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A Central Limit Theorem with Explicit Lyapunov Exponent and Variance for Products of 2 × 2 Random Non-invertible Matrices

Audrey Benson[1] · Hunter Gould[1] · Phanuel Mariano[1] · Grace Newcombe[1] · Joshua Vaidman[1]

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Abstract

The theory of products of random matrices and Lyapunov exponents have been widely studied and applied in the fields of biology, dynamical systems, economics, engineering and statistical physics. We consider the product of an i.i.d. sequence of 2×2 random non-invertible matrices with real entries. Given some mild moment assumptions we prove an explicit formula for the Lyapunov exponent and prove a central limit theorem with an explicit formula for the variance in terms of the entries of the matrices. We also give examples where exact values for the Lyapunov exponent and variance are computed. An important example where non-invertible matrices are essential is the random Hill's equation, which has numerous physical applications, including the astrophysical orbit problem.

Keywords Products of random matrices · Lyapunov exponent · Central limit theorem · CLT · m-Dependent sequences · Non-invertible matrices · Singular matrices

Mathematics Subject Classification Primary 37H15 , 60F05 · Secondary 60B20 , 60G10 , 34F05

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✉ Phanuel Mariano
marianop@union.edu

Audrey Benson
bensona2@union.edu

Hunter Gould
gouldh2@union.edu

Grace Newcombe
newcombg@union.edu

Joshua Vaidman
vaidmanj@union.edu

[1] Department of Mathematics, Union College, Schenectady, NY 12308, USA

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1 Introduction and Main Results

Let $Y_1, Y_2, Y_3, \ldots$ be i.i.d. random matrices from the ring $M_2(\mathbb{R})$ of 2×2 matrices with real entries. Let $S_n = Y_n \cdots Y_2 Y_1$ to be their matrix product and define the (top) **Lyapunov exponent** to be

$$\lambda := \lim_{n\to\infty} \frac{\mathbb{E}\left[\log \|S_n\|\right]}{n}, \tag{1.1}$$

where $\|\cdot\|$ is any matrix norm. A theorem of Fursternberg–Kesten [18, Theorem 2] proves the analog of the Law of Large Numbers for the product of random matrices by showing that

$$\lim_{n\to\infty} \frac{\log \|S_n\|}{n} = \lambda \ a.s. \tag{1.2}$$

as long as $\mathbb{E} \log^+ \|Y_i\| < \infty$ where $\log^+ x := \max\{0, \log x\}$. A useful formula that can be used in obtaining exact values for λ for matrices Y_i with some distribution μ on a group $G \subset \mathrm{SL}(d, \mathbb{R})$ was found by Furstenberg [19]

$$\lambda = \int_{\mathrm{SL}(d,\mathbb{R})} \int_{\mathbb{P}^{d-1}} \log \|Ax\| \, d\nu(x) \, d\mu(A), \tag{1.3}$$

where $\mathbb{P}^{d-1}$ is the projective space, and ν is a maximal μ-invariant measure.

We also point to the classical works of [25, 38] (see also [7, 8, 10]) where conditions are given on the distribution of $\{Y_j\}_{j\geq 1}$ so that the following central limit theorem holds

$$\frac{1}{\sqrt{n}} (\log \|S_n\| - n\lambda) \xrightarrow{\mathcal{L}} N\left(0, \sigma^2\right) \tag{1.4}$$

for some $\sigma^2 \geq 0$ where the convergence holds in law.

There has been several works in obtaining explicit formulas for the Lyapunov exponent given a random matrix distribution $\{Y_j\}_{j\geq 1}$ such as in the works of [6, 8, 11, 14, 15, 24, 27, 30, 32, 33]. But in many cases, one cannot obtain explicit formulas for the Lyapunov exponent given a particular random matrix ensemble, even for 2×2 random matrices. In fact, finding an exact μ-invariant measure ν in (1.4) is a difficult problem in itself [32]. In situations where explicit expressions aren't available, one can approximate the Lyapunov exponent such as in the works of [23, 26, 29, 34, 35, 37, 39]. Explicit formulas for the variance σ^2 are also known though less studied, see [9, 16, 36] for example. We point to the work of [9, Proposition 2.1] where explicit formulas are given for the Lyapunov exponent and variance given a condition on the distribution of the random matrices. The condition given in [9, Proposition 2.1] is only valid for special cases and does not hold for general random matrices.

