

THE DELANNOY CATEGORY

NATE HARMAN, ANDREW SNOWDEN, and NOAH SNYDER

Abstract

Let G be the group of all order-preserving self-maps of the real line. In previous work, the first two authors constructed a pre-Tannakian category $\underline{\text{Rep}}(G)$ associated to G . The present paper is a detailed study of this category, which we name the Delannoy category. We classify the simple objects, determine branching rules to open subgroups, and give a combinatorial rule for tensor products. The Delannoy category has some remarkable features: it is semisimple in all characteristics, all simples have categorical dimension ± 1 , and the Adams operations on its Grothendieck group are trivial. We also give a combinatorial model for $\underline{\text{Rep}}(G)$ based on Delannoy paths.

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1. Introduction

Pre-Tannakian tensor categories provide a natural generalization of representation categories of algebraic groups or supergroups. A topic of recent interest is to understand to what extent pre-Tannakian categories go beyond those coming from algebraic supergroups. An important theorem of Deligne [16] states that any pre-Tannakian category of moderate growth in characteristic 0 comes from an algebraic supergroup, and

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there has been much work devoted to generalizing this result to positive characteristic (see, e.g., [2], [3], [8]–[11], [22]–[24], [38]). Deligne [17] also introduced the first examples of pre-Tannakian categories not associated to algebraic supergroups: the interpolation categories $\underline{\text{Rep}}(\mathfrak{S}_t)$, $\underline{\text{Rep}}(\mathbf{GL}_t)$, $\underline{\text{Rep}}(\mathbf{O}_t)$ (see [6], [7], [20], [28], [29], [33], [34] for subsequent work).

In recent work [30], the first two authors introduced a general construction of pre-Tannakian categories associated to oligomorphic groups with an appropriate measure. In the case of the infinite symmetric group, this construction recovers Deligne’s category $\underline{\text{Rep}}(\mathfrak{S}_t)$. The construction also applies to the oligomorphic group $G = \text{Aut}(\mathbf{R}, <)$ of order-preserving self-bijections of the real line. The resulting category $\underline{\text{Rep}}(G)$ is a fundamentally new example of a tensor category: for example, in positive characteristic, it is the first known example of a semisimple pre-Tannakian category that is not of moderate growth. We note that [30] is mostly concerned with developing the general theory of pre-Tannakian categories associated to oligomorphic groups, and proved little about $\underline{\text{Rep}}(G)$ in particular.

This article is a detailed study of the category $\underline{\text{Rep}}(G)$. We christen this the *Delannoy category*, for reasons that will be shortly apparent. Our results show that it is a remarkable object in possession of a number of features distinguishing it from other known pre-Tannakian categories. In the rest of the introduction, we briefly describe what $\underline{\text{Rep}}(G)$ is, and then explain a number of our results about it.

1.1. The Delannoy category

From here on, we fix a field k and write G for $\text{Aut}(\mathbf{R}, <)$. We now give a brief discussion of what $\underline{\text{Rep}}(G)$ is. We hope this provides the reader with enough of a sense of the category to appreciate our main results. A more detailed discussion of the construction of $\underline{\text{Rep}}(G)$ is provided in Sections 2 and 3.

Let $\mathbf{R}^{(n)}$ be the subset of $\mathbf{R}^n$ consisting of tuples $(x_1, \dots, x_n)$ with $x_1 < \dots < x_n$. The group G acts transitively on $\mathbf{R}^{(n)}$. We say that a function $\varphi: \mathbf{R}^{(n)} \rightarrow k$ is a *Schwartz function* if it has finitely many level sets, all of which can be defined by first-order formulas using $<$, $=$, and finitely many real constants; equivalently, this means that the subgroup of G stabilizing φ is open, with respect to the natural topology on G . We define the *Schwartz space* $\mathcal{C}(\mathbf{R}^{(n)})$ to be the k -vector space of all Schwartz functions.

A concept of fundamental importance to this paper is integration with respect to Euler characteristic, as developed by Schapira and Viro (see, e.g., [47]). Precisely, if φ is a Schwartz function on $\mathbf{R}^{(n)}$ that takes the values $a_1, \dots, a_r$, then we define

$$\int_{\mathbf{R}^{(n)}} \varphi(x) dx = \sum_{i=1}^r a_i \cdot \chi_c(\varphi^{-1}(a_i)),$$

where χ_c is the Euler characteristic of compactly supported cohomology. The value of the integral belongs to k . For example, if $\varphi \in \mathcal{C}(\mathbf{R})$ is the characteristic function of an open interval, then its integral is -1 .

Suppose that Ψ is a Schwartz function on $\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}$ (defined in the obvious manner). We can then define an integral operator

$$A_\Psi: \mathcal{C}(\mathbf{R}^{(m)}) \rightarrow \mathcal{C}(\mathbf{R}^{(n)}), \quad (A_\Psi \varphi)(x) = \int_{\mathbf{R}^{(m)}} \Psi(x, y) \varphi(y) dy.$$

The composition of two such integral operators is an integral operator of the same kind. (One can think of Ψ as the matrix of the operator A_Ψ . The composition of two operators is given by an analogue of matrix multiplication.)

We can use the above constructions to define a category $\underline{\text{Perm}}(G)$ of “permutation modules.” The objects are formal direct sums of $\mathcal{C}(\mathbf{R}^{(n)})$ ’s. The morphisms of basic objects are G -invariant integral operators. This category carries a symmetric tensor structure “ $\underline{\otimes}$ ” defined on the basic objects by

$$\mathcal{C}(\mathbf{R}^{(n)}) \underline{\otimes} \mathcal{C}(\mathbf{R}^{(m)}) = \mathcal{C}(\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}).$$

The space $\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}$ decomposes into G -orbits isomorphic to $\mathbf{R}^{(s)}$ (for various s), and the right side above is defined to be the corresponding sum of Schwartz spaces. The tensor category $\underline{\text{Perm}}(G)$ is rigid: in fact, every object is self-dual.

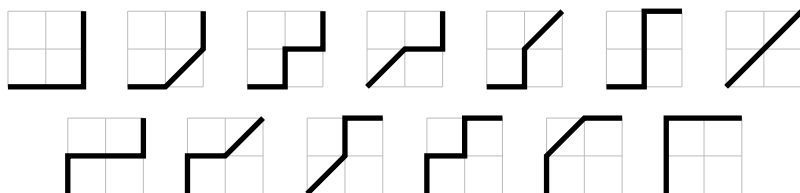
The category $\underline{\text{Rep}}^f(G)$ of finite length objects in $\underline{\text{Rep}}(G)$ is equivalent to the Karoubian envelope of $\underline{\text{Perm}}(G)$, and $\underline{\text{Rep}}(G)$ is equivalent to the ind-completion of $\underline{\text{Rep}}^f(G)$; these statements rely on nontrivial results of [30], but for the present purposes one can take them as definitions of $\underline{\text{Rep}}^f(G)$ and $\underline{\text{Rep}}(G)$. As we have already indicated, $\underline{\text{Rep}}^f(G)$ is a semisimple pre-Tannakian tensor category; this is again a nontrivial result of [30]. We emphasize that semisimplicity holding in all characteristics is rather remarkable.

1.2. The path model

Before stating our results, we give another description of the Delannoy category that highlights the relevant combinatorics. This description is new to this paper, and it is developed in more detail in Section 9.

An (n, m) -*Delannoy path* is a path in the plane from $(0, 0)$ to (n, m) composed of steps of the form $(1, 0)$, $(0, 1)$, and $(1, 1)$. The *Delannoy number* $D(n, m)$ is the number of (n, m) -Delannoy paths, and the *central Delannoy number* $D(n)$ is $D(n, n)$. For example, $D(2) = 13$ (see Figure 1). The Delannoy numbers are well known in the literature (see, e.g., [1]).

We introduce an algebra $\mathcal{D}(n)$, called the *Delannoy algebra*, which is modeled on Delannoy paths. The elements of $\mathcal{D}(n)$ are k -linear combinations of (n, n) -Delannoy

Figure 1. The thirteen $(2, 2)$ -Delannoy paths.

paths; we write $[p]$ for the basis element of $\mathcal{D}(n)$ corresponding to the path p . Suppose that p_1 , p_2 , and p_3 are three (n, n) -Delannoy paths. It turns out that there is at most one 3-dimensional (n, n, n) -Delannoy path q whose three projections are p_1 , p_2 , and p_3 . (See Section 9.1 for the definition of higher dimensional paths.) We define

$$\epsilon(p_1, p_2, p_3) = \begin{cases} (-1)^{\ell(q) + \ell(p_3)} & \text{if } q \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\ell(q)$ is the length of q . Multiplication in $\mathcal{D}(n)$ is then defined by

$$[p_1] \cdot [p_2] = \sum_{p_3} \epsilon(p_1, p_2, p_3) [p_3].$$

This product is associative and unital. We show that $\mathcal{D}(n)$ is isomorphic to the endomorphism algebra of $\mathcal{C}(\mathbf{R}^{(n)})$ in $\underline{\text{Rep}}(G)$; in particular, the dimension of this endomorphism algebra is the central Delannoy number $D(n)$. The key point here is that G -orbits on $\mathbf{R}^{(n)} \times \mathbf{R}^{(n)}$ are naturally parameterized by (n, n) -Delannoy paths (see Proposition 3.5).

More generally, define $\mathcal{D}(n, m)$ to be the space spanned by (n, m) -Delannoy paths. A similar combinatorial rule to the above defines a composition

$$\mathcal{D}(n, m) \times \mathcal{D}(m, \ell) \rightarrow \mathcal{D}(n, \ell).$$

We define a category with objects indexed by natural numbers and where the Hom sets are $\mathcal{D}(n, m)$. We show that the additive envelope of this category is $\underline{\text{Perm}}(G)$. We also explain how to construct the tensor product from the combinatorial point of view. (See Section 9 for details.)

1.3. Main results

Having given some sense of $\underline{\text{Rep}}(G)$, we now explain our results.

(a) *Classification of simples.* A *weight* is a word in the alphabet $\{\bullet, \circ\}$. Given a weight λ of length n , we define a submodule L_λ of $\mathcal{C}(\mathbf{R}^{(n)})$ by writing down an explicit family of Schwartz functions and taking the submodule they generate. We

show that the L_λ 's are exactly the simple objects of $\underline{\text{Rep}}(G)$ (Theorem 4.3, Corollary 4.12).

(b) *Decomposition of Schwartz space.* We determine the simple decomposition of $\mathcal{C}(\mathbf{R}^n)$ (Theorem 4.7). In particular, we find that it has length 3^n . This decomposition gives a representation-theoretic meaning to a classical formula for the central Delannoy number $D(n)$ (Remark 4.9).

(c) *Projectors and dimension.* Let λ be a weight of length n . From (b), it follows that L_λ has multiplicity 1 in $\mathcal{C}(\mathbf{R}^n)$. We determine the projection operator $\mathcal{C}(\mathbf{R}^n) \rightarrow L_\lambda$ explicitly (Proposition 5.6). As a corollary, we compute the categorical dimension of L_λ : it is $(-1)^n$ (Corollary 5.7). It is a remarkable feature of $\underline{\text{Rep}}(G)$ that every simple object has dimension ± 1 . Outside of some trivial cases, the only other pre-Tannakian category we know with this property is the representation category of the supergroup $\text{SOSp}(1, 2)$. (We thank the referee for this observation.)

(d) *Branching rules.* Let $G(0) \subset G$ be the stabilizer of $0 \in \mathbf{R}$, which is an open subgroup of G . There are induction and restriction operations between G and $G(0)$. A thematic problem in this situation is to determine the branching rules, that is, the simple decompositions of the induction or restriction of simple objects. We solve this problem completely. A convenient feature of the present situation is that $G(0)$ is isomorphic to $G \times G$, so we know exactly what its irreducible representations are.

To give a little more detail, we consider the induction

$$\text{Ind}_{G(0)}^G(L_\lambda \boxtimes L_\mu)$$

and completely describe its decomposition into simples (Theorem 6.1); an interesting phenomenon here is that this induction always has length 3. Similarly, we consider the restriction

$$\text{Res}_{G(0)}^G(L_\lambda)$$

and completely determine its decomposition into simples (Theorem 6.9). We note that one can deduce the branching rules between G and any open subgroup from the $G(0)$ case. The computation of restriction plays an especially important role in the remainder of the paper; see Example 6.11 for a demonstration of how it can be a powerful tool.

(e) *Tensor products.* Another thematic problem throughout representation theory is to describe the tensor product of irreducible representations. We solve this problem completely for $\underline{\text{Rep}}(G)$. Precisely, given weights λ and μ , we consider the tensor product

$$L_\lambda \underline{\otimes} L_\mu$$

and give an explicit combinatorial rule for its simple decomposition (Theorem 7.2). Roughly speaking, the simples that appear are indexed by weights obtained by shuffling the words λ and μ ; however, in these shuffles we allow letters to collide, and there are some rules for what happens in this case.

(f) *The Grothendieck group.* Let K be the Grothendieck group of the category $\underline{\text{Rep}}^f(G)$. We let $a_\lambda \in K$ be the class of the simple L_λ ; these elements form a $\mathbf{Z}$ -basis for K . The tensor product on $\underline{\text{Rep}}(G)$ endows K with the structure of a commutative ring. The restriction functor

$$\text{Res}: \underline{\text{Rep}}(G) \rightarrow \underline{\text{Rep}}(G(0)) \cong \underline{\text{Rep}}(G \times G)$$

defines a (non-cocommutative) comultiplication on K . We show that this gives K the structure of a Hopf algebra (see Section 6.3). The Grothendieck group K carries a natural ascending filtration, which is compatible with the Hopf algebra structure. We show that the associated graded is isomorphic (as a Hopf algebra) to the shuffle algebra (see Section 7.6). In particular, we find that $K \otimes \mathbf{Q}$ is a polynomial ring on the a_λ , where λ is a Lyndon word (Corollary 7.21).

(f) *Adams operations.* The Grothendieck group K naturally carries the structure of a λ -ring, and thus admits Adams operations ψ^n . We prove that these operations are trivial, that is, ψ^n is the identity for all n (Theorem 8.2); this implies that K is a binomial ring (Corollary 8.3). As a consequence, it is particularly easy to compute the action of Schur functors on $\underline{\text{Rep}}(G)$ (see Section 8.5). The triviality of all Adams operations is another remarkable feature of $\underline{\text{Rep}}(G)$: the only semisimple pre-Tannakian categories we know with this property are $\underline{\text{Rep}}(G^n)$ with $n \in \mathbf{N}$, and Czenky has recently shown that ψ^2 is nontrivial in any nontrivial symmetric fusion category of characteristic not equal to 2 (see [13, Theorem 4.8]).

(g) *The path model.* As stated above, we show that $\underline{\text{Rep}}(G)$ is equivalent to a category defined using the combinatorics of Delannoy paths (Theorem 9.8).

1.4. Connection to other work

This paper relates to a few other topics.

- Representations of other oligomorphic groups, such as the infinite symmetric group, coincide with Deligne's interpolation categories. There is a substantial literature on them, such as [6], [7], [15]–[17], [20], [28], [29], [33], and [34].
- Let $\mathbf{OI}$ be the category of finite totally ordered sets and monotonic injections. There is a natural functor $\mathbf{OI}^{\text{op}} \rightarrow \underline{\text{Perm}}(G)$ given on objects by $[n] \mapsto \mathcal{C}(\mathbf{R}^{(n)})$. It follows that $\mathbf{OI}$ -modules, that is, functors $\mathbf{OI} \rightarrow \text{Vec}$, are closely related to $\underline{\text{Rep}}(G)$. These have been studied in [26], [27], [39], and [41].
- Our work studies a modified notion of representation for the group G . Ordinary representations of G have been studied in [18], [36], and [46].

- The category $\underline{\text{Rep}}(G)$ is closely connected to the combinatorics of Delannoy paths. These have been studied extensively in the literature, for example, [1], [5], [12], [37], [42], [43], and [44].

1.5. Outline

We give an outline of the paper.

- In Section 2, we briefly review the general theory developed in [30].
- In Section 3, we specialize the discussion from Section 2 to the case of $\text{Aut}(\mathbf{R}, <)$.
- In Section 4, we classify the simple objects of $\underline{\text{Rep}}(G)$ and determine the simple decomposition of the Schwartz space $\mathcal{C}(\mathbf{R}^{(n)})$.
- In Section 5, we analyze the invariant spaces of simple objects under open subgroups, determine the projection operator from $\mathcal{C}(\mathbf{R}^{(n)})$ to L_λ , and compute the categorical dimension of L_λ .
- In Section 6, we study the induction and restriction operations between G and its open subgroup $G(0) \cong G \times G$. We also show that K is a Hopf algebra.
- In Section 7, we give a combinatorial rule for the tensor product in $\underline{\text{Rep}}(G)$ and show that $\text{gr}(K)$ is isomorphic to the shuffle algebra.
- In Section 8, we prove that the Adams operations are trivial on the Grothendieck group.
- In Section 9, we discuss the path model for $\underline{\text{Rep}}(G)$.

1.6. Notation

We list some of the important notation here:

k	coefficient ring (usually a field)
$\mathbb{1}$	unit object of a tensor category (e.g., the trivial representation)
$\mathbf{1}$	one-point set
$D(n, m)$	(n, m) -Delannoy number (see Section 1.2)
$\mathcal{C}(X)$	Schwartz space of X (see Section 2.2)
$\underline{\text{Perm}}(G)$	category of permutation modules (see Section 2.3)
$A(G)$	completed group algebra of G (see Section 2.4)
$\underline{\text{Rep}}(G)$	category of smooth $A(G)$ -modules (see Section 2.4)
G	group $\text{Aut}(\mathbf{R}, <)$ (from Section 3 onward)
$G(A)$	subgroup of G fixing the set A pointwise
$\mathbf{R}^{(n)}$	set of $x \in \mathbf{R}^n$ with $x_1 < \cdots < x_n$ (see Section 3.1)
$\text{vol}(X)$	volume of X under the principal measure (see Section 3.2)
$\underline{G}$	another notation for $A(G)$ (more or less; see Section 3.3)
K	Grothendieck group of $\underline{\text{Rep}}^f(G)$ (see Section 3.5)
res	restriction map on K (see Section 3.5)

ind	induction map on K (see Section 3.5)
λ	weight, that is, a word in the alphabet $\{\bullet, \circ\}$ (see Section 4.1)
L_λ	simple of weight λ (see Definition 4.2)
a_λ	class in K of the simple L_λ (see Definition 4.2)
$\odot$	concatenation product on K (see Section 4.6)
$\lambda[i, j]$	substring of λ (see Section 6.2)
$\pi(n)$	length- n weight with all letters $\bullet$ (see Section 7.2)

2. Generalities on oligomorphic groups

In this section, we review the construction and basic properties of the category $\text{Rep}(G)$ for a general oligomorphic group G . This discussion is a summary of the main points of [30], except for Section 2.5, which is new.

2.1. Oligomorphic groups

An *oligomorphic group* is a pair (G, Ω) consisting of a group G and a set Ω equipped with a faithful action of G such that G has finitely many orbits on Ω^n for all n . For a finite subset A of Ω , let $G(A)$ be the subgroup of G fixing each element of A . The $G(A)$'s form a neighborhood basis for a topology on G . This topology has the following three properties (see [30, Proposition 2.4]).

- (a) It is Hausdorff.
- (b) It is non-Archimedean: open subgroups form a neighborhood basis of the identity.
- (c) It is Roelcke precompact: if U and V are open subgroups, then $U \backslash G / V$ is finite.

An *admissible group* is a topological group satisfying (a), (b), and (c). Thus every oligomorphic group comes with an admissible topology. Although we are most interested in oligomorphic groups, our constructions depend only on the topology and not the particular set Ω . For this reason, it is most natural to work with admissible groups.

Let G be an admissible group. We say that an action of G on a set is *smooth* if every stabilizer is open. We use the term G -set for a set equipped with a smooth action of G . We say that a G -set is *finitary* if it has finitely many orbits. (See [30, Section 2.3] for basic properties of G -sets.)

We define a $\hat{G}$ -set to be a set X equipped with a smooth action of some open subgroup of G , which we call the *group of definition* of X . Shrinking the group of definition does not change the $\hat{G}$ -set. Formally, the category of $\hat{G}$ -sets is the 2-colimit of the categories of U -sets for open subgroups U (see [30, Section 2.5] for additional details). The notion of finitary is well defined for $\hat{G}$ -sets. The symbol $\hat{G}$ does not have any rigorous meaning on its own, but intuitively we think of $\hat{G}$ as an infinitesimal neighborhood of the identity in G .

Example 2.1

Any finite group is admissible, under the discrete topology.

Example 2.2

Let $\mathfrak{S}$ be the infinite symmetric group, that is, the group of all permutations of $\Omega = \{1, 2, \dots\}$. This is oligomorphic. Let $\mathfrak{S}(n)$ be the subgroup of $\mathfrak{S}$ fixing each of $1, \dots, n$. The open subgroups of $\mathfrak{S}$ are those that contain some $\mathfrak{S}(n)$; in fact, every open subgroup is conjugate to one of the form $H \times \mathfrak{S}(n)$, where H is a subgroup of the finite symmetric group $\mathfrak{S}_n$ (see [30, Proposition 15.1]). The $\hat{\mathfrak{S}}$ -subsets of Ω are the finite and cofinite subsets. More generally, the $\hat{\mathfrak{S}}$ -subsets of Ω^n are those that can be defined by a first-order formula using equality and finitely many constants.

Example 2.3

Many oligomorphic groups can be constructed using Fraïssé limits; for instance, the automorphism group of the Rado graph is oligomorphic. (See [30, Section 6.2] for a quick summary, and [4] for a detailed treatment.)

2.2. Measures and integration

Fix an admissible group G and a commutative ring k . We now come to a key concept introduced in [30, Section 3.1].

Definition 2.4

A *measure* for G valued in k is a rule μ that assigns to every finitary $\hat{G}$ -set a quantity $\mu(X)$ in k such that the following axioms hold (in which X and Y denote finitary $\hat{G}$ -sets):

- (a) Isomorphism invariance: $\mu(X) = \mu(Y)$ if $X \cong Y$.
- (b) Normalization: $\mu(\mathbf{1}) = 1$, where $\mathbf{1}$ is the one-point $\hat{G}$ -set.
- (c) Conjugation invariance: $\mu(X^g) = \mu(X)$, where X^g is the conjugate of the $\hat{G}$ -set X by $g \in G$ (see [30, Section 2.5] for the definition of X^g).
- (d) Additivity: $\mu(X \amalg Y) = \mu(X) + \mu(Y)$.
- (e) Multiplicativity in fibrations: if $X \rightarrow Y$ is a map of transitive U -sets, for some open subgroup U , with fiber F (over some point), then $\mu(X) = \mu(F) \cdot \mu(Y)$.

We note that a measure automatically satisfies $\mu(X \times Y) = \mu(X) \cdot \mu(Y)$. (See [30, Section 3.1] for additional details.) There is a universal measure valued in a ring $\Theta(G)$. To construct $\Theta(G)$, start with the polynomial ring in symbols $[X]$, where X is a finitary G -set (up to isomorphism), and impose relations corresponding to (b)–(e) above. (See [30, Section 4] for details.) Computing $\Theta(G)$ is an important (and often difficult) problem.

