The K-theory of perfectoid rings

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Abstract

We establish various properties of the p-adic algebraic K-theory of smooth algebras over perfectoid rings living over perfectoid valuation rings. In particular, the p-adic K-theory of such rings is homotopy invariant, and coincides with the p-adic K-theory of the p-adic generic fibre in high degrees. In the case of smooth algebras over perfectoid valuation rings of mixed characteristic the latter isomorphism holds in all degrees and generalises a result of Nizioł.

1 Introduction

In this note, we record some results concerning the p-adic K-theory of certain p-adic rings. Our starting point is the following result.

Theorem 1.1 (Quillen, Hiller [Hil81], Kratzer [Kra80])**.** *If A is a perfect $\mathbb{F}_p$-algebra, then $\mathrm{K}_i(A)$ is a $\mathbb{Z}[1/p]$-module for all $i > 0$.*

Theorem 1.1 is proved using the action of the Adams operations on K-theory. In particular, one shows that ψ^p is given by the Frobenius, and therefore is an isomorphism. The mixed characteristic analog of a perfect $\mathbb{F}_p$-algebra is a perfectoid ring, and many foundational results for perfect $\mathbb{F}_p$-algebras can be generalised to perfectoid rings. We begin by giving the following generalisation of Theorem 1.1 to mixed characteristic. In the statement, $\mathrm{K}(-;\mathbb{F}_p)$ denotes the cofiber $\mathrm{K}(-)/p$ of multiplication by p on non-connective K-theory $\mathrm{K}(-)$.

Theorem 1.2 (Theorem 5.10)**.** *If $\mathcal{O}$ is a perfectoid valuation ring and A is a perfectoid $\mathcal{O}$-algebra,*[1] *then the map $\mathrm{K}(A;\mathbb{F}_p) \to \mathrm{K}(A[1/p];\mathbb{F}_p)$ is 0-truncated.*[2]

Theorem 1.2 holds more generally for any ring A whose derived p-completion satisfies the hypotheses of the theorem, as the conclusion of the theorem is insensitive to replacing A by its derived p-completion. In the case where A is the absolute integral closure of a complete discrete valuation ring of mixed characteristic, Theorem 1.2 is proved by different methods in [Niz98] and [Hes06]. In fact, the result in [Niz98] works more generally for smooth algebras,

[1] For us, a "perfectoid valuation ring" is a valuation ring $\mathcal{O}$ which is simultaneously a perfectoid ring in the sense of [BMS18, Def. 3.5]: $\mathcal{O}$ is p-adically complete, there is an element ϖ in $\mathcal{O}$ such that ϖ^p divides p, the Frobenius map on $\mathcal{O}/p$ is surjective, and the kernel of $\theta\colon \mathrm{A}_{\mathrm{inf}}(\mathcal{O}) \to \mathcal{O}$ is principal; such a valuation ring is p-adically complete and separated. Equivalently, it is either a perfect valuation ring of characteristic p, or its field of fractions is a perfectoid field of characteristic 0 having ring of integers $\mathcal{O}_{\sqrt{p\mathcal{O}}}$. A "perfectoid $\mathcal{O}$-algebra" is a perfectoid ring A equipped with the structure of a $\mathcal{O}$-algebra.

[2] Recall that a map $f\colon M \to N$ of spectra is d-truncated if the fiber $\mathrm{fib}(f)$ satisfies $\pi_i\mathrm{fib}(f) = 0$ for $i > d$. Equivalently, $\pi_i f\colon \pi_i X \to \pi_i Y$ is an isomorphism for $i > d+1$ and an injection for $i = d+1$.

and plays a crucial role in the approach to the crystalline conjecture of p-adic Hodge theory in *op. cit.*

Theorem 1.3 (Nizioł [Niz98, Lem. 3.1])**.** *Let $\mathcal{O}_K$ be a complete discrete valuation ring with field of fractions K; let $\overline{K}$ be an algebraic closure of K, and $\mathcal{O}_{\overline{K}}$ the integral closure of $\mathcal{O}_K$ in $\overline{K}$. For any smooth $\mathcal{O}_{\overline{K}}$-algebra R, the canonical map $\mathrm{K}(R) \to \mathrm{K}(R \otimes_{\mathcal{O}_{\overline{K}}} \overline{K})$ becomes an equivalence after profinite completion.*

Theorem 1.3 is proved using localisation sequences in K-theory after descending R to the integral closure of $\mathcal{O}_K$ in a finite extension of K; more generally, one can replace $\mathcal{O}_{\overline{K}}$ by any absolutely integrally closed valuation ring (Corollary 3.3). Combining this argument with the tilting correspondence [Sch12] to reduce to characteristic p and the inseparable local uniformisation of Temkin [Tem13, Tem17], we give the following generalisation of (the p-adic case of) Theorem 1.3 as well as of Theorem 1.2. We remark that the theorem holds more generally for algebras which are p-completely smooth in a suitable sense (see Corollary 3.10 and Remark 5.13).

Theorem 1.4 (Theorem 3.9 and Theorem 5.10)**.** *Let $\mathcal{O}$ be a perfectoid valuation ring.*

1. *If $p \neq 0$ in $\mathcal{O}$, then for any smooth $\mathcal{O}$-algebra R, the canonical map $\mathrm{K}(R; \mathbb{F}_p) \to \mathrm{K}(R[1/p]; \mathbb{F}_p)$ is an equivalence.*
2. *For any perfectoid $\mathcal{O}$-algebra A and smooth A-algebra R of relative dimension $\leq d$, the canonical map $\mathrm{K}(R; \mathbb{F}_p) \to \mathrm{K}(R[1/p]; \mathbb{F}_p)$ is d-truncated.*

For any ring R, the canonical map $\mathrm{K}(R) \to \mathrm{K}(R[1/p])$ becomes an equivalence after $K(1)$-localisation [BCM20, LMMT20], but in general the conclusion that they agree in sufficiently high degrees requires further assumptions. The proof deduces part 2 from part 1 using cdh-descent and separate arguments in the case of valuation rings. These arguments also lead to the following result comparing algebraic and homotopy K-theory for perfectoid rings; since the comparison between K and KH of a noetherian ring is known to be related to singularities, the following result gives a sense in which perfectoid rings behave like regular ones.

Theorem 1.5 (Proposition 5.1, Corollary 5.6, and Theorem 5.12)**.** *Let R be a smooth algebra over either*

1. *a perfect $\mathbb{F}_p$-algebra; or*
2. *$W(A)$ where A is a perfect $\mathbb{F}_p$-algebra; or*
3. *a perfectoid ring which is an algebra over some perfectoid valuation ring.*

Then the map $\mathrm{K}(R; \mathbb{F}_p) \to \mathrm{KH}(R; \mathbb{F}_p)$ is an equivalence (in case 1, even $\mathrm{K}(R) \to \mathrm{KH}(R)$ is an equivalence).

Notation

We let $\mathrm{Cat}_\infty^{\mathrm{perf}}$ denote the ∞-category of small, stable idempotent-complete ∞-categories, and exact functors between them. We denote by K the nonconnective K-theory functor, defined on $\mathrm{Cat}_\infty^{\mathrm{perf}}$ as in [BGT13] and taking values in spectra. Similarly, we denote by KH the homotopy K-theory of [Wei89], defined more generally on $\mathbb{Z}$-linear ∞-categories [Tab15].

Throughout, we let $\mathrm{K}(-;\mathbb{Z}_p)$ and $\mathrm{KH}(-;\mathbb{Z}_p)$ denote the p-completions of K-theory and homotopy invariant K-theory. Similarly, we denote by $\mathrm{K}(-;\mathbb{F}_p)$ and $\mathrm{KH}(-;\mathbb{F}_p)$ their mod p reductions.

All rings in this paper will be commutative. Given a ring R, we let $\mathcal{D}(R)$ denote its derived ∞-category. Given a ring R (or more generally an $\mathbb{E}_\infty$-ring) and an ideal $I \subset R$, we let $\mathrm{Perf}(R \text{ on } I)$ denote the ∞-category of perfect R-module spectra M which are I-power torsion: in other words, for any $x \in I$, $M[x^{-1}] = 0$. We will only use this definition when I is the radical of a finitely generated ideal J, in which case $\mathrm{Perf}(R \text{ on } I)$ is the kernel of $\mathrm{Perf}(\mathrm{Spec}(R)) \to \mathrm{Perf}(\mathrm{Spec}(R) \setminus V(J))$. Given a localizing invariant E (cf. [BGT13, Def. 8.1]), we write $E(R \text{ on } I) = E(\mathrm{Perf}(R \text{ on } I))$. Given a map of pairs $(R, I) \to (S, J)$ such that $\mathrm{rad}(IS) = J$, base-change induces a functor $\mathrm{Perf}(R \text{ on } I) \to \mathrm{Perf}(S \text{ on } J)$ and a consequent map in any localizing invariant.

We adopt the convention in this paper that localizing invariants commute with filtered colimits.

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2 Localisation sequences

In this section, we review some basic properties of coherent rings and their K-theory, in particular proving the localisation results Proposition 2.5 and Proposition 2.6. All of these are direct analogs of standard properties [Qui73] of the K-theory and G-theory of noetherian schemes; we will need to apply them to valuation rings. In Appendix A, we indicate how to prove these results and some generalisations using dévissage results about the K-theory of stable ∞-categories (which will not be used in the rest of the paper); in this section we only use classical dévissage theorems.

A ring is said to be *coherent* if every finitely generated ideal is finitely presented, cf. [Sta18, Tag 05CU], which implies that the category of finitely presented modules is abelian. We will say that a coherent ring R is *weakly regular* if R has finite flat (or weak) dimension. Equivalently, by [Gla89, Cor. 2.5.6], the projective dimensions of finitely presented R-modules are uniformly bounded (necessarily by the flat dimension). A ring R is said to be *stably coherent* if every finitely presented R-algebra is coherent. It is sufficient to check coherence of finitely generated polynomial algebras over R. The class of stably coherent rings is closed under localisations, quotients by finitely generated ideals, and finitely presented extensions [Gla89, Thms. 2.4.1 & 2.4.2]. We will primarily be interested in weakly regular stably coherent rings; this includes all regular[3] rings of finite Krull dimension, but also valuation rings by the following results.

[3]Throughout the article, we adopt the convention that regular rings are assumed to be Noetherian.

Proposition 2.1. *Any valuation ring is stably coherent and of flat dimension* $\leqslant 1$.

Proof. The stable coherence is [Gla89, Th. 7.3.3]. Every torsion-free module over a valuation ring is flat [Sta18, Tag 0549], whence the second claim. □

Proposition 2.2. *Let A be a stably coherent ring with flat dimension d_0, and let R be a smooth algebra of relative dimension*[4] $\leq d$ *over A. Then R has flat dimension $\leqslant d + d_0$; in particular, R is weakly regular.*

Proof. Let M be a finitely presented R-module. It suffices to show that if M is flat as an A-module, then M has projective dimension $\leqslant d$ as an R-module. By [Gla89, Cor. 2.5.10], it suffices to show that for every maximal ideal $\mathfrak{m}$ of R, one has $\mathrm{Tor}_R^{d+1}(M, R/\mathfrak{m}) = 0$. Now $R/\mathfrak{m}$ pulls back to a prime ideal $\mathfrak{p} \subset A$ with residue field $\kappa(\mathfrak{p})$. Since M is flat over A, we have $\mathrm{Tor}_R^{d+1}(M, R/\mathfrak{m}) = \mathrm{Tor}^{d+1}_{R\otimes_A \kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p}), R/\mathfrak{m})$. However, this vanishes since $R \otimes_A \kappa(\mathfrak{p})$ is a smooth algebra over the field $\kappa(\mathfrak{p})$ of dimension $\leq d$ and consequently it has global dimension $\leq d$. □

Corollary 2.3. *Any smooth algebra over a valuation ring is weakly regular stably coherent.*

The K-theory of weakly regular stably coherent rings behaves in a similar way to that of regular Noetherian rings, as exemplified by the following result. Given a coherent ring R, we define the G-*theory* $\mathrm{G}(R)$ to be the connective K-theory of the abelian category of finitely presented R-modules. The first two parts of the next result appear as [Wei89, Ex. 1.4]; compare also [AGH19, Th. 3.33] and [KM21, Th. 3.3] for treatments, and [BS22] for some similar results mainly in the case of rings of weak global dimension at most one.