Most of the theory for products of random matrices require the matrices to come from $\mathrm{GL}(d, \mathbb{R})$ and often require conditions on the subgroup generated by the support of the distribution for the given random matrix model. An example is the well-known formula for computing Lyapunov exponents by Furstenberg ((1.3), see also [8, Theorem II.4.1]). If the matrices are non-invertible (also defined as singular), then other than (1.2), most of the results of the existing literature are not applicable. This holds especially true for central limit theorems with explicit variance for the products of non-invertible matrices. We do point to the works of [13, 17, 27] for some results for non-invertible matrices. The main goal of this paper is to find explicit formulas for the Lyapunov exponent λ and prove a central limit theorem with an explicit variance formula σ^2 in terms of the distribution of the matrix entries for the products of general 2×2 non-invertible matrices. Our work fills this gap in the literature by finding explicit formulas, such as in (1.3) for invertible matrices, for both λ and σ^2.

In particular, we consider the products of random matrices of the form

$$Y_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}, \quad j \geq 1,$$

where Y_j are non-invertible matrices in $M_2(\mathbb{R})$ and $\{(a_j, b_j, c_j, d_j)\}_{j\geq 1}$ are i.i.d. sequences of random vectors. We will assume a_j does not have an atom at zero. Since the Y_j are non-invertible then $a_j d_j - b_j c_j = 0$ for $j \geq 1$ and it suffices to consider Y_j of the form

$$Y_j = \begin{bmatrix} a_j & b_j \\ c_j & \frac{b_j c_j}{a_j} \end{bmatrix}, \quad j \geq 1. \tag{1.5}$$

As far as the authors know, the only paper to prove a CLT with an explicit variance for the products of 2×2 random non-invertible matrices is the special case in [31, Theorem 2.2]. In particular, in [31] the authors prove a CLT with an exact formula for σ^2 for matrices of the form

$$Y_j = \begin{bmatrix} 1 & x_j \\ \frac{1}{x_j} & 1 \end{bmatrix}, \tag{1.6}$$

where $\{x_j\}_{j\geq 1}$ is an i.i.d. sequence of random variables atomless at zero. Unlike in the invertible matrix case, there is an interesting and precise nondegeneracy condition that characterizes the entries $\{x_j\}_{j\geq 1}$ that give $\sigma^2 = 0$. These matrices are related to the random Hill's equation studied in [1–4]. In [5], Adams–Bloch–Lagarias were the first to prove an exact formula for the Lyapunov exponent. The Hill's equation has appeared in the literature with various applications such as in the modeling of lunar orbits [20] and various other astrophysical orbit problems [1]. See [28] for various physical and engineering applications of the Hill's equation.

We are now ready to state our main theorem. We prove an explicit formula for the Lyapunov exponent λ and a central limit theorem with an explicit formula for the variance σ^2 which is given in terms of the entries $\{(a_j, b_j, c_j)\}_{j\geq 1}$ of Y_j. We point out that no assumption is given on the distribution of the entries $\{(a_j, b_j, c_j)\}_{j\geq 1}$ other than a mild moment condition and that the entries are atomless at zero. Our results are the first to cover general 2×2 non-invertible matrices which includes the special case given in (1.6).

Theorem 1.1 *Consider the random non-invertible matrices of the form*

$$Y_j = \begin{bmatrix} a_j & b_j \\ c_j & \frac{b_j c_j}{a_j} \end{bmatrix}, \quad j \geq 1,$$

where $\left\{\left(a_j, b_j, c_j\right)\right\}_{j\geq 1}$ *is an i.i.d. sequence of* $\mathbb{R}\setminus\{0\}\times\mathbb{R}^2$*-valued random variables such that*

$$\mathbb{E}\left[\log^+\left(|a_1|+|c_1|\right)\right]<\infty \quad \textit{and} \quad \mathbb{E}\left[\log\left(1+\frac{|b_1|}{|a_1|}\right)\right]<\infty. \tag{1.7}$$

Then (1.2) *holds and the Lyapunov exponent has the explicit expression*

$$\lambda=\mathbb{E}\left[\log\left|a_1+\frac{b_2c_1}{a_2}\right|\right]. \tag{1.8}$$

