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Quantum state preparation in Jaynes-Cummings lattices

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Abstract. One of the key questions in quantum information is the preparation of desired multipartite quantum states with high fidelity. Adiabatic evolution has been widely explored to achieve state preparation in quantum many-body systems. However, in noisy quantum systems, the adiabatic approach faces a dilemma: either extending the evolution timescales to reduce diabatic transitions or shortening the timescales to mitigate decoherence effects. Various quantum control approaches have been studied to resolve this dilemma. In a few recent works, we utilized Jaynes-Cummings (JC) lattices as a platform to investigate the potential of several quantum control techniques in preparing quantum many-body states, including the optimized adiabatic evolution approach, the quantum optimal control technique, and quantum shortcuts to adiabaticity. Here we first give an overview of our previous results on utilizing quantum optimal control in JC lattices with unit filling and utilizing local counterdiabatic driving in JC lattices with a single excitation. Then we present our results on the energy costs and energy fluctuations in these approaches. Our studies give insights into the implementation of different approaches in practical quantum devices and the connection between the energy costs and the quantum speed limit in preparing desired quantum many-body states for quantum simulation and quantum computation.

1 Introduction

Preparing correlated quantum many-body states or multipartite entanglement is essential for quantum information and quantum computation. In the past few decades, various approaches have been investigated for high-fidelity quantum state preparation. One of these approaches utilizes adiabatic evolution to transform an initial state to the desired target state by slowly varying the Hamiltonian of given quantum systems [1, 2]. Despite intensive efforts, the effectiveness of the adiabatic approach is often affected by its extended timescales, which are required for retaining adiabaticity and achieving high fidelity for the prepared states, especially complex multipartite states. The extended timescales can result in severe decoherence in noisy quantum devices. Various control techniques have been studied to overcome this hurdle, including the quantum optimal control approach [3, 4, 5, 6] and quantum shortcuts to adiabaticity [7, 8, 9, 10]. Each approach has their respective advances and drawbacks in the requirements for experimental implementation.

Jaynes-Cummings (JC) lattices, constructed by connecting JC models into arrays of different configurations [11, 12, 13], provide an excellent platform to study the potential of these control techniques for quantum state preparation, given the rich controllability and coupling schemes of JC lattices in different physical platforms [14, 15, 16, 17, 18]. When occupied by polariton excitations, JC lattices can demonstrate novel quantum many-body effects such as quantum phase transitions as well as multipartite entanglement. With the advances in quantum technology, the JC model and JC lattices have been intensively experimented in various quantum platforms such as superconducting qubits coupled to microwave cavities, the internal states of trapped ions coupled to their motional states, and defect spins coupled to

nanocavities [19, 20, 21]. In recent works, we investigated quantum many-body state preparation in JC lattices with several quantum control techniques including the optimized adiabatic evolution approach, the quantum optimal control (QOC) technique and quantum shortcuts to adiabaticity (STA) [22, 23, 24, 25]. Our results provide insights into the high-fidelity state preparation in physical systems. Here we will first give an overview of our recent works using QOC and counterdiabatic (CD) driving – one of the STA approaches – for the generation of quantum many-body states with high fidelity. Then we will present numerical results on the energy costs and energy fluctuations in these approaches, which can provide insights into the relation between the energy costs and the required evolution time in preparing desired quantum states with high fidelity.

2 Jaynes-Cummings lattice

A Jaynes-Cummings (JC) model is made up of a qubit coupled to a cavity mode and is a cornerstone in quantum optics [12, 13]. JC lattices can be constructed by connecting adjacent cavities in JC models via photon hopping [22] or by connecting neighboring qubits and cavities in alternative configurations [16, 17]. The Hamiltonian of a one-dimensional JC lattice (assuming $\hbar = 1$) can be written as $H_{\rm r} = \sum_j \Delta a_j^\dagger a_j + gV_g + JV_J$, where $V_g = \sum_{j=1}^N (a_j^\dagger \sigma_{j-} + \sigma_{j+} a_j)$ is the onsite JC coupling between the qubits and the cavity modes on the same site and $V_J = -\sum_{i=1}^N (a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i)$ is the photon hopping between neighboring cavity modes. Under periodic boundary conditions, $a_{N+1} \equiv a_1$, with N being the number of sites in the JC lattice. This Hamiltonian is written in the rotating frame of the Hamiltonian $H_0^{(\rm rot)} = \omega_z \sum_j (a_j^\dagger a_j + \frac{1}{2}\sigma_{jz})$. Here ω_z is the qubit energy splitting, σ_{jz} and $\sigma_{j\pm}$ are the Pauli operators of the qubits, a_j ($a_j^\dagger$) is the annihilation (creation) operator of the cavity modes, g is the strength of the qubit-cavity coupling, J is the photon hopping rate, $\Delta = \omega_c - \omega_z$ is the detuning between the cavity modes and the qubits, and ω_c is the cavity frequency.