Proposition 2.4. *If R is a weakly regular stably coherent ring, then*

1. $\mathrm{K}_{-i}(R) = 0$ *for $i > 0$; that is, the canonical map $\mathrm{K}_{\geq 0}(R) \to \mathrm{K}(R)$ from connective* K*-theory to* K*-theory is an equivalence;*
2. *the canonical map $\mathrm{K}(R) \to \mathrm{KH}(R)$ is an equivalence;*
3. *the canonical map $\mathrm{K}_{\geq 0}(R) \to \mathrm{G}(R)$ is an equivalence.*

Proof. We have already explained that the first two parts may be found in [Wei89, Ex. 1.4], where we implicitly use Proposition 2.2. They are also special cases of Theorem A.1 and Corollary A.2.

The third part follows from Quillen's dévissage theorem, as the hypotheses imply that any object in the abelian category of finitely presented R-modules admits a finite length resolution by finite projective modules. Alternatively it is a special case of Theorem A.1. □

We may now present the localisation sequences which will be required later.

Proposition 2.5. *Let R be a weakly regular stably coherent ring and let $I \subset R$ a finitely generated ideal. Then there is a natural fiber sequence $\mathrm{G}(R/I) \to \mathrm{K}(R) \to \mathrm{K}(\mathrm{Spec}(R) \setminus V(I))$.*

Proof. We write $\mathrm{G}(\mathrm{Spec}(R) \setminus V(I))$ for the connective K-theory of the abelian category of finitely presented quasi-coherent sheaves on $\mathrm{Spec}(R) \setminus V(I)$. This abelian category is the Serre quotient of the abelian category of finitely presented R-modules by the subcategory of those

[4] We say that a smooth A-algebra R has *relative dimension* $\leq d$ if all fibers $R \otimes_A \kappa(\mathfrak{p})$, where $\kappa(\mathfrak{p})$ runs over the residue fields of A, have Krull dimension $\leq d$.

objects which are I-power torsion; see in particular [Sta18, Tag 01PD] for the result that finitely presented quasi-coherent sheaves can be extended so that $\mathrm{K}_0(R) \to \mathrm{K}_0(\mathrm{Spec}(R) \setminus V(I))$ is surjective.

The classical localisation and dévissage theorems [Qui73, Thms. 4 and 5] therefore provide a fiber sequence $\mathrm{G}(R/I) \to \mathrm{G}(R) \to \mathrm{G}(\mathrm{Spec} R \setminus V(I))$. Finally, Proposition 2.4 implies that $\mathrm{G}(R) \simeq \mathrm{K}(R)$ and that $\mathrm{G}(\mathrm{Spec}(R) \setminus V(I)) \simeq \mathrm{K}(\mathrm{Spec}(R) \setminus V(I))$ (in the latter case use induction on the s of a finite affine open cover of $\mathrm{Spec}(R) \setminus V(I)$, again using [Sta18, Tag 01PD] to eliminate any possible problem with failure of surjectivity on K_0). □

Proposition 2.6. *If R is a ring and $t \in R$ is a nonzerodivisor such that R/t is weakly regular stably coherent, then there are natural fiber sequences $\mathrm{K}(R/t) \to \mathrm{K}(R) \to \mathrm{K}(R[1/t])$ and $\mathrm{KH}(R/t) \to \mathrm{KH}(R) \to \mathrm{KH}(R[1/t])$. Consequently, we have a pullback square*

$$\begin{array}{ccc} \mathrm{K}(R) & \longrightarrow & \mathrm{K}(R[1/t]) \\ \downarrow & & \downarrow \\ \mathrm{KH}(R) & \longrightarrow & \mathrm{KH}(R[1/t]). \end{array} \tag{1}$$

Proof. Note first that R/t^n is stably coherent for each $n \geqslant 1$ by [BMS18, Lem. 3.26]. By the localisation theorem of [Gra76], the connective cover of the fiber of $\mathrm{K}(R) \to \mathrm{K}(R[1/t])$ (i.e., the connective cover of $\mathrm{K}(R \text{ on } tR)$) is the K-theory of the exact category $\mathcal{E}$ of finitely presented R-modules M such that $M[1/t] = 0$ and such that M has Tor-dimension $\leqslant 1$.

Consider the category $\mathcal{A}$ of all finitely presented R-modules which are t-power torsion. In other words, $\mathcal{A}$ is the union of the categories of finitely presented R/t^n-modules over all $n \geqslant 0$. Our coherence hypotheses thus show that $\mathcal{A}$ is an abelian category, and $\mathcal{A}$ contains $\mathcal{E}$ as an exact subcategory.

We observe that every object in $\mathcal{A}$ has finite Tor-dimension as an R-module. Indeed, suppose $M \in \mathcal{A}$ is a finitely presented R-module with $M[1/t] = 0$. We may assume $tM = 0$. Then our weak regularity hypothesis implies that M has finite Tor-dimension as an R/t-module, and hence as an R-module. Using this, we can show that every object in $\mathcal{A}$ admits a finite resolution by objects in $\mathcal{E}$. If $M \in \mathcal{A}$ has Tor-dimension $\geqslant 2$, then we can choose a surjection $(R/t^i)^n \twoheadrightarrow M$ (for appropriate $i, n \gg 0$); the kernel K will belong to $\mathcal{A}$ and have Tor-dimension at least one less, whence the claim by induction.

Thus we can apply the dévissage theorem in the form of [Qui73, Sec. 4] to see that $\mathrm{K}(\mathcal{E}) \xrightarrow{\sim} \mathrm{K}(\mathcal{A})$. By dévissage again, we have $\mathrm{K}(\mathcal{A}) = \mathrm{G}(R/t)$, which is $\mathrm{K}(R/t)$ by Proposition 2.4 because R/t is stably coherent and weakly regular. In conclusion, we have shown that the canonical map $\mathrm{K}(R/t) \to \mathrm{K}(R \text{ on } tR)$ identifies the left side with the connective cover of the right side.

To obtain the result in nonconnective degrees, and so complete the proof, we claim that the canonical map $\mathrm{K}_i(R/t) \to \mathrm{K}_i(R \text{ on } tR)$ is an isomorphism for all $i \leq 0$. The case $i = 0$ has already been proved, so we proceed inductively by Bass delooping via the fundamental theorem of K-theory [TT90, Thm. 6.1]; this gives exact sequences

$$\mathrm{K}_i(A[u]) \oplus \mathrm{K}_i(A[u^{-1}]) \to \mathrm{K}_i(A[u^{\pm 1}]) \to \mathrm{K}_{i-1}(A) \to 0$$

for any ring A, and more generally

$$\mathrm{K}_i(A[u] \text{ on } IA[u]) \oplus \mathrm{K}_i(A[u^{-1}] \text{ on } IA[u^{-1}]) \to \mathrm{K}_i(A[u^{\pm 1}] \text{ on } IA[u^{\pm 1}]) \to \mathrm{K}_{i-1}(A \text{ on } I) \to 0$$

for any finitely generated ideal $I \subseteq A$. Assuming that the claim has been proved in any fixed degree $i \leq 0$ (for all pairs R, t as in the statement of the proposition), one immediately obtains the claim in degree $i-1$ by comparing the Bass exact sequences for $\mathrm{K}(R/t)$ and $\mathrm{K}(R \text{ on } tR)$, noting that the inductive hypothesis for the pair $R[u], t$ implies that $\mathrm{K}_i(R[u]/t) \to \mathrm{K}_i(R[u] \text{ on } tR[u])$ is an isomorphism, and similarly for $R[u^{-1}]$ and $R[u^{\pm 1}]$. □

3 Smooth algebras over valuation rings

In this section we study the K-theory of smooth algebras over valuation rings, and in particular prove Theorem 1.4(1).

The proof of the following lemma is close to that of [Niz98, Lem. 3.1]:

Lemma 3.1. *Let $(A_1, \mathfrak{m}_1) \to (A_2, \mathfrak{m}_2)$ be a finite flat map of regular local rings; let R_1 be a smooth A_1-algebra and set $R_2 = R_1 \otimes_{A_1} A_2$. Then the canonical map (given by extension of scalars) $\mathrm{K}(R_1 \text{ on } \mathfrak{m}_1 R_1) \to \mathrm{K}(R_2 \text{ on } \mathfrak{m}_2 R_2)$ is divisible by the integer $d = \mathrm{len}_{A_2}(A_2/\mathfrak{m}_1 A_2)$.*

Proof. Consider the functor of stable ∞-categories

$$F \colon \mathrm{Perf}(R_1/\mathfrak{m}_1 R_1) \to \mathrm{Perf}(R_1 \text{ on } \mathfrak{m}_1 R_1) \xrightarrow{(-)\otimes_{A_1} A_2} \mathrm{Perf}(R_2 \text{ on } \mathfrak{m}_2 R_2).$$

By dévissage, the first map induces an equivalence on K-theory. The composite functor $F \colon \mathrm{Perf}(R_1/\mathfrak{m}_1) \to \mathrm{Perf}(R_2 \text{ on } \mathfrak{m}_2 R_2)$ is equivalently given by the tensor product functor $(-) \otimes_{A_1/\mathfrak{m}_1} A_2/\mathfrak{m}_1 A_2$. We need to show that F induces a map on K-theory which is divisible by d; it suffices to show that in $\mathrm{K}_0(\mathrm{Fun}(\mathrm{Perf}(R_1/\mathfrak{m}_1), \mathrm{Perf}(R_2 \text{ on } \mathfrak{m}_2 R_2)))$, the class $[F]$ is divisible by d.

Any finite length $A_2/\mathfrak{m}_1 A_2$-module M induces a functor $(-) \otimes_{A_1/\mathfrak{m}_1} M \colon \mathrm{Perf}(R_1/\mathfrak{m}_1) \to \mathrm{Perf}(R_2 \text{ on } \mathfrak{m}_2 R_2)$, from which we obtain a class in $\mathrm{K}_0(\mathrm{Fun}(\mathrm{Perf}(R_1/\mathfrak{m}_1), \mathrm{Perf}(R_2 \text{ on } \mathfrak{m}_2 R_2)))$; moreover, this process takes short exact sequences of modules to sums in the K_0-group. Since $A_2/\mathfrak{m}_1 A_2$ has a finite filtration with associated graded given by d copies of $A_2/\mathfrak{m}_2$, it follows that $[F]$ is equal to d times the class of the functor $(-) \otimes_{A_1/\mathfrak{m}_1} A_2/\mathfrak{m}_2$. □

An ind-regular local ring is a local ring which is a filtered colimit of regular rings (without loss of generality, one can take a filtered colimit of regular local rings under local homomorphisms).

Proposition 3.2. *Let A be an ind-regular local ring with maximal ideal $\mathfrak{m}_A$, and let $d \geq 1$; assume that $\mathfrak{m}_A$ is the radical of a finitely generated ideal in A, and that every element of $\mathfrak{m}_A$ is a d^{th} power. Then $\mathrm{K}(R \text{ on } \mathfrak{m}_A R)/d = 0$ for every smooth A-algebra R, i.e., $\mathrm{K}(R)/d \xrightarrow{\sim} \mathrm{K}(\mathrm{Spec}(R) \setminus V(\mathfrak{m}_A R))/d$.*

Proof. Let x be a class in $\pi_n(\mathrm{K}(R \text{ on } \mathfrak{m}_A R)/d)$ for some integer n; we need to show that x vanishes. By assumption A is a filtered colimit of regular local rings, so there exists a regular local ring $(A_0, \mathfrak{m}_0)$, a map of local rings $(A_0, \mathfrak{m}_0) \to (A, \mathfrak{m}_A)$, a smooth A_0-algebra R_0 with $R_0 \otimes_{A_0} A \simeq R$, and a class $x_0 \in \pi_n(K(R_0 \text{ on } \mathfrak{m}_0 R_0)/d)$ such that x_0 is carried to x under the canonical map

$$\mathrm{K}(R_0 \text{ on } \mathfrak{m}_0 R_0)/d \to \mathrm{K}(R \text{ on } \mathfrak{m}_A R)/d. \tag{2}$$

As $\mathfrak{m}_A$ is the radical of a finitely generated ideal, we may further assume that $\mathrm{rad}(\mathfrak{m}_0 A) = \mathfrak{m}_A$.