Moreover, if we further assume that

$$\mathbb{E}\left[\left(\log\left|a_1+\frac{b_2c_1}{a_2}\right|\right)^2\right]<\infty, \tag{1.9}$$

then $\frac{1}{\sqrt{n}}\left(\log\|S_n\|-n\lambda\right)\xrightarrow{\mathcal{L}}N\left(0,\sigma^2\right)$ *and the variance has the explicit expression*

$$\sigma^2=\mathbb{E}\left[\left(\log\left|a_1+\frac{b_2c_1}{a_2}\right|\right)^2\right]+2\mathbb{E}\left[\log\left|a_1+\frac{b_2c_1}{a_2}\right|\log\left|a_2+\frac{b_3c_2}{a_3}\right|\right]-3\lambda^2. \tag{1.10}$$

The proof of Theorem 1.1 relies on an explicit product formula for S_n, which will be given in Lemma 2.1. With this product formula we can then compute λ and use the theory of m-dependent sequences of random variables to prove the CLT using a theorem of Diananda [12, Theorem 2].

As demonstrated by the special case of the matrices related to the random Hill's equation given in (1.6), there are non-trivial distributions $\left\{\left(a_j, b_j, c_j\right)\right\}_{j\geq 1}$ where the related CLT is degenerate with $\sigma^2=0$. In the case of the random Hill's matrices, a precise non-degeneracy condition was given in [31, Theorem 2.2] that completely characterizes the distribution of the entries $\left\{x_j\right\}_{j\geq 1}$ that gives $\sigma^2=0$. Unfortunately, such a precise characterization result does not seem tractable for general non-invertible matrices. We use a result of Janson [22] (see also [21]) to give a non-degeneracy condition and prove some properties of the degenerate case.

Proposition 1.2 *Consider the setting of Theorem* 1.1. *We have that* $\sigma^2=0$ *if and only if there exists a function* $\varphi:\mathbb{R}^3\to\mathbb{R}$ *such that for any* $n\in\mathbb{N}$,

$$\sum_{j=1}^{n}\left(\log\left[\left|a_j+\frac{b_{j+1}c_j}{a_{j+1}}\right|\right]-\lambda\right)=\varphi\left(\xi_{n+1}\right)-\varphi\left(\xi_1\right),\ a.s.$$

where $\left\{\xi_j\right\}_{j\geq 1}=\left\{\left(a_j, b_j, c_j\right)\right\}_{j\geq 1}$. *Moreover, if* $\sigma^2=0$ *and the distribution of* $\left(a_j, b_j, c_j\right)$ *has an atom at* (a,b,c), *then*

$$\lambda=\log\left|a+\frac{bc}{a}\right|$$

and for all $j\geq 1$,

$$\left(a+\frac{b_jc}{a_j}\right)^2\left(a_j+\frac{bc_j}{a}\right)^2=\left(a+\frac{bc}{a}\right)^4,\ a.s.$$

We organize the paper as follows. In Sect. 2, we prove the product formula for S_n and give the proof for the explicit formula of the Lyapunov exponent in Eq. (1.8). In Sect. 3.1 we give the proof of the CLT with explicit variance. We prove Proposition 1.2 in Sect. 3.2. Finally, we use Theorem 1.1 to give examples with exact values of λ and σ^2.

2 Proof of Theorem 1.1: Explicit λ

2.1 Product Formula

We first start by proving the following product formula.

Lemma 2.1 *The product of the matrices of the form*

$$Y_j = \begin{bmatrix} a_j & b_j \\ c_j & \frac{b_j c_j}{a_j} \end{bmatrix}, \quad j \geq 1$$

can be expressed as

$$S_n = \beta_n \begin{bmatrix} a_n & \frac{b_1}{a_1} a_n \\ c_n & \frac{b_1}{a_1} c_n \end{bmatrix} \tag{2.1}$$

where $\beta_1 = 1$ and

$$\beta_n = \prod_{j=1}^{n-1} \left(a_j + \frac{b_{j+1} c_j}{a_{j+1}} \right), \quad n \geq 1.$$

Proof The representation given in (2.1) is clear when $n = 1$ since

$$S_1 = \beta_1 \begin{bmatrix} a_1 & \frac{b_1}{a_1} a_1 \\ c_1 & \frac{b_1}{a_1} c_1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & \frac{b_1 c_1}{a_1} \end{bmatrix}.$$