We denote the basis states of the JC model at the jth site by $|n, s\rangle_j$, where $n \geq 0$ is the photon number of the cavity mode and $s = g$, e are the ground and excited states of the qubit, respectively. The eigenstates of the JC model include the ground state $|0, g\rangle_j$ and the doublets $|n, \pm\rangle_j$ (for $n \geq 1$), which are superpositions of the basis states $|n, g\rangle_j$ and $|n-1, e\rangle_j$ and describe eigenstates of n total excitations. In the thermodynamic limit with $N \to \infty$, JC lattices can exhibit quantum many-body effects such as the quantum and dissipative phase transitions between the Mott insulating and the superfluid phases [14, 15] when the number of excitations is an integer multiple of the number of lattice sites. For JC lattices of a few sites, entangled states can be generated by applying control pulses [18]. However, the preparation of quantum many-body states in this system can be challenging due to the lack of knowledge of such states or the requirement of nonlocal interactions.

3 Quantum optimal control

The quantum optimal control (QOC) technique provides a computational framework to generate desired quantum states or quantum processes by searching for optimal, time-dependent control parameters under given constraints. It has been applied to diverse questions in quantum computation such as engineering desired quantum states and quantum dynamics, suppression of environmental noise, and control of the quantum transduction processes [3, 4, 5, 6]. We apply QOC to the quantum state preparation in a four-site JC lattice with unit filling, i.e., the number of excitations is equal to the number of sites in the lattice, where the ground state exhibits quantum phase transition between the Mott insulating phase and the superfluid phase. We choose the initial parameters of this system to be $g(0) = 0$ and $J(0) = 0.5$, and the target parameters at the final time T of the evolution to be $g(T) = 1$ and $J(T) = 0.02$. The initial state is the ground state of the initial parameters at $g(0) = 0$, which is in the superfluid phase with all excitations occupying the lowest collective photonic mode. The target state with $J(T) \ll g(T)$ is in the Mott insulating phase. The coupling strengths $g(t)$ and $J(t)$ are time-dependent and are optimized using QOC. We adopt the CRAB algorithm that parameterizes the couplings with truncated Fourier series [26, 27], and apply the Nelder-Mead method for the optimization. In [23], the time-dependent couplings $g(t)$ and $J(t)$ are parameterized with truncated Fourier series to the $8^{\rm th}$ harmonics and can be written as

$$g(t) = g_0(t)\,[1 + s(t) f_1(t)] \;\text{ and }\; J(t) = J_0(t)\,[1 + s(t) f_2(t)]\,, \tag{1}$$

with

(2a)

(2b)

where $c_{i,k}$ a
$J(t)$ respect
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(t) and
t $\delta\omega_{i,k}$,
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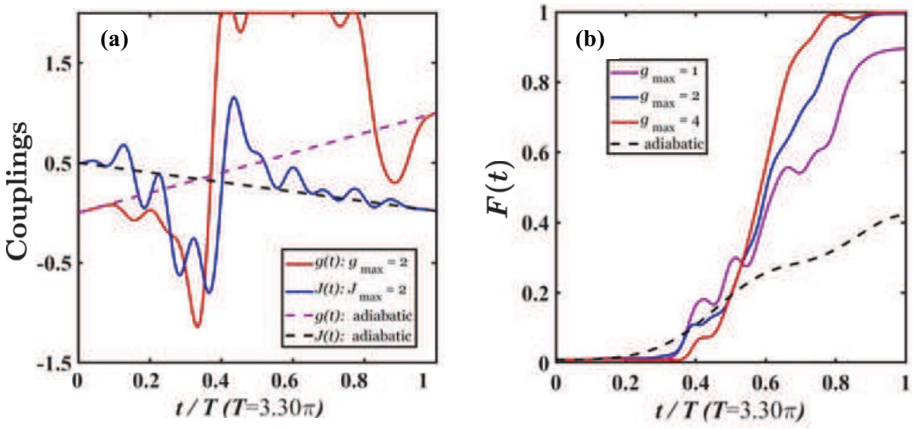

