i \in I} c_i \alpha_i \in Q^+$, we deduce that

$$(-1)^{1+\ell(w)+\ell(w)+\deg(Q^0)} N_{s_k,w}^{w,0} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^{\gamma} - 1 \mid \gamma \in -\Delta],$$

as desired.

Next, take $A \in \mathcal{A}(w, \Gamma(k)) \setminus \{\emptyset\}$. Then we have

$$\begin{aligned} &(-1)^{|A|} Q^{\mathrm{down}(w,A)} \mathbf{e}^{-\mathrm{wt}(w,A)} [\mathcal{O}^{\mathrm{end}(w,A)}] \\ &\quad = (-1)^{\ell(\mathrm{end}(w,A)) - \ell(w)} Q^{\mathrm{down}(w,A)} \mathbf{e}^{-\mathrm{wt}(w,A)} [\mathcal{O}^{\mathrm{end}(w,A)}]. \end{aligned}$$

Also, by Lemma 10, we have $\mathrm{wt}(w, A) \in -\varpi_k + Q^+$ for $A \in \mathcal{A}(w, \Gamma(k))$. Here we set

$$\mathcal{A}(w, \Gamma(k))_{u,\xi,\lambda} := \{A \in \mathcal{A}(w, \Gamma(k)) \mid \mathrm{end}(w, A) = u,\ \mathrm{down}(w, A) = \xi,\ \mathrm{wt}(w, A) = \lambda\},$$

for $u \in W$, $\xi \in Q^{\vee,+}$, and $\lambda \in -\varpi_k + Q^+$. Then by Theorem 7, we have

$$\begin{aligned} C_w^{u,\xi} &= \sum_{\lambda \in -\varpi_k + Q^+} \sum_{A \in \mathcal{A}(w,\Gamma(k))_{u,\xi,\lambda}} (-1)^{|A|} \mathbf{e}^{-\mathrm{wt}(w,A)} \\ &= (-1)^{\ell(u)-\ell(w)} \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))_{u,\xi,\lambda}| \mathbf{e}^{-\lambda}. \end{aligned}$$

Since $\deg(Q_j) = 2\langle \rho, \alpha_j^\vee \rangle = 2$ for all $j \in I$, we have $\deg(Q^\xi) \in 2\mathbb{Z}$. Therefore, we see that

$$N_{s_k,w}^{u,\xi} = -\mathbf{e}^{-\varpi_k} \cdot (-1)^{\ell(u)-\ell(w)} \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))_{u,\xi,\lambda}| \mathbf{e}^{-\lambda}$$
$$= (-1)^{1+\ell(w)+\ell(u)+\deg(Q^\xi)} \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))_{u,\xi,\lambda}| \mathbf{e}^{-\varpi_k - \lambda}.$$

This implies that

$$(-1)^{1+\ell(w)+\ell(u)+\deg(Q^\xi)} N_{s_k,w}^{u,\xi} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^\gamma - 1 \mid \gamma \in -\Delta],$$

as desired. This proves the corollary. $\square$

3. Quantum K-theory Chevalley formulas in the maximal parabolic case

Given a maximal parabolic subgroup P_J for $J = I \setminus \{k\}$, we will derive cancellation-free parabolic Chevalley formulas for the quantum multiplication in $QK_T(G/P_J)$ with $[\mathcal{O}(-\varpi_k)] := [\mathcal{O}_{G/P_J}(-\varpi_k)]$. Based on Theorem 6 explained in Section 2.3, we obtain certain formulas from equation (4) in Theorem 7 for $QK_T^{\mathrm{poly}}(G/B) \subset QK_T(G/B)$ by applying Φ_J; this argument works for an arbitrary fundamental weight ϖ_k of G of any type. However, upon applying Φ_J, there are many terms to be canceled in the corresponding formula in $QK_T^{\mathrm{poly}}(G/P_J) \subset QK_T(G/P_J)$. For any fundamental weight ϖ_k in types A and C, we cancel out all these terms via a sign-reversing involution, and obtain a cancellation-free formula. We rely on the structure of the corresponding $(-\varpi_k)$-chain of roots $\Gamma(k)$ in Section 2.2, as well as the quantum Bruhat graph criteria in Section 2.1.

Remark 12. Upon applying the above procedure, there are no cancellations among the terms corresponding to w-admissible subsets A with $A^- = \emptyset$, by Remark 8.

Remark 13. If G is of type A_{n-1}, then the partial flag manifold G/P_J for $J = I \setminus \{k\}$ is isomorphic to the *Grassmannian* $\mathrm{Gr}(k, n)$ defined as:

$$\mathrm{Gr}(k, n) := \{V \mid V \text{ is a subspace of } \mathbb{C}^n \text{ such that } \dim V = k\}.$$

Also, if G is of type C_n, then the partial flag manifold G/P_J for $J = I \setminus \{k\}$ is isomorphic to the *isotropic Grassmannian* $\mathrm{IG}(k, 2n)$ defined as:

$$\mathrm{IG}(k, 2n) := \left\{ V \;\middle|\; \begin{array}{l} V \text{ is a subspace of } \mathbb{C}^{2n} \text{ such that } \dim V = k, \text{ and} \\ V \text{ is isotropic with respect to } (-,-) \end{array} \right\};$$

where $(-,-)$ denotes a non-degenerate skew symmetric bilinear form on $\mathbb{C}^{2n}$.

3.1. Type A_{n-1}

We start with type A_{n-1}, and we fix the anti-dominant fundamental weight $-\varpi_k$. Note that $w \in W^J$ is equivalent to $w[1,k]$ and $w[k+1,n]$ being increasing sequences.

Lemma 14. *Consider $w \in W^J$. We have an edge $w \xrightarrow{(i,j)} w(i,j)$ in the quantum Bruhat graph on S_n, with $i \le k < j$, if and only if one of the following two conditions holds:*

(1) *the edge is a Bruhat cover, with $w(i) = a$, $w(j) = a+1$, and $w(i,j) \in W^J$;*
(2) *the edge is a quantum one, and $(i,j) = \alpha_k$.*

Proof. We implicitly use several times the quantum Bruhat graph criterion in Proposition 2, as well as the fact that $w[1,k]$ and $w[k+1,n]$ are increasing sequences. Letting $a := w(i)$, and assuming that the edge is a Bruhat cover, we cannot have $w(j) > a+1$ because the value $a+1$ would be straddled by the transposition (i,j). Indeed, this would happen irrespective of $a+1$ being in $w[1,k]$ or $w[k+1,n]$. So we must have $w(j) = a+1$. Now assume that $w(i) > w(j)$. If $i < k$, then the value $w(k)$ would be straddled, while if $j > k+1$, then the value $w(k+1)$ would be straddled. So we must have $i = k$ and $j = k+1$. □

We can now give a short proof of [12, Theorem I] in type A_{n-1}, which is restated below in terms of the quantum alcove model.

Theorem 15. *In type A_{n-1}, consider $1 \le k \le n-1$ and $w \in W^J$.*
(1) *If $w \ge \lfloor s_\theta \rfloor$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_k)] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_k} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k))} (-1)^{|A|} \left([\mathcal{O}^{\mathrm{end}(w,A)}] - Q_k [\mathcal{O}^{\lfloor \mathrm{end}(w,A) s_k \rfloor}] \right). \quad (5)$$

(2) *If $w \not\ge \lfloor s_\theta \rfloor$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_k)] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_k} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k))} (-1)^{|A|} [\mathcal{O}^{\mathrm{end}(w,A)}]. \quad (6)$$

Remark 16. As will be seen in the proof below, for $w \in W^J$, the condition $w \ge \lfloor s_\theta \rfloor$ is equivalent to the condition $w(k) = n$ and $w(k+1) = 1$.

Example 17. We give some examples of the Chevalley formula in the case that $n = 4$ and $k = 2$. Note that $\lfloor s_\theta \rfloor = s_3 s_1 s_2$. Also, we have $\Gamma(2) = ((1,4),(1,3),(2,4),(2,3))$ (with all roots negated).

(1) Let $w = s_3 s_1 s_2 = \lfloor s_\theta \rfloor$. Table 1 is the list of all admissible subsets $A \in \mathcal{A}(w, \Gamma(2))$ and their statistics $\mathrm{end}(w,A)$, $\mathrm{down}(w,A)$, together with $\lfloor \mathrm{end}(w,A) \rfloor$; note that $\mathrm{wt}(w,A) = -s_2\varpi_2$ for all $A \in \mathcal{A}(w,\Gamma(2))$.

Table 1
The list of all admissible subsets $A \in \mathcal{A}(s_3s_1s_2, \Gamma(2))$.

A	$\mathrm{end}(w, A)$	$\lfloor \mathrm{end}(w, A) \rfloor$	$\mathrm{down}(w, A)$
$\emptyset$	$s_3s_1s_2$	$s_3s_1s_2$	0
$\{1\}$	$s_2s_3s_1s_2$	$s_2s_3s_1s_2$	0
$\{4\}$	s_3s_1	e	$\alpha_2^\vee$
$\{1, 4\}$	$s_2s_3s_1$	s_2	$\alpha_2^\vee$

Table 2
The list of all admissible subsets $A \in \mathcal{A}(s_2, \Gamma(2))$.

A	$\mathrm{end}(w, A)$	$\lfloor \mathrm{end}(w, A) \rfloor$	$\mathrm{down}(w, A)$
$\emptyset$	s_2	s_2	0
$\{2\}$	s_1s_2	s_1s_2	0
$\{3\}$	s_3s_2	s_3s_2	0
$\{4\}$	e	e	$\alpha_2^\vee$
$\{2, 3\}$	$s_3s_1s_2$	$s_3s_1s_2$	0
$\{2, 4\}$	s_1	e	$\alpha_2^\vee$
$\{3, 4\}$	s_3	e	$\alpha_2^\vee$
$\{2, 3, 4\}$	s_3s_1	e	$\alpha_2^\vee$

By Theorem 7, in $QK_T^{\mathrm{poly}}(G/B)$, we have:

$$\begin{aligned}&[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2s_3s_1s_2}] \\ &\quad = \mathbf{e}^{s_2s_3s_1s_2\varpi_2} \left([\mathcal{O}^{s_3s_1s_2}] - [\mathcal{O}^{s_2s_3s_1s_2}] - Q_2[\mathcal{O}^{s_3s_1}] + Q_2[\mathcal{O}^{s_2s_3s_1}]\right).\end{aligned} \tag{7}$$

By applying the surjection $\Phi_J : QK_T^{\mathrm{poly}}(G/B) \to QK_T^{\mathrm{poly}}(G/P_J)$, explained in Theorem 6, to equation (7), we obtain the following cancellation-free formula in $QK_T^{\mathrm{poly}}(G/P_J) \subset QK_T(G/P_J)$:

$$[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2s_3s_1s_2}] = \mathbf{e}^{s_2s_3s_1s_2\varpi_2} \left([\mathcal{O}^{s_3s_1s_2}] - [\mathcal{O}^{s_2s_3s_1s_2}] - Q_2[\mathcal{O}^{e}] + Q_2[\mathcal{O}^{s_2}]\right).$$

Also, we deduce that $\mathcal{A}_{\lessdot}(w, \Gamma(2)) = \{\emptyset, \{1\}\}$. Therefore, we see that

$$\begin{aligned}&(\text{RHS of equation (5)}) \\ &= \mathbf{e}^{s_2s_3s_1s_2\varpi_2} \left(\left([\mathcal{O}^{s_3s_1s_2}] - Q_2[\mathcal{O}^{\lfloor s_3s_1 \rfloor}]\right) - \left([\mathcal{O}^{s_2s_3s_1s_2}] - Q_2[\mathcal{O}^{\lfloor s_2s_3s_1 \rfloor}]\right)\right) \\ &= \mathbf{e}^{s_2s_3s_1s_2\varpi_2} \left([\mathcal{O}^{s_3s_1s_2}] - Q_2[\mathcal{O}^{e}] - [\mathcal{O}^{s_2s_3s_1s_2}] + Q_2[\mathcal{O}^{s_2}]\right) \\ &= [\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2s_3s_1s_2}].\end{aligned}$$

Thus Theorem 15 (1) holds in this case.

(2) Let $w = s_2$; note that $w \not\geq \lfloor s_\theta \rfloor$. Then we can give the list of all admissible subsets $A \in \mathcal{A}(w, \Gamma(2))$ and their statistics $\mathrm{end}(w, A)$, $\mathrm{down}(w, A)$, together with $\lfloor \mathrm{end}(w, A) \rfloor$, as in Table 2. Note that $\mathrm{wt}(w, A) = -s_2\varpi_2$ for all $A \in \mathcal{A}(w, \Gamma(2))$.

By Theorem 7, in $QK_T^{\mathrm{poly}}(G/B)$, we have:

$$[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2}] = \mathbf{e}^{s_2\varpi_2} \left([\mathcal{O}^{s_2}] - [\mathcal{O}^{s_1s_2}] - [\mathcal{O}^{s_3s_2}] - Q_2[\mathcal{O}^{e}] \right.$$
$$\left. + [\mathcal{O}^{s_3s_1s_2}] + Q_2[\mathcal{O}^{s_1}] + Q_2[\mathcal{O}^{s_3}] - Q_2[\mathcal{O}^{s_3s_1}]\right). \tag{8}$$

By applying the surjection $\Phi_J : QK_T^{\mathrm{poly}}(G/B) \to QK_T^{\mathrm{poly}}(G/P_J)$ to equation (8), we obtain the following cancellation-free formula in $QK_T^{\mathrm{poly}}(G/P_J) \subset QK_T(G/P_J)$; here, the underlined terms in the first equality are canceled out:

$$\begin{aligned}[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2}] &= \mathbf{e}^{s_2\varpi_2}([\mathcal{O}^{s_2}] - [\mathcal{O}^{s_1s_2}] - [\mathcal{O}^{s_3s_2}]\underline{- Q_2[\mathcal{O}^{e}]} \\ &\quad + [\mathcal{O}^{s_3s_1s_2}]\underline{+ Q_2[\mathcal{O}^{e}] + Q_2[\mathcal{O}^{e}] - Q_2[\mathcal{O}^{e}]}) \\ &= \mathbf{e}^{s_2\varpi_2}([\mathcal{O}^{s_2}] - [\mathcal{O}^{s_1s_2}] - [\mathcal{O}^{s_3s_2}] + [\mathcal{O}^{s_3s_1s_2}]).\end{aligned}$$

Also, we deduce that $\mathcal{A}_{\lessdot}(w, \Gamma(2)) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. Therefore, we see that

$$\begin{aligned}(\text{RHS of equation (6)}) &= \mathbf{e}^{s_2\varpi_2} \left([\mathcal{O}^{s_2}] - [\mathcal{O}^{s_1s_2}] - [\mathcal{O}^{s_3s_2}] + [\mathcal{O}^{s_3s_1s_2}]\right) \\ &= [\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2}].\end{aligned}$$

Thus Theorem 15 (2) holds in this case.

Proof of Theorem 15. The result is clear when w is the identity (indeed, a w-admissible subset is either empty or consists only of the transposition $(k, k+1)$); so we can assume that $w(k) > w(k+1)$.

Let A be a generic w-admissible subset in $\mathcal{A}(w, \Gamma(k))$. Given the structure of the $(-\varpi_k)$-chain $\Gamma(k)$ in (1) and Lemma 14, we can see that a quantum step in a path $\Pi(w, A)$ must correspond to the transposition $\alpha_k = (k, k+1)$, which is the last one in $\Gamma(k)$. All other steps are Bruhat covers of the form specified in Lemma 14 (1). Moreover, the structure of $\Gamma(k)$ combined with the fact that $w \in W^J$ imply that A contains at most one root labeling a Bruhat cover in $\Pi(w, A)$ of the following forms: $(i, \cdot)$ for each $i \leq k$, and $(\cdot, j)$ for each $j > k$. All these facts will be used implicitly.

By Deodhar's criterion for the Bruhat order on the symmetric group [3, Theorem 2.6.3], we can see that $w \geq \lfloor s_\theta \rfloor = [2, 3, \ldots, k, n, 1, k+1, \ldots, n-1]$ (in one-line notation) if and only if $w(k) = n$ and $w(k+1) = 1$. Thus, we consider the following cases; whenever there are terms to be canceled, we describe the sign-reversing involution mentioned above.

Case 1: $w(k) < n$. Let $q > k+1$ be such that $w(q) = w(k) + 1 \leq n$. We pair every A containing $(k, k+1)$, but not (k, j) with $j > k+1$, with $A' := A \cup \{(k, q)\}$. It is clear that A' is also w-admissible, and in fact the root (k, q) is the predecessor of $(k, k+1)$ in A'. Moreover, we have

$$\lfloor \mathrm{end}(w, A') \rfloor = \lfloor \cdots (k, q)(k, k+1) \rfloor = \lfloor \cdots (k, k+1)(k+1, q) \rfloor = \lfloor \mathrm{end}(w, A) \rfloor,$$

as well as $\mathrm{down}(w, A) = \mathrm{down}(w, A')$ and $\mathrm{wt}(w, A) = \mathrm{wt}(w, A')$. The latter property is a consequence of the fact that all the affine reflections in the definition of $\mathrm{wt}(w, \cdot)$ in [19, Equation (12)] fix ϖ_k; for more details, see [22, Corollary 8.2] and the discussion preceding it. Finally, as the cardinalities of A and A' differ by 1, their contributions to the parabolic Chevalley formula for G/P_J have opposite signs. We have thus proved that the involution $A \leftrightarrow A'$ is sign-reversing.

Case 2: $w(k) = n$ and $w(k+1) > 1$. Let $p < k$ be such that $w(p) = w(k+1)-1 \geq 1$. We pair every A containing $(k, k+1)$, but not $(i, k+1)$ with $i < k$, with $A' := A \cup \{(p, k+1)\}$. We continue the reasoning like in Case 1.