Suppose (2.1) holds for n, then we have that

$$\begin{aligned}
S_{n+1} &= Y_{n+1} S_n \\
&= \beta_n \begin{bmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & \frac{b_{n+1} c_{n+1}}{a_{n+1}} \end{bmatrix} \begin{bmatrix} a_n & \frac{b_1}{a_1} a_n \\ c_n & \frac{b_1}{a_1} c_n \end{bmatrix} \\
&= \beta_n \begin{bmatrix} a_n a_{n+1} + b_{n+1} c_n & a_{n+1} \frac{b_1}{a_1} a_n + b_{n+1} \frac{b_1}{a_1} c_n \\ c_{n+1} a_n + \frac{b_{n+1} c_{n+1}}{a_{n+1}} c_n & c_{n+1} \frac{b_1}{a_1} a_n + \frac{b_{n+1} c_{n+1}}{a_{n+1}} \frac{b_1}{a_1} c_n \end{bmatrix} \\
&= \beta_n \begin{bmatrix} a_{n+1} \left(a_n + \frac{b_{n+1} c_n}{a_{n+1}} \right) & a_{n+1} \frac{b_1}{a_1} \left(a_n + \frac{b_{n+1} c_n}{a_{n+1}} \right) \\ c_{n+1} \left(a_n + \frac{b_{n+1} c_n}{a_{n+1}} \right) & \frac{b_1}{a_1} c_{n+1} \left(a_n + \frac{b_{n+1} c_n}{a_{n+1}} \right) \end{bmatrix} \\
&= \prod_{j=1}^{n} \left(a_j + \frac{b_{j+1} c_j}{a_{j+1}} \right) \begin{bmatrix} a_{n+1} & \frac{b_1}{a_1} a_{n+1} \\ c_{n+1} & \frac{b_1}{a_1} c_{n+1} \end{bmatrix} = \beta_{n+1} \begin{bmatrix} a_{n+1} & \frac{b_1}{a_1} a_{n+1} \\ c_{n+1} & \frac{b_1}{a_1} c_{n+1} \end{bmatrix}.
\end{aligned}$$

□

2.2 Proof of Exact λ in Eq. (1.8)

Proof We first take the log of the norm of the product representation of S_n given in (2.1) of Lemma 2.1 to get

$$\log \|S_n\| = \sum_{j=1}^{n-1} \log \left| a_j + \frac{b_{j+1} c_j}{a_{j+1}} \right| + \log \left\| \begin{bmatrix} a_n & \frac{b_1}{a_1} a_n \\ c_n & \frac{b_1}{a_1} c_n \end{bmatrix} \right\|. \tag{2.2}$$

Consider the degenerate case where there is a positive probability where $a_j + \frac{b_{j+1}c_j}{a_{j+1}} = 0$. Note that in this case we have $\log \|S_n\| = -\infty$ hence $\lambda = \mathbb{E}\left[\log \left|a_1 + \frac{b_2 c_1}{a_2}\right|\right] = -\infty$ as needed.

Let $\|\cdot\|$ denote the Hilbert–Schmidt norm and compute

$$\|Y_1\| = \sqrt{a_1^2 + b_1^2 + c_1^2 + \frac{b_1^2 c_1^2}{a_1^2}} = \sqrt{a_1^2\left(1 + \frac{c_1^2}{a_1^2}\right)\left(1 + \frac{b_1^2}{a_1^2}\right)}.$$

Using the moment condition (1.7) with the property that $\log^+(xy) \le \log^+(x) + \log^+(y)$ for any $x, y > 0$ we have

$$\begin{aligned}
\mathbb{E}\left[\log^+ \|Y_1\|\right] &= \mathbb{E}\left[\log^+ \sqrt{\left(a_1^2 + c_1^2\right)\left(1 + \frac{b_1^2}{a_1^2}\right)}\right] \\
&\le \mathbb{E}\left[\log^+ \sqrt{\left(a_1^2 + c_1^2\right)} + \mathbb{E}\left[\log^+ \sqrt{\left(1 + \frac{b_1^2}{a_1^2}\right)}\right]\right] \\
&\le \mathbb{E}\left[\log^+ (|a_1| + |c_1|)\right] + \mathbb{E}\left[\log\left(1 + \frac{|b_1|}{|a_1|}\right)\right] < \infty.
\end{aligned}$$

Hence by a direct application of [18, Theorem 2] we have that $\lambda = \lim_{n\to\infty} \frac{\log\|S_n\|}{n} < \infty$ almost surely.