Figure 1: Quantum optimal control [23]. (a) The optimized couplings (solid lines) and the couplings in the adiabatic evaluation (dashed lines) vs t/T for $g_{\max} = 2$. (b) The fidelity $F(t)$ vs t/T for the maximal parameter magnitude $g_{\max} = 1, 2$, and 4, respectively. Here $J_{\max} = 2$ and the evolution time $T = 3.30\pi$. All parameters are in dimensionless units.

We plot the optimized couplings $g(t)$ and $J(t)$ vs t/T under the maximal coupling magnitudes $J_{\max} = 2$ and $g_{\max} = 2$ for the total evolution time $T = 3.30\pi$ in Fig. 1(a). The optimized couplings change continuously over the course of the evolution, where $g(t)$ exhibits plateau areas due to the parameter constraint $g_{\max} = 2$ and $J(t)$ has no plateau. This is because $J_{\max} = 1$ is sufficiently large for the system to explore the deep superfluid regime. In Fig. 1(b), the fidelity of the prepared state is plotted. Here the fidelity is defined as $F(t) = |\langle\psi(T)|\psi_{\rm T}\rangle|^2$, which corresponds to the overlap between the quantum state of this system $|\psi(T)\rangle$ at the final time T and the target state $|\psi_{\rm T}\rangle$. Our numerical results show that the fidelity from the QOC approach far exceeds that from the adiabatic evolution, indicating that QOC can greatly outperform the adiabatic approach. The fidelity of the prepared state increases with the maximal coupling magnitude $g_{\max}$, as a larger $g_{\max}$ allows the system to explore the parameter space in deeper Mott insulating regime.

4 Local counterdiabatic driving

Quantum shortcuts to adiabaticity, where reverse engineering of diabatic transitions is employed to accelerate a slow adiabatic process via nonadiabatic shorcuts, have been developed to improve quantum state preparation using adiabatic process. One of the STA approaches is the counter-diabatic (CD) driving [28, 29, 30, 31, 32], where a CD Hamiltonian is applied to instantaneously eliminate all diabatic transitions during the adiabatic evolution. The exact form of the CD Hamiltonian is [28, 29]

$$H_{\rm CD} = i \sum_{(k,l)\neq(k',l')} \frac{\vec{w}_{k,l}\vec{w}^{\dagger}_{k,l}\partial_t H_{\rm r}\vec{w}_{k',l'}\vec{w}^{\dagger}_{k',l'}}{\lambda_{k',l'} - \lambda_{k,l}} \tag{3}$$

in terms of the instantaneous eigenstates $\vec{w}_{k,l}$ and eigenenergies $\lambda_{k,l}$ of the adiabatic Hamiltonian $H_{\rm r}$ at time t. However, this exact CD Hamiltonian almost always contains non-local or multipartite interactions

that are difficult to implement in realistic quantum systems. Various approaches have been explored in recent years to overcome this issue by deriving approximate local CD driving [33, 34, 35].

For JC lattices with a single excitation, the Hilbert space is $2N$-dimensional with the basis set $\{|1,g\rangle_j \prod_{j'\neq j}|0,g\rangle_{j'}, |0,e\rangle_j \prod_{j'\neq j}|0,g\rangle_{j'}\}$ and $j\in[1,N]$, which corresponds to the excitation being either a cavity photon or a qubit spin flip on the j-th site [24, 25]. We refer to this basis set as the real-space basis, where the excitation occupies a local state. Solving the adiabatic Hamiltonian $H_{\rm r}$, we find that the eigenstates of this system have the form $\vec{w}_{k,\pm}=\vec{R}_k\otimes\vec{v}_{k,\pm}$ in the real-space basis, where

