



Infinite Energy Harmonic Maps from Riemann Surfaces to $\text{Cat}(0)$ Spaces

Georgios Daskalopoulos¹ · Chikako Mese² 

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Abstract

We present results about harmonic maps with possibly infinite energy from punctured Riemann surfaces to $\text{CAT}(0)$ spaces. In particular, we give precise estimates of their energy growth near the punctures and prove uniqueness.

Keywords Harmonic maps · Uniqueness · $\text{CAT}(0)$ · Riemann surface · Infinite energy

Mathematics Subject Classification 53C43

1 Introduction

This is the first in a series of papers where we develop the techniques to study non-abelian Hodge theory on quasi-projective varieties. The sequel is [5], [6], and [2]. More specifically, in this first paper, we study harmonic maps with possibly infinite energy from punctured Riemann surfaces into complete $\text{CAT}(0)$ spaces, and we give a precise estimate of their energy growth near the punctures. We also prove a uniqueness result. Both results are crucial for the next papers.

Throughout this paper, we use the following notation unless explicitly stated otherwise:

- $\tilde{X}$ is a complete $\text{CAT}(0)$ -space
- $\tilde{\mathcal{R}}$ is a compact Riemann surface

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✉ Chikako Mese
cmese@math.jhu.edu

Georgios Daskalopoulos
daskal@math.brown.edu

¹ Department of Mathematics, Brown University, Providence, RI, USA

² Department of Mathematics, Johns Hopkins University, Baltimore, MD, USA

- $\mathcal{R} = \tilde{\mathcal{R}} \setminus \{p^1, \dots, p^n\}$ is a punctured Riemann surface
- $\Pi : \tilde{\mathcal{R}} \rightarrow \mathcal{R}$ is the universal covering map
- $\rho : \pi_1(\mathcal{R}) \rightarrow \text{Isom}(\tilde{X})$ is a homomorphism.

For each puncture p^j , $j = 1, \dots, n$,

- $\mathbb{D}^j$ is a conformal disk in $\mathcal{R}$ centered at the puncture p^j ($\mathbb{D}^j \cap \mathbb{D}^i = \emptyset$ for $i \neq j$)
- $\mathbb{D}_r^j = \{z \in \mathbb{D} : |z| < r\}$
- $\lambda^j \in \pi_1(\mathcal{R})$ is the conjugacy class associated to the loop in $\mathcal{R}$ around p^j ,
- $I^j := \rho(\lambda^j) \in \text{Isom}(\tilde{X})$, and
- $\Delta_{I^j} := \inf_{P \in \tilde{X}} d(I^j(P), P)$ is the translation length of I^j .

Recall that I^j is said to be *semisimple* if there exists $P_* \in \tilde{X}$ such that $d(I^j(P_*), P_*) = \Delta_{I^j}$.

We prove the following two theorems.

Theorem 1.1 (Existence) *Assume*

- (A) *The action of $\rho(\pi_1(\mathcal{R}))$ does not fix a point at infinity.*
 (B) *The isometry I^j is semisimple, or there exists a geodesic ray $c : [0, \infty) \rightarrow \tilde{X}$ and constants $a, b > 0$ such that*

$$d(I^j(c(t)), c(t)) \leq \Delta_{I^j} + be^{-at}.$$

Then, there exists a ρ -equivariant harmonic map $u : \tilde{\mathcal{R}} \rightarrow \tilde{X}$ with logarithmic energy growth towards the punctures, i.e.,

$$\sum_{j=1}^n \frac{\Delta_{I^j}^2}{2\pi} \log \frac{1}{r} \leq E^u[\mathcal{R}_r] \leq \sum_{j=1}^n \frac{\Delta_{I^j}^2}{2\pi} \log \frac{1}{r} + C \quad (1.1)$$

where $\mathcal{R}_r = \mathcal{R} \setminus \bigcup_{j=1}^n \mathbb{D}_r^j$ and $E^u[\mathcal{R}_r]$ is the energy of u in $\mathcal{R}_r$. The constant C is dependent only on ρ if I is semisimple and also on a, b from assumption (B) if I is not semisimple.

For smooth targets, the existence essentially follows from [13], [17], and [12]. The new statement in Theorem 1.1 is the precise growth estimate towards the punctures of the harmonic map.

We also prove the following uniqueness result.

Theorem 1.2 (Uniqueness) *Assume (A) and (B) of Theorem 1.1. If the ρ -equivariant harmonic map $\tilde{u} : \tilde{\mathcal{R}} \rightarrow \tilde{X}$ has logarithmic energy growth (cf. (1.1)), then it is the unique harmonic map with logarithmic energy growth in the following cases:*

- $\tilde{X}$ is a negatively curved space (i.e., for any $P \in \tilde{X}$, there exists a neighborhood U of P which is $\text{CAT}(-\kappa)$ for some $\kappa > 0$),*
- $\tilde{X}$ is an irreducible symmetric space of non-compact type, or*
- $\tilde{X}$ is an irreducible locally finite Euclidean building such that the action of $\rho(\pi_1(\mathcal{R}))$ does not fix an unbounded closed convex strict subset of $\tilde{X}$.*

In the case when the harmonic map has finite energy, uniqueness follows from previous works of the authors and Corlette ([3, 4, 14] for cases (i), (ii), (iii), respectively). In [2], we use the results of this paper to develop non-Abelian Hodge theory techniques extending the work by Gromov-Schoen [7] from the finite to the infinite energy case and Mochizuki [15] from the Archimedean to the non-Archimedean case.

Remark 1.3 Assumption (B) is a natural condition for many interesting examples. In particular, it is satisfied for symmetric spaces of non-compact type, Euclidean buildings, and Teichmüller space endowed with the Weil-Petersson metric.

Summary of the Paper For the purpose of explaining the main ideas, we consider the simple case when $\tilde{X}$ is a universal cover of a compact manifold X and $\gamma : \mathbb{S}^1 \rightarrow X$ is an arc length parameterization of a minimizing closed geodesic in X . Denote the energy of γ by E^γ . Note that $E^\gamma = \frac{L_\gamma^2}{2\pi}$ where L_γ is the length of γ . A prototype map v corresponding to γ is

$$v : [0, \infty) \times \mathbb{S}^1 \rightarrow X$$

which is equal to γ on slices $\{t\} \times \mathbb{S}^1$ for all sufficiently large t . (We note that the idea of the prototype map first appears in [13].) The energy of v on the finite cylinder $[0, T] \times \mathbb{S}^1$ is bounded from above by $T \cdot E^\gamma + C$ where C is a constant independent of T . Let

$$u_T : [0, T] \times \mathbb{S}^1 \rightarrow X$$

be the Dirichlet solution on the finite cylinder $[0, T] \times \mathbb{S}^1$ with boundary values equal to v . Since u_T is an energy minimizing map, the energy of u_T is bounded from above by $T \cdot E^\gamma + C$. Since γ is also an energy minimizing map, the energy of u_T in any finite cylinder $[0, t] \times \mathbb{S}^1$ for $0 < t \leq T$ is bounded from below by $t \cdot E^\gamma$. From this, we conclude that the energy of u_T on any finite cylinder $[t_1, t_2] \times \mathbb{S}^1$ has an upper bound independent of T ; namely, $(t_2 - t_1) \cdot E^\gamma + C$. Thus, the regularity theory for harmonic maps implies that the family of maps $\{u_T\}_{T \geq 1}$ has a uniform Lipschitz bound on any compact subset of $[0, \infty) \times \mathbb{S}^1$. This implies that there exists a sequence $T_i \rightarrow \infty$ such that u_{T_i} converges locally uniformly to a harmonic map $u : [0, \infty) \times \mathbb{S}^1 \rightarrow X$. The conformal equivalence of the punctured disk $\mathbb{D}^*$ and the infinite cylinder $[0, \infty) \times \mathbb{S}^1$, thus, defines a harmonic map on $\mathbb{D}^*$ with lower and upper energy bound

$$E^\gamma \log \frac{1}{r} \leq E^u[\mathbb{D}_r] \leq E^\gamma \log \frac{1}{r} + C. \quad (1.2)$$

We extend this idea to prove the existence of equivariant harmonic maps from any punctured Riemann surface Σ .

For finite energy harmonic maps, the uniqueness can be derived from the fact that the energy functional is a convex function along a geodesic interpolation. Namely, if u_t is the geodesic interpolation map (i.e., $t \mapsto u_t(x)$ is a geodesic), then $E^{u_t} \leq (1 - t)E^{u_0} + tE^{u_1}$. If u_0 and u_1 are energy minimizing maps, then the convexity

implies that $t \mapsto E^{u_t}$ is a constant function. This idea does not directly generalize to our situation because we deal with infinite energy maps. The main idea to prove the uniqueness assertion in our setting is to introduce a modified energy functional by subtracting off the logarithmic energy growth near the punctures. We then prove that the modified energy is constant along the geodesic interpolation of harmonic maps. This allows us to apply the argument used by the authors in [4, 14].

2 Preliminaries

2.1 Maps into CAT(0) Spaces

Definition 2.1 A space $\tilde{X}$ is a *complete CAT(0) space* if it is a complete geodesic space that satisfies the following condition: For any three points $P, R, Q \in \tilde{X}$ and an arclength parameterized geodesic $c : [0, l] \rightarrow \tilde{X}$ with $c(0) = Q$ and $c(l) = R$,

$$d^2(P, Q_t) \leq (1-t)d^2(P, Q) + td^2(P, R) - t(1-t)d^2(Q, R)$$

where $Q_t = c(tl)$. (We refer to [1] for more details.)

