RECIPROCITY AND THE KERNEL OF DEDEKIND SUMS

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ABSTRACT. We use the action of Atkin-Lehner operators to generate a family of reciprocity formulas for newform Dedekind sums. This family of reciprocity formulas provides symmetries which we use to investigate the kernel of these Dedekind sums.

1. INTRODUCTION

1.1. **Motivation.** Dedekind sums were introduced as a means of expressing the transformation formula of the Dedekind eta function. These sums appear in numerous contexts, including topology, quadratic reciprocity, and modular forms. More background information on the classical Dedekind sum and Dedekind eta function can be found in [Apo90].

Newform Dedekind sums S_{χ_1,χ_2} associated to a pair of primitive Dirichlet characters χ_1, χ_2 (modulo q_1, q_2, respectively) were defined in [SVY20], and many of their basic properties were developed therein. The most important property is that each Dedekind sum is a group homomorphism on $\Gamma_1(q_1 q_2)$ (see Lemma 2.3). In addition, there is a reciprocity formula relating S_{χ_1,χ_2} to S_{χ_2,χ_1} (see [SVY20, Thm 1.3] or (4) below for the precise statement). This reciprocity formula generalizes the well-known reciprocity formula for the classical Dedekind sum, which has numerous applications including a proof of quadratic reciprocity, fast calculation of Dedekind sums, etc. The main result in this paper, stated in Theorem 1.1 below, further generalizes this by giving a *family* of reciprocity formulas.

The kernel of a Dedekind sum, i.e., the set of elements of $\Gamma_1(q_1 q_2)$ for which the Dedekind sum vanishes, was introduced and studied in [NRY21]. The reciprocity formula from [SVY20] played a crucial role in [NRY21] in understanding certain experimentally-observed patterns in the kernel. However, there were other observed patterns that were not explained in [NRY21]. As an application of our new reciprocity formulas, our second main result, stated in Theorem 1.3 below, identifies additional families of elements lying in the kernel.

The key tool in the proof of the reciprocity formula in [SVY20] is understanding the action of the Fricke involution on Dedekind sums. Here we extend this by developing the action of all the Atkin-Lehner operators on the newform Dedekind sums.

1.2. **The reciprocity formula.** We begin with some notation. Let $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ be primitive Dirichlet characters, with $\chi_1 \chi_2(-1) = 1$. Let B_1 be the first Bernoulli function defined by

$$B_1(x) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \in \mathbb{R} \backslash \mathbb{Z} \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases}$$

Suppose $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q_1 q_2)$ with $c \geq 1$. For $q_1, q_2 > 1$, define the Dedekind sum by

$$(1) \qquad S_{\chi_1,\chi_2}(\gamma) = S_{\chi_1,\chi_2}(a,c) = \sum_{j \bmod c} \sum_{n \bmod q_1} \overline{\chi_2}(j)\overline{\chi_1}(n) B_1\left(\frac{j}{c}\right) B_1\left(\frac{n}{q_1} + \frac{aj}{c}\right).$$

To be clear, we will use a different definition of the Dedekind sum (see Section 2.3) which is better-suited for our purposes. In fact, (1) is derived in [SVY20] only after substantial work.

Let W_Q be an Atkin-Lehner operator on $\Gamma_0(N)$ as defined in Section 2.1, with $N = q_1 q_2 = QR$, and with $(Q, R) = 1$. A Dirichlet character $\chi_i \pmod{q_i}$ factors uniquely as $\chi_i = \chi_i^{(Q)} \chi_i^{(R)}$ where $\chi_i^{(Q)}$ has conductor $(q_i, Q) =: q_i^{(Q)}$ and $\chi_i^{(R)}$ has conductor $(q_i, R) =: q_i^{(R)}$. Define χ_1', χ_2' by

$$(2) \qquad \chi_1' = \chi_2^{(Q)} \chi_1^{(R)} \qquad \text{and} \qquad \chi_2' = \chi_1^{(Q)} \chi_2^{(R)}.$$

In words, χ_1' and χ_2' result from exchanging the Q-portions of χ_1 and χ_2. Here χ_i' has conductor q_i', where $q_1' = q_2^{(Q)} q_1^{(R)}$ and $q_2' = q_1^{(Q)} q_2^{(R)}$.

In simplified cases, our new family of reciprocity formulas takes the following form.