$$\vec{v}_{k,+}=\begin{pmatrix}\sqrt{\frac{\chi_k+\Delta_k}{2\chi_k}}\\ \sqrt{\frac{\chi_k-\Delta_k}{2\chi_k}}\end{pmatrix},\quad \vec{v}_{k,-}=\begin{pmatrix}-\sqrt{\frac{\chi_k-\Delta_k}{2\chi_k}}\\ \sqrt{\frac{\chi_k+\Delta_k}{2\chi_k}}\end{pmatrix},$$

and $\vec{R}_k$ is a N-dimensional vector with its j-th element $(\vec{R}_k)_j=\frac{1}{\sqrt{N}}e^{i\frac{2\pi k(j-1)}{N}}$ under the periodic boundary condition. Here $\chi_k=\sqrt{\Delta_k^2+4g^2}$, $\Delta_k=\Delta-2J\cos\frac{2\pi k}{N}$, and the wave vector $k\in[0,N-1]$. The corresponding eigenenergies are $\lambda_{k,\pm}=\frac{1}{2}(\Delta_k\pm\chi_k)$, and the ground state $\vec{w}_{\bar{k},-}$ has the wave vector $\bar{k}=0$. We refer to the basis set formed by these eigenstates as the k-space basis. Consider an adiabatic evolution with the detuning Δ time-independent and the couplings g and J linearly tuned between the initial and the target parameters. It can be shown that the exact CD Hamiltonian for this system is block diagonal in the k-space basis with the block matrices $H_{\rm CD}^{(k)}=(0,g_{\rm CD}^{(k)};g_{\rm CD}^{(k)^\star},0)$ for the subspace of the wave vector k and

$$g_{\rm CD}^{(k)}=-i\frac{\Delta_k\partial_t g}{\chi_k^2}+i\frac{g\left(\Delta_k-\Delta\right)\partial_t J}{J\chi_k^2} \tag{4}$$

in terms of the time derivatives $\partial_t g$ of the qubit-cavity coupling and $\partial_t J$ of the photon hoping rate. Hence, when the initial state is the ground state $\vec{w}_{\bar{k},-}$, the only allowable diabatic transition is to the excited state $\vec{w}_{\bar{k},+}$ of the same wave vector $\bar{k}$, and the only matrix element in $H_{\rm CD}$ that will affect the dynamics of the system is $g_{\rm CD}^{(\bar{k})}$.

To implement the exact CD Hamiltonian, we need to know its expression in the real-space basis, which connects directly to the physical operators of the qubits and the cavities. It can be shown that the exact CD Hamiltonian in the real-space basis contains nonzero off-diagonal matrices, which correspond to nonlocal couplings between qubits and cavities at different and distant sites. Such couplings are hard to implement in practical systems.

Exploring the symmetry of the eigenstates in the JC lattice, we can construct a completely local CD Hamiltonian. Starting from a block-diagonal Hamiltonian of the general form $H_{\rm L}=I_N\otimes H_{\rm L0}$ in the real-space basis with I_N a $N\times N$ identity matrix, $H_{\rm L0}=(\delta,g_{\rm L};g_{\rm L}^\star,-\delta)$ a 2×2 matrix, δ a real parameter, and $g_{\rm L}$ a complex parameter, we derive that $H_{\rm L}$ can be written as a block-diagonal matrix in the k-space basis with the block matrix $G^{(k)}$ for the subspace of the wave vector k. By choosing $\delta=0$ and $g_{\rm L}=g_{\rm CD}^{(\bar{k})}$ ($\mathrm{Re}[g_{\rm L}]=0$), we find that $G^{(k)}=(0,g_{\rm CD}^{(\bar{k})};g_{\rm CD}^{(\bar{k})\star},0)=H_{\rm CD}^{(\bar{k})}$, i.e., the block matrices for all wave vectors are equal to $H_{\rm CD}^{(\bar{k})}$ for the $\bar{k}$-subspace of the exact CD Hamiltonian $H_{\rm CD}$. The Hamiltonian $H_{\rm L}$ will hence generate the same dynamics as the exact CD Hamiltonian when the initial state is $\vec{w}_{\bar{k},-}$. Meanwhile, $H_{\rm L}$ only contains local interaction between qubits and cavities at the same site.

