On Complexity of Finding Strong-Weak Solutions in Bilevel Linear Programming

Tomas Lagos and Oleg A. Prokopyev*

Department of Industrial Engineering, University of Pittsburgh
3700 O'Hara Street, Pittsburgh, PA 15213, USA
E-mails: {tol28,droleg}@pitt.edu

Abstract

We consider bilevel linear programs (BLPs) that model hierarchical decision-making settings with two independent decision-makers (DMs), referred to as a *leader* (an upper-level DM) and a *follower* (a lower-level DM). BLPs are strongly NP-hard. In general, the follower's rational reaction (i.e., a set that contains optimal solutions of the lower-level problem for a given leader's decision) is not a singleton. If we assume that, for a given leader's decision, the follower always selects a solution from the rational reaction set that is most (least) favorable to the leader, then we obtain the optimistic (pessimistic) model. It is known that the optimistic (pessimistic) model remains NP-hard even if an optimal pessimistic (optimistic) solution of the same BLP is known. One interesting generalization of these two approaches studied in the related literature is to consider the so-called α-strong-weak model, where parameter $\alpha \in [0,1]$ controls the leader's level of conservatism. That is, α provides the probability of the follower's "cooperation" in a sense that, the lower-level's rational decisions that are most and least favorable to the leader are picked with probabilities α and $1-\alpha$, respectively. In this note we show that for any fixed $\alpha \in (0,1)$ the problem of finding an optimal α-strong-weak solution remains strongly NP-hard, even when both optimal optimistic and pessimistic solutions for the same BLP are known.

Keywords: bilevel linear programs, computational complexity, strong-weak approach, pessimistic bilevel programs

1 Introduction

Bilevel programming problems (BPPs) model hierarchical decision-making settings with two independent decision-makers (DMs), referred to as a *leader* and a *follower*, also known as an *upper-level* DM and a *lower-level* DM, respectively [7]. The considered DMs make their decisions sequentially, and the leader, whose perspective is modeled and optimized, acts first. Given the leader's decision, the follower solves the lower-level optimization problem that is parameterized by the leader's variables. The leader's objective function and, possibly, constraints at the upper level involve both the leader's and follower's variables; hence, the leader needs to take into account the follower's *rational response* when making its decisions. BPPs provide a flexible modeling framework for decentralized decision-making in multiple application domains including network design [13], pricing [14], energy [15], and defense [5]; see surveys in [2, 12].

Formally, we consider bilevel linear programs (BLPs) of the form:

$$[\text{BLP}]: \quad \text{“}\min_{x}\text{”} \quad c^\top x + d_1^\top y$$

*Corresponding author.

$$
\begin{aligned}
\text{subject to}\quad & A_1 x \leq b_1, \\
& x \geq 0, \\
& y \in \underset{y}{\operatorname{argmax}} \quad d_2^\top y \\
& \quad \text{subject to} \quad A_2 x + By \leq b_2, \\
& \qquad\qquad\qquad\quad y \geq 0,
\end{aligned}
$$

where $A_1 \in \mathbb{R}^{m_1 \times n_1}$, $A_2 \in \mathbb{R}^{m_2 \times n_1}$, $B \in \mathbb{R}^{m_2 \times n_2}$, $c \in \mathbb{R}^{n_1}$, $d_1 \in \mathbb{R}^{n_2}$, $d_2 \in \mathbb{R}^{n_2}$, $b_1 \in \mathbb{R}^{m_1}$ and $b_2 \in \mathbb{R}^{m_2}$. The leader's and the follower's decision variables are denoted by x and y, respectively. Finally, we note that we consider BLPs without coupling constraints, i.e., the follower's variables do not appear in the upper-level constraints.

We use "min" (with quotes) in the leader's objective function of BLP to reflect some uncertainty with respect to how the follower's select the lower-level decisions in BLP. Specifically, define set:

$$\mathcal{R}(x) := \operatorname{argmin}\{d_2^\top y \mid A_2 x + By \leq b, \ y \geq 0\}, \tag{1}$$

which is referred to as the *follower's (lower-level) rational reaction set* for a leader's decision $x \in \mathbb{X}$, where

$$\mathbb{X} := \{x \mid A_1 x \leq b_1, \ x \geq 0\}.$$

There may exist multiple optimal solutions to the follower's problem for a given leader's decision, i.e., $\mathcal{R}(x)$ is not necessarily a singleton. If we assume that the follower selects a solution from the rational reaction set that is the most favorable for the leader, then we obtain the *optimistic* formulation of BLP:

$$[\text{BLP}^{\text{opt}}]: \qquad f^* := \min_{x \in \mathbb{X}, \ y \in \mathcal{R}(x)} \{c^\top x + d_1^\top y\},$$

which is the most typical approach in the related bilevel optimization literature. This approach assumes that the follower is, in a sense, collaborative with the leader. Alternatively, one can say that the upper-level decision-maker is optimistic with respect to the follower's reaction; hence, the moniker optimistic is used.