Suppose now that we have a measure μ for G valued in k . We then obtain a theory of integration, as follows. Let X be a $\hat{G}$ -set. We say that a function $\varphi: X \rightarrow k$ is *smooth* if it is invariant under some open subgroup of G (contained in the group of definition for X), and we say φ has *finitary support* if its support is a finitary $\hat{G}$ -set. A *Schwartz function* on X is a smooth k -valued function on X with finitary support, and the space of all Schwartz functions is called *Schwartz space* and denoted $\mathcal{C}(X)$. Let φ be a Schwartz function on X . The definition ensures that φ attains only finitely many nonzero values, say $c_1, \dots, c_n$ (see [30, Proposition 3.6]). Let $A_i = \varphi^{-1}(c_i)$. We define the *integral* of φ by

$$\int_X \varphi(x) dx = \sum_{i=1}^n c_i \mu(A_i).$$

Integration defines a k -linear map $\mathcal{C}(X) \rightarrow k$. (See [30, Section 3.3] for more details.)

If $f: X \rightarrow Y$ is a smooth map of $\hat{G}$ -sets, then there is an associated pushforward map $f_*: \mathcal{C}(X) \rightarrow \mathcal{C}(Y)$ given by integrating over fibers. This satisfies the expected properties; for example, if $g: Y \rightarrow Z$ is a second map, then $(gf)_* = g_* f_*$. If X and Y are themselves finitary, then there is also a pullback map $f^*: \mathcal{C}(Y) \rightarrow \mathcal{C}(X)$. (See [30, Section 3.4] for more on pushforwards.)

Example 2.5

A finite group G admits a unique $\mathbf{Z}$ -valued measure via $\mu(X) = \#X$. This identifies $\Theta(G)$ with $\mathbf{Z}$.

Example 2.6

For each complex number t , there is a unique $\mathbf{C}$ -valued measure μ_t for $\mathfrak{S}$ with the property that $\mu_t(\Omega) = t$. This measure satisfies $\mu_t(\Omega^{(n)}) = \binom{t}{n}$, where $\Omega^{(n)}$ denotes the set of n -element subsets of Ω . A Schwartz function $\varphi: \Omega \rightarrow \mathbf{C}$ is simply one that is eventually constant. If $\varphi(n) = a$ for all $n > N$, then

$$\int_{\Omega} \varphi(x) dx = (t - N)a + \sum_{i=1}^N \varphi(i).$$

The ring $\Theta(G)$ is identified with the ring of integer-valued polynomials; in particular, the μ_t account for all complex measures. (See [30, Section 15] for more details on this case.)

2.3. Permutation representations

Fix an admissible group G and a k -valued measure μ . If X and Y are finitary $\hat{G}$ -sets, then a $Y \times X$ *matrix* A is a Schwartz function $A: Y \times X \rightarrow k$. We write $\text{Mat}_{Y,X}$ for the space of all $Y \times X$ matrices. We also write Mat_X for $\text{Mat}_{X,X}$, and Vec_X for

$\text{Mat}_{X,1}$. If A is a $Z \times Y$ matrix and B is a $Y \times X$ matrix, then we define the *product matrix* AB to be the $Z \times X$ matrix given by

$$AB(z, x) = \int_Y A(z, y)B(y, x) dy.$$

We note that AB is indeed a Schwartz function (see [30, Section 7.1]). Matrix multiplication gives Mat_X the structure of an associative and unital ring (see [30, Proposition 7.2]); the identity element is the *identity matrix* I_X , which is simply the characteristic function of the diagonal. Much of ordinary linear algebra can be generalized to this setting (see [30, Section 7]) for details.

We now introduce the category of “permutation modules” $\underline{\text{Perm}}(G)$; we refer to [30, Section 8] for details. The category is defined as follows.

- The objects are Vec_X , for X a finitary G -set.
- A morphism $\text{Vec}_X \rightarrow \text{Vec}_Y$ is a G -invariant $Y \times X$ matrix.
- Composition is given by matrix multiplication.

The category $\underline{\text{Perm}}(G)$ depends on the measure μ , even though this is absent from the notation. The category $\underline{\text{Perm}}(G)$ is naturally k -linear. It is also additive: we have

$$\text{Vec}_X \oplus \text{Vec}_Y = \text{Vec}_{X \sqcup Y}.$$

The structure maps here are given by the usual projection matrices.

The category $\underline{\text{Perm}}(G)$ carries a natural monoidal structure $\underline{\otimes}$. On objects, it is given by

$$\text{Vec}_X \underline{\otimes} \text{Vec}_Y = \text{Vec}_{X \times Y}.$$

On morphisms, it is given by the usual Kronecker product of matrices. The monoidal unit is $\mathbb{1} = \text{Vec}_1$. This monoidal structure is naturally symmetric; the symmetric structure is induced by the symmetric structure on the category of G -sets.

The tensor category $\underline{\text{Perm}}(G)$ is rigid. Every object is self-dual. The evaluation and coevaluation morphisms for Vec_X are both given by the characteristic function of the diagonal. The categorical trace of a morphism $A \in \text{Mat}_{X,X}^G$ is the “usual” matrix trace

$$\int_X A(x, x) dx.$$

In particular, the categorical dimension of Vec_X is $\mu(X)$. (See [21, Section 4.7] for the definition of categorical trace and dimension.)

Example 2.7

If G is a finite group, then $\underline{\text{Perm}}(G)$ is equivalent to the usual category of permutation G -modules.

2.4. General representations

Let G and μ be as above. The category $\underline{\text{Perm}}(G)$ is typically not abelian. We now explain how we can produce an abelian representation category $\underline{\text{Rep}}(G)$, which is well behaved in certain circumstances. We note that the category $\underline{\text{Perm}}(G)$ has a rather formal definition, while the category $\text{Rep}(G)$ will be an actual module category; relating these two perspectives requires nontrivial results from [30].

The *completed group algebra* of G , denoted $A(G)$, is the inverse limit of the Schwartz spaces $\mathcal{C}(G/U)$ over open subgroups U ; the transition map $\mathcal{C}(G/U) \rightarrow \mathcal{C}(G/V)$ is a pushforward along the map $G/U \rightarrow G/V$. Convolution endows $A(G)$ with the structure of an associative and unital algebra (see [30, Section 10.3]). The algebra structure on $A(G)$ depends on the measure μ , even though it is absent from the notation. An $A(G)$ -module M is called *smooth* if for every element x there is an open subgroup U such that the action of $A(G)$ on x factors through $\mathcal{C}(G/U)$. We define $\underline{\text{Rep}}(G)$ to be the full subcategory of $\text{Mod}_{A(G)}$ spanned by smooth modules. The category $\underline{\text{Rep}}(G)$ is always a Grothendieck abelian category (see [30, Proposition 10.7]).

In order for $\underline{\text{Rep}}(G)$ to be well behaved, we must make some assumptions on the measure μ . We say that μ is *regular* if $\mu(X)$ is a unit in k for every transitive G -set X , and *quasiregular* if there is an open subgroup U of G such that $\mu|_U$ is regular. The complex-valued measure μ_t in Example 2.6 is quasiregular for all t , and regular if and only if $t \notin \mathbb{N}$. There is another condition on measures used in [30] called *normal*, which is weaker than quasiregular, but we will not need it here. We assume in the following discussion that μ is quasiregular.

The most important objects of $\underline{\text{Rep}}(G)$ are the Schwartz spaces $\mathcal{C}(X)$, where X is a G -set. The $A(G)$ -module structure on $\mathcal{C}(X)$ is defined via convolution (see [30, Proposition 10.11]). The Schwartz spaces $\mathcal{C}(X)$ have a mapping property (see [30, Corollary 11.19]), and every object of $\underline{\text{Rep}}(G)$ is a quotient of some $\mathcal{C}(X)$ (see [30, Proposition 11.14]). There is a functor

$$\Phi: \underline{\text{Perm}}(G) \rightarrow \underline{\text{Rep}}(G), \quad \text{Vec}_X \mapsto \mathcal{C}(X)$$

that is fully faithful (see [30, Proposition 11.12]). Note that Vec_X and $\mathcal{C}(X)$ are the same vector space; we use different notation to emphasize that $\mathcal{C}(X)$ is thought of as in $\underline{\text{Rep}}(G)$.

Although we will not precisely define the completed group algebra or its action on $\mathcal{C}(X)$, we can give a direct description of submodules of $\mathcal{C}(X)$. Let φ belong to $\mathcal{C}(X)$, and suppose that φ is invariant under the open subgroup $U \subset G$. Given $\psi \in \mathcal{C}(G/U)$, we define $\psi * \varphi$ to be the function on X given by

$$(\psi * \varphi)(x) = \int_{G/U} \psi(g) \varphi(g^{-1}x) dg.$$

We then have the following criterion: a k -subspace V of $\mathcal{C}(X)$ is an $A(G)$ -submodule if and only if it is closed under the above convolutions; that is, given $\varphi \in V$ that is U -invariant and $\psi \in \mathcal{C}(G/U)$, the element $\psi * \varphi$ belongs to V . This also describes how to form the $A(G)$ -submodule of $\mathcal{C}(X)$ generated by a set of elements (take all such convolutions).

Since the Schwartz spaces generate $\underline{\text{Rep}}(G)$, there is at most one symmetric monoidal structure on $\underline{\text{Rep}}(G)$ that is compatible with Φ and cocontinuous in each variable. Such a symmetric monoidal structure is constructed in [30, Theorem 12.9], and we denote it by $\underline{\otimes}$. It satisfies $\mathcal{C}(X) \underline{\otimes} \mathcal{C}(Y) = \mathcal{C}(X \times Y)$ for arbitrary G -sets X and Y .

In fact, $\underline{\otimes}$ satisfies a mapping property: giving a map from $M \underline{\otimes} N$ is equivalent to giving a G -equivariant bilinear map from $M \times N$ that commutes with certain integrals (see [30, Section 12.2]). This permits us to give a more concrete description of the self-duality of $\mathcal{C}(X)$: the pairing $\mathcal{C}(X) \underline{\otimes} \mathcal{C}(X) \rightarrow \mathbb{1}$ is induced from the bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{C}(X) \times \mathcal{C}(X) \rightarrow k, \quad \langle \varphi, \psi \rangle = \int_X \varphi(x) \psi(x) dx.$$

This pairing will be used several times throughout this paper.

Remark 2.8

We warn the reader that the forgetful functor from $\underline{\text{Rep}}(G)$ to k -modules is not monoidal. For example, $\mathcal{C}(X) \underline{\otimes} \mathcal{C}(X) = \mathcal{C}(X \times X)$, which is typically larger than $\mathcal{C}(X) \otimes_k \mathcal{C}(X)$ since one usually cannot express the characteristic function of the diagonal as a finite sum of pure tensors. One should think of $\mathcal{C}(X) \underline{\otimes} \mathcal{C}(Y)$ as a kind of completion of the algebraic tensor product.

We recall the following general definition.

Definition 2.9

Suppose that k is a field. We say that a k -linear symmetric monoidal category $\mathcal{C}$ is *pre-Tannakian* if it is abelian, all objects have finite length, all Hom spaces are finite-dimensional, all objects have duals, and $\text{End}_{\mathcal{C}}(\mathbb{1}) = k$, where $\mathbb{1}$ is the unit object.

The following is the most important general theorem on the structure of $\underline{\text{Rep}}(G)$. It uses the somewhat technical condition *Property (P)* on the measure μ . This property essentially means that μ is valued in a subring of k that maps to enough fields of positive characteristic. Since the property holds in all cases of interest to this paper, we do not bother to explain it in detail (see [30, Definition 7.17] for details).

THEOREM 2.10 ([30, Theorem 13.2])

Assume that k is a field, μ is quasiregular, and Property (P) holds.

- (a) Every object of $\underline{\text{Rep}}(G)$ is the union of its finite length subobjects.
- (b) The category $\underline{\text{Rep}}^f(G)$ of finite length objects is pre-Tannakian.
- (c) If μ is regular, then $\underline{\text{Rep}}^f(G)$ is semisimple.

Remark 2.11

One can view part (c) above as an analogue of Maschke's theorem.

As a corollary of the theorem, we find that in the regular case one can abstractly recover the category $\underline{\text{Rep}}^f(G)$ without going through the completed group algebra.

COROLLARY 2.12

If μ is regular and satisfies (P), and k is a field, then $\underline{\text{Rep}}^f(G)$ is the Karoubian envelope of $\underline{\text{Perm}}(G)$.

2.5. Induction and restriction

Let G be an admissible group equipped with a quasiregular measure μ , and let U be an open subgroup of G . The measure μ restricts to a quasiregular measure on U (see [30, Proposition 3.23]), and there is a restriction functor (see [30, Section 10.5])

$$\text{Res}_U^G: \underline{\text{Rep}}(G) \rightarrow \underline{\text{Rep}}(U).$$

This functor is simply a restriction of scalars along the natural homomorphism $A(U) \rightarrow A(G)$.

We now discuss induction (which was not treated in [30]). Let N be an object of $\underline{\text{Rep}}(U)$. We define $\text{Ind}_U^G(N)$ to be the space of all functions $\varphi: G \rightarrow N$ satisfying the following two conditions:

- φ is left U -equivariant, that is, $\varphi(ug) = u\varphi(g)$ for all $u \in U$ and $g \in G$;
- φ is right G -smooth, that is, there is an open subgroup V of G such that $\varphi(gv) = \varphi(g)$ for all $v \in V$ and $g \in G$.

Given $a \in A(G)$ and $\varphi \in \text{Ind}_U^G(N)$, we define a function $a\varphi: G \rightarrow N$ by

$$(a\varphi)(g) = \int_{G/V} a_V(h)\varphi(gh) dh,$$

where V is an open subgroup of G for which φ is right-invariant. (Here a_V is the component of a in $\mathcal{C}(G/V)$; recall that $A(G)$ is the inverse limit of these spaces.) The function $a\varphi$ is well defined (i.e., independent of the choice of V), and belongs to $\text{Ind}_U^G(N)$; furthermore, this construction defines on $\text{Ind}_U^G(N)$ the structure of a smooth $A(G)$ -module. We omit the verification of these statements. We call $\text{Ind}_U^G(N)$

the *induction* of N from U to G . One easily sees that induction defines an additive functor

$$\mathrm{Ind}_U^G: \underline{\mathrm{Rep}}(U) \rightarrow \underline{\mathrm{Rep}}(G).$$

The following proposition gives its main properties.

PROPOSITION 2.13

We have the following.

- (a) *The induction functor is both left and right adjoint to the restriction functor.*
- (b) *The induction functor is continuous and cocontinuous (and thus exact).*
- (c) *For a U -set X , we have $\mathrm{Ind}_U^G(\mathcal{C}(X)) = \mathcal{C}(I_U^G(X))$, where $I_U^G(X) = G \times^U X$ is the induction of the U -set X to G .*

Proof

(a) Let M and N be objects of $\underline{\mathrm{Rep}}(G)$ and $\underline{\mathrm{Rep}}(U)$, respectively. To show that induction is right adjoint to restriction, we must give a natural isomorphism

$$\mathrm{Hom}(M, \mathrm{Ind}_U^G(N)) = \mathrm{Hom}(\mathrm{Res}_U^G(M), N).$$

Suppose that $\alpha: M \rightarrow \mathrm{Ind}_U^G(N)$ is given. We define $\alpha': \mathrm{Res}_U^G(M) \rightarrow N$ by $\alpha'(m) = \alpha(m)(1)$, where the right side indicates the value of the function $\alpha(m)$ at $1 \in G$. Now suppose that $\beta: \mathrm{Res}_U^G(M) \rightarrow N$ is given. We define $\beta': M \rightarrow \mathrm{Ind}_U^G(N)$ by $\beta'(m)(g) = \beta(gm)$. We leave to the reader the verification that these constructions are well defined and mutually inverse.

To show that induction is left adjoint to restriction, we must give a natural isomorphism

$$\mathrm{Hom}(\mathrm{Ind}_U^G(N), M) = \mathrm{Hom}(N, \mathrm{Res}_U^G(M)).$$

Suppose that $\alpha: \mathrm{Ind}_U^G(N) \rightarrow M$ is given. We define $\alpha': N \rightarrow \mathrm{Res}_U^G(M)$ by $\alpha'(n) = \alpha(\gamma_n)$, where $\gamma_n: G \rightarrow N$ is the function defined by

$$\gamma_n(g) = \begin{cases} gn & \text{if } g \in U, \\ 0 & \text{otherwise.} \end{cases}$$

Now suppose that $\beta: N \rightarrow \mathrm{Res}_U^G(M)$ is given. We define $\beta': \mathrm{Ind}_U^G(N) \rightarrow M$ by $\beta'(\varphi) = \int_{G/U} g\varphi(g^{-1}) dg$, where here we are integrating a module-valued function (see [30, Section 11.9]). We again leave the necessary verifications to the reader.

(b) This follows from (a) and general properties of adjoints.

(c) Let M be an object of $\underline{\mathrm{Rep}}(G)$. We have

$$\begin{aligned}\mathrm{Hom}_{A(G)}(\mathrm{Ind}_U^G(\mathcal{C}(X)), M) &= \mathrm{Hom}_{A(U)}(\mathcal{C}(X), M) = \mathrm{Hom}_U(X, M) \\ &= \mathrm{Hom}_G(I_U^G(X), M) = \mathrm{Hom}_{A(G)}(\mathcal{C}(I_U^G(X)), M).\end{aligned}$$

In the first step, we used (a); in the second, the mapping property for Schwartz space (see [30, Corollary 11.19]); in the third, adjunction for I_U^G (at the G -set level); and in the fourth, the property for Schwartz space again. The result now follows from Yoneda's lemma. $\square$

There is one more general fact about induction we require, namely, the projection formula.

PROPOSITION 2.14

Let M and N be objects of $\underline{\mathrm{Rep}}(G)$ and $\underline{\mathrm{Rep}}(U)$, respectively. Then there is a natural isomorphism

$$\mathrm{Ind}_U^G(\mathrm{Res}_U^G(M) \otimes N) \cong M \otimes \mathrm{Ind}_U^G(N).$$

Proof

This can be deduced from general considerations (see, e.g., [31, Proposition 4.6]), but we will give a direct argument. There is a natural map (from the left side to the right side) coming from adjunctions; we must show that it is an isomorphism. Since all functors involved are right exact, it suffices (by choosing presentations) to verify this when $M = \mathcal{C}(X)$ and $N = \mathcal{C}(Y)$, where X and Y are G - and U -sets. This follows from the projection formula at the set level; that is, we have a natural isomorphism of G -sets

$$I_U^G(R_U^G(X) \times Y) = X \times I_U^G(Y),$$

where R_U^G is the restriction functor from G -sets to U -sets. $\square$

3. Automorphisms of the line

In this section, we introduce the main group of interest: the automorphism group of the line. We recall some results about this case from [30], and develop some additional basic theory.

3.1. The group

Let $G = \mathrm{Aut}(\mathbf{R}, <)$ be the group of all order-preserving bijections $\mathbf{R} \rightarrow \mathbf{R}$. The symbol G will denote this group for the remainder of the paper. It is easy to see that G is oligomorphic with respect to its action on $\mathbf{R}$.

Let $\mathbf{R}^{(n)}$ denote the set of n -element subsets of $\mathbf{R}$. We identify $\mathbf{R}$ with the subset of $\mathbf{R}^n$ consisting of tuples $(x_1, \dots, x_n)$ satisfying $x_1 < x_2 < \dots < x_n$. One easily sees

that G acts transitively on $\mathbf{R}^{(n)}$. For $a \in \mathbf{R}^{(n)}$, we let $G(a)$ denote the subgroup of G consisting of elements that fix each a_i (which is equivalent to fixing the set a). Recall that a subgroup of G is open if it contains some $G(a)$. In fact, we have the following.

PROPOSITION 3.1

Every open subgroup of G is of the form $G(a)$ for some $a \in \mathbf{R}^{(n)}$ and $n \in \mathbf{N}$.

Proof

See [30, Proposition 17.1]. □

COROLLARY 3.2

Every transitive G -set is isomorphic to $\mathbf{R}^{(n)}$, for some n .¹

For an open interval $I \subset \mathbf{R}$, let G_I be the group of order-preserving bijections $I \rightarrow I$. Choosing an order-preserving bijection $i: I \rightarrow \mathbf{R}$ yields an isomorphism $f: G_I \rightarrow G$ of topological groups. Different choices of i lead to different choices of f ; however, the various f 's only differ by an inner automorphism of G . Thus we have a well-defined identification $G_I \cong G$ up to inner automorphisms.

For $a \in \mathbf{R}^{(n)}$, the set $\mathbf{R} \setminus a$ is a disjoint union of $n + 1$ open intervals $I_0, \dots, I_n$. The group $G(a)$ preserves these intervals, and so there is an induced homomorphism

$$G(a) \rightarrow G_{I_0} \times \cdots \times G_{I_n}.$$

One easily sees that this map is an isomorphism of topological groups. Thus $G(a)$ is isomorphic to G^{n+1} . This self-similarity present in the structure of G will play an important role in its representation theory.

Since G_I is isomorphic to G , Corollary 3.2 implies that every transitive G_I -set is isomorphic to $I^{(n)}$ for some n . Combined with the above classification and description of open subgroups of G , we obtain the following description of $\hat{G}$ -sets.

PROPOSITION 3.3

Every finitary $\hat{G}$ -set is a finite disjoint union of sets of the form $I_1^{(n_1)} \times \cdots \times I_r^{(n_r)}$, where the I_i 's are disjoint open intervals in $\mathbf{R}$ and the n_i 's are natural numbers.

3.2. Measure and integration

Let k be a commutative ring. By [30, Theorem 17.7], the group G admits a unique k -valued measure μ satisfying

$$\mu(I_1^{(n_1)} \times \cdots \times I_r^{(n_r)}) = (-1)^{n_1 + \cdots + n_r},$$

¹Recall our convention that “ G -set” means “set with a smooth G -action.”

where the I_i 's are disjoint open intervals of $\mathbf{R}$ and the n_i 's are natural numbers. For a general $\hat{G}$ -set X , we have $\mu(X) = \chi_c(X)$, where χ_c is a compactly supported Euler characteristic; one can define $\chi_c(X)$ by appealing to Proposition 3.3. Since $\mu(\mathbf{R}^{(n)}) = (-1)^n$, it follows that the measure μ is regular. We refer to μ as the *principal measure* on G , and it is the only measure for G that we will use. In what follows, we simply write $\text{vol}(X)$ in place of $\mu(X)$, and we no longer use the symbol μ for a measure.

Integration with respect to the principal measure, as defined in Section 2.2, is essentially the Euler calculus of Schapira and Viro (see, e.g., [47]). Here are some important examples of integrals:

$$\int_{\mathbf{R}} \delta_p(x) dx = 1, \quad \int_{\mathbf{R}} 1_{(p,q)}(x) dx = -1, \quad \int_{\mathbf{R}} 1_{[p,q]}(x) dx = 0.$$

Here $p < q$ are real numbers, δ_p is the point mass at p , and 1_X is the characteristic function of the set X .

Remark 3.4

The group G admits essentially four measures in total; more precisely, the universal ring $\Theta(G)$ (discussed after Definition 2.4) is isomorphic to $\mathbf{Z}^4$ (see [30, Theorem 17.7]). The three nonprincipal measures are not quasiregular though.

3.3. Representation categories

We now come to the main object of study of this paper: the category $\underline{\text{Rep}}(G)$, taken with respect to the principal measure on G over the field k . One can consider this category over a general commutative ring, but we confine our attention to the field case. As discussed in Section 2.4, $\underline{\text{Rep}}(G)$ is a semisimple Grothendieck abelian category equipped with a tensor product $\underline{\otimes}$, and the category $\underline{\text{Rep}}^f(G)$ of finite length objects is pre-Tannakian.