Theorem 1.1 (Simplified Reciprocity Formula). *Let q_1, q_2 be such that $q_1 q_2 > 1$. Let $N = q_1 q_2$, suppose $\gamma \in \Gamma_1(N)$, and define $\gamma' \in \Gamma_1(N)$ by $W_Q \gamma = \gamma' W_Q$. Then*

$$(3) \qquad S_{\chi_1, \chi_2}(\gamma') = \xi S_{\chi_1', \chi_2'}(\gamma),$$

where ξ has absolute value one and depends on χ_1, χ_2, Q, and the entries of W_Q.

The more general statement of this reciprocity formula, including an explicit value of ξ, is presented in Theorem 3.1.

A special case of Theorem 1.1 occurs with $W_N = \left(\begin{smallmatrix} 0 & -1 \\ q_1 q_2 & 0 \end{smallmatrix} \right)$ chosen to be the Fricke involution. If $\gamma = \left(\begin{smallmatrix} a & b \\ c q_1 q_2 & d \end{smallmatrix} \right) \in \Gamma_1(q_1 q_2)$, then $\gamma' = \left(\begin{smallmatrix} d & -c \\ -b q_1 q_2 & a \end{smallmatrix} \right)$. Moreover, (2) says $\chi_1' = \chi_2$ and $\chi_2' = \chi_1$, and (3) becomes

$$(4) \qquad S_{\chi_1, \chi_2}(\gamma) = \pm S_{\chi_2, \chi_1}(\gamma')$$

where $\pm = +1$ if both χ_i are even, and $\pm = -1$ if both χ_i are odd. This is essentially Theorem 1.3 in [SVY20].

Note that [SVY20] and [NRY21] considered Dedekind sums with $q_1 > 1$ and $q_2 > 1$. It can happen in (3) that $q_1 > 1$ and $q_2 > 1$, but that $q_1' = 1$ or $q_2' = 1$. For this reason, we were naturally led to extend the definition of Dedekind sums to allow at most one q_i to equal 1, and to develop some of their properties generalizing results from [SVY20] and [NRY21]. The reader should beware that the formula (1) no longer holds when some $q_i = 1$, and in fact $S_{1, \chi_2}(\gamma)$ does not depend only on the first column of γ (e.g., see (21) below). Some of the forthcoming results, such as Theorem 1.3 below, have some hypotheses ultimately due to subtleties arising from $q_i = 1$.

Since [SVY20] treated properties of the newform Dedekind sum when $q_1 > 1$ and $q_2 > 1$, and the present work treats $q_1 > 1$ or $q_2 > 1$, it is natural to ask about the case $q_1 = q_2 = 1$. There are indeed many works that develop properties of the classical Dedekind sum from the level 1 Eisenstein series; for instance, [Gol73] treats the Eisenstein series attached to a cusp for a general Fuchsian group of the first kind. One of the pleasant features of the newform Eisenstein series (see Section 2.2 for the definition) is that it does not have pole at $s = 1$, except when $q_1 = q_2 = 1$. This pole at $s = 1$ is responsible for a term of the form $\log y$ in the Laurent expansion around $s = 1$, which in turn has a dramatic effect on the resulting formulas for the Dedekind sums. Since the classical case with $q_1 = q_2 = 1$ has a voluminous literature, our assumption that $q_1 > 1$ or $q_2 > 1$ is not a significant logical restriction. Moreover, this assumption allows for some pleasant simplifications due to the holomorphy of the Eisenstein series at $s = 1$.

1.3. **The kernel of Dedekind sums.**

Definition 1.2 ([NRY21] Defn. 1.4)**.** *Let χ_1 and χ_2 be primitive Dirichlet characters modulo q_1 and q_2, respectively, with $q_1 > 1$ and $q_2 > 1$. Then let*

$$K_{\chi_1,\chi_2} = \{\gamma \in \Gamma_0(N) \mid S_{\chi_1,\chi_2}(\gamma) = 0\}, \quad K_{q_1,q_2} = \cap_{\chi_1,\chi_2} K_{\chi_1,\chi_2}, \quad K^1_{q_1,q_2} = K_{q_1,q_2} \cap \Gamma_1(N).$$