If A_0 is a field then we automatically have $x = 0$, so suppose $\dim(A_0) \geqslant 1$. Let $\alpha \in \mathfrak{m}_0 \setminus \mathfrak{m}_0^2$, and define the local ring $(A_1, \mathfrak{m}_1)$ via $A_1 = A_0[t]/(t^{d^2} - \alpha)$. Note that A_1 is

a regular local ring (with t part of a system of parameters), and that $A_0 \to A_1$ is finite flat with $d^2 = \mathrm{len}(A_1/\mathfrak{m}_0 A_1)$. Our assumptions imply that the map $(A_0, \mathfrak{m}_0) \to (A, \mathfrak{m}_A)$ factors over the inclusion $(A_0, \mathfrak{m}_0) \to (A_1, \mathfrak{m}_1)$. Therefore, the map (2) factors over the map $\mathrm{K}(R_0 \text{ on } \mathfrak{m}_0 R_0)/d \to \mathrm{K}(R_1 \text{ on } \mathfrak{m}_1 R_1)/d$, where $R_1 = R_0 \otimes_{A_0} A_1$. But this map is zero by Lemma 3.1. □

Consequently we recover [Niz98, Lem. 3.1], generald to arbitrary absolutely integrally closed valuation rings.

Corollary 3.3. *Let V be an absolutely integrally closed valuation ring, and let R be a smooth V-algebra. Then the canonical map $\mathrm{K}(R) \to \mathrm{K}(R \otimes_V \mathrm{Frac}(V))$ is an equivalence after profinite completion.*

Proof. Writing V as a filtered colimit of finite rank absolutely integrally closed valuation rings, we may assume that V has finite rank. In particular, in this case the maximal ideal $\mathfrak{m}_V \subset V$ is the radical of (t), for any $t \in \mathfrak{m}_V$ which does not belong to a smaller prime ideal. Using induction on the rank of the valuation (note by elementary properties of valuation rings that $V[1/t]$ is a valuation ring of rank one lower than V, unless V is a field in which case we are done), we are therefore reduced to showing that $\mathrm{K}(R \text{ on } \mathfrak{m}_V R) = \mathrm{fib}(\mathrm{K}(R) \to \mathrm{K}(R[1/t]))$ vanishes after profinite completion. But V is ind-regular by a result of Temkin [Tem17] as observed by Elmanto and Hoyois (see [AD21, Cor. 4.2.4] for a discussion), so Proposition 3.2 applies for all $d \geqslant 1$ to complete the proof. □

Corollary 3.4. *Let V be a perfect valuation ring of characteristic p, and let R be a smooth V-algebra. Then $\mathrm{K}(R; \mathbb{Z}_p) \to \mathrm{K}(R \otimes_V \mathrm{Frac}(V); \mathbb{Z}_p)$ is an equivalence.*

Proof. This is proved exactly as in Corollary 3.3, where we use Proposition 3.2 with $d = p$. Here we use purely inseparable local uniformisation [Tem13] to see that V is ind-smooth over $\mathbb{F}_p$. □

Remark 3.5 (Motivic refinements)**.** Let X be a qcqs scheme over $\mathbb{F}_p$, and suppose that X is the filtered limit of a diagram of smooth $\mathbb{F}_p$-schemes along affine transition maps. Then, defining motivic cohomology $\mathbb{Z}(i)^{\mathrm{mot}}(X)$ as the filtered colimit of the motivic cohomologies of the smooth $\mathbb{F}_p$-schemes, we see from [FS02] and [Lev08] (again by taking the filtered colimit), that $\mathrm{K}(X)$ admits a "motivic filtration" whose graded pieces $\mathrm{gr}^i \mathrm{K}(X)$ are $\mathbb{Z}(i)^{\mathrm{mot}}(X)[2i]$ for $i \geq 0$. When K-theory admits such a motivic filtration it is natural to ask whether our results can be upgraded to filtered equivalences. For example, for R a smooth algebra over a perfect valuation ring V of characteristic p, the motivic refinement of Corollary 3.4 states that $\mathbb{Z}(i)^{\mathrm{mot}}(R)/p \xrightarrow{\sim} \mathbb{Z}(i)^{\mathrm{mot}}(R \otimes_V \mathrm{Frac}(V))/p$ for all $i \geq 0$; in this remark we show that this is indeed true.

Repeating the proof of Corollary 3.4, it is enough to establish the following motivic variant of Proposition 3.2 (in the characteristic p context): let A be an ind-smooth local $\mathbb{F}_p$-algebra, such that A is perfect and its maximal ideal $\mathfrak{m}_A$ is the radical of a finitely generated ideal. Then, for any smooth (or ind-smooth) A-algebra R, the canonical maps $\mathbb{Z}(i)^{\mathrm{mot}}(R)/p \to \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Spec}(R) \setminus V(\mathfrak{m}_A R))/p$ are equivalences for all $i \geq 0$.

To prove this, we can assume that R is local and essentially smooth over A. Then, by the Geisser–Levine theorem [GL00], the motivic filtration on $\mathrm{K}(R; \mathbb{F}_p)$ is just the Postnikov filtration. Since we already know that $\mathrm{K}(R; \mathbb{F}_p) \xrightarrow{\sim} \mathrm{K}(\mathrm{Spec}(R) \setminus V(\mathfrak{m}_A R); \mathbb{F}_p)$ by Proposition 3.2, it remains to to show that the motivic filtration on $\mathrm{K}(\mathrm{Spec}(R) \setminus V(\mathfrak{m}_A R); \mathbb{F}_p)$ is also the Postnikov filtration, or in other words that $\mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Spec}(R) \setminus V(\mathfrak{m}_A R))/p$ is

concentrated in cohomological degree i. In one direction, we know (again by [GL00]) that $\mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Spec}(R) \setminus V(\mathfrak{m}_A R))/p$ is concentrated in cohomological degrees $\geqslant i$; it remains to prove the bound in the other direction. Now we can write the pair $(A, \mathfrak{m}_A)$ as a filtered colimit of of essentially smooth, local $\mathbb{F}_p$-algebras $(A_0, \mathfrak{m}_{A_0})$ with maps $(A_0, \mathfrak{m}_{A_0}) \to (A, \mathfrak{m}_A)$ such that $\mathfrak{m}_{A_0}$ generates $\mathfrak{m}_A$ up to radical; we similarly write R as a filtered colimit of algebras R_0 which are essentially smooth and local over such A_0. Then the Gysin sequence in motivic cohomology (see [MVW06, Thm. 15.15])

$$\mathbb{Z}(i-d)^{\mathrm{mot}}(R_0/\mathfrak{m}_{A_0})[-d]/p \to \mathbb{Z}(i)^{\mathrm{mot}}(R_0)/p \to \mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Spec}(R_0) \setminus V(\mathfrak{m}_{A_0} R_0))/p$$

(where $d = \dim(A_0)$) shows that $\mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Spec}(R_0) \setminus V(\mathfrak{m}_{A_0} R_0))/p$ is concentrated in cohomological degrees $\leq i$. Passing to the limit yields the same bound for $\mathbb{Z}(i)^{\mathrm{mot}}(\mathrm{Spec}(R) \setminus V(\mathfrak{m}_A R))/p$ and so completes the proof.

Although we will not need it, we record a final corollary of Proposition 3.2 which extends Corollary 3.4 to smooth algebras over arbitrary ind-smooth perfect domains. To bridge the gap between the punctured spectrum of Proposition 3.2 and the full field of fractions we must first prove the next lemma. For a noetherian spectral space X of finite Krull dimension and $x \in X$, we let X_x denote the space of all generisations of x, i.e., the intersection of all open subsets containing x; note that X_x is itself a noetherian spectral space with the subspace topology. We recall also if X is irreducible, then constant sheaves on X have no higher cohomology [Sta18, Tag 02UU], so constant sheaves and presheaves of spectra are the same, and constant sheaves are pushed forward from the generic point.

Lemma 3.6. *Let X be an irreducible, noetherian spectral space of finite Krull dimension. Let $\mathcal{G}$ be a sheaf of spectra on X; we extend $\mathcal{G}$ by continuity to all pro-open subsets of X. Suppose that for each $x \in X$, we have $\mathcal{G}(X_x) \xrightarrow{\sim} \mathcal{G}(X_x \setminus \{x\})$. Then $\mathcal{G}$ is constant. That is, for every nonempty open subset $U \subset X$, we have $\mathcal{G}(X) \xrightarrow{\sim} \mathcal{G}(U)$.*

Proof. Let η be the generic point of X, and let $\mathcal{G}_\eta$ denote the stalk of $\mathcal{G}$ at the generic point. We claim that $\mathcal{G}$ is the constant sheaf (or presheaf) with value $\mathcal{G}_\eta$. Note also that equivalences of sheaves of spectra can be detected on stalks by our assumptions and [Lur09, Cor. 7.2.4.20].

To prove the claim, we induct on the Krull dimension of X. For each $x \in X$ with $x \neq \eta$, we need to see that the generisation map $\mathcal{G}_x = \mathcal{G}(X_x) \to \mathcal{G}_\eta$ is an equivalence. However, by induction on the dimension of X, we find that $\mathcal{G}$ defines the constant presheaf on the pro-open subset $X_x \setminus \{x\} \subset X$, which is an irreducible, noetherian spectral space of smaller Krull dimension.[5] Therefore, by induction, $\mathcal{G}_x \xrightarrow{\sim} \mathcal{G}(X_x \setminus \{x\}) \xrightarrow{\sim} \mathcal{G}_\eta$ as desired. □

Corollary 3.7. *Let A be a perfect integral domain which is ind-smooth over $\mathbb{F}_p$, and let R be a smooth A-algebra. Then $\mathrm{K}(R; \mathbb{Z}_p) \to \mathrm{K}(R \otimes_A \mathrm{Frac}(A); \mathbb{Z}_p)$ is an equivalence.*

Proof. Equivalently, the assertion is that $\mathrm{K}(R; \mathbb{F}_p) \xrightarrow{\sim} \mathrm{K}(R[1/t]; \mathbb{F}_p)$ for any nonzero $t \in A$. As such, we may reduce to the case where A is the perfection of a smooth domain over $\mathbb{F}_p$. Then $\mathrm{Spec}(A)$ is a noetherian, irreducible spectral space of finite Krull dimension. We have a sheaf of spectra $\mathcal{F}$ on $\mathrm{Spec}(A)$ which sends an open subset $U \subset \mathrm{Spec}(A)$ to $\mathcal{F}(U) = \mathrm{K}(\mathrm{Spec}(R) \times_{\mathrm{Spec}(A)} U; \mathbb{F}_p)$. Proposition 3.2 (with $d = p$) and Lemma 3.6 imply that $\mathcal{F}$ is a constant presheaf. □

[5] Geometrically, $X = \mathrm{Spec}(A)$ for some domain A, in which case A, in which case $X_x \setminus \{x\}$ corresponds to $\mathrm{Spec}(A_{\mathfrak{p}}) \setminus \{\mathfrak{p}\}$ for some $\mathfrak{p} \in \mathrm{Spec}(A)$.

We now return to smooth algebras over valuation rings and prove the main results of the section.

Proposition 3.8. *Let V be a perfect valuation ring of characteristic p, let $t \in V$ be nonzero, and let $\overline{R}$ be a smooth V/t-algebra. Then $\mathrm{G}(\overline{R}; \mathbb{Z}_p) = 0$.*

Proof. We can lift $\overline{R}$ to a smooth V-algebra R by [Sta18, Tag 07M8]. The V-algebra R is weakly regular stably coherent (Corollary 2.3). By Proposition 2.5 we have a localisation sequence $\mathrm{G}(\overline{R}) \to \mathrm{K}(R) \to \mathrm{K}(R[1/t])$, and the result now follows from Corollary 3.4 (for the valuation rings V and $V[1/t]$) which shows $\mathrm{K}(R; \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{K}(R[1/t]; \mathbb{Z}_p)$. □

The next result establishes Theorem 1.4(1).

