Microscopic Model for Fractional Quantum Hall Nematics

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Geometric fluctuations of the density mode in a fractional quantum Hall (FQH) state can give rise to a nematic FQH phase, a topological state with a spontaneously broken rotational symmetry. While experiments on FQH states in the second Landau level have reported signatures of putative FQH nematics in anisotropic transport, a realistic model for this state has been lacking. We show that the standard model of particles in the lowest Landau level interacting via the Coulomb potential realizes the FQH nematic transition, which is reached by a progressive reduction of the strength of the shortest-range Haldane pseudopotential. Using exact diagonalization and variational wave functions, we demonstrate that the FQH nematic transition occurs when the system's neutral gap closes in the long-wavelength limit while the charge gap remains open. We confirm the symmetry-breaking nature of the transition by demonstrating the existence of a "circular moat" potential in the manifold of states with broken rotational symmetry, while its geometric character is revealed through the strong fluctuations of the nematic susceptibility and Hall viscosity.

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Introduction.—Nematicity in the fractional quantum Hall (FQH) effect provides an intriguing link between the notions of topology and spontaneous symmetry breaking [1]. The FQH nematic (FQHN) phase [2–4] has a finite charge gap that leads to a quantized plateau in the Hall resistance, reminiscent of incompressible FQH fluids [5]. However, while the latter are also gapped to *neutral* excitations, such as a pair of fundamental quasielectron and quasihole, the neutral gap in the FQHN phase is expected to vanish as a consequence of the spontaneous breaking of continuous rotational symmetry. Symmetry breaking occurs in many QH systems with the prominent examples being skyrmions [6–9] and charge-density-wave (CDW) phases, such as stripes, bubbles, and Wigner crystals [10–25]. Unlike these, the FQHN maintains translation invariance but only breaks rotational symmetry about the z axis perpendicular to the two-dimensional electron gas (2DEG) [26]. The breaking of a continuous symmetry distinguishes the FQHN from other examples of discrete nematics [27,28] in multivalley materials [29–31], frustrated magnets [32], and moiré superlattices [33–35].

A general mechanism believed to give rise to a FQHN from a proximate incompressible FQH state is the softening of the magnetoroton mode in the long-wavelength limit [3,4], Fig. 1. The magnetoroton mode is a collective density wave excitation [36–38] that occurs in many FQH states, including the Laughlin [39] and Moore-Read [40] states. At long wavelengths, the mode is described by the Girvin, MacDonald, and Platzman (GMP) ansatz [41,42]. Unlike the topological properties of FQH states, such as the

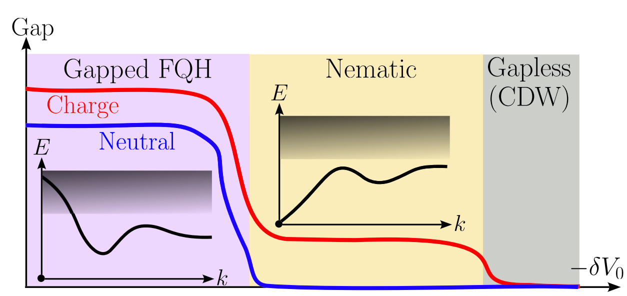

FIG. 1. Schematic of the phase diagram and the FQHN transition for bosons, driven by varying the shortest-range potential from its Coulomb value by δV_0. In the nematic phase, the charge gap remains open, while the neutral gap has closed due to the presence of a Goldstone mode associated with the spontaneous breaking of continuous rotation symmetry. Upon further softening of the shortest-range repulsion, the system becomes fully gapless, e.g., by also breaking translation symmetry (CDW).

Hall conductance, the long-wavelength limit of the GMP mode has a *geometric* character: in an effective field theory, it behaves as a quadrupole that can be described by a quantum metric [43–50].

