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Improved critical drift estimates for the frog model on trees[*]

Poly Mathews Jr.[†]

Abstract

Place an active particle at the root of the infinite d-ary tree and dormant particles at each non-root site. Active particles move towards the root with probability p and otherwise move to a uniformly sampled child vertex. When an active particle moves to a site containing dormant particles, all the particles at the site become active. The critical drift p_d is the infimum over all p for which infinitely many particles visit the root almost surely. We give improved bounds on $\sup_{d \geq m} p_d$ and prove monotonicity of critical values associated to a self-similar variant.

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1 Introduction

The *frog model with drift* on the infinite rooted d-ary tree $\mathbb{T}_d$ with initial probability measure ν supported on the nonnegative integers, denoted $\mathrm{FM}(d,p,\nu)$, is defined as follows. Initially one active particle is at the root $\varnothing \in \mathbb{T}_d$ and the other vertices have independent and identically ν-distributed (i.i.d.) many dormant particles. Active particles perform independent p-biased random walk, i.e., moving towards the root with probability p and otherwise moving to a uniformly random child vertex. When an active particle moves to an unvisited site, any dormant particles there become active. These dynamics capture aspects of spatial processes with activation such as combustion, rumor spread, and infection [16]. Due to the chaotic nature in which the model propagates, researchers referred to particles as awake and sleeping frogs. The zoomorphism has persisted, so we will often use that terminology.

A *root visit* occurs each time that an awake frog moves to $\varnothing$. Let $V = V_{\mathrm{FM}(d,p,\nu)}$ be the total number of root visits. Call the process *recurrent* if $V = \infty$ almost surely. It was proven in [4] that V satisfies a 0-1 law. Accordingly, we call the process *transient* in the other case that $V < \infty$ almost surely.

The *critical drift*

$$p_d(\nu) := \inf\{p\colon\ \mathrm{FM}(d,p,\nu) \text{ is recurrent}\}$$

is the minimal drift below which the process is transient. It is known that $p_d(\nu)$ is sensitive to more than just the mean of ν [12, 13]. Most interest has been in one-per-site

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[†] E-mail: `Matthew.Junge@baruch.cuny.edu,sit218@lehigh.edu`

and Poisson-distributed initial configurations. We will write $\mathrm{FM}(d,p,1)$ to denote the one-per-site case $\nu(1)=1$ and $\mathrm{FM}(d,p,\mathrm{Poi}(\mu))$ to denote the case ν has a Poisson distribution with mean μ. To reduce the number of parameters when discussing the model, it is useful to standardize the density of sleeping frogs. Unless otherwise indicated we write

$$p_d \coloneqq \inf\{p\colon\ \mathrm{FM}(d,p,1) \text{ is recurrent}\}$$

for the critical drift in the one-per-site frog model.

The frog model was first studied on $\mathbb{Z}^d$. The frog model on $\mathbb{Z}^d$ is related to branching random walks on polynomially-growing graphs. Telcs and Wormald [17] proved that the one-per-site version is recurrent in all dimensions. Telcs and Wormald [17] and Popov [14] proved that it is recurrent for any ν with $\nu(0) \neq 1$. Alves, Machado, and Popov as well as Ramírez and Sidoravicius showed that the set of visited sites has a limiting shape for the one-per-site model [1, 16]. There has also been interest in the frog model on $\mathbb{Z}^d$ in which particles have a drift in one coordinate direction [6, 5].

On $\mathbb{T}_d$, a simple random walk corresponds to $p=\tilde{p}\coloneqq 1/(d+1)$. Determining the transience/recurrence behavior of $\mathrm{FM}(d,\tilde{p},1)$ was open for over a decade [15]. The question was partially answered in [9] by Hoffman, Johnson, and Junge who proved that $\mathrm{FM}(d,\tilde{p},1)$ is recurrent for $d=2$ and transient for $d\geq 5$. Simulations suggest that the process is recurrent when $d=3$ and transient when $d=4$. Later, the same authors in [8, 11] proved that there is a critical $\mu_c(d)=\Theta(d)$ above which $\mathrm{FM}(d,\tilde{p},\mathrm{Poi}(\mu))$ is recurrent and below which it is transient.

Wanting to isolate the role of the drift from the tree structure in the phase transition, Beckman, Frank, Jiang, Junge, and Tang introduced $\mathrm{FM}(d,p,1)$ and its critical value p_d [4]. It is easy to see that the stochastically larger process with all particles initially active is transient whenever $p<1/(d+1)$. Moreover, the process with no activation is recurrent for $p\geq 1/2$. So, $p_d\in[1/(d+1),1/2]$. An intriguing aspect of $\mathrm{FM}(d,p,1)$ is that several intuitive monotonicity statements have evaded proof. We say that Y *stochastically dominates* X if there is a coupling such that $X\leq Y$ almost surely. This is denoted by $X\preceq Y$.

Conjecture 1.1 ([3, 7, 2])**.**

(i) If $d\leq d'$ and $p\leq p'$, then $V_{\mathrm{FM}(d,p,\nu)}\preceq V_{\mathrm{FM}(d',p',\nu)}$.[1]

(ii) $p_{d+1}<p_d$.[2]

(iii) $\lim_{d\to\infty} p_d=(2-\sqrt{2})/4\coloneqq q^*$ (≈ 0.1464), the critical drift for a branching random walk that doubles only when moving away from the root (with probability $1-p$).[3]

The main result of [8] can be restated as $p_2=1/3$. However, the current lack of the monotonicity claimed in Conjecture (i) makes it unclear if

$$S_m\coloneqq \sup_{d\geq m} p_d$$

is bounded from above by $1/3$. In testament to this uncertainty, the first bound proven on S_m was that $S_3\leq 0.4155$ [4, Theorem 1.1]. Guo, Tang, and Wei later established the bound $S_3\leq 1/3$ which implies the “sharp” statement $S_2=1/3$ [7]. This was further improved by Bailey, Junge, and Liu to $S_3\leq 5/17$ [2] which implies that $p_d<p_2$ for $d\geq 3$. Bailey, Junge, and Liu also proved that $S_4\leq 27/100$ and outlined a computer-assisted method for obtaining better bounds for larger m.

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[1] Conjecture 1.1 (i) is known in the special case that $d'=kd$ and $p'=p$ [4, Proposition 1.2].