On the other hand, the *pessimistic* formulation of BLP assumes that the follower selects the least favorable solution for the leader that is:

$$[\text{BLP}^{\text{pes}}]: \qquad f_p^\star := \min_{x \in \mathbb{X}} \quad \{c^\top x + \max_{y \in \mathcal{R}(x)} d_1^\top y\},$$

where the leader acts in a conservative manner by taking into account the worst possible (for the leader) rational reaction by the follower. Note that BLP^{pes} can be viewed as the three-level optimization problem. Naturally, whenever $d_1 = d_2$ we obtain the case of max-min (or min-max) problems (sometime referred to as the "symmetric" case of BLP) for which the outlined optimistic and pessimistic approaches coincide.

Standard (single-level) LPs are known to be solvable in polynomial time [11]. In contrast, BLPs are computationally difficult. Ben-Ayed and Blair [3] show that BLPs are NP-hard, while Hansen et al. [9] strengthen this result by showing that BLPs are strongly NP-hard even for the min-max case. Finally, one should also refer to Jeroslow [10] who establishes a complexity hierarchy for general multilevel LPs.

This study is motivated by a relatively recent study in [16], where it is shown that BLP^{opt} remains NP-hard, even if an optimal solution of BLP^{pes} is known, and vice versa, i.e., BLP^{pes} is NP-hard, even if an optimal solution of BLP^{opt} is known. Simply speaking, if the decision-maker knows an optimal optimistic (or pessimistic) solution to a BLP, then the problem of finding an optimal pessimistic (or optimistic, respectively) solution to the same BLP remains a computationally difficult problem in the worst case.

The outlined two approaches (namely, the optimistic and pessimistic models), can be generalized to consider the so-called *strong-weak* (or, *α-strong-weak*) model (see, e.g., [6]):

$$[\alpha\text{-BLP}]: \qquad f_{\alpha}^{\star} := \min_{x \in \mathbb{X}} \quad \{c^{\top}x + \alpha \cdot \min_{y \in \mathcal{R}(x)} d_1^{\top} y + (1-\alpha) \cdot \max_{y \in \mathcal{R}(x)} d_1^{\top} y\},$$

where by using the parameter $\alpha \in [0, 1]$ the decision-maker controls the leader's level of conservatism. Furthermore, the value of α can be interpreted as the probability of the follower's cooperation by selecting a rational reaction most favorable to the leader.

Naturally, if $\alpha = 1$, then α-BLP reduces to BLP^{opt}, and if $\alpha = 0$, then α-BLP reduces to BLP^{pes}. That is, both BLP^{opt} and BLP^{pes} can be viewed as two extreme special cases of α-BLP. We also refer the reader to a recent survey in [2] that overviews BPPs under uncertainty in a more general context, including the bilevel problems that involve uncertainty with respect to the follower's decision-making approach.

Our *main result* is given as follows:

Theorem 1. *For any fixed $\alpha \in (0, 1)$, α-BLP is strongly NP-hard even if optimal optimistic and optimal pessimistic solutions of the same BLP are known to the decision-maker.*

Loosely speaking, our result can be viewed as a generalization of the observation in [16], where the theoretical computational complexity of BLP^{opt} and BLP^{pes} is considered whenever an optimal solution of one of these two models is known; recall our brief discussion above. Here, we show that for a given fixed value of $\alpha \in (0, 1)$ solving α-BLP, i.e., finding an optimal α-strong-weak solution, remains a difficult problem even when the decision-maker is aware of optimal solutions for both BLP^{opt} and BLP^{pes}.

The remainder of this note is organized as follows. In Section 2 we provide the proof of the main result. Then in Section 3 we provide some additional comments and conclude the note.

Finally, with respect to the notation, in our derivations we use $f(x)$, $f_p(x)$ and $f_{\alpha}(x)$ to denote the leader's objective function values given the leader's decision $x \in \mathbb{X}$ whenever the follower reacts in the optimistic, pessimistic and α-strong-weak manners, respectively. Hence, we have $f^* := \min_{x \in \mathbb{X}} f(x)$, $f_p^{\star} := \min_{x \in \mathbb{X}} f_p(x)$ and $f_{\alpha}^{\star} := \min_{x \in \mathbb{X}} f_{\alpha}(x)$.