Theorem 3.9. *Let $\mathcal{O}$ be a perfectoid valuation ring and let R be a smooth $\mathcal{O}$-algebra. Then the map $\mathrm{K}(R; \mathbb{Z}_p) \to \mathrm{K}(R \otimes_{\mathcal{O}} \mathrm{Frac}(\mathcal{O}); \mathbb{Z}_p)$ is an equivalence.*

Proof. We may assume that $\mathcal{O}$ is of mixed characteristic, since the positive characteristic case has already been handled in Corollary 3.4. Let $t = up \in \mathcal{O}$ be a unit multiple of p admitting a compatible sequence of p-power roots (see [BMS18, Lem. 3.9]) and let $t^\flat = (t, t^{1/p}, t^{1/p^2}, \dots)$ be the corresponding element of the tilt $\mathcal{O}^\flat$. Note that $\mathrm{Frac}(\mathcal{O}) = \mathcal{O}[1/p] = \mathcal{O}[1/t]$. Then $\mathcal{O}^\flat$ is a perfect valuation ring of characteristic p and the multiplicative untilting map $\#\colon \mathcal{O}^\flat \to \mathcal{O}$ induces an isomorphism of rings $\mathcal{O}^\flat/t^\flat\mathcal{O}^\flat \cong \mathcal{O}/p\mathcal{O}$ [ČS19, (2.1.2.2)]. So we may view $R/tR \cong R/pR$ as a smooth $\mathcal{O}^\flat/t^\flat\mathcal{O}^\flat$-algebra, whence $\mathrm{G}(R/tR; \mathbb{Z}_p) = 0$ by Proposition 3.8. Then the localisation sequence of Proposition 2.5 completes the proof. □

We also record the following strengthening of Theorem 3.9.

Corollary 3.10. *Let $\mathcal{O}$ be a perfectoid valuation ring, $t \in \mathcal{O}$ a nonzero element, and R a t-torsion-free $\mathcal{O}$-algebra such that $\mathcal{O}/t\mathcal{O} \to R/tR$ is smooth. Then the map $\mathrm{K}(R; \mathbb{Z}_p) \to \mathrm{K}(R[1/t]; \mathbb{Z}_p)$ is an equivalence.*

Proof. We may lift R/tR to a smooth $\mathcal{O}$-algebra R' by [Sta18, Tag 07M8]. Then, by the infinitesimal lifting criterion for smoothness, we may lift the identification $R'/tR' = R/tR$ to a morphism $R' \to \widehat{R}$ where the hat denotes t-adic completion. This in turn induces a morphism $\widehat{R'} \to \widehat{R}$, which is an isomorphism since both sides are t-torsion-free and it is an isomorphism modulo t. Considering the following diagram in which the outer two squares

$$\begin{array}{ccccccc}
\mathrm{K}(R') & \longrightarrow & \mathrm{K}(\widehat{R'}) & = & \mathrm{K}(\widehat{R}) & \longleftarrow & \mathrm{K}(R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{K}(R'[1/t]) & \longrightarrow & \mathrm{K}(\widehat{R'}[1/t]) & = & \mathrm{K}(\widehat{R}[1/t]) & \longleftarrow & \mathrm{K}(R[1/t])
\end{array}$$

are cartesian, the problem reduces to showing that $\mathrm{K}(R'; \mathbb{Z}_p) \xrightarrow{\sim} \mathrm{K}(R'[1/t]; \mathbb{Z}_p)$. If $\mathcal{O}$ has positive characteristic then this follows from Corollary 3.4 for both $\mathcal{O}$ and $\mathcal{O}[1/t]$. If $\mathcal{O}[1/t] = \mathcal{O}[1/p]$ then this follows from Theorem 3.9. It remains to treat the case that $\mathcal{O}$ is of mixed characteristic and that $t \notin \sqrt{p\mathcal{O}}$. In this case, $p\mathcal{O} \subseteq p\mathcal{O}[1/t] \subseteq \mathcal{O}$, whence $\mathcal{O}[1/t]$ is p-adically complete and separated and thus is a perfectoid valuation ring. Therefore we conclude in this case by applying Theorem 3.9 to both $\mathcal{O}$ and $\mathcal{O}[1/t]$. □

4 Cdh sheaves on perfect schemes

In this section we present some cdh-descent properties for localizing invariants on perfect schemes and on their Witt vectors. We first recall the definition of the cdh-topology in the non-noetherian setting. Throughout, we will use the Nisnevich topology in the non-noetherian setting, cf. [Lur18, Sec. 3.7].

Definition 4.1 (The cdh-topology)**.** An *abstract blow-up square* of schemes

$$\begin{array}{ccc} Y' & \longrightarrow & X' \\ \downarrow & & \downarrow f \\ Y & \xrightarrow[i]{} & X \end{array} \tag{3}$$

is a cartesian square where i is a finitely presented closed embedding and f is a proper finitely presented morphism inducing an isomorphism $X' \setminus Y' \xrightarrow{\sim} X \setminus Y$. The *cdh topology* on the category $\mathrm{Sch}^{\mathrm{qcqs}}$ of qcqs schemes is the topology generated by the Nisnevich topology and by $\{Y \to X,\, X' \to X\}$ as one runs over all abstract blow-up squares of qcqs schemes. We will also work with the restriction of these topologies to $\mathrm{Sch}^{\mathrm{qcqs}}_A$, the category of qcqs schemes over a base ring A.

Proposition 4.2. *Let A be a base ring and let $\mathcal{D}$ be a complete ∞-category. If $E\colon \mathrm{Sch}^{\mathrm{qcqs,op}}_A \to \mathcal{D}$ is a Nisnevich sheaf, then the following are equivalent:*

1. *E is a cdh sheaf;*
2. *E sends all abstract blow-up squares in $\mathrm{Sch}^{\mathrm{qcqs}}_A$ to cartesian squares of $\mathcal{D}$;*

Proof. Since we are already assuming that E is a Nisnevich sheaf, the equivalence of (1) and (2) is a result of Voevodsky about cd-structures [Voe10, Cor. 5.10]. We refer to [AHW17, Thm. 3.2.5] for a modern treatment. □

The cdh sheaves of interest to us will satisfy the following excision property.

Definition 4.3 (Excision)**.** Given a base ring A and a functor $\mathcal{F}$ from the category of A-algebras to some stable ∞-category $\mathcal{D}$, we say that $\mathcal{F}$ *satisfies excision* if the following holds. For every map of pairs $f\colon (B, I) \to (C, J)$, where B, C are A-algebras and $I \subset B$, $J \subset C$ are ideals such that f carries I isomorphically onto J, then $\mathcal{F}$ carries the square (usually called a *Milnor square*)

$$\begin{array}{ccc} B & \longrightarrow & C \\ \downarrow & & \downarrow \\ B/I & \longrightarrow & C/J, \end{array}$$

to a cartesian square in $\mathcal{D}$.

Given a ring A (commutative as always), we will study certain localizing invariants $\mathrm{Mod}_{\mathrm{Perf}(A)}(\mathrm{Cat}^{\mathrm{perf}}_\infty) \to \mathcal{D}$, where $\mathcal{D}$ is a stable ∞-category and $\mathrm{Mod}_{\mathrm{Perf}(A)}(\mathrm{Cat}^{\mathrm{perf}}_\infty)$ is the ∞-category of A-linear stable ∞-categories, i.e., modules over the stably symmetric monoidal ∞-category $\mathrm{Perf}(A)$ viewed as an algebra object of $\mathrm{Cat}^{\mathrm{perf}}_\infty$. See, for example, [LT19, Rmk. 1.7]

or [CMNN20, Sec. 3.2] for further details. In this section we will be interested in the case where A is a perfect ring or its ring of Witt vectors.

Recall that an $\mathbb{F}_p$-scheme X is called *perfect* if the absolute Frobenius $\varphi\colon X \to X$ is an isomorphism. Given an arbitrary $\mathbb{F}_p$-scheme X, its *perfection* is the scheme $X_{\mathrm{perf}} := \varprojlim_{\varphi} X$. See [BS17, Sec. 3] for an account of the theory of perfect schemes.

Here is the main theorem of this section, showing that localizing invariants of perfections are cdh sheaves.

Theorem 4.4. *Let A be a perfect $\mathbb{F}_p$-algebra, $\mathcal{C}$ a $W(A)$-linear stable ∞-category such that $\mathcal{C}[\frac{1}{p}] \simeq 0$, and $E\colon \mathrm{Mod}_{\mathrm{Perf}(W(A))}(\mathrm{Cat}_\infty^{\mathrm{perf}}) \to \mathcal{D}$ a localizing invariant of $W(A)$-linear stable ∞-categories valued in a stable ∞-category $\mathcal{D}$. Then there is a cdh sheaf $\mathcal{F}\colon \mathrm{Sch}_A^{\mathrm{qcqs,op}} \to \mathcal{D}$ characterised by*

$$\mathcal{F}(\operatorname{Spec} B) = E(\mathcal{C} \otimes_{\mathrm{Perf}(W(A))} \mathrm{Perf}(W(B_{\mathrm{perf}})))$$

for all A-algebras B; moreover, $\mathcal{F}$ satisfies excision.

Proof. Firstly, by writing $\mathcal{C}$ as a colimit as in [BCM20, Prop. 2.15], we can assume that $\mathcal{C}$ is a $W_n(A)$-linear ∞-category for some $n \gg 0$; here we implicitly use that a filtered colimit of cdh sheaves is a cdh sheaf, which follows from Proposition 4.2 as cartesian squares are preserved under filtered colimits, and since Nisnevich descent can be tested via Nisnevich excision [Lur18, Th. 3.7.5.1]. The goal is to construct a cdh sheaf given on affines by $\operatorname{Spec} B \mapsto E(\mathcal{C} \otimes_{\mathrm{Perf}(W_n(A))} \mathrm{Perf}(W_n(B_{\mathrm{perf}})))$; this has the advantage that it extends to qcqs A-schemes X by replacing $W_n(B_{\mathrm{perf}})$ by the scheme $W_n(X_{\mathrm{perf}})$.

We may moreover replace E by $E(\mathcal{C} \otimes_{\mathrm{Perf}(W_n(A))} -)$ so that E is now a localizing invariant of $\mathrm{Perf}(W_n(A))$-linear ∞-categories and our goal is to show simply that the functor

$$\mathrm{Sch}_A^{\mathrm{qcqs}} \to \mathcal{D}, \quad X \mapsto E(W_n(X_{\mathrm{perf}})) := E(\mathrm{Perf}(W_n(X_{\mathrm{perf}})))$$

is a cdh sheaf which satisfies excision.

The construction $X \mapsto E(W_n(X_{\mathrm{perf}}))$ satisfies Nisnevich descent by a result of Thomason–Trobaugh [TT90], cf. [CMNN20, App. A]. To verify cdh-descent, we need to show that any abstract blow-up square (3) is carried to a cartesian square in $\mathcal{D}$.

We prove this using Bhatt–Scholze's v-descent for quasi-coherent complexes on perfect schemes [BS17, Cor. 11.28]. Since $Y \to X$ and $X' \to X$ are finitely presented, *loc. cit.* implies that the square

$$\begin{array}{ccc} \mathrm{QCoh}(W_n(X_{\mathrm{perf}})) & \longrightarrow & \mathrm{QCoh}(W_n(X'_{\mathrm{perf}})) \\ \downarrow & & \downarrow{\scriptstyle j^*} \\ \mathrm{QCoh}(W_n(Y_{\mathrm{perf}})) & \longrightarrow & \mathrm{QCoh}(W_n(Y'_{\mathrm{perf}})) \end{array}$$

is a pullback of ∞-categories, where j^* is pullback along $Y'_{\mathrm{perf}} \to X'_{\mathrm{perf}}$. Furthermore, the right adjoint j_* of j^* is fully faithful: indeed, $j^* j_* \simeq \mathrm{id}$ since $C \otimes_B^L C \simeq C$ for any surjection of perfect rings $B \to C$ [BS17, Lem 3.16]. That is, the square

$$\begin{array}{ccc} \mathrm{Perf}(W_n(X_{\mathrm{perf}})) & \longrightarrow & \mathrm{Perf}(W_n(X'_{\mathrm{perf}})) \\ \downarrow & & \downarrow \\ \mathrm{Perf}(W_n(Y_{\mathrm{perf}})) & \longrightarrow & \mathrm{Perf}(W_n(Y'_{\mathrm{perf}})) \end{array}$$

in $\mathrm{Mod}_{\mathrm{Perf}(W_n(A))}(\mathrm{Cat}_\infty^{\mathrm{perf}})$ is excisive in the sense of [Tam18, Def. 14] (note that all our schemes are qcqs, so $\mathrm{Ind}(\mathrm{Perf}(-)) = \mathrm{QCoh}(-)$), and so Tamme's excision criterion [Tam18, Thm. 18] shows that

$$\begin{array}{ccc} E(W_n(X_{\mathrm{perf}})) & \longrightarrow & E(W_n(X'_{\mathrm{perf}})) \\ \downarrow & & \downarrow \\ E(W_n(Y_{\mathrm{perf}})) & \longrightarrow & E(W_n(Y'_{\mathrm{perf}})) \end{array}$$

is indeed cartesian, as required to complete the proof of cdh-descent.