[2] More generally, it should hold that $p_{d+1}(\nu)<p_d(\nu)$ so long as ν has finite mean.

[3] The analogue should hold for general ν with a branching random walk that produces on average the mean of ν offspring only when moving away from the root.

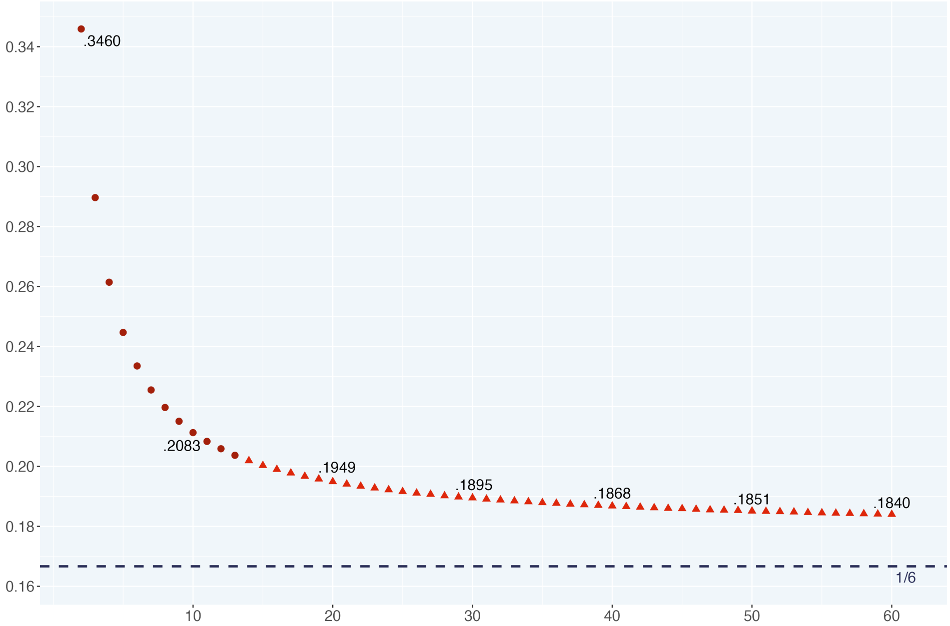

Figure 1: Upper bounds on S_m for $2 \leq m \leq 60$. The horizontal axis is the value of m and the vertical axis is the bound we get for S_m. Circles are rigorous bounds from Theorem 1.2 and triangles are numerically approximated bounds. The dashed line at $1/6$ is the known limiting critical drift for the self-similar frog model, but is larger than the conjectured limit of S_m (stated in Conjecture 1.1 (iii)).

1.1 Results

Our results provide further insight into Conjecture (i), (ii), and (iii). The first uses a computer-assisted proof to carry out the proposed method from [2] for bounding S_m. The general idea is to bound the critical drift for a non-backtracking variant that has stochastically fewer root visits (see Theorem 1.3).

Theorem 1.2. *S_m satisfies the bounds in the table below.*

m	2	3	4	5	6	7	8	9	10	11	12	13
$S_m \leq$	$\frac{55}{159}$	$\frac{42}{145}$	$\frac{40}{153}$	$\frac{23}{94}$	$\frac{46}{197}$	$\frac{23}{102}$	$\frac{38}{173}$	$\frac{20}{93}$	$\frac{15}{71}$	$\frac{5}{24}$	$\frac{7}{34}$	$\frac{11}{54}$
$\approx$	.346	.290	.261	.245	.234	.225	.220	.215	.211	.208	.206	.204

Beyond $m = 13$ we encounter runtime issues (even getting to $m = 13$ requires the efficiency-boosting inductive scheme described in Lemma 3.2). However, we are able to use the method to make non-rigorous approximations of bounds for larger m. See Figure 1. It was proven in [2] that the upper bounds on S_m will converge to $1/6$ as $m \to \infty$ (although S_m ought to converge to q^* from Conjecture 1.1 (iii)). The convergence in Figure 1 appears slow. This leads to a refinement to Conjecture 1.1 (iii). We find it plausible that $p_d - q^* = \Theta(d^{-1/2})$. This might occur because for large d the frog model behaves like the branching random walk that doubles when moving away from the root until vertices start getting revisited. The birthday paradox tells us that repeated visits to child vertices of $\varnothing$ occur after $O(\sqrt{d})$ visits to $\varnothing$. The branching random walk visits $\varnothing$ at a constant (linear in time) rate, as does the frog model [10] in certain regimes. This suggests that $d^{-1/2}$ may play a role in the point at which the frog model begins to lag behind the branching random walk.

The rest of our results are for an important variant of $\mathrm{FM}(d, p, \nu)$ called the *self-similar* *frog model*, which we denote as $\mathrm{SFM}(d,p,\nu)$. This process, introduced in [9], is the only known tool for proving recurrence of a frog model on trees. Put briefly, frogs in the self-similar frog model are restricted to the non-backtracking (loop-erased) portion of their random walk paths, and only one frog is allowed to move away from the root to each subtree. This results in a stochastically smaller model in terms of the root visits that is more amenable to anaylsis.

In $\mathrm{SFM}(d,p,\nu)$ the jump distribution is different for frogs that just woke up versus for those that have already taken a step. This arises from the fact that the law for non-backtracking paths is influenced by both the drift and the degree of the tree. Let

$$p_d^* = p_d^*(p) := \frac{p(d-1)}{d-(d+1)p} \text{ and } \hat{p} = \hat{p}(p) := \frac{p}{1-p}. \tag{1.1}$$

Initially, there is one active frog at the root. It moves to a uniformly sampled child vertex in the first step. Just-activated frogs move towards the root with probability p_d^*, and otherwise away from the root to a uniformly sampled child vertex. For subsequent steps, if the previous step was towards the root, then the next step will be towards the root with probability $\hat{p}$. If the previous step was away from the root, all subsequent steps will be away from the root to uniformly sampled child vertices. Any particles that visit the root are killed there and no longer participate in the process. The last modification is that particles moving away from the root are killed upon visiting a vertex that has already been visited. If multiple active particles attempt to move away from the root to the same unvisited vertex, then one is chosen to continue its path and the others are killed.