Case 3: $w(k) = n$ and $w(k+1) = 1$. It is clear that no w-admissible subset A can contain transpositions of the form $(i, k+1)$ with $i < k$, and (k, j) with $j > k+1$. Furthermore, there is a 2-to-1 correspondence between $\mathcal{A}(w, \Gamma(k))$ and $\mathcal{A}_{\lessdot}(w, \Gamma(k))$: every $A \in \mathcal{A}_{\lessdot}(w, \Gamma(k))$ corresponds to itself and $A \cup \{(k, k+1)\}$. Like above, we can check that $\mathrm{wt}(w, A) = \mathrm{wt}(w, A \cup \{(k, k+1)\})$. Finally, based on the above facts and Remark 12, we can see that there are no cancellations of terms corresponding to the elements of either $A_{\lessdot}(w, \Gamma(k))$ or $\mathcal{A}(w, \Gamma(k)) \setminus A_{\lessdot}(w, \Gamma(k))$.

It is now easy to see that the uncanceled terms in the resulting combinatorial formula are precisely those in (5) in Case 3, and those in (6) in Cases 1 and 2. □

3.2. Type C_n

As we move beyond type A, we note that the following analogue of Lemma 14 exists: [12, Lemma 5.1] for any simply laced type and ϖ_k minuscule. Below we present the corresponding result in type C_n, which works for any ϖ_k; this result is easily proved based on the quantum Bruhat graph criterion in Section 2.1.

Lemma 18. *Consider $1 \leq k \leq n$ and $w \in W^J$ in type C_n. We have a quantum edge $w \xrightarrow{\alpha} ws_\alpha$ in $\mathrm{QB}(W)$, with $\alpha \in \Phi^+ \setminus \Phi_J^+$, if and only if $w \neq e$ and one of the following two conditions holds:*

(1) *$\alpha = \alpha_k$;*
(2) *$\alpha = (k, \overline{k})$, $w(k) = \overline{a}$ for $1 \leq a \leq n$, and $w[k+1, n] \subseteq \{a+1, \ldots, n\}$ if $k < n$.*

Let us now turn to a short proof in the case of ϖ_k in type C_n, where $1 \leq k \leq n$. Note that $w \in W^J$ is equivalent to $w[1, k]$ and $w[k+1, n]$ being increasing sequences (with respect to the total order on $[\overline{n}]$), as well as $w[k+1, n]$ consisting of positive entries. We also need to introduce more notation. The $(-\varpi_k)$-chain $\Gamma(k)$ in (2) has an obvious splitting $\Gamma(k) = \Gamma^1(k)\Gamma^2(k)$, where $\Gamma^1(k) := \Gamma_2' \cdots \Gamma_k'$ and $\Gamma^2(k) := \Gamma_1(k) \cdots \Gamma_k(k)$. This induces a splitting $A = A^1 \sqcup A^2$ of any w-admissible subset A, where $A^i = A \cap \Gamma^i(k)$, for $i = 1, 2$.

Table 3
The list of all admissible subsets $A \in \mathcal{A}(s_2s_3s_2, \Gamma(2))$.

A	$\mathrm{end}(w, A^1)$	$\mathrm{end}(w, A)$	$\lfloor \mathrm{end}(w, A) \rfloor$	$\mathrm{down}(w, A)$
$\emptyset$	$s_2s_3s_2$	$s_2s_3s_2$	$s_2s_3s_2$	0
$\{1\}$	$s_1s_2s_3s_2$	$s_1s_2s_3s_2$	$s_1s_2s_3s_2$	0
$\{4\}$	$s_2s_3s_2$	$s_2s_3s_1s_2$	$s_2s_3s_1s_2$	0
$\{5\}$	$s_2s_3s_2$	$s_1s_2s_3s_2$	$s_1s_2s_3s_2$	0
$\{7\}$	$s_2s_3s_2$	e	e	$\alpha_2^\vee + \alpha_3^\vee$
$\{8\}$	$s_2s_3s_2$	s_2s_3	s_2	$\alpha_2^\vee$
$\{1,4\}$	$s_1s_2s_3s_2$	$s_1s_2s_3s_1s_2$	$s_1s_2s_3s_1s_2$	0
$\{1,7\}$	$s_1s_2s_3s_2$	s_1	e	$\alpha_2^\vee + \alpha_3^\vee$
$\{1,8\}$	$s_1s_2s_3s_2$	$s_1s_2s_3$	s_1s_2	$\alpha_2^\vee$
$\{4,6\}$	$s_2s_3s_2$	$s_1s_2s_3s_1s_2$	$s_1s_2s_3s_1s_2$	0
$\{4,8\}$	$s_2s_3s_2$	$s_2s_3s_1$	s_2	$\alpha_2^\vee$
$\{5,7\}$	$s_2s_3s_2$	s_1	e	$\alpha_2^\vee + \alpha_3^\vee$
$\{5,8\}$	$s_2s_3s_2$	$s_1s_2s_3$	s_1s_2	$\alpha_2^\vee$
$\{7,8\}$	$s_2s_3s_2$	s_2	s_2	$\alpha_2^\vee + \alpha_3^\vee$
$\{1,4,7\}$	$s_1s_2s_3s_2$	s_2s_1	s_2	$\alpha_2^\vee + \alpha_3^\vee$
$\{1,4,8\}$	$s_1s_2s_3s_2$	$s_1s_2s_3s_1$	s_1s_2	$\alpha_2^\vee$
$\{1,7,8\}$	$s_1s_2s_3s_2$	s_1s_2	s_1s_2	$\alpha_2^\vee + \alpha_3^\vee$
$\{4,6,7\}$	$s_2s_3s_2$	s_2s_1	s_2	$\alpha_2^\vee + \alpha_3^\vee$
$\{4,6,8\}$	$s_2s_3s_2$	$s_1s_2s_3s_1$	s_1s_2	$\alpha_2^\vee$
$\{5,7,8\}$	$s_2s_3s_2$	s_1s_2	s_1s_2	$\alpha_2^\vee + \alpha_3^\vee$
$\{1,4,7,8\}$	$s_1s_2s_3s_2$	$s_1s_2s_1$	s_1s_2	$\alpha_2^\vee + \alpha_3^\vee$
$\{4,6,7,8\}$	$s_2s_3s_2$	$s_1s_2s_1$	s_1s_2	$\alpha_2^\vee + \alpha_3^\vee$

Theorem 19. *In type C_n, given $w \in W^J$, we have the following cancellation-free formula in $QK_T^{\mathrm{poly}}(G/P_J) \subset QK_T(G/P_J)$:*

$$[\mathcal{O}(-\varpi_k)] \cdot [\mathcal{O}^w] = \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k))} (-1)^{|A|} \mathbf{e}^{-\mathrm{wt}(w,A)} [\mathcal{O}^{\mathrm{end}(w,A)}] \tag{9}$$
$$- Q_k \sum_{\substack{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k)) \\ \mathrm{end}(w,A^1) \geq \lfloor s_\theta \rfloor}} (-1)^{|A|} \mathbf{e}^{-\mathrm{wt}(w,A)} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) s_{2\varepsilon_k} \rfloor}].$$

Remark 20. As will be seen in the proof below, for $w \in W^J$, the condition $w \geq \lfloor s_\theta \rfloor$ is equivalent to the condition $w(k) = \overline{1}$.

Example 21. In this example, we consider the case that $n = 3$ and $k = 2$. Note that $\lfloor s_\theta \rfloor = s_1s_2s_3s_2$. Recall that $\Gamma(2) = ((1,\overline{2}), (1,\overline{3}), (1,\overline{1}), (1,3), (1,\overline{2}), (2,\overline{3}), (2,\overline{2}), (2,3))$ (with all roots negated). Let $w = s_2s_3s_2$. Then the list of all admissible subsets $A \in \mathcal{A}(w, \Gamma(2))$ and their statistics $\mathrm{end}(w, A)$, $\mathrm{down}(w, A)$, together with $\mathrm{end}(w, A^1)$, $\lfloor \mathrm{end}(w, A) \rfloor$, is given in Table 3. Note that $\mathrm{wt}(w, A) = -s_2s_3s_2\varpi_2$ for all $A \in \mathcal{A}(w, \Gamma(2))$.

By Theorem 7, in $QK_T^{\mathrm{poly}}(G/B)$, we have:

$$\begin{aligned}
&[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2s_3s_2}] \\
&= \mathbf{e}^{s_2s_3s_2\varpi_2}([\mathcal{O}^{s_2s_3s_2}] - [\mathcal{O}^{s_1s_2s_3s_2}] - [\mathcal{O}^{s_2s_3s_1s_2}] - [\mathcal{O}^{s_1s_2s_3s_2}] - Q_2Q_3[\mathcal{O}^{e}] - Q_2[\mathcal{O}^{s_2s_3}] \\
&\quad + [\mathcal{O}^{s_1s_2s_3s_1s_2}] + Q_2Q_3[\mathcal{O}^{s_1}] + Q_2[\mathcal{O}^{s_1s_2s_3}] + [\mathcal{O}^{s_1s_2s_3s_1s_2}] + Q_2[\mathcal{O}^{s_2s_3s_1}] + Q_2Q_3[\mathcal{O}^{s_1}] \\
&\quad + Q_2[\mathcal{O}^{s_1s_2s_3}] + Q_2Q_3[\mathcal{O}^{s_2}] - Q_2Q_3[\mathcal{O}^{s_2s_1}] - Q_2[\mathcal{O}^{s_1s_2s_3s_1}] - Q_2Q_3[\mathcal{O}^{s_1s_2}]
\end{aligned}$$

$$- Q_2Q_3[\mathcal{O}^{s_2s_1}] - Q_2[\mathcal{O}^{s_1s_2s_3s_1}] - Q_2Q_3[\mathcal{O}^{s_1s_2}] + Q_2Q_3[\mathcal{O}^{s_1s_2s_1}] + Q_2Q_3[\mathcal{O}^{s_1s_2s_1}]). \tag{10}$$

By applying the surjection $\Phi_J : QK_T^{\mathrm{poly}}(G/B) \to QK_T^{\mathrm{poly}}(G/P_J)$ to equation (10), we obtain the following cancellation-free formula in $QK_T^{\mathrm{poly}}(G/P_J) \subset QK_T(G/P_J)$; here, the underlined terms in the first equality are canceled out:

$$\begin{aligned}
&[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2s_3s_2}] \\
&= \mathbf{e}^{s_2s_3s_2\varpi_2}([\mathcal{O}^{s_2s_3s_2}] - [\mathcal{O}^{s_1s_2s_3s_2}] - [\mathcal{O}^{s_2s_3s_1s_2}] - [\mathcal{O}^{s_1s_2s_3s_2}] \underline{- Q_2[\mathcal{O}^{e}] - Q_2[\mathcal{O}^{s_2}]} \\
&\quad + [\mathcal{O}^{s_1s_2s_3s_1s_2}] \underline{+ Q_2[\mathcal{O}^{e}] + Q_2[\mathcal{O}^{s_1s_2}]} + [\mathcal{O}^{s_1s_2s_3s_1s_2}] \underline{+ Q_2[\mathcal{O}^{s_2}]} + Q_2[\mathcal{O}^{e}] \\
&\quad \underline{+ Q_2[\mathcal{O}^{s_1s_2}] + Q_2[\mathcal{O}^{s_2}] - Q_2[\mathcal{O}^{s_2}] - Q_2[\mathcal{O}^{s_1s_2}] - Q_2[\mathcal{O}^{s_1s_2}]} \\
&\quad - Q_2[\mathcal{O}^{s_2}] \underline{- Q_2[\mathcal{O}^{s_1s_2}] - Q_2[\mathcal{O}^{s_1s_2}] + Q_2[\mathcal{O}^{s_1s_2}] + Q_2[\mathcal{O}^{s_1s_2}]}) \\
&= \mathbf{e}^{s_2s_3s_2\varpi_2}([\mathcal{O}^{s_2s_3s_2}] - 2[\mathcal{O}^{s_1s_2s_3s_2}] - [\mathcal{O}^{s_2s_3s_1s_2}] + 2[\mathcal{O}^{s_1s_2s_3s_1s_2}] + Q_2[\mathcal{O}^{e}] - Q_2[\mathcal{O}^{s_2}]).
\end{aligned}$$

Also, we deduce that $\mathcal{A}_{\lessdot}(w, \Gamma(2)) = \{\emptyset, \{1\}, \{4\}, \{5\}, \{1,4\}, \{4,6\}\}$; note that only two elements $A = \{1\}, \{1,4\}$ of $\mathcal{A}_{\lessdot}(w, \Gamma(2))$ satisfy $\mathrm{end}(w, A^1) \geq \lfloor s_\theta \rfloor$. Therefore, we see that

$$\begin{aligned}
&(\text{RHS of equation (9)}) \\
&= \mathbf{e}^{s_2s_3s_2\varpi_2}([\mathcal{O}^{s_2s_3s_2}] - [\mathcal{O}^{s_1s_2s_3s_2}] - [\mathcal{O}^{s_2s_3s_1s_2}] \\
&\qquad\qquad - [\mathcal{O}^{s_1s_2s_3s_2}] + [\mathcal{O}^{s_1s_2s_3s_1s_2}] + [\mathcal{O}^{s_1s_2s_3s_1s_2}]) \\
&\quad - Q_2\mathbf{e}^{s_2s_3s_2\varpi_2}(-[\mathcal{O}^{\lfloor s_1 \rfloor}] + [\mathcal{O}^{\lfloor s_2s_1 \rfloor}]) \\
&= \mathbf{e}^{s_2s_3s_2\varpi_2}([\mathcal{O}^{s_2s_3s_2}] - 2[\mathcal{O}^{s_1s_2s_3s_2}] - [\mathcal{O}^{s_2s_3s_1s_2}] + 2[\mathcal{O}^{s_1s_2s_3s_1s_2}] + Q_2[\mathcal{O}^{e}] - Q_2[\mathcal{O}^{s_2}]) \\
&= [\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_2s_3s_2}];
\end{aligned}$$

here, we have used $s_{2\varepsilon_2} = s_2s_3s_2$ for the first equality. Thus Theorem 19 holds in this case.

Proof of Theorem 19. We assume that w is not the identity, as this case is trivial. We follow the same procedure outlined above, and describe the sign-reversing involution canceling terms obtained from the Chevalley formula for G/B.

We carry out the proof in the case $k < n$, and refer to $k = n$ at the end. Consider a generic w-admissible subset A in $\mathcal{A}(w, \Gamma(k))$, corresponding to a term in the Chevalley formula for G/B. Like in type A, the structure of $\Gamma(k)$ combined with the fact that $w \in W^J$ imply that A contains at most one root labeling a Bruhat cover in $\Pi(w, A)$ from each row in the display of $\Gamma_j(k)$ in (3).

We focus on those A with $A^- \neq \emptyset$. By Lemma 18, we have $A^- \subseteq \{\alpha_k = (k, k+1), 2\varepsilon_k = (k, \overline{k})\}$. Note that both of these roots appear only once in the $(-\varpi_k)$-chain $\Gamma(k)$, with $(k, k+1)$ being the last one, while $(k, \overline{k})$ appears in the last segment $\Gamma_k(k)$.

In fact, we have either $A^- = \{(k, k+1)\}$ or $A^- = \{(k, \overline{k})\}$. Indeed, assuming that $(k, \overline{k}) \in A^-$, and considering the signed permutation u in $\Pi(w, A)$ to which $(k, \overline{k})$ is applied, we have $u \in W^J$ and $u[k+1, n] \subseteq \{a+1, \dots, n\}$, where $a := |u(k)| = u(\overline{k})$; therefore, it is impossible for $(k, k+1)$ to correspond to a quantum step in $\Pi(w, A)$.

Now assume that $A^- = \{(k, k+1)\}$, and let $v := \mathrm{end}(w, A \setminus \{(k, k+1)\})$. We clearly have $v \in W^J$. We will pair A with another w-admissible subset A', such that their contributions to the parabolic Chevalley formula for G/P_J cancel out. We must have one of the following cases, where $1 \le a < b \le n$.

Case 1: $v(k) = b$, $v(k+1) = a$, and A does not contain (k, j) with $j > k+1$.

Subcase 1.1: $b < n$. This case is completely similar to Case 1 in the type A proof. Indeed, there clearly exists $q > k+1$ such that $v(q) = b+1 \le n$. We let $A' := A \cup \{(k, q)\}$, so $(A')^- = \{(k, k+1)\}$, and continue the reasoning as above.

Subcase 1.2: $b = n$. We let $A' := A \cup \{(k, \overline{k})\}$, and we have $(A')^- = \{(k, k+1)\}$.

Case 2: $v(k) = \overline{a}$, $v(k+1) = b$. We let $A' := A \cup \{(k, \overline{k})\}$, and we have $(A')^- = \{(k, \overline{k})\}$.

Case 3: $v(k) = \overline{b}$, $v(k+1) = a$, and A contains neither $(k, \overline{k})$, nor $(k, \overline{\jmath})$ for $j > k+1$.

Subcase 3.1: $k+2 \le n$ and $v(k+2) < b$. Consider $q > k+1$ largest such that $v(q) < b$. We let $A' := A \cup \{(k, \overline{q})\}$, and we have $(A')^- = \{(k, k+1)\}$.

Subcase 3.2: $k+2 > n$ or $v(k+2) > b$. We let $A' := (A \setminus \{(k, k+1)\}) \cup \{(k, \overline{k+1}), (k, \overline{k})\}$, and we have $(A')^- = \{(k, \overline{k})\}$.

We claim that in all cases,

$$A' \in \mathcal{A}(w, \Gamma(k)), \quad \lfloor \mathrm{end}(w, A) \rfloor = \lfloor \mathrm{end}(w, A') \rfloor, \quad \text{and} \quad \mathrm{wt}(w, A) = \mathrm{wt}(w, A').$$

Furthermore, it is not hard to check that these cases completely pair up all w-admissible subsets A with $A^- = \{(k, k+1)\}$, either among themselves (in Cases 1.1, 1.2, and 3.1), or with A satisfying $A^- = \{(k, \overline{k})\}$ (in Cases 2 and 3.2); see below for a discussion of the latter A which are not paired up above.