Our numerical simulations of the local CD driving on a four-site JC lattice confirm the above analytical results. Consider an adiabatic evolution where only the photon hopping is ramped linearly from zero to a target value, with $g\equiv1$ and $\Delta\equiv1$ through the evolution and $J(t)=J_0+\frac{t}{T}(J_f-J_0)$, $J_0=0$ and $J_f=2$. The initial state is the ground state of the initial parameters. In Fig. 2(a), we plot the couplings $g(t)$, $J(t)$ and the local CD coupling $\mathrm{Im}[g_{\rm CD}]$. The fidelity is defined as $F(t)=|\vec{w}^\dagger(t)\vec{w}_{\bar{k},-}(t)|^2$ with $\vec{w}(t)$ the system state at time t and $\vec{w}_{\bar{k},-}(t)$ the instantaneous ground state of the Hamiltonian $H_{\rm r}$. In Fig. 2(b), we plot the infidelity $1-F(t)$ vs time t for the process governed by the adiabatic Hamiltonian $H_{\rm r}$ and by the total Hamiltonian $H_{\rm t}=H_{\rm r}+H_{\rm L}$, respectively. It can be seen that the infidelity under the local CD driving is negligibly small during the entire evolution, demonstrating the effectiveness of the local CD driving in cancelling the diabatic transitions. The numerical results of the infidelity under the exact CD Hamiltonian $H_{\rm CD}$ (not shown) are identical to that of $H_{\rm L}$ up to a small numerical error below 10^{-12}. These results confirm that the local CD driving can overcome the requirement of long timescales in an adiabatic process and can generate desired quantum states within short timescales, making the system resilient to decoherence caused by environmental noises.

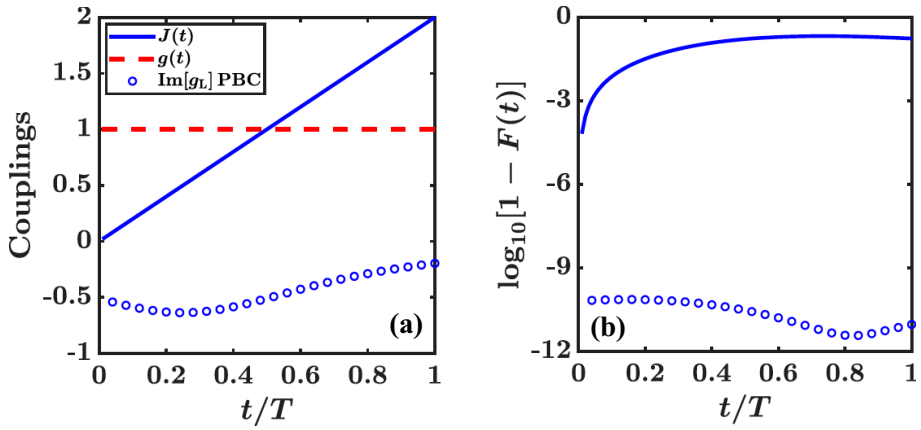

Figure 2: Local counterdiabatic driving [25]. (a) Couplings $J(t)$, $g(t)$, and $\mathrm{Im}[g_{\mathrm{L}}]$ vs t/T for $T = 0.5\pi$. (b) Infidelity $1 - F(t)$ vs t/T for $T = 0.5\pi$, where the solid line is for the system under H_{r} only; and the circles are for the system when H_{L} is applied.

5 Energy cost and energy fluctuation

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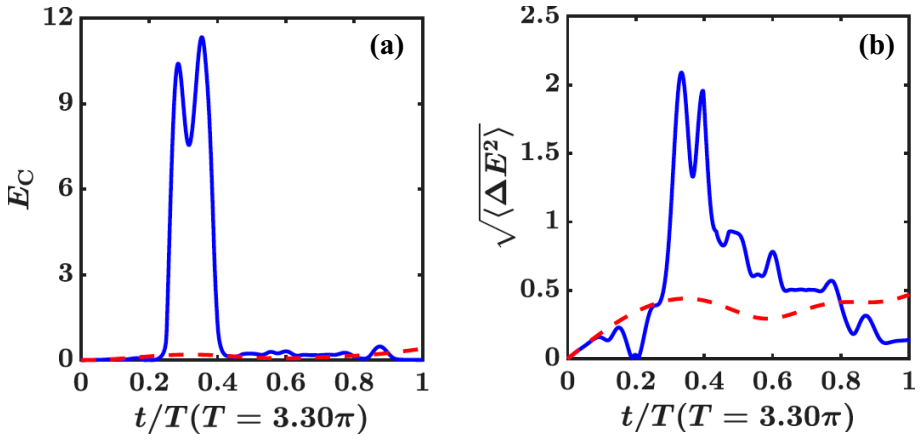