Given an excision situation of A-algebras, $(B, I) \to (C, J)$, then $(B_{\mathrm{perf}}, I' := \sqrt{IB_{\mathrm{perf}}}) \to (C_{\mathrm{perf}}, J' := \sqrt{JC_{\mathrm{perf}}})$ is an excision situation (note that the radical I' is precisely the kernel of $B_{\mathrm{perf}} \to (B/I)_{\mathrm{perf}}$, and similarly for J'), and then so is $(W_n(B_{\mathrm{perf}}), W_n(I')) \to (W_n(C_{\mathrm{perf}}), W_n(J'))$. We claim that $W_n(B_{\mathrm{perf}}) \to W_n(B_{\mathrm{perf}})/W_n(I') = W_n((B/I)_{\mathrm{perf}})$ is Tor-unital in the sense of [Tam18, Def. 21]. Indeed, to show that the canonical map $W_n((B/I)_{\mathrm{perf}}) \otimes^L_{W_n(B_{\mathrm{perf}})} W_n((B/I)_{\mathrm{perf}}) \to W_n((B/I)_{\mathrm{perf}})$ is an equivalence it is enough to check after base change along $\mathbb{Z}/p^n \to \mathbb{Z}/p$, as which point we obtain the equivalence $(B/I)_{\mathrm{perf}} \otimes^L_{B_{\mathrm{perf}}} (B/I)_{\mathrm{perf}} \simeq (B/I)_{\mathrm{perf}}$ of [BS17, Lem 3.16]. It now follows from Tamme's excision condition [Tam18, Thm. 28] that

$$\begin{array}{ccc} E(W_n(B_{\mathrm{perf}})) & \longrightarrow & E(W_n(C_{\mathrm{perf}})) \\ \downarrow & & \downarrow \\ E(W_n((B/I)_{\mathrm{perf}})) & \longrightarrow & E(W_n((C/J)_{\mathrm{perf}})) \end{array}$$

is indeed cartesian, as desired. □

Question 4.5. Given an abstract blow-up square of qcqs $\mathbb{F}_p$-schemes as in Definition 4.1 but without the assumption that $f\colon X' \to X$ is finitely presented, is it true that applying $\mathrm{QCoh}(-_{\mathrm{perf}})$ gives a pullback square of ∞-categories? Under the additional assumption that $f : X' \to X$ is finitely presented this is precisely [BS17, Cor. 11.28], which was used above.

The above result will be useful in reducing questions to henselian valuation rings in light of the next result, which we quote for convenience. We refer to [EHIK21, Sec. 2.3] for an introduction to Jaffard's notion of valuative dimension; note that the perfection of any finitely generated $\mathbb{F}_p$-algebra has finite valuative dimension, given by its Krull dimension (indeed, there is a one-to-one correspondence between the valuation subrings of a field of characteristic p and of its perfection, or else more generally we could apply [Jaf60, Prop. 4, p. 54]). A cdh-sheaf on $\mathrm{Sch}_A^{\mathrm{qcqs}}$ is said to be *finitary* if it preserves filtered colimits of A-algebras.

Proposition 4.6 ([EHIK21, Cor. 2.4.19])**.** *If A is an $\mathbb{F}_p$-algebra of finite valuative dimension, then a map of finitary cdh-sheaves $\mathrm{Sch}_A^{\mathrm{qcqs,op}} \to \mathcal{S}$ (for $\mathcal{S}$ the ∞-category of spaces) which is an equivalence on henselian valuation rings is an equivalence.*

5 Applications

In this section, we complete the proofs of our main theorems concerning algebraic K-theory.

5.1 Characteristic p

We begin with two propositions about the K-theory of perfect schemes; the results in mixed characteristic will then follow by an elaboration of the arguments.

Proposition 5.1. *Let R be a smooth algebra over a perfect $\mathbb{F}_p$-algebra A. Then the canonical map $\mathrm{K}(R) \to \mathrm{KH}(R)$ is an equivalence.*

Proof. Taking a filtered colimit, it suffices to prove the result when A is the perfection of a finitely generated $\mathbb{F}_p$-algebra. Theorem 4.4, with $\mathcal{C} = \mathrm{Perf}(R)$ and $\mathrm{Perf}(R[T])$ respectively, implies that the functors $X \mapsto \mathrm{K}(X_{\mathrm{perf}} \otimes_A R)$ and $X \mapsto \mathrm{K}(X_{\mathrm{perf}} \otimes_A R[T])$, from finitely presented A-schemes to spectra, are cdh sheaves; recall here that $\mathrm{Perf}(X_{\mathrm{perf}}) \otimes_{\mathrm{Perf}(A)} \mathrm{Perf}(R) \simeq \mathrm{Perf}(X_{\mathrm{perf}} \otimes_A R)$ and similarly for $R[T]$, by e.g., [Lur17, Th. 4.8.4.6]. To check that the canonical map $\mathrm{K}(R) \to \mathrm{K}(R[T])$ is an equivalence, we therefore reduce by Proposition 4.6 to proving that $\mathrm{K}(V_{\mathrm{perf}} \otimes_A R) \xrightarrow{\sim} \mathrm{K}(V_{\mathrm{perf}} \otimes_A R[T])$ for all henselian valuation rings V under A; but that is a special case of Proposition 2.4(2) since V_{perf} is a valuation ring. □

Question 5.2 (Cartier smooth algebras)**.** Can Proposition 5.1 be extended to those $\mathbb{F}_p$-algebras R which are Cartier smooth in the sense of [KM21], i.e., those for which the cotangent complex $L_{R/\mathbb{F}_p}$ is a flat R-module and the Cartier isomorphism for de Rham cohomology holds?

Question 5.3 (Motivic refinement)**.** Let R be as in Proposition 5.1, or more generally Cartier smooth as in the previous question. Then the Geisser–Levine theorem [GL00] holds Zariski locally on $\mathrm{Spec}\, R$ by [KM21], and so the Zariski local Postnikov filtration on $\mathrm{K}(R; \mathbb{F}_p)$ deserves to be termed the "motivic filtration". Does the equivalence $\mathrm{K}(R; \mathbb{F}_p) \xrightarrow{\sim} \mathrm{K}(R[T]; \mathbb{F}_p)$ (following from Proposition 5.1) upgrade to a filtered equivalence, i.e., are the canonical maps $R\Gamma_{\mathrm{Zar}}(\mathrm{Spec}\, R, \Omega^i_{\log}) \to R\Gamma_{\mathrm{Zar}}(\mathrm{Spec}\, R[T], \Omega^i_{\log})$ equivalences? When A is a perfect valuation ring, the answer is 'yes': perfect valuation rings are ind-smooth [Tem13], so the motivic filtration defined at the top of Remark 3.5 is $\mathbb{A}^1$-invariant.

Regarding the K-theory of perfect schemes themselves (rather than smooth schemes over them), we record the following calculation since it has not explicitly appeared previously; the cdarc topology, which is a completely decomposed analogue of the arc topology [BM21], is defined in [EHIK21].

Corollary 5.4. *1. For any perfect qcqs $\mathbb{F}_p$-scheme X, there is a natural equivalence $\mathrm{K}(X; \mathbb{Z}/p^r) \simeq R\Gamma_{\mathrm{cdh}}(X, \mathbb{Z}/p^r)$, where the right side denotes cdh cohomology on $\mathrm{Sch}^{\mathrm{qcqs}}_X$ of the constant sheaf $\mathbb{Z}/p^r$.*

2. The presheaf $X \mapsto \mathrm{K}(X_{\mathrm{perf}}; \mathbb{Z}/p^r)$ on $\mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}_p}$ satisfies cdarc descent.

Proof. Theorem 4.4 implies that $\mathrm{K}((-)_{\mathrm{perf}}; \mathbb{Z}/p^r)$ is a cdh sheaf on $\mathrm{Sch}^{\mathrm{qcqs,op}}_{\mathbb{F}_p}$. For any qcqs $\mathbb{F}_p$-scheme X there is a map $\mathbb{Z}/p^r = \mathrm{K}(\mathbb{F}_p; \mathbb{Z}/p^r) \to \mathrm{K}(X_{\mathrm{perf}}; \mathbb{Z}/p^r)$, which induces a map $R\Gamma_{\mathrm{cdh}}(-, \mathbb{Z}/p^r) \to \mathrm{K}((-)_{\mathrm{perf}}; \mathbb{Z}/p^r)$ of cdh sheaves.

Note that both these sheaves commute with filtered colimits of rings, the latter because K-theory is finitary and the former by [Sta18, Tag 0737]; furthermore, both these sheaves take values in $\mathcal{D}(\mathbb{Z}/p^r)_{\leqslant 0}$, the latter by Theorem 1.1. Since both these cdh sheaves satisfy Milnor excision (the latter by Theorem 4.4), "(1) implies (3)" of the main theorem of [EHIK21] therefore implies that they are both in fact cdarc sheaves. This completes the proof of part (2).

We have also reduced part (1) to the case that $X = \operatorname{Spec} A$ is the spectrum of a perfect $\mathbb{F}_p$-algebra; writing A as a filtered colimit of perfections of finite type $\mathbb{F}_p$-algebras then allows us to even assume that A has finite valuative dimension. Then viewing $R\Gamma_{\mathrm{cdh}}(-, \mathbb{Z}/p^r) \to \mathrm{K}((-)_{\mathrm{perf}}; \mathbb{Z}/p^r)$ as a map of cdh sheaves on $\mathrm{Sch}_A^{\mathrm{qcqs}}$, Proposition 4.6 reduces the problem to checking that $\mathbb{Z}/p^r \to \mathrm{K}(V_{\mathrm{perf}}; \mathbb{Z}/p^r)$ is an equivalence for all henselian valuation rings V under A. But V_{perf} is again a valuation ring, hence weakly regular stably coherent by Proposition 2.1 and so has no negative K-groups by Proposition 2.4; then Theorem 1.1 shows that indeed $\mathbb{Z}/p^r \xrightarrow{\sim} \mathrm{K}(V_{\mathrm{perf}}; \mathbb{Z}/p^r)$, as desired. □

In [BS17, Corollary 5.6], Bhatt and Scholze prove that if A is the perfection of a regular $\mathbb{F}_p$-algebra, then there is a localisation fiber sequence

$$\mathrm{K}(A) \to \mathrm{K}(W(A)) \to \mathrm{K}(W(A)[1/p]).$$

More precisely, they show that the canonical map $\mathrm{K}(A) \to \mathrm{K}(\mathrm{W}(A) \text{ on } pW(A))$ is an equivalence; this is used to define their determinant line bundle on the Witt vector affine Grassmannian. We will prove more generally that this assertion is true for any perfect $\mathbb{F}_p$-algebra A, and even for algebras over $W(A)$ satisfying a "p-smoothness" condition as in Corollary 3.10.