Indeed, let us consider, for instance, Case 2. We cannot have $(k, \overline{k}) \in A$, because the corresponding up step in Bruhat order would not be a cover (by the classical part of the criterion in Proposition 3 (3)). Moreover, A cannot contain any root of the form (k, j) with $j > k+1$, as the corresponding reflection would bring a positive entry to position k, whereas $v(k)$ is negative. Therefore, the roots $(k, \overline{k})$ and $(k, k+1)$ are the last two in A', while the step corresponding to $(k, \overline{k})$ is a quantum one (by the criterion in Proposition 3 (3)). Moreover, we have

$$\lfloor \mathrm{end}(w, A') \rfloor = \lfloor v(k, \overline{k})(k, k+1) \rfloor = \lfloor v(k, k+1)(k+1, \overline{k+1}) \rfloor = \lfloor \mathrm{end}(w, A) \rfloor.$$

The weight preservation is verified by noting that all affine hyperplanes corresponding to the roots in $\Gamma^2(k)$ contain ϖ_k; so the corresponding affine reflections fix ϖ_k, and are thus irrelevant for the weight computation.

On another hand, in the Chevalley formula for G/B, the quantum steps corresponding to both roots $(k, k+1)$ and $(k, \overline{k})$ contribute the variable Q_k. Indeed, as indicated above, we have the following coroot splitting: $(2\varepsilon_k)^\vee = \alpha_k^\vee + (\alpha_{k+1}^\vee + \cdots + \alpha_n^\vee)$. Finally, since the cardinalities of A and A' differ by 1, we conclude that the involution $A \leftrightarrow A'$ is indeed sign-reversing. In this way, the contributions to the parabolic Chevalley formula for G/P_J of all A with $A^- = \{(k, k+1)\}$ are canceled.

We have now exhausted all w-admissible sets A with $A^- = \{(k, k+1)\}$. Thus, it remains to discuss the contributions of the remaining A with $A^- \neq \emptyset$, i.e., $A^- = \{(k, \overline{k})\}$ and A is not among the A' in Cases 2 and 3.2. So from now on we work under this assumption. We previously considered the signed permutation $u \in W^J$ in $\Pi(w, A)$ to which $(k, \overline{k})$ is applied, and observed that $u[k+1, n] \subseteq \{a+1, \ldots, n\}$, where $a := |u(k)|$. If $(k, \overline{k})$ is followed by another root in A, then this can only be $(k, k+1)$; but this situation was considered in Case 2 above, which means that $(k, \overline{k})$ must be the last root in A. Moreover, A cannot contain any root of the form $(k, \overline{\jmath})$ with $j > k$, because we would be in Case 3.2. The following two cases cover all remaining possibilities, and we continue to use the above notation.

Case 4: $u(k) \neq \overline{1}$ (i.e., $a \neq 1$), and A^2 contains no root $(i, \overline{k})$ with $i < k$. There clearly exists $p < k$ such that $u(p) = a - 1$. We let $A' := A \cup \{(p, \overline{k})\}$, where the root $(p, \overline{k})$ is taken from $\Gamma^2(k)$. We have $(A')^- = \{(k, \overline{k})\}$. Like above, we verify that the terms corresponding to A and A' cancel out, so we can extend the sign-reversing involution above by pairing A with A'.

Now recall that, in general, A^2 contains at most one root $(i, \overline{k})$ with $i < k$. Whenever it contains one, the values in positions i and k of the signed permutation to which this reflection is applied are of the form $b - 1$ and $\overline{b}$, respectively. Thus, the remaining case consists of the following w-admissible subsets A.

Case 5: $u(k) = \overline{1}$ (i.e., $a = 1$), and A^2 contains no root $(i, \overline{k})$ with $i < k$. We clearly have $A \setminus \{(k, \overline{k})\} \in \mathcal{A}_{\lessdot}(w, \Gamma(k))$, where we recall that $(k, \overline{k})$ is the last root in A. Now let $u' := \mathrm{end}(w, A^1)$. Based on the structures of $\Gamma(k)$ and A, we have $u'(k) = u(k) = \overline{1}$. But this is equivalent to $u' \geq \lfloor s_\theta \rfloor = [2, 3, \ldots, k, \overline{1}, k+1, \ldots, n]$ (in the window notation), by Deodhar's criterion for the type C Bruhat order [3, Chapter 8, Exercise 6]. In the same way as above, we can see that $\mathrm{wt}(w, A) = \mathrm{wt}(A \setminus \{(k, \overline{k})\})$. The above facts imply that the terms corresponding to this case make up the second sum in (9). By Remark 12, there are no cancellations between these terms.

We conclude by considering $k = n$, and noting that the proof reduces to Cases 4 and 5 above. □

We now prove the positivity property of structure constants for isotropic Grassmannians as a corollary of Theorem 19.

Corollary 22. *Let G be of type C_n, and $J = I \setminus \{k\}$ for an arbitrary fixed $1 \leq k \leq n$. Then, for $w, u \in W^J$ and $\xi \in Q^{\vee,+}_{I \setminus J}$, we have*

$$(-1)^{1+\ell(w)+\ell(u)+\deg(Q^\xi)} N^{u,\xi}_{s_k,w} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^\gamma - 1 \mid \gamma \in -\Delta].$$

Proof. Take $A \in \mathcal{A}_{\lessdot}(w, \Gamma(k))$ such that $\mathrm{end}(w, A^1) \geq \lfloor s_\theta \rfloor$, and set $v := \mathrm{end}(w, A)$. Recall from the proof of Theorem 19 that there exists a quantum edge $v \xrightarrow{2\varepsilon_k} vs_{2\varepsilon_k} \in \mathrm{QB}(W)$. Also, by Case 5 in the proof of Theorem 19, we have $v(k) = \overline{1}$. Note that $v \in W^J$, and hence that $v(1) < \cdots < v(k)$, $v(k+1) < \cdots < v(n)$. It follows that $1 = vs_{2\varepsilon_k}(k) < vs_{2\varepsilon_k}(1) < \cdots < vs_{2\varepsilon_k}(k-1)$ and $vs_{2\varepsilon_k}(k+1) < \cdots < vs_{2\varepsilon_k}(n)$. Therefore, if we take a cyclic permutation $\sigma := (1, k, k-1, \dots, 2) \in W$, then we have $\lfloor vs_{2\varepsilon_k} \rfloor = vs_{2\varepsilon_k}\sigma$. Hence we see that

$$\begin{aligned}
|A| + 1 = |A \cup \{(k, \overline{k})\}| \\
&\equiv \ell(vs_{2\varepsilon_k}) - \ell(w) \\
&\equiv (\ell(vs_{2\varepsilon_k}\sigma) - \ell(\sigma)) - \ell(w) \\
&\equiv \ell(\lfloor vs_{2\varepsilon_k} \rfloor) - (k-1) - \ell(w) \\
&\equiv \ell(w) + \ell(\lfloor vs_{2\varepsilon_k} \rfloor) + k - 1
\end{aligned}$$

modulo 2. Thus, we obtain

$$(-1)^{|A|+1} = (-1)^{\ell(w)+\ell(\lfloor vs_{2\varepsilon_k} \rfloor)+k-1}.$$

It is easy to check (see, for example, [9, Section 3.1.5, Exercise 4]) that

$$2\rho_J = \begin{cases} \displaystyle\sum_{i=1}^{k-1} i(k-1)\alpha_i \\ \displaystyle\quad + \sum_{i=1}^{n-k-1} i(2(n-k) - i + 1)\alpha_{k+1} + \frac{(n-k)(n-k+1)}{2}\alpha_n & \text{if } k \neq n, \\ \displaystyle\sum_{i=1}^{n-1} i(n-i)\alpha_i & \text{if } k = n. \end{cases}$$

Since

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1 \text{ and } i \neq n, \\ -2 & \text{if } i = n \text{ and } j = n-1, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$2\langle \rho_J, \alpha_k^\vee \rangle = \begin{cases} -(k-1) - 2(n-k) & \text{if } k \neq n-1, n, \\ -(n-2) - 2 \cdot \dfrac{1 \cdot 2}{2} & \text{if } k = n-1, \\ -(n-1) & \text{if } k = n \end{cases}$$

$$\equiv k-1 \mod 2.$$

In addition, we have $2\langle \rho,\, \alpha_k^\vee \rangle = 2$. Therefore, we see that

$$\deg(Q_k) = 2\langle \rho,\, \alpha_k^\vee \rangle - 2\langle \rho_J,\, \alpha_k^\vee \rangle \equiv 2-(k-1) \equiv k-1 \mod 2.$$

We set

$$\mathcal{A}(w, \Gamma(k))^0_{u,\alpha_k^\vee,\lambda} := \left\{ A \in \mathcal{A}_{\lessdot}(w, \Gamma(k)) \;\middle|\; \begin{array}{l} \mathrm{end}(w, A^{(1)}) \geq \lfloor s_\theta \rfloor,\ \lfloor \mathrm{end}(w, A) s_{2\varepsilon_k} \rfloor = u, \\ \mathrm{wt}(A \cup \{(k, \overline{k})\}) = \lambda \end{array} \right\}.$$

Then, since $\mathrm{wt}(w, A) \in -\varpi_k + Q^+$ by Lemma 10, we deduce from Theorem 19 that

$$\begin{aligned} C_w^{u,\alpha_k^\vee} &= \sum_{\lambda \in -\varpi_k + Q^+} \sum_{A \in \mathcal{A}(w,\Gamma)^0_{u,\alpha_k^\vee,\lambda}} (-1)^{|A|+1} Q_k \mathbf{e}^{-\mathrm{wt}(w, A \cup \{(k,\overline{k})\})} \\ &= (-1)^{\ell(w)+\ell(u)+k-1} Q_k \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))^0_{u,\alpha_k^\vee,\lambda}| \mathbf{e}^{-\lambda}. \end{aligned}$$

Therefore, we obtain

$$N_{s_k,w}^{u,\alpha_k^\vee} = (-1)^{1+\ell(w)+\ell(u)+k-1} Q_k \sum_{\lambda \in -\varpi_k + Q^+} |\mathcal{A}(w, \Gamma(k))^0_{u,\alpha_k^\vee,\lambda}| \mathbf{e}^{-\varpi_k-\lambda}.$$

This implies that

$$(-1)^{1+\ell(w)+\ell(u)+\deg(Q_k)} N_{s_k,w}^{u,\alpha_k^\vee} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^\gamma - 1 \mid \gamma \in -\Delta], \tag{11}$$

as desired. Equation (11), together with the positivity property of $N_{u,v}^{w,0}$ for $u, v, w \in W^J$, proves the corollary. □

4. Quantum K-theory Chevalley formulas for two-step flag manifolds

In this section, we concentrate on the case of type A_{n-1}; note that $I = [n-1]$ in this case. Let us consider the (standard) parabolic subgroup $P_J \supset B$ corresponding to $J = I \setminus \{k_1, k_2\}$ for some $k_1, k_2 \in I$ with $k_1 < k_2$; the resulting partial flag manifold G/P_J is isomorphic to a *two-step flag manifold* $\mathrm{Fl}(k_1, k_2; n)$ defined as:

$$\mathrm{Fl}(k_1, k_2; n) := \left\{ (V_1, V_2) \;\middle|\; \begin{array}{l} V_1 \text{ and } V_2 \text{ are subspaces of } \mathbb{C}^n \text{ such that } V_1 \subset V_2, \\ \dim V_1 = k_1, \text{ and } \dim V_2 = k_2 \end{array} \right\}.$$

The purpose of this section is to derive cancellation-free parabolic Chevalley formulas for the quantum multiplication in $QK_T(G/P_J)$ with $[\mathcal{O}(-\varpi_k)]$, for $k = k_1$ and $k = k_2$. For this purpose, as in Section 3, we examine all the terms to be canceled in certain formulas obtained from equation (4) in Theorem 7, in $QK_T^{\mathrm{poly}}(G/B) \subset QK_T(G/B)$, by applying the map $\Phi_J : QK_T^{\mathrm{poly}}(G/B) \to QK_T^{\mathrm{poly}}(G/P_J)$.

4.1. Some lemmas on admissible subsets

Note that for $w \in W = W(A_{n-1}) = S_n$, $w \in W^J$ is equivalent to $w[1, k_1]$, $w[k_1+1, k_2]$, and $w[k_2 + 1, n]$ being increasing sequences (see [3, Lemma 2.4.7]).

We first consider the case $k = k_1$. We will make repeated use of the following.

Lemma 23. *Consider $w \in W^J$. We have an edge $w \xrightarrow{(i,j)} w(i, j)$ in the quantum Bruhat graph $\mathrm{QB}(W)$, with $i \leq k_1 < j$, if and only if one of the following two conditions holds:*

(1) *the edge above is a Bruhat cover, and $w(i, j) \in W^J$;*
(2) *the edge above is a quantum one, and $(i, j) = (k_1, k_2 + 1)$ or $(i, j) = \alpha_{k_1}$.*

Proof. As in the proof of Lemma 14, we implicitly use Proposition 2, as well as the fact that $w[1, k_1]$, $w[k_1 + 1, k_2]$, and $w[k_2 + 1, n]$ are increasing sequences. Assume first that the edge above is a Bruhat cover. Then, since $(i, j) \notin W_J$, [3, Corollary 2.5.2] implies that $w(i, j) \in W^J$, as desired. Assume next that the edge above is a quantum one; note that $w(i) > w(j)$ in this case. If $i < k_1$, then the value $w(k_1)$ would be straddled between $w(i)$ and $w(j)$. Hence we must have $i = k_1$. Also, if $k_1 + 1 < j \leq k_2$, then the value $w(k_1 + 1)$ would be straddled between $w(k_1)$ and $w(j)$; if $j > k_2 + 1$, then the value $w(k_2 + 1)$ would be straddled between $w(k_1)$ and $w(j)$. Hence we must have $j = k_1 + 1$ or $j = k_2 + 1$. This proves the lemma. □

Lemma 24. *Consider $w \in W^J$, and assume that we have a quantum edge $w \xrightarrow{(k_1,k_2+1)} w(k_1, k_2 + 1)$ in $\mathrm{QB}(W)$. Then, for $k_1 + 1 \leq j \leq k_2$, we have an edge $w(k_1, k_2 + 1) \xrightarrow{(k_1,j)} w(k_1, k_2 + 1)(k_1, j)$ in $\mathrm{QB}(W)$ if and only if $j = k_1 + 1$. In this case, the edge $w(k_1, k_2 + 1) \xrightarrow{(k_1,j)} w(k_1, k_2 + 1)(k_1, j)$ is a Bruhat cover.*

Proof. Set $v := w(k_1, k_2 + 1)$. Since we have a quantum edge $w \xrightarrow{(k_1,k_2+1)} v$ in $\mathrm{QB}(W)$, Proposition 2 implies the following:

$$v(k_1) < v(k_1 + 1) < v(k_1 + 2) < \cdots < v(k_2) < v(k_2 + 1).$$

If $k_1 + 1 < j \leq k_2$, then the value $v(k_1 + 1)$ would be straddled between $v(k_1)$ and $v(j)$. Hence we must have $j = k_1 + 1$. In this case, by Proposition 2, we have a Bruhat edge $v \xrightarrow{(k_1,k_1+1)} v(k_1, k_1 + 1)$. This proves the lemma. □

As a corollary of Lemmas 23 and 24, we immediately obtain the following.

Lemma 25. *Let $w \in W^J$, and take $A = \{j_1 < \cdots < j_s\} \in \mathcal{A}(w, \Gamma(k_1))$. If the directed path $\Pi(w, A)$ contains a quantum edge, then $\Pi(w, A)$ is one of the following forms; here $\xrightarrow[\mathsf{B}]{}$ indicates a Bruhat edge, while $\xrightarrow[\mathsf{Q}]{}$ indicates a quantum edge:*

(1) $\Pi(w, A) : w = w_0 \underset{\mathsf{B}}{\rightarrow} \cdots \underset{\mathsf{B}}{\rightarrow} w_{s-1} \xrightarrow[\mathsf{Q}]{(k_1, k_1+1)} w_s$;
(2) $\Pi(w, A) : w = w_0 \underset{\mathsf{B}}{\rightarrow} \cdots \underset{\mathsf{B}}{\rightarrow} w_{s-1} \xrightarrow[\mathsf{Q}]{(k_1, k_2+1)} w_s$;
(3) $\Pi(w, A) : w = w_0 \underset{\mathsf{B}}{\rightarrow} \cdots \underset{\mathsf{B}}{\rightarrow} w_{s-2} \xrightarrow[\mathsf{Q}]{(k_1, k_2+1)} w_{s-1} \xrightarrow[\mathsf{B}]{(k_1, k_1+1)} w_s$.

In view of this lemma, we divide the set $\mathcal{A}(w, \Gamma(k_1))$ into the disjoint union of the following four subsets:

(1) $\mathcal{A}_{\lessdot}(w, \Gamma(k_1))$ (defined in Section 2.1);
(2) $\mathcal{A}_1(w, \Gamma(k_1)) := \{A \in \mathcal{A}(w, \Gamma(k_1)) \mid \Pi(w, A)$ is of the form (1) in Lemma 25$\}$;
(3) $\mathcal{A}_2(w, \Gamma(k_1)) := \{A \in \mathcal{A}(w, \Gamma(k_1)) \mid \Pi(w, A)$ is of the form (2) in Lemma 25$\}$;
(4) $\mathcal{A}_3(w, \Gamma(k_1)) := \{A \in \mathcal{A}(w, \Gamma(k_1)) \mid \Pi(w, A)$ is of the form (3) in Lemma 25$\}$.