Figure 3: (a) Energy cost E_{C} and (b) Energy fluctuation $\sqrt{\Delta E^2}$ vs t/T. The solid lines are from QOC and the dashed lines are from the adiabatic evolution. Here the parameter constraints $g_{\max} = 2$, $J_{\max} = 2$, and the evolution time $T = 3.30\pi$.

We first consider a JC lattice with unit filling as described in Sec. 3, where the many-body state is prepared with both the adiabatic and the QOC approaches. In Fig. 3(a) and (b), we plot the energy

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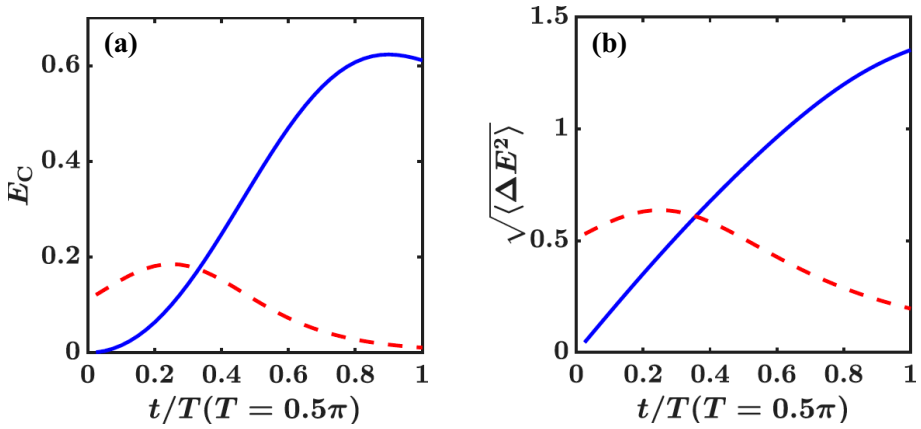

Figure 4: (a) Energy cost E_C and (b) Energy fluctuation $\sqrt{\Delta E^2}$ vs t/T. The solid lines are from the adiabatic evolution and the dashed lines are with the local counterdiabatic driving applied. Here the evolution time $T = 0.5\pi$.

Next we calculate the energy cost and the energy fluctuation for a JC lattice with a single excitation under the adiabatic evolution both without and with the local CD driving applied. As shown in Fig. 4(a) and (b), in the adiabatic evolution, the energy cost and the energy fluctuation both have an increasing trend during the majority part of the evolution. For the adiabatic process under the Hamiltonian H_r, $\Delta E = \sqrt{\Delta E^2} = 0$ at time $t = 0$, because the initial state is the ground state of the adiabatic Hamiltonian. As t increases, these quantities increase because the system has an increasing probability to be in an excited state due to the diabetic transitions. On the contrary, when the local CD Hamiltonian is applied, the energy cost and the energy fluctuation at $t = 0$ are not zero any more, because the initial state is not the ground state of the total Hamiltonian at $t = 0$. For $t > 0$, the energy cost and the energy fluctuation do not increase significantly because the diabatic transitions are eliminated by the CD driving.

6 Conclusions

We utilize the JC lattices as a platform to study various quantum control techniques in preparing quantum many-body states. Our results show that both QOC and CD driving can achieve high fidelity for the prepared quantum states at significantly shortened timescales, which overcomes the requirement of long timescales in adiabatic evolution. Moreover, we developed a local CD driving scheme that only requires local interactions between qubits and cavities, and hence can be implemented in practical quantum devices. These studies shed light on the application of control techniques in preparing quantum many-body states and can lead to future endeavors in studying these approaches in quantum simulation and quantum computation.

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