Experiments in the second LL with tilted magnetic fields [51] and hydrostatic pressure [52,53] have observed anisotropic resistance that was attributed to nematicity. These experiments indeed suggest that nematic order can be proximate to incompressible FQH states or even *coexist* with them. In addition, a Raman scattering experiment at filling factors $\nu = 5/2$ and $7/2$ has provided evidence of an interplay between nematicity and electron pairing [54]. A major challenge in bridging the gap between these experimental observations and effective theories of FQHNs has been the lack of a microscopic model. Early work [55] introduced a class of variational Laughlin-like wave functions that break the continuous rotational symmetry. More recently, Ref. [56] proposed a toy model exhibiting some signatures of the FQHN. For a similar class of short-range interacting models, Ref. [57] formulated conditions for the quadrupole excitation to become gapless. However, the identification of FQHNs in realistic models has been lacking.

In this Letter, we consider a quintessential model of the FQH effect—Coulomb interaction projected to the lowest LL (LLL) [58–60]. Contrary to the general belief that the softening of the shortest range component of the Coulomb interaction results in a destruction of the FQH liquid and direct transition to a CDW, we argue that this model realizes the scenario of the FQHN transition as shown in Fig. 1. Using exact diagonalization (ED), we demonstrate the hallmarks of the FQHN in this model: the GMP mode goes soft at long wavelengths and the neutral gap closes, while the charge gap remains open. Furthermore, we demonstrate the symmetry-breaking nature of the transition and show that the nematic susceptibility and the Hall viscosity both diverge in its vicinity. The results of exact numerics are supported by estimates of the gaps in larger systems using the GMP ansatz and composite fermion (CF) theory [61]. Below we present illustrative results for the Laughlin state of bosons at filling factor $\nu_b = 1/2$, which is relevant for recent experiments in cold atom setups [62]. However, in the Supplemental Material [63], we demonstrate that our results equally apply to the $\nu = 1/3$ Laughlin state of electrons.

Model.—We consider a standard model for the FQH effect where N electrons, interacting pairwise via the Coulomb potential, are confined to a spherical surface [86], with a Dirac monopole at the center, emanating a radial magnetic flux of strength $2Qhc/e$. The radius of the sphere is $R = \sqrt{Q}\ell_B$, where $\ell_B = \sqrt{\hbar c/eB}$ is the magnetic length. The Hamiltonian is translation and rotation invariant, which implies that the total orbital angular momentum L and its z component M are good quantum numbers. The magnitude of the planar wave vector k is given by $k = L/R$. The rotational invariance of the interaction implies that it can be expressed in terms of Haldane pseudopotentials $\{V_m\}$, where $m = 0, 1, 2, 3, \dots$ is the relative angular momentum of any pair of particles [58,86]. Hence, m is constrained by the statistics of the particles: for bosons, only even pseudopotentials V_0, V_2, V_4, etc., are relevant.

We will consider the filling factor $\nu_b = 1/2$ for bosons, at which the Laughlin state [39] is realized if we set the magnetic monopole flux to $2Q = 2N - \mathcal{S}$, with $\mathcal{S} = 2$ denoting the Wen-Zee shift [87]. Starting from the Coulomb interaction, we soften the shortest-range component of the interaction potential and therefore the model is parametrized by a single number $\delta V_0 = V_0 - V_0^{\mathrm{C}}$, where V_0^{C} is the Coulomb interaction's V_0 value [88].

Spectral properties.—We first analyze the energy spectra of the $\nu_b = 1/2$ bosonic state obtained using ED, Figs. 2(a) and 2(b). Throughout, energies are expressed in units of Coulomb energy, $E_C \equiv e^2/\epsilon\ell_B$. While there is a gapped magnetoroton mode for the pure Coulomb interaction [Fig. 2(a)], the dispersion of the collective mode is significantly altered by reducing the V_0 component of the interaction. As we add a delta interaction of strength $\delta V_0 = -0.4$, the wave number corresponding to the roton minimum changes from $k\ell_B \sim 1.5$ to $k \to 0$ [Fig. 2(b)].