Then it follows that

$$\mathcal{A}(w, \Gamma(k_1)) = \mathcal{A}_{\lessdot}(w, \Gamma(k_1)) \sqcup \mathcal{A}_1(w, \Gamma(k_1)) \sqcup \mathcal{A}_2(w, \Gamma(k_1)) \sqcup \mathcal{A}_3(w, \Gamma(k_1)).$$

Also, we can verify the following:

- if $A \in \mathcal{A}_{\lessdot}(w, \Gamma(k_1))$, then $\mathrm{down}(w, A) = 0$, and hence $Q^{[\mathrm{down}(w,A)]^J} = 0$;
- if $A \in \mathcal{A}_1(w, \Gamma(k_1))$, then $\mathrm{down}(w, A) = \alpha_{k_1}^{\vee}$, and hence $Q^{[\mathrm{down}(w,A)]^J} = Q_{k_1}$;
- if $A \in \mathcal{A}_2(w, \Gamma(k_1))$ or $A \in \mathcal{A}_3(w, \Gamma(k_1))$, then $\mathrm{down}(w, A) = \alpha_{k_1}^{\vee} + \cdots + \alpha_{k_2}^{\vee}$, and hence $Q^{[\mathrm{down}(w,A)]^J} = Q^{\alpha_{k_1}^{\vee} + \alpha_{k_2}^{\vee}} = Q_{k_1} Q_{k_2}$.

Therefore, by equation (4), we deduce that

$$\begin{aligned}
[\mathcal{O}(-\varpi_{k_1})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_1}} & \sum_{A \in \mathcal{A}_{\lessdot}(w, \Gamma(k_1))} (-1)^{|A|} [\mathcal{O}^{\mathrm{end}(w,A)}] \\
& + \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_1(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] \\
& + \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] \\
& + \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_3(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}].
\end{aligned}$$

Next, we consider the case $k = k_2$. In this case, we use $\Gamma^*(k_2)$ instead of $\Gamma(k_2)$. From Lemma 25, by applying the diagram automorphism ω, we obtain the following.

Lemma 26. *Let $w \in W^J$, and take $A = \{j_1 < \cdots < j_s\} \in \mathcal{A}(w, \Gamma^*(k_2))$. If the directed path $\Pi(w, A)$ contains a quantum edge, then $\Pi(w, A)$ is one of the following forms; here, $\underset{\mathsf{B}}{\rightarrow}$ indicates a Bruhat edge, while $\underset{\mathsf{Q}}{\rightarrow}$ indicates a quantum edge:*

(1) $\Pi(w,A) : w = w_0 \xrightarrow[\mathsf{B}]{} \cdots \xrightarrow[\mathsf{B}]{} w_{s-1} \xrightarrow[\mathsf{Q}]{(k_2,k_2+1)} w_s$;
(2) $\Pi(w,A) : w = w_0 \xrightarrow[\mathsf{B}]{} \cdots \xrightarrow[\mathsf{B}]{} w_{s-1} \xrightarrow[\mathsf{Q}]{(k_1,k_2+1)} w_s$;
(3) $\Pi(w,A) : w = w_0 \xrightarrow[\mathsf{B}]{} \cdots \xrightarrow[\mathsf{B}]{} w_{s-2} \xrightarrow[\mathsf{Q}]{(k_1,k_2+1)} w_{s-1} \xrightarrow[\mathsf{B}]{(k_2,k_2+1)} w_s$.

In view of this lemma, we divide the set $\mathcal{A}(w,\Gamma^*(k_2))$ into the disjoint union of the following four subsets:

(1) $\mathcal{A}_{\lessdot}(w,\Gamma^*(k_2))$ (already defined);
(2) $\mathcal{A}_1(w,\Gamma^*(k_2)) := \{A \in \mathcal{A}(w,\Gamma^*(k_2)) \mid \Pi(w,A)$ is of the form (1) in Lemma 26$\}$;
(3) $\mathcal{A}_2(w,\Gamma^*(k_2)) := \{A \in \mathcal{A}(w,\Gamma^*(k_2)) \mid \Pi(w,A)$ is of the form (2) in Lemma 26$\}$;
(4) $\mathcal{A}_3(w,\Gamma^*(k_2)) := \{A \in \mathcal{A}(w,\Gamma^*(k_2)) \mid \Pi(w,A)$ is of the form (3) in Lemma 26$\}$.

4.2. Parabolic Chevalley formulas for two-step flag manifolds

We state cancellation-free parabolic Chevalley formulas for the equivariant quantum K-theory of the two-step flag manifold G/P_J; the proofs of these results will be given in Sections 4.3 and 4.4. First, we assume that $k = k_1$. Take and fix $w \in W^J$.

Theorem 27. *If $w(k_1) < w(k_1+1)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_1})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|} [\mathcal{O}^{\mathrm{end}(w,A)}].$$

We consider the following condition:

(Q) $w(k_1) > w(k_2)$ and $w(k_1+1) > w(k_2+1)$.

Remark 28. As mentioned at the beginning of Section 4.1, $w[k_1+1,k_2]$ is an increasing sequence for $w \in W^J$. Hence condition (Q) implies that $w(k_1) > w(k_2) \geq w(k_1+1) > w(k_2+1)$.

Theorem 29. *Assume that $w(k_1) > w(k_1+1)$, and assume that condition* (Q) *does not hold.*

(1) *If $w(1) < w(k_1+1)$ or $w(k_1) < w(k_2)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_1})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|} [\mathcal{O}^{\mathrm{end}(w,A)}].$$

(2) *If $w(1) > w(k_1+1)$ and $w(k_1) > w(k_2)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_1})]\cdot[\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|}\left([\mathcal{O}^{\mathrm{end}(w,A)}] - Q_{k_1}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)s_{k_1}\rfloor}]\right).$$

Also, we consider the following condition:

(Full) both of the following hold:
(1) $w(k_1) = n$ and $w(k_2+1) = 1$; and
(2) $w(k_1+1)$ is the minimum element in the sequence $w[1,k_2]$.

Remark 30. Condition (Full) holds if and only if condition (Q) holds and $w(1) > w(k_1+1)$, $w(k_1) > w(n)$; note that the inequality $w(1) > w(k_1+1)$, together with condition (Q), implies that $w(1) > w(k_2+1)$.

Theorem 31. *Assume condition* (Q)*.*

(1) *Assume that $w(k_1) < w(n)$.*
(a) *If $w(1) < w(k_1+1)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_1})]\cdot[\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|}[\mathcal{O}^{\mathrm{end}(w,A)}].$$

(b) *If $w(1) > w(k_1+1)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_1})]\cdot[\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|}\left([\mathcal{O}^{\mathrm{end}(w,A)}] - Q_{k_1}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)s_{k_1}\rfloor}]\right).$$

(2) *Assume that $w(k_1) > w(n)$.*
(a) *If $w(1) < w(k_2+1)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_1})]\cdot[\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|}[\mathcal{O}^{\mathrm{end}(w,A)}].$$

(b) *If $w(k_2+1) < w(1) < w(k_1+1)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_1})]\cdot[\mathcal{O}^w] \\ = \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|}\left([\mathcal{O}^{\mathrm{end}(w,A)}] - Q_{k_1}Q_{k_2}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1,k_2+1)\rfloor}]\right).$$

(3) *If condition* (Full) *holds, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_1})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|} \Big([\mathcal{O}^{\mathrm{end}(w,A)}] - Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) s_{k_1} \rfloor}] \\ - Q_{k_1} Q_{k_2} \Big([\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1,k_2+1) \rfloor}] - [\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1,k_2+1) s_{k_1} \rfloor}] \Big) \Big) .$$

Next, we assume that $k = k_2$.

Theorem 32. *If $w(k_2) < w(k_2+1)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_2})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_2}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma^*(k_2))} (-1)^{|A|} [\mathcal{O}^{\mathrm{end}(w,A)}] .$$

Recall condition (Q) above.

Theorem 33. *Assume that $w(k_2) > w(k_2+1)$, and assume that condition* (Q) *does not hold.*

(1) *If $w(k_2) < w(n)$ or $w(k_1+1) < w(k_2+1)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_2})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_2}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma^*(k_2))} (-1)^{|A|} [\mathcal{O}^{\mathrm{end}(w,A)}] .$$

(2) *If $w(k_2) > w(n)$ and $w(k_1+1) > w(k_2+1)$, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_2})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_2}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma^*(k_2))} (-1)^{|A|} \Big([\mathcal{O}^{\mathrm{end}(w,A)}] - Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) s_{k_2} \rfloor}] \Big) .$$

We consider the following analog of condition (Full):

(Full)* both of the following hold:
  (1) $w(k_1) = n$ and $w(k_2+1) = 1$; and
  (2) $w(k_2)$ is the maximum element in the sequence $w[k_1+1, n]$.

Remark 34. Condition (Full)* holds if and only if condition (Q) holds and $w(n) < w(k_2)$, $w(k_2+1) < w(1)$; note that the inequality $w(n) < w(k_2)$, together with condition (Q), implies that $w(n) < w(k_1)$.

Theorem 35. *Assume condition* (Q)*.*

(1) *Assume that* $w(1) < w(k_2+1)$.

(a) *If* $w(k_2) < w(n)$*, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_2})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_2}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma^*(k_2))} (-1)^{|A|} [\mathcal{O}^{\mathrm{end}(w,A)}].$$

(b) *If* $w(k_2) > w(n)$*, then we have the following cancellation-free formula:*

$$\begin{aligned}&[\mathcal{O}(-\varpi_{k_2})] \cdot [\mathcal{O}^w] \\ &\quad = \mathbf{e}^{w\varpi_{k_2}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma^*(k_2))} (-1)^{|A|} \left([\mathcal{O}^{\mathrm{end}(w,A)}] - Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) s_{k_2} \rfloor}] \right).\end{aligned}$$

(2) *Assume that* $w(1) > w(k_2+1)$.

(a) *If* $w(k_1) < w(n)$*, then we have the following cancellation-free formula:*

$$[\mathcal{O}(-\varpi_{k_2})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_2}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma^*(k_2))} (-1)^{|A|} [\mathcal{O}^{\mathrm{end}(w,A)}].$$

(b) *If* $w(k_2) < w(n) < w(k_1)$*, then we have the following cancellation-free formula:*

$$\begin{aligned}&[\mathcal{O}(-\varpi_{k_2})] \cdot [\mathcal{O}^w] \\ &\quad = \mathbf{e}^{w\varpi_{k_2}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma^*(k_2))} (-1)^{|A|} \left([\mathcal{O}^{\mathrm{end}(w,A)}] - Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1,k_2+1) \rfloor}] \right).\end{aligned}$$

(3) *If condition* (Full)* *holds, then we have the following cancellation-free formula:*

$$\begin{aligned}[\mathcal{O}(-\varpi_{k_2})] \cdot [\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_2}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma^*(k_2))} (-1)^{|A|} \Big(&[\mathcal{O}^{\mathrm{end}(w,A)}] - Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) s_{k_2} \rfloor}] \\ &- Q_{k_1} Q_{k_2} \left([\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1,k_2+1) \rfloor}] - [\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1,k_2+1) s_{k_2} \rfloor}] \right) \Big).\end{aligned}$$

Example 36. In this example, we consider the case that $n = 6$ and $(k_1, k_2) = (2, 4)$. Let $w = s_4 s_1 s_2 s_3 s_5 s_4 s_3 s_2$. Then, w satisfies condition (Q), and we see that $w(k_2+1) (= w(5)) < w(1) < w(k_1+1) (= w(3))$. Recall that $\Gamma(2) = ((1,6), (1,5), (1,4), (1,3), (2,6), (2,5), (2,4), (2,3))$. Then Table 4 is the list of all admissible subset $A \in \mathcal{A}(w, \Gamma(5))$ and their statistics $\mathrm{end}(w, A)$, $\mathrm{down}(w, A)$, together with $\lfloor \mathrm{end}(w, A) \rfloor$.

Table 4
The list of all admissible subsets $A \in \mathcal{A}(s_4s_1s_2s_3s_5s_4s_3s_2, \Gamma(2))$.

A	$\mathrm{end}(w, A)$	$\lfloor \mathrm{end}(w, A) \rfloor$	$\mathrm{down}(w, A)$
$\emptyset$	$s_4s_5s_1s_2s_3s_4s_3s_2$	$s_4s_5s_1s_2s_3s_4s_3s_2$	0
$\{4\}$	$s_4s_5s_1s_2s_3s_4s_3s_1s_2$	$s_4s_5s_1s_2s_3s_4s_3s_1s_2$	0
$\{6\}$	$s_4s_5s_1$	s_4	$\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee$
$\{8\}$	$s_4s_5s_1s_2s_3s_4s_3$	$s_4s_5s_1s_2s_3s_4$	$\alpha_2^\vee$
$\{4, 6\}$	$s_4s_5s_2s_1$	s_4s_2	$\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee$
$\{4, 8\}$	$s_4s_5s_1s_2s_3s_4s_3s_1$	$s_4s_5s_1s_2s_3s_4$	$\alpha_2^\vee$
$\{6, 8\}$	$s_4s_5s_1s_2$	$s_4s_1s_2$	$\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee$
$\{4, 6, 8\}$	$s_4s_5s_1s_2s_1$	$s_4s_1s_2$	$\alpha_2^\vee + \alpha_3^\vee + \alpha_4^\vee$

By Theorem 7, in $QK_T^{\mathrm{poly}}(G/B)$, we have:

$$\begin{aligned}
&[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_2}] \\
&= \mathbf{e}^{s_4s_5s_1s_2s_3s_4s_3s_2\varpi_2}\big([\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_2}] - [\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_1s_2}] \\
&\quad - Q_2Q_3Q_4[\mathcal{O}^{s_4s_5s_1}] - Q_2[\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3}] + Q_2Q_3Q_4[\mathcal{O}^{s_4s_5s_2s_1}] \\
&\quad + Q_2[\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_1}] + Q_2Q_3Q_4[\mathcal{O}^{s_4s_5s_1s_2}] - Q_2Q_3Q_4[\mathcal{O}^{s_4s_5s_1s_2s_1}]\big).
\end{aligned} \tag{12}$$

By applying the surjection $\Phi_J : QK_T^{\mathrm{poly}}(G/B) \to QK_T^{\mathrm{poly}}(G/P_J)$ to equation (12), we obtain the following cancellation-free formula in $QK_T^{\mathrm{poly}}(G/P_J) \subset QK_T(G/P_J)$; here, the underlined terms in the first equality are canceled out:

$$\begin{aligned}
&[\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_2}] \\
&= \mathbf{e}^{s_4s_5s_1s_2s_3s_4s_3s_2\varpi_2}\big([\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_2}] - [\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_1s_2}] \\
&\quad - Q_2Q_3Q_4[\mathcal{O}^{s_4}] \underline{- Q_2[\mathcal{O}^{s_4s_5s_1s_2s_3s_4}]} + Q_2Q_3Q_4[\mathcal{O}^{s_4s_2}] \\
&\quad \underline{+ Q_2[\mathcal{O}^{s_4s_5s_1s_2s_3s_4}] + Q_2Q_3Q_4[\mathcal{O}^{s_4s_1s_2}] - Q_2Q_3Q_4[\mathcal{O}^{s_4s_1s_2}]}\big). \\
&= \mathbf{e}^{s_4s_5s_1s_2s_3s_4s_3s_2\varpi_2}\big([\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_2}] \\
&\quad - [\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_1s_2}] - Q_2Q_4[\mathcal{O}^{s_4}] + Q_2Q_4[\mathcal{O}^{s_4s_2}]\big).
\end{aligned}$$

Also, we deduce that $\mathcal{A}_{\lessdot}(w, \Gamma(2)) = \{\emptyset, \{4\}\}$. Therefore, we see that

$$\begin{aligned}
&(\text{RHS of the equation in Theorem 31 (2)(b)}) \\
&= \mathbf{e}^{s_4s_5s_1s_2s_3s_4s_3s_2\varpi_2}\big(([\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_2}] - Q_2Q_4[\mathcal{O}^{\lfloor s_4s_5s_1 \rfloor}]) \\
&\quad - ([\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_1s_2}] - Q_2Q_4[\mathcal{O}^{\lfloor s_4s_5s_2s_1 \rfloor}])\big) \\
&= \mathbf{e}^{s_4s_5s_1s_2s_3s_4s_3s_2\varpi_2}\big([\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_2}] \\
&\quad - Q_2Q_4[\mathcal{O}^{s_4}] - [\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_1s_2}] + Q_2Q_4[\mathcal{O}^{s_4s_2}]\big) \\
&= [\mathcal{O}(-\varpi_2)] \cdot [\mathcal{O}^{s_4s_5s_1s_2s_3s_4s_3s_2}].
\end{aligned}$$

Thus Theorem 31 (2)(b) holds in this case.

Remark 37. In [27, Theorem 4.5], Xu obtained a Chevalley formula for *incidence varieties*, that is, for the two-step flag manifold G/P_J in the case that $J = I \setminus \{1, n-1\}$, by a completely different method of proof than ours of Theorems 27, 29, 31, 32, 33, and 35. We can verify that in this case, our Chevalley formula coincides with the one in [27, Theorem 4.5] for incidence varieties. As an example, we compare Theorem 29 (2) with [27, Equation (9) of Theorem 4.5]; this is a most complicated case. As for Theorems 27 and 29 (1), we can also compare our formulas and Xu's ones by the same argument as below. As for Theorem 31, w should be the unique element of W^J such that $w(1) = n$ and $w(n) = 1$, and hence we can compare the formulas by direct calculation. As for Theorems 32, 33, and 35, we can show the coincidence of the formulas from that of the formulas in Theorems 27, 29, and 31 by applying the diagram automorphism ω (see Section 4.3).

Throughout this remark, we assume that $k_1 = 1$, $k_2 = n-1$. Note that under this assumption, for $1 \le i, j \le n$ with $i \neq j$, there exists a unique $w \in W^J$ such that $w(1) = i$ and $w(n) = j$; in such a case, we write $w = [i, j]$, as in [27].

We assume that $w \in W^J$ satisfies the following:

- $w(k_1) > w(k_1 + 1)$,
- condition (Q) does not hold,
- $w(1) > w(k_1 + 1)$,
- $w(k_1) > w(k_2)$,

and set $i := w(1)$, $j := w(n)$ (i.e., $w = [i, j]$). Under these assumptions, we see that $i + 1 \equiv j \mod n$ if and only if $i = n-1$ and $j = n$ (i.e., $w = [n-1, n]$).