Notation 2.2 It follows immediately from Definition 2.1 that, given $P, Q \in \tilde{X}$ and $t \in [0, 1]$, there exists a *unique* point with distance from P equal to $td(P, Q)$ and the distance from Q equal to $(1-t)d(P, Q)$. We denote this point by

$$(1-t)P + tQ.$$

Fix a smooth conformal Riemannian metric g on $\mathcal{R}$ (i.e., g is given by $\rho(z) dz d\bar{z}$ for some smooth function ρ in any holomorphic coordinate z of $\mathcal{R}$). We refer to [10] for the notion of energy E^f and energy density function $|\nabla f|^2$ of a map $f : \mathcal{R} \rightarrow \tilde{X}$. The (Korevaar-Schoen) energy of f in a domain $\Omega \subset \mathcal{R}$ is

$$E^f[\Omega] = \int_{\Omega} |\nabla f|^2 d\text{vol}_g.$$

It is well known that $E^f[\Omega]$ and $|\nabla f|^2 d\text{vol}_g$ are invariant of the choice of the smooth conformal Riemannian metric g .

We say a continuous map $u : \mathcal{R} \rightarrow \tilde{X}$ is *harmonic* if it is locally energy minimizing; more precisely, at each $p \in \mathcal{R}$, there exists a neighborhood Ω of p so that all continuous comparison maps which agree with u outside of this neighborhood have no less energy.

For $V \in \Gamma\Omega$ where $\Gamma\Omega$ is the set of Lipschitz vector fields on Ω , $|f_*(V)|^2$ is similarly defined. The real-valued L^1 function $|f_*(V)|^2$ generalizes the norm squared on the directional derivative of f . The generalization of the pullback metric is the continuous, symmetric, bilinear, non-negative and tensorial operator

$$\pi_f(V, W) = \Gamma\Omega \times \Gamma\Omega \rightarrow L^1(\Omega, \mathbf{R})$$

where

$$\pi_f(V, W) = \frac{1}{2}|f_*(V + W)|^2 - \frac{1}{2}|f_*(V - W)|^2.$$

We refer to [10] for more details.

Fix a local chart $(U; z)$. By the conformal invariance of the energy, we can assume that the metric g the Euclidean metric $dzd\bar{z}$ in U . We extend π_f linearly over $\mathbb{C}$. Then energy density function of f is given by

$$\frac{1}{4}|\nabla f|^2 = \pi_f \left(\frac{\partial f}{\partial z^i}, \frac{\partial f}{\partial \bar{z}^j} \right).$$

Furthermore, set $z = re^{i\theta}$ to define polar coordinates (r, θ) . We define

$$\left| \frac{\partial u}{\partial r} \right|^2 := \pi_f \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \quad \text{and} \quad \left| \frac{\partial u}{\partial \theta} \right|^2 := \pi_f \left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right).$$

2.2 Equivariant Maps and Sections of the Flat $\tilde{X}$ -Bundle

Definition 2.3 Let $\rho : \pi_1(M) \rightarrow \text{Isom}(\tilde{X})$. A map $\tilde{f} : \tilde{M} \rightarrow \tilde{X}$ is said to be ρ -equivariant if

$$\tilde{f}(\gamma p) = \rho(\gamma)\tilde{f}(p), \quad \forall \gamma \in \pi_1(M), \quad p \in \tilde{M}.$$

It is convenient to replace equivariant maps with sections of an associated fiber bundle. The quotient under the action of $\pi_1(M)$ on the product $\tilde{M} \times \tilde{X}$ is the *twisted product*

$$\tilde{M} \times_{\rho} \tilde{X}.$$

In other words, $\tilde{M} \times_{\rho} \tilde{X}$ is the set of orbits $[(p, x)]$ of a point $(p, x) \in \tilde{M} \times \tilde{X}$ under the action of $\gamma \in \pi_1(M)$ via the deck transformation on the first component and the isometry $\rho(\gamma)$ on the second component. The fiber bundle $\tilde{M} \times_{\rho} \tilde{X}$ is called the *flat $\tilde{X}$ -bundle* over M defined by ρ .

There is a one-to-one correspondence between sections of this fibration and ρ -equivariant maps

$$\tilde{f} : \tilde{M} \rightarrow \tilde{X} \quad \longleftrightarrow \quad f : M \rightarrow \tilde{M} \times_{\rho} \tilde{X}$$

by

$$[(\tilde{p}, \tilde{f}(\tilde{p}))] \leftrightarrow f(p) \quad \text{where } \Pi(\tilde{p}) = p.$$

Since the energy density function $|\nabla \tilde{f}|^2$ of $\tilde{f}$ is a ρ -invariant function, we can define

$$|\nabla f|^2(p) := |\nabla \tilde{f}|^2(\tilde{p}).$$

We can similarly define the pullback inner product and directional energy density functions of f by using the corresponding notions for $\tilde{f}$ given in Sect. 2.1. For $U \subset M$, the energy of a section f is

$$E^f[U] = \int_U |\nabla f|^2 d\text{vol}_g. \quad (2.1)$$

Furthermore, for sections f_1, f_2 , we define

$$d(f_1(p), f_2(p)) := d(\tilde{f}_1(\tilde{p}), \tilde{f}_2(\tilde{p})) \quad (2.2)$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are the corresponding ρ -equivariant maps.

2.3 Sublogarithmic Growth

We will

$$\text{fix a conformal disk } \mathbb{D}^j \subset \tilde{\mathcal{R}} \text{ centered at each puncture } p^j \quad (2.3)$$

such that $\mathbb{D}^i \cap \mathbb{D}^j = \emptyset$ for $i \neq j$. Furthermore, let $\mathbb{D}^{j*} = \mathbb{D}^j \setminus \{0\}$.

Fix $P_0 \in \tilde{X}$ and a fundamental domain F of $\tilde{\mathcal{R}}$. Let f_0 be the section of the fiber bundle $\tilde{\mathcal{R}} \times_\rho \tilde{X} \rightarrow \mathcal{R}$ such that, for any $p \in \mathcal{R} \cap \Pi(F)$, $f_0(p) = [(\tilde{p}, P_0)]$ where $\tilde{p} = \Pi^{-1}(p) \cap F$. (Note that $\Pi(F)$ is of full measure in $\mathcal{R}$.)

For a given section $f : \mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_\rho \tilde{X}$, define

$$\delta_j : \mathbb{D}^{j*} \rightarrow [0, \infty), \quad \delta_j(z) = d(f(z), f_0(z)). \quad (2.4)$$

Recall that $d(f(z), f_0(z))$ is defined by (2.2).

Definition 2.4 We say a section $f : \mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_\rho \tilde{X}$ (or its associated equivariant map) has *sublogarithmic growth* if for any $j = 1, \dots, n$

$$\lim_{|z| \rightarrow 0} \delta_j(z) + \epsilon \log |z| = -\infty \text{ in } \mathbb{D}^{j*}.$$

By the triangle inequality, this definition is independent of the choice of $P_0 \in \tilde{X}$. We say that f has *logarithmic energy growth* if near the punctures satisfies condition (1.1) in Theorem 1.1.

3 Harmonic Maps from a Punctured Disk

Before we consider the domain to be a general punctured Riemann surface, we first start by studying harmonic maps from a punctured disk $\bar{\mathbb{D}}^*$. Denote

$$\begin{aligned}\mathbb{D}_r^* &= \{z \in \mathbb{D} : |z| < r\} \\ \mathbb{D}_{r,r_0} &= \mathbb{D}_{r_0} \setminus \mathbb{D}_r.\end{aligned}$$

Identify the boundary of the disk with the circle; i.e., $\partial\mathbb{D} = \mathbb{S}^1$.

This induces a natural identification

$$2\pi\mathbb{Z} \simeq \pi_1(\mathbb{S}^1) \simeq \pi_1(\bar{\mathbb{D}}^*).$$

The main result of this section is the following.

Theorem 3.1 (Existence and Uniqueness of the Dirichlet solution on $\mathbb{D}^*$) *Assume the following:*

- $\rho : 2\pi\mathbb{Z} \simeq \pi_1(\mathbb{S}^1) \simeq \pi_1(\bar{\mathbb{D}}^*) \rightarrow \text{Isom}(\tilde{X})$ is a homomorphism
- $k : \bar{\mathbb{D}}^* \rightarrow \bar{\mathbb{D}}^* \times_\rho \tilde{X}$ is a locally Lipschitz section
- $I := \rho([\mathbb{S}^1])$ satisfies condition (B) of Theorem 1.1 with I^j replaced by I .

Then there exists a harmonic section

$$u : \bar{\mathbb{D}}^* \rightarrow \bar{\mathbb{D}}^* \times_\rho \tilde{X} \text{ with } u|_{\mathbb{S}^1} = k|_{\mathbb{S}^1}.$$

Furthermore, there exists a constant $C > 0$ that depends only on Δ_I , k if I is semisimple and also on a , b from assumption (B) if I is not semisimple such that u satisfies the following properties:

- $\frac{\Delta_I^2}{2\pi} \log \frac{1}{r} \leq E^u[\mathbb{D}_{r,1}] \leq \frac{\Delta_I^2}{2\pi} \log \frac{1}{r} + C, \quad 0 < r \leq 1$
- $\left| \frac{\partial u}{\partial r} \right|^2 \leq \frac{C}{r^2(-\log r)} \quad \text{and} \quad \left(\left| \frac{\partial u}{\partial \theta} \right|^2 - \frac{\Delta_I^2}{4\pi^2} \right) \leq \frac{C}{-\log r} \quad \text{in } \mathbb{D}_{\frac{1}{4}}^*$
- u has sub-logarithmic growth.