2 Main result: proof of Theorem 1

The proof relies on the **3-SATISFIABILITY (3SAT)** problem that is defined as follows: Given a collection $C = \{c_1, c_2 \ldots, c_m\}$ of clauses on a finite set $U = \{u_1, u_2, \ldots, u_n\}$ of variables such that $|c_j| = 3$ for all $j \in \{1, 2, \ldots, m\}$, does there exist a truth assignment for U that satisfies all the clauses in C? The **3SAT** problem is NP-complete [8]. It is well known that **3SAT** can be represented as a linear mixed-integer feasibility problem. Next, for the completeness of our discussion we outline one such approach, which is commonly in the related literature; see, e.g., [10].

Formally, for every clause $c_j \in C$ we define variable y_j, and for every $u_i \in U$ we define variable x_i, where $j \in \{1, 2, \ldots, m\}$ and $i \in \{1, 2, \ldots, n\}$, respectively. For any clause $c_j \in C$ of the form, say, $c_j = u_{i_1} \vee u_{i_2} \vee u_{i_3}$, we introduce the following set of linear constraints:

$$y_j \geq x_{i_1}, \tag{2a}$$
$$y_j \geq x_{i_2}, \tag{2b}$$
$$y_j \geq x_{i_3}, \tag{2c}$$
$$y_j \leq x_{i_1} + x_{i_2} + x_{i_3}, \tag{2d}$$
$$0 \leq y_j \leq 1, \tag{2e}$$

and whenever the negation, say $\bar{u}_i$, appears in a clause, then we replace the respective variable x_i by $1-x_i$ in (2). For example, any clause $c_j \in C$ of the form, say, $c_j = \bar{u}_{i_1} \vee \bar{u}_{i_2} \vee u_{i_3}$, corresponds to the following set of linear constraints:

$$
\begin{aligned}
y_j &\geq 1 - x_{i_1}, && \text{(3a)}\\
y_j &\geq 1 - x_{i_2}, && \text{(3b)}\\
y_j &\geq x_{i_3}, && \text{(3c)}\\
y_j &\leq (1 - x_{i_1}) + (1 - x_{i_2}) + x_{i_3}, && \text{(3d)}\\
0 \leq y_j &\leq 1, && \text{(3e)}
\end{aligned}
$$

and we observe from the outlined construction that if $x_i \in \{0, 1\}$ for all $i \in \{1, 2, \ldots, n\}$, then $y_j \in \{0, 1\}$ for all $j \in \{1, 2, \ldots, m\}$. Clearly, setting variable u_i to either "true" or "false" corresponds to $x_i = 1$ or $x_i = 0$, respectively. Then variable y_j is equal to 1 if and only if clause c_j is satisfied.

Denote by $Y(x)$ the feasible set of y defined by the constructed set of linear constraints of the form as outlined in two examples given by (2) and (3). The answer to **3SAT** is "yes" if and only if there exists $x \in \{0, 1\}^n$ such that the corresponding set $Y(x) = \{(1, 1, \ldots, 1)^\top\}$, i.e., all variables y_j, $j \in \{1, 2, \ldots, m\}$, are forced to be 1 by the respective linear constraints. Moreover, we observe that for any 0–1 vector x, set $Y(x)$ is a *singleton*, which is also a 0–1 vector. Hence, for simplicity, below we use $y := Y(x)$, to denote the corresponding 0–1 vector y, whenever x is also binary.

Next, consider some fixed $\alpha \in (0, 1)$. Then we construct the following BLP instance, where variables (x, y, z) are controlled by the leader and variables (v, u, w) are controlled by the follower:

$$
\begin{aligned}
\text{“}\min_{x,y,z}\text{”} \quad & \sum_{j=1}^{m} z_j + m z_{m+1} + (m + M) z_{m+2} + \frac{1 - \frac{1}{2}\alpha}{(1-\alpha)n} M \sum_{i=1}^{n} (v_i + u_i) + M w_1 && \text{(4a)}\\
\text{s.t.} \quad & \sum_{j=1}^{m} z_j + m z_{n+1} + m z_{m+2} \geq m, && \text{(4b)}\\
& 0 \leq z_j \leq y_j, \quad \forall j \in \{1, 2 \ldots, m\}, && \text{(4c)}\\
& z_{m+1} + z_{m+2} \leq 1,\ z_{m+1} \geq 0,\ z_{m+2} \geq 0, && \text{(4d)}\\
& 0 \leq x_i \leq 1, \quad \forall i \in \{1, 2, \ldots, n\}, && \text{(4e)}\\
& y \in Y(x), && \text{(4f)}\\
& (u, v, w) \in \underset{v,u,w}{\operatorname{argmax}} \quad \sum_{i=1}^{n} (u_i - v_i) + (w_1 - w_2) && \text{(4g)}\\
& \qquad \text{s.t.} \quad u_i - v_i \leq 1 - x_i, \qquad \forall i \in \{1, 2, \ldots, n\}, && \text{(4h)}\\
& \qquad\qquad u_i - v_i \leq x_i, \qquad \forall i \in \{1, 2, \ldots, n\}, && \text{(4i)}\\
& \qquad\qquad v_i \leq 1 - \frac{1}{2} z_{m+2}, \qquad \forall i \in \{1, 2, \ldots, n\}, && \text{(4j)}\\
& \qquad\qquad w_1 \leq z_{m+1}, && \text{(4k)}\\
& \qquad\qquad w_1 - w_2 \leq 1 - z_{m+1}, && \text{(4l)}\\
& \qquad\qquad w_1 - w_2 \leq z_{m+1}, && \text{(4m)}\\
& \qquad\qquad u_i \geq 0,\ v_i \geq 0, \quad \forall i \in \{1, 2, \ldots, n\}, && \text{(4n)}\\
& \qquad\qquad w_1 \geq 0,\ w_2 \geq 0, && \text{(4o)}
\end{aligned}
$$