Proposition 5.5. *Let A be any perfect $\mathbb{F}_p$-algebra, and let R be a p-torsion-free $W(A)$-algebra such that $A \to R/pR$ is smooth. Then the canonical map $\mathrm{K}(R/pR) \to \mathrm{K}(R \text{ on } pR)$ is an equivalence, so there is a localisation sequence $\mathrm{K}(R/pR) \to \mathrm{K}(R) \to \mathrm{K}(R[1/p])$.*

Proof. Since we can replace R by its p-completion, it suffices to assume that R is p-complete. The functor that sends the perfect $\mathbb{F}_p$-algebra A to isomorphism classes of p-complete $W(A)$-algebras R with R/p smooth over A commutes with filtered colimits in A, since it is also isomorphic to the functor of smooth A-algebras. Using this observation, and the fact that $\mathrm{K}(R \text{ on } pR)$ commutes with filtered colimits in p-complete R, we may reduce to the case where A is the perfection of a finitely generated $\mathbb{F}_p$-algebra.

Theorem 4.4, with $\mathcal{C} = \mathrm{Perf}(R/pR)$ and $\mathrm{Perf}(R \text{ on } pR)$ respectively, implies that there are cdh sheaves on $\mathrm{Sch}_A^{\mathrm{qcqs}}$ given on affines by $\operatorname{Spec} B \mapsto \mathrm{K}(B_{\mathrm{perf}} \otimes_A R/pR)$ and $\mathrm{K}(W(B_{\mathrm{perf}}) \otimes_{W(A)} R \text{ on } p)$.[6] To prove that the canonical map $\mathrm{K}(R/pR) \to \mathrm{K}(R \text{ on } pR)$ is an equivalence, we therefore reduce by Proposition 4.6 to proving that $\mathrm{K}(V_{\mathrm{perf}} \otimes_A R/pR) \xrightarrow{\sim} \mathrm{K}(W(V_{\mathrm{perf}}) \otimes_{W(A)} R \text{ on } p)$ for all henselian valuation rings V under A. This follows from Proposition 2.6 with $t = p$. □

Corollary 5.6. *Let A and R be as in the statement of Proposition 5.5.*

1. *There is a localisation sequence $\mathrm{KH}(R/pR) \to \mathrm{KH}(R) \to \mathrm{KH}(R[\frac{1}{p}])$.*
2. *The canonical map $\mathrm{K}(R; \mathbb{Z}_p) \to \mathrm{KH}(R; \mathbb{Z}_p)$ is an equivalence.*

Proof. Part (1) follows by applying Proposition 5.5 to $R[T_0, \dots, T_n]$ for all $n \geq 0$ and taking the geometric realisation.

Part (2) then follows by comparing the localisation sequences in $\mathrm{K}(-; \mathbb{Z}_p)$ and $\mathrm{KH}(-; \mathbb{Z}_p)$. Indeed we know that $\mathrm{K}(R/pR) \xrightarrow{\sim} \mathrm{KH}(R/pR)$ by Proposition 5.1, while $\mathrm{K}(R[1/p]; \mathbb{Z}_p) \to \mathrm{KH}(R[1/p]; \mathbb{Z}_p)$ is an equivalence thanks to [Wei89, Prop. 1.6]. □

[6] Here, and on occasion below, we break with our convention and replace for example $\mathrm{K}(W(B_{\mathrm{perf}}) \otimes_{W(A)} R \text{ on } p(W(B_{\mathrm{perf}}) \otimes_{W(A)} R))$ by $\mathrm{K}(W(B_{\mathrm{perf}}) \otimes_{W(A)} R \text{ on } p)$ for readability.

Remark 5.7. Proposition 5.1 and Proposition 5.5 belong to a general class of results: for many purposes, perfect rings and their rings of Witt vectors behave similarly to regular rings. For example, in [BS17, Prop. 11.31], it is shown that perfectly finitely presented perfect $\mathbb{F}_p$-algebras have finite global dimension. The above results are indications of a similar phenomenon in algebraic K-theory. Note however that perfect rings can have nontrivial nonconnective K-theory.

The next result refines the classical Theorem 1.1.

Proposition 5.8. *Let R be a smooth algebra of relative dimension $\leq d$ over a perfect $\mathbb{F}_p$-algebra. Then $\mathrm{K}(R;\mathbb{F}_p)$ is d-truncated.*

Proof. We may assume by passage to a filtered colimit that the perfect base algebra A is the perfection of a finitely generated $\mathbb{F}_p$-algebra. In this case, the Zariski topos of R is hypercomplete since $\mathrm{Spec}(R)$ has finite Krull dimension and is noetherian, [Lur09, Cor. 7.2.4.20]. We therefore reduce to showing that $\mathrm{K}(R_{\mathfrak{p}};\mathbb{F}_p)$ is d-truncated for each prime ideal $\mathfrak{p} \subset R$. This may be shown in two ways. Either one has by [KM21, Th. 2.1] the analog of the Geisser–Levine theorem [GL00], computing the mod-p K-groups of $R_{\mathfrak{p}}$ as the module of logarithmic differential forms, i.e., $\mathrm{K}_n(R_{\mathfrak{p}};\mathbb{F}_p) \simeq \Omega^n_{R_{\mathfrak{p}},\log}$ for $n \geq 0$; but these vanish when $n > d$. Alternatively, we could avoid [KM21, Th. 2.1] by instead using Theorem 4.4 and Proposition 4.6 to reduce to the case that A is a perfect valuation ring, hence ind-smooth [Tem13], and apply the usual Geisser–Levine theorem. □

5.2 Mixed characteristic

Next we treat the mixed characteristic results. Given a perfectoid valuation ring $\mathcal{O}$ of mixed characteristic we briefly recall the tilting correspondence, for which we refer to [BMS18, Sec. 3] and [ČS19, Sec. 2.1] for further details. As in the proof of Theorem 3.9 we may rescale p by a unit so that it admits a compatible sequence of p-power roots, and we let $t^\flat$ be the corresponding of the tilt $\mathcal{O}^\flat$ so that the untilting map induces an isomorphism $\mathcal{O}^\flat/t^\flat \xrightarrow{\sim} \mathcal{O}/p$. For a perfect $\mathcal{O}^\flat$-algebra A, we let $A^\sharp = W(A) \otimes_{W(\mathcal{O}^\flat),\theta} \mathcal{O}$ denote its *untilt*. The untilt depends only on the (classical) $t^\flat$-adic completion of A, and $A \mapsto A^\sharp$ establishes an equivalence between $t^\flat$-adically complete and separated perfect $\mathcal{O}^\flat$-algebras and perfectoid $\mathcal{O}$-algebras; this restricts to an equivalence between perfect $\mathcal{O}^\flat$-algebras in which $t^\flat = 0$ and perfectoid (equivalently, perfect) $\mathcal{O}$-algebras in which $p = 0$.

The following result exemplifies the approach through which we may study localizing invariants over a mixed characteristic perfectoid valuation ring via the cdh topology over its tilt.

Proposition 5.9. *Let E a localizing invariant of $\mathbb{Z}$-linear stable ∞-categories valued in spectra, and fix $d, m \in \mathbb{Z}$; make the following assumption:*

> *For every perfectoid valuation ring V and every smooth V-algebra R_V such that $V/pV \to R_V/pR_V$ has relative dimension $\leq d$, the spectrum $E(R_V \text{ on } pR_V)$ is m-truncated.*

Then, for every perfectoid valuation ring $\mathcal{O}$, every perfectoid $\mathcal{O}$-algebra A, and every smooth A-algebra R such that $A/pA \to R/pR$ has relative dimension $\leq d$, the spectrum $E(R \text{ on } pR)$ is m-truncated.

Proof. We treat a series of cases, culminating in a complete proof. For $\mathcal{O}$, A, R as in the statement of the proposition, Theorem 4.4 with $\mathcal{C} = \mathrm{Perf}(R \text{ on } pR)$ implies that there exists a cdh sheaf $\mathcal{F}_{\mathcal{O},A,R}$ on $\mathrm{Sch}^{\mathrm{qcqs}}_{A^\flat}$ given on affines by

$$\mathrm{Spec}\, B \mapsto E(\mathrm{Perf}(R \text{ on } pR) \otimes_{\mathrm{Perf}(W(A^\flat))} \mathrm{Perf}(W(B_{\mathrm{perf}}))) = E(B^{\#}_{\mathrm{perf}} \otimes_A R \text{ on } p),$$

and that this sheaf satisfies excision.

Case 1: $p = 0$ in A, i.e., A is a perfect $\mathbb{F}_p$-algebra. Replacing $\mathcal{O}$ by $\mathcal{O}/\sqrt{p\mathcal{O}}$ we may clearly also suppose that $p = 0$ in $\mathcal{O}$, i.e., that $\mathcal{O}$ is a perfect valuation ring over $\mathbb{F}_p$. The assertion to be proved is that $E(R)$ is m-truncated. Writing $\mathcal{O}$ as a filtered colimit of perfect valuation rings $\mathcal{O}_i$ of finite rank, and then writing A as a filtered colimit of perfectly finitely presented $\mathcal{O}_i$-algebras for varying i, we reduce to the case that $\mathcal{O}$ has finite rank and A is a perfectly finitely presented $\mathcal{O}$-algebra. Therefore A has finite valuative dimension [EHIK21, Corol. 2.3.3] and Proposition 4.6 applies to the cdh sheaf $\Omega^{m+1}\Omega^\infty \mathcal{F}_{\mathcal{O},A,R}$: so the desired m-truncatedness of $E(R)$ follows from the assumed m-truncatedness of $E(V_{\mathrm{perf}} \otimes_A R)$, as V varies over henselian valuation rings under A.

Case 2: $\mathcal{O}$ is mixed characteristic of rank one, and A is p-torsion-free. Similarly to the previous case, we begin by reducing to a finitely presented situation. Write the tilt $A^\flat$ as a filtered colimit of perfectly finitely presented $\mathcal{O}^\flat$-algebras $B_i^\flat$, so that A is the (classical) p-completion of the filtered colimit $\mathrm{colim}_i B_i^\sharp$ of the perfectoid $\mathcal{O}$-algebras $B_i^\sharp$. Since $\mathrm{colim}_i B_i^\sharp$ is p-torsion-free (otherwise its p-completion A would also contain p-torsion; this implicitly uses that the p-power torsion in any perfectoid is killed by p, so that $\mathrm{colim}_i B_i^\sharp$ has bounded p-power torsion), we may argue just as at the beginning of the proof of Corollary 3.10 to find a smooth $\mathrm{colim}_i B_i^\sharp$-algebra R' equipped with an isomorphism $\widehat{R'} \cong \widehat{R}$ of A-algebras, where the hats denote p-adic completions. Since $\mathrm{Perf}(R' \text{ on } pR') \simeq \mathrm{Perf}(\widehat{R'} \text{ on } p\widehat{R'}) \simeq \mathrm{Perf}(\widehat{R} \text{ on } p\widehat{R}) \simeq \mathrm{Perf}(\widehat{R} \text{ on } p\widehat{R})$, the first and third isomorphisms being [Bha16, Lem. 5.12], the problem reduces to showing that $E(R' \text{ on } pR')$ is m-truncated. Descending R' to a smooth algebra over $B_i^\sharp$ for $i \gg 0$ and taking the filtered colimit, we may henceforth assume in this case that A is the untilt of a perfectly finitely presented $\mathcal{O}^\flat$-algebra B.[7] Since $\mathcal{O}^\flat$ has finite rank (even rank one), therefore B has finite valuative dimension; then the m-truncatedness of $E(R \text{ on } pR)$ follows as in Case 1, namely by applying Proposition 4.6 and using the hypothesis that $E(V^{\#}_{\mathrm{perf}} \otimes_A R \text{ on } p)$ is m-truncated as V varies over henselian valuation rings under A.

Case 3: $\mathcal{O}$ is mixed characteristic of rank one, but no conditions on A. The perfectoid ring A fits into a Milnor square with surjective arrows

$$\begin{array}{ccc} A & \longrightarrow & \overline{A} \\ \downarrow & & \downarrow \\ A_0 & \longrightarrow & \overline{A}_0 \end{array}$$

where $\overline{A}$ is a p-torsion-free perfectoid ring, and A_0 and $\overline{A}_0$ are perfect $\mathbb{F}_p$-algebras [ČS19, 2.1.3]; tilting each term forms a Milnor square of perfect $\mathbb{F}_p$-algebras [loc. cit]. The sheaf

[7] $A^\flat$ is not in general equal to B, but rather to its $t^\flat$-adic completion in the notation of the opening paragraph of the subsection. We also remark that by replacing A by $B_i^\sharp$ in this fashion might mean that A is no longer p-torsion-free, but this condition is not required in the remainder of the proof.