Let $V_{\mathrm{SFM}(d,p,\nu)}$ denote the total number of root visits. The self-similar frog model is dominated by the usual frog model. Indeed, [7] worked out the transition probabilities p_d^* and $\hat{p}$ so that $V_{\mathrm{SFM}(d,p,\nu)} \preceq V_{\mathrm{FM}(d,p,\nu)}$. In support of Conjecture 1.1 (i), we prove that $V_{\mathrm{SFM}(d,p,\nu)}$ is monotone in d.

Theorem 1.3. *$V_{\mathrm{SFM}(d,p,\nu)} \preceq V_{\mathrm{SFM}(d+1,p,\nu)}$ for all $d \geq 2$, $p \in (0,1/2)$ and ν.*

Theorem 1.3 is the strongest contribution of this work. It is interesting foremost because it supports Conjecture 1.1 (i). Another benefit of Theorem 1.3 is that the main results from [4] and [7] are immediate corollaries. For example, since [9] proved that $\mathrm{SFM}(2,1/3,1)$ is recurrent, Theorem 1.3 implies that $S_3 \leq 1/3$. Also, the bounds in [2] are strengthened after applying Theorem 1.3. For example, together with Theorems 1.2 and 1.3, the bound $S_4 \leq 0.27$ from [2] is improved to $S_4 \leq 0.262$.

The main difficulty with proving Theorem 1.3 is that $\mathrm{SFM}(d,p,\nu)$ and $\mathrm{SFM}(d+1,p,\nu)$ have different probabilities that the first step taken by a newly activated frogs is towards the root ($p_d^* < p_{d+1}^*$). In one way this is good for $\mathrm{SFM}(d+1,p,\nu)$ since frogs are more likely to move towards the root. However, it is not monotonically helpful since moving away from the root sometimes comes with the benefit of waking more frogs.

A fortunate inequality, that had previously gone unnoticed, is that so long as at least one vertex below a just-activated site, say v, has been visited, the probability a particle activated at v moves away from the root to a new site in $\mathrm{SFM}(d+1,p,\nu)$ is larger than the probability in $\mathrm{SFM}(d,p,\nu)$. An innovation in our coupling is to modify the order in which frogs jump in a way that ensures that newly awoken frogs will have at least one visited child vertex below them. The basic idea is to allow frogs that have jumped away from the root to keep jumping until reaching a freezing barrier at distance n. After sometimes eliminating frogs from $\mathrm{SFM}(d+1,p,\nu)$, we arrive at a coupling that gives a one-to-one correspondence between activated frogs at all distances from the root.

Our last result is more technical and involves Conjecture 1.1 (ii). We define a new critical value for $\mathrm{SFM}(d,p,\mathrm{Poi}(1))$ and prove that it is strictly monotone in d. Informally speaking, the critical value is the smallest value of p such that the only known approach

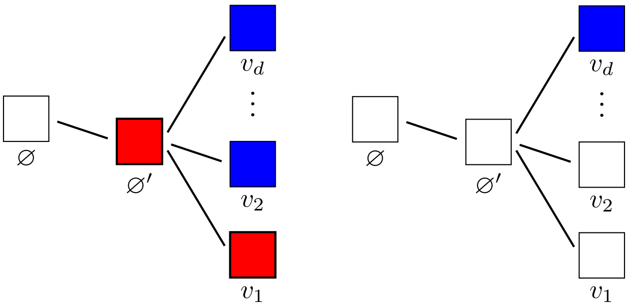

Figure 2: The process used to define $U(d,p,\lambda)$. Red sites contain particles that are initially active and blue sites contain initially dormant particles. $U(d,p,\lambda)$ is the number of vertices among $v_2,\dots,v_d$ that are ever visited. Empty boxes on the right at $v_1,\dots,v_d$ represent sites at which particles have been activated.

for proving $\mathrm{SFM}(d,p,\mathrm{Poi}(1))$ is recurrent applies. To define this formally takes some extra notation.

We now define the *frog star process*. Consider a star graph with root $\varnothing$, central vertex $\varnothing'$, and leaves $v_1,\dots,v_d$ (see Figure 2). There is a $\mathrm{Poi}(1)$ number of active particles at $\varnothing'$ and an independent $\mathrm{Poi}(\lambda)$ number of active particles at v_1. An independent $\mathrm{Poi}(\lambda)$-distributed number of dormant particles is placed at each of $v_2,\dots,v_d$.

The active particles started at $\varnothing'$ move to $\varnothing$ independently with probability p_d^* and otherwise each moves to an independently and uniformly sampled vertex from $v_1,\dots,v_d$. Active particles at v_i move to $\varnothing'$ with probability 1, and then to either $\varnothing$ with probability $\hat{p}$ or otherwise to a uniformly sampled vertex among $\{v_1,\dots,v_d\}\setminus\{v_i\}$. Whenever active particles encounter dormant particles, the dormant particles become active. When a particle moves to a leaf or to $\varnothing$, it remains frozen there for all subsequent time steps. We define $U=U(d,p,\lambda)$ to be how many of $v_2,\dots,v_d$ have been visited after all particles are either frozen or dormant. The random variable U is important because understanding its distribution leads to a sufficient condition for recurrence.

Define

$$M^{d,p} \coloneqq \sup_{\lambda\geq 0} \mathbf{E}[e^{\lambda - p_d^* - \hat{p}(1+U)\lambda}].$$

It follows from [2, Proposition 2.6] that

$$M^{d,p} < 1 \text{ implies that } \mathrm{SFM}(d,p,\mathrm{Poi}(1)) \text{ is recurrent.} \tag{1.2}$$

We define the critical value

$$q_d \coloneqq \inf\{p\colon M^{d,p} < 1\}$$

that corresponds to the threshold at which the proof technique in [2, 8, 11] fails. We are unsure whether or not q_d is equal to the more natural critical value

$$\inf\{p\colon \mathrm{SFM}(d,p,\mathrm{Poi}(1)) \text{ is recurrent}\}.$$

In favor of equality, the “bootstrapping” proof technique fairly accurately captures how root visits behave in the self-similar model. Roughly, the proof goes by showing that if the total number of root visits dominates a Poisson random variable of mean λ, then it dominates a Poisson random variable of mean $\lambda+\epsilon$. This could be a sharp approximation since the total number of root visits in the self-similar frog model is a Poisson with random mean.

Theorem 1.4. $q_{d+1} < q_d$ *for all* $d \geq 2$.