An investigation of these kernels was undertaken in [NRY21]. The authors used SageMath [SD21] and (1) to compute $S_{\chi_1,\chi_2}(a,c)$ for all a (mod c) and $c \leq 10q_1q_2$, and various small values of q_1, q_2. Using this data, they recorded those elements lying in K_{χ_1,χ_2}, K_{q_1,q_2}, and so on. In Figure 1, we have reproduced the elements of $K_{3,5}$ from [NRY21], where the horizontal axis corresponds to the a-value, and the vertical axis corresponds to the c-value. In the graph of $K_{3,5}$ in Figure 1, we have highlighted elements in the kernel that were proved to exist in [NRY21] and which crucially used (4). By applying Theorem 3.1, we additionally explain all but two of the remaining elements in Figure 1 (see Theorem 1.3 and (27)).
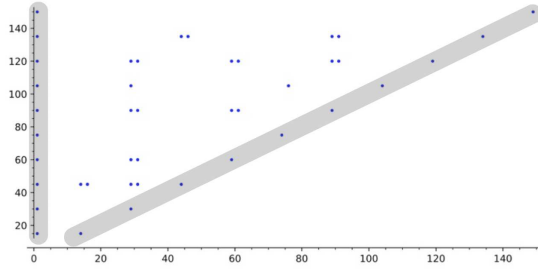
Figure 1. Elements of $K_{3,5}$ with $1 \leq c \leq 10q_1q_2$

The following relationship follows from Theorem 1.1:

(5) $$S_{\chi_1',\chi_2'}(\gamma) = 0 \iff S_{\chi_1,\chi_2}(\gamma') = 0.$$

Technically, the simplified version in Theorem 1.1 gives this relation if $\gamma, \gamma' \in \Gamma_1(N)$, but the more general reciprocity formula in Theorem 3.1 implies (5) for $\gamma, \gamma' \in \Gamma_0(N)$. Using (5), we derive the following theorem (see Section 4 for the proof).

Theorem 1.3. *Let $q_1q_2 = N = QR$ with $(Q,R) = 1$, $q_1 \neq 1$, and $q_2 \neq R$. Suppose $r, u \in \mathbb{Z}$ with $(r,R) = 1$ and $(u,Q) = 1$. Then for any $k \in \mathbb{Z}$, we have*

$$S_{\chi_1,\chi_2}(\pm 1 + Nkur, NRku^2) = 0.$$

Example 1.4. We illustrate Theorem 1.3's ability to justify many new elements displayed in Figure 1. Let $N = 15$, $R = 3$, $k = u = 1$. Taking $r = 1, 2$ reveals that $(\pm 1 + 15, 45)$ and $(\pm 1 + 30, 45)$ are in the kernel. Taking $k = 2, 3$ demonstrates the same for the non-highlighted points in Figure 1 with $c = 90$ and $c = 135$. Now let $R = 1$. Then, taking $r = 1$, $u = 2$, and $k = 1, 2$ explains that the non-highlighted points with $c = 60$ and $c = 120$ are also in the kernel. The only remaining unexplained points from Figure 1 are those such that $c = 105$ and $a = 29, 76$. These points are not obtainable from Theorem 1.3. This can be seen without computation by noting that the points proved to be in the kernel via Theorem 1.3 come in pairs with the values of a separated by two, yet the points in the row $c = 105$ do not occur in such pairs.

Also note that since $a^2 \equiv 1$ (mod 105) for $a = 29, 76$, then [NRY21, Prop. 2.2] may be used to explain these remaining points appearing in the figure; we leave the details for

an interested reader. We caution the reader that there is no reason to expect that all the elements of $K_{3,5}$ should be understood by results of these types.

2. BACKGROUND

2.1. **Atkin-Lehner Operators.** Let $N = QR$ with $(Q, R) = 1$. Following [AL78], we define an Atkin-Lehner operator W_Q on $\Gamma_0(N)$ as follows. Let $r_0, u_0 \in \mathbb{Z}$ be such that $(r_0, R) = 1$ and $(u_0, Q) = 1$. Let $r, t, u, v \in \mathbb{Z}$ with $r \equiv r_0 \pmod{R}$, $u \equiv u_0 \pmod{Q}$, and $Qrv - Rut = 1$. Then we define

$$W_Q^{(r_0, u_0)} = W_Q = \begin{pmatrix} Qr & t \\ Nu & Qv \end{pmatrix}.$$

This definition preserves the essential properties (see Lemma 2.1 below) of the operators as given in [AL78], which took $u_0 = r_0 = 1$. The added flexibility in u_0 and r_0 will be helpful in Section 4. The relaxed restrictions also mean the Fricke involution is a specialization of the Atkin-Lehner operators: take $Q = N$, $R = 1$, $u = u_0 = 1$, $t = -1$, $r = v = r_0 = 0$, so $W_Q = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, the Fricke involution.