Associated to each G -set X , we have the Schwartz space $\mathcal{C}(X)$ in $\underline{\text{Rep}}(G)$. By the classification of transitive G -sets (Corollary 3.2), $\mathcal{C}(X)$ decomposes into a direct sum of $\mathcal{C}(\mathbf{R}^{(n)})$'s. Thus every simple object of $\underline{\text{Rep}}(G)$ is a summand of some $\mathcal{C}(\mathbf{R}^{(n)})$. The Schwartz spaces $\mathcal{C}(\mathbf{R}^{(n)})$ will play a central role in our analysis of $\underline{\text{Rep}}(G)$.

We introduce one additional piece of notation/terminology: we use the term “ $\underline{G}$ -module” or “representation of $\underline{G}$ ” (or similar) for objects of the category $\underline{\text{Rep}}(G)$, and “ $\underline{G}$ -map” (or similar) for morphisms in $\underline{\text{Rep}}(G)$. Thus $\underline{G}$ is essentially synonymous with the completed group algebra $A(G)$, with the convention that “ $\underline{G}$ -module” means “smooth module over $A(G)$.” This terminology makes the language closer to classical representation theory, which we find helpful.

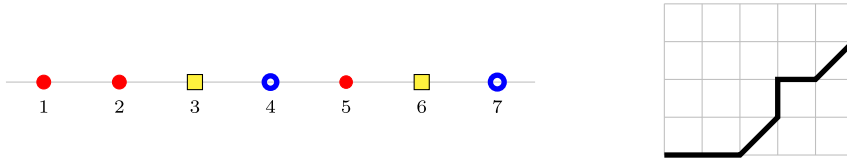


Figure 2. (Color online) An illustration of the proof of Proposition 3.5, with $x = (1, 2, 3, 5, 6)$ and $y = (3, 4, 6, 7)$. The solid dots are placed at lone x_i 's, the open circles are placed at lone y_j 's, and the squares are placed at points that are both of the form x_i and y_j .

3.4. Orbits on products

We now arrive at the connection between $\text{Rep}(G)$ and Delannoy paths. For the moment, we establish a simple bijection; in Section 9, we will find deeper connections.

PROPOSITION 3.5

The G -orbits on $\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}$ are naturally in bijective correspondence with (n, m) -Delannoy paths. In particular, the number of orbits is the Delannoy number $D(n, m)$.

Proof

Let (x, y) be an element of $\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}$. We associate to (x, y) an (n, m) -Delannoy path $p(x, y)$ as follows. Put a red dot on the real line at each x_i and a blue dot at each y_j ; a red and blue dot at the same point make a yellow dot. Now start in the plane at $(0, 0)$ and read the real line from $-\infty$ to ∞ . Whenever a red dot is encountered, take a $(1, 0)$ step; for blue dots, take a $(0, 1)$ step; and for yellow, a $(1, 1)$ step. The resulting path is $p(x, y)$. See Figure 2 for an example. Note that words in red, blue, and yellow dots correspond bijectively to Delannoy paths.

Let (x', y') be a second element of $\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}$. It is clear that if (x, y) and (x', y') belong to the same G -orbit, then $p(x, y) = p(x', y')$. Conversely, suppose that $p(x, y) = p(x', y')$; note that this means that (x, y) and (x', y') define the same word in colored dots. Regarding x and y as subsets of $\mathbf{R}$ size n and m , let z be their union; similarly, let z' be the union of x' and y' . The cardinality of z is the length (i.e., number of steps) of the path $p(x, y)$, and similarly for z' . We thus see that z and z' have the same cardinality, say ℓ . Since G acts transitively on $\mathbf{R}^{(\ell)}$, there is $g \in G$ such that $gz = z'$, that is, $gz_r = z'_r$ for all $1 \leq r \leq \ell$. Now, whether z_r is an x_i , y_j , or both can be determined from $p(x, y)$ (or, equivalently, the word of colored dots). Since $p(x, y) = p(x', y')$, it follows that $gx = x'$ and $gy = y'$, and so (x, y) and (x', y') are in the same orbit. $\square$

COROLLARY 3.6

We have

$$\dim \operatorname{Hom}_{\underline{G}}(\mathcal{C}(\mathbf{R}^{(n)}), \mathcal{C}(\mathbf{R}^{(m)})) = D(n, m).$$

Proof

Recall that $\underline{G}$ -maps $\mathcal{C}(\mathbf{R}^{(n)}) \rightarrow \mathcal{C}(\mathbf{R}^{(m)})$ are given by G -invariant $\mathbf{R}^{(m)} \times \mathbf{R}^{(n)}$ matrices. Thus the space of maps is isomorphic to the space of k -valued functions on the orbit space $G \backslash (\mathbf{R}^{(m)} \times \mathbf{R}^{(n)})$, and so the result follows. $\square$

Remark 3.7

The G -orbits on $\mathbf{R}^{(n_1)} \times \cdots \times \mathbf{R}^{(n_d)}$ can similarly be parameterized by d -dimensional Delannoy paths, as defined in Section 9.1.

3.5. The Grothendieck group

Let K be the Grothendieck group of the category $\underline{\operatorname{Rep}}^f(G)$. For a finite length $\underline{G}$ -representation V , we let $[V]$ denote its class in K . Classes of the form $[V]$ are called *effective*. The classes $[V]$, with V a simple representation, form a basis of K ; we will explicitly describe this basis in Corollary 4.13 below. We now explain a number of features of K .

(a) *The unit class.* We let $1 \in K$ be the class of the trivial representation.

(b) *The ring structure.* The tensor product $\otimes$ on $\underline{\operatorname{Rep}}(G)$ induces a product on K , which we call the *standard product*. (We use this term to distinguish it from two other products discussed below: the induction and concatenation products.) Under the standard product, K is an associative, unital, and commutative ring; the unit element is 1 .

(c) *The Hom pairing.* We let

$$\langle \cdot, \cdot \rangle : K \times K \rightarrow \mathbf{Z}$$

be the Hom pairing, defined on effective classes by

$$\langle [V], [W] \rangle = \dim_k \operatorname{Hom}_{\underline{G}}(V, W).$$

It is symmetric since $\underline{\operatorname{Rep}}(G)$ is semisimple. The classes of simple objects form an orthogonal basis; in fact, it follows from Theorem 4.3 that the simples are absolutely simple (i.e., they remain irreducible over $\bar{k}$) and so the classes of simples form an orthonormal basis.

(d) *Induction and restriction.* Let $G(0) \subset G$ be the stabilizer of the point $0 \in \mathbf{R}$. By definition, $G(0)$ is an open subgroup of G . We thus have functors

$$\operatorname{Ind} : \underline{\operatorname{Rep}}(G(0)) \rightarrow \underline{\operatorname{Rep}}(G), \quad \operatorname{Res} : \underline{\operatorname{Rep}}(G) \rightarrow \underline{\operatorname{Rep}}(G(0))$$

as in Section 2.5. The group $G(0)$ is isomorphic to $G \times G$, and so the Grothendieck group of $\underline{\text{Rep}}^f(G(0))$ is isomorphic to $K \otimes K$. The above functors therefore induce additive maps

$$\text{ind}: K \otimes K \rightarrow K, \quad \text{res}: K \rightarrow K \otimes K.$$

These maps are associative and coassociative, respectively (see below). The product ind is not unital or commutative. We will see that res is counital, but it is not cocommutative. We note that res is a ring homomorphism (with respect to the standard product), since Res is a tensor functor.

We briefly sketch a proof of associativity. Let $G(0, 1) \subset G$ be the stabilizer of $\{0, 1\} \in \mathbf{R}^{(2)}$, which is isomorphic to $G \times G \times G$. Induction from $G(0, 1)$ to G defines a map $K \otimes K \otimes K \rightarrow K$. One can factor this induction in two ways: first induce from $G(0, 1)$ to $G(0)$, and then to G ; or first induce to $G(1)$, and then to G . These equivalent factorizations prove that ind is associative. We note that associativity also follows from our explicit computation of ind in Theorem 6.1.

(e) *Frobenius reciprocity.* For $x, y, z \in K$, we have

$$\langle \text{ind}(x \otimes y), z \rangle = \langle x \otimes y, \text{res}(z) \rangle.$$

On the right side, $\langle \cdot, \cdot \rangle$ is the Hom pairing on $K \otimes K$, which is simply the tensor product of the Hom pairing on K with itself. This follows from Frobenius reciprocity at the representation level, as discussed in Section 2.5.

(f) *The projection formula.* For $x, y, z \in K$, we have

$$\text{ind}(\text{res}(x) \cdot (y \otimes z)) = x \cdot \text{ind}(y \otimes z).$$

This follows from Proposition 2.14.

(g) *Mackey theory.* For $x, y \in K$, we have

$$\text{res}(\text{ind}(x \otimes y)) = x \otimes y + \text{ind}_{1,2}(x \otimes \text{res}(y)) + \text{ind}_{2,3}(\text{res}(x) \otimes y),$$

where $\text{ind}_{1,2}$ denotes the induction product on the first two tensor factors. Since we will not use this formula, we omit the proof.

(h) *The filtration.* We let $K_{\leq n}$ be the $\mathbf{Z}$ -submodule of K spanned by the simple classes appearing in $\mathcal{C}(\mathbf{R}^{(m)})$ for some $0 \leq m \leq n$ (see Remark 4.14 for an explicit basis of $K_{\leq n}$). The $K_{\leq n}$ define an increasing exhaustive filtration of K . This filtration is compatible with multiplication in the sense that

$$K_{\leq n} \cdot K_{\leq m} \subset K_{\leq n+m}.$$

Indeed, if N and M are simple $\underline{G}$ -submodules of $\mathcal{C}(\mathbf{R}^{(n)})$ and $\mathcal{C}(\mathbf{R}^{(m)})$, then $N \otimes M$ is a $\underline{G}$ -submodule of

$$\mathcal{C}(\mathbf{R}^{(n)}) \underline{\otimes} \mathcal{C}(\mathbf{R}^{(m)}) = \mathcal{C}(\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}),$$

and $\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}$ decomposes into orbits of the form $\mathbf{R}^{(s)}$ with $s \leq n + m$. The filtration is also compatible with the comultiplication res , in the sense that

$$\text{res}(\mathbf{K}_{\leq n}) \subset \sum_{i+j=n} \mathbf{K}_{\leq i} \otimes \mathbf{K}_{\leq j}.$$

One can prove this directly by examining the restriction of $\mathcal{C}(\mathbf{R}^{(n)})$, or deduce it from our explicit computation of res in Theorem 6.9.

(i) *Other structure.* There is some other structure on $\mathbf{K}$ that we are not yet ready to define in detail, but we mention here. In Section 4.6, we define a third product on $\mathbf{K}$, called the concatenation product, and denoted $\odot$. We show that res admits a counit δ (Propositions 5.9 and 6.12). We also show that $\mathbf{K}$ forms a Hopf algebra under the standard product and the restriction coproduct (Proposition 6.13). Finally, we show that $\mathbf{K}$ has the structure of a λ -ring (see Section 8.1); in fact, it is a binomial ring (see Corollary 8.3).

4. Simple representations

In this section, we construct and classify the simple objects of $\underline{\text{Rep}}(G)$. We also determine the simple decomposition of the Schwartz space $\mathcal{C}(\mathbf{R}^{(n)})$.

4.1. Construction of simples

To begin, we construct certain simples, which will turn out to be all of them. To parameterize these objects, we introduce the following notion.

Definition 4.1

A *weight* is a word λ in the alphabet $\{\bullet, \circ\}$. We write $\ell(\lambda)$ for the length λ , and let Λ denote the set of all weights.

By a half-open interval in $\mathbf{R}$, we mean a nonempty interval of the form $(b, a]$ or $[a, b)$, where $a \in \mathbf{R}$ and $b \in \mathbf{R} \cup \{\pm\infty\}$. We define the *type* of a half-open interval to be $\bullet$ if its right endpoint is included, and $\circ$ if its left endpoint is included. For two subsets I and J of $\mathbf{R}$, we write $I < J$ to mean that every element of I is less than every element of J . For example, we have $[a, b) < [b, c)$ if $a < b < c$.

Now consider a tuple $\mathbf{I} = (I_1, \dots, I_n)$ of half-open intervals. We define the *type* of $\mathbf{I}$ to be the weight $\lambda_1 \cdots \lambda_n$, where λ_i is the type of I_i . We write $\mathbf{I}$ still for the hypercube $I_1 \times \cdots \times I_n$ in $\mathbf{R}^n$, and we let $\varphi_{\mathbf{I}} \in \mathcal{C}(\mathbf{R}^n)$ be its characteristic function. We say that $\mathbf{I}$ is *ordered* if $I_1 < \cdots < I_n$. In this case, $\mathbf{I}$ is contained in $\mathbf{R}^{(n)}$, and we regard $\varphi_{\mathbf{I}}$ as an element of $\mathcal{C}(\mathbf{R}^{(n)})$. We can now introduce some extremely important representations.



Figure 3. Two $(4, 4)$ -Delannoy paths. The first is quasidiagonal and turns left, diagonally, right, and diagonally. The second is not quasidiagonal.

Definition 4.2

For a weight λ of length n , we define L_λ to be the $\underline{G}$ -submodule of $\mathcal{C}(\mathbf{R}^{(n)})$ generated by the $\varphi_{\mathbf{I}}$, where $\mathbf{I}$ is an ordered tuple of half-open intervals of type λ . We also let a_λ be the class of L_λ in the Grothendieck group K .

When $n = 0$, we take $\varphi_\emptyset$ to be the constant function 1 on the point $\mathbf{R}^{(0)}$, so that $L_\emptyset = \mathcal{C}(\mathbf{R}^{(0)})$ is the 1-dimensional trivial representation. The following is the main theorem of Section 4.1.

THEOREM 4.3

We have the following.

- (a) The module L_λ is simple, and $\text{End}_{\underline{G}}(L_\lambda) = k$.
- (b) If λ and μ are distinct weights, then L_λ and L_μ are nonisomorphic.

We require two lemmas before proving the theorem. For an (n, n) -Delannoy path p , let $A_p \in \text{Mat}_{\mathbf{R}^{(n)}}$ be the indicator function of the orbit on $\mathbf{R}^{(n)} \times \mathbf{R}^{(n)}$ corresponding to p via Proposition 3.5. We say that p is *quasidiagonal* if it passes through every vertex along the main diagonal. Suppose that p is quasidiagonal. At the i th square along the main diagonal, p can behave in three possible ways:

- p turns diagonally if it goes from $(i-1, i-1)$ to (i, i) ;
- p turns right if it goes from $(i-1, i-1)$ to $(i-1, i)$ and then to (i, i) ; or
- p turns left if it goes from $(i-1, i-1)$ to $(i, i-1)$ and then to (i, i) .

See Figure 3 for an illustration.

LEMMA 4.4

Let $\lambda \in \Lambda$ have length n , let p be an (n, n) -Delannoy path, and let $\varphi_{\mathbf{I}}$ be one of the generators of L_λ . If p is not quasidiagonal, then $A_p \cdot \varphi_{\mathbf{I}} = 0$. Suppose that p is quasidiagonal. For $1 \leq i \leq n$, define

$$\epsilon_i(\lambda, p) = \begin{cases} 1 & \text{if } p \text{ turns diagonally in the } (i, i) \text{ square,} \\ -1 & \text{if } \lambda_i = \bullet \text{ and } p \text{ turns right in the } (i, i) \text{ square,} \\ -1 & \text{if } \lambda_i = \circ \text{ and } p \text{ turns left in the } (i, i) \text{ square,} \\ 0 & \text{otherwise} \end{cases}$$

and define $\epsilon(\lambda, p) = \prod_{i=1}^n \epsilon_i(\lambda, p)$. Then

$$A_p \cdot \varphi_{\mathbf{I}} = \epsilon(\lambda, p) \cdot \varphi_{\mathbf{I}}.$$

Proof

Let $X \subset \mathbf{R}^{(n)} \times \mathbf{R}^{(n)}$ be the orbit corresponding to p . We have

$$(A_p \varphi_{\mathbf{I}})(y) = \text{vol}(\{x \in \mathbf{R}^{(n)} \mid x \in \mathbf{I} \text{ and } (y, x) \in X\}).$$

The set on the right side above factors as $J_1 \times \cdots \times J_n$, where J_i is a subset of I_i . To be a bit more precise, fix y and i . Then the condition $(y, x) \in X$ puts us in one of two cases, as far as x_i is concerned:

- If p steps diagonally in the i th column, then $x_i = y_j$ for some j ; in this case, J_i is $\{y_j\} \cap I_i$.
- If p steps horizontally in the i th column, then $y_j < x_i < y_{j+1}$ for some j ; in this case, J_i is $(y_j, y_{j+1}) \cap I_i$. (We use the convention $y_0 = -\infty$ and $y_{n+1} = +\infty$.)

Suppose that $y \notin \mathbf{I}$. We claim that $(A_p \varphi_{\mathbf{I}})(y) = 0$. Indeed, if any J_i is empty or all of I_i , then the total volume is 0 since each I_i is half open. To ensure that J_i is nonempty and not all of I_i , it must contain some y_j . If this holds for all i , then each I_i contains exactly one y_j . Since the I_i 's and y_j 's are both in increasing order, we must have $y_i \in I_i$ for each i , that is, $y \in \mathbf{I}$.

Now suppose that $y \in \mathbf{I}$. First we claim that if p is not quasidiagonal, then $(A_p \varphi_{\mathbf{I}})(y) = 0$. Indeed, if $(A_p \varphi_{\mathbf{I}})(y) \neq 0$, then each I_i contains both an x_i and y_i , so before we enter the interval the path is at the vertex $(i-1, i-1)$ and as we exit it is at (i, i) . Second, if p is quasidiagonal, then we want to show that $\epsilon_i(\lambda, p) = \text{vol}(J_i)$. Let a_i and b_i denote the left and right endpoints of I_i . We split into cases accordingly using the above description of J_i (with $j = i$ or $j = i+1$ according to how the path turns in the i th square):

- If the path turns diagonally in the i th square, then $\epsilon_i(\lambda, p) = 1$ and $J_i = \{y_i\} \cap I_i = \{y_i\}$, which has volume 1.
- If $\lambda_i = \bullet$ and the path turns right in the i th square, then $\epsilon_i(\lambda, p) = -1$ and $J_i = (y_{i-1}, y_i) \cap (a_i, b_i] = (a_i, y_i)$ has volume -1 .
- If $\lambda_i = \circ$ and the path turns right in the i th square, then $\epsilon_i(\lambda, w) = 0$ and $J_i = (y_{i-1}, y_i) \cap [a_i, b_i) = [a_i, y_i)$ has volume 0.

- If $\lambda_i = \bullet$ and the path turns left in the i th square, then $\epsilon_i(\lambda, w) = 0$ and $J_i = (y_i, y_{i+1}) \cap (a_i, b_i] = (y_i, b_i]$ has volume 0.
- If $\lambda_i = \circ$ and the path turns right in the i th square, then $\epsilon_i(\lambda, w) = -1$ and $J_i = (y_i, y_{i+1}) \cap [a_i, b_i) = (y_i, b_i)$ has volume -1 . $\square$

LEMMA 4.5

Let $p: \mathbf{R}^{(n)} \rightarrow \mathbf{R}^{(n-1)}$ be the projection away from the last coordinate. Then $p_*(L_\lambda) = 0$.

Proof

It suffices to show that $p_*(\varphi_{\mathbf{I}}) = 0$ for all $\varphi_{\mathbf{I}}$ in L_λ . Fix $(x_1, \dots, x_{n-1})$ in $\mathbf{R}^{(n-1)}$, and let $X \subset \mathbf{R}$ be the set of x_n 's such that $(x_1, \dots, x_n) \in \mathbf{I}$. Then the value of $p_*(\varphi_{\mathbf{I}})$ at $(x_1, \dots, x_{n-1})$ is the volume of X . If $x_i \notin I_i$ for some $1 \leq i \leq n-1$, then X is empty and the volume is 0. If $x_i \in I_i$ for all i , then $X = I_n$, which also has volume 0. $\square$

Proof of Theorem 4.3

(a) Lemma 4.4 shows that A_p acts by a scalar on L_λ . Since the A_p 's are the standard basis of the endomorphism algebra of $\mathcal{C}(\mathbf{R}^{(n)})$ by Proposition 3.5, it follows that every endomorphism acts as a scalar on L_λ . Since $\mathcal{C}(\mathbf{R}^{(n)})$ is semisimple, every endomorphism of L_λ is induced by an endomorphism of $\mathcal{C}(\mathbf{R}^{(n)})$. We thus find $\text{End}_{\underline{G}}(L_\lambda) = k$. Since L_λ is semisimple, it follows that L_λ is simple. Note that since every endomorphism of $\mathcal{C}(\mathbf{R}^{(n)})$ preserves L_λ , it follows that L_λ has multiplicity 1 in $\mathcal{C}(\mathbf{R}^{(n)})$. (We will prove a more general statement in Theorem 4.3.)

(b) Let λ and μ be distinct weights; we show that L_λ and L_μ are nonisomorphic. Put $n = \ell(\lambda)$ and $m = \ell(\mu)$, and assume that $m \leq n$ without loss of generality. First suppose that $m = n$. By Lemma 4.4, we see that A_p acts by distinct scalars on L_λ and L_μ for appropriate p : in fact, note that there is a unique quasidiagonal path p having no diagonal turns for which A_p is nonzero on L_λ , and this path determines λ . Thus L_λ and L_μ are distinct subspaces of $\mathcal{C}(\mathbf{R}^{(n)})$. Since each has multiplicity 1 in $\mathcal{C}(\mathbf{R}^{(n)})$, it follows that they are nonisomorphic.

Now suppose that $m < n$. We first claim that if L is a simple appearing in $\mathcal{C}(\mathbf{R}^{(n-1)})$, then L is not isomorphic to L_λ . Indeed, let p be the projection in Lemma 4.5, and note that $p^*: \mathcal{C}(\mathbf{R}^{(n-1)}) \rightarrow \mathcal{C}(\mathbf{R}^{(n)})$ is injective. If L were isomorphic to L_λ , then we would have $L_\lambda = p^*(L)$ since L_λ has multiplicity 1 in $\mathcal{C}(\mathbf{R}^{(n)})$. But $p_*(L_\lambda) = 0$ by Lemma 4.5, while $p_*p^* = -1$ (since the fibers of p are open intervals), a contradiction. This proves the claim. Now simply note that L_μ appears in $\mathcal{C}(\mathbf{R}^{(n-1)})$, as the standard projection $\mathbf{R}^{(n-1)} \rightarrow \mathbf{R}^{(m)}$ gives an injection $\mathcal{C}(\mathbf{R}^{(m)}) \rightarrow \mathcal{C}(\mathbf{R}^{(n-1)})$. $\square$

4.2. Evaluation maps

For $a \in \mathbf{R}^{(n)}$, define a map

$$\mathrm{ev}_a: \mathcal{C}(\mathbf{R}^{(n)}) \rightarrow k, \quad \mathrm{ev}_a(\varphi) = \varphi(a).$$

We call ev_a the *evaluation at a* map. It is clear that ev_a is a map of $\underline{G}(a)$ -representations. We will require the following simple observation.