This result is interesting because it supports the strict monotonicity claimed in Conjecture 1.1 (ii). Moreover, it suggests that proving Conjecture 1.1 (ii) may be difficult, since a much simpler monotonicity claim requires non-trivial analysis. The advantage of q_d is that it is defined in terms of a concrete inequality satisfied by $M^{d,p}$. The self-similar nature of $\mathrm{SFM}(d, p, \mathrm{Poi}(1))$ as well as computational advantages unique to a Poisson-distributed number of frogs let us reduce this to analyzing functions. Even with these advantages the argument is not straightforward. The general case is unlikely to reduce to such tractable analysis.

1.2 Organization

We begin by proving Theorem 1.3 in Section 2. The result is needed to deduce Theorem 1.2. In the next two sections we prove Theorem 1.2 and Theorem 1.4, respectively. The appendix contains additional details on the code we used to obtain Theorem 1.2 and Figure 1.

2 Proof of Theorem 1.3

We first provide a sketch of the main idea. We will fix a killing barrier at the vertices of distance $n+1$ from the root of $\mathbb{T}_d$. At each time step s, a single frog in $\mathrm{SFM}(d, p, \nu)$ is sampled uniformly. The selected frog will either move a single step back towards the root, or enter an already activated subtree and die, or follow a uniformly random non-backtracking path until being killed at the barrier. Any sleeping frogs visited become active, but do not yet move. For step $s+1$, another active frog will be sampled. This continues until all frogs are either killed or sleeping. The advantage of this procedure is that, after each time step, the tree is left in a configuration for which every visited vertex within distance n of the root has at least one already-visited child. We then couple this process with a sub-process of $\mathrm{SFM}(d+1, p, \nu)$ that also has a killing barrier at distance $n+1$. Inducting over $s \geq 0$, we will describe how to maintain a bijection between the movement and waking behavior frogs in $\mathrm{SFM}(d+1, p, \nu)$ with what occurs in $\mathrm{SFM}(d, p, \nu)$.

A major issue in coupling the two processes is that just-activated frogs move towards the root with different probabilities $p_d^* < p_{d+1}^*$ on $\mathbb{T}_d$ and $\mathbb{T}_{d+1}$. This has the potential to cause the two processes decouple when a given frog moves away from the root on $\mathbb{T}_d$, while its coupled frog does not. However, since our modified manner in which frogs move ensures that all active frogs have at least one visited child, the inequality

$$(1 - p_d^*)\frac{m}{d} < (1 - p_{d+1}^*)\frac{m+1}{d+1}, \qquad 0 \leq m < d$$

with m the number of unvisited child vertices of a given vertex, allows us to kill the frog in $\mathrm{SFM}(d+1, p, \nu)$ whenever the corresponding frog in $\mathrm{SFM}(d, p, \nu)$ would be killed by moving to an already visited child vertex. This observation, plus some additional killing of frogs in $\mathrm{SFM}(d+1, p, \nu)$, makes it possible to maintain a bijective correspondence between active frogs and their locations at all time steps.

Proof of Theorem 1.3. Fix $n \geq 1$ and let $\mathrm{SFM}_n(d, p, \nu)$ denote the self-similar frog model with sleeping frogs placed at vertices within distance n of the root. We will couple $\mathrm{SFM}_n(d, p, \nu)$ with a sub-process $\mathrm{SFM}'_n(d+1, p, \nu)$, dominated by $\mathrm{SFM}_n(d+1, p, \nu)$, that sometimes kills additional frogs. For $s \geq 0$, consider a time-changed version of the self-similar frog model that moves a randomly sampled awake frog at each step s. If the frog moves towards the root, then it takes just that one step. Note that frogs are still killed upon visiting the root. If the frog moves away from the root, then it either dies (if the vertex it jumps to is the root of an already activated subtree), or it samples a non-backtracking path to distance $n+1$ from the root and is killed there. Any sleeping frogs it visits become active, but do not yet move. For step $s+1$, a new active frog, that has not been killed, is sampled to move. First we must prove that this modified frog process has the same limiting number of visits as $\mathrm{SFM}(d,p,\nu)$.

Claim 2.1. *Let $V_{\mathrm{SFM}_n(d,p,\nu)}$ be the total number of visits to the root once the process $\mathrm{SFM}_n(d,p,\nu)$ has no active frogs. It holds almost surely that*

$$\lim_{n\to\infty} V_{\mathrm{SFM}_n(d,p,\nu)} = V_{\mathrm{SFM}(d,p,\nu)}.$$

Proof. In both processes, we may couple the number of sleeping particles at each site and the paths they follow upon waking by assigning sleeping particle counts and stacks of independent non-backtracking paths to each particle in advance. It then suffices to prove that a site v is visited in $\mathrm{SFM}(d,p,\nu)$ if and only if v is visited in $\mathrm{SFM}_n(d,p,\nu)$ for all large enough n almost surely. In what follows, we will refer to a *waking chain* as a sequence of frogs $f_0, f_1, \dots, f_k$ for which f_0 starts at $\varnothing$ and each f_j is woken by f_{j-1} for $1 \le j \le k$.

Suppose that v is visited at step t in $\mathrm{SFM}(d,p,\nu)$. This visit happens from an active frog f_k in an almost surely finite waking chain. As the chain involves finitely many frogs, the starting vertices are within an almost surely finite distance N of the root. In $\mathrm{SFM}_N(d,p,\nu)$, all frogs within distance N of the root move until reaching distance $N+1$, thus v will be visited in $\mathrm{SFM}_n(d,p,\nu)$ for all $n \ge N$.

Fix $N \ge 1$ and suppose that v is visited in $\mathrm{SFM}_n(d,p,\nu)$ for all $n \ge N$. This means that there is a frog f_k in some waking chain that visits v and the starting locations of the frogs in the chain are all within distance N of the root. Since only finitely many frogs are involved, that same waking chain occurs in $\mathrm{SFM}(d,p,\nu)$ and f_k reaches v after an almost surely finite number of time steps. ☐

Continuing towards Theorem 1.3, let $\mathcal{T}_n^s(d,p,\nu)$ be the random subtree of sites within distance n of the root that have been visited in $\mathrm{SFM}_n(d,p,\nu)$ after s such steps. We take as our inductive hypothesis that there is an embedding $\psi^s\colon \mathcal{T}_n^s(d,p,\nu) \to \mathbb{T}_{d+1}$ that maps root to root ($\psi^s(\varnothing_d) = \varnothing_{d+1}$) and for each vertex $v \in \mathcal{T}_n^s(d,p,\nu)$

(i) $\psi^s(v)$ has been visited in $\mathrm{SFM}_n'(d+1,p,\nu)$.