Suppose that W_Q and W_Q' are Atkin-Lehner operators (with possibly different values of r, t, u, v, but the same choice of r_0, u_0). For $\gamma \in \Gamma_0(N)$, define γ' via

$$(6) \qquad\qquad\qquad\qquad W_Q \gamma = \gamma' W_Q'.$$

Lemma 2.1. *With γ' defined as in (6), we have $\gamma' \in \Gamma_0(N)$. Let d_γ and $d_{\gamma'}$ be the lower-right entries of γ and γ', respectively. Then*

$$(7) \qquad\qquad\qquad\qquad d_{\gamma'} \equiv \begin{cases} d_\gamma & \pmod{R}, \\ d_\gamma^{-1} & \pmod{Q}. \end{cases}$$

Proof. Say $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$. By brute force, we compute $\gamma' = W_Q \gamma W_Q'^{-1}$ as

$$(8) \qquad \gamma' = \begin{pmatrix} Qrv'a + Ntv'c - Nru'b - Rtu'd & -rt'a - Rtt'c + Qrr'b + tr'd \\ N(uv'a + Qvv'c - Ruu'b - vu'd) & -Rut'a - Nvt'c + Nur'b + Qvr'd \end{pmatrix}.$$

Since $\det(W_Q) = \det(W_Q') = Q$, then $\det(\gamma') = 1$. By inspection, γ' has integer entries, and its lower-left entry is $\equiv 0 \pmod{N}$. Modulo R, the lower-right entry is

$$d_\gamma' \equiv Qvr'd_\gamma \equiv (Qvr - Rut)d_\gamma \equiv d_\gamma \pmod{R},$$

where we have used $r' \equiv r_0 \equiv r \pmod{R}$. Similarly, using $a \equiv d_\gamma^{-1} \pmod{N}$, we have

$$d_\gamma' \equiv -Rut'a \equiv (Qvr - Rut)d_\gamma^{-1} \equiv d_\gamma^{-1} \pmod{Q}. \qquad \square$$

Remark 2.2. By taking $\gamma \in \Gamma_1(N)$, Lemma 2.1 gives that γ' defined as in (6) satisfies $\gamma' \in \Gamma_1(N)$. In particular, taking $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ shows $W_Q = \alpha W_Q'$ for some $\alpha \in \Gamma_1(N)$.

The Atkin-Lehner operators act on modular functions on $\Gamma_1(N)$ as follows. Suppose g is on $\Gamma_0(N)$ with Dirichlet character ψ, meaning $g(\gamma z) = \psi(\gamma)g(z)$ for all $\gamma \in \Gamma_0(N)$ and $z \in \mathbb{H}$; here $\psi(\gamma) = \psi(d_\gamma)$, where d_γ is the lower-right entry of γ as above. Write $\psi = \psi^{(Q)}\psi^{(R)}$, where $\psi^{(Q)}$ has modulus Q and $\psi^{(R)}$ has modulus R. Let $h(z) = g(W_Q z) = g(W_Q' z)$ (the latter equality following from Remark 2.2). Then $h(\gamma z) = g(W_Q \gamma z) = g(\gamma' W_Q' z) = \psi(\gamma')h(z)$. By Lemma 2.1, $\psi(\gamma') = \psi'(\gamma)$, where

$$(9) \qquad\qquad\qquad\qquad \psi' := \overline{\psi}^{(Q)}\psi^{(R)}.$$

2.2. **Eisenstein Series.** Let χ_1, χ_2 be primitive Dirichlet characters modulo q_1, q_2, respectively, such that $\chi_1\chi_2(-1) = 1$. By convention, we allow $q_i = 1$, in which case $\chi_i(n) = 1$ for all $n \in \mathbb{Z}$. The *newform Eisenstein series* attached to χ_1, χ_2 is defined as

$$E_{\chi_1,\chi_2}(z, s) = \frac{1}{2} \sum_{(c,d)=1} \frac{(q_2 y)^s \chi_1(c) \chi_2(d)}{|cq_2 z + d|^{2s}}, \quad \text{Re}(s) > 1.$$

The function E_{χ_1,χ_2} is automorphic on $\Gamma_0(q_1 q_2)$ with character $\psi = \chi_1\overline{\chi_2}$, and we refer the reader to [You19] as a convenient reference for this fact and others to follow.