Moreover, u is the only harmonic section satisfying $u|_{\partial\mathbb{D}} = k|_{\partial\mathbb{D}}$ and property (iii).

Proof Combine Lemmas 3.5, 3.7, 3.9, and 3.10 below. $\square$

The purpose for the rest of this subsection is to prove the lemmas that comprise the proof of Theorem 3.1. We start with the following preliminary lemma about subharmonic functions.

Lemma 3.2 *For $r > 0$, let $v : \mathbb{D}_r^* \rightarrow \mathbb{R}$ be a subharmonic function that extends as a continuous function to $\bar{\mathbb{D}}_r^* = \bar{\mathbb{D}}_r \setminus \{0\}$. If we assume that*

$$\lim_{z \rightarrow 0} v(z) + \epsilon \log |z| = -\infty, \quad \forall \epsilon > 0, \quad (3.1)$$

then

$$\sup_{z \in \mathbb{D}_r^*} v(z) \leq \max_{\zeta \in \partial \mathbb{D}_r} v(\zeta).$$

In particular, v extends to a subharmonic function on $\mathbb{D}_r$ (cf. [9]).

Proof Let $h : \mathbb{D}_r \rightarrow \mathbb{R}$ be the unique Dirichlet solution with $h|_{\partial \mathbb{D}_r} = v|_{\partial \mathbb{D}_r}$. By the maximum principle, h is non-negative and bounded on $\mathbb{D}_r$. The function

$$f_\epsilon(z) = v(z) - h(z) + \epsilon(\log |z| - \log r)$$

is subharmonic on $\mathbb{D}_r^*$ with $f_\epsilon|_{\partial \mathbb{D}_r} \equiv 0$ on $\partial \mathbb{D}_r$. Furthermore, since h is a bounded function and v satisfies (3.1),

$$\lim_{z \rightarrow 0} f_\epsilon(z) = -\infty.$$

By the maximum principle, $f_\epsilon \leq 0$. By letting $\epsilon \rightarrow 0$, we conclude $v(z) - h(z) \leq 0$ for all $z \in \mathbb{D}_r^*$. Thus,

$$\sup_{z \in \mathbb{D}_r^*} v(z) \leq \sup_{z \in \mathbb{D}_r} h(z) \leq \max_{\zeta \in \partial \mathbb{D}_r} h(\zeta) = \max_{\zeta \in \partial \mathbb{D}_r} v(\zeta).$$

□

Lemma 3.3 If ρ and I are as in Theorem 3.1, then for any section $f : \mathbb{D}_{r,r_0} \rightarrow \widetilde{\mathbb{D}}^* \times_\rho \tilde{X}$,

$$\frac{\Delta_I^2}{2\pi} \log \frac{r_0}{r} \leq \int_{\mathbb{D}_{r,r_0}} \frac{1}{r^2} \left| \frac{\partial f}{\partial \theta} \right|^2 r dr \wedge d\theta \leq E^f[\mathbb{D}_{r,r_0}], \quad 0 < r < r_0 < 1.$$

Proof By the definition of Δ_I , any section $c : \mathbb{S}^1 \rightarrow \mathbb{R} \times_\rho \tilde{X}$ satisfies

$$\frac{\Delta_I^2}{2\pi} \leq \int_0^{2\pi} \left| \frac{\partial c}{\partial \theta} \right|^2 d\theta.$$

After identifying $\mathbb{S}^1$ with $\partial \mathbb{D}_r$, we can view the restriction $f|_{\partial \mathbb{D}_r}$ as a section $\mathbb{S}^1 \rightarrow \mathbb{R} \times_\rho \tilde{X}$. Thus,

$$\begin{aligned} \frac{\Delta_I^2}{2\pi} \log \frac{r_0}{r} &= \frac{\Delta_I^2}{2\pi} \int_r^{r_0} \frac{1}{r^2} r dr \wedge d\theta \\ &\leq \int_0^{2\pi} \int_r^{r_0} \frac{1}{r^2} \left| \frac{\partial f}{\partial \theta}(r, \theta) \right|^2 r dr \wedge d\theta \\ &\leq E^f[\mathbb{D}_{r,r_0}]. \end{aligned}$$

□

Lemma 3.4 *If ρ , I , and $k : \mathbb{D}^* \rightarrow \widetilde{\mathbb{D}}^* \times_\rho \tilde{X}$ are as in Theorem 3.1, then there exists a constant $C > 0$ and a Lipschitz section $v : \mathbb{D}^* \rightarrow \widetilde{\mathbb{D}}^* \times_\rho \tilde{X}$ with $v|_{\partial\mathbb{D}} = k|_{\partial\mathbb{D}}$ such that*

$$\frac{\Delta_I^2}{2\pi} \log \frac{r_0}{r} \leq E^v[\mathbb{D}_{r,r_0}] \leq \frac{\Delta_I^2}{2\pi} \log \frac{r_0}{r} + C, \quad 0 < r < r_0 \leq \frac{1}{2}. \quad (3.2)$$

Moreover, the Lipschitz constants of v and C are dependent only on Δ_I , k if I is semisimple and also on a, b from assumption (B) if I is not semisimple.

Proof We will first construct an one-parameter family of sections $\gamma_s : \mathbb{S}^1 \mapsto \mathbb{R} \times_\rho \tilde{X}$ as follows:

- If I is semisimple, let $P_\star \in \tilde{X}$ be a point where the infimum Δ_I is attained. For any $s \in [0, \infty)$, define $\tilde{\gamma}_s : \mathbb{R} \rightarrow \tilde{X}$ to be the ρ -equivariant geodesic (a constant map if I is elliptic) such that $\tilde{\gamma}_s(0) = P_\star$ and $\tilde{\gamma}_s(2\pi) = I(P_\star)$ and let $\gamma_s : \mathbb{S}^1 \rightarrow \tilde{\mathbb{S}}^1 \times_\rho \tilde{X}$ be the associated section.
- Otherwise, let $c : [0, \infty) \rightarrow \tilde{X}$ be a geodesic ray defined in assumption (B). Define the ρ -equivariant curve $\tilde{\gamma}_s : \mathbb{R} \rightarrow \tilde{X}$ such that $\tilde{\gamma}_s|_{[0,2\pi]}$ is the geodesic from $c(s)$ and $I(c(s))$ and let $\gamma_s : \mathbb{S}^1 \rightarrow \tilde{\mathbb{S}}^1 \times_\rho \tilde{X}$ be the associated section.

We note that by the quadrilateral comparison of CAT(0) spaces (cf. [16]), for $\theta \in [0, 2\pi]$,

$$d(\tilde{\gamma}_{s_1}(\theta), \tilde{\gamma}_{s_2}(\theta)) \leq \left(1 - \frac{\theta}{2\pi}\right) d(c(s_1), c(s_2)) + \frac{\theta}{2\pi} d(I \circ c(s_1), I \circ c(s_2)) = |s_1 - s_2|.$$

Thus,

$$\left| \frac{d}{ds}(\tilde{\gamma}_s(\theta)) \right| \leq 1, \quad \forall s \in (0, \infty), \theta \in \mathbb{S}^1. \quad (3.3)$$

If I is semisimple, then

$$\int_{\mathbb{S}^1} \left| \frac{\partial \gamma_s}{\partial \theta} \right|^2 d\theta = \int_0^{2\pi} \left(\frac{\Delta_I}{2\pi} \right)^2 d\theta = \frac{\Delta_I^2}{2\pi}.$$

Otherwise, by the assumption on $c(s)$, we have (by modifying the constants a, b in assumption (B)) that

$$\int_{\mathbb{S}^1} \left| \frac{\partial \gamma_s}{\partial \theta} \right|^2 d\tau \leq \frac{\Delta_I^2}{2\pi} + be^{-as}. \quad (3.4)$$

Define a ρ -equivariant map $\tilde{v} : \mathbb{D}^* \rightarrow \tilde{X}$ as follows: First, for $(r, t) \in \mathbb{D}^*$, let

$$\tilde{v}(r, t) := \tilde{\gamma}_{(-\log r - \log 2)^{\frac{1}{3}}}(t) \text{ for } r \in (0, \frac{1}{2}] \text{ and } \tilde{v}(1, t) := \tilde{k}(1, t).$$

Next, for each $t \in \mathbb{R}$, let

$$r \mapsto \tilde{v}(r, t) \text{ for } r \in \left[\frac{1}{2}, 1\right]$$

to be the arclength parameterization of the geodesic between $\tilde{v}(\frac{1}{2}, t)$ and $\tilde{k}(1, t)$. Finally, let $v : \mathbb{D}^* \rightarrow \widetilde{\mathbb{D}^*} \times_{\rho} \tilde{X}$ be the section associated to the ρ -equivariant map $\tilde{v}$.