(ii) There is a bijection between the frogs moving towards and away from the root at v and $\psi^s(v)$.

We will induct over the values $s \ge 0$. Once established, the inductive claim implies Theorem 1.3. This is because for any fixed n the algorithm will terminate with all frogs sleeping or killed after finitely many steps s. Thus the terminated algorithm produces the total number of visits to the root in $\mathrm{SFM}_n(d,p,\nu)$ which we have coupled to be identical as the process $\mathrm{SFM}_n'(d+1,p,\nu)$ that kills frogs and thus produces fewer root visits than $\mathrm{SFM}_n(d+1,p,\nu)$. It follows from Claim 2.1 that $V_{\mathrm{SFM}(d,p,\nu)} \preceq V_{\mathrm{SFM}(d+1,p,\nu)}$.

We now prove the inductive claims. Clearly (i) and (ii) are satisfied at $s=0$. It suffices to assume (i) and (ii) hold after s steps and prove that they continue to hold after moving any one of the active frogs. Suppose that a frog f at v is selected to move on $\mathbb{T}_d$. Let $v' = \psi^s(v)$ and f' be the corresponding frog at v'. Since any frogs that move away from the root either die immediately or are allowed to jump until being killed at distance $n+1$ from the root, we need only consider the cases that the previous jump of f was towards the root or that f was just awoken.

Suppose that the last step f took was towards the root. Both f and its corresponding f' at v' will take another step towards the root with probability $\hat{p}$. We may then couple them to move in the same direction. If they move towards the root, then (i) and (ii) are preserved. Suppose f and f' move away from the root and that there are m child vertices of v that are yet to be visited. As ψ^s is an embedding, there are $m+1$ child vertices of v' that are yet to be visited. If f does not move to a new site, then kill f and f'. The frog f moves to a new site with probability m/d. The frog f' moves to a new site with probability $(m+1)/(d+1) \geq m/d$. Thus, whenever f moves to a new site u, there is a coupling that preserves the random walk law for f' and has it move to a new site u'. We set $\psi^{s+1}(u) = u'$ and otherwise $\psi^{s+1} = \psi^s$. Further, we couple the number of particles discovered at u and u' to be the same. We then have f and f' repeat this coupling process until they reach distance $n+1$.

Suppose that the frogs f and f' were just woken at v and v', respectively. Suppose that v has m child vertices that are yet to be visited. Necessarily v' has $m+1$ such vertices. Moreover, our requirement that frogs which have moved away from the root continue doing so until reaching distance $m+1$ from the root ensures that at least one child vertex below v and v' has been visited so that $m < d$. The frog f will move towards $\varnothing$ with probability $p_d^* < p_{d+1}^*$. So, if f moves towards the root, we may couple f' to do the same preserving (i) and (ii). If f does not move towards $\varnothing$ but f' does (with probability $p_{d+1}^* - p_d^*$), then we kill f'. The frog f will move away from the root to a new vertex with probability $(1-p_d^*)\frac{m}{d}$. It is easy to verify that, so long as $m < d$, this is strictly less than the probability $(1-p_{d+1}^*)\frac{m+1}{d+1}$ that f' moves away to a new vertex. Thus, we may couple f' to visit a new vertex whenever f does. We then couple the number of frogs activated at the two sites. Lastly, if f moves away from $\varnothing$ to an already-visited vertex, then we kill both f and f'. These rules were defined so that both f and f' move with the appropriate random walks, but f' only discovers a new vertex when f does. This preserves the embedding as well as (i) and (ii). ☐

3 Proof of Theorem 1.2

First, we describe an efficient way to inductively compute the distribution of $U(d,p,\lambda)$ as d is increased.

Given $\Phi, \Lambda \in [0,1]$ and $d \geq 1$, define the *pulse process* $L(\Phi,\Lambda,d)$ as follows. Start with a set of d dormant vertices $v_1,\dots,v_d$. At $t=0$, there is an initial pulse which causes the vertices $v_2,\dots,v_d$ to each have an independent probability $1-\Phi$ of being activated. Every time a dormant vertex is activated, it sends out another pulse that causes all of the other vertices to have an independent $1-\Lambda$ probability of being activated. (If two vertices are activated at the same time, they both send out a pulse, and these pulses are independent.) If an already activated vertex is activated again, nothing happens. At $t=0$, the vertex v_1 activates itself. Define the random variable $U_{L(\Phi,\Lambda,d)}$ to be the number of activated vertices in $v_2,\cdots,v_d$ after the process has finished.

Claim 3.1. *The frog star process that is used to define the random variable $U(d,p,\lambda)$ (see Figure 2) is equivalent to the pulse process L in the sense that when $\Phi_{d,p} = \exp\left(-\frac{1-p_d^*}{d}\right)$ and $\Lambda_{d,p} = \exp\left(-\frac{1-\hat{p}}{d-1}\lambda\right)$ we have*

$$U_{L(\Phi_{d,p},\Lambda_{d,p},d)} \overset{\mathrm{D}}{=} U(d,p,\lambda).$$

Proof. It suffices to establish the following two facts about the frog star process:

1. The probability that $\varnothing'$ sends at least one frog to v_i is $1-\exp\left(-\frac{1-p_d^*}{d}\right)$, and these events are independent for each i.
2. Upon the activation of the vertex v_j, the probability that it sends at least one frog to another frog v_i is $1-\exp\left(-\frac{1-\hat{p}}{d-1}\lambda\right)$, and these events are independent for each i.

Both of these facts are true due to Poisson thinning. □

For all $d \geq 1$, $0 \leq u \leq d-1$, let $s_{d,u}\colon [0,1]^2 \to [0,1]$ be defined by

$$s_{d,u}(\Phi,\Lambda) \coloneqq \mathbf{P}(U_{L(\Phi,\Lambda,d)} = u).$$

By Claim 3.1, we then have

$$s_{d,u}\Big(\exp\Big(-\frac{1-p_d^*}{d}\Big), \exp\Big(-\frac{1-\hat{p}}{d-1}\lambda\Big)\Big) = \mathbf{P}(U(d,p,\lambda) = u).$$

Thus, if we know the functions $s_{d,u}$, then we know the distribution of U.