Define the "completed" Eisenstein series by

(10) $$E^*_{\chi_1,\chi_2}(z, s) = \frac{(q_2/\pi)^s}{\tau(\chi_2)} \Gamma(s) L(2s, \chi_1\chi_2) E_{\chi_1,\chi_2}(z, s),$$

where $\tau(\chi_2)$ denotes the Gauss sum, and $L(s, \chi_1\chi_2) = \sum_{n \geq 1} \chi_1\chi_2(n) n^{-s}$. The Fourier expansion takes the form

(11) $$E^*_{\chi_1,\chi_2}(z, s) = e^*_{\chi_1,\chi_2}(y, s) + 2\sqrt{y} \sum_{n \neq 0} \lambda_{\chi_1,\chi_2}(n, s) e(nx) K_{s-\frac{1}{2}}(2\pi|n|y),$$

where $z = x + iy$, K_ν is the K-Bessel function,

(12) $$\lambda_{\chi_1,\chi_2}(n, s) = \chi_2(\text{sgn}(n)) \sum_{ab=|n|} \chi_1(a)\overline{\chi_2}(b)(b/a)^{s-\frac{1}{2}},$$

and the constant term $e^*_{\chi_1,\chi_2}(y, s)$ is given by

(13) $$\delta_{q_1=1} q_2^{2s} \frac{\pi^{-s}}{\tau(\chi_2)} \Gamma(s) L(2s, \chi_2) y^s + \delta_{q_2=1} q_1^{2-2s} \frac{\pi^{-(1-s)}}{\tau(\overline{\chi_1})} \Gamma(1-s) L(2-2s, \overline{\chi_1}) y^{1-s},$$

where $\delta_{q=1}$ is the delta function equalling 1 if $q = 1$ and 0 otherwise. Throughout the remainder of the paper, we assume that $q_1 q_2 > 1$. Combined with (11), this condition ensures that $e^*_{\chi_1,\chi_2}(y, s)$ and $E^*_{\chi_1,\chi_2}(z, s)$ are analytic for all $s \in \mathbb{C}$. With some simplifications (see [SVY20, (1.5), (1.6)]), (11) specializes as

(14) $$E^*_{\chi_1,\chi_2}(z, 1) = F_{\chi_1,\chi_2}(z) + \chi_2(-1)\overline{F}_{\overline{\chi_1},\overline{\chi_2}}(z),$$

where

(15) $$F_{\chi_1,\chi_2}(z) = c_1 z + c_0 + \sum_{n=1}^{\infty} \frac{\lambda_{\chi_1,\chi_2}(n, 1)}{\sqrt{n}} e(nz)$$

and where

(16) $$c_1 = \delta_{q_1=1} \frac{q_2^2 L(2, \chi_2)}{2\pi i \tau(\chi_2)} = \delta_{q_1=1} \pi i L(-1, \overline{\chi_2}), \quad \text{and} \quad c_0 = \delta_{q_2=1} \frac{1}{2} L(1, \chi_1).$$

Note that the calculation of both constants c_1 and c_0 used the functional equation of the Dirichlet L-function.

A key feature of the newform Eisenstein series is that they are pseudo-eigenfunctions of the Atkin-Lehner operators. By [You19, (9.3)] (see also [Wei77]),

(17) $$E_{\chi_1,\chi_2}(W_Q z, s) = C E_{\chi'_1,\chi'_2}(z, s),$$

where $C = \chi_1^{(Q)}(-1)\psi^{(Q)}(q_1^{(R)} u_0)\overline{\psi}^{(R)}(q_2^{(Q)} r_0)$. This evaluation of C can be found in [You19, Section 9.1] with $r_0, u_0 = 1$, but it is easy to extend the calculation for general r_0 and u_0.

Note that ψ' defined by (9) equals $\chi_1' \overline{\chi_2}'$. Similarly, for the completed Eisenstein series, we derive

$$(18) \qquad E_{\chi_1,\chi_2}^*(W_Q z, 1) = \beta E_{\chi_1',\chi_2'}^*(z, 1), \qquad \text{where} \qquad \beta = \frac{q_2 \tau(\chi_2')}{q_2' \tau(\chi_2)} C.$$

2.3. **Dedekind Sums.** We now turn to the newform Dedekind sums that are constructed with the newform Eisenstein series. These Dedekind sums were defined in [SVY20], though with the assumption that $q_1 \neq 1$ *and* $q_2 \neq 1$, in which case (1) was derived after some extensive calculations. We will extend their definition to cover the cases where at most one of the q_i is 1. Some of the proofs from [SVY20] carry over nearly verbatim, in which case we will omit the details here and refer the reader to [SVY20].