For $0 < r < r_0 < \frac{1}{2}$, we have

$$\begin{aligned} \frac{\Delta_I^2}{2\pi} \log \frac{r_0}{r} &\leq \int_0^{2\pi} \int_r^{r_0} \frac{1}{r^2} \left| \frac{\partial v}{\partial \theta} \right|^2 r dr \wedge d\theta \quad (\text{by Lemma 3.3}) \\ &\leq \int_0^{2\pi} \int_r^{r_0} \frac{1}{r^2} \left| \frac{\partial(\gamma_{(-\log r - \log 2)^{\frac{1}{3}}})}{\partial \theta} \right|^2 r dr \wedge d\theta \\ &= \int_0^{2\pi} \int_{-\log r_0}^{-\log r} \left| \frac{\partial(\gamma_{(t - \log 2)^{\frac{1}{3}}})}{\partial \theta} \right|^2 dt d\theta \\ &\leq \int_{-\log r_0}^{-\log r} \left(\frac{\Delta_I^2}{2\pi} + b e^{-a(t - \log 2)^{\frac{1}{3}}} \right) dt \quad (\text{by (3.4)}) \\ &= \frac{\Delta_I^2}{2\pi} \log \frac{r_0}{r} + C. \end{aligned}$$

By (3.3),

$$\begin{aligned} \left| \frac{\partial v}{\partial r} \right|^2(r, \theta) &= \left| \frac{\partial}{\partial s} \Big|_{s=(-\log r - \log 2)^{\frac{1}{3}}} \gamma_s(\theta) \right|^2 \left| \frac{\partial((-\log r - \log 2)^{\frac{1}{3}})}{\partial r} \right|^2 \\ &\leq \frac{1}{r^2(-\log r - \log 2)^{\frac{4}{3}}}. \end{aligned}$$

Thus,

$$\int_0^{2\pi} \int_r^{r_0} \left| \frac{\partial v}{\partial r} \right|^2 r dr \wedge d\theta = \int_0^{2\pi} \int_r^{r_0} \frac{1}{r(-\log r - \log 2)^{\frac{4}{3}}} dr \wedge d\theta \leq C.$$

□

Lemma 3.5 (Existence and property (i) of Theorem 3.1) *If $\tilde{X}$, ρ , Δ_I , and k as in Theorem 3.1, then there exists a harmonic section $u : \mathbb{D}^* \rightarrow \mathbb{D}^* \times_{\rho} \tilde{X}$ with $u|_{\partial \mathbb{D}} = k|_{\partial \mathbb{D}}$ and a constant $C > 0$ that depends only on Δ_I , k if I is semisimple and also on a, b from assumption (B) if I is not semisimple such that*

$$\frac{\Delta_I^2}{2\pi} \log \frac{1}{r} \leq E^u[\mathbb{D}_{r,1}] \leq \frac{\Delta_I^2}{2\pi} \log \frac{1}{r} + C, \quad 0 < r \leq 1.$$

Proof Let v be as in Lemma 3.4. By [10, Theorem 2.7.2] (or more specifically, by the proof of this theorem which solves the equivariant problem), there exists a unique harmonic section

$$u_r : \mathbb{D}_{r,1} \rightarrow \widetilde{\mathbb{D}_{r,1}} \times_{\rho} \tilde{X} \text{ with } u_r|_{\partial \mathbb{D}_{r,1}} = v|_{\partial \mathbb{D}_{r,1}}.$$

We have that

$$\begin{aligned} \frac{\Delta_I^2}{2\pi} \log \frac{r_0}{r} + E^{u_r}[\mathbb{D}_{r_0,1}] &\leq E^{u_r}[\mathbb{D}_{r,r_0}] + E^{u_r}[\mathbb{D}_{r_0,1}] \quad (\text{by Lemma 3.3}) \\ &= E^{u_r}[\mathbb{D}_{r,1}] \\ &\leq E^v[\mathbb{D}_{r,1}] \quad (\text{since } u_r \text{ is minimizing}) \\ &= E^v[\mathbb{D}_{r,r_0}] + E^v[\mathbb{D}_{r_0,1}] \\ &\leq \frac{\Delta_I^2}{2\pi} \log \frac{r_0}{r} + C + E^v[\mathbb{D}_{r_0,1}] \quad (\text{by (3.2)}). \end{aligned}$$

Therefore,

$$E^{u_r}[\mathbb{D}_{r_0,1}] \leq C + E^v[\mathbb{D}_{r_0,1}]. \quad (3.5)$$

Note that the right-hand side of the above is independent of $r \in (0, \frac{1}{2}]$. Thus, $\{u_r|_{\mathbb{D}_{2r_0,1}}\}$ is an equicontinuous family of sections (cf. [10, Theorem 2.4.6]). Consequently, there exists a sequence $r_i \rightarrow 0$ and a harmonic section

$$u : \tilde{\mathbb{D}}^* \rightarrow \tilde{\mathbb{D}}^* \times_{\rho} \tilde{X} \text{ with } u|_{\partial \mathbb{D}} = k|_{\partial \mathbb{D}}. \quad (3.6)$$

such that the sequence $\{u_{r_i}\}$ converges uniformly on compact subsets of $\mathbb{D}^*$ to u .

To prove property (i), we note that for $0 < r < r_1 < 1$,

$$\begin{aligned} E^{u_r}[\mathbb{D}_{r,r_1}] + E^{u_r}[\mathbb{D}_{r_1,1}] &= E^{u_r}[\mathbb{D}_{r,1}] \\ &\leq E^v[\mathbb{D}_{r,1}] \\ &= E^v[\mathbb{D}_{r,r_1}] + E^v[\mathbb{D}_{r_1,1}] \end{aligned}$$

which, combined with Lemma 3.3 and (3.2), implies that

$$E^{u_r}[\mathbb{D}_{r_1,1}] \leq E^v[\mathbb{D}_{r_1,1}] + C.$$

Letting $r \rightarrow 0$ and applying the lower semicontinuity of energy [10, Theorem 1.6.1], we obtain

$$E^u[\mathbb{D}_{r_1,1}] \leq E^v[\mathbb{D}_{r_1,1}] + C.$$

Applying Lemma 3.3 and (3.2), we obtain

$$\frac{\Delta_I^2}{2\pi} \log \frac{1}{r_1} \leq E^u[\mathbb{D}_{r_1,1}] \leq E^v[\mathbb{D}_{r_1,1}] + C \leq \frac{\Delta_I^2}{2\pi} \log \frac{1}{r_1} + C.$$

By (3.2), the constant C is dependent on Δ_I , k if I is semisimple and also on a , b from assumption (B) if I is not semisimple. $\square$

Lemma 3.6 *If $u : \mathbb{D}^* \rightarrow \widetilde{\mathbb{D}}^* \times_\rho \tilde{X}$ is a harmonic section with logarithmic energy growth, then*

$$\int_{\mathbb{D}^*} \left| \frac{\partial u}{\partial r} \right|^2 r dr \wedge d\theta \leq C \text{ and } \int_{\mathbb{D}^*} \frac{1}{r^2} \left(\left| \frac{\partial u}{\partial \theta} \right|^2 - \frac{\Delta_I^2}{4\pi^2} \right) r dr \wedge d\theta \leq C$$

where C here depends on C from (1.1).

Proof Follows immediately from the inequality (1.1) in the Definition 2.4 and Lemma 3.3. $\square$

Lemma 3.7 (Property (ii) of Theorem 3.1) *If $u : \mathbb{D}^* \rightarrow \widetilde{\mathbb{D}}^* \times_\rho \tilde{X}$ is a harmonic section with logarithmic energy growth, then*

$$\left(\left| \frac{\partial u}{\partial \theta} \right|^2 - \frac{\Delta_I^2}{4\pi^2} \right) \leq \frac{C}{(-\log r)} \text{ in } \mathbb{D}_{\frac{1}{4}}^* \quad (3.7)$$

$$\lim_{r \rightarrow 0} (-\log r) \left(\left| \frac{\partial u}{\partial \theta} \right|^2 (r, \theta) - \frac{\Delta_I^2}{4\pi^2} \right) = 0 \quad (3.8)$$

$$\left| \frac{\partial u}{\partial r} \right|^2 \leq \frac{C}{r^2(-\log r)} \text{ in } \mathbb{D}_{\frac{1}{4}}^* \quad (3.9)$$

$$\lim_{r \rightarrow 0} (-\log r) r^2 \left| \frac{\partial u}{\partial r} \right|^2 (r, \theta) = 0. \quad (3.10)$$

where C here depends on C from (1.1).

Proof Consider the cylinder

$$\mathcal{C} = (0, \infty) \times \mathbb{S}^1$$

and let

$$\Phi : \mathcal{C} \rightarrow \mathbb{D}^*, \quad (t, \psi) = (r = e^{-t}, \theta = \psi). \quad (3.11)$$

Since Φ is a conformal map, $u \circ \Phi$ is harmonic. Thus, the directional energy density functions $\left| \frac{\partial(u \circ \Phi)}{\partial t} \right|^2$ and $\left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2$ are subharmonic by [10, Remark 2.4.3]. Furthermore, $\left| \frac{\partial(u \circ \Phi)}{\partial t} \right|^2$ and $\left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 - \frac{\Delta_I^2}{4\pi^2}$ are integrable in $\mathcal{C} = (0, \infty) \times \mathbb{S}^1$ by Lemma 3.6 and the chain rule.