PROPOSITION 4.6

Let $a_1, \dots, a_r \in \mathbf{R}^{(n)}$ be distinct, and let λ be a weight of length n . Then $\mathrm{ev}_{a_1}, \dots, \mathrm{ev}_{a_r}$ restrict to linearly independent functionals on L_λ .

Proof

Fix $1 \leq i \leq r$. We can then find $\mathbf{I} = (I_1 < \dots < I_n)$ of type λ such that $a_i \in \mathbf{I}$, but $a_j \notin \mathbf{I}$ for all $j \neq i$. It follows that $\mathrm{ev}_{a_i}(\varphi_{\mathbf{I}}) = 1$, but $\mathrm{ev}_{a_j}(\varphi_{\mathbf{I}}) = 0$ for $j \neq i$. This completes the proof. $\square$

4.3. Decomposition of Schwartz space

We now determine the simple decomposition of Schwartz space.

THEOREM 4.7

For $n \geq 0$, we have an isomorphism

$$\mathcal{C}(\mathbf{R}^{(n)}) = \bigoplus_{\ell(\lambda) \leq n} L_\lambda^{\oplus m_\lambda}, \quad m_\lambda = \binom{n}{\ell(\lambda)}.$$

Proof

Define m_λ to be the multiplicity of L_λ in $\mathcal{C}(\mathbf{R}^{(n)})$, for any weight λ . We first give a lower bound for m_λ . Let $a \in \mathbf{R}^{(n)}$, and put $s = \ell(\lambda)$. We have

$$m_\lambda = \dim \mathrm{Hom}_{\underline{G}}(L_\lambda, \mathcal{C}(\mathbf{R}^{(n)})) = \dim \mathrm{Hom}_{\underline{G}(a)}(L_\lambda, k),$$

where the second step is Frobenius reciprocity. The functionals ev_b , with $b \subset a$ of cardinality s , are $\underline{G}(a)$ -equivariant and linearly independent by Proposition 4.6. Thus $m_\lambda \geq \binom{n}{s}$.

We have

$$\mathcal{C}(\mathbf{R}^{(n)}) = \left(\bigoplus_{\lambda} L_\lambda^{\oplus m_\lambda} \right) \oplus X,$$

where X contains the simples not of the form L_λ . We now compare the endomorphism rings of the two sides. The dimension of $\mathrm{End}_{\underline{G}}(\mathcal{C}(\mathbf{R}^{(n)}))$ is the central Delany number $D(n)$ (Corollary 3.6), while $\mathrm{End}_{\underline{G}}(L_\lambda)$ is 1-dimensional (Theorem 4.3).

We thus find

$$D(n) = \sum_{\lambda} m_{\lambda}^2 + d,$$

where d is the dimension of $\text{End}_{\underline{G}}(X)$. We have

$$\sum_{\lambda} m_{\lambda}^2 = \sum_{s \geq 0} \sum_{\ell(\lambda)=s} m_{\lambda}^2 \geq \sum_{s \geq 0} 2^s \binom{n}{s}^2 = D(n),$$

where in the final step we used a well-known formula for $D(n)$ (see [42, Example 2]). It follows that we must have $m_{\lambda} = \binom{n}{s}$ where $s = \ell(\lambda)$, and $d = 0$ (which implies that $X = 0$). $\square$

Example 4.8

For $n = 1, 2$, the theorem gives the following:

$$\mathcal{C}(\mathbf{R}) = L_{\bullet} \oplus L_{\circ} \oplus \mathbb{1},$$

$$\mathcal{C}(\mathbf{R}^{(2)}) = L_{\bullet\bullet} \oplus L_{\bullet\circ} \oplus L_{\circ\bullet} \oplus L_{\circ\circ} \oplus L_{\bullet}^{\oplus 2} \oplus L_{\circ}^{\oplus 2} \oplus \mathbb{1}.$$

Here $\mathbb{1}$ is the trivial representation. The first decomposition above was obtained “by hand” in [30, Section 17.6].

Remark 4.9

The following two quantities are equal to each other, and to $D(n)$ (see [42, Examples 1, 3]):

$$\sum_{k=0}^n \frac{(2n-k)!}{(n-k)!(n-k)!k!} = \sum_{k=0}^n \binom{n}{k}^2 2^k.$$

In light of Theorem 4.7, the above equality can be seen as analogous to the Selberg trace formula. The left side is the geometric side, as one obtains it by counting the orbits of G on the geometric object $\mathbf{R}^{(n)} \times \mathbf{R}^{(n)}$. The right side is the spectral side, as one easily obtains it from the simple decomposition of $\mathcal{C}(\mathbf{R}^{(n)})$.

The theorem has a number of corollaries.

COROLLARY 4.10

The length of $\mathcal{C}(\mathbf{R}^{(n)})$ is 3^n .

Proof

From the decomposition of $\mathcal{C}(\mathbf{R}^{(n)})$, we find that its length is

$$\sum_{\lambda} \binom{n}{\ell(\lambda)} = \sum_{s \geq 0} \sum_{\ell(\lambda)=s} \binom{n}{s} = \sum_{s \geq 0} \binom{n}{s} 2^s = 3^n,$$

where in the third step we use that there are 2^s weights of length s . $\square$

COROLLARY 4.11

Let λ be a weight of length s , and let $a \in \mathbf{R}^{(n)}$. Then the dimension of the $G(a)$ -invariants of L_{λ} is $\binom{n}{s}$.

Proof

The dimension of $L_{\lambda}^{G(a)}$ is the multiplicity of L_{λ} in $\mathcal{C}(\mathbf{R}^{(n)})$ by Frobenius reciprocity, which is $\binom{n}{s}$. (We note that G -invariants and $\underline{G}$ -invariants coincide by [30, Section 11.7].) $\square$

COROLLARY 4.12

Every simple object of $\underline{\text{Rep}}(G)$ is isomorphic to L_{λ} for some λ .

Proof

Every object of $\underline{\text{Rep}}(G)$ is a quotient of a sum of $\mathcal{C}(\mathbf{R}^{(n)})$'s, so all simples must appear in some Schwartz space $\mathcal{C}(\mathbf{R}^{(n)})$. The theorem shows that the only simples in $\mathcal{C}(\mathbf{R}^{(n)})$ are those of the form L_{λ} . $\square$

Recall that a_{λ} is the class of the simple L_{λ} in the Grothendieck group $\mathbf{K}$. We note one final corollary that we will use constantly (often without mention).

COROLLARY 4.13

The elements $\{a_{\lambda}\}_{\lambda \in \Lambda}$ are a $\mathbf{Z}$ -basis of $\mathbf{K}$.

Remark 4.14

We also see that the elements a_{λ} with $\ell(\lambda) \leq n$ form a $\mathbf{Z}$ -basis of $\mathbf{K}_{\leq n}$.

Remark 4.15

Write $\mathbf{K}(k)$ for the Grothendieck group, indicating the dependence on the field k . If k' is a second field, Corollary 4.13 gives an isomorphism $\iota: \mathbf{K}(k) \rightarrow \mathbf{K}(k')$ via $a_{\lambda} \mapsto a_{\lambda}$. In fact, ι is canonical and respects the extra structure on $\mathbf{K}$, as we now explain.

There are two main cases to discuss. First, if k is the prime subfield of k' , then it follows from the construction of L_{λ} that extension of scalars induces ι . Second, we have a natural map $\mathbf{K}(\mathbf{Q}) \rightarrow \mathbf{K}(\mathbf{F}_p)$ obtained by picking an integral lattice and

reducing mod p . One can show that this induces ι ; here one must use the existence of a good integral form of $\underline{\text{Rep}}(G)$, and Proposition 4.20 is helpful too.

4.4. Duality

For a weight λ , let $\lambda^\vee$ be the weight obtained by interchanging $\bullet$ and $\circ$.

PROPOSITION 4.16

Let λ be a weight of length n . The pairing

$$\langle \cdot, \cdot \rangle : L_\lambda \times L_{\lambda^\vee} \rightarrow k, \quad \langle \varphi, \psi \rangle = \int_{\mathbf{R}^{(n)}} \varphi(x) \psi(x) dx$$

identifies $L_{\lambda^\vee}$ with the dual of L_λ .

Proof

The pairing is clearly $\underline{G}$ -equivariant, and thus it induces a map of $\underline{G}$ -modules $L_{\lambda^\vee} \rightarrow L_\lambda^\vee$. Since both objects are simple, this map is an isomorphism provided that it is nonzero. It thus suffices to exhibit one nonzero pairing between functions in L_λ and $L_{\lambda^\vee}$.

Choose real numbers

$$a_1 < b_1 < c_1 < d_1 < \cdots < a_n < b_n < c_n < d_n.$$

Define intervals I_i and J_i as follows:

- If $\lambda_i = \bullet$, then $I_i = (a_i, c_i]$ and $J_i = [b_i, d_i)$.
- If $\lambda_i = \circ$, then $I_i = [b_i, d_i)$ and $J_i = (a_i, c_i]$.

Then $\mathbf{I} = (I_1 < \cdots < I_n)$ and $\mathbf{J} = (J_1 < \cdots < J_n)$ have types λ and $\lambda^\vee$, and so $\varphi_{\mathbf{I}}$ and $\varphi_{\mathbf{J}}$ belong to L_λ and $L_{\lambda^\vee}$. The pairing $\langle \varphi_{\mathbf{I}}, \varphi_{\mathbf{J}} \rangle$ is equal to the volume of $\mathbf{I} \cap \mathbf{J}$. By our construction, this set factors as $\prod_{i=1}^n (I_i \cap J_i)$, and $I_i \cap J_i = [b_i, c_i]$ is a closed interval. Thus the volume is 1, which completes the proof. $\square$

Remark 4.17

The group G admits an outer automorphism σ obtained by reversing the real line. This induces an involution of $\mathbf{K}$ that is different from the one induced by duality: we have $a_\lambda^\sigma = a_\mu$, where μ is the reverse of $\lambda^\vee$.

4.5. A dual description of the simples

We defined L_λ as the subrepresentation of $\mathcal{C}(\mathbf{R}^{(n)})$ generated by certain elements. We now give a dual description of L_λ , that is, we describe it as the subrepresentation of $\mathcal{C}(\mathbf{R}^{(n)})$ consisting of elements satisfying certain linear equations. We fix $n \geq 0$ in what follows.

To begin, we explicitly describe the embeddings of simple objects into Schwartz space. For a subset S of $[n]$ of cardinality m , let $p_S: \mathbf{R}^{(n)} \rightarrow \mathbf{R}^{(m)}$ be the projection onto the S coordinates. This map induces a map of representations $p_S^*: \mathcal{C}(\mathbf{R}^{(m)}) \rightarrow \mathcal{C}(\mathbf{R}^{(n)})$.

PROPOSITION 4.18

Let λ be a weight of length m with $m \leq n$. For $S \subset [n]$ of cardinality m , let $i_S: L_\lambda \rightarrow \mathcal{C}(\mathbf{R}^{(n)})$ be the restriction of p_S^* to L_λ . Then the i_S form a basis of the multiplicity space $\text{Hom}_{\underline{G}}(L_\lambda, \mathcal{C}(\mathbf{R}^{(n)}))$.

Proof

By Theorem 4.7, the multiplicity space in question has dimension $\binom{n}{m}$. It thus suffices to prove that the elements i_S are linearly independent. To see this, pick $a \in \mathbf{R}^{(n)}$. Then the composition

$$L_\lambda \xrightarrow{i_S} \mathcal{C}(\mathbf{R}^{(n)}) \xrightarrow{\text{ev}_a} k$$

is equal to $\text{ev}_{p_S(a)}$. Since these functionals are linearly independent on L_λ (Proposition 4.6), it follows that the i_S 's are linearly independent. $\square$

For $1 \leq i \leq n$, let $p_i: \mathbf{R}^{(n)} \rightarrow \mathbf{R}^{(n-1)}$ be the projection away from the i th coordinate; this is just p_S , where $S = [n] \setminus \{i\}$.

PROPOSITION 4.19

Define subrepresentations X and Y of $\mathcal{C}(\mathbf{R}^{(n)})$ by

$$X = \bigcap_{i=1}^n \ker((p_i)_*), \quad Y = \sum_{i=1}^n \text{im}(p_i^*).$$

Then X (resp., Y) is the sum of the L_λ -isotypic spaces of $\mathcal{C}(\mathbf{R}^{(n)})$, where $\ell(\lambda) = n$ (resp., $\ell(\lambda) < n$). In particular, $\mathcal{C}(\mathbf{R}^{(n)}) = X \oplus Y$.

Proof

It is clear that Y only contains simples L_λ with $\ell(\lambda) < n$. Now, if S is a proper subset of $[n]$, then it does not contain some element i , and so p_S factors through p_i ; it follows that $\text{im}(p_S^*) \subset \text{im}(p_i^*)$. It thus follows from Proposition 4.18 that Y contains every copy of L_λ in $\mathcal{C}(\mathbf{R}^{(n)})$ if $\ell(\lambda) < n$. We have thus verified the statement about Y .

The maps $(p_i)_*$ and p_i^* are adjoint with respect to the standard pairings on $\mathcal{C}(\mathbf{R}^{(n)})$ and $\mathcal{C}(\mathbf{R}^{(n-1)})$. It follows that X is exactly the orthogonal complement of Y . This yields the statement about X . $\square$

We now come to the desired description of L_λ .

PROPOSITION 4.20

Let λ be a weight of length n . Then L_λ consists of those functions φ in $\mathcal{C}(\mathbf{R}^{(n)})$ satisfying the following two conditions:

- (a) We have $(p_i)_*(\varphi) = 0$ for all $1 \leq i \leq n$.
- (b) For every weight $\mu \neq \lambda^\vee$ of length n and every $\psi \in L_\mu$, we have $\langle \varphi, \psi \rangle = 0$. Moreover, in (b) it is enough to consider functions ψ of the form $\varphi_{\mathbf{I}}$.

Proof

The first statement follows immediately from Propositions 4.16 and 4.19. We now explain the second statement (about the $\varphi_{\mathbf{I}}$ being enough). Suppose that $\varphi \in \mathcal{C}(\mathbf{R}^{(n)})$ is orthogonal to all $\varphi_{\mathbf{I}}$ in some L_μ . Let U be an open subgroup of G such that φ is U -invariant. The $\varphi_{\mathbf{I}}$ generate L_μ as a representation of U (see [30, Corollary 11.23] and the preceding discussion). It follows that φ is orthogonal to all functions in L_μ . $\square$

4.6. The concatenation product

We define the *concatenation product* to be the unique bilinear map

$$\odot : \mathbf{K} \times \mathbf{K} \rightarrow \mathbf{K}$$

satisfying

$$a_\lambda \odot a_\mu = a_{\lambda\mu}$$

for all weights λ and μ ; here $\lambda\mu$ simply denotes the concatenation of the words λ and μ . This definition is justified since the a_λ 's form a $\mathbf{Z}$ -basis of $\mathbf{K}$ (Corollary 4.13). The concatenation product gives $\mathbf{K}$ the structure of an associative unital ring; in fact, it is simply the noncommutative polynomial ring on the variables $a_\bullet$ and $a_\circ$.

5. Invariants

In the previous section, we computed the dimension of the space of $G(a)$ -invariants on the simple L_λ (Corollary 4.11). In this section, we explicitly identify a basis of this space. Using this, we write down the projection operator for L_λ in $\mathcal{C}(\mathbf{R}^{(n)})$, which allows us to compute the categorical dimension of L_λ (Corollary 5.7). We close this section by showing how one can determine the structure of an arbitrary representation by analyzing its invariant spaces.

5.1. The key set

Let λ be a weight of length n . We define

$$\Psi_\lambda \subset \mathbf{R}^{(n)} \times \mathbf{R}^{(n)}$$

to be the subset consisting of all pairs (x, y) satisfying the following conditions, for $1 \leq i \leq n$:

- If $\lambda_i = \bullet$, then $x_i \leq y_i$.
- If $\lambda_i = \circ$, then $y_i \leq x_i$.
- If $i < n$, then $x_i < y_{i+1}$ and $y_i < x_{i+1}$.

Here are some examples:

- When λ is the empty word, $\Psi_\lambda = \mathbf{R}^{(0)} \times \mathbf{R}^{(0)}$ is a point.
- For $\lambda = \bullet\bullet$, we have $x_1 \leq y_1 < x_2 \leq y_2$.
- For $\lambda = \bullet\circ$, we have $x_1 \leq y_1 < y_2 \leq x_2$.
- For $\lambda = \circ\bullet$, we have $y_1 \leq x_1 < x_2 \leq y_2$.
- For $\lambda = \circ\circ$, we have $y_1 \leq x_1 < y_2 \leq x_2$.

It is clear that Ψ_λ is a G -stable set; however, it is not a single orbit. The set Ψ_λ will play a key role in the rest of this section. Note that if τ is the map on $\mathbf{R}^{(n)} \times \mathbf{R}^{(n)}$ given by $\tau(x, y) = (y, x)$, then $\tau(\Psi_\lambda) = \Psi_{\lambda^\vee}$.

Remark 5.1

Let p be the quasidiagonal (n, n) -Delannoy path where the i th turn is right (resp., left) if $\lambda_i = \bullet$ (resp., $\lambda_i = \circ$). The G -orbit on $\mathbf{R}^{(n)} \times \mathbf{R}^{(n)}$ corresponding to p in Proposition 3.5 is defined just like Ψ_λ , except with all $\leq$ changed to $<$.

5.2. Invariant functions

For $a = (a_1, a_2, \dots, a_n) \in \mathbf{R}^{(n)}$ and λ a weight of length n , define $\Psi_\lambda^a \subset \mathbf{R}^{(n)}$ to be the set of points x such that (x, a) belongs to Ψ_λ , and let $\psi_\lambda^a \in \mathcal{C}(\mathbf{R}^{(n)})$ be the characteristic function of Ψ_λ^a . The main interest in these functions lies in the following result.

PROPOSITION 5.2

The space of $G(a)$ -invariants in L_λ is spanned by ψ_λ^a .

Proof

Since Ψ_λ^a is stable by $G(a)$, it follows that ψ_λ^a is a $G(a)$ -invariant element of $\mathcal{C}(\mathbf{R}^{(n)})$. Since we already know that $G(a)$ -invariants of L_λ are 1-dimensional (Corollary 4.11), it is enough to show that ψ_λ^a belongs to L_λ . For $1 \leq i \leq n$, let $p_i: \mathbf{R}^{(n)} \rightarrow \mathbf{R}^{(n-1)}$ be the projection away from the i th coordinate. Recall (Proposition 4.20) that an element θ of $\mathcal{C}(\mathbf{R}^{(n)})$ belongs to L_λ if and only if the following two conditions hold:

- (a) $(p_i)_*(\theta) = 0$ for all $1 \leq i \leq n$, and
- (b) θ is orthogonal to all L_μ 's with $\ell(\mu) = n$, except for $\mu = \lambda^\vee$.

We verify these two conditions for $\theta = \psi_\lambda^a$.

(a) Fix $(x_1, \dots, \hat{x}_i, \dots, x_n)$ in $\mathbf{R}^{(n-1)}$. Let $X \subset \mathbf{R}$ be the set of x_i 's such that $(x_1, \dots, x_i, \dots, x_n)$ belongs to Ψ_λ^a . Then the value of $(p_i)_* \psi_\lambda^a$ at $(x_1, \dots, \hat{x}_i, \dots, x_n)$ is the volume of X . We claim that X is a half-open interval (or empty); this will prove the statement as such a set has volume 0. First suppose that $\lambda_i = \bullet$. There are three cases:

- If $i = 1$, then $X = (-\infty, a_1]$.
- If $i > 1$ and $\lambda_{i-1} = \bullet$, then $X = (a_{i-1}, a_i]$.
- If $i > 1$ and $\lambda_{i-1} = \circ$, then $X = (x_{i-1}, a_i]$.

The case $\lambda_i = \circ$ is similar. This proves the claim.

(b) Suppose that μ is a weight of length n that is not equal to $\lambda^\vee$; this precisely means that there is some $1 \leq i \leq n$ such that $\mu_i = \lambda_i$. To show that ψ_λ^a is orthogonal to L_μ , it suffices to show that it is orthogonal to the generators $\varphi_{\mathbf{I}}$. Thus let $\varphi_{\mathbf{I}}$ be given, where $\mathbf{I} = (I_1 < \dots < I_n)$ are intervals of type μ . By Fubini's theorem (see [30, Corollary 3.19]), we have

$$\langle \psi_\lambda^a, \varphi_{\mathbf{I}} \rangle = \int_{\mathbf{R}^{(n)}} \psi_\lambda^a(x) \varphi_{\mathbf{I}}(x) dx = \int_{I_1} \cdots \int_{I_n} \psi_\lambda^a(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

In fact, we can put the integrals on the right side in whatever order we prefer. We do the i th integral first. Thus fix $x_j \in I_j$ for all $j \neq i$, and consider the integral

$$\int_{I_i} \psi_\lambda^a(x_1, \dots, x_i, \dots, x_n) dx_i.$$

It suffices to show that this integral vanishes. Let $X \subset \mathbf{R}$ be defined as in (a). Then the above integral is simply the volume of $X \cap I_i$. Suppose that $\lambda_i = \bullet$. Then, as we saw in (a), X is a half-open interval containing its right endpoint (or empty). Since I_i has type $\mu_i = \lambda_i$, it is a half-open interval containing its right endpoint. Thus $X \cap I_i$ is also a half-open interval containing its right endpoint (or empty), and therefore has volume 0. The case $\lambda_i = \circ$ is similar. $\square$

Since the $\underline{G}(a)$ -invariant space of L_λ is 1-dimensional, it follows from semisimplicity that the $\underline{G}(a)$ -coinvariant space is also 1-dimensional, that is, up to scalar multiples, there is a unique $\underline{G}(a)$ -equivariant functional $L_\lambda \rightarrow k$. We know two such functionals, namely, the evaluation map ev_a (see Section 4.2) and the pairing $\langle -, \psi_{\lambda^\vee}^a \rangle$. These two functionals are necessarily linearly dependent; in fact, they are equal.

PROPOSITION 5.3

For any $\varphi \in L_\lambda$, we have

$$\varphi(a) = \int_{\mathbf{R}^{(n)}} \varphi(x) \psi_{\lambda^\vee}^a(x) dx.$$

In other words, $\text{ev}_a = \langle -, \psi_{\lambda^\vee}^a \rangle$.

Proof

By the above discussion, we have $\text{ev}_a = \beta \cdot \langle -, \psi_{\lambda^\vee}^a \rangle$, for some scalar β . Let $\mathbf{I} = (I_1 < \dots < I_n)$ be intervals of type λ such that a_i is the closed endpoint of I_i for all i . We have $\text{ev}_a(\varphi_{\mathbf{I}}) = 1$. To show that $\beta = 1$, it suffices to show that $\langle \varphi_{\mathbf{I}}, \psi_{\lambda^\vee}^a \rangle = 1$. The value of this pairing is the volume of $X = \Psi_{\lambda^\vee}^a \cap \mathbf{I}$. We claim that $X = \{a\}$, which will complete the proof. It is clear that $a \in X$. Conversely, suppose that $b \in X$. Say $\lambda_i = \bullet$. Since $b_i \in I_i$, we have $b_i \leq a_i$, and since $b \in \Psi_{\lambda^\vee}^a$, we have $a_i \leq b_i$. Thus $b_i = a_i$. The case $\lambda_i = \circ$ is similar. Thus $a = b$, as required. $\square$

COROLLARY 5.4

Let $a^1, \dots, a^r$ be distinct elements of $\mathbf{R}^{(n)}$. Then $\psi_{\lambda}^{a^1}, \dots, \psi_{\lambda}^{a^r}$ are linearly independent functions.