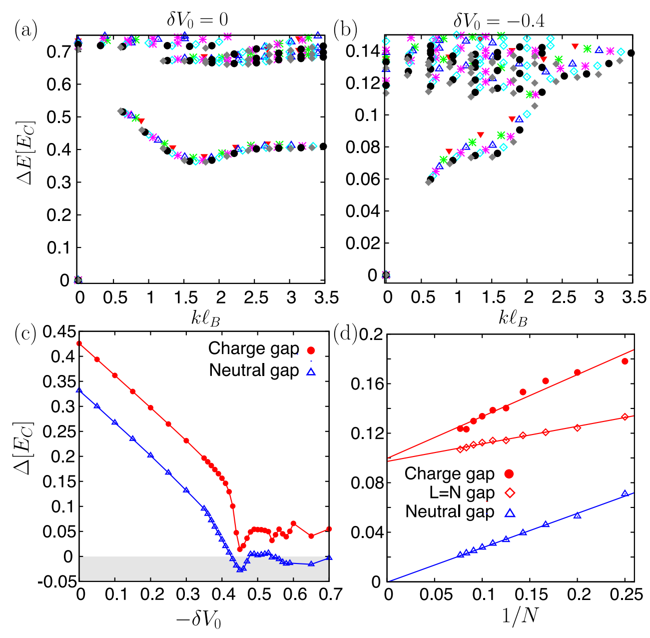

FIG. 2. (a)–(b) Energy spectrum for bosons at $\nu_b = 1/2$ in the LLL, with pure Coulomb potential (a) and softened by adding $\delta V_0 = -0.4$ pseudopotential (b), which pushes the system close to a critical point. Data is for system sizes $N = 6$–12. All energies include the rescaling of the magnetic length, see Supplemental Material [63]. (c) Charge and neutral gaps (see text for the definition) of the bosonic $\nu_b = 1/2$ state as a function of δV_0. (d) The scaling of the gaps as a function of $1/N$ for the fixed value $\delta V_0 = -0.43$.

The softening of the magnetoroton mode implies that the neutral gap decreases as V_0 is reduced and potentially closes in the long-wavelength limit.

In Figs. 2(c) and 2(d) we show the gaps as a function of δV_0. We evaluate two types of gaps for accessible system sizes and perform their extrapolation to the thermodynamic limit in $1/N$: (i) the neutral gap is defined as the difference in the lowest two energies at the ground state flux, $2Q = 2N - 2$; (ii) the charge or transport gap, i.e., the energy to create an individual quasihole (occurs at flux $2Q + 1$) and quasiparticle excitation (occurs at flux $2Q - 1$). An alternative way to estimate the charge gap is to extrapolate the gap $\Delta^{L=N}$, the energy difference between the ground state and the lowest-lying state with orbital angular momentum $L = N$, at the ground state flux. The $L = N$ roton state is formed by a quasiparticle and a quasihole on opposite poles of the sphere. The interaction between these localized excitations vanishes in the thermodynamic limit, hence the charge gap is simply the sum of their individual energies.

The results in Fig. 2(c) show a sharp drop in both the charge and neutral gaps as the interaction is softened by $\delta V_0 \approx -0.4$. More precisely, at $\delta V_0 = -0.43$, the neutral gap drops to zero (or slightly below, due to the uncertainty of the extrapolation), however, the charge gap, while significantly reduced compared to the pure Coulomb point, remains nonzero. This is demonstrated by the raw scaling data for gaps as a function of $1/N$ in Fig. 2(d). The reliability of the extrapolation is confirmed by independently calculating $\Delta^{L=N}$, which extrapolates to the same value as the charge gap, Fig. 2(d).

Variational wave functions.—To access system sizes beyond ED, we employ GMP [41] and composite fermion (CF) [89] ansatz wave functions. Figure 3 shows the gap estimates from such wave functions. We approximate the ground state by the bosonic Laughlin wave function [39] which has a high overlap (upwards of 0.85 for all $N \leq 14$) with the exact ground state across the range $\delta V_0 \in [-0.43, 0]$. The exact spectra, Fig. 2(b), show that the gap can be upper bounded by the GMP excitation that carries angular momentum $L = 2$ on the sphere [90]. On the other hand, to estimate the charge gap, we use the CF exciton wave functions, which accurately describe the entire magnetoroton branch [91,92]. The CF exciton wave function of the Laughlin state in the long wavelength limit is identical to the GMP ansatz. We estimate the charge gap by the energy of the $L = N$ member of the CF-exciton mode [61,93].