We are left to compute the Lyapunov exponent λ explicitly. First we show some finite moment conditions. Note that for all $n \ge 1$,

$$\begin{aligned}
\mathbb{E}\left[\log \left\| \begin{bmatrix} a_n & \frac{b_1}{a_1} a_n \\ c_n & \frac{b_1}{a_1} c_n \end{bmatrix} \right\| \right] &= \mathbb{E}\left[\log \sqrt{a_n^2 + \frac{b_1^2}{a_1^2} a_n^2 + c_n^2 + \frac{b_1^2}{a_1^2} c_n^2}\right] \\
&= \mathbb{E}\left[\log \sqrt{\left(a_n^2 + c_n^2\right)\left(1 + \frac{b_1^2}{a_1^2}\right)}\right] \\
&\le \mathbb{E}\left[\log^+ (|a_n| + |c_n|)\right] + \mathbb{E}\left[\log\left(1 + \frac{|b_1|}{|a_1|}\right)\right] < \infty,
\end{aligned}$$

where we used (1.7). Moreover, we have that for all $j \ge 1$, $-\infty \le \mathbb{E}\log\left[\left|a_j + \frac{b_{j+1}c_j}{a_{j+1}}\right|\right] < \infty$ since

$$\begin{aligned}
\mathbb{E}\left[\log\left[\left|a_j + \frac{b_{j+1}c_j}{a_{j+1}}\right|\right]\right] &\le \mathbb{E}\left[\log\left[\left(|a_j| + |c_j|\right)\left(1 + \frac{|b_{j+1}|}{|a_{j+1}|}\right)\right]\right] \\
&\le \mathbb{E}\left[\log^+\left[\left(|a_j| + |c_j|\right)\right]\right] + \mathbb{E}\left[\log\left[\left(1 + \frac{|b_{j+1}|}{|a_{j+1}|}\right)\right]\right] < \infty.
\end{aligned}$$

Taking the expected value of (2.2) and using the finite moment conditions shown above we have that

$$\lambda = \lim_{n\to\infty} \frac{1}{n}\left(\mathbb{E}\left[\sum_{j=1}^{n-1}\log\left[\left|a_j + \frac{b_{j+1}c_j}{a_{j+1}}\right|\right]\right] + \mathbb{E}\left[\log\left[\left\|\begin{bmatrix} a_n & \frac{b_1}{a_1}a_n \\ c_n & \frac{b_1}{a_1}c_n \end{bmatrix}\right\|\right]\right]\right)$$
$$= \lim_{n\to\infty} \frac{1}{n}\left(\sum_{j=1}^{n-1}\mathbb{E}\left[\log\left[\left|a_1 + \frac{b_2c_1}{a_2}\right|\right]\right] + \mathbb{E}\left[\log\left[\left\|\begin{bmatrix} a_2 & \frac{b_1}{a_1}a_2 \\ c_2 & \frac{b_1}{a_1}c_2 \end{bmatrix}\right\|\right]\right]\right)$$
$$= \mathbb{E}\left[\log\left|a_1 + \frac{b_2c_1}{a_2}\right|\right],$$

as needed. □

3 Proof of Theorem 1.1: The Explicit CLT and Non-degeneracy Condition

3.1 Proof of the CLT and Eq. (1.10)

Proof First note that by the condition (1.7) combined with the L^2-moment condition in (1.9) we have that

$$|\lambda| = \left|\mathbb{E}\left[\log\left|a_1 + \frac{b_2c_1}{a_2}\right|\right]\right| \leq \sqrt{\mathbb{E}\left[\left(\log\left|a_1 + \frac{b_2c_1}{a_2}\right|\right)^2\right]} < \infty.$$

By (2.2) we can write

$$\log\|S_n\| - n\lambda = \sum_{j=1}^{n-1}\left(\underbrace{\log\left[\left|a_j + \frac{b_{j+1}c_j}{a_{j+1}}\right|\right] - \lambda}_{A_j}\right) + \underbrace{\log\left[\left\|\begin{bmatrix} a_n & \frac{b_1}{a_1}a_n \\ c_n & \frac{b_1}{a_1}c_n \end{bmatrix}\right\|\right] - \lambda}_{B_n}. \quad (3.1)$$