Proof

This follows from the above proposition and the fact that $\text{ev}_{a^1}, \dots, \text{ev}_{a^r}$ are linearly independent functionals on $L_{\lambda^\vee}$ (Proposition 4.6). $\square$

COROLLARY 5.5

Let $m \geq n$, and let $b \in \mathbf{R}^{(m)}$. Then the functions ψ_{λ}^a , with a an n -element subset of b , form a basis of $L_{\lambda}^{G(b)}$.

Proof

The ψ_{λ}^a belong to $L_{\lambda}^{G(b)}$ by Proposition 5.2. By Corollary 5.4, these functions are linearly independent. Since $L_{\lambda}^{G(b)}$ has dimension $\binom{m}{n}$ (see Corollary 4.11), the result follows. $\square$

5.3. The projection map

Define

$$\pi_{\lambda}: \mathbf{R}^{(n)} \times \mathbf{R}^{(n)} \rightarrow k$$

to be the characteristic function of Ψ_{λ} . We regard π_{λ} as an $\mathbf{R}^{(n)} \times \mathbf{R}^{(n)}$ matrix; as such it defines an endomorphism of $\mathcal{C}(\mathbf{R}^{(n)})$.

PROPOSITION 5.6

The matrix π_λ defines the projection map $\mathcal{C}(\mathbf{R}^{(n)}) \rightarrow L_\lambda$, that is, π_λ is idempotent and its image is exactly L_λ .

Proof

The “columns” of the matrix π_λ are the functions ψ_λ^a considered above. Since these belong to L_λ (see Proposition 5.2), it follows that the image of π_λ is contained in L_λ . To complete the proof, it suffices to show that π_λ acts by the identity on elements of L_λ . This is exactly Proposition 5.3; note that $\psi_{\lambda \vee}^a(x) = \pi_\lambda(a, x)$. $\square$

Recall that an object in a pre-Tannakian category has a “categorical dimension” (see [21, Section 4.7]).

COROLLARY 5.7

The categorical dimension of L_λ is $(-1)^{\ell(\lambda)}$.

Proof

The categorical dimension d of L_λ is equal to the categorical trace of π_λ , which is just the integral of the diagonal of π_λ (see [30, Proposition 8.11]). Since Ψ_λ contains the diagonal, we find that $d = \text{vol}(\mathbf{R}^{(n)})$, which is $(-1)^n$, as claimed. $\square$

Remark 5.8

Suppose that k has positive characteristic p . The paper [22] defines a notion of “ p -adic dimension” for objects in (certain) pre-Tannakian categories over k . The p -adic dimension of L_λ is $(-1)^{\ell(\lambda)}$. One can see this by using the integral version of $\underline{\text{Rep}}(G)$, and the fact that L_λ exists there.

PROPOSITION 5.9

There exists a unique ring homomorphism $\delta: \mathbf{K} \rightarrow \mathbf{Z}$ satisfying $\delta(a_\lambda) = (-1)^{\ell(\lambda)}$.

Proof

Suppose that k has characteristic 0. Then the map $\mathbf{K} \rightarrow k$ assigning to each class its categorical dimension is a ring homomorphism. Since this takes values in $\mathbf{Z} \subset k$ by Corollary 5.7, the result follows. In positive characteristic, one can either use p -adic dimension, or simply appeal to the fact that $\mathbf{K}$ is independent of the field k (see Remark 4.15). $\square$

5.4. Operators on invariants

Let V be a simple $\underline{G}$ -module. We know that V is isomorphic to L_λ for some weight λ . By Corollary 4.11, the length of λ is the minimal n for which $V^{G(a)}$ is nonzero for $a \in \mathbf{R}^{(n)}$. We now show how one can recover λ itself via certain operators on this invariant space.

Fix a $\underline{G}$ -module V , and let $a \in \mathbf{R}^{(n)}$. For every λ , we define an operator

$$e_\lambda: V^{G(a)} \rightarrow V^{G(a)}$$

as follows. Let $x \in V^{G(a)}$ be given. Then

$$e_\lambda(x) = \int_{G/G(a)} \psi_\lambda^a(ga) gx \, dg.$$

The element $e_\lambda(x)$ is $G(a)$ -invariant since the function ψ_λ^a is $G(a)$ -invariant. The following is our main result.

PROPOSITION 5.10

The operator e_λ is idempotent. Its rank is the multiplicity of L_λ in V .

Proof

Since e_λ is natural in V , it suffices to treat the case where V is simple. We show that $e_\lambda = 0$ if V is not isomorphic to L_λ , and e_λ is the identity if $V = L_\lambda$.

First suppose that V is not isomorphic to L_λ . Let $x \in V^{G(a)}$. There is a unique map of $\underline{G}$ -modules $f: \mathcal{C}(\mathbf{R}^{(n)}) \rightarrow V$ satisfying $f(\delta_a) = x$ (see [30, Corollary 11.19]). For $\varphi \in \mathcal{C}(\mathbf{R}^{(n)})$, we have

$$f(\varphi) = \int_{G/G(a)} \varphi(ga) gx \, dg.$$

Thus $e_\lambda(x) = f(\psi_\lambda^a)$. Since V is not isomorphic to L_λ , we see that f restricts to the zero map $L_\lambda \rightarrow V$. Since ψ_λ^a belongs to L_λ , it follows that $e_\lambda(x) = 0$.

Now suppose that $V = L_\lambda$. The space $V^{G(a)}$ is 1-dimensional and spanned by ψ_λ^a . It follows that $e_\lambda(\psi_\lambda^a) = \alpha \cdot \psi_\lambda^a$ for some scalar α , and we must show that $\alpha = 1$. Since $\psi_\lambda^a(a) = 1$, we have $\alpha = (e_\lambda \psi_\lambda^a)(a)$. For $g \in G$, we have $g\psi_\lambda^a = \psi_\lambda^{g a}$. Thus

$$\begin{aligned} \alpha &= \int_{G/G(a)} \psi_\lambda^a(ga) \psi_\lambda^{g a}(a) \, dg = \int_{\mathbf{R}^{(n)}} \psi_\lambda^a(b) \psi_\lambda^b(a) \, db \\ &= \int_{\mathbf{R}^{(n)}} \psi_\lambda^a(b) \psi_{\lambda^\vee}^a(b) \, db = \psi_\lambda^a(a) = 1, \end{aligned}$$

where in the fourth step we used Proposition 5.3. □

6. Induction and restriction

In this section, we study induction and restriction between G and its open subgroups. Our main results give explicit formulas for these operations. As applications, we show that K is a Hopf algebra and we classify its primitive elements.

6.1. Induction

Recall (Section 3.5) the induction product

$$\text{ind}: K \otimes K \rightarrow K,$$

which is induced by the induction functor from $G(0) \cong G \times G$ to G . We sometimes write $\text{ind}(x, y)$ in place of $\text{ind}(x \otimes y)$ for readability. We now compute this operation explicitly, in terms of the concatenation product $\odot$ (see Section 4.6).

THEOREM 6.1

For $x, y \in K$, we have

$$\text{ind}(x, y) = x \odot [\mathcal{C}(\mathbf{R})] \odot y,$$

where $[\mathcal{C}(\mathbf{R})] = a_{\bullet} + a_{\circ} + 1$.

See Example 4.8 for the formula for $[\mathcal{C}(\mathbf{R})]$. We note two corollaries of the theorem.

COROLLARY 6.2

For weights λ and μ , we have

$$\text{ind}(a_{\lambda}, a_{\mu}) = a_{\lambda \bullet \mu} + a_{\lambda \circ \mu} + a_{\lambda \mu}.$$

COROLLARY 6.3

For $x, y, z \in K$, we have

$$\text{ind}(x \odot y, z) = x \odot \text{ind}(y, z), \quad \text{ind}(x, y \odot z) = \text{ind}(x, y) \odot z.$$

Remark 6.4

The Schwartz space $\mathcal{C}(\mathbf{R}^{(n)})$ is the induction to G of the trivial representation of $G(a)$, where $a \in \mathbf{R}^{(n)}$. The theorem therefore gives $[\mathcal{C}(\mathbf{R}^{(n)})] = [\mathcal{C}(\mathbf{R})]^{\odot n}$. This agrees with the decomposition of $\mathcal{C}(\mathbf{R}^{(n)})$ given in Theorem 4.7.

Remark 6.5

We have defined the concatenation product on the Grothendieck group, but not at the level of representations. Corollary 6.2 suggests a possible definition of $L_{\lambda} \odot L_{\mu}$,

namely, it can be realized as the $\underline{G}$ -submodule of $\text{Ind}(L_\lambda \boxtimes L_\mu)$ generated by the 1-dimensional space of $G(a)$ -invariants, where $a \in \mathbf{R}^{(n)}$ with $n = \ell(\lambda) + \ell(\mu)$.

We now prove Theorem 6.1. The proof has essentially two steps. First, we show that $\text{Ind}(L_\lambda \boxtimes L_\mu)$ has length 3 by computing the dimensions of invariant spaces. Second, we identify the three simple constituents using the operators studied in Section 5.4.

We begin by introducing a convenient device for counting invariants. Let V be a finite length object of $\text{Rep}(G)$. We define the *Hilbert function* h_V of V by letting $h_V(n)$ be the dimension of the $G(a)$ -invariants of V , for $a \in \mathbf{R}^{(n)}$; this is well defined since the various $G(a)$ subgroups are all conjugate. We also define the *Hilbert series* of V by

$$H_V(t) = \sum_{n \geq 0} h_V(n) t^n.$$

The following result tells us what these invariants look like.

LEMMA 6.6

Let V be a finite length object of $\text{Rep}(G)$. Let m_λ be the multiplicity of L_λ in V , and let $m_n = \sum_{\ell(\lambda)=n} m_\lambda$. Then

$$H_V(t) = \sum_{n \geq 0} m_n \cdot \frac{t^n}{(1-t)^{n+1}}.$$

Proof

It suffices to treat the case where V is simple; this follows from Corollary 4.11. $\square$

Write $\mathbf{R} = \mathbf{R}_- \sqcup \{0\} \sqcup \mathbf{R}_+$, where $\mathbf{R}_- = (-\infty, 0)$ and $\mathbf{R}_+ = (0, \infty)$, and let $G_\pm$ be the group of order-preserving automorphisms of $\mathbf{R}_\pm$. These groups are isomorphic to G , and we have $G(0) = G_- \times G_+$. Via the isomorphism $G_\pm \cong G$, we can apply concepts about G -representations to $G_\pm$ -representations, such as Hilbert series.

LEMMA 6.7

Let W_1 and W_2 be representations of $\underline{G}_-$ and $\underline{G}_+$, and let V be the induction of $W_1 \boxtimes W_2$ from $G(0)$ to G . Then

$$H_V(t) = (1+t)H_{W_1}(t)H_{W_2}(t).$$

Proof

Recall that $h_V(n)$ is the dimension of the invariant space $V^{G(a)}$, where $a \in \mathbf{R}^{(n)}$. By

Frobenius reciprocity, this is equal to the dimension of $\text{Hom}_{\underline{G}}(\mathcal{C}(\mathbf{R}^{(n)}), V)$. Applying Frobenius reciprocity in the other argument, we find that

$$h_V(n) = \dim \text{Hom}_{\underline{G}(0)}(\mathcal{C}(\mathbf{R}^{(n)}), W_1 \boxtimes W_2).$$

Now, we have

$$\mathbf{R}^{(n)} = \bigsqcup_{r+s+t=n, s=0,1} \mathbf{R}_-^{(r)} \times \mathbf{R}_+^{(t)},$$

and so

$$\mathcal{C}(\mathbf{R}^{(n)}) = \bigoplus_{r+s+t=n, s=0,1} \mathcal{C}(\mathbf{R}_-^{(r)}) \boxtimes \mathcal{C}(\mathbf{R}_+^{(t)}).$$

Thus we have

$$h_V(n) = \sum_{r+t=n} h_{W_1}(r) h_{W_2}(t) + \sum_{r+t=n-1} h_{W_1}(r) h_{W_2}(t),$$

where the second term is omitted if $n = 0$. This yields the stated formula. $\square$

LEMMA 6.8

Let W_1 , W_2 , and V be as in the previous lemma, and suppose that W_1 and W_2 are simple. Then V has length 3.

Proof

Suppose that W_1 and W_2 are associated to weights of length n and m . By Lemmas 6.6 and 6.7, we have

$$\begin{aligned} H_V(t) &= (1+t)H_{W_1}(t)H_{W_2}(t) = (1+t) \cdot \frac{t^n}{(1-t)^{n+1}} \cdot \frac{t^m}{(1-t)^{m+1}} \\ &= ((1-t) + 2t) \cdot \frac{t^{n+m}}{(1-t)^{n+m+2}} \\ &= \frac{t^{n+m}}{(1-t)^{n+m+1}} + 2 \frac{t^{n+m+1}}{(1-t)^{n+m+2}}. \end{aligned}$$

By Lemma 6.6 again, we thus see that V has three simples, with one of length $n + m$ and two of length $n + m + 1$. $\square$

Proof of Theorem 6.1

Write L_λ^+ for the G_+ version of L_λ . This is the $\underline{G}_+$ -submodule of $\mathcal{C}(\mathbf{R}_+^{(n)})$ generated by the $\varphi_{\mathbf{I}}$'s, where $\mathbf{I} = (I_1 < \dots < I_n)$ are intervals of type λ contained in $\mathbf{R}_+$ (here n is the length of λ). We similarly define L_λ^- .

Let $W_1 = L_\lambda^-$ and $W_2 = L_\mu^+$, and let V be the induction of $W_1 \boxtimes W_2$. We show that

$$V = L_{\lambda\mu} \oplus L_{\lambda\bullet\mu} \oplus L_{\lambda\circ\mu},$$

and this will complete the proof. Since we know that V has length 3 by Lemma 6.8, it is enough to show that each simple on the right side appears in V . By Frobenius reciprocity, it is enough to show that $L_\lambda^- \boxtimes L_\lambda^+$ appears in the restriction to $G(0)$ of each of these simples.

Let $n = \ell(\lambda)$ and $m = \ell(\mu)$. We have natural maps

$$L_\lambda^- \boxtimes L_\mu^+ \rightarrow \mathcal{C}(\mathbf{R}_-^{(n)}) \boxtimes \mathcal{C}(\mathbf{R}_+^{(m)}) \rightarrow \text{Res}_{G(0)}^G \mathcal{C}(\mathbf{R}^{(n+m)}),$$

where the first comes from the inclusions, and the second takes the product of the pullbacks of the two functions. This map carries $\varphi_{\mathbf{I}} \boxtimes \varphi_{\mathbf{J}}$ to $\varphi_{(\mathbf{I}, \mathbf{J})}$, where $(\mathbf{I}, \mathbf{J})$ means $(I_1, \dots, I_n, J_1, \dots, J_m)$. Since $\varphi_{(\mathbf{I}, \mathbf{J})}$ belongs to $L_{\lambda\mu}$, we thus have a nonzero $G(0)$ -map $L_\lambda^- \boxtimes L_\mu^+ \rightarrow L_{\lambda\mu}$, as desired.

Let $v = \lambda \bullet \mu$, and fix $a \in \mathbf{R}_-^{(n)}$ and $b \in \mathbf{R}_+^{(m)}$. Let e_λ^- be the operator on the $G(a)$ -invariants of a $\underline{G}_-$ -representation defined in Section 5.4, and similarly define e_μ^+ . To show that $L_\lambda^- \boxtimes L_\mu^+$ appears in $\text{Res}_{G(0)}^G(L_v)$, it is enough (by Proposition 5.10) to show that $e_\lambda^- e_\mu^+$ is nonzero on the $(G_-(a) \times G_+(b))$ -invariants. Note that $G_-(a) \times G_+(b) = G(c)$, where $c = (a, 0, b) \in \mathbf{R}^{(n+m+1)}$. We must therefore show that $e_\lambda^- e_\mu^+$ is nonzero on $L_v^{G(c)}$. This space is 1-dimensional, and spanned by ψ_v^c (see Proposition 5.2). Thus it is enough to show that e_λ^- and e_μ^+ are separately nonzero on ψ_v^c .

The argument here is similar to the final paragraph of the proof of Proposition 5.10. We have $e_\lambda^-(\psi_v^c) = \alpha \cdot \psi_v^c$ for some scalar α , and we must show that $\alpha \neq 0$. We have

$$\alpha = \int_{\mathbf{R}_-^{(n)}} \psi_\lambda^a(p) \cdot \psi_v^{(p, 0, b)}(c) dp,$$

and so $\alpha = \text{vol}(X)$, where

$$X = \{p \in \mathbf{R}_-^{(n)} \mid p \in \Psi_\lambda^a \text{ and } (a, 0, b) \in \Psi_v^{(p, 0, b)}\}.$$

One finds that $X = \{a\}$, and so $\alpha = 1$. The case of e_μ^+ is similar.

We have thus shown that $L_\lambda^- \boxtimes L_\mu^+$ appears in the restriction of $L_{\lambda\bullet\mu}$. The case of $L_{\lambda\circ\mu}$ is similar. $\square$

6.2. Restriction

Recall (Section 3.5) that we also have a map

$$\text{res}: \mathbf{K} \rightarrow \mathbf{K} \otimes \mathbf{K}$$

given by restricting from G to $G \times G$. We now determine this map explicitly. We first introduce a piece of notation: for a weight $\lambda = \lambda_1 \cdots \lambda_n$ of length n , we let $\lambda[i, j]$ denote the substring of λ between indices i and j (inclusively), that is, $\lambda_i \cdots \lambda_j$. As with intervals, we use parentheses to exclude the edge values, for example, $\lambda[i, j) = \lambda_i \cdots \lambda_{j-1}$. Empty intervals yield the empty word, for example, $\lambda[1, 0] = \emptyset$.

THEOREM 6.9

Let λ be a word of length n . Then

$$\text{res}(a_\lambda) = \sum_{i=0}^n a_{\lambda[1,i]} \otimes a_{\lambda(i,n]} + \sum_{i=1}^n a_{\lambda[1,i)} \otimes a_{\lambda(i,n]}.$$

In words, the first sum is over all ways of breaking λ into two between letters (or at the ends), while the second sum is over all ways of breaking λ into two by deleting letters.

Proof

Let α and β be weights, and let $i = \ell(\alpha)$. We have

$$\begin{aligned} \langle \text{res}(a_\lambda), a_\alpha \otimes a_\beta \rangle &= \langle a_\lambda, \text{ind}(a_\alpha \otimes a_\beta) \rangle \\ &= \langle a_\lambda, a_{\alpha \bullet \beta} \rangle + \langle a_\lambda, a_{\alpha \circ \beta} \rangle + \langle a_\lambda, a_{\alpha \beta} \rangle, \end{aligned}$$

where in the first step we used Frobenius reciprocity (see Section 3.5), and in the second Corollary 6.2. The first two terms above, counted together, give 1 if $\alpha = \lambda[1, i]$ and $\beta = \lambda[i + 2, n]$ and $0 \leq i \leq n - 1$, and 0 otherwise. The third term above gives 1 if $\alpha = \lambda[1, i]$ and $\beta = \lambda[i + 1, n]$, and 0 otherwise. This yields the stated formula. $\square$

The theorem gives us the following very important corollary for how restriction interacts with the filtration on K .

COROLLARY 6.10

Let $x \in K_{\leq n}$. Then

$$\text{res}(x) = x \otimes 1 + 1 \otimes x + y,$$

where y belongs to $\sum_{i+j < n} K_{\leq i} \otimes K_{\leq j}$.

The point here is that y is built out of elements in smaller filtration degree; thus $\text{res}(x)$ has a very simple form up to an error that can be controlled. This observation is used in several proofs in the remainder of the paper; the following example illustrates the basic mechanism used in these arguments.

Example 6.11

Theorem 6.9 gives an alternative approach to computing the categorical dimension of simple objects. Let λ of length n be given, and suppose we know that a_μ has dimension $(-1)^{\ell(\mu)}$ for $\ell(\mu) < n$. From the theorem, we have

$$\text{res}(a_\lambda) = (a_\lambda \otimes 1 + 1 \otimes a_\lambda) + \sum_{i=1}^{n-1} a_{\lambda[1,i]} \otimes a_{\lambda(i,n)} + \sum_{i=1}^n a_{\lambda[1,i]} \otimes a_{\lambda(i,n)}.$$

Categorical dimension is invariant under restriction, and multiplicative for the external tensor product. Thus if d is the dimension of a_λ , then the above equation gives

$$d = 2d + (n-1)(-1)^n + n(-1)^{n-1},$$

and so $d = (-1)^n$.

6.3. The Hopf algebra structure on K

Recall (Proposition 5.9) that we have an algebra homomorphism $\delta: K \rightarrow \mathbf{Z}$ defined by $\delta(a_\lambda) = (-1)^{\ell(\lambda)}$.

PROPOSITION 6.12

The map δ is a counit for res.

Proof

We must show that the composition

$$K \xrightarrow{\text{res}} K \otimes K \xrightarrow{\delta \otimes 1} K$$

is the identity, and similarly for $1 \otimes \delta$. Let λ be a weight of length n . Applying Theorem 6.9, we have

$$\begin{aligned} (\delta \otimes 1)(\text{res}(a_\lambda)) &= \sum_{i=0}^n \delta(a_{\lambda[1,i]}) a_{\lambda(i,n)} + \sum_{i=1}^n \delta(a_{\lambda[1,i]}) a_{\lambda(i,n)} \\ &= a_\lambda + \sum_{i=1}^n (\delta(a_{\lambda[1,i]}) + \delta(a_{\lambda[1,i]}) a_{\lambda(i,n)}). \end{aligned}$$

The a_λ here is the $i = 0$ term of the first sum. Since the lengths of $\lambda[1, i]$ and $\lambda[1, i]$ differ by one, the δ 's in the second line cancel, and so the final result is simply a_λ . This shows that $(\delta \otimes 1) \circ \text{res}$ is the identity. The $1 \otimes \delta$ case is similar. $\square$

We now know that K is a bialgebra, under the standard multiplication m and the comultiplication given by res ; the unit is 1 and the counit is δ . In fact, we have the following.

PROPOSITION 6.13

The bialgebra K is a Hopf algebra.

Proof

Recall that if $f, g: K \rightarrow K$ are additive maps, then their *convolution* $f * g$ is the composition

$$K \xrightarrow{\text{res}} K \otimes K \xrightarrow{f \otimes g} K \otimes K \xrightarrow{m} K.$$

Convolution gives the set $\text{Hom}(K, K)$ of all additive maps the structure of a unital and associative algebra (see [14, Section 4.2]). The unit is the composition $\eta\delta$, where $\eta: \mathbf{Z} \rightarrow K$ is the unit map. An *antipode* is an additive map $S: K \rightarrow K$ that is a two-sided inverse to the identity map under convolution, that is, we have $S * \text{id} = \text{id} * S = \eta\delta$. To prove the proposition, we must show that K admits an antipode.