Let us compute the product $[\mathcal{O}^{s_1}] \cdot [\mathcal{O}^{w}]$ by our Chevalley formula. Recall that

$$\Gamma(1) = ((1, n), (1, n-1), \dots, (1, 2))$$

(with all roots negated). First, assume that $w = [n-1, n]$. Then, by Proposition 2, we deduce that $\mathcal{A}_{\lessdot}([n-1, n], \Gamma(1)) = \{\emptyset, \{(1, n)\}\}$. By Theorem 29 (2), we compute:

$$\begin{aligned}
&[\mathcal{O}(-\varpi_1)] \cdot [\mathcal{O}^{[n-1,n]}] \\
&= \mathbf{e}^{[n-1,n]\varpi_1} \sum_{A \in \mathcal{A}_{\lessdot}([n-1,n],\Gamma(1))} (-1)^{|A|} \left([\mathcal{O}^{\mathrm{end}([n-1,n],A)}] - Q_1 [\mathcal{O}^{\lfloor \mathrm{end}([n-1,n],A) s_1 \rfloor}] \right) \\
&= \mathbf{e}^{\varepsilon_{n-1}} \left(\underbrace{\left([\mathcal{O}^{[n-1,n]}] - Q_1 [\mathcal{O}^{[1,n]}] \right)}_{A=\emptyset} - \underbrace{\left([\mathcal{O}^{[n,n-1]}] - Q_1 [\mathcal{O}^{[1,n-1]}] \right)}_{A=\{(1,n)\}} \right).
\end{aligned}$$

By the well-known formula $[\mathcal{O}^{s_1}] = 1 - \mathbf{e}^{-\varpi_1} [\mathcal{O}(-\varpi_1)]$ (cf., [4, Section 4.1]), we see that

$$[\mathcal{O}^{s_1}] \cdot [\mathcal{O}^{[n-1,n]}]$$

$$
\begin{aligned}
&= (1 - \mathbf{e}^{-\varepsilon_1}[\mathcal{O}(-\varpi_1)]) \cdot [\mathcal{O}^{[n-1,n]}] \\
&= [\mathcal{O}^{[n-1,n]}] - \mathbf{e}^{-\varepsilon_1}[\mathcal{O}(-\varpi_1)] \cdot [\mathcal{O}^{[n-1,n]}] \\
&= [\mathcal{O}^{[n-1,n]}] - \mathbf{e}^{\varepsilon_{n-1}-\varepsilon_1} \left(\left([\mathcal{O}^{[n-1,n]}] - Q_1[\mathcal{O}^{[1,n]}] \right) - \left([\mathcal{O}^{[n,n-1]}] - Q_1[\mathcal{O}^{[1,n-1]}] \right) \right) \\
&= (1 - \mathbf{e}^{\varepsilon_{n-1}-\varepsilon_1})[\mathcal{O}^{[n-1,n]}] + \mathbf{e}^{\varepsilon_{n-1}-\varepsilon_1} \left(Q_1[\mathcal{O}^{[1,n]}] + [\mathcal{O}^{[n,n-1]}] - Q_1[\mathcal{O}^{[1,n-1]}] \right).
\end{aligned}
$$

This result agrees with the second equation of [27, Equation (9) of Theorem 4.5].

Next, we consider the case $w = [i, j] \neq [n-1, n]$. In this case, we see that $i + 1 \not\equiv j \mod n$. Since condition (Q) does not hold, we have $w(n) \neq 1$. Also, we have $w(n) \neq n$; this is because if $w(n) = n$, then w must be $[n-1, n]$ under our assumptions. These facts imply that $w(1) = i = n$ and $w(2) = 1$. By Proposition 2, we deduce that $\mathcal{A}_{\lessdot}(w, \Gamma(1)) = \{\emptyset\}$. Therefore, we compute:

$$
\begin{aligned}
&[\mathcal{O}(-\varpi_1)] \cdot [\mathcal{O}^{[n,j]}] \\
&= \mathbf{e}^{w\varpi_1} \sum_{A \in \mathcal{A}_{\lessdot}([n,j],\Gamma(1))} (-1)^{|A|} \left([\mathcal{O}^{\mathrm{end}([n,j],A)}] - Q_1[\mathcal{O}^{\lfloor \mathrm{end}([n,j],A)s_1 \rfloor}] \right) \\
&= \mathbf{e}^{\varepsilon_n} \left([\mathcal{O}^{[n,j]}] - Q_1[\mathcal{O}^{[1,j]}] \right).
\end{aligned}
$$

Again, since $[\mathcal{O}^{s_1}] = 1 - \mathbf{e}^{-\varpi_1}[\mathcal{O}(-\varpi_1)]$, we see that

$$
\begin{aligned}
&[\mathcal{O}^{s_1}] \cdot [\mathcal{O}^{[n,j]}] \\
&= (1 - \mathbf{e}^{-\varpi_1}[\mathcal{O}(-\varpi_1)]) \cdot [\mathcal{O}^{[n,j]}] \\
&= [\mathcal{O}^{[n,j]}] - \mathbf{e}^{-\varepsilon_1}[\mathcal{O}(-\varpi_1)] \cdot [\mathcal{O}^{[n,j]}] \\
&= [\mathcal{O}^{[n,j]}] - \mathbf{e}^{\varepsilon_n-\varepsilon_1} \left([\mathcal{O}^{[n,j]}] - Q_1[\mathcal{O}^{[1,j]}] \right) \\
&= (1 - \mathbf{e}^{\varepsilon_n-\varepsilon_1})[\mathcal{O}^{[n,j]}] + \mathbf{e}^{\varepsilon_n-\varepsilon_1} Q_1[\mathcal{O}^{[1,j]}].
\end{aligned}
$$

This result agrees with the first equation of [27, Equation (9) of Theorem 4.5].

4.3. Proofs of parabolic Chevalley formulas: part 1

In this and the next subsection, we give proofs of the results stated in the previous subsection. Since Theorems 32, 33, and 35 follow from Theorems 27, 29, and 31, respectively, by applying the diagram automorphism $\omega : [n-1] \to [n-1]$, it suffices to prove Theorems 27, 29, and 31. Note that the diagram automorphism ω induces a group automorphism $\omega : W \xrightarrow{\sim} W$, $s_l \mapsto s_{\omega(l)}$, together with a linear automorphism $\omega : \mathfrak{h}_{\mathbb{R}}^* \xrightarrow{\sim} \mathfrak{h}_{\mathbb{R}}^*$, $\varpi_l \mapsto \varpi_{\omega(l)}$, and also an isomorphism $\omega : G/P_J \xrightarrow{\sim} G/P_{\omega(J)}$ of varieties; recall that G is simply-connected. Hence, as mentioned in [24, Sections 8.1 and 8.3], we see that there exists a $\mathbb{Z}$-module isomorphism $\omega : QK_T(G/P_J) \xrightarrow{\sim} QK_T(G/P_{\omega(J)})$ such that

$$\mathbf{e}^{\mu}[\mathcal{O}^{w}] \mapsto \mathbf{e}^{\omega(\mu)}[\mathcal{O}^{\omega(w)}]$$

for $\mu \in \Lambda$, $w \in W^J$, and such that $\omega(Q_l) = Q_{\omega(l)}$ for $l \in I \setminus J$. In this subsection, we give proofs of Theorems 27 and 29.

By Remark 8, we obtain the following.

Lemma 38. *The sum*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|}[\mathcal{O}^{\mathrm{end}(w,A)}]$$

is cancellation-free.

Also, by making use of Proposition 2, we can verify the following.

Lemma 39. *The following hold.*

(1) *We have $\mathcal{A}_1(w,\Gamma(k_1)) \neq \emptyset$ if and only if $w(k_1) > w(k_1+1)$.*
(2) *We have $\mathcal{A}_2(w,\Gamma(k_1)) \neq \emptyset$ if and only if condition* (Q) *holds.*

Remark 40. It is obvious that $\mathcal{A}_2(w,\Gamma(k_1)) \neq \emptyset$ if and only if $\mathcal{A}_3(w,\Gamma(k_1)) \neq \emptyset$.

Remark 41. If $w(k_1) > w(k_1+1)$, then we have

$$\mathcal{A}_1(w,\Gamma(k_1)) = \{A \sqcup \{(k_1,k_1+1)\} \mid A \in \mathcal{A}_{\lessdot}(w,\Gamma(k_1))\}. \tag{13}$$

Also, if condition (Q) holds, then we have

$$\mathcal{A}_2(w,\Gamma(k_1)) = \{A \sqcup \{(k_1,k_2+1)\} \mid A \in \mathcal{A}_{\lessdot}(w,\Gamma(k_1))\}, \tag{14}$$
$$\mathcal{A}_3(w,\Gamma(k_1)) = \{A \sqcup \{(k_1,k_2+1),(k_1,k_1+1)\} \mid A \in \mathcal{A}_{\lessdot}(w,\Gamma(k_1))\}. \tag{15}$$

Proof of Theorem 27. By Lemma 39, we have $\mathcal{A}(w,\Gamma(k_1)) = \mathcal{A}_{\lessdot}(w,\Gamma(k_1))$. Therefore, the theorem follows from Lemma 38. □

In the rest of this subsection, we assume that $w(k_1) > w(k_1+1)$, and assume that condition (Q) does not hold. Hence we have $\mathcal{A}_2(w,\Gamma(k_1)) = \mathcal{A}_3(w,\Gamma(k_1)) = \emptyset$.

First, assume that $w(1) < w(k_1+1)$. Take the maximal $1 \leq p \leq k_1$ such that $w(p) < w(k_1+1)$. Then, we can define an involution ι on $\mathcal{A}_1(w,\Gamma(k_1))$ as follows: set

$$\mathcal{A}_1^1(w,\Gamma(k_1)) := \{A \in \mathcal{A}_1(w,\Gamma(k_1)) \mid (p,k_1+1) \in A\},$$
$$\mathcal{A}_1^2(w,\Gamma(k_1)) := \{A \in \mathcal{A}_1(w,\Gamma(k_1)) \mid (p,k_1+1) \notin A\},$$

and define ι by

$$A \in \mathcal{A}_1^2(w,\Gamma(k_1)) \;\mapsto\; \iota(A) := A \sqcup \{(p,k_1+1)\} \in \mathcal{A}_1^1(w,\Gamma(k_1))\,,$$
$$A \in \mathcal{A}_1^1(w,\Gamma(k_1)) \;\mapsto\; \iota(A) := A \setminus \{(p,k_1+1)\} \in \mathcal{A}_1^2(w,\Gamma(k_1))\,.$$

This ι has the following properties:

- $\mathrm{end}(w,\iota(A)) = \mathrm{end}(w,A)(p,k_1)$ (and hence $\lfloor \mathrm{end}(w,\iota(A))\rfloor = \lfloor \mathrm{end}(w,A)\rfloor$);
- $|\iota(A)| = |A| \pm 1$.

By using the involution ι, we obtain the following.

Lemma 42. *Assume that $w(k_1) > w(k_1+1)$, and assume that condition* (Q) *does not hold. If $w(1) < w(k_1+1)$, then*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_1(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)\rfloor}] = 0\,.$$

Remark 43. Even if we assume condition (Q), the identity in Lemma 42 is still valid, if all the conditions of this lemma other than the negation of condition (Q) hold. This is because the involution ι above is well-defined whether or not condition (Q) holds.

Next, assume that $w(k_1) < w(k_2)$. Take the minimal $k_1+1 \le q \le k_2$ such that $w(k_1) < w(q)$. Then, we can define an involution ι on $\mathcal{A}_1(w,\Gamma(k_1))$ as follows: set

$$\mathcal{A}_1^1(w,\Gamma(k_1)) := \{A \in \mathcal{A}_1(w,\Gamma(k_1)) \mid (k_1,q) \in A\}\,,$$
$$\mathcal{A}_1^2(w,\Gamma(k_1)) := \{A \in \mathcal{A}_1(w,\Gamma(k_1)) \mid (k_1,q) \notin A\}\,,$$

and define ι by

$$A \in \mathcal{A}_1^2(w,\Gamma(k_1)) \;\mapsto\; \iota(A) := A \sqcup \{(k_1,q)\} \in \mathcal{A}_1^1(w,\Gamma(k_1))\,,$$
$$A \in \mathcal{A}_1^1(w,\Gamma(k_1)) \;\mapsto\; \iota(A) := A \setminus \{(k_1,q)\} \in \mathcal{A}_1^2(w,\Gamma(k_1))\,.$$

This ι has the following properties:

- $\mathrm{end}(w,\iota(A)) = \mathrm{end}(w,A)(k_1+1,q)$ (and hence $\lfloor \mathrm{end}(w,\iota(A))\rfloor = \lfloor \mathrm{end}(w,A)\rfloor$);
- $|\iota(A)| = |A| \pm 1$.

By using the involution ι, we obtain the following.

Lemma 44. *Assume that $w(k_1) > w(k_1+1)$, and assume that condition* (Q) *does not hold. If $w(k_1) < w(k_2)$, then*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_1(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0 .$$

Proof of Theorem 29 (1). By Lemmas 42 and 44, we deduce that

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_1(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0 .$$

Therefore, we obtain the desired cancellation-free formula from Lemma 38, together with the fact that $\mathcal{A}_2(w, \Gamma(k_1)) = \mathcal{A}_3(w, \Gamma(k_1)) = \emptyset$. □

We assume that $w(1) > w(k_1+1)$ and $w(k_1) > w(k_2)$ until the end of this subsection. Let $A \in \mathcal{A}_1(w, \Gamma(k_1))$, and set $y := \mathrm{end}(w, A \setminus \{(k_1, k_1+1)\})$. Since $A \setminus \{(k_1, k_1+1)\}$ contains only Bruhat steps, we see that $y(k_1+1) < y(1)$ and $y(k_2) < y(k_1)$. Therefore, if we set $z := ys_{k_1} = \mathrm{end}(w, A)$, then we have

- $z(k_1) < z(1) < z(2) < \cdots < z(k_1 - 1)$,
- $z(k_1+2) < z(k_1+3) < \cdots < z(k_2) < z(k_1+1)$, and
- $z(k_2+1) < z(k_2+2) < \cdots < z(n)$;

hence, if we take cyclic permutations $\sigma_1 := (1, k_1, k_1 - 1, \dots, 2)$ (if $k_1 = 1$, then we take $\sigma_1 := e$, the identity permutation) and $\sigma_2 := (k_1+1, k_1+2, \dots, k_2)$ (if $k_1 + 1 = k_2$, then we take $\sigma_2 := e$), then we deduce that $\lfloor z \rfloor = z\sigma_1\sigma_2$. Note that the definitions of σ_1 and σ_2 do not depend on the choice of A. Thus, for $A, B \in \mathcal{A}_1(w, \Gamma(k_1))$ with $A \neq B$, it follows that

$$\lfloor \mathrm{end}(w, A) \rfloor = \mathrm{end}(w, A)\sigma_1\sigma_2 \neq \mathrm{end}(w, B)\sigma_1\sigma_2 = \lfloor \mathrm{end}(w, B) \rfloor .$$

Since the right-hand side of equation (4) in Theorem 7 is cancellation-free, as mentioned in Remark 8, this proves the following.

Lemma 45. *Assume that $w(k_1) > w(k_1 + 1)$, and assume that condition* (Q) *does not hold. If $w(1) > w(k_1 + 1)$ and $w(k_1) > w(k_2)$, then the sum*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_1(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] \tag{16}$$

is cancellation-free.

Remark 46. Note that we do not use the negation of condition (Q) in the proof of Lemma 45. Hence the sum (16) is cancellation-free whether or not we assume condition (Q), if all the conditions of Lemma 45 other than the negation of (Q) hold.

Remark 47. If $\mathcal{A}_1(w, \Gamma(k_1)) \neq \emptyset$ (or equivalently, $w(k_1) > w(k_1+1)$), then equation (13) shows that

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_1(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}]$$
$$= -\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) s_{k_1} \rfloor}] .$$

Proof of Theorem 29 (2). The desired identity follows from Lemmas 38, 45, and Remark 47, together with the fact that $\mathcal{A}_2(w, \Gamma(k_1)) = \mathcal{A}_3(w, \Gamma(k_1)) = \emptyset$. □

4.4. *Proofs of parabolic Chevalley formulas: part 2*

In this subsection, we give a proof of Theorem 31; since we assume condition (Q), we have $w(k_1) > w(k_1 + 1)$; see Remark 28.

First, assume that $w(k_1) < w(n)$. Then, we can take the minimal $k_2 + 1 \leq q \leq n$ such that $w(k_1) < w(q)$, and define an involution ι on $\mathcal{A}_l(w, \Gamma(k_1))$, $l = 2, 3$, as follows: for each $l = 2, 3$, we set

$$\mathcal{A}_l^1(w, \Gamma(k_1)) := \{A \in \mathcal{A}_l(w, \Gamma(k_1)) \mid (k_1, q) \in A\} ,$$
$$\mathcal{A}_l^2(w, \Gamma(k_1)) := \{A \in \mathcal{A}_l(w, \Gamma(k_1)) \mid (k_1, q) \notin A\} ,$$

and define ι by

$$A \in \mathcal{A}_l^2(w, \Gamma(k_1)) \;\mapsto\; \iota(A) := A \sqcup \{(k_1, q)\} \in \mathcal{A}_l^1(w, \Gamma(k_1)) ,$$
$$A \in \mathcal{A}_l^1(w, \Gamma(k_1)) \;\mapsto\; \iota(A) := A \setminus \{(k_1, q)\} \in \mathcal{A}_l^2(w, \Gamma(k_1)) .$$

This ι has the following properties:

- $\mathrm{end}(w, \iota(A)) = \mathrm{end}(w, A)(k_2 + 1, q)$ (and hence $\lfloor \mathrm{end}(w, \iota(A)) \rfloor = \lfloor \mathrm{end}(w, A) \rfloor$);
- $|\iota(A)| = |A| \pm 1$.