Lemma 3.2. *The functions $s_{d,u}$ satisfy the following relations:*

$$s_{1,0}(\Phi,\Lambda) = 1, \tag{3.1}$$

$$s_{d,u}(\Phi,\Lambda) = \binom{d-1}{u}\big(\Phi\Lambda^{u+1}\big)^{d-u-1} s_{u+1,u}(\Phi,\Lambda), \tag{3.2}$$

$$s_{d,d-1}(\Phi,\Lambda) = 1 - \sum_{i=0}^{d-2} s_{d,i}(\Phi,\Lambda). \tag{3.3}$$

Proof. Equation (3.1) is satisfied because

$$s_{1,0}(\Phi,\Lambda) = \mathbf{P}(U_{L(\Phi,\Lambda,1)} = 0) = 1.$$

$U_{L(\Phi,\Lambda,1)}$ must be 0 since there are no vertices to activate.

To see that equation (3.2) is satisfied, we define two events:

- `SelfActivate` is the event that all of $v_1, v_2, \ldots, v_{u+1}$ are activated using only pulses from $v_1, v_2, \ldots, v_{u+1}$, and the initial pulse.
- `NoVisit` is the event that neither the initial pulse nor any of the pulses that $v_1, v_2, \ldots, v_{u+1}$ send out activate any of $v_{u+2}, \ldots, v_d$.

Notice that for $0 \leq u \leq d-1$,

$$s_{d,u}(\Phi,\Lambda) = \binom{d-1}{u}\mathbf{P}(\texttt{NoVisit} \mid \texttt{SelfActivate})\,\mathbf{P}(\texttt{SelfActivate}).$$

The choice of $v_2, \ldots, v_{u+1}$ can be replaced by any u vertices, leading to the $\binom{d-1}{u}$ factor at the front. In the case of `SelfActivate`, since a pulse affects each vertex independently in a way not dependent on d, we can pretend like the vertices $v_{u+2}, \ldots, v_d$ don't exist, which lets us deduce that

$$\mathbf{P}(\texttt{SelfActivate}) = s_{u+1,u}(\Phi,\Lambda).$$

In the case of $\mathbf{P}(\texttt{NoVisit} \mid \texttt{SelfActivate})$, we realize that this is the probability that none of the $u+1$ pulses sent out by $v_1, \ldots, v_{u+1}$ activate any of the $d-u-1$ vertices $v_{u+2}, \ldots, v_d$. This is equal to $(\Lambda^{u+1})^{d-u-1}$. The initial pulse must also not activate any of $v_{u+2}, \ldots, v_d$, which gives us an additional factor of Φ^{d-u-1}. Multiplying these together, we get $(\Phi\Lambda^{u+1})^{d-u-1}$. This proves (3.2).

Lastly, equation (3.3) follows from the observation that $s_{d,d-1} = \mathbf{P}(U_{L(\Phi,\Lambda,d)} = d-1)$ is complementary to $\mathbf{P}(U_{L(\Phi,\Lambda,d)} < d-1) = \sum_{i=0}^{d-2} s_{d,i}(\Phi,\Lambda)$. □

Proof of Theorem 1.2. Let $f(\lambda) = f^{d,p}(\lambda) = \mathbf{E}[e^{\lambda - p_d^* - \hat{p}(1+U)\lambda}]$. Suppose we prove that

$$M^{d,p} := \sup_{\lambda \geq 0} f^{d,p}(\lambda) < 1.$$

Then, (1.2) implies that $\mathrm{SFM}(d, p, \mathrm{Poi}(1))$ is recurrent. The stochastic comparison result in [12] implies that $\mathrm{SFM}(d, p, 1)$ is recurrent. The monotonicity result in Theorem 1.3 further ensures that $\mathrm{SFM}(d', p, 1)$ is recurrent for all $d' \geq d$. We then apply the observation that $V_{\mathrm{SFM}(d,p,1)} \preceq V_{\mathrm{FM}(d,p,1)}$ in [7] to conclude that $\mathrm{FM}(d, p, 1)$ is recurrent and thus $S_d \leq p$.

Lemma 3.2 allows us to efficiently compute the distribution of U as we increase d. This formula only involves powers of exponential functions of λ. The method devised in [2] gives a way to rigorously prove that f has a single maximum on $[0, \infty)$. To accomplish this we make the change of variables $f(-c \log(y)) = g(y)$. If c is an appropriately chosen rational number, then g is a polynomial. The `CountRoots` function in Mathematica helps us to confirm with Sturm's theorem that g' has a single root in $(0, 1]$. So, g (and thus f) has a unique global maximum which we prove is strictly less than 1. We implemented this approach for $2 \leq d \leq 13$. For larger d we experience runtime issues with this computer-assisted rigorous proof. More details can be found in the Appendix and the code documentation at https://github.com/fredcheng02/frog-model. ☐

4 Proof of Theorem 1.4

We first prove that U is monotone in d. A similar, but less general observation was made in [2].

Lemma 4.1. $U(d, p, \lambda) \preceq U(d + 1, p, \lambda)$.

Proof. Recall that a coupon collector process on m coupons is a process that at each step independently and uniformly samples one of m distinct coupons. Typically we track the number of distinct coupons sampled after t selections.

We start with a general observation about coupon collectors collecting from different sized sets of coupons. Namely, whoever has a larger set of coupons collects at least as many distinct coupons as the collector with less options. To be more precise, suppose that two coupon collectors are independently collecting coupons. The first collector collects from a set of n coupons and the second collector collects from a set of $n + 1$ coupons. Let C_t^i be the number of distinct coupons collected by collector $i = 1, 2$ after each takes t uniform draws from their set of coupons.

Claim 4.2. *There is a coupling so that* $C_t^1 \leq C_t^2$ *for all* $t \geq 0$.