Consider the equation $\text{id} * S = \eta\delta$ on a map S . Let λ be a word of length n . Evaluating at a_λ , this takes the form

$$m((\text{id} \otimes S)(\text{res}(a_\lambda))) = (-1)^n.$$

Appealing to our computation of res (Theorem 6.9), this becomes

$$\sum_{i=0}^n a_{\lambda[1,i]} S(a_{\lambda(i,n)}) + \sum_{i=1}^n a_{[1,i]} S(a_{\lambda(i,n)}) = (-1)^n.$$

The $i = 0$ term in the first sum is $S(a_\lambda)$. Every other input to S in the above equation has the form a_μ , where $\ell(\mu) < n$. Thus the above equation allows us to recursively solve for $S(a_\lambda)$.

The above reasoning shows that there is a unique right inverse to id under convolution. A similar argument shows that there is a unique left inverse. Since convolution is associative, these two inverses are equal and are thus the antipode. $\square$

Example 6.14

Proposition 6.13 shows how to recursively compute the antipode S . Here are a few examples:

$$S(a_\bullet) = -a_\bullet - 2,$$

$$S(a_{\bullet\bullet}) = a_{\bullet\bullet} + 3a_\bullet + 3,$$

$$S(a_{\bullet\circ}) = a_{\circ\bullet} + 2a_\bullet + 2a_\circ + 4.$$

We do not know the general formula for $S(a_\lambda)$. However, the leading term is always $(-1)^{\ell(\lambda)} a_\mu$, where μ is the reverse of λ (see Remark 7.22).

6.4. Classification of primitive elements

An element $x \in K$ is *primitive* if

$$\text{res}(x) = (x \otimes 1) + (1 \otimes x).$$

The collection of primitive elements of K forms an additive subgroup, which we denote by K^{prim} . We now determine it.

PROPOSITION 6.15

The group K^{prim} has a $\mathbf{Z}$ -basis consisting of $a_{\bullet} + 1$ and $a_{\circ} + 1$.

Proof

From Theorem 6.9, we have

$$\text{res}(a_{\bullet} + 1) = (a_{\bullet} \otimes 1) + (1 \otimes a_{\bullet}) + 2;$$

note that $\text{res}(1) = 1$. We also have

$$(a_{\bullet} + 1) \otimes 1 + 1 \otimes (a_{\bullet} + 1) = (a_{\bullet} \otimes 1) + (1 \otimes a_{\bullet}) + 2.$$

Thus $a_{\bullet} + 1$ is primitive, and similarly for $a_{\circ} + 1$.

Let $\gamma: K \otimes K \rightarrow K$ be the concatenation product, that is, $\gamma(a_{\lambda} \otimes a_{\mu}) = a_{\lambda\mu}$. From Theorem 6.9, we have $\gamma(\text{res}(a_{\lambda})) = (n+1)a_{\lambda} + y$, where y belongs to $K_{\leq n-1}$. It follows that for any $x \in K_{\leq n}$, we have $\gamma(\text{res}(x)) = (n+1)x + y$ for some $y \in K_{\leq n-1}$. On the other hand, if x is primitive, then clearly $\gamma(\text{res}(x)) = 2x$. We thus see that x can be primitive only if x belongs to $K_{\leq 1}$; in other words, $K^{\text{prim}} \subset K_{\leq 1}$.

Now, the $\mathbf{Z}$ -module $K_{\leq 1}$ is free of rank 3. Since no multiple of 1 is primitive, it follows that K^{prim} has rank at most 2. But we have already exhibited two independent elements, so K^{prim} has rank 2. Since these two elements generate a saturated $\mathbf{Z}$ -submodule of $K_{\leq 1}$, they must generate all of K^{prim} . $\square$

Let $\Delta: K \rightarrow K \otimes K$ be the map given by

$$\Delta(x) = \text{res}(x) - x \otimes 1 - 1 \otimes x.$$

Thus x is primitive if and only if $\Delta(x) = 0$. Since there are very few primitive elements, Δ is nearly injective. On the other hand, Δ maps $K_{\leq n}$ into $K_{\leq n-1} \otimes K_{\leq n-1}$ by Corollary 6.10. This provides a very useful setup for inductive arguments, which we employ a few times.

7. Tensor products

In this section, we give a combinatorial rule for the decomposition of a tensor product of two simple representations. As an application, we show that $K \otimes \mathbf{Q}$ is a polynomial ring with variables indexed by Lyndon words (Corollary 7.21).

7.1. The tensor product formula

Let λ and μ be two weights of lengths n and m . A *ruffle*, or (n, m) -*ruffle*, is a surjective function $r: [n] \amalg [m] \rightarrow [\ell]$ that is injective and order-preserving on $[n]$ and on $[m]$. We think of a ruffle like a shuffle except that there can be collisions. Fix a ruffle r , and let r_1 and r_2 be the restrictions of r to $[n]$ and $[m]$. We say that there is a *collision* at $i \in [\ell]$ if i belongs to the images of r_1 and r_2 . Suppose that $i = r_1(j) = r_2(\ell)$ is a collision. We say that the collision is *positive* if $\lambda_j = \mu_\ell = \bullet$, *negative* if $\lambda_j = \mu_\ell = \circ$, and *neutral* otherwise. We define the *preproduct* of λ and μ with respect to r to be the length ℓ word defined as follows. At the i th spot, we put:

- the letter λ_j if $i = r_1(j)$ is not a collision,
- the letter μ_ℓ if $i = r_2(\ell)$ is not a collision,
- the letter $\bullet$ if i is a positive collision,
- the letter $\circ$ if i is a negative collision,
- the letter $?$ if i is a neutral collision.

We now define a multiset $P_r(\lambda, \mu)$ by replacing each $?$ in the preproduct with either $\bullet$, $\circ$, or the empty string (in which case the $?$ character is simply deleted), in all possible ways. Thus if there are q question marks, then $P_r(\lambda, \mu)$ has cardinality 3^q (as a multiset). Finally, we define

$$m'(a_\lambda, a_\mu) = \sum_r \sum_{v \in P_r(\lambda, \mu)} a_v,$$

where the outer sum is over all (n, m) -ruffles. We extend m' bilinearly to a product on K .

Example 7.1

Consider the two weights $\lambda = \bullet$ and $\mu = \bullet\circ$. There are five ruffles. The corresponding preproducts are

$$\underline{\bullet}\bullet\circ, \quad \bullet\underline{\bullet}\circ, \quad \bullet\circ\underline{\bullet}, \quad \underline{\bullet}\circ, \quad \bullet\underline{?}.$$

We have underlined the position that λ occupies in the above words. In the first three cases above there are no collisions (i.e., they are shuffles); in the fourth case there is a positive collision; and in the fifth case there is a neutral collision. In the first four cases, $P_r(\lambda, \mu)$ contains just the indicated word; in the final case, $P_r(\lambda, \mu) = \{\bullet, \bullet\bullet, \bullet\circ\}$. Thus

$$m'(a_\bullet, a_{\bullet\circ}) = 2a_{\bullet\bullet\circ} + a_{\bullet\circ\bullet} + 2a_{\bullet\circ} + a_{\bullet\bullet} + a_{\bullet}.$$

Throughout this section, we write m for the standard product on K , that is, the one induced by tensor products. The following is our main result on tensor products.

THEOREM 7.2

We have $m = m'$.

The theorem gives the following combinatorial rule for computing tensor products.

COROLLARY 7.3

Let λ and μ be weights of lengths n and m . Then

$$L_\lambda \underline{\otimes} L_\mu \cong \bigoplus_r \bigoplus_{v \in P_r(\lambda, \mu)} L_v,$$

where the outer sum is over all (n, m) -ruffles r .

Example 7.4

Combining Example 7.1 and Corollary 7.3, we find that

$$L_\bullet \underline{\otimes} L_{\bullet\bullet} \cong L_{\bullet\bullet\bullet}^{\oplus 2} \oplus L_{\bullet\bullet\bullet} \oplus L_{\bullet\bullet\bullet}^{\oplus 2} \oplus L_{\bullet\bullet} \oplus L_\bullet.$$

To prove the theorem, we compute one special type of product for m (Proposition 7.5) and establish the projection formula for m' (Proposition 7.14); this turns out to be enough to deduce that $m = m'$ (see Proposition 7.9).

7.2. Some simple tensor products

We now compute a family of particularly simple products. Let $\pi(n)$ be a string of length n consisting of all $\bullet$'s.

PROPOSITION 7.5

For $n \geq 0$, we have

$$a_{\pi(1)} \cdot a_{\pi(n)} = (n + 1)a_{\pi(n+1)} + na_{\pi(n)}.$$

We first give a nonrigorous but illuminative argument. The tensor product $L_{\pi(1)} \underline{\otimes} L_{\pi(n)}$ is the subrepresentation of $\mathcal{C}(\mathbf{R} \times \mathbf{R}^{(n)})$ generated by the functions $\varphi_{(J, I_1, \dots, I_n)}$, where J and each I_i has type $\bullet$, and $I_1 < \dots < I_n$. The functions with $J = I_i$ generate a copy of $L_{\pi(n)}$, while the functions with $I_i < J < I_{i+1}$ (or $J < I_1$ or $I_n < J$) generate a copy of $L_{\pi(n+1)}$. This gives the stated decomposition.

One can make the above argument rigorous; however, there are many details to handle. Instead, we give a different proof simply by computing in the Grothendieck group, which avoids any subtleties. This proof is perhaps less insightful, but demonstrates how rigid the various structures on $\mathbf{K}$ are.

Put $x_n = a_{\pi(1)}a_{\pi(n)}$ and $y_n = (n+1)a_{\pi(n+1)} + na_{\pi(n)}$. Thus we must show that $x_n = y_n$. We will eventually do this by induction. We prove two lemmas first. Recall from Section 6.4 that $\Delta(x) = \text{res}(x) - x \otimes 1 - 1 \otimes x$.

LEMMA 7.6

Let $n > 0$ be given, and suppose that $x_i = y_i$ for $0 \leq i < n$. Then $\Delta(x_n) = \Delta(y_n)$.

Proof

Put

$$\Sigma_n = \sum_{i+j=n} a_{\pi(i)} \otimes a_{\pi(j)}$$

so that $\text{res}(a_{\pi(n)}) = \Sigma_n + \Sigma_{n-1}$ by Theorem 6.9. Since res is a ring homomorphism, we have

$$\text{res}(x_n) = (\Sigma_1 + 1) \cdot (\Sigma_n + \Sigma_{n-1}).$$

We have

$$\begin{aligned} (a_{\pi(1)} \otimes 1) \cdot \Sigma_n &= \sum_{i+j=n} x_i \otimes a_{\pi(j)} \\ &= (x_n - y_n) \otimes 1 + \sum_{i+j=n} ((i+1)a_{\pi(i+1)} + ia_{\pi(i)}) \otimes a_{\pi(j)} \\ &= (x_n - y_n) \otimes 1 + \sum_{i+j=n+1} ia_{\pi(i)} \otimes a_{\pi(j)} + \sum_{i+j=n} ia_{\pi(i)} \otimes a_{\pi(j)}. \end{aligned}$$

There is a similar formula for $(1 \otimes a_{\pi(1)}) \cdot \Sigma_n$. Summing these, we find that

$$\Sigma_1 \cdot \Sigma_n = (x_n - y_n) \otimes 1 + 1 \otimes (x_n - y_n) + (n+1)\Sigma_{n+1} + n\Sigma_n.$$

A similar computation gives

$$\Sigma_1 \cdot \Sigma_{n-1} = n\Sigma_n + (n-1)\Sigma_{n-1}.$$

Combining all the above, we find that

$$\text{res}(x_n) = (x_n - y_n) \otimes 1 + 1 \otimes (x_n - y_n) + (n+1)\Sigma_{n+1} + (2n+1)\Sigma_n + n\Sigma_{n-1}.$$

We also have

$$\begin{aligned} \text{res}(y_n) &= (n+1)(\Sigma_{n+1} + \Sigma_n) + n(\Sigma_n + \Sigma_{n-1}) \\ &= (n+1)\Sigma_{n+1} + (2n+1)\Sigma_n + n\Sigma_{n-1}. \end{aligned}$$

Thus

$$\text{res}(x_n) - \text{res}(y_n) = (x_n - y_n) \otimes 1 + 1 \otimes (x_n - y_n),$$

and so $\Delta(x_n) = \Delta(y_n)$, as required. $\square$

LEMMA 7.7

We have

$$a_{\bullet}a_{\bullet} = 2a_{\bullet\bullet} + a_{\bullet}, \quad a_{\bullet}a_{\circ} = a_{\bullet\circ} + a_{\circ\bullet} + a_{\bullet} + a_{\circ} + 1, \quad a_{\circ}a_{\circ} = 2a_{\circ\circ} + a_{\circ}.$$

Proof

From Lemma 7.6, we have $\Delta(x_1) = \Delta(y_1)$. It follows that $x_1 - y_1$ belongs to the kernel of Δ and is therefore primitive. Thus by Proposition 6.15, we have

$$a_{\bullet}a_{\bullet} = 2a_{\bullet\bullet} + a_{\bullet} + p(a_{\bullet} + 1) + q(a_{\circ} + 1)$$

for integers p and q . Since this is an effective class, we have $p \geq -1$ and $q \geq 0$. We also have

$$p + q = \langle a_{\bullet}a_{\bullet}, 1 \rangle = \langle a_{\bullet}, a_{\circ} \rangle = 0,$$

where in the second step we used adjunction. We thus have

$$a_{\bullet}a_{\bullet} = 2a_{\bullet\bullet} + (1 - q)a_{\bullet} + qa_{\circ}$$

and $q \in \{0, 1\}$. Next,

$$\begin{aligned} a_{\bullet} \cdot (a_{\bullet} + a_{\circ} + 1) &= a_{\bullet} \cdot \text{ind}(1) = \text{ind}(\text{res}(a_{\bullet})) \\ &= \text{ind}(a_{\bullet} \otimes 1 + 1 \otimes a_{\bullet} + 1) \\ &= 2a_{\bullet\bullet} + a_{\bullet\circ} + a_{\circ\bullet} + 3a_{\bullet} + a_{\circ} + 1, \end{aligned}$$

where we have used Theorems 6.1 and 6.9 and the projection formula. Thus

$$a_{\bullet}a_{\circ} = a_{\bullet\circ} + a_{\circ\bullet} + (1 + q)a_{\bullet} + (1 - q)a_{\circ} + 1.$$

Since the above element is obviously self-dual, we must have $q = 0$. This yields the first two formulas, while the third follows from the first by duality. $\square$

Proof of Proposition 7.5

We must show that $x_n = y_n$ for all n . The $n = 0$ case is trivial, while the $n = 1$ case follows from Lemma 7.7. Now let $n \geq 2$ be given, and suppose that $x_i = y_i$ for

$0 \leq i < n$. By Lemma 7.6, we have $\Delta(x_n) = \Delta(y_n)$. Thus $x_n - y_n$ is primitive, and so by Proposition 6.15 we have

$$x_n = y_n + p(a_\bullet + 1) + q(a_\circ + 1)$$

for integers p and q . We have

$$p + q = \langle x_n, 1 \rangle = \langle a_{\pi(n)}, a_\circ \rangle = 0,$$

where in the second step we used adjunction. We also have

$$q = \langle x_n, a_\circ \rangle = \langle a_{\pi(n)}, a_\circ a_\circ \rangle,$$

where again we have used adjunction. From the computation of $a_\circ a_\circ$ in Lemma 7.7, the above pairing vanishes. Thus $p = q = 0$, and so $x_n = y_n$ as required. $\square$

Let $\mathbf{Z}\langle x \rangle$ be the ring of integer-valued polynomials in the variable x . Put $b_n = \binom{x}{n}$. The elements $\{b_n\}_{n \geq 0}$ form a $\mathbf{Z}$ -basis of $\mathbf{Z}\langle x \rangle$.

COROLLARY 7.8

The additive map $\varphi: \mathbf{Z}\langle x \rangle \rightarrow \mathbf{K}$ defined by $\varphi(b_n) = a_{\pi(n)}$ is a ring homomorphism.

Proof

We have $b_1 b_n = (n+1)b_{n+1} + n b_n$. Since b_1 generates $\mathbf{Z}\langle x \rangle \otimes \mathbf{Q}$ as a $\mathbf{Q}$ -algebra, it follows that $\varphi \otimes \mathbf{Q}$ is a ring homomorphism. Since $\mathbf{K}$ is torsion-free, it follows that φ is a ring homomorphism. $\square$

7.3. Characterization of m

Let

$$q: \mathbf{K} \times \mathbf{K} \rightarrow \mathbf{K}$$

be a bilinear map, which we view as a product on $\mathbf{K}$. We write q still for the induced product on $\mathbf{K} \otimes \mathbf{K}$. We consider the following conditions on μ :

- (a) The product q is commutative and unital with unit $a_\emptyset$.
- (b) The projection formula holds: for $x, y, z \in \mathbf{K}$, we have

$$\text{ind}(q(\text{res}(x), y \otimes z)) = q(x, \text{ind}(y \otimes z)).$$

- (c) For any $n \geq 0$, we have

$$q(a_{\pi(1)}, a_{\pi(n)}) = (n+1)a_{\pi(n+1)} + n a_{\pi(n)}.$$

We characterize the standard product m via these conditions.

PROPOSITION 7.9

The standard product is the unique product satisfying (a)–(c) above.

We will first require a lemma. Put

$$I_{\leq n} = \sum_{i+j=n-1} \text{ind}(K_{\leq i} \otimes K_{\leq j}).$$

Note that $I_{\leq n}$ is contained in $K_{\leq n}$ by Theorem 6.1. We define the *Hamming distance* between two weights of the same length to be the number of positions at which they differ.

LEMMA 7.10

Let λ and μ be two weights of length n with Hamming distance d . Then in the quotient

$$K_{\leq n} / (K_{\leq n-1} + I_{\leq n})$$

we have $a_\lambda = (-1)^d a_\mu$. In particular, the above group is cyclic and generated by $a_{\pi(n)}$.

Proof

It suffices to treat the case $d = 1$. Thus, after swapping λ and μ if necessary, we have $\lambda = \alpha \bullet \beta$ and $\mu = \alpha \circ \beta$ for two weights α and β whose lengths sum to $n - 1$. In the quotient group we have

$$\text{ind}(a_\alpha \otimes a_\beta) = a_\lambda + a_\mu + a_{\alpha\beta} = 0,$$

and also $a_{\alpha\beta} = 0$, and so the result follows. $\square$

Proof of Proposition 7.9

We have seen in Section 3.5 that m satisfies (a) and (b), while (c) is given by Proposition 7.5. Suppose now that q is an arbitrary product satisfying (a)–(c). We must show that $q = m$. Consider the following statement:

(S_n) : If $x \in K_{\leq r}$ and $y \in K_{\leq s}$ with $r + s \leq n$, then $q(x, y) = m(x, y)$.

We prove (S_n) by induction on n . The statement (S_0) follows from (a), since $K_{\leq 0}$ is spanned by 1. Suppose now that (S_{n-1}) holds, and let us prove (S_n) .

Let x and y as in (S_n) be given. We must show that

$$q(x, y) = m(x, y). \quad (7.11)$$

If y belongs to $K_{\leq s-1}$, then (7.11) holds by (S_{n-1}) . Suppose now that $y = \text{ind}(a_\alpha \otimes a_\beta)$, where $\ell(\alpha) + \ell(\beta) \leq s - 1$. We have

$$q(x, y) = \text{ind}(q(\text{res}(x), a_\alpha \otimes a_\beta)),$$

and similarly for m ; this is where (b) is used. Note that $\text{res}(x)$ is contained in $(K \otimes K)_{\leq r}$ by Theorem 6.9. By (S_{n-1}) , we thus have

$$q(\text{res}(x), a_\alpha \otimes a_\beta) = m(\text{res}(x), a_\alpha \otimes a_\beta).$$

Applying ind , we find that (7.11) holds. We thus see that (7.11) holds if y belongs to $K_{\leq s-1} + I_{\leq s}$; of course, there is an analogous statement for x (since q is commutative).

Applying Lemma 7.10, write

$$x = ua_{\pi(r)} + x', \quad y = va_{\pi(s)} + y',$$

where $u, v \in \mathbf{Z}$ and $x' \in K_{\leq r-1} + I_{\leq r}$ and $y' \in K_{\leq s-1} + I_{\leq s}$. We have

$$q(x, y) = uvq(a_{\pi(r)}, a_{\pi(s)}) + uq(a_{\pi(r)}, y') + vq(x', a_{\pi(s)}) + q(x', y').$$

Call the four terms above $q_1, \dots, q_4$. We have a similar expression for m , yielding terms $m_1, \dots, m_4$. We have $q_1 = m_1$ by an easy inductive argument, or, more directly, by the reasoning in Corollary 7.8; in any case, this is where assumption (c) is used. In each of the remaining terms, we have $q_i = m_i$ by the previous paragraph. Thus (7.11) holds, which completes the proof. $\square$

Example 7.12

The proof of Proposition 7.9 explains how to actually compute arbitrary products using the projection formula and Proposition 7.5. We illustrate the simplest case by computing $a_\bullet \cdot a_\circ$. First we express $a_\circ$ in terms of inductions and $a_{\pi(n)}$'s. By Theorem 6.1, we have

$$a_\circ = \text{ind}(1) - a_\bullet - 1.$$

Multiplying by $a_\bullet$, we find that

$$a_\bullet \cdot a_\circ = \text{ind}(\text{res}(a_\bullet)) - a_\bullet \cdot a_\bullet - a_\bullet.$$

We have $a_\bullet \cdot a_\bullet = 2a_{\bullet\bullet} + a_\bullet$ by Proposition 7.5. We have

$$\text{res}(a_\bullet) = (a_\bullet \otimes 1) + (1 \otimes a_\bullet) + 1$$

by Theorem 6.9, and so

$$\text{ind}(\text{res}(a_\bullet)) = 2a_{\bullet\bullet} + a_{\bullet\circ} + a_{\circ\bullet} + 3a_\bullet + a_\circ + 1$$

by Theorem 6.1. Combining the above, we thus find that

$$a_{\bullet} \cdot a_{\circ} = a_{\bullet\circ} + a_{\circ\bullet} + a_{\bullet} + a_{\circ} + 1.$$

Note that we computed the above product in Lemma 7.7, which was used in the proof of Proposition 7.5, so the above computation is somewhat circular; it is just intended to illustrate the procedure provided by the proof of Proposition 7.9.

Remark 7.13

Lemma 7.10 implies that the elements $a_{\pi(n)}$ generate K under the induction product (which is a nonunital ring).

7.4. The projection formula for m'

We would like to apply Proposition 7.9 to show that $m' = m$. For this, we need to know that m' satisfies the projection formula. We now prove this.

PROPOSITION 7.14

For $x, y, z \in K$, we have

$$m'(x, \text{ind}(y \otimes z)) = \text{ind}(m'(\text{res}(x), y \otimes z)).$$

Before giving the proof, we require two lemmas. We begin by verifying the projection formula in a simple case.

LEMMA 7.15

For $x \in K$, we have $m'(x, \text{ind}(1)) = \text{ind}(\text{res}(x))$.

Proof

It suffices to treat the case $x = a_{\lambda}$. Let $n = \ell(\lambda)$. Define elements x_2, x_3, x_4, x_5 as follows:

- x_2 is the sum of all a_{μ} 's where μ is obtained by inserting a $\bullet$ in between two letters of λ , or at the start or end of λ ; this sum has $n + 1$ terms.
- x_3 is defined like x_2 , except that we insert $\circ$.
- x_4 is the sum of all a_{μ} 's where μ is obtained by toggling a single letter of λ (switching a $\bullet$ to a $\circ$, or vice versa); this sum has n terms.
- x_5 is the sum of all a_{μ} 's where μ is obtained by deleting a letter from λ ; this sum has n terms.