Our goal will be to apply [12, Theorem 2] to the sequence $\{A_j\}_{j\geq 1}$ to obtain an explicit CLT. First, it is clear that $\{A_j\}_{j\geq 1}$ is a 1-dependent sequence of random variables. One-dependence means that $\{A_1, \ldots, A_j\}$ is independent of $\{A_k, A_{k+1}, \ldots\}$ whenever $k - j > 1$. Define C_j by $C_{i-j} = \mathbb{E}\left[A_iA_j\right]$ for all $i, j \in \mathbb{Z}$. We then have that $\mathbb{E}\left[A_j\right] = 0$,

$$C_0 = \mathbb{E}\left[A_j^2\right] = \mathbb{E}\left[\left(\log\left|a_1 + \frac{b_2c_1}{a_2}\right|\right)^2\right] - \lambda^2,$$

and

$$C_1 = C_{-1} = \mathbb{E}\left[A_2A_1\right] = \mathbb{E}\left[\log\left|a_1 + \frac{b_2c_1}{a_2}\right| \log\left|a_2 + \frac{b_3c_2}{a_3}\right|\right] - \lambda^2.$$

By 1-dependence it is clear that $C_i = 0$ for all $|i| > 1$.

Applying Diananda's CLT for stationary m-dependent sequences from [12, Theorem 2] we have

$$\frac{1}{\sqrt{n}}A_j \xrightarrow{\mathcal{L}} N\left(0, \sigma^2\right),$$

where

$$\sigma^2 = \sum_{j=-1}^{1} C_j = \mathbb{E}\left[\left(\log\left|a_1 + \frac{b_2c_1}{a_2}\right|\right)^2\right] + 2\mathbb{E}\left[\log\left|a_1 + \frac{b_2c_1}{a_2}\right|\log\left|a_2 + \frac{b_3c_2}{a_3}\right|\right] - 3\lambda^2.$$

By the moment condition (1.7) , (1.9) and Cauchy–Schwarz we have that $\sigma^2 < \infty$. A standard application of Markov's inequality gives that $\frac{1}{\sqrt{n}}B_n \to 0$ in probability. Then, by Slutsky's theorem, we have

$$\frac{1}{\sqrt{n}}\left(\log\|S_n\| - n\lambda\right) \xrightarrow{\mathcal{L}} N\left(0, \sigma^2\right),$$

as desired. □

3.2 Proof of Proposition 1.2

Proof Suppose $\sigma^2 = 0$ in (1.10). We then apply a theorem of Janson [22, Theorem 8.1] to obtain a characterization of the degenerate case. We follow the notation from [22]. Let $\{\xi_i\}_{i\geq 1}$ be the i.i.d. sequence of random variables given by $\{(a_i, b_i, c_i)\}_{i\geq 1}$ in $\mathcal{S}_0 = \mathbb{R}\setminus\{0\} \times \mathbb{R}^2$ as in the setting of Theorem 1.1. Let $X_i := A_i$ as in (3.1). Recall that X_i is a two-block factor of ξ_i since $X_i = h\left(\xi_i, \xi_{i+1}\right)$ where $h : \mathcal{S}_0^2 \to \mathbb{R}$ is given by

$$h\left((a_i, b_i, c_i), (a_{i+1}, b_{i+1}, c_{i+1})\right) = \log\left[\left|a_i + \frac{b_{i+1}c_i}{a_{i+1}}\right|\right] - \lambda.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be the identity with $\mathcal{S} = \mathbb{R}$, $\ell = 1$ in the notation of [22] . The U-statistic U_n in [22, Eq. (3.1)] is the usual sum

$$S_n = \sum_{i=1}^{n} X_j.$$

By [22, Theorem 8.1(vi)] we have that $\sigma^2 = 0$ if and only if there exists a function $\varphi : \mathcal{S}_0 \to \mathbb{R}$ such that

$$\sum_{j=1}^{n} X_j = \varphi\left(\xi_{n+1}\right) - \varphi\left(\xi_1\right)$$

for all $n \geq 1$ so that

$$\sum_{j=1}^{n}\left(\log\left[\left|a_j + \frac{b_{j+1}c_j}{a_{j+1}}\right|\right] - \lambda\right) = \varphi\left(a_{j+1}, b_{j+1}, c_{j+1}\right) - \varphi\left(a_1, b_1, c_1\right) \tag{3.2}$$

for all $n \geq 1$.