$\mathcal{F}_{\mathcal{O},A,R}$ satisfies excision, so the square

$$\begin{array}{ccc} E(R \text{ on } pR) & \longrightarrow & E(\overline{A} \otimes_A R \text{ on } p) \\ \downarrow & & \downarrow \\ E(A_0 \otimes_A R) & \longrightarrow & E(\overline{A}_0 \otimes_A R) \end{array}$$

is cartesian. The top right term is m-truncated by case 2, and the bottom two terms are m-truncated by case 1. Therefore the top left term is m-truncated.

Case 4: The general case. If $p = 0$ in $\mathcal{O}$ then we may appeal to case 1. Otherwise, similarly to case 3, we use an excision trick, this time applied to the Milnor square of valuation rings

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{O}_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ \mathcal{O}/\mathfrak{p} & \longrightarrow & k(\mathfrak{p}) \end{array}$$

where $\mathfrak{p} = \sqrt{p\mathcal{O}}$. Note that $\mathcal{O}_{\mathfrak{p}}$ is a perfectoid valuation ring of mixed characteristic of rank one (argue as at the end of Corollary 3.10 to see p-completeness), and the bottom two terms are perfect valuation rings. Base changing along $\mathcal{O} \to A$, we claim that

$$\begin{array}{ccc} A & \longrightarrow & A \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}} \\ \downarrow & & \downarrow \\ A \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} & \longrightarrow & A \otimes_{\mathcal{O}} k(\mathfrak{p}) \end{array}$$

is an Milnor square of perfectoid rings. Firstly, the terms are perfectoid rings after p-adic completion by [ČS19, Prop. 2.1.11(2)]; so the bottom two terms, which are $\mathbb{F}_p$-algebras, are perfect. But $A \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}}$ is already p-adically complete since $p(A \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}}) \subseteq A$ and A is p-adically complete; so $A \otimes_{\mathcal{O}} \mathcal{O}_{\mathfrak{p}}$ is perfectoid. Secondly, the square is a Milnor square because it is a base-change of a Milnor square and since $A \otimes^{\mathbb{L}}_{\mathcal{O}} \mathcal{O}/\mathfrak{p}$ is discrete.[8] Tilting each term forms a Milnor square of perfect $\mathbb{F}_p$-algebras [ČS19, Prop. 2.1.4]. The sheaf $\mathcal{F}_{\mathcal{O},A,R}$ satisfies excision, so the square

$$\begin{array}{ccc} E(R \text{ on } pR) & \longrightarrow & E(\overline{A} \otimes_A R \text{ on } p) \\ \downarrow & & \downarrow \\ E(A_0 \otimes_A R) & \longrightarrow & E(\overline{A}_0 \otimes_A R) \end{array}$$

is cartesian. The top right term is m-truncated by case 2, and the bottom two terms are m-truncated by case 1. Therefore the top left term is m-truncated. □

Theorem 5.10. *Let $\mathcal{O}$ be a perfectoid valuation ring, A a perfectoid $\mathcal{O}$-algebra, and R a smooth A-algebra such that $A/pA \to R/pR$ has relative dimension $\leq d$. Then $\mathrm{K}(R; \mathbb{F}_p) \to \mathrm{K}(R[1/p]; \mathbb{F}_p)$ is d-truncated.*

[8] In fact, if B_0, B_1, B_2 are perfectoid rings with maps $B_0 \to B_1, B_0 \to B_2$, the derived tensor product $B_1 \otimes^{\mathbb{L}}_{B_0} B_1$ has discrete derived p-completion; in fact, this follows from the result for perfect $\mathbb{F}_p$-algebras [BS17, Lem. 3.16], which implies the analogous results for their Wit vectors, and [BMS18, Lem. 3.13].

Proof. We are claiming that $\mathrm{K}(R \text{ on } pR; \mathbb{F}_p)$ is d-truncated. It suffices to check that the hypothesis of Proposition 5.9 are satisfied, namely that $\mathrm{K}(R_V \text{ on } pR_V; \mathbb{F}_p)$ is d-truncated whenever R_V is a smooth algebra over a perfectoid valuation ring V such that $V/pV \to R_V/pR_V$ has relative dimension $\leq d$. This follows from Theorem 3.9 (in the case of mixed characteristic V) and Proposition 5.8 (in the case of V of characteristic p). □

Lemma 5.11. *Let V be a valuation ring and R a smooth V-algebra. Then, for any $t \in V$, the map $\mathrm{K}(R \text{ on } tR) \to \mathrm{KH}(R \text{ on } tR)$ is an equivalence.*

Proof. By comparing R and $R[1/t]$, we see that it suffices to prove the stronger statement that $\mathrm{K}(R) \xrightarrow{\sim} \mathrm{KH}(R)$ and $\mathrm{K}(R[\frac{1}{t}]) \xrightarrow{\sim} \mathrm{KH}(R[\frac{1}{t}])$. This follows from Corollary 2.3 and Proposition 2.4. □

Theorem 5.12. *Let $\mathcal{O}$ be a perfectoid valuation ring, A a perfectoid $\mathcal{O}$-algebra, and R a smooth A-algebra. Then the canonical map $\mathrm{K}(R; \mathbb{Z}_p) \to \mathrm{KH}(R; \mathbb{Z}_p)$ is an equivalence.*

Proof. We consider the spectra-valued localizing invariant NK of $\mathbb{Z}$-linear stable ∞-categories

$$\mathrm{NK}\colon \mathcal{C} \mapsto \mathrm{hocofib}(\mathrm{K}(\mathcal{C}; \mathbb{F}_p) \to \mathrm{K}(\mathcal{C} \otimes_{\mathrm{Perf}(\mathbb{Z})} \mathrm{Perf}(\mathbb{Z}[T]); \mathbb{F}_p).$$

Lemma 5.11 (with $t = p$) shows that $\mathrm{NK}(V \text{ on } pV)$ vanishes on any valuation ring V, whence we may apply Proposition 5.9 with any value of m and so deduce that $\mathrm{NK}(R \text{ on } pR)$ vanishes. In other words, the square

$$\begin{array}{ccc} \mathrm{K}(R; \mathbb{Z}_p) & \longrightarrow & \mathrm{K}(R[T]; \mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \mathrm{K}(R[\frac{1}{p}]; \mathbb{Z}_p) & \longrightarrow & \mathrm{K}(R[\frac{1}{p}][T]; \mathbb{Z}_p) \end{array}$$

is cartesian. The bottom horizontal arrow is an equivalence since $\mathrm{K}(-; \mathbb{Z}_p)$ is homotopy invariant on $\mathbb{Z}[\frac{1}{p}]$-algebras [Wei89, Prop. 1.6], so the top horizontal arrow is also an equivalence. □

Remark 5.13 (The p-smooth case)**.** Similarly to Corollary 3.10, Theorems 5.10 and 5.12 remain true more generally if the A-algebra R is merely required to be p-smooth rather than smooth. Here we say that an A-algebra R is *p-smooth* if it satisfies the following equivalent conditions:

1. there exists a smooth A-algebra R' such that the derived p-adic completions[9] of R and R' are equivalent as animated A-algebras;
2. the A/pA-algebra $R \otimes^{\mathbb{L}}_A A/pA$ is discrete and smooth.

To obtain the stronger versions of the theorems from the smooth versions, we use characterisation (1) and the equivalence $\mathrm{Perf}(R \text{ on } pR) \xrightarrow{\sim} \mathrm{Perf}(\widehat{R} \text{ on } p\widehat{R})$, resulting from the equivalence $\widehat{R} \otimes^{\mathbb{L}}_R R/pR \simeq R/pR$ and a strong form of Thomason's excision theorem [Bha16, Lem. 5.12(2)] (and similarly for R' in place of R); here hats denote derived p-adic completions. In the case of Theorem 5.12 we also use the analogous equivalences for $R[T]$ and $R'[T]$, as well as the fact that $\mathrm{K}(-; \mathbb{Z}_p)$ is homotopy invariant on $\mathbb{Z}[\frac{1}{p}]$-algebras.

[9] For the sake of clarify, we remark that the derived p-completion of an A-module M is $\mathrm{holim}_n M/^{\mathbb{L}}p^n$ where $M/^{\mathbb{L}}p^n := M \otimes^{\mathbb{L}}_{\mathbb{Z}[T], T \mapsto p^n} \mathbb{Z}$ is the Koszul complex associated to multiplication by p^n

Lacking a simple reference, we explain why conditions (1) and (2) are indeed equivalent.[10] It is clear that (1) implies (2), so suppose that R is an A-algebra satisfying condition (2). Let R' be any smooth A-algebra lifting the smooth A/pA-algebra $R \otimes^{\mathbb{L}}_A A/pA$ [Sta18, Tag 07M8]. By derived deformation theory (cf. [Lur, Sec. 3.4]) the canonical map $R' \to R'/pR' = R \otimes^{\mathbb{L}}_A A/pA = \widehat{R} \otimes^{\mathbb{L}}_A A/pA$ may be lifted to a morphism $R' \to \widehat{R}$, which induces a morphism $\widehat{R'} \to \widehat{R}$. The fibre F of the latter morphism satisfies $F \otimes^{\mathbb{L}}_A A/pA \simeq 0$; therefore $F \otimes^{\mathbb{L}}_A A/^{\mathbb{L}}p \simeq 0$ (as $A/^{\mathbb{L}}p$ is a bounded complex with cohomologies being A/pA-modules), and so $F \simeq 0$ since it is derived p-complete.

Question 5.14. Do Theorems 5.10 and 5.12 hold for any perfectoid ring A, without assuming that it is an algebra over a perfectoid valuation ring $\mathcal{O}$? This assumption appeared in the proof of case 2 of Proposition 5.9, where it was used to reduce to a situation of finite valuative dimension. Indeed, the theorems therefore remain true for any perfectoid ring A which receives a map $A_0 \to A$ from another perfectoid ring A_0 having the property that $A_0^\flat$ has finite valuative dimension.

A Localisation sequences via t-structures

In the appendix, we explain how to deduce the dévissage results of Section 2 (and slight generalisations) from the theorem of the heart due to Barwick [Bar15] and Antieau–Gepner–Heller [AGH19]. Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a t-structure that is compatible with filtered colimits. We say that $\mathcal{C}$ is *regular coherent* if $\mathcal{C}$ is compactly generated and the t-structure on $\mathcal{C}$ restricts to a *bounded* t-structure on the compact objects $\mathcal{C}^\omega \subset \mathcal{C}$. In particular, this implies that all compact objects are truncated, and that the truncations of a compact object remain compact. Consequently, the compact objects of the heart $\mathcal{C}^\heartsuit$ form an abelian category $\mathcal{C}^{\heartsuit\omega}$, and $\mathcal{C}^\heartsuit$ is the Ind-completion of $\mathcal{C}^{\heartsuit\omega}$. For example, if R is a weakly regular coherent ring, then the derived ∞-category $\mathcal{D}(R)$ of R-module spectra with its usual t-structure is regular coherent.

For each connective $\mathbb{E}_\infty$-ring S, one has the presentable stable ∞-category $\mathcal{C} \otimes S$ of S-module objects in $\mathcal{C}$, which also inherits a t-structure where connectivity and coconnectivity are checked along restriction of scalars $\mathcal{C} \otimes S \to \mathcal{C}$. Let $\mathbb{S}[t]$ denote the suspension spectrum of the commutative monoid $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{S}[t_1, \dots, t_n]$ be the n-fold smash product of $\mathbb{S}[t]$; similarly we define $\mathbb{S}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ by inverting the generators. We say that $\mathcal{C}$ is *stably regular coherent* if $\mathcal{C} \otimes \mathbb{S}[t_1, \dots, t_n]$ is regular coherent for each $n \geqslant 0$. For example, if R is a weakly regular stably coherent ring, then $\mathcal{D}(R)$ is stably regular coherent. An important example is that there is an analog of Hilbert's basis theorem: if $\mathcal{C}$ is regular coherent and $\mathcal{C}^{\heartsuit\omega}$ is noetherian, then $\mathcal{C}$ is stably regular coherent [AGH19, Cor. 3.17].