We will show that each side of the equation in the statement of the lemma is equal to

$$(2n + 1)x + x_2 + x_3 + x_4 + x_5, \tag{7.16}$$

which will complete the proof.

We begin with the left side. From the computation of $\text{ind}(1)$, we have

$$m'(x, \text{ind}(1)) = m'(x, a_\bullet) + m'(x, a_\circ) + x.$$

We claim that this is equal to (7.16). The terms in $m'(x, a_\bullet)$ without collisions produce x_2 ; similarly, the terms in $m'(x, a_\circ)$ without collisions produce x_3 . We now consider collisions at the i th letter of λ , for both products at once. Together, these produce

$$a_{\lambda[1,i]} \odot (2a_{\lambda_i} + a_{\lambda_i}^\vee + 1) \odot a_{\lambda(i,n)},$$

where $a_{\lambda_i}^\vee$ just means that we toggle λ_i . Indeed, when we collide with λ_i , we just get λ_i , while in the other case we get $a_\bullet + a_\circ + 1$, which is $a_{\lambda_i} + a_{\lambda_i}^\vee + 1$. Summing over i , we obtain $2nx + x_4 + x_5$. This proves the claim.

We next claim that the right side is also equal to (7.16). By Theorem 6.9, we have $\text{res}(x) = y_1 + y_2$, where

$$y_1 = \sum_{i=0}^n a_{\lambda[1,i]} \otimes a_{\lambda(i,n)}, \quad y_2 = \sum_{i=1}^n a_{\lambda[1,i]} \otimes a_{\lambda(i,n)}.$$

Thus y_1 is the sum of ways of breaking λ into two pieces between letters (or at the ends), while y_2 is the sum of ways of breaking λ into two pieces by deleting letters. By Theorem 6.1, we have

$$\text{ind}(y_1) = \sum_{i=0}^n (a_{\lambda[1,i]} \odot (1 + a_\bullet + a_\circ) \odot a_{\lambda(i,n)}) = (n+1)x + x_2 + x_3.$$

Similarly,

$$\text{ind}(y_2) = \sum_{i=1}^n (a_{\lambda[1,i]} \odot (1 + a_\bullet + a_\circ) \odot a_{\lambda(i,n)}) = x_5 + nx + x_4.$$

In each term in the above sum, one choice of $a_\bullet$ or $a_\circ$ will replace the deleted letter, and these terms amount to nx ; the terms corresponding to the other choice amount to x_4 . This proves the claim, and completes the proof. $\square$

For $\alpha, \beta \in \{\bullet, \circ\}$, we define a quantity $c(\alpha, \beta) \in \mathbf{K}$ as follows:

$$c(\alpha, \beta) = \begin{cases} a_\bullet & \text{if } \alpha = \beta = \bullet, \\ a_\bullet + a_\circ + 1 & \text{if } \alpha \neq \beta, \\ a_\circ & \text{if } \alpha = \beta = \circ. \end{cases}$$

Note that $c(\alpha, \beta)$ is exactly the quantity used in the definition of m' when a collision is encountered. The following lemma gives a recursive characterization of m' .

LEMMA 7.17

Let λ and ρ be weights of lengths r and 1, and let $x \in K$. We then have

$$\begin{aligned} m'(a_\lambda, a_\rho \odot x) &= \sum_{i=0}^r a_{\lambda[1,i]} \odot a_\rho \odot m'(a_{\lambda(i,r]}, x) \\ &\quad + \sum_{i=1}^r a_{\lambda[1,i]} \odot c(\lambda_i, \rho) \odot m'(a_{\lambda(i,r]}, x). \end{aligned}$$

There is a similar formula when the order of $a_\rho \odot x$ is reversed.

Proof

The first sum is the case where the ruffle has no collision and i is taken to be maximal such that $[1, i]$ in the first variable is mapped to $[1, i]$ by the ruffle. The second sum is the case where the ruffle has a collision at i . $\square$

Proof of Proposition 7.14

Consider the following statement:

(S_n) Given $x \in K$ and $y \otimes z \in (K \otimes K)_{\leq n}$, we have

$$m'(x, \text{ind}(y \otimes z)) = \text{ind}(m'(\text{res}(x), y \otimes z)).$$

We prove (S_n) by induction on n . The $n = 0$ case was established in Lemma 7.15. Suppose now that $n \geq 1$ and (S_{n-1}) holds; we will prove (S_n) .

It suffices to treat the case where x , y , and z are basis vectors. Moreover, we can assume that y and z are not both 1 since this is covered by (S_0) ; since y and z play symmetric roles, without loss of generality we can assume that $y \neq 1$. We can thus write $x = a_\lambda$, $y = a_{\rho\alpha}$, and $z = a_\beta$, where ρ is a single letter, and $\ell(\alpha) + \ell(\beta) \leq n - 1$. Put $w = \text{ind}(a_\alpha \otimes a_\beta)$. Note that $y = a_\rho \odot a_\alpha$, and $\text{ind}(y \otimes z) = a_\rho \odot w$ by Corollary 6.3. Put

$$A = m'(a_\lambda, a_\rho \odot w), \quad B = m'(\text{res}(a_\lambda), a_{\rho\alpha} \otimes a_\beta).$$

We must prove that $A = \text{ind}(B)$.

Computing $\text{res}(a_\lambda)$ via Theorem 6.9, we find that $B = X + Y$, where

$$\begin{aligned} X &= \sum_{j=0}^r m'(a_{\lambda[1,j]}, a_\rho \odot a_\alpha) \otimes m'(a_{\lambda(j,r]}, a_\beta), \\ Y &= \sum_{j=1}^r m'(a_{\lambda[1,j]}, a_\rho \odot a_\alpha) \otimes m'(a_{\lambda(j,r]}, a_\beta). \end{aligned}$$

Applying Lemma 7.17 to the first m' in X , we obtain $X = X_1 + X_2$ where

$$X_1 = \sum_{j=0}^r \sum_{i=0}^j (a_{\lambda[1,i]} \odot a_\rho \odot m'(a_{\lambda(i,j)}, a_\alpha)) \otimes m'(a_{\lambda(j,r)}, a_\beta),$$

$$X_2 = \sum_{j=0}^r \sum_{i=1}^j (a_{\lambda[1,i]} \odot c(\lambda_i, \rho) \odot m'(a_{\lambda(i,j)}, a_\alpha)) \otimes m'(a_{\lambda(j,r)}, a_\beta).$$

Similarly, we obtain $Y = Y_1 + Y_2$ where

$$Y_1 = \sum_{j=1}^r \sum_{i=0}^{j-1} (a_{\lambda[1,i]} \odot a_\rho \odot m'(a_{\lambda(i,j)}, a_\alpha)) \otimes m'(a_{\lambda(j,r)}, a_\beta),$$

$$Y_2 = \sum_{j=1}^r \sum_{i=1}^{j-1} (a_{\lambda[1,i]} \odot c(\lambda_i, \rho) \odot m'(a_{\lambda(i,j)}, a_\alpha)) \otimes m'(a_{\lambda(j,r)}, a_\beta).$$

Reversing the order of the sums in $X_1 + Y_1$ and using Theorem 6.9, we obtain

$$\begin{aligned} & \sum_{i=0}^r ((a_{\lambda[1,i]} \odot a_\rho) \otimes 1) \odot \left(\sum_{j=i}^r m'(a_{\lambda(i,j)}, a_\alpha) \right) \otimes m'(a_{\lambda(j,r)}, a_\beta) \\ & + \sum_{j=i+1}^r m'(a_{\lambda(i,j)}, a_\alpha) \otimes m'(a_{\lambda(j,r)}, a_\beta) \\ & = \sum_{i=0}^r ((a_{\lambda[1,i]} \odot a_\rho) \otimes 1) \odot m'(\text{res}(a_{\lambda(i,r)}), a_\alpha \otimes a_\beta). \end{aligned}$$

Applying ind to this, and appealing to Corollary 6.3 and (S_{n-1}) , we obtain

$$\text{ind}(X_1 + Y_1) = \sum_{i=0}^r a_{\lambda[1,i]} \odot a_\rho \odot m'(a_{\lambda(i,r)}, w).$$

Similarly, we find that

$$X_2 + Y_2 = \sum_{i=1}^r (a_{\lambda[1,i]} \odot c(\lambda_i, \rho) \otimes 1) \odot m'(\text{res}(\lambda_{(i,r)}), a_\alpha \otimes a_\beta),$$

and so

$$\text{ind}(X_2 + Y_2) = \sum_{i=1}^r a_{\lambda[1,i]} \odot c(\lambda_i, \rho) \odot m'(\lambda_{(i,r)}, w).$$

Putting this all together, we obtain

$$\text{ind}(B) = \sum_{i=0}^r a_{\lambda[1,i]} \odot a_{\rho} \odot m'(a_{\lambda(i,r]}, w) + \sum_{i=1}^r a_{\lambda[1,i)} \odot c(\lambda_i, \rho) \odot m'(\lambda_{(i,r]}, w).$$

This is equal to A by Lemma 7.17. $\square$

7.5. Proof of Theorem 7.2

We now prove $m' = m$ by showing that m' satisfies conditions (a)–(c) of Proposition 7.9. It is clear that $a_{\emptyset}$ is the unit for m' , and that m' is commutative (since the notion of ruffle is symmetric); thus condition (a) holds. We have verified that m' satisfies the projection formula (Proposition 7.14), which gives (b). Finally, (c) is a simple combinatorial exercise, carried out in the following lemma.

LEMMA 7.18

We have $m'(a_{\pi(1)}, a_{\pi(n)}) = (n+1)a_{\pi(n+1)} + na_{\pi(n)}$.

Proof

Let $\lambda = \pi(1)$ and $\mu = \pi(n)$. There are $n+1$ shuffles (i.e., ruffles without collisions), and for each one r the set $P_r(\lambda, \mu)$ is the singleton containing $\pi(n+1)$. There are n ruffles where the letter from λ collides with one of the letters from μ ; for such ruffles r , the set $P_r(\lambda, \mu)$ is the singleton containing $\pi(n)$. We thus see that

$$m(a_{\lambda}, a_{\mu}) = (n+1)a_{\pi(n+1)} + na_{\pi(n)},$$

which verifies (c). $\square$

7.6. The structure of K as a ring

Now that we have an explicit formula for the product on K , we can describe K as a ring. Recall that K is filtered (as a ring) by the $K_{\leq n}$. We let $\text{gr}(K)$ be the associated graded ring. For a weight λ of length n , we let $\bar{a}_{\lambda}$ be the degree n element of $\text{gr}(K)$ defined by a_{λ} ; these elements form a $\mathbf{Z}$ -basis of $\text{gr}(K)$.

Recall that an (n, m) -shuffle is an (n, m) -ruffle $s: [n] \amalg [m] \rightarrow [\ell]$ with $\ell = n + m$, that is, s is bijective. If s is such a shuffle and λ and μ are words of lengths n and m , then $P_s(\lambda, \mu)$ contains a unique weight, which we denote by $s(\lambda, \mu)$; it has length $n + m$.

PROPOSITION 7.19

For weights λ and μ of lengths n and m , we have

$$\bar{a}_{\lambda} \cdot \bar{a}_{\mu} = \sum_s \bar{a}_{s(\lambda, \mu)},$$

where the sum is over all (n, m) -shuffles s .

Proof

According to Theorem 7.2, we have

$$a_\lambda \cdot a_\mu = \sum_r \sum_{v \in P_r(\lambda, \mu)} a_v,$$

where the sum is over all (n, m) -ruffles r . We thus find that

$$\bar{a}_\lambda \cdot \bar{a}_\mu = \sum_r \sum_{\substack{v \in P_r(\lambda, \mu), \\ \ell(v) = n+m}} \bar{a}_v,$$

as all other terms in the first formula become 0 in $\text{gr}(K)$. If r is a ruffle that is not a shuffle, then there is necessarily a collision, and so $\ell(v) < n + m$ for any $v \in P_r(\lambda, \mu)$. Thus in the above formula, it suffices to sum over shuffles. If r is a shuffle, then $P_r(\lambda, \mu)$ contains a unique element $r(\lambda, \mu)$, and it has length $n + m$. This gives the stated result. $\square$

Recall that the *shuffle algebra* on an alphabet has a $\mathbf{Z}$ -basis consisting of all words, and the product of two words is the sum of all shuffles (see [40, Section 1.4]). The above proposition thus yields the following corollary.

COROLLARY 7.20

The ring $\text{gr}(K)$ is the shuffle algebra on the alphabet $\{\bullet, \circ\}$.

Now, define an order on $\{\bullet, \circ\}$ by $\bullet < \circ$, and use this to lexicographically order the set of weights Λ . We say that $\lambda \in \Lambda$ is a *Lyndon weight* if it is nonempty and lexicographically smaller than its proper suffixes; that is, whenever $\lambda = \mu\nu$ with μ nonempty we have $\lambda < \nu$.

COROLLARY 7.21

The ring $K \otimes \mathbf{Q}$ is the polynomial algebra in the elements a_λ with λ a Lyndon weight.

Proof

It is a well-known property of the shuffle algebra that $\text{gr}(K) \otimes \mathbf{Q}$ is a polynomial ring in the elements $\bar{a}_\lambda$ with λ a Lyndon weight (see [40, Theorem 6.1]). The stated result follows from this by a standard argument. $\square$

Remark 7.22

The restriction map is also compatible with filtrations. By Theorem 6.9, the induced comultiplication on $\text{gr}(K)$ agrees with the usual one on the shuffle algebra (see [40, Section 1.5]). Thus $\text{gr}(K)$ is isomorphic to the shuffle algebra as a bialgebra, and

hence as a Hopf algebra. In particular, writing S for the antipode on K , the leading term of $S(a_\lambda)$ is $(-1)^{\ell(\lambda)} a_\mu$, where μ is the reverse of λ , as this is the formula for the antipode on the shuffle algebra.

8. Adams operations

In this section, we show that the Adams operations are trivial on the Grothendieck group K . As an application, we compute the action of Schur functors on K (Proposition 8.10).

8.1. The λ - and Adams operations

Recall that a λ -ring is a commutative ring R equipped with operations $\lambda^i: R \rightarrow R$, for $i \in \mathbb{N}$, satisfying some conditions, and a *special λ -ring* is one satisfying some further conditions. These definitions are reviewed in [30, Section 5.2], and general background can be found in [45, Section 3] or [35, Section I]. (Note: [35] uses the terminology “pre- λ -ring” and “ λ -ring” in place of our “ λ -ring” and “special λ -ring.”) Given a λ -ring R , one defines the *Adams operations* $\psi^i: R \rightarrow R$ in terms of the λ^j ’s (see [45, Section 3.4] or [35, Section I.4]).

Suppose now that $\mathcal{C}$ is a k -linear pre-Tannakian category with k of characteristic 0. Then the Grothendieck group $K(\mathcal{C})$ admits the structure of a special λ -ring, with $\lambda^i([X]) = [\bigwedge^i(X)]$ (see [22, Section 3.3]). In the case where $\mathcal{C}$ is the category of complex representations of a finite group, the Adams operations are given by $\chi_{\psi^i([V])}(g) = \chi_{[V]}(g^i)$, where χ denotes virtual character. In positive characteristic, $K(\mathcal{C})$ does not carry a natural λ -ring structure in general (see [22, Example 3.3]); it does if $\mathcal{C}$ is semisimple, though even then the λ -ring structure may not be special (as the Verlinde category shows; see [22, Section 3.3]).

By the above, we see that our Grothendieck group K is a special λ -ring via $\lambda^i([V]) = [\bigwedge^i V]$ if k has characteristic 0. In fact, this is true in positive characteristic as well; one can prove this directly or by appealing to the fact that K is independent of k (see Remark 4.15). The tensor product $K \otimes K$ carries a natural special λ -ring structure as it is the Grothendieck group of $\underline{\text{Rep}}(G \times G)$.

PROPOSITION 8.1

The map $\text{res}: K \rightarrow K \otimes K$ is a map of λ -rings.

Proof

The map res is induced by the restriction functor

$$\underline{\text{Rep}}(G) \rightarrow \underline{\text{Rep}}(G(0)) \cong \underline{\text{Rep}}(G \times G),$$

which is a symmetric tensor functor; thus the result follows. $\square$

8.2. The main theorem

The following is the main result of this section.

THEOREM 8.2

The Adams operations are trivial; that is, ψ^i is the identity for all i .

We again emphasize that this is an extremely special property of the Delannoy category: the only semisimple categories we know with this property are $\underline{\text{Rep}}(G^n)$ for $n \in \mathbf{N}$. In fact, the theorem is a formal consequence of results we have already proved, as the proof will show. The proof will take most of the remainder of the section.

We observe one corollary of Theorem 8.2 here. Recall that a *binomial ring* is a commutative ring that is $\mathbf{Z}$ -torsion-free and closed under the operations $x \mapsto \binom{x}{n}$ for all $n \geq 0$.

COROLLARY 8.3

K is a binomial ring, and $\lambda^i(x) = \binom{x}{i}$ for all $x \in K$.

Proof

This follows from a result of Elliott [19, Proposition 8.3] and Wilkerson [48], which states that a λ -ring with trivial Adams operations is binomial, and $\lambda^i(x) = \binom{x}{i}$ identically. $\square$

In particular, we see that one can compute the simple decomposition of exterior powers purely in terms of the ring structure on K . In fact, in Section 8.5 we will see that this is true for arbitrary Schur functors as well.

Remark 8.4

Many elements of G are conjugate to their powers, and this perhaps provides some intuition for Theorem 8.2. To be a little more precise, suppose we had a character theory for $\underline{G}$ -modules that behaved like that for finite groups. Let V be a finite length $\underline{G}$ -module. If $g \in G$ is a piecewise linear map, then g is conjugate to any power g^i with $i \geq 1$. Thus $\chi_{\psi^i([V])}(g) = \chi_{[V]}(g^i) = \chi_{[V]}(g)$. Since the piecewise linear elements of G are dense, we find that $\chi_{\psi^i([V])}(g) = \chi_{[V]}(g)$ for all g , and so $\psi^i([V]) = [V]$. We have not attempted to turn this argument into a rigorous proof.

8.3. A preliminary computation

We begin by verifying that the Adams operations are trivial on the elements $a_\bullet$ and $a_\circ$. We deduce this from the following result.

PROPOSITION 8.5

We have $\bigwedge^n(L_\bullet) \cong L_{\pi(n)}$ for all $n \geq 0$, and so $\lambda^n(a_\bullet) = a_{\pi(n)}$.

Proof

As with Proposition 7.5, there is an illuminative proof at the representation level (see Remark 8.7), but we opt for a proof in the Grothendieck group which involves fewer subtleties. Let $z_n = \lambda^n(a_\bullet)$. We must show that $z_n = a_{\pi(n)}$. This is clear for $n \leq 1$. We proceed by induction on n . Thus let $n \geq 2$ be given, and suppose that $z_i = a_{\pi(i)}$ holds for $0 \leq i \leq n-1$. We prove $z_n = a_{\pi(n)}$ by combining two constraints on z_n .

We have

$$\bigwedge^{n-1}(L_\bullet) \otimes L_\bullet \cong L_{\pi(n-1)} \otimes L_{\pi(1)} \cong L_{\pi(n)}^{\oplus n} \oplus L_{\pi(n-1)}^{\oplus(n-1)},$$

where in the first step we used the inductive hypothesis, and in the second step we used the tensor product rule (Theorem 7.2). Since $\bigwedge^n(L_\bullet)$ is a summand of the above, we find that

$$z_n = r a_{\pi(n)} + s a_{\pi(n-1)}$$

for integers $0 \leq r \leq n$ and $0 \leq s \leq n-1$. This is our first constraint.

Since res is a map of λ -rings (Proposition 8.1), we have

$$\text{res}(z_n) = \lambda^n(\text{res}(a_\bullet)) = \lambda^n(a_\bullet \otimes 1 + 1 \otimes a_\bullet + 1) = \sum_{i+j=n} z_i \otimes z_j + \sum_{i+j=n-1} z_i \otimes z_j.$$

Recall that $\Delta(x) = \text{res}(x) - x \otimes 1 - 1 \otimes x$. Applying the inductive hypothesis to the above formula, we find that

$$\Delta(z_n) = \sum_{i+j=n, i,j \neq 0} a_{\pi(i)} \otimes a_{\pi(j)} + \sum_{i+j=n-1} a_{\pi(i)} \otimes a_{\pi(j)}.$$

The right side above is equal to $\Delta(a_{\pi(n)})$ by the restriction formula (Theorem 6.9). Thus $z_n - a_{\pi(n)}$ is in the kernel of Δ , and therefore a primitive element. It thus follows from the classification of primitive elements (Proposition 6.15) that

$$z_n = a_{\pi(n)} + p(a_\bullet + 1) + q(a_\circ + 1)$$

for integers p and q . This is our second constraint.

The two constraints combined yield $z_n = a_{\pi(n)}$. Indeed, there is no $a_\circ$ in the first constraint, so $q = 0$. Since $n \geq 2$, there is no 1 in the first constraint, so $p = 0$. (In fact, even for $n = 1$ there is no 1 in the first constraint as $s \leq n-1$.) $\square$

COROLLARY 8.6

We have $\psi^i(a_\bullet) = a_\bullet$ and $\psi^i(a_\circ) = a_\circ$ for all $i \geq 1$.

Proof

Recall that $\mathbf{Z}\langle x \rangle$ is the ring of integer-valued polynomials, which has a $\mathbf{Z}$ -basis b_n given by $b_n = \binom{x}{n}$. This ring is a special λ -ring with $\lambda^i(b_1) = b_i$. We have already seen that the map $\varphi: \mathbf{Z}\langle x \rangle \rightarrow \mathbf{K}$ given by $\varphi(b_n) = a_{\pi(n)}$ is a ring homomorphism (Corollary 7.8). By the previous corollary, we see that $\varphi(\lambda^i(b_1)) = \lambda^i(\varphi(b_1))$. This implies that φ is a λ -ring homomorphism (see [30, Lemmas 5.26, 5.27]). Since the Adams operations are trivial on $\mathbf{Z}\langle x \rangle$, it follows that they are trivial on $a_\bullet$ as well. The case $a_\circ$ follows from symmetry. $\square$

Remark 8.7

Here is the basic idea of how to prove Proposition 8.5 at the representation level (in characteristic not equal to 2). Identify $L_\bullet^{\otimes n}$ with the submodule of $\mathcal{C}(\mathbf{R}^n)$ generated by functions $\varphi_{\mathbf{I}}$, where $\mathbf{I}$ is an (unordered) tuple of type $\pi(n)$. We thus see that $\bigwedge^n(L_\bullet)$ is generated by functions $\varphi_{\mathbf{I}}$, subject to relations $\varphi_{\sigma\mathbf{I}} = \text{sgn}(\sigma) \cdot \varphi_{\mathbf{I}}$ for $\sigma \in \mathfrak{S}_n$. When the intervals in $\mathbf{I}$ are disjoint, we may as well put them in order, and we thus obtain a copy of $L_{\pi(n)}$. If two intervals in $\mathbf{I}$ are equal, then $\varphi_{\mathbf{I}} = 0$ in $\bigwedge^n(L_\bullet)$. One can show that any $\varphi_{\mathbf{I}}$ can be decomposed into $\varphi_{\mathbf{J}}$'s with $\mathbf{J}$ disjoint or two components equal, which yields the result.