By using the involution ι, we obtain the following.

Lemma 48. *Assume condition* (Q)*. If* $w(k_1) < w(n)$*, then for* $l = 2, 3$*,*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_l(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0 .$$

Proof of Theorem 31 (1). By Lemma 48, we deduce that

$$[\mathcal{O}(-\varpi_{k_1})]\cdot[\mathcal{O}^w] = \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|}[\mathcal{O}^{\mathrm{end}(w,A)}] \\ + \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_1(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}[\mathcal{O}^{\lfloor\mathrm{end}(w,A)\rfloor}].$$

If $w(1) < w(k_1+1)$, then

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_1(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}[\mathcal{O}^{\lfloor\mathrm{end}(w,A)\rfloor}] = 0$$

by Remark 43. Therefore, Theorem 31 (1) (a) follows from Lemma 38.

Assume now that $w(1) > w(k_1+1)$. Note that $w(k_1) > w(k_2)$ by condition (Q). Hence Remark 46 implies that the sum

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_1(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}[\mathcal{O}^{\lfloor\mathrm{end}(w,A)\rfloor}]$$

is cancellation-free. Therefore, Theorem 31 (1) (b) follows from Lemma 38 and Remark 47. □

Next, assume that $w(k_1) > w(n)$. We consider the following auxiliary condition:

(Q-A) there exists $1 \le l \le k_1$ such that $w(k_2+1) < w(l) < w(k_1+1)$.

Assume condition (Q-A), and that $w(1) < w(k_2+1)$. We take the maximal $1 \le p_{\mathcal{A}_2} \le k_1$ such that $w(p_{\mathcal{A}_2}) < w(k_2+1)$. Then, we can define an involution $\iota_{\mathcal{A}_2}$ on $\mathcal{A}_2(w,\Gamma(k_1))$ as follows: set

$$\mathcal{A}_2^1(w,\Gamma(k_1)) := \{A \in \mathcal{A}_2(w,\Gamma(k_2)) \mid (p_{\mathcal{A}_2}, k_2+1) \in A\},$$
$$\mathcal{A}_2^2(w,\Gamma(k_1)) := \{A \in \mathcal{A}_2(w,\Gamma(k_2)) \mid (p_{\mathcal{A}_2}, k_2+1) \notin A\},$$

and define $\iota_{\mathcal{A}_2}$ by

$$A \in \mathcal{A}_2^2(w,\Gamma(k_1)) \;\mapsto\; \iota_{\mathcal{A}_2}(A) := A \sqcup \{(p_{\mathcal{A}_2}, k_2+1)\} \in \mathcal{A}_2^1(w,\Gamma(k_1)),$$
$$A \in \mathcal{A}_2^1(w,\Gamma(k_1)) \;\mapsto\; \iota_{\mathcal{A}_2}(A) := A \setminus \{(p_{\mathcal{A}_2}, k_2+1)\} \in \mathcal{A}_2^2(w,\Gamma(k_1)).$$

Remark 49. If condition (Q-A) does not hold, then the above $\iota_{\mathcal{A}_2} : \mathcal{A}_2^1(w,\Gamma(k_1)) \to \mathcal{A}_2^2(w,\Gamma(k_1))$ is not well-defined; we will explain this situation later.

This $\iota_{\mathcal{A}_2}$ has the following properties:

- $\mathrm{end}(w, \iota_{\mathcal{A}_2}(A)) = \mathrm{end}(w, A)(p_{\mathcal{A}_2}, k_1)$ (and hence $\lfloor \mathrm{end}(w, \iota_{\mathcal{A}_2}(A)) \rfloor = \lfloor \mathrm{end}(w, A) \rfloor$);
- $|\iota_{\mathcal{A}_2}(A)| = |A| \pm 1$.

By using the involution $\iota_{\mathcal{A}_2}$, we obtain the following.

Lemma 50. *Assume condition* (Q)*. If* $w(1) < w(k_2 + 1)$ *and condition* (Q-A) *hold, then*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0 .$$

Also, we take the maximal $1 \leq p_{\mathcal{A}_3} \leq k_1$ such that $w(p_{\mathcal{A}_3}) < w(k_1 + 1)$. We can define an involution $\iota_{\mathcal{A}_3}$ on $\mathcal{A}_3(w, \Gamma(k_1))$ as follows: set

$$\mathcal{A}_3^1(w, \Gamma(k_1)) := \{A \in \mathcal{A}_3(w, \Gamma(k_1)) \mid (p_{\mathcal{A}_3}, k_1 + 1) \in A\} ,$$
$$\mathcal{A}_3^2(w, \Gamma(k_1)) := \{A \in \mathcal{A}_3(w, \Gamma(k_1)) \mid (p_{\mathcal{A}_3}, k_1 + 1) \notin A\} ,$$

and define $\iota_{\mathcal{A}_3}$ by

$$A \in \mathcal{A}_3^2(w, \Gamma(k_1)) \ \mapsto \ \iota_{\mathcal{A}_3}(A) := A \sqcup \{(p_{\mathcal{A}_3}, k_1 + 1)\} \in \mathcal{A}_3^1(w, \Gamma(k_1)) ,$$
$$A \in \mathcal{A}_3^1(w, \Gamma(k_1)) \ \mapsto \ \iota_{\mathcal{A}_3}(A) := A \setminus \{(p_{\mathcal{A}_3}, k_1 + 1)\} \in \mathcal{A}_3^2(w, \Gamma(k_1)) .$$

Remark 51. If condition (Q-A) does not hold, then the above $\iota_{\mathcal{A}_3} : \mathcal{A}_3^2(w, \Gamma(k_1)) \to \mathcal{A}_3^1(w, \Gamma(k_1))$ is not well-defined for the same reason as $\iota_{\mathcal{A}_2}$.

This $\iota_{\mathcal{A}_3}$ has the following properties:

- $\mathrm{end}(w, \iota_{\mathcal{A}_3}(A)) = \mathrm{end}(w, A)(p_{\mathcal{A}_3}, k_1)$ (and hence $\lfloor \mathrm{end}(w, \iota_{\mathcal{A}_3}(A)) \rfloor = \lfloor \mathrm{end}(w, A) \rfloor$);
- $|\iota_{\mathcal{A}_3}(A)| = |A| \pm 1$.

By using the involution $\iota_{\mathcal{A}_3}$, we obtain the following.

Lemma 52. *Assume condition* (Q)*. If* $w(1) < w(k_2 + 1)$ *and condition* (Q-A) *holds, then*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_3(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0 .$$

Next, assume that condition (Q-A) does not hold, but assume that $w(1) < w(k_2 + 1)$. Take the maximal $1 \leq p \leq k_1$ such that $w(p) < w(k_2 + 1)$. Set

$$\mathcal{A}_2'(w, \Gamma(k_1)) := \{A \in \mathcal{A}_2(w, \Gamma(k_1)) \mid (p, k_1 + 1) \in A\} ,$$
$$\mathcal{A}_2'^{C}(w, \Gamma(k_1)) := \mathcal{A}_2(w, \Gamma(k_1)) \setminus \mathcal{A}_2'(w, \Gamma(k_1)) .$$

Observe that if $A \in \mathcal{A}'_2(w, \Gamma(k_1))$, then we must have $(p, k_2+1) \in A$; if not, then A cannot contain a quantum step (k_1, k_2+1), which contradicts the definition of $\mathcal{A}_2(w, \Gamma(k_1))$. Thus, the above $\iota_{\mathcal{A}_2} : \mathcal{A}^1_2(w, \Gamma(k_1)) \to \mathcal{A}^2_2(w, \Gamma(k_1))$ is not well-defined. Hence we need another involution.

In fact, we can define an involution on $\mathcal{A}'^C_2(w, \Gamma(k_1))$ similar to $\iota_{\mathcal{A}_2}$ as follows. We set

$$\begin{aligned}
\mathcal{A}'^{C,1}_2(w, \Gamma(k_1)) &:= \{A \in \mathcal{A}'^C_2(w, \Gamma(k_1)) \mid (p, k_2+1) \in A\}, \\
\mathcal{A}'^{C,2}_2(w, \Gamma(k_1)) &:= \{A \in \mathcal{A}'^C_2(w, \Gamma(k_1)) \mid (p, k_2+1) \notin A\}.
\end{aligned}$$

Then we can define an involution ι on $\mathcal{A}'^C_2(w, \Gamma(k_1))$ by

$$\begin{aligned}
A \in \mathcal{A}'^{C,2}_2(w, \Gamma(k_1)) &\mapsto \iota(A) := A \sqcup \{(p, k_2+1)\} \in \mathcal{A}'^{C,1}_2(w, \Gamma(k_1)), \\
A \in \mathcal{A}'^{C,1}_2(w, \Gamma(k_1)) &\mapsto \iota(A) := A \setminus \{(p, k_2+1)\} \in \mathcal{A}'^{C,2}_2(w, \Gamma(k_1)).
\end{aligned}$$

This ι has the following properties:

- $\mathrm{end}(w, \iota(A)) = \mathrm{end}(w, A)(p, k_1)$ (and hence $\lfloor \mathrm{end}(w, \iota(A)) \rfloor = \lfloor \mathrm{end}(w, A) \rfloor$);
- $|\iota(A)| = |A| \pm 1$.

By using the involution ι, we obtain the following.

Lemma 53. *Assume condition* (Q). *If* $w(1) < w(k_2+1)$, *and if condition* (Q-A) *does not hold, then*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}'^C_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0.$$

Similarly, we set

$$\begin{aligned}
\mathcal{A}'_3(w, \Gamma(k_1)) &:= \{A \in \mathcal{A}_3(w, \Gamma(k_1)) \mid (p, k_2+1) \notin A\}, \\
\mathcal{A}'^C_3(w, \Gamma(k_1)) &:= \mathcal{A}_3(w, \Gamma(k_1)) \setminus \mathcal{A}'_3(w, \Gamma(k_1)).
\end{aligned}$$

Observe that if $A \in \mathcal{A}'_3(w, \Gamma(k_1))$, then we must have $(p, k_1+1) \notin A$; if not, then A cannot contain a quantum step (k_1, k_2+1). However, we can define an involution on $\mathcal{A}'_3(w, \Gamma(k_1))$ similar to $\iota_{\mathcal{A}_3}$ as follows. We set

$$\begin{aligned}
\mathcal{A}'^{C,1}_3(w, \Gamma(k_1)) &:= \{A \in \mathcal{A}'^C_3(w, \Gamma(k_1)) \mid (p, k_1+1) \in A\}, \\
\mathcal{A}'^{C,2}_3(w, \Gamma(k_1)) &:= \{A \in \mathcal{A}'^C_3(w, \Gamma(k_1)) \mid (p, k_1+1) \notin A\}.
\end{aligned}$$

Then we can define an involution ι on $\mathcal{A}'^C_3(w, \Gamma(k_1))$ by

$$A \in \mathcal{A}_3'^{C,2}(w, \Gamma(k_1)) \;\mapsto\; \iota(A) := A \sqcup \{(p, k_1+1)\} \in \mathcal{A}_3'^{C,1}(w, \Gamma(k_1)),$$
$$A \in \mathcal{A}_3'^{C,1}(w, \Gamma(k_1)) \;\mapsto\; \iota(A) := A \setminus \{(p, k_1+1)\} \in \mathcal{A}_3'^{C,2}(w, \Gamma(k_1)).$$

This ι has the following properties:

- $\mathrm{end}(w, \iota(A)) = \mathrm{end}(w, A)(p, k_1)$ (and hence $\lfloor \mathrm{end}(w, \iota(A)) \rfloor = \lfloor \mathrm{end}(w, A) \rfloor$);
- $|\iota(A)| = |A| \pm 1$.

By using the involution ι, we obtain the following.

Lemma 54. *Assume condition* (Q). *If* $w(1) < w(k_2+1)$, *and if condition* (Q-A) *does not hold, then*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_3'^{C}(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0.$$

It remains to examine cancellations for the set

$$\mathcal{A}_{23}'(w, \Gamma(k_1)) := \mathcal{A}_2'(w, \Gamma(k_1)) \sqcup \mathcal{A}_3'(w, \Gamma(k_1)).$$

The desired involution on $\mathcal{A}_{23}'(w, \Gamma(k_1))$ is given as follows:

$$\begin{aligned} &A \in \mathcal{A}_2'(w, \Gamma(k_1)) \\ &\qquad \mapsto\; \iota(A) := (A \setminus \{(p, k_2+1), (p, k_1+1)\}) \sqcup \{(k_1, k_1+1)\} \in \mathcal{A}_3'(w, \Gamma(k_1)), \\ &A \in \mathcal{A}_3'(w, \Gamma(k_1)) \\ &\qquad \mapsto\; \iota(A) := (A \setminus \{(k_1, k_1+1)\}) \sqcup \{(p, k_2+1), (p, k_1+1)\} \in \mathcal{A}_2'(w, \Gamma(k_1)). \end{aligned}$$

This ι has the following properties:

- $\mathrm{end}(w, \iota(A)) = \mathrm{end}(w, A)(p, k_1)$ (and hence $\lfloor \mathrm{end}(w, \iota(A)) \rfloor = \lfloor \mathrm{end}(w, A) \rfloor$);
- $|\iota(A)| = |A| \pm 1$.

By using the involution ι, we obtain the following.

Lemma 55. *Assume condition* (Q). *If* $w(1) < w(k_2+1)$, *and if condition* (Q-A) *does not hold, then*

$$\begin{aligned} &\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_2'(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] \\ &\qquad + \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_3'(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0. \end{aligned}$$

Proof of Theorem 31 (2) (a). By Lemmas 50, 52, 53, 54, and 55, we have

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_2(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] + \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_3(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0\,.$$

Also, since $w(1) < w(k_2+1) < w(k_1+1)$, Remark 43 implies that

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_1(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0\,.$$

These observations, together with Lemma 38, prove the desired cancellation-free identity. □

Remark 56. In the proof of Theorem 31 (2) (a), we do not use the assumption that $w(k_1) > w(n)$.

Now, we assume that $w(k_1) > w(n)$ and $w(k_2+1) < w(1) < w(k_1+1)$, which are the assumptions of Theorem 31 (2) (b); note that $w(1) < w(k_1+1) < w(k_1)$ by condition (Q), and hence $k_1 \neq 1$. In this case, the same proof as that of Lemma 52 yields the following.

Lemma 57. *Assume condition* (Q). *If* $w(k_2+1) < w(1) < w(k_1+1)$, *then*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_3(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0\,.$$

Remark 58. We do not need the assumption that $w(k_1) > w(n)$ for Lemma 57.

In contrast, the sum over $\mathcal{A}_2(w,\Gamma(k_1))$ is cancellation-free. Indeed, let $A \in \mathcal{A}_2(w,\Gamma(k_1))$, and set $y := \mathrm{end}(w, A \setminus \{(k_1, k_2+1)\})$. Note that $A \setminus \{(k_1, k_2+1)\}$ contains only Bruhat steps. Hence we see that $y(k_2+1) < y(1)$ and $y(n) < y(k_1)$. Therefore, if we set $z := y(k_1, k_2+1) = \mathrm{end}(w, A)$, then

- $z(k_1) < z(1) < z(2) < \cdots < z(k_1-1)$,
- $z(k_1+1) < z(k_1+2) < \cdots < z(k_2)$, and
- $z(k_2+2) < z(k_2+3) < \cdots < z(n) < z(k_2+1)$;

hence, if we take cyclic permutations $\sigma_1 := (1, k_1, k_1-1, \ldots, 2)$ and $\sigma_2 := (k_2+1, k_2+2, \ldots, n)$ (if $k_2+1 = n$, then we take $\sigma_2 := e$), then we have $\lfloor \mathrm{end}(w,A) \rfloor = \mathrm{end}(w,A)\sigma_1\sigma_2$. Note that these σ_1 and σ_2 do not depend on the choice of A. Thus, for $A, B \in \mathcal{A}_2(w,\Gamma(k_1))$ with $A \neq B$, it follows that

$$\lfloor \mathrm{end}(w, A) \rfloor = \mathrm{end}(w, A)\sigma_1\sigma_2 \neq \mathrm{end}(w, B)\sigma_1\sigma_2 = \lfloor \mathrm{end}(w, B) \rfloor .$$

This, together with Remark 8, proves the following.

Lemma 59. *Assume condition* (Q)*. If $w(k_1) > w(n)$ and $w(k_2 + 1) < w(1) < w(k_1 + 1)$, then the sum*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}]$$

is cancellation-free.

Remark 60. If $\mathcal{A}_2(w, \Gamma(k_1)) \neq \emptyset$ (or equivalently, if condition (Q) holds), then equation (14) shows that

$$\begin{aligned} &\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_2(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] \\ &= -\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_{\lessdot}(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} Q_{k_2} [\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1, k_2+1) \rfloor}] . \end{aligned}$$

Proof of Theorem 31 (2) (b). Since $w(1) < w(k_1 + 1)$, Remark 43 implies that

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_1(w, \Gamma(k_1))} (-1)^{|A|} Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] = 0 .$$

Hence, by Lemmas 38, 57, 59, and Remark 60, we obtain the desired cancellation-free formula. □

It only remains to prove Theorem 31 (3). To do so, we assume condition (Full). By the same argument as in the proof of Lemma 59, we see that for $A \in \mathcal{A}_2(w, \Gamma(k_1))$, $\lfloor \mathrm{end}(w, A) \rfloor = \mathrm{end}(w, A)\sigma_1\sigma_2$, where σ_1 and σ_2 are the cyclic permutations defined above (if $k_1 = 1$, then we take $\sigma_1 := e$). In addition, since $w(k_1 + 1) < w(1)$ by condition (Full) (2), it follows that $\mathrm{end}(w, A)(k_1) < \mathrm{end}(w, A)(k_1 + 1) < \mathrm{end}(w, A)(1)$ (if $k_1 = 1$, then we need only the inequality $\mathrm{end}(w, A)(k_1 + 1) < \mathrm{end}(w, A)(1)$). Therefore, for $A \in \mathcal{A}_3(w, \Gamma(k_1))$ (note that $A \setminus \{(k_1, k_1 + 1)\} \in \mathcal{A}_2(w, \Gamma(k_1))$), the following hold:

- $\mathrm{end}(w, A)(k_1) < \mathrm{end}(w, A)(1) < \mathrm{end}(w, A)(2) < \cdots < \mathrm{end}(w, A)(k_1 - 1)$,
- $\mathrm{end}(w, A)(k_1 + 1) < \mathrm{end}(w, A)(k_1 + 2) < \cdots < \mathrm{end}(w, A)(k_2)$, and
- $\mathrm{end}(w, A)(k_2 + 2) < \mathrm{end}(w, A)(k_2 + 3) < \cdots < \mathrm{end}(w, A)(n) < \mathrm{end}(w, A)(k_2 + 1)$.