where M is a sufficiently large strictly positive constant parameter. For our discussion below, we can simply set $M = 1$. However, having general parameter M allows us to provide some additional interesting insights; see Section 3.

Our proof of Theorem 1 is based on the combination of the ideas from [1,16] and relies on the following three lemmas about the structure of the BLP model given by (4). More specifically, we prove Theorem 1 in three steps. First, in Lemma 1, we show the optimal solution for the optimistic version of (4) is readily available and computable in polynomial time. The second step, provided by Lemma 2, is similar and provides the same result for the pessimistic version of (4). Finally, in Lemma 3, we show that the answer to the corresponding instance of the 3SAT problem is "yes" if and only if the optimal objective function value for the strong-weak version of (4) is equal to some particular pre-defined value.

Lemma 1. *Let $\alpha \in (0,1)$. Setting $x^* = (0,0,\ldots,0)^\top$, $y^* = Y(x^*)$, $z^* = (0,0,\ldots,0,1,0)^\top$, $u^* = (0,0,\ldots,0)^\top$, $v^* = (0,0,\ldots,0)^\top$, and $w_1^* = w_2^* = 0$ forms an optimal optimistic solution for model (4), and $f^* = m$.*

Proof. First, consider the leader's decision (x^*, y^*, z^*) given by $x_1^* = \ldots = x_n^* = 0$, $y^* = Y(x^*)$, $z_1^* = \ldots = z_m^* = 0$, $z_{m+1}^* = 1$ and $z_{m+2}^* = 0$. We can verify that the upper-level constraints (4b)-(4f) hold for these settings of the leader's variables. Moreover, due to (4i) and (4l), we observe that the corresponding follower's optimal objective function value in (4g) is equal to zero.

Next, given the leader's decision (x^*, y^*, z^*) consider the corresponding follower's solution (v^*, u^*, w^*) defined as $u_1^* = \ldots = u_n^* = 0$, $v_1^* = \ldots = v_n^* = 0$ and $w_1^* = w_2^* = 0$. Clearly, in view of the above observations, these variables' settings are optimal for the follower. At the same time, (v^*, u^*, w^*) provides the follower's response that is most favorable to the leader given the last two terms in the leader's objective function in (4a).

Finally, note that $f^* \geq m$ due to (4b). Hence, we conclude that the constructed leader's, (x^*, y^*, z^*), and the follower's, (v^*, u^*, w^*), decisions form an optimal optimistic solution for model (4). ■

Lemma 2. *Let $\alpha \in (0,1)$. Setting $x^\star = (0,0,\ldots,0)^\top$, $y^\star = Y(x^\star)$, $z^\star = (0,0,\ldots,0,0,1)^\top$, $u^\star = (1/2,\ldots,1/2)^\top$, $v^\star = (1/2,\ldots,1/2)^\top$, and $w_1^\star = w_2^\star = 0$ forms an optimal pessimistic solution for model (4), and $f_p^\star = m + (1 + \frac{1-\frac{1}{2}\alpha}{1-\alpha})M$.*

Proof. Recall that we use $f_p(\cdot)$ to denote the leader's objective function value given that the follower's is pessimistic. It is sufficient to consider two cases for variable z_{m+2}, namely, either (i) $z_{m+2} = 1$ or (ii) $z_{m+2} = \beta$, where $0 \leq \beta < 1$.