Atomic Case:

Now suppose ξ_j has an atom, that is, there exists a $(a, b, c) \in \mathcal{S}_0$ so that $\mathbb{P}\left(\left(a_j, b_j, c_j\right) = (a, b, c)\right) > 0$. Using $n = 1$ in (3.2), there exists a function $\varphi : (\mathbb{R}\setminus\{0\})^3 \to \mathbb{R}$ such that

$$\log\left[\left|a_1 + \frac{b_2c_1}{a_2}\right|\right] - \lambda = \varphi\left((a_2, b_2, c_2)\right) - \varphi\left((a_1, b_1, c_1)\right),\ a.s.$$

By independence we know $\mathbb{P}\left((a_1, b_1, c_1) = (a, b, c), (a_2, b_2, c_2) = (a, b, c)\right) > 0$ and

$$\begin{aligned}
&\mathbb{P}\left((a_1, b_1, c_1) = (a, b, c), (a_2, b_2, c_2) = (a, b, c)\right) \\
&\quad = \mathbb{P}\left((a_1, b_1, c_1) = (a, b, c), (a_2, b_2, c_2) = (a, b, c)\right) \mathbb{P}\left(\lambda = \log\left|a + \frac{bc}{a}\right|\right)
\end{aligned}$$

so that $\mathbb{P}\left(\lambda = \log\left|a + \frac{bc}{a}\right|\right) = 1$. Hence

$$\lambda = \log\left|a + \frac{bc}{a}\right|.$$

Now let $n = 2$ in (3.2), so that by rewriting we have

$$\left|\left(a_1 + \frac{b_2 c_1}{a_2}\right)\left(a_2 + \frac{b_3 c_2}{a_3}\right)\right| = e^{2\lambda + \varphi((a_3, b_3, c_3)) - \varphi((a_1, b_1, c_1))}, \quad a.s.$$

Define $\alpha = \mathbb{P}\left((a_j, b_j, c_j) = (a, b, c)\right) > 0$ and use independence with the equation above to obtain

$$\begin{aligned}
\alpha^2 &= \mathbb{P}\left((a_1, b_1, c_1) = (a, b, c), (a_3, b_3, c_3) = (a, b, c)\right) \\
&= \mathbb{P}\left((a_1, b_1, c_1) = (a, b, c), (a_3, b_3, c_3) = (a, b, c),\right. \\
&\qquad \left.\left|\left(a_1 + \frac{b_2 c_1}{a_2}\right)\left(a_2 + \frac{b_3 c_2}{a_3}\right)\right| = e^{2\lambda + \varphi((a_3, b_3, c_3)) - \varphi((a_1, b_1, c_1))}\right) \\
&= \mathbb{P}\left((a_1, b_1, c_1) = (a, b, c), (a_3, b_3, c_3) = (a, b, c)\right) \\
&\quad \times \mathbb{P}\left(\left|\left(a + \frac{b_2 c}{a_2}\right)\left(a_2 + \frac{b c_2}{a}\right)\right| = \left|a + \frac{bc}{a}\right|^2\right) \\
&= \alpha^2 \mathbb{P}\left(\left|\left(a + \frac{b_2 c}{a_2}\right)\left(a_2 + \frac{b c_2}{a}\right)\right| = \left|a + \frac{bc}{a}\right|^2\right)
\end{aligned}$$

so that

$$\left|\left(a + \frac{b_2 c}{a_2}\right)\left(a_2 + \frac{b c_2}{a}\right)\right| = \left|a + \frac{bc}{a}\right|^2, \quad a.s.$$

or

$$\left(a + \frac{b_2 c}{a_2}\right)^2\left(a_2 + \frac{b c_2}{a}\right)^2 = \left(a + \frac{bc}{a}\right)^4, \quad a.s.$$

□

4 Examples: Exact Results

We use the results in Theorem 1.1 to compute the exact Lyapunov exponent λ and variance σ^2 for the central limit theorem for the products of the following random matrices.