Thus we conclude that $\lfloor \mathrm{end}(w, A) \rfloor = \mathrm{end}(w, A)\sigma_1\sigma_2$, and hence that for $A, B \in \mathcal{A}_2(w, \Gamma(k_1)) \sqcup \mathcal{A}_3(w, \Gamma(k_1))$ with $A \neq B$,

$$\lfloor \mathrm{end}(w, A) \rfloor = \mathrm{end}(w, A)\sigma_1\sigma_2 \neq \mathrm{end}(w, B)\sigma_1\sigma_2 = \lfloor \mathrm{end}(w, B) \rfloor .$$

This, together with Remark 8, proves the following.

Lemma 61. *Assume conditions* (Q) *and* (Full). *Then the sum*

$$\mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_2(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}Q_{k_2}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)\rfloor}] + \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_3(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}Q_{k_2}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)\rfloor}]$$

is cancellation-free.

Remark 62. If $\mathcal{A}_3(w,\Gamma(k_1)) \neq \emptyset$ (or equivalently, if condition (Q) holds), then equation (15) shows that

$$\begin{aligned}\mathbf{e}^{w\varpi_{k_1}} &\sum_{A\in\mathcal{A}_3(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}Q_{k_2}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)\rfloor}] \\ &= \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}Q_{k_2}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1,k_2+1)s_{k_1}\rfloor}].\end{aligned}$$

Therefore, by Remark 60, we deduce that

$$\begin{aligned}&\mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_2(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}Q_{k_2}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)\rfloor}] \\ &+ \mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_3(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}Q_{k_2}[\mathcal{O}^{\lfloor \mathrm{end}(w,A)\rfloor}] \\ &= -\mathbf{e}^{w\varpi_{k_1}} \sum_{A\in\mathcal{A}_{\lessdot}(w,\Gamma(k_1))} (-1)^{|A|} Q_{k_1}Q_{k_2} \left([\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1,k_2+1)\rfloor}] - [\mathcal{O}^{\lfloor \mathrm{end}(w,A)(k_1,k_2+1)s_{k_1}\rfloor}]\right).\end{aligned}$$

Proof of Theorem 31 (3). The desired identity follows from Lemmas 38, 61 and Remarks 46, 47, 62. □

4.5. The positivity property

We prove the positivity property of structure constants for two-step flag manifolds in type A, which is a corollary of Chevalley formulas (Theorems 27, 29, 31, 32, 33, and 35).

Corollary 63. *Let G be of type A_{n-1}, $J = I \setminus \{k_1, k_2\}$ for arbitrarily fixed $1 \leq k_1 < k_2 \leq n-1$, and $k = k_1$ or $k = k_2$. Then, for $w, u \in W^J$ and $\xi \in Q^{\vee,+}_{I\setminus J}$, we have*

$$(-1)^{1+\ell(w)+\ell(u)+\deg(Q^{\xi})} N^{u,\xi}_{s_k,w} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^{\gamma} - 1 \mid \gamma \in -\Delta].$$

Proof. We give a proof of Corollary 63 under the assumptions of Theorems 27, 29 and 31. The positivity property under the assumptions of Theorems 32, 33, and 35 follows by the same arguments as those for Theorems 27, 29 and 31. Note that the positivity property under the assumptions of Theorems 27, 29 (1), 31 (1) (a) and (2) (a) has already been known because of the positivity property of $N_{u,v}^{w,0}$ for $u, v, w \in W^J$. First, it is easy to check (see, for example, [9, Section 3.1.5, Exercise 4]) that

$$2\rho_J = \sum_{i=1}^{k_1-1} i(k_1 - i)\alpha_i + \sum_{i=1}^{k_2-k_1-1} i(k_2 - k_1 - i)\alpha_{k_1+i} + \sum_{i=1}^{n-k_2-1} i(n - k_2 - i)\alpha_{k_2+i}.$$

Since

$$\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} 2\langle \rho_J, \alpha_{k_1}^\vee \rangle &= -(k_1 - 1) - (k_2 - k_1 - 1) = 2 - k_2, \\ 2\langle \rho_J, \alpha_{k_2}^\vee \rangle &= -(k_2 - k_1 - 1) - (n - k_2 - 1) = 2 - n + k_1. \end{aligned}$$

In addition, we know that $2\langle \rho, \alpha_{k_1}^\vee \rangle = 2$. Therefore,

$$\begin{aligned} \deg(Q_{k_1}) &= 2\langle \rho - \rho_J, \alpha_{k_1}^\vee \rangle = -k_2 \\ \deg(Q_{k_2}) &= 2\langle \rho - \rho_J, \alpha_{k_2}^\vee \rangle = n - k_1, \end{aligned}$$

and hence

$$\deg(\underbrace{Q_{k_1} Q_{k_2}}_{=Q^{\alpha_{k_1}^\vee + \alpha_{k_2}^\vee}}) = 2\langle \rho - \rho_J, \alpha_{k_1}^\vee + \alpha_{k_2}^\vee \rangle = n - k_1 + k_2.$$

Let us consider the structure constants $N_{s_{k_1},w}^{u,\xi}$ with $\xi \neq 0$ under the assumptions of Theorems 29 (2) and 31 (1) (b). We maintain the setting of Lemma 45 except for the negation of condition (Q) (see Remark 46). Take $A \in \mathcal{A}_1(w, \Gamma(k_1))$, and set $u := \lfloor \mathrm{end}(w, A) \rfloor$, $u_0 := \mathrm{end}(w, A)$. Then, by the proof of Lemma 45, we have $u = u_0 \sigma_1 \sigma_2$, where $\sigma_1 = (1, k_1, k_1 - 1, \ldots, 2)$ and $\sigma_2 = (k_1 + 1, k_1 + 2, \ldots, k_2)$. Therefore, we see that

$$\begin{aligned} &(-1)^{|A|} \mathbf{e}^{w\varpi_{k_1}} Q_{k_1} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] \\ &= (-1)^{\ell(u_0) - \ell(w)} \mathbf{e}^{w\varpi_{k_1}} Q_{k_1} [\mathcal{O}^u] \\ &= (-1)^{(\ell(u_0\sigma_1\sigma_2) - \ell(\sigma_1) - \ell(\sigma_2)) - \ell(w)} \mathbf{e}^{w\varpi_{k_1}} Q_{k_1} [\mathcal{O}^u] \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\ell(u)-(k_1-1)-(k_2-k_1-1)-\ell(w)}\mathbf{e}^{w\varpi_{k_1}}Q_{k_1}[\mathcal{O}^u] \\
&= (-1)^{\ell(w)+\ell(u)+k_2}\mathbf{e}^{w\varpi_{k_1}}Q_{k_1}[\mathcal{O}^u] \\
&= (-1)^{\ell(w)+\ell(u)+\deg(Q_{k_1})}\mathbf{e}^{w\varpi_{k_1}}Q_{k_1}[\mathcal{O}^u].
\end{aligned}
$$

We set

$$\mathcal{A}_1(w,\Gamma(k_1))_u := \{A \in \mathcal{A}_1(w,\Gamma(k_1)) \mid \lfloor \mathrm{end}(w,A) \rfloor = u\}$$

for $u \in W^J$. Then, for $u \in W^J$, we deduce from Theorems 29 (2) and 31 (1) (b) that

$$C_w^{u,\alpha_{k_1}^\vee} = \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_1(w,\Gamma(k_1))_u} (-1)^{|A|} = (-1)^{\ell(u)+\ell(w)+\deg(Q_{k_1})}|\mathcal{A}_1(w,\Gamma(k_1))_u|\mathbf{e}^{w\varpi_{k_1}},$$

and hence that

$$N_{s_{k_1},w}^{u,\alpha_{k_1}^\vee} = (-1)^{1+\ell(w)+\ell(u)+\deg(Q_{k_1})}|\mathcal{A}_1(w,\Gamma(k_1))_u|\mathbf{e}^{w\varpi_{k_1}-\varpi_{k_1}}.$$

Since $w\varpi_{k_1} - \varpi_{k_1} \in -Q^+$ and hence $\mathbf{e}^{w\varpi_{k_1}-\varpi_{k_1}} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^\gamma - 1 \mid \gamma \in -\Delta]$, we conclude that

$$(-1)^{1+\ell(w)+\ell(u)+\deg(Q_{k_1})} N_{s_{k_1},w}^{u,\alpha_{k_1}^\vee} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^\gamma - 1 \mid \gamma \in -\Delta], \tag{17}$$

as desired. Equation (17), together with the positivity property of $N_{u,v}^{w,0}$ for $u, v, w \in W^J$, implies Corollary 63 under the assumptions of Theorems 29 (2) and 31 (1) (b).

Next, we consider the structure constants $N_{s_{k_1},w}^{u,\xi}$ with $\xi \neq 0$ under the assumption of Theorem 31 (2) (b). We maintain the setting of Lemma 59. Take $A \in \mathcal{A}_2(w,\Gamma(k_1))$, and set $u := \lfloor \mathrm{end}(w,A) \rfloor$, $u_0 := \mathrm{end}(w,A)$. Then, by the proof of Lemma 59, we have $u = u_0\sigma_1\sigma_2$, where $\sigma_1 = (1, k_1, k_1 - 1, \dots, 2)$ and $\sigma_2 = (k_2 + 1, k_2 + 2, \dots, n)$. Therefore, we see that

$$
\begin{aligned}
&(-1)^{|A|}\mathbf{e}^{w\varpi_{k_1}}Q_{k_1}Q_{k_2}[\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] \\
&= (-1)^{\ell(u_0)-\ell(w)}\mathbf{e}^{w\varpi_{k_1}}Q_{k_1}Q_{k_2}[\mathcal{O}^u] \\
&= (-1)^{(\ell(u_0\sigma_1\sigma_2)-\ell(\sigma_1)-\ell(\sigma_2))-\ell(w)}\mathbf{e}^{w\varpi_{k_1}}Q_{k_1}Q_{k_2}[\mathcal{O}^u] \\
&= (-1)^{\ell(u)-(k_1-1)-(n-k_2-1)-\ell(w)}\mathbf{e}^{w\varpi_{k_1}}Q_{k_1}Q_{k_2}[\mathcal{O}^u] \\
&= (-1)^{\ell(w)+\ell(u)+(n-k_1+k_2)}\mathbf{e}^{w\varpi_{k_1}}Q_{k_1}Q_{k_2}[\mathcal{O}^u] \\
&= (-1)^{\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2})}\mathbf{e}^{w\varpi_{k_1}}Q_{k_1}Q_{k_2}[\mathcal{O}^u].
\end{aligned}
$$

We set

$$\mathcal{A}_2(w,\Gamma(k_1))_u := \{A \in \mathcal{A}_2(w,\Gamma(k_1)) \mid \lfloor \mathrm{end}(w,A) \rfloor = u\}$$

for $u \in W^J$. Then, for $u \in W^J$, we deduce from Theorem 31 (2) (b) that

$$\begin{aligned} C_w^{u,\alpha_{k_1}^\vee+\alpha_{k_2}^\vee} &= \mathbf{e}^{w\varpi_{k_1}} \sum_{A \in \mathcal{A}_2(w,\Gamma(k_1))_u} (-1)^{|A|} \\ &= (-1)^{\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2})} |\mathcal{A}_2(w,\Gamma(k_1))_u| \mathbf{e}^{w\varpi_{k_1}}, \end{aligned}$$

and hence that

$$N_{s_{k_1},w}^{u,\alpha_{k_1}^\vee+\alpha_{k_2}^\vee} = (-1)^{1+\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2})} |\mathcal{A}_2(w,\Gamma(k_1))_u| \mathbf{e}^{w\varpi_{k_1}-\varpi_{k_1}}. \tag{18}$$

Again, since $\mathbf{e}^{w\varpi_{k_1}-\varpi_{k_1}} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^\gamma - 1 \mid \gamma \in -\Delta]$, we conclude that

$$(-1)^{1+\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2})} N_{s_{k_1},w}^{u,\alpha_{k_1}^\vee+\alpha_{k_2}^\vee} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^\gamma - 1 \mid \gamma \in -\Delta], \tag{19}$$

as desired. Equation (19), together with the positivity property of $N_{u,v}^{w,0}$ for $u, v, w \in W^J$, implies Corollary 63 under the assumption of Theorem 31 (2) (b).

It remains to consider the structure constants $N_{s_{k_1},w}^{u,\xi}$ with $\xi \neq 0$ under the assumption of Theorem 31 (3) and consider the structure constants $N_{s_{k_1},w}^{u,\xi}$ for $\xi \neq 0$. The positivity property in the case $\xi = \alpha_{k_1}^\vee$ has already been proved by equation (17). Hence it suffices to consider the case $\xi = \alpha_{k_1}^\vee + \alpha_{k_2}^\vee$. We maintain the setting of Lemma 61. We set

$$\mathcal{A}_{23}(w,\Gamma(k_1))_u := \{A \in \mathcal{A}_2(w,\Gamma(k_1)) \sqcup \mathcal{A}_3(w,\Gamma(k_1)) \mid \lfloor \mathrm{end}(w,A) \rfloor = u\}$$

for $u \in W^J$. Then, by the same argument as that for equation (18), we deduce from Theorem 31 (3) that

$$N_{s_{k_1},w}^{u,\alpha_{k_1}^\vee+\alpha_{k_2}^\vee} = (-1)^{1+\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2})} |\mathcal{A}_{23}(w,\Gamma(k_1))_u| \mathbf{e}^{w\varpi_{k_1}-\varpi_{k_1}},$$

and hence conclude that

$$(-1)^{1+\ell(w)+\ell(u)+\deg(Q_{k_1}Q_{k_2})} N_{s_{k_1},w}^{u,\alpha_{k_1}^\vee+\alpha_{k_2}^\vee} \in \mathbb{Z}_{\geq 0}[\mathbf{e}^\gamma - 1 \mid \gamma \in -\Delta], \tag{20}$$

as desired. Equations (17), (20), together with the positivity property of $N_{u,v}^{w,0}$ for $u, v, w \in W^J$, imply Corollary 63 under the assumption of Theorem 31 (3). This completes the proof of the corollary. □

Data availability

No data was used for the research described in the article.

Appendix A. Another proof of the existence of the multiplicative surjection Φ_J

In this appendix, we mainly use the notation of Section 2.3. In addition, we set $QK_T^{\mathrm{poly}}(G/B) := K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q]$, where $\mathbb{Z}[\Lambda][Q]$ is the polynomial ring with coefficients in $\mathbb{Z}[\Lambda]$ in the (Novikov) variables $Q_i = Q^{\alpha_i^\vee}$, $i \in I$; also, for an arbitrary subset $J \subset I$, we set $QK_T^{\mathrm{poly}}(G/P_J) := K_T(G/P_J) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q_K]$, with $K := I \setminus J$, where $\mathbb{Z}[\Lambda][Q_K]$ is the polynomial ring with coefficients in $\mathbb{Z}[\Lambda]$ in the variables Q_k, $k \in K$. It is known (see [10]) that there exists a surjective $\mathbb{Z}[\Lambda]$-algebra homomorphism Φ_J from $QK_T^{\mathrm{poly}}(G/B)$ onto $QK_T^{\mathrm{poly}}(G/P_J)$ such that $\Phi_J(Q^\xi[\mathcal{O}^w]) = Q^{[\xi]^J}[\mathcal{O}_J^{\lfloor w \rfloor}]$ for $w \in W$ and $\xi \in Q^{\vee,+}$, where $[\xi]^J := \sum_{k \in I \setminus J} c_k \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$. In this appendix, based on results in [6], we give another (short) proof of the existence of such a $\mathbb{Z}[\Lambda]$-algebra homomorphism. First of all, we note that $QK_T^{\mathrm{poly}}(G/B)$ is a $\mathbb{Z}[\Lambda]$-subalgebra of $QK_T(G/B) = K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][\![Q]\!]$ by [6, Corollary 1.2].