For $\gamma \in \Gamma_0(N)$, define the function

$$(19) \qquad \phi_{\chi_1,\chi_2}(\gamma, z) = F_{\chi_1,\chi_2}(\gamma z) - \psi(\gamma) F_{\chi_1,\chi_2}(z),$$

where recall $\psi = \chi_1 \overline{\chi_2}$ and $\chi_1 \chi_2(-1) = 1$. This definition of ϕ_{χ_1,χ_2} extends that of [SVY20] by allowing at most one of c_1 or c_0 to be nonzero. The function $\phi_{\chi_1,\chi_2}(\gamma, z)$ is constant in terms of z; see [SVY20, Lemma 2.1] for a proof that applies to this extended definition. Therefore we write $\phi_{\chi_1,\chi_2}(\gamma, z)$ more simply as $\phi_{\chi_1,\chi_2}(\gamma)$. As in [SVY20], we define the newform Dedekind sum by

$$(20) \qquad S_{\chi_1,\chi_2}(\gamma) = \frac{\tau(\overline{\chi_1})}{\pi i} \phi_{\chi_1,\chi_2}(\gamma).$$

The most important property of the newform Dedekind sum is that it is a group homomorphism on $\Gamma_1(N)$. Precisely, we have

Lemma 2.3 ([SVY20] Lemma 2.2). *Let $\gamma_1, \gamma_2 \in \Gamma_0(N)$. Then*

$$S_{\chi_1,\chi_2}(\gamma_1 \gamma_2) = S_{\chi_1,\chi_2}(\gamma_1) + \psi(\gamma_1) S_{\chi_1,\chi_2}(\gamma_2).$$

The proof in [SVY20] carries over identically to the case where $q_1 = 1$ or $q_2 = 1$.

Note that $S_{\chi_1,\chi_2}(\gamma) = 0$ if and only if $F_{\chi_1,\chi_2}(\gamma z) = \psi(\gamma) F_{\chi_1,\chi_2}(z)$ for all $z \in \mathbb{H}$. So, the elements of K_{χ_1,χ_2} are those for which F_{χ_1,χ_2} transforms like an automorphic form. This is some motivation for studying the kernel of the Dedekind sum.

Proposition 2.4. *Suppose $\gamma_1, \gamma_2 \in \Gamma_0(N)$ have the same left column and suppose $q_1 \neq 1$. Then $S_{\chi_1,\chi_2}(\gamma_1) = S_{\chi_1,\chi_2}(\gamma_2)$.*

Proof. It follows from our conditions on γ_1, γ_2 that $\gamma_1 = \gamma_2 \omega$ with $\omega = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ for some $b \in \mathbb{Z}$. Now, by Lemma 2.3, $S_{\chi_1,\chi_2}(\gamma_1) = S_{\chi_1,\chi_2}(\gamma_2 \omega) = S_{\chi_1,\chi_2}(\gamma_2) + \psi(\gamma_2) S_{\chi_1,\chi_2}(\omega)$. By (19) and (15), if $q_1 \neq 1$ then $S_{\chi_1,\chi_2}(\omega) = 0$. □

Proposition 2.4 justifies our earlier notation $S_{\chi_1,\chi_2}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = S_{\chi_1,\chi_2}(a, c)$, provided $q_1 \neq 1$. On the other hand, we have

$$(21) \qquad S_{1,\chi_2}\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = b L(-1, \chi_2),$$

showing the condition $q_1 \neq 1$ is necessary in Proposition 2.4. Now define

$$(22) \qquad \phi_{\chi_1,\chi_2}(W_Q, z) = \phi_{\chi_1,\chi_2}(W_Q) = F_{\chi_1,\chi_2}(W_Q z) - \beta F_{\chi_1',\chi_2'}(z)$$

for $z \in \mathbb{H}$. Note, $\phi_{\chi_1,\chi_2}(W_Q)$ may depend on the choice of r, t, u, and v in W_Q, but β does not (though β does depend on u_0, r_0, the characters χ_i, and so on).

Lemma 2.5. *The function $\phi_{\chi_1,\chi_2}(W_Q, z)$ is independent of z.*

Proof. From (14) and (18), it immediately follows that

$$\phi_{\chi_1,\chi_2}(W_Q, z) = -\chi_2(-1)\overline{\phi_{\overline{\chi_1},\overline{\chi_2}}}(W_Q, z).$$

Since $\phi_{\chi_1,\chi_2}(W_Q, z)$ is holomorphic and $\overline{\phi_{\overline{\chi_1},\overline{\chi_2}}}(W_Q, z)$ is anti-holomorphic, $\phi_{\chi_1,\chi_2}(W_Q, z)$ must be constant in z. □

Lemma 2.5 justifies writing $\phi_{\chi_1,\chi_2}(W_Q, z) = \phi_{\chi_1,\chi_2}(W_Q)$. Analogously to (20), we define the Dedekind sum S_{χ_1,χ_2} associated to W_Q as

$$(23) \qquad S_{\chi_1,\chi_2}(W_Q) = \frac{\tau(\overline{\chi_1})}{\pi i}\phi_{\chi_1,\chi_2}(W_Q).$$