Theorem A.1 (Barwick [Bar15] and Antieau–Gepner–Heller [AGH19])**.** *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a t-structure which is compatible with filtered colimits. Suppose $\mathcal{C}$ is stably regular coherent. Then the canonical map identifies the (connective) K-theory of the abelian category $(\mathcal{C}^\heartsuit)^\omega$ with the K-theory of the stable ∞-category $\mathcal{C}^\omega$. In particular, $\mathrm{K}_i(\mathcal{C}^\omega) = 0$ for $i < 0$.*

Proof. Our assumptions yield that $\mathcal{C}^\omega$ admits a bounded t-structure with heart $\mathcal{C}^{\heartsuit\omega}$. By the connective theorem of the heart [Bar15], it follows that $\mathrm{K}(\mathcal{C}^{\heartsuit\omega}) = \mathrm{K}_{\geqslant 0}(\mathcal{C}^\omega)$. It remains to

[10] The proof works over any base ring A, though we remark that if its p-power torsion is not bounded then the derived p-completions appearing in (1) might not be discrete.

show that $\mathrm{K}_{-i}(\mathcal{C}^\omega) = 0$ for $i < 0$. This is proved exactly as in the proof of [AGH19, Th. 3.6]; we reproduce a sketch of the argument for the convenience of the reader.

We prove $\mathrm{K}_i(\mathcal{C}^\omega) = 0$ for $i < 0$ by induction. First, by [AGH19, Th. 2.35], it follows that $\mathrm{K}_{-1}(\mathcal{C}^\omega) = 0$. Suppose $i < -1$. Consider the localisation sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$ given by $\mathrm{Perf}(\mathbb{S}[x] \text{ on } 0) \to \mathrm{Perf}(\mathbb{S}[x]) \to \mathrm{Perf}(\mathbb{S}[x^{\pm 1}])$. Tensoring with $\mathcal{C}^\omega$, we obtain another localisation sequence in $\mathrm{Cat}_\infty^{\mathrm{perf}}$, leading to a fiber sequence

$$\mathrm{K}(\mathrm{Perf}(\mathbb{S}[x] \text{ on } 0) \otimes \mathcal{C}^\omega) \to \mathrm{K}(\mathrm{Perf}(\mathbb{S}[x]) \otimes \mathcal{C}^\omega) \to \mathrm{K}(\mathrm{Perf}(\mathbb{S}[x^{\pm 1}]) \otimes \mathcal{C}^\omega).$$

As in the proof of [AGH19, Th. 3.6], it follows that $\mathrm{K}(\mathcal{C}^\omega)$ is a retract of $\mathrm{K}(\mathrm{Perf}(\mathbb{S}[x] \text{ on } 0) \otimes \mathcal{C}^\omega)$ and that this summand maps to zero in $\mathrm{K}(\mathrm{Perf}(\mathbb{S}[x]) \otimes \mathcal{C}^\omega)$. In particular, it follows that $\mathrm{K}_i(\mathcal{C}^\omega)$ is a summand of a quotient of $\mathrm{K}_{i+1}(\mathrm{Perf}(\mathbb{S}[x^{\pm 1}]) \otimes \mathcal{C}^\omega)$. However, our hypotheses imply that $\mathrm{Perf}(\mathbb{S}[x^{\pm 1}]) \otimes \mathcal{C}^\omega$ is the subcategory of compact objects in $\mathcal{C} \otimes \mathbb{S}[x^{\pm 1}]$, which is stably regular coherent by assumption. Inductively, it follows $\mathrm{K}_{i+1}(\mathrm{Perf}(\mathbb{S}[x^{\pm 1}]) \otimes \mathcal{C}^\omega) = 0$, whence $\mathrm{K}_i(\mathcal{C}^\omega) = 0$ as desired. □

Corollary A.2. *Let $\mathcal{C}$ be a presentable $\mathbb{Z}$-linear stable ∞-category equipped with a t-structure which is compatible with filtered colimits, and suppose $\mathcal{C}$ is stably regular coherent. Then $\mathrm{K}(\mathcal{C}^\omega) \xrightarrow{\sim} \mathrm{KH}(\mathcal{C}^\omega)$.*

Proof. It suffices to show that $\mathrm{K}(\mathcal{C}^\omega) \xrightarrow{\sim} \mathrm{K}((\mathcal{C} \otimes_{\mathbb{Z}} \mathbb{Z}[x])^\omega)$, and the classical proof in algebraic K-theory works. We switch now to geometric notation (writing schemes instead of rings). By our assumptions above, we have a fiber sequence

$$\mathrm{K}(\mathcal{C}^\omega) \to \mathrm{K}(\mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{C}^\omega) \to \mathrm{K}(\mathrm{Perf}(\mathbb{A}^1_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{C}^\omega), \tag{4}$$

where the first map is obtained from the pushforward $\mathrm{Perf}(\mathbb{Z}) \to \mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}})$ at the section ∞. Indeed, this follows from Thomason–Trobaugh localisation [TT90] and Theorem A.1, since $\mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}} \text{ on } \infty) \otimes_{\mathbb{Z}} \mathcal{C}^\omega$ is the compact objects of the stably regular coherent ∞-category of y-torsion objects in $\mathcal{C} \otimes_{\mathbb{Z}} \mathbb{Z}[y]$ (for y a coordinate near ∞ on $\mathbb{P}^1_{\mathbb{Z}}$). In particular, this gives a bounded t-structure on $\mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}} \text{ on } \infty) \otimes_{\mathbb{Z}} \mathcal{C}^\omega$ whose heart is the category of objects in $\mathcal{C}^{\heartsuit\omega}$ with a nilpotent endomorphism; now use Quillen dévissage [Qui73, Sec. 5] to identify its K-theory with that of $\mathcal{C}^\omega$.

Now we have two maps $f_1, f_2 \colon \mathcal{C}^\omega \to \mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{C}^\omega$ given by tensoring with the structure sheaf and given by pushing forward at ∞. By the projective bundle formula for $\mathbb{P}^1_{\mathbb{Z}}$, these establish an equivalence $(f_1, f_2) \colon \mathrm{K}(\mathcal{C}^\omega)^{\oplus 2} \xrightarrow{\sim} \mathrm{K}(\mathcal{C}^\omega \otimes_{\mathbb{Z}} \mathrm{Perf}(\mathbb{P}^1_{\mathbb{Z}}))$. Combining this with (4), we find that pullback induces an equivalence $\mathrm{K}(\mathcal{C}^\omega) \xrightarrow{\sim} \mathrm{K}(\mathcal{C}^\omega \otimes_{\mathbb{Z}} \mathrm{Perf}(\mathbb{A}^1_{\mathbb{Z}}))$ as desired. □

Finally, we explain how to deduce Proposition 2.5 and a generalisation of Proposition 2.6.

Alternative proof of Proposition 2.5. By Thomason–Trobaugh localisation [TT90], the fiber of $\mathrm{K}(R) \to \mathrm{K}(\mathrm{Spec} R \setminus V(I))$ is given by the K-theory of the stable ∞-category $\mathrm{Perf}(R \text{ on } I)$ of perfect R-module spectra which are I-power torsion. These are the compact objects in the presentable stable ∞-category $\mathcal{D}(R)_{I-\mathrm{tors}} \subset \mathcal{D}(R)$ of I-power torsion objects. Our assumption implies that the usual t-structure on $\mathcal{D}(R)_{I-\mathrm{tors}}$ is stably regular coherent in the above sense. In particular, $\mathrm{Perf}(R \text{ on } I)$ has a bounded t-structure with heart given by the abelian category $\mathrm{Mod}^{\mathrm{fp}}(R)_{I-\mathrm{tors}}$ of finitely presented R-modules which are I-power torsion. It follows from Theorem A.1 and Quillen dévissage [Qui73, Sec. 5] that $\mathrm{G}(R/I) \simeq \mathrm{K}(\mathrm{Mod}^{\mathrm{fp}}(R)_{I-\mathrm{tors}}) \simeq \mathrm{K}(\mathrm{Perf}(R \text{ on } I))$. □

Proposition A.3. *Let R be a ring and let $I \subset R$ be a finitely generated regular ideal. Suppose R/I is stably coherent and weakly regular. Then there is a fiber sequence of spectra*

$$\mathrm{K}(R/I) \to \mathrm{K}(R) \to \mathrm{K}(\mathrm{Spec}(R) \setminus V(I)),$$

and similarly in KH.

Proof. We have a fiber sequence $\mathrm{K}(R \text{ on } I) \to \mathrm{K}(R) \to \mathrm{K}(\mathrm{Spec}(R) \setminus V(I))$. Thus, it suffices to show that $\mathrm{K}(R/I) \xrightarrow{\sim} \mathrm{K}(R \text{ on } I)$.

Consider the presentable stable ∞-categories $\mathcal{D}(R)_{I-\mathrm{tors}}$ of I-power torsion objects in $\mathcal{D}(R)$ and $\mathcal{D}(R/I)$. Then $\mathrm{Perf}(R \text{ on } I)$ is the compact objects in $\mathcal{D}(R)_{I-\mathrm{tors}}$ and $\mathrm{Perf}(R/I)$ is the compact objects in $\mathcal{D}(R/I)$. We show that $\mathcal{D}(R)_{I-\mathrm{tors}}, \mathcal{D}(R/I)$ (with the natural t-structures) are stably regular coherent. For $\mathcal{D}(R/I)$, this is part of our assumption. Since the hypotheses of the result apply to a polynomial ring over R, it suffices to show that $\mathcal{D}(R)_{I-\mathrm{tors}}$ is regular coherent, i.e., that it admits a bounded t-structure.

First, since I is a regular ideal, I/I^2 is a finitely generated projective R/I-module, and $\mathrm{Sym}^i_{R/I}(I/I^2) \xrightarrow{\sim} I^i/I^{i+1}$ for each $i \geqslant 0$. We argue that R/I^i is a coherent ring for each $i \geqslant 1$ by induction. For $i = 1$ this is the assumption. If R/I^i is coherent, then consider the short exact sequence $0 \to I^i/I^{i+1} \to R/I^{i+1} \to R/I^i \to 0$. Our assumptions give that I^i/I^{i+1} is a finitely presented (indeed, finitely generated projective) R/I-module, whence a finitely presented R/I^i-module [BMS18, Lem. 3.25(i)]. Therefore, by [BMS18, Lem. 3.26], it follows that R/I^{i+1} is coherent.

It follows that we have an abelian category $\mathcal{A}$ of finitely presented R-modules which are annihilated by I^n for some $n \gg 0$ (it is then equivalent to assuming they are finitely presented R/I^n-modules, [BMS18, Lem. 3.25(i)]). We claim that the usual t-structure on $\mathcal{D}(R)_{I-\mathrm{tors}}$ restricts to a bounded t-structure on $\mathrm{Perf}(R \text{ on } I)$ with heart $\mathcal{A}$. We observe that any object $M \in \mathcal{A}$ is perfect as an R-module; this reduces to the case where M is an R/I-module, when it follows because R/I is regular coherent and is perfect as an R-module by regularity of I. This easily implies that the homology groups of any object of $\mathrm{Perf}(R \text{ on } I)$ belong to $\mathrm{Perf}(R \text{ on } I)$, whence the claim.

Thus, we have shown that $\mathcal{D}(R)_{I-\mathrm{tors}}, \mathcal{D}(R/I)$ are stably regular coherent. The hearts in the compact objects $\mathrm{Perf}(R \text{ on } I)$ and $\mathrm{Perf}(R/I)$ are given by $\mathcal{A}$ and the category of finitely presented R/I-modules respectively. Using Quillen dévissage and Theorem A.1, we conclude. □

Remark A.4. Much of the argument in the proof of Proposition A.3 can be established instead with the recent main theorem of [BL21], which gives a criterion for when, for a coconnective ring spectrum A, the canonical map $\pi_0(A) \to A$ induces an equivalence on K-theory. The hypotheses of their theorem holds in the setting of the proposition for the canonical map $R/I \to R\mathrm{Map}_R(R/I, R/I)$, the derived endomorphism spectrum of the R-module R/I. But, $\mathrm{Perf}(R\mathrm{Map}_R(R/I, R/I)) \simeq \mathrm{Perf}(R \text{ on } I)$ by derived Morita theory.

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The K-theory of perfectoid rings

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