Case (i): Let $(\bar{x}, \bar{y}, \bar{z})$ be a feasible solution of the leader with $\bar{z}_{m+2} = 1$. Then, we observe that (4b) holds regardless of the other variables' settings. Therefore, in view of the first two terms in the upper-level objective function (4a), it is most favorable for the leader to set $\bar{z}_1 = \ldots = \bar{z}_{m+1} = 0$; then, due to (4k)-(4m) and (4o) we have $\bar{w}_1 = \bar{w}_2 = 0$.

Assume that the leader implements a solution with all variables x to be binary, i.e., $\bar{x} \in \{0,1\}^n$. Then, let $\bar{y} = Y(\bar{x})$, where both (4c) and (4d) hold. Given constraints (4h) and (4i) as well as the aforementioned values of $\bar{w}_1$ and $\bar{w}_2$, the follower's optimal objective function is equal to zero. Therefore, due to (4j) and the fourth term in the upper-level objective function (4a) that involves variables u and v, the pessimistic follower sets $\bar{v} = (1/2,\ldots,1/2)^\top$ and $\bar{u} = (1/2,\ldots,1/2)^\top$. We can verify that for these settings of the follower's variables, constraints (4h)-(4j) hold.

Thus, we have

$$f_p(\bar{x}, \bar{y}, \bar{z}) = m + \left(1 + \frac{1 - \frac{1}{2}\alpha}{1 - \alpha}\right) M. \tag{5}$$

Next, assume that the leader implements a solution $(\widetilde{x}, \widetilde{y}, \widetilde{z})$, where $\widetilde{x}$ is not a 0–1 vector. That is, there exists $i \in \{1, 2, \ldots, n\}$ such that $\widetilde{x}_i = \gamma \in (0, 1)$. Then in contrast to the previous discussion, the follower's optimal objective function is not necessarily zero; see (4h) and (4i) as well as the corresponding terms in the follower's objective function in (4g). Moreover, in the pessimistic case the follower sets $\widetilde{v}_i = \frac{1}{2}$ and $\widetilde{u}_i = \frac{1}{2} + \min\{\gamma, 1-\gamma\}$, while having (4h)-(4j) satisfied. Therefore, $\widetilde{u}_i + \widetilde{v}_i > 1$, and

$$f_p(\widetilde{x}, \widetilde{y}, \widetilde{z}) > m + \left(1 + \frac{1 - \frac{1}{2}\alpha}{1-\alpha}\right) M,$$

which, in view of (5), implies that the case of not-binary setting for x can be disregarded.

Case (ii): Let $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ be a feasible solution of the leader with $\widetilde{z}_{m+2} = \beta$, where $0 \leq \beta < 1$. Consider any $i \in \{1, 2, \ldots, n\}$. Similar to the above discussion of case (i), we consider the binary and non-binary settings for the leader's variable x_i as well as the contributions of the corresponding follower's variables u_i and v_i to the leader's objective function; see the fourth term in (4a). Note that the arguments below mirror those used in the above discussion of case (i) and, hence, most of them are omitted for brevity.

Specifically, if $\widetilde{x}_i \in \{0, 1\}$, then in the pessimistic case the follower sets $\widetilde{u}_i = \widetilde{v}_i = 1 - \frac{1}{2}\beta$, while having (4h)-(4j) satisfied. If $\widetilde{x}_i = \gamma \in (0, 1)$, then in the pessimistic case the follower sets $\widetilde{v}_i = 1 - \frac{1}{2}\beta$ and $\widetilde{u}_i = 1 - \frac{1}{2}\beta + \min\{\gamma, 1-\gamma\}$, while having (4h)-(4j) satisfied. For both of these cases we have that $\widetilde{u}_i + \widetilde{v}_i \geq 2(1 - \frac{1}{2}\beta)$ for all $i \in \{1, 2, \ldots, n\}$.

Therefore, we obtain that:

$$\begin{aligned} f_p(\widetilde{x}, \widetilde{y}, \widetilde{z}) &\geq m + M\beta + \frac{1 - \frac{1}{2}\alpha}{(1-\alpha)n} M \cdot 2(1 - \frac{1}{2}\beta)n = m + \frac{\beta - \beta\alpha + 2 - \beta - \alpha + \frac{1}{2}\alpha\beta}{1-\alpha} \cdot M \\ &= m + \frac{1 - \alpha + 1 - \frac{1}{2}\alpha\beta}{1-\alpha} \cdot M = m + \left(1 + \frac{1 - \frac{1}{2}\alpha\beta}{1-\alpha}\right) \cdot M > f_p(\bar{x}, \bar{y}, \bar{z},), \end{aligned}$$

where the last inequality holds because $\beta \in [0, 1)$, and the required result is established. ■

Lemma 3. *Consider a fixed $\alpha \in (0, 1)$. Then for model (4) the optimal objective function value for its α-strong-weak version is given by*

$$f^\star_\alpha = m + (2 - \alpha)M \tag{6}$$

if and only if the answer to the corresponding instance of the ***3SAT*** *problem is "yes."*