3. Proof of the generalized reciprocity formula

Theorem 3.1 (Generalized Reciprocity Formula)**.** *Let χ_1, χ_2 be primitive Dirichlet characters with moduli q_1, q_2, respectively, such that $q_1 q_2 > 1$, and $\chi_1\chi_2(-1) = 1$. Let $N = q_1 q_2$ and let W_Q and W_Q' be Atkin-Lehner operators as described in Section 2.1. Then for $\gamma, \gamma' \in \Gamma_0(N)$ related by $W_Q\gamma = \gamma' W_Q'$, we have the following reciprocity formula:*

$$(24) \qquad S_{\chi_1,\chi_2}(W_Q) + \xi S_{\chi_1',\chi_2'}(\gamma) = \psi'(\gamma)S_{\chi_1,\chi_2}(W_Q') + S_{\chi_1,\chi_2}(\gamma'),$$

where $\xi = \frac{\tau(\overline{\chi_1})}{\tau(\overline{\chi_1'})}\beta$ and $\psi' = \chi_1'\overline{\chi_2'}$. Here $|\xi| = 1$.

Proof. Recall (19) and (22). Now we calculate

$$(25) \qquad F_{\chi_1,\chi_2}(W_Q\gamma z) - \beta\psi'(\gamma)F_{\chi_1',\chi_2'}(z)$$

in two ways. First, note that it equals

$$\underbrace{F_{\chi_1,\chi_2}(W_Q\gamma z) - \beta F_{\chi_1',\chi_2'}(\gamma z)}_{\phi_{\chi_1,\chi_2}(W_Q)} + \beta\underbrace{\left(F_{\chi_1',\chi_2'}(\gamma z) - \psi'(\gamma)F_{\chi_1',\chi_2'}(z)\right)}_{\phi_{\chi_1',\chi_2'}(\gamma)}.$$

Alternatively, we use $W_Q\gamma = \gamma' W_Q'$ to see that (25) equals

$$F_{\chi_1,\chi_2}(\gamma' W_Q' z) - \psi'(\gamma)F_{\chi_1,\chi_2}(W_Q' z) + \psi'(\gamma)(F_{\chi_1,\chi_2}(W_Q' z) - \beta F_{\chi_1',\chi_2'}(z)).$$

Using $\psi'(\gamma) = \psi(\gamma')$, this becomes

$$\underbrace{F_{\chi_1,\chi_2}(\gamma' W_Q' z) - \psi(\gamma')F_{\chi_1,\chi_2}(W_Q' z)}_{\phi_{\chi_1,\chi_2}(\gamma')} + \psi'(\gamma)\underbrace{\left(F_{\chi_1,\chi_2}(W_Q' z) - \beta F_{\chi_1',\chi_2'}(z)\right)}_{\phi_{\chi_1,\chi_2}(W_Q')}.$$

Since the value of β depends solely on r_0 and u_0 and not r, t, u, and v, it remains unchanged between the formulas involving W_Q and W_Q'. Equating the two expressions for (25), we infer

$$\phi_{\chi_1,\chi_2}(W_Q) + \beta\phi_{\chi_1',\chi_2'}(\gamma) = \phi_{\chi_1,\chi_2}(\gamma') + \psi'(\gamma)\phi_{\chi_1,\chi_2}(W_Q').$$

By (20) and (23), we deduce (24). Note that $|\beta| = (q_2/q_2')^{1/2}$, so $|\xi| = 1$, since $q_1 q_2 = q_1' q_2'$.

Finally, we deduce Theorem 1.1, which is the special case that $\gamma \in \Gamma_1(N)$ and $W_Q' = W_Q$. In this case, $\gamma' \in \Gamma_1(N)$ as well, and (24) simplifies via $\psi'(\gamma) = 1$. In addition, the term $S_{\chi_1,\chi_2}(W_Q)$ appears on both sides of (24) and may be cancelled. □

4. Kernel

In this section, we use Theorem 3.1 to further the study initiated in [NRY21] of the kernel of Dedekind sums.

Proof of Theorem 1.3. Given r, u as stated in the theorem, let $W_Q = \left(\begin{smallmatrix} Qr & t \\ Nu & Qv \end{smallmatrix} \right)$ be an Atkin-Lehner operator (the conditions $(r, R) = 1$ and $(u, Q) = 1$ are sufficient to ensure such a matrix exists, and then r_0, u_0 are determined).