The subharmonicity of the directional energy density functions implies

$$0 \leq \int_{(t_1, t_2) \times \mathbb{S}^1} \Delta \varphi \left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 dt d\psi, \quad \forall \varphi \in C_c^\infty((t_1, t_2) \times \mathbb{S}^1), \quad \varphi \geq 0.$$

Letting φ approximate the characteristic function of $(t_1, t_2) \times \mathbb{S}^1$, we obtain

$$0 \leq \frac{d}{dt} \Big|_{t=t_2} \left(\int_{\{t\} \times \mathbb{S}^1} \left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 d\psi \right) - \frac{d}{dt} \Big|_{t=t_1} \left(\int_{\{t\} \times \mathbb{S}^1} \left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 d\psi \right).$$

In other words,

$$F(t) := \int_{\{t\} \times \mathbb{S}^1} \left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 - \frac{\Delta_I^2}{4\pi^2} d\psi, \quad t \in (0, \infty)$$

is a convex function. By Lemma 3.6, $\int_0^\infty F(t) dt < \infty$. Since $F(t)$ is convex and integrable, $F(t)$ is a decreasing. Thus, we obtain

$$tF(t) \leq 2 \int_{\frac{t}{2}}^t F(\tau) d\tau \leq 2 \int_{\frac{t}{2}}^\infty F(\tau) d\tau. \quad (3.12)$$

With $\mathcal{B}_1(t_0, \psi_0)$ denoting the unit disk centered at $(t_0, \psi_0) \in \mathcal{C}$, the mean value inequality implies

$$\begin{aligned} & t_0 \left(\left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 - \frac{\Delta_I^2}{4\pi^2} \right) (t_0, \psi_0) \\ & \leq \frac{t_0}{\pi} \int_{\mathcal{B}_1(t_0, \psi_0)} \left(\left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 - \frac{\Delta_I^2}{4\pi^2} \right) dt d\psi \\ & \leq \frac{t_0}{\pi} \int_{(t_0-1, t_0+1) \times \mathbb{S}^1} \left(\left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 - \frac{\Delta_I^2}{4\pi^2} \right) dt d\psi \\ & \leq \frac{1}{\pi} \frac{t_0}{t_0-1} \int_{(t_0-1, t_0+1) \times \mathbb{S}^1} t \left(\left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 - \frac{\Delta_I^2}{4\pi^2} \right) dt d\psi \\ & = \frac{1}{\pi} \frac{t_0}{t_0-1} \int_{t_0-1}^{t_0+1} t F(t) dt \\ & \leq \frac{1}{\pi} \frac{t_0}{t_0-1} \int_{t_0-1}^{t_0+1} 2 \int_{\frac{t}{2}}^\infty F(\tau) d\tau dt \quad (\text{by (3.12)}). \end{aligned}$$

Since $F(\tau) \geq 0$,

$$\int_{\frac{t}{2}}^\infty F(\tau) d\tau \leq \int_{\frac{t_0-1}{2}}^\infty F(\tau) d\tau, \quad t \in (t_0-1, t_0+1).$$

Combining the above two inequalities and assuming $t_0 \geq 2$ (and thus $\frac{t_0}{t_0-1} \leq 2$),

$$t_0 \left(\left| \frac{\partial(u \circ \Phi)}{\partial \psi} \right|^2 - \frac{\Delta_I^2}{4\pi^2} \right) (t_0, \psi_0) \leq \frac{4}{\pi} \int_{\frac{t_0-1}{2}}^\infty F(\tau) d\tau. \quad (3.13)$$

Estimates (3.7) and (3.8) both follow from (3.13) by the chain rule and the fact that $F(\tau)$ is integrable on $(0, \infty)$ with integral bound given by Lemma 3.6.

Similar argument for

$$G(t) := \int_{\{t\} \times \mathbb{S}^1} \left| \frac{\partial(u \circ \Phi)}{\partial t} \right|^2 d\psi, \quad t \in (0, \infty)$$

proves (3.9) and (3.10). $\square$

Lemma 3.8 *Let $u : \bar{\mathbb{D}}^* \rightarrow \bar{\mathbb{D}}^* \times_{\rho} \tilde{X}$ be a harmonic section with logarithmic energy growth. For any $\epsilon > 0$, there exists $\rho_0 > 0$ such that*

$$d^2(u(r_1 e^{i\theta}), u(r_0 e^{i\theta})) \leq -\epsilon \log r_1, \quad \forall 0 < r_1 < r_0 \leq \rho_0, \quad 0 < \theta < 2\pi.$$

Proof Fix $\epsilon > 0$. The convergence of $(-\log r)r^2 \left| \frac{\partial u}{\partial r} \right|^2(r, \theta)$ to 0 as $r \rightarrow 0$ (cf. (3.10)) implies that there exists $\rho_0 > 0$ such that

$$\left| \frac{\partial u}{\partial r} \right| \leq \frac{\sqrt{\epsilon}}{2r(-\log r)^{\frac{1}{2}}}, \quad r \in (0, \rho_0].$$

Thus, for $0 < r_1 < r_0 \leq \rho_0$, and noting the ρ -equivariance of $\tilde{u}$, we have

$$\begin{aligned} d^2(u(r_1 e^{i\theta}), u(r_0 e^{i\theta})) &\leq \left(\int_{r_0}^{r_1} \frac{d}{dr} d(u(re^{i\theta}), u(r_0 e^{i\theta})) dr \right)^2 \\ &\leq \left(\int_{r_1}^{r_0} \left| \frac{\partial u}{\partial r} \right| (re^{i\theta}) dr \right)^2 \\ &\leq \frac{\epsilon}{4} \left(\int_{r_1}^{r_0} \frac{dr}{r(-\log r)^{\frac{1}{2}}} \right)^2 \\ &\leq \epsilon \left((-\log r_1)^{\frac{1}{2}} - (-\log r_0)^{\frac{1}{2}} \right)^2 \\ &\leq -\epsilon \log r_1. \end{aligned}$$

$\square$

Lemma 3.9 (Property (iii) of Theorem 3.1) *If $u : \bar{\mathbb{D}}^* \rightarrow \bar{\mathbb{D}}^* \times_{\rho} \tilde{X}$ is a harmonic section with logarithmic energy growth, then u has sub-logarithmic growth.*

Proof By Lemma 3.8, for $\epsilon > 0$, there exists $r_0 > 0$ sufficiently small such that

$$d^2(u(re^{i\theta}), u(r_0 e^{i\theta})) \leq -\frac{\epsilon}{4} \log r, \quad r \in (0, r_0).$$

Set $P_0 = \tilde{u}(r_0 e^{i\theta_0}) \in \tilde{X}$ and define $f_0(p) = [(\tilde{p}, P_0)]$ as in Definition 2.4. For $z = re^{i\theta}$,

$$\begin{aligned} d^2(u(z), f_0(z)) + \epsilon \log |z| &= d^2(u(re^{i\theta}), u(r_0e^{i\theta_0})) + \epsilon \log r \\ &\leq 2d^2(u(re^{i\theta}), u(r_0e^{i\theta})) + 2d^2(u(r_0e^{i\theta}), u(r_0e^{i\theta_0})) + \epsilon \log r \\ &\leq \frac{\epsilon}{2} \log r + d^2(u(r_0e^{i\theta}), u(r_0e^{i\theta_0})). \end{aligned}$$

$r \rightarrow 0$. □

Lemma 3.10 (Uniqueness for sub-logarithmic growth maps) *If $u, v : \tilde{\mathbb{D}}^* \rightarrow \tilde{\mathbb{D}}^* \times_{\rho} \tilde{X}$ are harmonic sections with sub-logarithmic growth with $u = v$ on $\partial\mathbb{D}$, then $u = v$ on $\mathbb{D}^*$.*

Proof The function $d^2(u, v)$ (as defined by (2.2)) is a continuous subharmonic function (cf. [10, Remark 2.4.3]). Furthermore, $d^2(u, v) \equiv 0$ on $\partial\mathbb{D}$. Since u, v both have sub-logarithmic growth, the triangle inequality implies

$$\lim_{|z| \rightarrow 0} d^2(u(z), v(z)) + \epsilon \log |z| = -\infty.$$

Thus, we can apply Lemma 3.2 to conclude $d^2(u(z), v(z)) \equiv 0$ on $\mathbb{D}^*$ which implies $u \equiv v$. □

Corollary 3.11 *Any harmonic section $v : \tilde{\mathbb{D}}^* \rightarrow \tilde{\mathbb{D}}^* \times_{\rho} \tilde{X}$ with sub-logarithmic growth satisfies properties (i), (ii) and (iii) of Theorem 3.1.*

Proof By the uniqueness assertion of Lemma 3.10, v must be the harmonic section u constructed in Theorem 3.1. □

Remark 3.12 Together, Lemma 3.9 and Corollary 3.11 say that a harmonic section $u : \tilde{\mathbb{D}}^* \rightarrow \tilde{\mathbb{D}}^* \times_{\rho} \tilde{X}$ has logarithmic energy growth if and only if it has sub-logarithmic growth.