Proof. Denote the part of the leader's objective function that contains the follower's variables by:

$$g(u, v, w) = \frac{1 - \frac{1}{2}\alpha}{(1-\alpha)n} M \sum_{i=1}^{n} (v_i + u_i) + M w_1,$$

and, as in (1), we denote the follower's reaction set in (4) by $\mathcal{R}(x, y, z)$. Hence, we need to consider the following model:

$$f^*_\alpha = \min_{x,y,z} \quad \left\{ \sum_{j=1}^{m} z_j + m z_{m+1} + (m + M) z_{m+2} \right. \tag{7a}$$

$$\left. + \alpha \min_{(u,v,w) \in \mathcal{R}(x,y,z)} g(u, v, w) + (1 - \alpha) \max_{(u,v,w) \in \mathcal{R}(x,y,z)} g(u, v, w) \right\} \tag{7b}$$

$$
\begin{aligned}
\text{s.t.} \quad & \sum_{j=1}^{m} z_j + m z_{m+1} + m z_{m+2} \geq m, && \text{(7c)}\\
& 0 \leq z_j \leq y_j, \quad \forall j \in \{1, 2, \ldots, m\}, && \text{(7d)}\\
& z_{m+1} + z_{m+2} \leq 1, \ z_{m+1} \geq 0, \ z_{m+2} \geq 0, && \text{(7e)}\\
& 0 \leq x_i \leq 1, \quad \forall i \in \{1, 2, \ldots, n\}, && \text{(7f)}\\
& y \in Y(x), && \text{(7g)}
\end{aligned}
$$

which is the strong-weak version of (4) for any $\alpha \in (0, 1)$.

$\Longleftarrow$ Suppose the answer to the **3SAT** problem is "yes." In our discussion below, we first construct a feasible leader's solution $(\bar{x}, \bar{y}, \bar{z})$ that satisfies the upper-level constraints and show that its α-strong-weak objective function value is given by (6). Then we show that this solution is also an optimal leader's decision for the BLP model given by (7).

Specifically, consider the leader's solution, where $\bar{x}_i = 1$ or $\bar{x}_i = 0$ whenever the respective variable u_i is set to "true" or false, respectively. As all clauses in C are satisfied, we can set $\bar{y}_j = 1$ for all $j \in \{1, 2, \ldots, m\}$. Finally, we set $\bar{z}_{m+1} = \bar{z}_{m+2} = 0$ and $\bar{z}_j = 1$ for all $j \in \{1, 2, \ldots, m\}$, and we observe that the leader's constraints (7c)-(7g) are satisfied.

Next, given the outlined leader's decision $(\bar{x}, \bar{y}, \bar{z})$, we need to compute the resulting optimistic and pessimistic responses by the follower. Given the follower's constraints in (4h)-(4o), for both responses we observe that the follower's optimal objective function (4g) is equal to zero. More specifically, similar to the proofs in Lemmas 1 and 2, we observe that $\hat{u}_1 = \ldots = \hat{u}_n = \hat{v}_1 = \ldots = \hat{v}_n = \hat{w}_1 = \hat{w}_2 = 0$ is the respective optimistic solution of the follower with $g(\hat{u}, \hat{v}, \hat{w}) = 0$, as it is the most favorable outcome for the leader. Similarly, we observe that $\breve{u}_1 = \ldots = \breve{u}_n = \breve{v}_1 = \ldots = \breve{v}_n = 1$ and $\breve{w}_1 = \breve{w}_2 = 0$ is the respective pessimistic solution of the follower with $g(\breve{u}, \breve{v}, \breve{w}) = \frac{2-\alpha}{1-\alpha} M$, as it is the least favorable outcome for the leader. Therefore, we have that

$$f_\alpha(\bar{x}, \bar{y}, \bar{z}) = m + (2 - \alpha) M,$$

where one should recall that we use $f_\alpha(\cdot)$ to denote the leader's objective function given that the follower "cooperates" with probability α, i.e., the most and least favorable decisions to the leader are picked by the follower (from the reaction set) with probabilities α and $1 - \alpha$, respectively.