Let $\gamma = \left(\begin{smallmatrix} 1 & -k \\ 0 & 1 \end{smallmatrix} \right)$. Recall γ' is defined by $W_Q \gamma = \gamma' W_Q$ and is calculated in (8), giving

$$(26) \qquad \gamma' = \begin{pmatrix} 1 + Nkur & * \\ NRku^2 & * \end{pmatrix}.$$

The strategy for the proof is now to show that the assumptions in Theorem 1.3 imply that $S_{\chi_1', \chi_2'}(\gamma) = 0$, and so the reciprocity formula Theorem 3.1 implies $S_{\chi_1, \chi_2}(\gamma') = 0$. The condition $q_1 \neq 1$ is required for $S_{\chi_1, \chi_2}(\gamma')$ to only depend on the left column of γ'.

We now show that the condition $q_2 \neq R$ implies that $q_1' \neq 1$. Recall that $q_1' = (q_2, Q)(q_1, R)$, so that $q_1' = 1$ implies $(q_2, Q) = 1$ and $(q_1, R) = 1$. However, these latter two conditions imply $q_2 = R$ (and also $q_1 = Q$), a contradiction.

Using $q_1' \neq 1$, and by (19) and (15), we conclude $S_{\chi_1', \chi_2'}(\gamma) = 0$. Therefore, $S_{\chi_1, \chi_2}(\gamma') = 0$, by the reciprocity formula. That is, $S_{\chi_1, \chi_2}(1 + Nkur, NRku^2) = 0$, where we have used $q_1 \neq 1$ in order to invoke Proposition 2.4.

Finally, by taking $\gamma = \left(\begin{smallmatrix} -1 & -k \\ 0 & -1 \end{smallmatrix} \right)$, we obtain

$$(27) \qquad S_{\chi_1, \chi_2}(-1 + Nkur, NRku^2) = 0. \qquad \square$$

In the following example, we elaborate on a special case of Theorem 1.3 to demonstrate its versatility.

Example 4.1. Consider $K_{7,11}$ displayed in Figure 2. It follows from Theorem 1.3 (with $R = 7$, $Q = 11$, $u = 1$, and $(r, 7) = 1$) that $(\pm 1 + 77kr, 539k)$ is in the kernel. This encompasses all elements with $c = 539k$, which are those circled in Figure 2 (note that due to the ± 1 term in Theorem 1.3, each circle contains a pair of kernel elements with close upper-left entries).
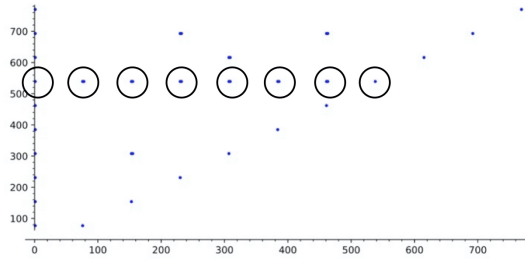
Figure 2. Elements of $K_{7,11}$ for $1 \leq c \leq 10 q_1 q_2$

Note that in Figures 1 and 2, the only kernel elements present in rows $c = Nkq_2$ are those equivalent to $(\pm 1, Nkq_2)$. We refer the reader to [NRY21] for additional examples of this phenomenon. The following proposition gives some evidence for this sparseness by demonstrating that no element of the form $(a, c) = (\pm 1 + Nkr, Nkq_2)$ with nonzero k is in the kernel.

Proposition 4.2. *Let $q_1 q_2 = N$ with $(q_1, q_2) = 1$, $q_1 \neq 1$. Suppose $r \in \mathbb{Z}$ with $(r, q_2) = 1$. Then for any nonzero $k \in \mathbb{Z}$, we have*

$$(28) \qquad\qquad S_{\chi_1, \chi_2}(\pm 1 + Nkr, Nkq_2) \neq 0.$$

Proof. The proof follows the same framework as Theorem 1.3. Let W_Q be an Atkin-Lehner operator with $Q = q_1$ and $R = q_2$, so $(q_1, R) = 1$ and $(q_2, Q) = 1$. The effect of W_Q on q_1, q_2 is such that $q_1' = 1$. Let $\gamma = \left(\begin{smallmatrix} 1 & -k \\ 0 & 1 \end{smallmatrix} \right)$. By (21), $S_{1, \chi_2'}(\gamma) = -kL(-1, \overline{\chi_2}') \neq 0$. Therefore, with γ' as in (26), we have $S_{\chi_1, \chi_2}(\gamma') \neq 0$. This shows the claim (28). □

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6. Statements and Declarations

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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