4 Existence of Infinite Energy Harmonic Maps

In this section, we prove existence of equivariant harmonic maps from the punctured Riemann surface $\mathcal{R}$. We use the following notation:

- $\mathbb{D}^{j*} = \mathbb{D}^j \setminus \{0\}$
- $\mathbb{D}_r^j = \{z \in \mathbb{D}^j : |z| < r\}$
- $\mathbb{D}_{r,r_0}^j = \mathbb{D}_{r_0}^j \setminus \mathbb{D}_r^j$
- $\mathcal{R}_r = \mathcal{R} \setminus \bigcup_{j=1}^n \mathbb{D}_r^j$
- $2\pi\mathbb{Z} \simeq \langle \lambda^j \rangle$ is the free group generated by a loop around the puncture p^j
- $\rho^j : \langle \lambda^j \rangle \rightarrow \text{Isom}(\tilde{X})$ is the restriction of $\rho : \pi_1(\mathcal{R}) \rightarrow \text{Isom}(\tilde{X})$
- $k : \mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$ is a locally Lipschitz section (cf. [10, Proposition 2.6.1])
- $k^j := k|_{\mathbb{D}^{j*}}$, is the restriction to the conformal disk around the puncture p^j

Applying Lemma 3.4 with $\rho = \rho^j$, $I = I^j$ and $k = k^j$ yields a prototype section

$$v^j : \mathbb{D}^* \rightarrow \tilde{\mathbb{D}}^* \times_{\rho^j} \tilde{X} \subset \tilde{\mathcal{R}} \times_{\rho^j} \tilde{X}.$$

The composition of v^j and the quotient map $\tilde{\mathcal{R}} \times_{\rho^j} \tilde{X} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$ defines a section of $\tilde{\mathcal{R}} \times_{\rho} \tilde{X} \rightarrow \mathcal{R}$ over $\mathbb{D}^{j*}$ which we call again v^j . We extend these local sections $v^j : \mathbb{D}^{j*} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$ for $j = 1, \dots, n$ to define a locally Lipschitz section $v : \mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$. By the construction and Lemma 3.4,

$$\sum_{j=1}^n \frac{\Delta_j^2}{2\pi} \log \frac{1}{r} \leq E^v[\mathcal{R}_r] \leq \sum_{j=1}^n \frac{\Delta_j^2}{2\pi} \log \frac{1}{r} + C, \quad 0 < r \leq 1. \quad (4.1)$$

The constant C is dependent only on Δ_I, k if I is semisimple and also on a, b from assumption (B) if I is not semisimple.

Definition 4.1 The locally Lipschitz section

$$v : \mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X} \quad (4.2)$$

constructed above is called the *prototype section* of the fiber bundle $\tilde{\mathcal{R}} \times_{\rho} \tilde{X} \rightarrow \mathcal{R}$. The associated ρ -equivariant map $\tilde{v} : \tilde{\mathcal{R}} \rightarrow \tilde{X}$ is called the *prototype map*.

Define

$$u_{\mathcal{R}_r} : \mathcal{R}_r \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$$

to be the unique harmonic section with boundary values equal to that of $v|_{\partial\mathcal{R}_r}$ (cf. [10, Theorem 2.7.2]). Fix $r_0 \in (0, \frac{1}{2}]$ and let $r \in (0, r_0)$. By Lemma 3.3,

$$\frac{\Delta_j^2}{2\pi} \log \frac{r_0}{r} \leq \int_{\mathbb{D}_{r,r_0}} \frac{1}{r^2} \left| \frac{\partial u_{\mathcal{R}_r}}{\partial \theta} \right|^2 r dr \wedge d\theta \leq E^{u_{\mathcal{R}_r}}[\mathbb{D}_{r,r_0}^j].$$

Thus, by (4.1),

$$E^v\left[\bigcup_{j=1}^n \mathbb{D}_{r,r_0}^j\right] \leq E^{u_{\mathcal{R}_r}}\left[\bigcup_{j=1}^n \mathbb{D}_{r,r_0}^j\right] + C$$

which implies

$$\begin{aligned} E^{u_{\mathcal{R}_r}}\left[\bigcup_{j=1}^n \mathbb{D}_{r,r_0}^j\right] + E^{u_{\mathcal{R}_r}}[\mathcal{R}_{r_0}] &= E^{u_{\mathcal{R}_r}}[\mathcal{R}_r] \\ &\leq E^v[\mathcal{R}_r] \\ &= E^v\left[\bigcup_{j=1}^n \mathbb{D}_{r,r_0}^j\right] + E^v[\mathcal{R}_{r_0}] \\ &\leq E^{u_{\mathcal{R}_r}}\left[\bigcup_{j=1}^n \mathbb{D}_{r,r_0}^j\right] + C + E^v[\mathcal{R}_{r_0}]. \end{aligned}$$

In other words,

$$E^{u_{\mathcal{R}_r}}[\mathcal{R}_{r_0}] \leq C + E^v[\mathcal{R}_{r_0}], \quad \forall r \in (0, r_0) \quad (4.3)$$

where C is as in (4.1). The right-hand side of the inequality (4.3) is independent of the parameter r ; i.e., once we fix r_0 , the quantity $E^{u_{\mathcal{R}_r}}[\mathcal{R}_{r_0}]$ is uniformly bounded for all $r \in (0, r_0)$. This implies a uniform Lipschitz bound, say L , of $u_{\mathcal{R}_r}$ for $r \in (0, r_0)$ in $\mathcal{R}_{2r_0}$ (cf. [10, Theorem 2.4.6]).

Let $\{\mu_1, \dots, \mu_N\}$ be a set of generators of $\pi_1(\mathcal{R})$ and $\tilde{u}_{\mathcal{R}_r}, \tilde{v}$ be the ρ -equivariant maps associated to sections $u_{\mathcal{R}_r}, v$ respectively. Thus,

$$d(\tilde{u}_{\mathcal{R}_r}(\mu_i p), \tilde{u}_{\mathcal{R}_r}(p)) \leq L d_{\tilde{\mathcal{R}}}(\mu_i p, p), \quad \forall p \in \tilde{\mathcal{R}}_{2r_0}, \quad i = 1, \dots, N, \quad r \in (0, r_0).$$

If we let

$$c = L \sup\{d_{\tilde{\mathcal{R}}}(\mu_i p, p) : i = 1, \dots, N, \quad p \in \tilde{\mathcal{R}}_{2r_0}\},$$

then by equivariance

$$d(\rho(\mu_i)\tilde{u}_{\mathcal{R}_r}(p), \tilde{u}_{\mathcal{R}_r}(p)) \leq c, \quad p \in \tilde{\mathcal{R}}_{2r_0}, \quad i = 1, \dots, N, \quad r \in (0, r_0).$$

In other words, $\delta(\tilde{u}_{\mathcal{R}_r}(p)) \leq c$ for all $p \in \tilde{\mathcal{R}}_{2r_0}$ and $r \in (0, r_0)$. By the properness of ρ , there exists $P_0 \in \tilde{X}$ and $R_0 > 0$ such that

$$\{\tilde{u}_{\mathcal{R}_r}(p) : p \in \tilde{\mathcal{R}}_{2r_0}, \quad r \in (0, r_0)\} \subset B_{R_0}(P_0).$$

Thus, by taking a compact exhaustion and applying [11, Theorem 2.1.3], we conclude that there exists a sequence $r_i \rightarrow 0$ and a ρ -equivariant harmonic map $\tilde{u} : \tilde{\mathcal{R}} \rightarrow \tilde{X}$ such that $u_{\mathcal{R}_{r_i}}$ converges to u in L^2 (i.e., $d^2(u_{\mathcal{R}_{r_i}}, u) \rightarrow 0$) on every compact subsets of $\mathcal{R}$. Let $u : \mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$ be the associated harmonic section. The lower semicontinuity of energy (cf. [10, Theorem 1.6.1]), (4.1) and (4.3) imply

$$E^u[\mathcal{R}_{r_0}] \leq \sum_{j=1}^n \frac{\Delta_j^2}{2\pi} \log \frac{1}{r_0} + C, \quad \forall r_0 \in (0, \frac{1}{2}].$$

By the use of (4.1) in the above argument, the constant C is dependent only on Δ_I, k if I is semisimple and also on a, b from assumption (B) if I is not semisimple. Note that k is dependent only on ρ (cf. [10, Proposition 2.6.1]). The above inequality and Lemma 3.3 imply that u has logarithmic energy growth (cf. Definition 2.4). The fact that u has sub-logarithmic growth follows from Lemma 3.9. This completes the proof of Theorem 1.1.

5 Uniqueness of Infinite Energy Harmonic Maps

In this section, we prove uniqueness of harmonic maps from punctured Riemann surfaces $\mathcal{R}$ (cf. Theorem 1.2). First, let

$$E^f(r) = E^f[\mathcal{R}_r] - \sum_{j=1}^n \frac{\Delta_j^2}{2\pi} \log \frac{1}{r}.$$

By Lemma 3.3, for $0 < r < r_0 < 1$ and any section $f : \mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$,

$$\sum_{j=1}^n \frac{\Delta_j^2}{2\pi} \log \frac{r_0}{r} \leq E^f\left[\bigcup_{j=1}^n \mathbb{D}_{r,r_0}^j\right]. \quad (5.1)$$

Thus, $r \mapsto E^f(r)$ is an increasing function.

Definition 5.1 The *modified energy* of f is

$$E^f(0) = \lim_{r \rightarrow 0} E^f(r).$$

Let $\mathcal{L}_{\rho}$ be the set of all sections $\mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$ such that $E^f(0) < \infty$. Let $\mathcal{H}_{\rho} \subset \mathcal{L}_{\rho}$ denote the set of all harmonic sections $u : \mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$ such that the associated ρ -equivariant map u is not identically constant or does not map into a geodesic.

Remark 5.2 By Lemma 3.9, $u \in \mathcal{H}_{\rho}$ has sub-logarithmic growth.

The goal is to prove that $\mathcal{H}_{\rho}$ contains only one element. We first prove the following series of preliminary lemmas.

Lemma 5.3 If $u \in \mathcal{H}_{\rho}$ and $\mathbb{D}^j \subset \tilde{\mathcal{R}}$ is the fixed conformal disk at the puncture p^j (cf. (2.3)), then the restriction map $u|_{\mathbb{D}^{j*}}$ satisfies the properties (i), (ii) and (iii) of Theorem 3.1.