Next, we need to establish that the constructed leader's feasible decision $(\bar{x}, \bar{y}, \bar{z})$ forms an optimal solution for (7). Specifically, it is sufficient to consider the following *four* cases for other possible feasible leader's decisions $(\widetilde{x}, \widetilde{y}, \widetilde{z})$: *(i)* $\widetilde{z}_{m+2} = \beta \in (0, 1]$, *(ii)* $\widetilde{z}_{m+1} = \beta \in (0, 1]$ and $\widetilde{z}_{m+2} = 0$, *(iii)* $\widetilde{z}_{m+1} = \widetilde{z}_{m+2} = 0$, but there exists $k \in \{1, 2, \ldots, n\}$ such that $\widetilde{x}_k = \beta \in (0, 1)$, and *(iv)* $\widetilde{z}_{m+1} = \widetilde{z}_{m+2} = 0$ and $\widetilde{x} \in \{0, 1\}^n$, where $\widetilde{x}$ does not correspond to a truth assignment of U. Note that these four cases are mutually exclusive and cover all possible situations.

Case *(i)*: Consider a leader's feasible solution $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ such that $\widetilde{z}_{m+2} = \beta \in (0, 1]$. Then in view of constraint (7c), the value of the terms of the leader's objective function in (7a) is at least $m + M\beta$. Furthermore, recall from the case *(ii)* in the proof of Lemma 2 that regardless whether x is a binary vector, if $\widetilde{z}_{m+2} = \beta \in (0, 1]$, then for the pessimistic response we have that $\widetilde{u}_i + \widetilde{v}_i \geq 2(1 - \frac{1}{2}\beta)$ for all $i \in \{1, 2, \ldots, n\}$. This fact can be used to estimate the leader's objective function terms appearing in (7b). Combining the above two observations, we obtain that:

$$
\begin{aligned}
f_\alpha(\widetilde{x}, \widetilde{y}, \widetilde{z}) &\geq m + M\beta + (1 - \alpha) \frac{1 - \frac{1}{2}\alpha}{1 - \alpha} \cdot 2(1 - \frac{1}{2}\beta) M = m + M\beta + M(2 - \alpha)(1 - \frac{1}{2}\beta)\\
&= m + \left(\beta + 2 - \beta - \alpha + \frac{1}{2}\alpha\beta\right) \cdot M = m + \left(2 - \alpha + \frac{1}{2}\alpha\beta\right) \cdot M > f_\alpha(\bar{x}, \bar{y}, \bar{z}),
\end{aligned}
$$

where the last inequality holds as $\alpha \in (0,1)$ and $\beta \in (0,1]$.

Case (ii): Consider a leader's feasible solution $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ such that $\widetilde{z}_{m+1} = \beta \in (0,1]$ and $\widetilde{z}_{m+2} = 0$. Then the first condition and (4k)-(4m) imply that the pessimistic follower should set $\widetilde{w}_1 = \beta$ and $\widetilde{w}_2 = \beta - \min\{\beta, 1-\beta\}$; the second condition imply that the pessimistic follower should set $\widetilde{u}_i = \widetilde{v}_i = 1$ for all $i \in \{1, 2, \ldots, n\}$, where one should recall the discussion from the case (ii) in the proof of Lemma 2. Finally, (7c) implies that the value of the first three terms of the leader's objective function in (7a) is at least m. Combining the above three observations, we obtain that:

$$f_\alpha(\widetilde{x}, \widetilde{y}, \widetilde{z}) \geq m + (1-\alpha)\left(\frac{1-\frac{1}{2}\alpha}{1-\alpha} \cdot 2M + M\beta\right) = m + (2 - \alpha + (1-\alpha)\beta) \cdot M > f_\alpha(\bar{x}, \bar{y}, \bar{z}),$$

where the last inequality holds as $\alpha \in (0,1)$ and $\beta \in (0,1]$.

Case (iii): Consider a leader's feasible solution $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ such that $\widetilde{z}_{m+1} = \widetilde{z}_{m+2} = 0$, but there exists $k \in \{1, \ldots, n\}$ with $\widetilde{x}_k = \beta \in (0,1)$. Let $\gamma = \min\{\beta, 1-\beta\} \in (0,1)$. Then in view of (4h)-(4j) the optimistic follower should set $\widetilde{u}_k = \gamma$ and $\widetilde{v}_k = 0$, which can be used to provide a lower bound for the first term in (7b). Also, given $\widetilde{z}_{m+2} = 0$, the pessimistic follower should set $\widetilde{u}_i = \widetilde{v}_i = 1$ for all $i \in \{1, 2, \ldots, n\}$ as in the case (ii) above, which can be used to provide a lower bound for the second term in (7b). Then combining these observations and (7c), we have that:

$$\begin{aligned} f_\alpha(\widetilde{x}, \widetilde{y}, \widetilde{z}) \geq & m + \alpha\left(\frac{1-\frac{1}{2}\alpha}{1-\alpha} \cdot \frac{\gamma}{n} M\right) + (1-\alpha)\left(\frac{1-\frac{1}{2}\alpha}{1-\alpha} \cdot 2M\right) \\ = & m + \alpha\left(\frac{1-\frac{1}{2}\alpha}{1-\alpha} \cdot \frac{\gamma}{n} M\right) + (2-\alpha)\, M > f_\alpha(\bar{x}, \bar{y}, \bar{z}), \end{aligned}$$

where the last inequality holds as $\alpha \in (0,1)$ and $\gamma \in (0,1)$.