Let us briefly recall the main result of [6]. Following [6], let Gr_G denote Pressley-Segal's model of the affine Grassmannian associated to a simple and simply-connected complex Lie group G; more precisely, let Gr_G be the space of polynomial based loops in a (fixed) maximal compact subgroup of G, equipped with an ind-variety structure (see [26, Chapter 8] for details). We denote by $K^T(Gr_G)$ the T-equivariant K-homology (in the topological sense) of the affine Grassmannian Gr_G, equipped with the Pontryagin product $\odot$ coming from the group product on the topological group Gr_G. Then, we have two bases. One is a basis (called the localization basis) $\mathcal{O}_\xi := [\mathcal{O}_{x_\xi}]$, $\xi \in Q^\vee$, of $K^T(Gr_G)$ over $\mathrm{Frac}(\mathbb{Z}[\Lambda])$, where x_ξ is the T-fixed point of Gr_G corresponding to the cocharacter of T associated to $\xi \in Q^\vee$. More precisely, if we consider the $\mathbb{Z}[\Lambda]$-algebra $\bigoplus_{\xi \in Q^\vee} \mathrm{Frac}(\mathbb{Z}[\Lambda])\mathcal{O}_\xi$ equipped with the product $\odot$ defined by $\mathcal{O}_{\xi_1} \odot \mathcal{O}_{\xi_2} := \mathcal{O}_{\xi_1+\xi_2}$, $\xi_1, \xi_2 \in Q^\vee$, then we have an injective $\mathbb{Z}[\Lambda]$-algebra homomorphism $K^T(Gr_G) \hookrightarrow \bigoplus_{\xi \in Q^\vee} \mathrm{Frac}(\mathbb{Z}[\Lambda])\mathcal{O}_\xi$ which fixes every $\mathcal{O}_\xi$. Another is indeed a basis of $K^T(Gr_G)$ over $\mathbb{Z}[\Lambda]$ given as follows. Let $W_{\mathrm{af}} = W \ltimes Q^\vee$ be the affine Weyl group of G, and let W_{af}^0 denote the set of minimal-length coset representatives for W_{af}/W. We know from [23, Section 3] that an element $wt_\xi \in W_{\mathrm{af}}$, with $w \in W$ and $\xi \in Q^\vee$, lies in W_{af}^0 if and only if $\xi \in Q^\vee$ is anti-dominant and w is of minimal length in its coset wW_ξ in W/W_ξ, where $W_\xi \subset W$ is the stabilizer of ξ in W; note that if $\xi \in Q^\vee$ is anti-dominant, then $\xi \in -Q^{\vee,+}$. In particular, if $\xi \in Q^\vee$ is regular anti-dominant, then $wt_\xi \in W_{\mathrm{af}}^0$ for all $w \in W$. For each $wt_\xi \in W_{\mathrm{af}}^0$, there exists a complex cell (called an affine Schubert cell) in Gr_G containing the T-fixed point $x_{w\xi} \in Gr_G$ of finite dimension; the class of the structure sheaf of the Zariski closure of this cell is denoted by $\mathcal{O}_{wt_\xi}$, and is called the affine Schubert class associated to $wt_\xi \in W_{\mathrm{af}}^0$. Then we know that the classes $\mathcal{O}_{wt_\xi}$, $wt_\xi \in W_{\mathrm{af}}^0$, form a $\mathbb{Z}[\Lambda]$-basis of $K^T(Gr_G)$.

Now the main result of [6] is stated as follows.

Theorem 64 *([6, Theorem 1.1]). Let J be an arbitrary subset of I. Then, there exists a $\mathbb{Z}[\Lambda]$-algebra homomorphism*

$$\Psi_J : K^T(Gr_G) \to QK_T(G/P_J)[(Q^{\vee,+})^{-1}],$$

where $QK_T(G/P_J)[(Q^{\vee,+})^{-1}]$ denotes the localization of $QK_T(G/P_J)$ with respect to the monomials in the Novikov variables $Q_i = Q^{\alpha_i^\vee}$, $i \in I$. Moreover, $\Psi_J(\mathcal{O}_{wt_\xi}) = Q^{[\xi]^J}[\mathcal{O}_J^{\lfloor w \rfloor}]$ for each $wt_\xi \in W_{\mathrm{af}}^0$, where $[\xi]^J = \sum_{k \in I \setminus J} c_k \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^\vee$, and $[\mathcal{O}_J^{\lfloor w \rfloor}]$ denotes the (opposite) Schubert class in $K_T(G/P_J)$ associated to the minimal-length coset representative $\lfloor w \rfloor \in W^J$ for the coset wW_J in W/W_J.

Note that in the case $J = \emptyset$, i.e., $P_J = B$, the $\mathbb{Z}[\Lambda]$-algebra homomorphism $\Psi := \Psi_\emptyset$ is injective since the affine Schubert classes $\mathcal{O}_{wt_\xi}$, $wt_\xi \in W_{\mathrm{af}}^0$, form a $\mathbb{Z}[\Lambda]$-basis of $K^T(Gr_G)$ and $\Psi([\mathcal{O}_{wt_\xi}]) = Q^\xi[\mathcal{O}^w]$.

We will construct a surjective $\mathbb{Z}[\Lambda]$-algebra homomorphism Φ_J from $QK_T^{\mathrm{poly}}(G/B)$ to $QK_T^{\mathrm{poly}}(G/P_J)$ such that $\Phi_J(Q^\xi[\mathcal{O}^w]) = Q^{[\xi]^J}[\mathcal{O}_J^{\lfloor w \rfloor}]$ for $w \in W$ and $\xi \in Q^{\vee,+}$, where $[\xi]^J = \sum_{k \in I \setminus J} c_k \alpha_k^\vee$ for $\xi = \sum_{i \in I} c_i \alpha_i^\vee \in Q^{\vee,+}$. We first note that for each element $v \in QK_T^{\mathrm{poly}}(G/B) = K_T(G/B) \otimes_{\mathbb{Z}[\Lambda]} \mathbb{Z}[\Lambda][Q]$, there exists a sufficiently regular anti-dominant coroot $\eta \in -Q^{\vee,+}$ such that $Q^\eta v \in QK_T(G/B)[(Q^{\vee,+})^{-1}]$ lies in the image of the map Ψ, i.e., $Q^\eta v \in \Psi(K^T(Gr_G))$; by the injectivity of Ψ, there exists a unique $u \in K^T(Gr_G)$ such that $\Psi(u) = Q^\eta v$. Indeed, we may assume that $v = Q^\xi[\mathcal{O}^w]$ for some $w \in W$ and $\xi \in Q^{\vee,+}$ since each $v \in QK_T^{\mathrm{poly}}(G/B)$ is, by its definition, a finite linear combination with coefficients in $\mathbb{Z}[\Lambda]$ of such elements. Hence we can take a sufficiently regular anti-dominant coroot $\eta \in Q^\vee$ such that $\xi + \eta \in Q^\vee$ is also regular anti-dominant; note that we have $\eta \in -Q^{\vee,+}$ since $\eta \in Q^\vee$ is anti-dominant. We set $u := \mathcal{O}_{wt_{\xi+\eta}}$, where $wt_{\xi+\eta}$ lies in W_{af}^0 since $\xi + \eta \in Q^\vee$ is regular anti-dominant. Then it follows that $\Psi(u) = Q^\eta v$ by Theorem 64. Now we define $\Phi_J(v) := Q^{[-\eta]^J}\Psi_J(u) \in QK_T(G/P_J)$. We can easily verify that the element $Q^{[-\eta]^J}\Psi_J(u)$ does not depend on the choice of (a sufficiently regular anti-dominant coroot) $\eta \in -Q^{\vee,+}$, and hence that Φ_J is a well-defined surjective $\mathbb{Z}[\Lambda]$-module homomorphism from $QK_T^{\mathrm{poly}}(G/B)$ onto $QK_T^{\mathrm{poly}}(G/P_J)$. Indeed, if $v = Q^\xi[\mathcal{O}^w]$ with $w \in W$ and $\xi \in Q^{\vee,+}$, then $\Phi_J(Q^\xi[\mathcal{O}^w]) = Q^{[\xi]^J}[\mathcal{O}_J^{\lfloor w \rfloor}]$.

Also, for $v_1, v_2 \in QK_T^{\mathrm{poly}}(G/B)$, we can take sufficiently regular anti-dominant coroots $\eta_1, \eta_2 \in -Q^{\vee,+}$ such that $Q^{\eta_1} v_1, Q^{\eta_2} v_2 \in \Psi(K^T(Gr_G))$; hence there exist uniquely $u_1, u_2 \in K^T(Gr_G)$ such that $\Psi(u_1) = Q^{\eta_1} v_1$ and $\Psi(u_2) = Q^{\eta_2} v_2$. Since $\Psi = \Psi_\emptyset$ is a $\mathbb{Z}[\Lambda]$-algebra homomorphism, we have $Q^{\eta_1+\eta_2}(v_1 \cdot v_2) = (Q^{\eta_1} v_1) \cdot (Q^{\eta_2} v_2) = \Psi(u_1) \cdot \Psi(u_2) = \Psi(u_1 \odot u_2)$ in $QK_T(G/B)[(Q^{\vee,+})^{-1}]$, where $u_1 \odot u_2 \in K^T(Gr_G)$. Therefore, we see that $\Phi_J(v_1 \cdot v_2) = Q^{[-\eta_1-\eta_2]^J}\Psi_J(u_1 \odot u_2) = Q^{[-\eta_1-\eta_2]^J}(\Psi_J(u_1) \cdot \Psi_J(u_2)) = (Q^{[-\eta_1]^J}\Psi_J(u_1)) \cdot (Q^{[-\eta_2]^J}\Psi_J) = \Phi_J(v_1) \cdot \Phi_J(v_2)$ in $QK_T(G/P_J)[(Q^{\vee,+})^{-1}]$ since Ψ_J is a $\mathbb{Z}[\Lambda]$-algebra homomorphism. This proves that the map Φ_J is a $\mathbb{Z}[\Lambda]$-algebra homomorphism from $QK_T^{\mathrm{poly}}(G/B)$ to $QK_T^{\mathrm{poly}}(G/P_J)$, as desired.

Finally, since $[\mathcal{O}^{s_i}] = 1 - \mathbf{e}^{-\varpi_i}[\mathcal{O}_{G/B}(-\varpi_i)]$ in $K_T(G/B)$ for all $i \in I$ and $[\mathcal{O}^{s_k}] = 1 - \mathbf{e}^{-\varpi_k}[\mathcal{O}_{G/P_J}(-\varpi_k)]$ in $K_T(G/P_J)$ for all $k \in K = I \setminus J$, it follows that $\Phi_J([\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_{G/P_J}(-\varpi_k)]$, and hence that $\Phi_J([\mathcal{O}^w] \cdot [\mathcal{O}_{G/B}(-\varpi_k)]) = \Phi_J([\mathcal{O}^w]) \cdot \Phi_J([\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_J^{\lfloor w \rfloor}] \cdot [\mathcal{O}_{G/P_J}(-\varpi_k)]$ for all $k \in K = I \setminus J$.

Thus, we have given a proof of the following fact; cf. Theorem 6, due to Kato ([10]).

Corollary 65. *Let J be an arbitrary subset of I. Then, there exists a surjective $\mathbb{Z}[\Lambda]$-algebra homomorphism*

$$\Phi_J : QK_T^{\mathrm{poly}}(G/B) \to QK_T^{\mathrm{poly}}(G/P_J)$$

such that $\Phi_J(Q^\xi[\mathcal{O}^w]) = Q^{[\xi]^J}[\mathcal{O}_J^{\lfloor w \rfloor}]$ for $w \in W$ and $\xi \in Q^{\vee,+}$. Also, for each $k \in K = I \setminus J$, the following equality holds for all $w \in W$:

$$\Phi_J([\mathcal{O}^w] \cdot [\mathcal{O}_{G/B}(-\varpi_k)]) = [\mathcal{O}_J^{\lfloor w \rfloor}] \cdot [\mathcal{O}_{G/P_J}(-\varpi_k)].$$

Appendix B. Weihong Xu's conjecture about a cancellation-free parabolic Chevalley formula in type A (with Weihong Xu)

In this appendix, we mention the relation between our results and a conjecture due to Weihong Xu, which is expected to be a cancellation-free Chevalley formula in type A for an arbitrary subset $J \subset I$.

Let G be of type A_{n-1}. Take $1 \le k_1 < k_2 < \cdots < k_m \le n-1$, and set $J := I \setminus \{k_1, \dots, k_m\}$. In this case, the partial flag manifold G/P_J is isomorphic to the m-step flag manifold $\mathrm{Fl}(k_1, \dots, k_m; n)$, defined as:

$$\mathrm{Fl}(k_1, \dots, k_m; n) := \left\{ (V_1, \dots, V_m) \;\middle|\; \begin{array}{l} V_l,\, l = 1, \dots, m, \text{ is a subspace of } \mathbb{C}^n \text{ such that} \\ \dim V_l = k_l, \text{ and } V_1 \subset V_2 \subset \cdots \subset V_m \end{array} \right\}.$$

For a directed path

$$\mathbf{p} : w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_r} w_r$$

in $\mathrm{QB}(W)$, we define $\ell(\mathbf{p}) \ge 0$, $\mathrm{end}(\mathbf{p}) \in W$, and $\mathrm{wt}(\mathbf{p}) \in Q^{\vee,+}$ by

$$\begin{aligned} \ell(\mathbf{p}) &:= r, \\ \mathrm{end}(\mathbf{p}) &:= w_r, \\ \mathrm{wt}(\mathbf{p}) &:= \sum_{\substack{1 \le k \le r \\ w_{k-1} \to w_k \text{is a quantum edge}}} \gamma_k^\vee. \end{aligned}$$

Also, for $1 \le a \le n-1$, the quantum a-Bruhat graph $\mathrm{QB}_a(W)$ is defined to be the subgraph of $\mathrm{QB}(W)$ having only those edges whose labels are of the form (i, j) such that $i \le a < j$. In addition, we define a total order $\lhd$ on Φ^+ as follows: for $1 \le i < j \le n$ and $1 \le k < l \le n$, we define $(i, j) \lhd (k, l)$ if $(j > l)$ or ($j = l$ and $i < k$).

Xu formulated the following conjecture on a cancellation-free Chevalley formula for $QK(G/P_J)$, the non-equivariant quantum K-theory of G/P_J, and checked it for all partial flag manifolds with $n \le 8$ and $m \le 4$ using a computer program.

Conjecture 66 *(Weihong Xu). In $QK(G/P_J)$, for $w \in W^J$, the following cancellation-free formula holds:*

$$[\mathcal{O}^{s_{k_l}}] \cdot [\mathcal{O}^w] = \sum_{\mathbf{p}} (-1)^{\ell(\mathbf{p})-1} Q^{[\mathrm{wt}(\mathbf{p})]^J} [\mathcal{O}^{\lfloor \mathrm{end}(\mathbf{p}) \rfloor}], \tag{21}$$

where the sum on the right-hand side is over all non-empty paths $\mathbf{p}$ in $\mathrm{QB}_{k_l}(W)$ of the form

$$\mathbf{p} : w = w_0 \xrightarrow{(i_1,j_1)} w_1 \xrightarrow{(i_2,j_2)} \cdots \xrightarrow{(i_r,j_r)} w_r$$

such that

(1) $(i_1, j_1) \lhd (i_2, j_2) \lhd \cdots \lhd (i_r, j_r)$,
(2) *for each $0 \le t \le r$ (regarding as $k_0 = 0$ and $k_{n+1} = n$) and an edge $v \xrightarrow{(i,j)} w$ in $\mathbf{p}$,*
   - *there does not exist any path of the form $v \xrightarrow{(i,j')} w'$ in $\mathrm{QB}_{k_l}(W)$ such that $k_t + 1 \le j < j' \le k_{t+1}$,*
   - *there does not exist any path of the form $v \xrightarrow{(i',j)} w'$ in $\mathrm{QB}_{k_l}(W)$ such that $k_t + 1 \le i' < i \le k_{t+1}$,*
(3) *if there are two edges $\xrightarrow{(i,j)}$ and $\xrightarrow{(i,j')}$ in $\mathbf{p}$ such that $(i,j) \lhd (i,j')$, then there exists $1 \le t \le n-1$ such that $j' \le k_t < j$,*
(4) *if there are two edges $\xrightarrow{(i,j)}$ and $\xrightarrow{(i',j)}$ in $\mathbf{p}$ such that $(i,j) \lhd (i',j)$, then there exists $1 \le t \le n-1$ such that $i \le k_t < i'$.*

We now compare Xu's conjectural formula in the case $m = 2$ with our cancellation-free Chevalley formula for two-step flag manifolds. For $w \in W^J$, we obtain the following formula in $QK(G/P_J)$ by applying the surjection Φ_J to equation (4) and specializing at $\mathbf{e}^\mu = 1$ for $\mu \in \Lambda$:

$$\begin{aligned}[\mathcal{O}(-\varpi_{k_1})] \cdot [\mathcal{O}^w] &= \sum_{A \in \mathcal{A}(w, \Gamma(k_1))} (-1)^{|A|} Q^{[\mathrm{down}(w,A)]^J} [\mathcal{O}^{\lfloor \mathrm{end}(w,A) \rfloor}] \\ &= \sum_{\mathbf{p}} (-1)^{\ell(\mathbf{p})} Q^{[\mathrm{wt}(\mathbf{p})]^J} [\mathcal{O}^{\lfloor \mathrm{end}(\mathbf{p}) \rfloor}],\end{aligned}$$

where the sum $\sum_{\mathbf{p}}$ is over all (possibly empty) directed paths in $\mathrm{QB}_{k_1}(W)$ satisfying (1) in Conjecture 66. By the formula $[\mathcal{O}^{s_{k_1}}] = 1 - [\mathcal{O}(-\varpi_{k_1})]$ in $QK(G/P_J)$, we deduce that

$$[\mathcal{O}^{s_{k_1}}] \cdot [\mathcal{O}^w] = \sum_{\mathbf{p}} (-1)^{\ell(\mathbf{p})-1} Q^{[\mathrm{wt}(\mathbf{p})]^J} [\mathcal{O}^{\lfloor \mathrm{end}(\mathbf{p}) \rfloor}],$$

where the sum $\sum_{\mathbf{p}}$ is over all non-empty directed paths in $\mathrm{QB}_{k_1}(W)$ satisfying (1) in Conjecture 66. Here, we can construct certain involutions among non-empty directed paths satisfying (1) but not (2), and those satisfying (1) but not (3) or (4). Furthermore, we can verify that such involutions agree with those constructed in Section 4 by direct calculation. Hence we deduce that equation (21) coincides with our cancellation-free Chevalley formula (Theorems 27, 29, and 31). We can also consider the product $[\mathcal{O}^{s_{k_2}}] \cdot [\mathcal{O}^{w}]$ by using the diagram automorphism ω and the result above for the product $[\mathcal{O}^{s_{k_1}}] \cdot [\mathcal{O}^{w}]$. In addition, we can verify that Xu's conjectural formula (21) coincides with our Chevalley formula for Grassmannians of type A (Theorem 15) in the same way as above.

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