Proof This follows immediately from Remark 5.2 and Corollary 3.11. $\square$

Lemma 5.4 If $u_0, u_1 \in \mathcal{H}_{\rho}$, then $d^2(u_0, u_1) = c$ for some constant c .

Proof We prove Lemma 5.4 by showing that $d^2(u_0, u_1)$ extends as a subharmonic function to $\tilde{\mathcal{R}}$. Since $\tilde{\mathcal{R}}$ is compact, this implies $d^2(u_0, u_1)$ is constant. Indeed, let $\mathbb{D} \subset \mathcal{R}$ be a holomorphic disk. By [10, Lemma 2.4.2 and Remark 2.4.3], the function $d^2(u_0, u_1)$ is subharmonic in $\mathbb{D}$. Since this statement is true for any holomorphic disk $\mathbb{D} \subset \mathcal{R}$, we conclude that $d^2(u_0, u_1)$ is subharmonic in $\mathcal{R}$.

Next, consider the fixed conformal disk $\mathbb{D}^j \subset \tilde{\mathcal{R}}$ centered at $p^j \in \mathcal{P}$ (cf. (2.3)). Since both u_0 and u_1 have sub-logarithmic growth, $d(u_0(z), u_1(z)) + \epsilon \log |z| \rightarrow -\infty$ as $z \rightarrow 0$ in $\mathbb{D}^j$ by the triangle inequality. Thus, by Lemma 3.2, $d^2(u_0, u_1)$ is bounded in $\mathbb{D}^{j*}$ and extends as a subharmonic function on $\mathbb{D}^j$. Hence, we conclude that $d^2(u_0, u_1)$ extends as a subharmonic function on $\tilde{\mathcal{R}}$, thereby proving Lemma 5.4. $\square$

For $u_0, u_1 \in \mathcal{H}_\rho$, let $\tilde{u}_0, \tilde{u}_1$ be the associated ρ -equivariant maps. For $s \in [0, 1]$, define $u_s : \mathcal{R} \rightarrow \tilde{\mathcal{R}} \times_\rho \tilde{X}$ to be the associated section of the ρ -equivariant map

$$\tilde{u}_s : \tilde{\mathcal{R}} \rightarrow \tilde{X}, \quad \tilde{u}_s(z) = (1-s)\tilde{u}_0(z) + s\tilde{u}_1(z) \quad (5.2)$$

where the sum on the right-hand side above denotes geodesic interpolation (cf. Notation 2.2). Since $\tilde{u}_0$ and $\tilde{u}_1$ are ρ -equivariant, $\tilde{u}_s$ is also ρ -equivariant.

From Lemmas 5.3 and 5.4, the convexity of the distance function and the convexity of energy (cf. [10, (2.2vi)]), it follows that $u_s|_{\mathbb{D}^{j*}}$ also satisfies the properties (i), (ii) and (iii) of Theorem 3.1 for all $s \in [0, 1]$; i.e.,

- $E^{u_s}[\mathbb{D}_{r,r_0}^j] \leq C + \frac{\Delta_j^2}{2\pi} \log \frac{r_0}{r}$
- $\lim_{r \rightarrow 0} E^{u_s|_{\partial \mathbb{D}_r^j}}[\mathbb{S}^1] = \frac{\Delta_j^2}{2\pi}$
- u_s has sub-logarithmic growth.

Lemma 5.5 *Let $u_0, u_1 \in \mathcal{H}_\rho$ and $\epsilon > 0$. For any $\rho_0 \in (0, 1)$, there exists $r_0 \in (0, \rho_0)$ such that*

$$\frac{1}{-\log r_0} \sum_{j=1}^n \int_0^{2\pi} d^2(u_0|_{\partial \mathbb{D}_{r_0}^j}, u_s|_{\partial \mathbb{D}_{r_0}^j}) d\theta < \epsilon, \quad \forall s \in [0, 1].$$

Proof It suffices to prove Lemma 5.5 for $s = 1$. Let $\rho_0 > 0$ be given. First, Lemma 3.8 asserts that there exists $\rho_1 \in (0, \rho_0)$ such that

$$\frac{1}{-\log r_0} \sum_{j=1}^n \int_0^{2\pi} d^2(u_1|_{\partial \mathbb{D}_{r_0}^j}, u_1|_{\partial \mathbb{D}_{r_0}^j}) d\theta < \frac{\epsilon}{4}, \quad \forall r_0 \in (0, \rho_1).$$

For $c > 0$ as in Lemma 5.4, choose $r_0 \in (0, \rho_1)$ such that $\frac{2\pi nc}{-\log r_0} < \frac{\epsilon}{4}$. Then

$$\frac{1}{-\log r_0} \sum_{j=1}^n \int_0^{2\pi} d^2(u_0|_{\partial \mathbb{D}_{r_0}^j}, u_1|_{\partial \mathbb{D}_{r_0}^j}) d\theta = \frac{2\pi nc}{-\log r_0} < \frac{\epsilon}{4}.$$

The inequality of Lemma 5.5 for $s = 1$ follows from the above two inequalities and the triangle inequality. $\square$

Lemma 5.6 *Let $u_0, u_1 \in \mathcal{H}_\rho$ and $s \in [0, 1]$. For $\epsilon > 0$, there exists $\rho_0 > 0$ sufficiently small such that*

$$-\log r_0 \sum_{j=1}^n \left(E^{u_0|_{\partial \mathbb{D}_{r_0}^j}}[\mathbb{S}^1] + E^{u_s|_{\partial \mathbb{D}_{r_0}^j}}[\mathbb{S}^1] \right) < -2 \log r_0 \sum_{j=1}^n \frac{\Delta_j^2}{2\pi} + \epsilon, \quad 0 < r_0 < \rho_0.$$

Proof For $s = 1$, Lemma 5.6 follows from Lemma 3.7. The general case of $s \in [0, 1]$ follows immediately by convexity of energy (cf. [10, (2.2vi)]). $\square$

Lemma 5.7 For $u_0, u_1 \in \mathcal{H}_\rho$ and $s \in [0, 1]$, we have $E^{u_0}(0) = E^{u_s}(0)$.

Proof It suffices to prove Lemma 5.7 for $s = 1$. Assume on the contrary that $E^{u_0}(0) \neq E^{u_1}(0)$. By changing the role of u_0 and u_1 if necessary, we can assume $E^{u_0}(0) < E^{u_1}(0)$. Let $\rho_0 > 0$ smaller of the ρ_0 in Lemmas 5.5 and 5.6. Choose $\epsilon > 0$ and $r_0 \in (0, \rho_0]$ such that

$$E^{u_0}(r_0) < E^{u_1}(r_0) - 2\epsilon$$

which implies (cf. Definition 5.1)

$$E^{u_0}[\mathcal{R}_{r_0}] < E^{u_1}[\mathcal{R}_{r_0}] - 2\epsilon. \quad (5.3)$$

Next fix $r_1 \in (0, r_0)$. Let $\tilde{u}_0, \tilde{u}_1 : \tilde{\mathcal{R}} \rightarrow \tilde{X}$ be the ρ -equivariant maps associated to sections u_0, u_1 . For the fixed conformal disk $\mathbb{D}^j \subset \tilde{\mathcal{R}}$ centered at the puncture p^j (cf. (2.3)), let $\tilde{\mathbb{D}}_{r_1, r_0}^j \subset \tilde{\mathcal{R}}$ the lift of $\mathbb{D}_{r_1, r_0}^j \subset \mathcal{R}$. We define a “bridge” between map $\tilde{u}_0$ and $\tilde{u}_1$ by setting

$$\tilde{b} : \bigcup_{j=1}^n \tilde{\mathbb{D}}_{r_1, r_0}^j \rightarrow \tilde{X}$$

to be the geodesic interpolation $(1-t)U_0(\theta) + tU_1(\theta)$ where

$$t = \frac{\log |z| - \log r_0}{\log r_1 - \log r_0}, \quad U_0(\theta) = \tilde{u}_0(r_0, \theta) \quad \text{and} \quad U_1(\theta) = \tilde{u}_1(r_1, \theta).$$

In other words,

$$\tilde{b}(r, \theta) = \frac{\log r_1 - \log r}{\log r_1 - \log r_0} \tilde{u}_0(r_0, \theta) + \frac{\log r - \log r_0}{\log r_1 - \log r_0} \tilde{u}_1(r_1, \theta)$$

for $(r, \theta) \in \tilde{\mathbb{D}}_{r_1, r_0}^j, j = 1, \dots, n$. Let

$$b : \bigcup_{j=1}^n \mathbb{D}_{r_1, r_0}^j \rightarrow \tilde{\mathcal{R}} \times_\rho \tilde{X}$$

be the local section associated with $\tilde{b}$.