Proof. The inequality holds at $t = 0$. Suppose it is true after collecting t coupons. If $C_t^1 < C_t^2$, then regardless of the $(t + 1)$th draw we will have $C_{t+1}^1 \leq C_{t+1}^2$. Suppose that $C_t^1 = C_t^2 = c \leq n$. For the $(t + 1)$th draw, the first collector has probability $(n - c)/n$ of collecting a new coupon. The second collector has probability $(n + 1 - c)/(n + 1)$. It is easy to check that the second probability is at least as large as the first. Thus, we may couple the two processes so that the second collector discovers a new coupon whenever the first collector does. This gives $C_{t+1}^1 \leq C_{t+1}^2$. ☐

Now we will describe $U(d, p, \lambda)$ and $U(d + 1, p, \lambda)$ in terms of coupon collector processes. Collector A is collecting coupons uniformly from coupons $\{1, \ldots, d\}$. Collector B is collecting coupons uniformly from coupons $\{1, \ldots, d + 1\}$. Suppose further that each collector has initially collected coupon 1.

At the first step Collector A samples $\mathrm{Poi}\big((1 - p_d^*)\frac{d-1}{d}\big)$ coupons uniformly from $\{2, \ldots, d\}$, and Collector B samples $\mathrm{Poi}\big((1 - p_{d+1}^*)\frac{d}{d+1}\big)$ coupons uniformly from $\{2, \ldots, d+$

$1\}$. Recall that $p_d^* = \frac{p(d-1)}{d-(d+1)p}$ as defined in (1.1). We have for $p < 1/2$ and $d \geq 2$,

$$(1-p_d^*)\frac{d-1}{d} < (1-p_{d+1}^*)\frac{d}{d+1}.$$

Thus, we may couple Collector B to collect at least as many total coupons from $\{2,\dots,d+1\}$ as Collector A does from $\{2,\dots,d\}$. By Claim 4.2 there is a coupling so that Collector B ends up with at least as many distinct coupons as Collector A.

For each distinct coupon collected, say coupon i, the collectors receive an independent $\mathrm{Poi}((1-\hat{p})\lambda)$ distributed number of additional uniform draws from the set of coupons minus coupon i. We may couple the number of uniform draws from each newly discovered coupon and repeatedly apply Claim 4.2 to ensure that Collector B always has at least as many distinct coupons as Collector A. In particular, this is true once no new coupons are discovered and the collecting ends. Comparing to the definition of U, the number of distinct coupons collected by Collectors A and B are distributed as $U(d,p,\lambda)$ and $U(d+1,p,\lambda)$, respectively. This gives the claimed stochastic dominance. ☐

Recall that $f^{d,p}(\lambda) = \mathbf{E}[e^{\lambda - p_d^* - \hat{p}(1+U)\lambda}]$ and $M^{d,p} = \sup_{\lambda\geq 0} f^{d,p}(\lambda)$.

Lemma 4.3. *For* $p \in (\frac{1}{d+1}, \frac{1}{2})$

- *(i)* $M^{d,p}$ *is continuous in* p*.*
- *(ii)* $M^{d+1,p} < M^{d,p}$*.*

Proof of (i). For now fix p and d. Let $g(y) = f(-\log(y))$. Recall

$$\mathbf{P}(U(d,p,\lambda) = u) = s_{d,u}(\Phi_d, \Lambda_d)$$

from Claim 3.1. Making this replacement, we have that $f\colon [0,\infty) \to \mathbb{R}$ given by

$$f(\lambda) = e^{-p_d^*}\sum_{u=0}^{d-1} e^{(1-\hat{p}(1+u))\lambda} s_{d,u}(\Phi_d, \Lambda_d)$$

becomes $g\colon (0,1] \to \mathbb{R}$ with

$$g(y) = e^{-p_d^*}\sum_{u=0}^{d-1} y^{\hat{p}(1+u)-1} s_{d,u}\big(\Phi_d, y^{\frac{1-\hat{p}}{d-1}}\big) \tag{4.1}$$

using $\Lambda_d = \exp\big(-\frac{(1-\hat{p})}{d-1}\lambda\big)$. Now $M^{d,p} = \sup_{y\in(0,1]} g^{d,p}(y)$.

By Lemma 3.2 it is easy to show via induction that the y terms in $s_{d,u}\big(\Phi_d, y^{\frac{1-\hat{p}}{d-1}}\big)$ above have nonnegative exponents. Also we replace Λ_d by $y^{\frac{1-\hat{p}}{d-1}}$ in equation (3.2) and obtain

$$s_{d,u}\big(\Phi_d, y^{\frac{1-\hat{p}}{d-1}}\big) = \binom{d-1}{u}\big(\Phi_d \cdot y^{\frac{1-\hat{p}}{d-1}(u+1)}\big)^{d-u-1} s_{u+1,u}\big(\Phi_d, y^{\frac{1-\hat{p}}{d-1}}\big). \tag{4.2}$$

Plugging this equation into (4.1), we see that the exponent of y in each summand of $g(y)$ as $0 \leq u \leq d-2$ is

$$(1-\hat{p})\frac{(d-1-u)(u+1)}{(d-1)} + \big[\text{exponent of } y \text{ in } s_{u+1,u}\big(\Phi_d, y^{\frac{1-\hat{p}}{d-1}}\big)\big] + [\hat{p}(1+u)-1].$$

The above expression is nonnegative since all three terms are nonnegative and equal zero if and only if $u = 0$. (Recall from equation (3.1) that $s_{1,0} \equiv 1$.) When $u = d-1$, our assumption that $p > \frac{1}{d+1}$ ensures that $\hat{p}(1+u) = \hat{p}\cdot d > 0$. Recall from (3.3) that

$$s_{d,d-1}\big(\Phi_d, y^{\frac{1-\hat{p}}{d-1}}\big) = 1 - \sum_{i=0}^{d-2} s_{d,i}\big(\Phi_d, y^{\frac{1-\hat{p}}{d-1}}\big),$$

which must have nonnegative y-exponents by our previous reasoning. So the y-exponents in the $u = d-1$ summand of (4.1) are strictly positive.

$g(y)$ is now a sum of terms, each with nonnegative y-exponents. Therefore $g(y)$ is continuous on the interval $[0,1]$.