8.4. Proof of Theorem 8.2

We now prove the theorem. We begin with a few lemmas. For letters $a, b \in \{\bullet, \circ\}$, let $S_{a,b}$ be the set of weights λ of length at least 2 that have first letter a and final letter b . Let $K_{a,b}$ be the $\mathbf{Z}$ -submodule of $\mathbf{K}$ spanned by elements a_λ with $\lambda \in S_{a,b}$.

LEMMA 8.8

$K_{a,b}$ is closed under multiplication.

Proof

Consider a product $a_\lambda a_\mu$, where λ and μ belong to $S_{a,b}$. We compute this product according to the combinatorial rule in Theorem 7.2. Consider a ruffle r and the preproduct of λ and μ associated to r . The first letter of the preproduct is necessarily a : indeed, it must be the first letter of λ (which is a), the first letter of μ (which is a), or the result of a collision of the first letters of λ and μ (which can only yield a). Similarly, the final letter of the preproduct is b . The length of the preproduct is always at least the minimum length of λ and μ , and thus at least 2 in this case. Thus the preproduct starts with a , ends with b , and has length at least 2. It follows that all elements of $P_r(\lambda, \mu)$ belong to $S_{a,b}$. Thus $a_\lambda a_\mu$ is a sum of a_ν 's with $\nu \in S_{a,b}$, and therefore belongs to $K_{a,b}$. $\square$

Note that $K_{a,b}$ is not a subring of K since it does not contain 1.

LEMMA 8.9

$K_{a,b}$ is closed under the λ^i and ψ^i operators for $i \geq 1$.

Proof

Let λ be a nonempty weight in $S_{a,b}$. By Lemma 8.8, $L_{\lambda}^{\otimes i}$ decomposes into a sum of simples L_{ν} with $\nu \in S_{a,b}$ (here $i \geq 1$). Since $\bigwedge^i(L_{\lambda})$ is a quotient of $L_{\lambda}^{\otimes i}$, the same is true for it. Thus $\lambda^i(a_{\lambda})$ is contained in $K_{a,b}$. It now follows that $K_{a,b}$ is closed under λ^i from Lemma 8.8 and the addition rule for λ^i . Finally, ψ^i is a homogeneous degree i polynomial in $\lambda^1, \dots, \lambda^i$, and thus it too preserves $K_{a,b}$. $\square$

Proof of Theorem 8.2

Fix $i \geq 1$. We show that ψ^i is trivial on K by inductively showing that it is trivial on $K_{\leq n}$ for all n . Triviality on $K_{\leq 1}$ follows from Corollary 8.6.

Suppose now that $n \geq 2$ and ψ^i is trivial on $K_{\leq n-1}$. Let λ be a word of length at least 2. By Theorem 6.9, we have

$$\text{res}(a_{\lambda}) = a_{\lambda} \otimes 1 + 1 \otimes a_{\lambda} + y,$$

where y belongs to $K_{\leq n-1} \otimes K_{\leq n-1}$. We now apply ψ^i . This commutes with res by Proposition 8.1 and fixes y by the inductive hypothesis. We thus find that

$$\text{res}(\psi^i(a_{\lambda})) = \psi^i(a_{\lambda}) \otimes 1 + 1 \otimes \psi^i(a_{\lambda}) + y.$$

It therefore follows that $\psi^i(a_{\lambda}) - a_{\lambda}$ is a primitive element. It thus belongs to $K_{\leq 1}$ by Proposition 6.15. On the other hand, every basis vector appearing in $\psi^i(a_{\lambda}) - a_{\lambda}$ is of the form a_{μ} where $\ell(\mu) \geq 2$ by Lemma 8.9. Thus $\psi^i(a_{\lambda}) - a_{\lambda}$ must vanish. It follows that ψ^i is trivial on $K_{\leq n}$, which completes the proof. $\square$

8.5. Application to Schur functors

We now assume that k is a field of characteristic 0. For a partition λ , let

$$s_{\lambda} : K \rightarrow K$$

be the action of the Schur functor $\mathbf{S}_{\lambda}$ on K . Thus if $x = [V]$ is an effective class, then $s_{\lambda}(x)$ is the class of $\mathbf{S}_{\lambda}(V)$. We note that s_{λ} is not linear, but it is some polynomial expression in the λ^i operators. We now determine the action of s_{λ} explicitly. To this end, let p_{λ} be the integer-valued polynomial such that

$$p_{\lambda}(n) = \dim \mathbf{S}_{\lambda}(k^n).$$

The polynomial p_λ is called the *hook-content* polynomial (see [25, Example 6.4] for an explicit formula). Our main result is the following.

PROPOSITION 8.10

We have $s_\lambda(x) = p_\lambda(x)$ for all $x \in K$.

We note that $p_\lambda(x)$ is simply the result of applying the polynomial p_λ to the ring element x . In particular, the proposition implies that the action of s_λ can be obtained from just the ring structure on K , which is not true for Grothendieck groups in general.

Proof

Fix $x \in K$, and let S be the ring of symmetric functions. We have a homomorphism of Λ -rings

$$\varphi: S \rightarrow K, \quad s_\lambda \mapsto s_\lambda(x).$$

Let $P_i \in S$ denote the i th power sum symmetric function. We have $\varphi(P_i) = \psi^i(x)$, essentially by definition of ψ^i . Since the Adams operations are trivial on K (Theorem 8.2), we thus see that $\varphi(P_i) = x = \varphi(P_1)$. Let $\overline{S}$ be the quotient of S by the ideal generated by the $P_i - P_1$ for $i \geq 1$. Thus φ induces a ring homomorphism $\overline{\varphi}: \overline{S} \rightarrow K$.

We have a ring homomorphism $\psi: S \otimes \mathbf{Q} \rightarrow \mathbf{Q}[t]$ induced by $\psi(s_\lambda) = p_\lambda(t)$. Explicitly, for $s \in S$ and $n \in \mathbf{N}$, we have

$$(\psi s)(n) = s(1, \dots, 1, 0, 0, \dots),$$

where there are n 1's. In particular, $\psi(P_i) = t$ for all $i \geq 1$. Thus ψ induces a ring homomorphism $\overline{\psi}: \overline{S} \otimes \mathbf{Q} \rightarrow \mathbf{Q}[t]$. Since $S \otimes \mathbf{Q}$ is a polynomial ring in the P_i 's, it follows that $\overline{S} \otimes \mathbf{Q}$ is a polynomial ring in P_1 , so $\overline{\psi}$ is a ring isomorphism. We have

$$\overline{\psi}(p_\lambda(P_1)) = p_\lambda(\overline{\psi}(P_1)) = p_\lambda(x) = \overline{\psi}(s_\lambda),$$

so $s_\lambda = p_\lambda(P_1)$ in $\overline{S} \otimes \mathbf{Q}$.

Applying $\overline{\varphi}$ to the identity just obtained, we see that $s_\lambda(x) = p_\lambda(x)$ holds in $K \otimes \mathbf{Q}$. Since K is torsion-free, this equality already holds in K . $\square$

We can use the proposition to decompose the action of a Schur functor on $\underline{\text{Rep}}(G)$, as we now explain. Since $p_\lambda(t)$ is an integer-valued polynomial, we have an expression

$$p_\lambda(t) = \sum_{i \geq 0} c(\lambda, i) \binom{t}{i}$$

for some integers $c(\lambda, i)$, almost all of which vanish. Recall that $\pi(n) \in \Lambda$ is the weight consisting of n $\bullet$'s.

COROLLARY 8.11

We have an irreducible decomposition

$$\mathbf{S}_\lambda(L_\bullet) = \bigoplus_{i \geq 0} L_{\pi(i)}^{\oplus c(\lambda, i)}.$$

In particular, the $c(\lambda, i)$ are nonnegative and the length of $\mathbf{S}_\lambda(L_\bullet)$ is $\sum_{i \geq 0} c(\lambda, i)$.

Proof

We have $\lambda^i(x) = \binom{x}{i}$ for $x \in K$ (Corollary 8.3), and $\lambda^i(a_\bullet) = a_{\pi(i)}$ (Proposition 8.5). By Proposition 8.10, we thus have

$$s_\lambda(a_\bullet) = \sum_{i \geq 0} c(\lambda, i) \cdot a_{\pi(i)}.$$

Since the $a_{\pi(i)}$'s are distinct basis vectors of K and $s_\lambda(a_\bullet)$ is the class of the representation $\mathbf{S}_\lambda(L_\bullet)$, it follows that $c(\lambda, i)$ is nonnegative. The result follows. $\square$

The above corollary provides explicit confirmation of the fact that $L_\bullet$ is not annihilated by any Schur functor. We already knew this for general reasons: it is not hard to verify that the length of tensor powers of $L_\bullet$ grows superexponentially, which implies this property by [16, Proposition 0.5].

COROLLARY 8.12

For any V in $\underline{\text{Rep}}(G)$, we have

$$\mathbf{S}_\lambda(V) \cong \bigoplus_{i \geq 0} \left(\bigwedge^i V \right)^{\oplus c(\lambda, i)}.$$

Proof

It suffices to treat the finite length case. Again, we have $\lambda^i(x) = \binom{x}{i}$ for $x \in K$ (Corollary 8.3). Applying Proposition 8.10 to $[V] \in K$, we find that

$$[\mathbf{S}_\lambda(V)] = \sum_{i \geq 0} c(\lambda, i) \cdot \left[\bigwedge^i (V) \right].$$

Since $\underline{\text{Rep}}(G)$ is semisimple, this equality in K yields the stated isomorphism in $\underline{\text{Rep}}(G)$. $\square$

Remark 8.13

Sam Hopkins has pointed out to us that $c(\lambda, i)$ is the number of semistandard Young tableaux with entries $1, \dots, i$ such that each number appears at least once (see [32]).

9. The path model*9.1. Delannoy paths*

Fix a vector $\underline{a} \in \mathbf{N}^s$. A *0-1 vector* in $\mathbf{R}^s$ is a vector whose coordinates are all either 0 or 1. An *$\underline{a}$ -Delannoy path* is a tuple $p = (p_1, \dots, p_\ell)$ where each p_i is a nonzero 0-1 vector and $a = p_1 + \dots + p_\ell$. We picture p as a path in $\mathbf{R}^s$ from 0 to a , composed of steps $p_1, \dots, p_\ell$. The *length* of the path p , denoted $\ell(p)$, is the number of steps, that is, its length as a tuple. We write $\Gamma(\underline{a})$ for the set of $\underline{a}$ -Delannoy paths.

Example 9.1

When $n = 2$ there are three nonzero 0-1 vectors, namely, $(1, 0)$, $(0, 1)$, and $(1, 1)$. Thus the above definition recovers the usual notion of planar Delannoy paths.

Remark 9.2

Combinatorial properties of higher dimensional Delannoy paths are studied in [5] and [44].

Let $i: [t] \rightarrow [s]$ be an injection of finite sets, and let $i^*: \mathbf{R}^s \rightarrow \mathbf{R}^t$ be the corresponding projection. Given an $\underline{a}$ -Delannoy path $p = (p_1, \dots, p_\ell)$, we define its *projection*, denoted $i^*(p)$, by taking the tuple $(i^*(p_1), \dots, i^*(p_\ell))$ and deleting any entries that are zero. This is an $i^*(\underline{a})$ -Delannoy path, possibly of shorter length. We thus have a function $i^*: \Gamma(\underline{a}) \rightarrow \Gamma(i^*(\underline{a}))$.

9.2. The path category

We now define a k -linear category based on Delannoy paths. Let $\mathcal{D}(n, m)$ be the vector space with basis indexed by the set of paths $\Gamma(n, m)$. We write $[p]$ for the basis vector of $\mathcal{D}(n, m)$ corresponding to $p \in \Gamma(n, m)$. We will define a composition law on the $\mathcal{D}$'s. Before doing so, we prove a proposition.

PROPOSITION 9.3

Let $\underline{a} = (a_1, a_2, a_3)$ be a vector in $\mathbf{N}^3$, and suppose that

$$p_{12} \in \Gamma(a_1, a_2), \quad p_{23} \in \Gamma(a_2, a_3), \quad p_{13} \in \Gamma(a_1, a_3)$$

are given. Then there is at most one $\underline{a}$ -Delannoy path q such that

$$\pi_{1,2}(q) = p_{1,2}, \quad \pi_{2,3}(q) = p_{2,3}, \quad \pi_{1,3}(q) = p_{1,3},$$

where $\pi_{i,j}$ is the usual projection map $\mathbf{R}^3 \rightarrow \mathbf{R}^2$.

Proof

Let q and q' be paths in $\Gamma(\underline{a})$ such that $\pi_{i,j}(q) = \pi_{i,j}(q')$ for all (i, j) as in the proposition statement. We show that $q = q'$. It suffices to show that $q_1 = q'_1$, for then we can remove the first step of each of q and q' and continue by induction.

First suppose that $q_1 = (1, 0, 0)$. We then know that the first steps in $\pi_{1,2}(q)$ and $\pi_{1,3}(q)$ are $(1, 0)$, and we have no information about the first step of $\pi_{2,3}(q)$. At most one of $\pi_{1,2}(q'_1)$ and $\pi_{1,3}(q'_1)$ vanishes; suppose the first one does not. Then $\pi_{1,2}(q'_1)$ is the first step in $\pi_{1,2}(q') = \pi_{1,2}(q)$, which we know to be $(1, 0)$; it follows that $q' = (1, 0, x)$. We thus find that $\pi_{1,3}(q'_1) = (1, x)$ does not vanish, and so it is the first step in $\pi_{1,3}(q') = \pi_{1,3}(q)$, which we know to be $(1, 0)$; thus $x = 0$. We have therefore shown that $q'_1 = (1, 0, 0)$ as required.

The other possibilities for q_1 are similar (or easier). Thus the result follows. $\square$

Now, let $p_1 \in \mathcal{D}(n, m)$, $p_2 \in \mathcal{D}(m, \ell)$, and $p_3 \in \mathcal{D}(n, \ell)$ be given. By the above proposition, there is at most one element $q \in \mathcal{D}(n, m, \ell)$ with $\pi_{1,2}(q) = p_1$, $\pi_{2,3}(q) = p_2$, and $\pi_{1,3}(q) = p_3$. Put

$$\epsilon(p_1, p_2, p_3) = \begin{cases} (-1)^{\ell(q)+\ell(p_3)} & \text{if } q \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

We define a k -bilinear composition law

$$\mathcal{D}(n, m) \times \mathcal{D}(m, \ell) \rightarrow \mathcal{D}(n, \ell)$$

by

$$[p_1] \circ [p_2] = \sum_{p_3 \in \Gamma(n, \ell)} \epsilon(p_1, p_2, p_3) [p_3].$$

We now come to the main definition of Section 9.

Definition 9.4

We define a k -linear category $\mathcal{D}$ as follows. For each nonnegative integer n , there is an object X_n . The space of morphisms $X_m \rightarrow X_n$ is $\mathcal{D}(n, m)$. The composition law is as defined above.

Definition 9.5

The *Delannoy algebra* $\mathcal{D}(n)$ is the algebra $\text{End}_{\mathcal{D}}(X_n)$.

Note that we have not yet shown that composition is associative or admits identity elements, so the above definition is somewhat incomplete. It is possible to establish these properties directly; however, we will deduce them in the course of proving Theorem 9.8.

Example 9.6

We describe the algebra $\mathcal{D}(1)$. There are three $(1, 1)$ -Delannoy paths: the diagonal path A , the path B that goes up and then right, and the path C that goes right and then up. The algebra $\mathcal{D}(1)$ is commutative, with identity element A . The other compositions are given by

$$B^2 = -B, \quad C^2 = -C, \quad BC = -A - B - C.$$

The third composition is significant since it shows that a product of basis elements need not be a scalar multiple of a basis element.

9.3. The equivalence theorem

Given an (n, m) -Delannoy path p , let $O_p \subset \mathbf{R}^{(n)} \times \mathbf{R}^{(m)}$ be the corresponding orbit, as defined in Proposition 3.5, and let A_p be the characteristic function of O_p , thought of as an $\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}$ matrix. We thus have a natural k -linear map

$$\mathcal{D}(n, m) \rightarrow \text{Hom}_{\underline{G}}(\mathcal{C}(\mathbf{R}^{(m)}), \mathcal{C}(\mathbf{R}^{(n)})), \quad [p] \mapsto A_p.$$

As this map is a bijection on bases, it is an isomorphism of k -vector spaces. We now verify that it is compatible with composition.

LEMMA 9.7

Given $p_1 \in \mathcal{D}(n, m)$ and $p_2 \in \mathcal{D}(m, \ell)$, we have

$$A_{[p_1] \circ [p_2]} = A_{p_1} A_{p_2},$$

where on the right side we use matrix multiplication.

Proof

Fix an (n, ℓ) -Delannoy path p_3 , and let us compute the coefficient c of A_{p_3} in the matrix product $A_{p_1} A_{p_2}$. By definition, this is the value of $A_{p_1} A_{p_2}$ at a point (z, x) in O_{p_3} . Thus for such a point (z, x) , we have

$$c = \int_{\mathbf{R}^{(m)}} A_{p_1}(z, y) A_{p_2}(y, x) dy.$$

Integrating both sides over $(z, x) \in O_{p_3}$, we find that

$$c \operatorname{vol}(O_{p_3}) = \int_{\mathbf{R}^{(n)} \times \mathbf{R}^{(m)} \times \mathbf{R}^{(\ell)}} A_{p_1}(z, y) A_{p_2}(y, z) A_{p_3}(z, x) dx dy dz.$$

The right side above is $\operatorname{vol}(X)$, where X is the set of points (x, y, z) such that $(z, y) \in O_{p_1}$, $(y, z) \in O_{p_2}$, and $(z, x) \in O_{p_3}$. In other words, X is the union of those orbits that project to O_{p_1} , O_{p_2} , and O_{p_3} under the three projections.

Now, the correspondence between orbits and Delannoy paths in Proposition 3.5 extends naturally to orbits on a threefold product. Moreover, this classification is compatible with projections, in the sense that a projection of an orbit corresponds to the projection of the Delannoy path. Furthermore, the volume of an orbit is easily seen to be $(-1)^\ell$, where ℓ is the length of the Delannoy path.

It follows that if q as in the definition of $\epsilon(p_1, p_2, p_3)$ exists, then $X = O_q$, and otherwise X is empty. In the first case, $c = \operatorname{vol}(O_q) / \operatorname{vol}(O_{p_3}) = (-1)^{\ell(q) + \ell(p_3)}$, and in the second case $c = 0$. In all cases, $c = \epsilon(p_1, p_2, p_3)$, which completes the proof. $\square$

We can now prove our main result about $\mathcal{D}$.

THEOREM 9.8

We have the following:

- (a) *The composition law in $\mathcal{D}$ is associative and has identity elements.*
- (b) *There is a fully faithful functor $\Phi: \mathcal{D} \rightarrow \underline{\operatorname{Rep}}(G)$ defined on objects by $\Phi(X_n) = \mathcal{C}(\mathbf{R}^{(n)})$ and on morphisms by $\Phi([p]) = A_p$.*
- (c) *Φ identifies $\mathcal{D}$ with the full subcategory of $\underline{\operatorname{Rep}}(G)$ spanned by the $\mathcal{C}(\mathbf{R}^{(n)})$'s.*
- (d) *Φ identifies the additive envelope of $\mathcal{D}$ with $\underline{\operatorname{Perm}}(G)$.*
- (e) *Φ identifies the additive-Karoubian envelope of $\mathcal{D}$ with $\underline{\operatorname{Rep}}^f(G)$.*

Proof

(a) Lemma 9.7 shows that composition in $\mathcal{D}$ matches with composition of matrices in $\underline{\operatorname{Perm}}(G)$, and is therefore associative and has identity elements.

(b) Lemma 9.7 shows that this is a functor, and we already know that it induces an isomorphism on Hom spaces.

(c) Let $\mathcal{C}_0$ be the full subcategory of $\underline{\operatorname{Rep}}(G)$ spanned by the objects $\mathcal{C}(\mathbf{R}^{(n)})$. As $\Phi: \mathcal{D} \rightarrow \mathcal{C}_0$ is fully faithful and essentially surjective, it is an equivalence.

(d) This follows since $\underline{\operatorname{Perm}}(G)$ is the additive envelope of $\mathcal{C}_0$.

(e) This follows since $\underline{\operatorname{Rep}}^f(G)$ is the additive-Karoubian envelope of $\mathcal{C}_0$. $\square$

Remark 9.9

The equivalence theorem, combined with results on $\underline{\operatorname{Rep}}(G)$ from [30], implies that the Delannoy algebra $\mathcal{D}(n)$ is semisimple and, more generally, that the additive-

Karoubian envelope of $\mathcal{D}$ is semisimple. It is possible to prove these statements directly, without relying on [30].

9.4. The tensor product

Let $\mathcal{D}^+$ be the additive envelope of $\mathcal{D}$. We have seen in Theorem 9.8 that $\mathcal{D}^+$ is equivalent to $\underline{\text{Perm}}(G)$. We can thus transfer the tensor product from $\underline{\text{Perm}}(G)$ to $\mathcal{D}^+$. We now say a few words about this.

First of all, in $\underline{\text{Perm}}(G)$ we have

$$\mathcal{C}(\mathbf{R}^{(n)}) \otimes \mathcal{C}(\mathbf{R}^{(m)}) = \mathcal{C}(\mathbf{R}^{(n)} \times \mathbf{R}^{(m)}) = \bigoplus_{p \in \Gamma(n,m)} \mathcal{C}(O_p),$$

where here O_p is the orbit corresponding to the Delannoy path p . It follows from the proof of Proposition 3.5 that O_p is isomorphic to $\mathbf{R}^{(\ell)}$, where $\ell = \ell(p)$. We thus obtain the following description of the tensor product on $\mathcal{D}^+$ at the level of objects:

$$X_n \otimes X_m = \bigoplus_{p \in \Gamma(n,m)} X_{\ell(p)}.$$

One can describe $\otimes$ on morphisms, as well as the associator for $\otimes$, directly in terms of Delannoy path, but we will not do that here. We note that the associator is nontrivial.

There is an alternate approach that is a bit more clean. We define a tensor category $\mathcal{D}'$ as follows. The objects are symbols $X_{\underline{n}}$, where $\underline{n}$ is a tuple of nonnegative integers; intuitively, $X_{\underline{n}}$ is the tensor product of the X_{n_i} 's. Morphisms $X_{\underline{n}} \rightarrow X_{\underline{m}}$ are given by linear combinations of $\underline{nm}$ Delannoy paths, where $\underline{nm}$ indicates concatenation of tuples. Composition is defined analogously to $\mathcal{D}$. The tensor product is defined on objects by $X_{\underline{n}} \otimes X_{\underline{m}} = X_{\underline{nm}}$, and by a simple formula on morphisms. The associator is trivial. One can then show that the additive envelopes of $\mathcal{D}$ and $\mathcal{D}'$ agree, by decomposing $X_{\underline{n}}$ with explicit idempotents. This allows one to transfer the tensor product from $\mathcal{D}'$ to $\mathcal{D}^+$.

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Harman

Department of Mathematics, University of Georgia, Athens, Georgia, USA; nharman@uga.edu

Snowden

Department of Mathematics, University of Michigan, Ann Arbor, Michigan, USA;
asnowden@umich.edu

Snyder

Department of Mathematics, Indiana University, Bloomington, Indiana, USA;
nsnyder1@indiana.edu