The CAT(0) condition implies (by an argument analogous to the proof of the bridge lemma [11, Lemma 3.12])

$$\begin{aligned} E^b[\bigcup_{j=1}^n \mathbb{D}_{r_1, r_0}^j] &\leq \frac{1}{2} \log \frac{r_0}{r_1} \sum_{j=1}^n \left(E^{u_0|_{\partial \mathbb{D}_{r_0}^j}}[\mathbb{S}^1] + E^{u_1|_{\partial \mathbb{D}_{r_1}^j}}[\mathbb{S}^1] \right) \\ &\quad + \frac{1}{\log \frac{r_0}{r_1}} \sum_{j=1}^n \int_0^{2\pi} d^2(u_0|_{\partial \mathbb{D}_{r_0}^j}, u_1|_{\partial \mathbb{D}_{r_1}^j}) d\theta. \end{aligned}$$

Choose $r_1 = r_0^2$ (cf. Lemmas 5.5 and 5.6) to obtain

$$E^b[\bigcup_{j=1}^n \mathbb{D}_{r_0^2, r_0}^j] < -\log r_0 \sum_{j=1}^n \frac{\Delta_j^2}{2\pi} + 2\epsilon. \quad (5.4)$$

Since the section

$$h : \mathcal{R}_{r_0^2} \rightarrow \tilde{\mathcal{R}} \times_{\rho} \tilde{X}$$

defined by setting

$$h = \begin{cases} u_0 & \text{in } \mathcal{R}_{r_0} \\ b & \text{in } \bigcup_{j=1}^n \mathbb{D}_{r_0^2, r_0}^j \end{cases}$$

is a competitor for u_1 , we have

$$\begin{aligned} E^{u_1}[\mathcal{R}_{r_0^2}] &\leq E^h[\mathcal{R}_{r_0^2}] \\ &= E^{u_0}[\mathcal{R}_{r_0}] + E^b[\bigcup_{j=1}^n \mathbb{D}_{r_0^2, r_0}^j] \\ &< E^{u_1}[\mathcal{R}_{r_0}] - 2\epsilon + E^b[\bigcup_{j=1}^n \mathbb{D}_{r_0^2, r_0}^j] \quad (\text{by (5.3)}) \\ &< E^{u_1}[\mathcal{R}_{r_0}] - \log r_0 \sum_{j=1}^n \frac{\Delta_j^2}{2\pi} \quad (\text{by (5.4)}). \end{aligned}$$

Thus,

$$E^{u_1}[\bigcup_{j=1}^n \mathbb{D}_{r_0^2, r_0}^j] < -\log r_0 \sum_{j=1}^n \frac{\Delta_j^2}{2\pi}.$$

This contradicts (5.1) and proves Lemma 5.7. $\square$

Lemma 5.8 *For $u_0, u_1 \in \mathcal{H}_{\rho}$, there exists a constant c such that*

$$\begin{aligned} d(u_s(p), u_1(p)) &\equiv cs, \forall p \in \mathcal{R} \\ |(u_s)_*(V)|^2(p) &= |(u_0)_*(V)|^2(p), \text{ for a.e. } s \in [0, 1], p \in \mathcal{R}, V \in T_p \tilde{\mathcal{R}}. \end{aligned} \quad (5.5)$$

Proof By the convexity of energy (cf. [10, (2.2vi)]),

$$E^{u_s}[\mathcal{R}_r] \leq (1-s)E^{u_0}[\mathcal{R}_r] + sE^{u_1}[\mathcal{R}_r] - s(1-s) \int_{\mathcal{R}_r} |\nabla d(u_0, u_1)|^2 d\text{vol}_M$$

for any $r > 0$. Letting $r \rightarrow 0$ and applying Lemma 5.7, we conclude

$$0 = \int_{\mathcal{R}} |\nabla d(u_0, u_1)|^2 d\text{vol}_M \quad (5.7)$$

$$E^{u_s} = E^{u_0}, \quad \forall s \in [0, 1]. \quad (5.8)$$

First, (5.7) implies that $\nabla d(u_0, u_1) = 0$ a.e. in M which in turn implies $d(u_0, u_1) \equiv c$ for some c . By the definition of the map u_s , (5.5) follows immediately.

Next, for $\{P, Q, R, S\} \subset \tilde{X}$, the quadrilateral comparison for CAT(0) spaces (cf. [16]) implies

$$d^2(P_t, Q_t) \leq (1-t)d^2(P, Q) + td^2(R, S)$$

where $P_t = (1-t)P + tS$ and $Q_t = (1-t)Q + tR$. Applying the above inequality with $P = u_0(p)$, $Q = u_1(p)$, $S = u_1(\exp_p(tV))$ and $R = u_0(\exp_t(tV))$ where $t > 0$ and $V \in T_p\tilde{M}$, dividing by t and letting $t \rightarrow 0$, we obtain (cf. [10, Theorem 1.9.6])

$$|(u_s)_*(V)|^2(p) \leq (1-s)|(u_0)_*(V)|^2(p) + s|(u_1)_*(V)|^2(p), \quad \text{a.e. } p \in \tilde{M}, V \in T_p\tilde{M}.$$

Integrating the above over all unit vectors $V \in T_p\tilde{M}$ and then over $p \in F$, we obtain

$$E^{u_s} \leq (1-s)E^{u_0} + sE^{u_1}.$$

Combining this with (5.8) implies (5.6). $\square$

Proof of Theorem 1.2 We assume there exist $u_0, u_1 \in \mathcal{H}_\rho$ such that $u_0 \not\equiv u_1$ and treat the different cases of Theorem 1.2 separately.

- *$\tilde{X}$ is a negatively curved space:* For $p \in \tilde{\mathcal{R}}$, choose $\kappa > 0$ and $R > 0$ such that $B_R(u_0(p))$ is a CAT($-\kappa$) space. Let $\mathcal{U}$ be a sufficiently small neighborhood of p and $s_0 \in [0, 1]$ sufficiently small such that $\tilde{u}_s(\mathcal{U}) \subset B_R(u(p))$ for $s \in [0, s_0]$. Thus, if $\tilde{u}_{s_0}(p) \neq \tilde{u}_0(p)$, then applying [14, Sect. 5] implies that the image under u_0 of a sufficiently small neighborhood of p is contained in an image $\sigma(\mathbb{R})$ of a geodesic line. (If $\tilde{X}$ is a smooth manifold, this follows by Hartman [8]). If $\tilde{u}_0 \neq \tilde{u}_1$, then $\tilde{u}_s(p) \neq \tilde{u}_0(p)$ for all $p \in \tilde{\mathcal{R}}$ and $s \in (0, s_0]$ by (5.5). Thus, we conclude that $\tilde{u}_0(\tilde{\mathcal{R}})$ is contained in $\sigma(\mathbb{R})$. Consequently, $\rho(\pi_1(\tilde{\mathcal{R}}))$ fixes $\sigma(\mathbb{R})$ which contradicts the fact that $\rho(\pi_1(\mathcal{R}))$ satisfies assumption (B).
- *$\tilde{X}$ is an irreducible symmetric space of non-compact type or a locally finite Euclidean building:* For these two target spaces, the conclusion of Lemma 5.8 is the same as that of [4, Lemma 3.1]. Thus, we can apply the arguments of [4, Sects. 3.2 and 3.3].

$\square$

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References

1. Bridson, M.R., Haefliger, A.: *Metric Spaces of Non-positive Curvature*. Springer, Berlin (1999)
2. Brotbek, D., Daskalopoulos, G., Deng, Y., Mese, C.: Representations of fundamental groups and logarithmic symmetric differential forms. arXiv Preprint [arXiv:2206.11835](https://arxiv.org/abs/2206.11835)
3. Corlette, K.: Flat G -bundles with anonical metrics. *J. Differ. Geom.* **28**, 361–382 (1988)
4. Daskalopoulos, G., Mese, C.: Uniqueness of equivariant harmonic maps to symmetric spaces and buildings. Accepted for publication, *Mathematical Research Letters*. [arXiv:2111.11422](https://arxiv.org/abs/2111.11422)
5. Daskalopoulos, G., Mese, C.: Infinite energy harmonic maps from Kähler surface to CAT(0) spaces. Preprint
6. Daskalopoulos, G., Mese, C.: Pluriharmonic maps from quasi-projective varieties into symmetric spaces. Preprint
7. Gromov, M., Schoen, R.: Harmonic maps into singular spaces and p -adic superrigidity for lattices in groups of rank one. *Publ. Math. IHES* **76**, 165–246 (1992)
8. Hartman, P.: On homotopic harmonic maps. *Can. J. Math.* **19**, 673–687 (1967)
9. Hayman, W.K., Kennedy, P.B.: *Harmonic Functions*, London Mathematical Society Monographs, Vol. 1, No. 9. Academic Press, London (1976)
10. Korevaar, N., Schoen, R.: Sobolev spaces and harmonic maps for metric space targets. *Commun. Anal. Geom.* **1**, 561–659 (1993)
11. Korevaar, N., Schoen, R.: Global existence theorem for harmonic maps to non-locally compact spaces. *Commun. Anal. Geom.* **5**, 333–387 (1997)
12. Koziarz, V., Maubon, J.: Harmonic maps and representations of non-uniform lattices of $PU(m, 1)$. *Annales de l'Institut Fourier* **58**, 507–558 (2008)
13. Lohkamp, J.: An existence theorem for harmonic maps. *Manuscripta Math.* **67**, 21–23 (1990)
14. Mese, C.: Uniqueness theorems for harmonic maps into metric spaces. *Commun. Contemp. Math.* **4**, 725–750 (2002)
15. Mochizuki, T.: Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D -modules. *Memoirs of the AMS* **185** (2007)
16. Reshetnyak, Y.G.: Nonexpanding maps in a space of curvature no greater than K . *Siberian Math. J.* **9**, 918–927 (1968)
17. Wolf, M.: Infinite energy harmonic maps and degeneration of hyperbolic surfaces in moduli spaces. *J. Differ. Geom.* **33**, 487–539 (1991)

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