Now define the function $G\colon (\frac{1}{d+1}, \frac{1}{2}) \times [0,1] \to \mathbb{R}$ by $G(p,y) = g^{d,p}(y)$ for every fixed d. One can show that G is continuous on its entire domain, including $(\frac{1}{d+1}, \frac{1}{2}) \times \{0\}$. Consider any $p_0 \in (\frac{1}{d+1}, \frac{1}{2})$. Take any $r_1 < p_0$ and $r_2 > p_0$ so that $p_0 \in [r_1, r_2] \subseteq (\frac{1}{d+1}, \frac{1}{2})$, then G is uniformly continuous on the compact set $[r_1, r_2] \times [0,1]$. In particular we have for all $\gamma > 0$, there exists some $0 < \delta \le \min\{p_0 - r_1, r_2 - p_0\}$ such that for any $y \in (0,1]$, if $|p - p_0| < \delta$, then $|G(p,y) - G(p_0,y)| < \gamma$.

To show $M^{d,p} = \sup_{y\in(0,1]} G(p,y)$ is continuous in p, note

$$\Big| \sup_{y\in(0,1]} G(p,y) - \sup_{y\in(0,1]} G(p_0,y) \Big| \le \sup_{y\in(0,1]} |G(p,y) - G(p_0,y)|.$$

By the uniform continuity argument above we know immediately that $M^{d,p}$ is continuous at all $p \in (\frac{1}{d+1}, \frac{1}{2})$. ☐

Proof of (ii). In part (i) we said for each fixed p that $g(y)$ is continuous on the compact interval $[0,1]$, which implies $g(y)$ attains its maximum on $[0,1]$.

We also mentioned that the $g(y)$ in (4.1) is a sum of terms with positive y exponents except for $u = 0$, which produces a constant term. Plug $y = 0$ into (4.1) and use (4.2) in the case $u = 0$, we have

$$g(0) = e^{-p_d^*} \cdot \Phi_d^{d-1} = \exp\Big(-p_d^* - \frac{d-1}{d}(1 - p_d^*)\Big) = \exp\Big(-\frac{1}{d}p_d^* - \frac{d-1}{d}\Big).$$

(Recall the definition of Φ_d from Claim 3.1.) Also from (4.1) we have

$$g(1) = e^{-p_d^*} \sum_{u=0}^{d-1} s_{d,u}\big(\Phi_d, 1\big) = e^{-p_d^*}.$$

By $p < 1/2$ we have $p_d^* < 1$, which implies $-\frac{1}{d}p_d^* - \frac{d-1}{d} < -p_d^*$. It follows that $g(1) > g(0)$, and hence $g(y)$ attains its maximum on $(0,1]$, i.e.,

$$M^{d,p} = \max_{y\in(0,1]} g^{d,p}(y).$$

We can write $f^{d,p}(\lambda) = e^{-p_d^* + \lambda}\, \mathbf{E}[e^{-\hat{p}(1+U)\lambda}]$. By Lemma 4.1 we have $U(d,p,\lambda) \preceq U(d+1,p,\lambda)$. Since $p_{d+1}^* > p_d^*$, it follows that $f^{d+1,p}(\lambda) < f^{d,p}(\lambda)$. Therefore $g^{d+1,p}(y) < g^{d,p}(y)$ for all $y \in (0,1]$. Since g attains its maximum on $(0,1]$, $M^{d+1,p} < M^{d,p}$, as desired. ☐

It is now quick to deduce our final result.

Proof of Theorem 1.4. Referring to the statements in Lemma 4.3. (i) implies that $M^{d,q_d} \le 1$. (ii) implies that $M^{d+1,q_d} < 1$. Another application of (i) implies that $M^{d+1,q_d-\epsilon} < 1$ for some $\epsilon > 0$. Hence $q_{d+1} \le q_d - \epsilon$. ☐

Appendix

We explain how the exact bounds in Theorem 1.2 and the approximate bounds in Figure 1 are obtained. The code and further explanation can be found in the Jupyter Notebooks at https://github.com/fredcheng02/frog-model. The code is written in SageMath, but we imported the `CountRoots` function from Mathematica at one point, as previously noted.

We mentioned in the proof of Theorem 1.2 that we make a change of variables $f(-c\log(y)) = g(y)$. The approach in [2] was to visually inspect the formula to determine which c should be chosen so that $g(y)$ gives us a polynomial (integer exponents). This is possible for $d \leq 4$, but problematic for larger d when the expressions involve an exponentially increasing number of terms. Our approach is to take p to be an irreducible fraction $\frac{a}{b} \in (\frac{1}{d+1}, \frac{1}{2})$ and set $c = (b-a)(d-1)$. Then

$$g(y) = \exp\Big(\frac{a - ad}{bd - ad - d}\Big) \sum_{u=0}^{d-1} y^{(d-1)[(u+2)a-b]} s_{d,u}(\Phi_d, y^{b-2a}),$$

where $\Phi_d = \exp\big(\frac{2a-b}{bd-ad-a}\big)$. Using the same argument as in Lemma 4.3 (i), one can show that this change of variables always makes $g(y)$ a polynomial. We then have the power of exact algorithms to show that g has a unique maximum on $(0, 1]$, which we prove is strictly less than 1. The bounds obtained in Theorem 1.2 are nearly the best possible using this method. We use continued fraction approximation to find the p's such that $0.9994 < \sup_{y\in(0,1]} g^{d,p}(y) < 1$ for all $2 \leq d \leq 13$.

In finding the maximum of g on $(0, 1]$ above, we need to use the `find_root` function in SageMath to obtain the root of g'. The Brent's method used in `find_root` only allows fixed machine precision, and fails to give a correct root when $d \geq 14$, given the fast-growing complexity of the exact symbolic expression for $g(y)$. This led us to numerically approximate the bounds for higher d's instead.

To obtain the values in Figure 1, we work with the original $f(\lambda)$. It appears that $\arg\max_{\lambda\geq 0} f^{d,p}(\lambda)$ converges monotonically down to some value in $[0, 1)$ as $d \to \infty$, and f has small curvature around its peak. For a chosen p, if we can verify that $f^{d,p}(0), f^{d,p}(0.01), \dots, f^{d,p}(0.99)$ are all strictly less than 1, then we check for a slightly smaller p (e.g., we decrease p by 0.0001 in our code) if they are still less than 1. If not, then we have just found the approximate p such that $\sup_{\lambda\in[0,\infty)} f^{d,p}(\lambda) \approx 1$.

Our code also lets us verify if a specific p can work as an upper bound to S_m. For large m we have to adjust the real number precision accordingly to avoid numerical errors. For example, we have checked that the upper bounds fall below 0.18 for the first time at around $m = 230$, which again supports our conjecture that p_d converges at rate $d^{-1/2}$.

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