Example 4.1 (Binary) Let $Y_j = \begin{bmatrix} x_j & \frac{1}{x_j} \\ 1 & \frac{1}{x_j^2} \end{bmatrix}$ where x_j are independent and identically distributed random variables taking values $a, b \neq 0, -1$ and $\left(ab^2 + 1\right)\left(a^2 b + 1\right) \neq 0$ with $\mathbb{P}\left(x_j = a\right) = p = 1 - \mathbb{P}\left(x_j = b\right)$. Then the Lyapunov exponent is given by

$$\lambda = p^2 \log\left|a + \frac{1}{a^2}\right| + p(1-p)\log\left(\left|a + \frac{1}{b^2}\right|\left|b + \frac{1}{a^2}\right|\right) + (1-p)^2 \log\left|b + \frac{1}{b^2}\right|$$

and the variance is

$$\begin{aligned}\sigma^2 &= p^2\left(1+(2-3p)\,p\right)\left(\log\left|a+\frac{1}{a^2}\right|\right)^2\\ &\quad -2\,(1-p)\,p^2\log\left|a+\frac{1}{a^2}\right|\left((3p-1)\log\left(\left|a+\frac{1}{b^2}\right|\left|b+\frac{1}{a^2}\right|\right)\right.\\ &\quad \left.+3\,(1-p)\,p\log\left(\left|b+\frac{1}{b^2}\right|\right)\right)\\ &\quad +(1-p)\left(1+3\,(p-1)\,p\right)p\left(\log\left(\left|a+\frac{1}{b^2}\right|\left|b+\frac{1}{a^2}\right|\right)\right)^2\\ &\quad +2\,(1-p)^2\,(3p-2)\,p\log\left(\left|a+\frac{1}{b^2}\right|\left|b+\frac{1}{a^2}\right|\right)\log\left|b+\frac{1}{b^2}\right|\\ &\quad -(1-p)^2\,(3p-4)\,p\left(\log\left|b+\frac{1}{b^2}\right|\right)^2.\end{aligned}$$

In the next few examples we consider the non-invertible matrix

$$Y_j = \begin{bmatrix} x_j & x_j \\ y_j & y_j \end{bmatrix},$$

which allow for a simplified computation of exact values. In particular, using Theorem 1.1 we have that

$$\lambda = \mathbb{E}\left[(\log|x_1+y_1|)\right].$$

Note that by independence we have

$$\mathbb{E}\left[\log\left|a_1+\frac{b_2c_1}{a_2}\right|\log\left|a_2+\frac{b_3c_2}{a_3}\right|\right] = \mathbb{E}\left[\log|x_1+y_1|\log|x_2+y_2|\right] = \lambda^2$$

which allows for the simplification of the variance formula

$$\sigma^2 = \mathbb{E}\left[(\log|x_1+y_1|)^2\right] - \lambda^2.$$

Example 4.2 (Uniform) Let $Y_j = \begin{bmatrix} x_j & x_j \\ y_j & y_j \end{bmatrix}$ where x_j, y_i are all independent and identically distributed over the interval $[-a, b]$ where $-a \le 0 < b$. Then the Lyapunov exponent and variance is given by

$$\lambda = \begin{cases} 2\log 2 - \frac{3}{2} + \log b, & a = 0, b > 0 \\ 2\log 2 - \frac{3}{2} + \log a, & -a < 0, b = 0 \\ \log(2b) - \frac{3}{2}, & -a < 0 < b, a = b \end{cases} \quad \text{and}$$

$$\sigma^2 = \begin{cases} \frac{5}{4} - 2(\log 2)^2, & a = 0, b > 0 \\ \frac{5}{4} - 2(\log 2)^2, & -a < 0, b = 0 \\ \frac{5}{4}, & -a < 0 < b, a = b. \end{cases}$$

Example 4.3 (Exponential) Let $Y_j = \begin{bmatrix} x_j & x_j \\ y_j & y_j \end{bmatrix}$ where x_j, y_i are all independent and identically distributed as exponential random variables with parameter $\theta > 0$. Then

$$\lambda = 1 - \gamma - \log\theta \quad \text{and} \quad \sigma^2 = \frac{\pi^2}{6} - 1,$$

where $\gamma \approx .57721$ is the Euler–Mascheroni constant.

Example 4.4 (Cauchy) Let $Y_j = \begin{bmatrix} x_j & x_j \\ y_j & y_j \end{bmatrix}$ where x_j, y_i are all independent and identically distributed as a standard Cauchy distribution. Then

$$\lambda = \log 2 \quad \text{and} \quad \sigma^2 = \frac{\pi^2}{4}.$$

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