Case (iv): Consider a leader's feasible solution $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ such that $\widetilde{z}_{m+1} = \widetilde{z}_{m+2} = 0$ and $\widetilde{x} \in \{0,1\}^n$, where $\widetilde{x}$ does not correspond to a truth assignment of U. Recall that for binary $\widetilde{x}$ set $Y(\widetilde{x})$ is a singleton; let $\widetilde{y}$ denote the corresponding 0–1 vector. As $\widetilde{x}$ does not correspond to a truth assignment, then there exists k such that $\widetilde{y}_k = 0$. Then (7d) forces $\widetilde{z}_k = 0$, which implies that (7c) cannot hold, i.e., $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ is not feasible, which is a contradiction.

$\Longrightarrow$ Suppose that $f^\star_\alpha = m + (2-\alpha)M$. Denote the corresponding leader's optimal α-strong-weak solution by (x^*, y^*, z^*). We need to demonstrate that the answer to the **3SAT** problem is "yes." From the above discussion, namely, cases (i)-(ii) we observe that whenever $z^*_{m+1} + z^*_{m+2} > 0$, then the corresponding leader's solution cannot be optimal for the strong-weak case as the respective objective function value is strictly greater than $m + (2-\alpha)M$. Similarly, from the discussion in (iii) we conclude that whenever $z^*_{m+1} = z^*_{m+2} = 0$ but x^* is not a 0–1 vector, then the respective objective function value is also strictly greater than $m + (2-\alpha)M$. Hence, we need to focus on the case of $z^*_{m+1} = z^*_{m+2} = 0$ and $x^* \in \{0,1\}^n$.

From (7c), we observe that $z^*_1 = z^*_2 = \ldots = z^*_m = 1$. Then constraints (7d) enforce $y^*_1 = y^*_2 = \ldots = y^*_m = 1$, which implies, by our construction of $Y(x^*)$, that the corresponding values of x^* result in a truth assignment of U for C and the answer to the **3SAT** problem is "yes." ■

In view of our Lemmas 1-3, we can provide the main result of this note:

Proof of Theorem 1. Given fixed $\alpha \in (0,1)$ we construct a BLP of the form (4). By Lemmas 1 and 2 we observe that its optimal optimistic and pessimistic solutions are known. Then applying Lemma 3 we obtain the necessary result by observing that the BLP of the form as in (4), is of polynomial size with respect to the size of the corresponding **3SAT** instance. ■

3 Concluding remarks

Let $s = (x, y, z)$ be the leader's feasible decisions in BLP given by (4). Denote by s^*_{OPT} and $s^\star_{PES}$ the optimal leader's decisions in the optimistic and pessimistic cases, respectively, that are provided by Lemmas 1 and 2, respectively. Also, denote by $s^\star_\alpha$ the optimal leader's decision for the α-strong-weak model when assuming that the answer to the corresponding instance of the **3SAT** problem is "yes"; see Lemma 3.

Then it is relatively easy to verify that

$$f^\star_p(s^*_{OPT}) = f^\star_p + \frac{1-\frac{1}{2}\alpha}{1-\alpha}M \qquad \text{and} \qquad f^\star_\alpha(s^*_{OPT}) = f^\star_\alpha + (1-\alpha)M,$$

while

$$f^*(s^\star_{PES}) = f^* + M \qquad \text{and} \qquad f^\star_\alpha(s^\star_{PES}) = f^\star_\alpha + \frac{\alpha}{2}M,$$

which implies that whenever the upper-level decision-maker (i.e., the leader) makes an incorrect assumption on the mode of the follower's response, then the difference of the resulting leader's objective function value from the optimal one, can be arbitrarily large in the worst-case scenario; recall that in our derivations in Section 2 we simply assume $M \geq 1$. Therefore, one may conclude that the obtained theoretical computational complexity results are relatively intuitive. Moreover, from a somewhat more practical (i.e., real-life applications' driven) perspective, we can argue that under the general "asymmetric" setting the decision-makers should be careful when introducing assumptions on the follower's response in their mathematical optimization models (see, e.g., [17] for an additional discussion on this issue). These observations echo the findings of [16], albeit in a more general setting that captures the strong-weak modeling approach.

Finally, for future research it is of interest to consider similar complexity issues whenever the global optimality of the lower-level solution is not required. We refer the reader to a related study in [4] and the aforementioned survey in [2].

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