We independently evaluate the Coulomb and $V_0 = 1$ pseudopotential gaps for many systems using the Monte Carlo method and extrapolate these gaps to the thermodynamic limit [63]. By superposing these gaps, we extract the variational gap for an arbitrary combination of the Coulomb and V_0 pseudopotential. As shown in Fig. 3, the neutral and charge gaps have different slopes as a function of δV_0, the former decaying faster. Thus, there is a region of parameter space $0.43 \leq -\delta V_0 \leq 0.84$ where the neutral gap vanishes while the charge gap remains finite. This variational estimate of the FQHN phase boundaries is consistent with the ED results above.

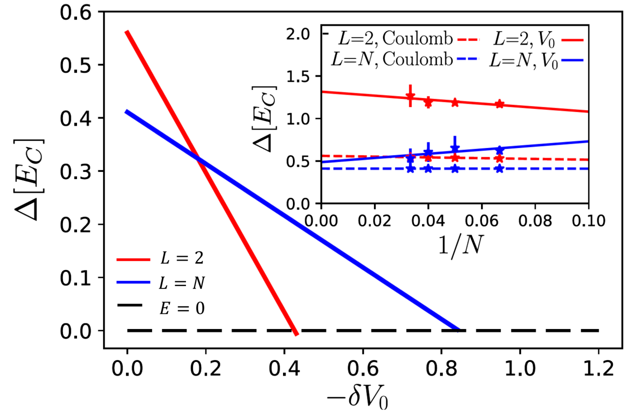

FIG. 3. Variational estimate of the neutral gap (i.e., the gap of the $L = 2$ GMP state) and charge gap (i.e., the gap of $L = N$ CF exciton state) as a function of δV_0 for bosons at $\nu_b = 1/2$. The inset shows the finite-size extrapolation of the Coulomb and V_0 gaps of $L = 2$ and $L = N$ trial states.

Symmetry breaking and geometric response.—As an order parameter for the FQHN transition, we utilize the deformed Laughlin states which partly break rotational symmetry [55,94]:

$$\Psi_{\nu_b=1/2,\alpha} = \prod_{i<j} (z_i - z_j - \alpha)(z_i - z_j + \alpha), \qquad (1)$$

where we have suppressed the usual Gaussian factor and the parameter α controls the breaking of rotational symmetry by "splitting" the zeros of the wave function. For small α, these wave functions, as well as related ones in Refs. [43,94], describe an incompressible fluid rather than an FQHN. Nevertheless, we can use their variational energy to define a "mean-field" order parameter: we keep the interaction fully rotation invariant, while we evaluate the expectation value of the energy for the anisotropic wave functions. For $\delta V_0 > \delta V_0^{\text{critical}}$, we expect the energy minimum to correspond to $\alpha = 0$, i.e., the isotropic Laughlin wave function, while for $\delta V_0 < \delta V_0^{\text{critical}}$ the minimum should shift to a nonzero $\alpha^* > 0$. In the vicinity of a second-order phase transition, mean-field theory predicts a critical component $\beta = 1/2$ for the order parameter $\alpha^* \propto (\delta V_0^{\text{critical}} - \delta V_0)^\beta$ [4].

This scenario is confirmed in Fig. 4(a), which shows the energy of $\Psi_{\nu_b=1/2,\alpha}$ states for the LLL-projected Coulomb interaction at several values of δV_0. The isotropic Laughlin state (recovered at $\alpha = 0$) indeed yields the minimum of energy when $|\delta V_0| \leq 0.44 = -\delta V_0^{\text{critical}}$. This estimate of the FQHN transition point is in good

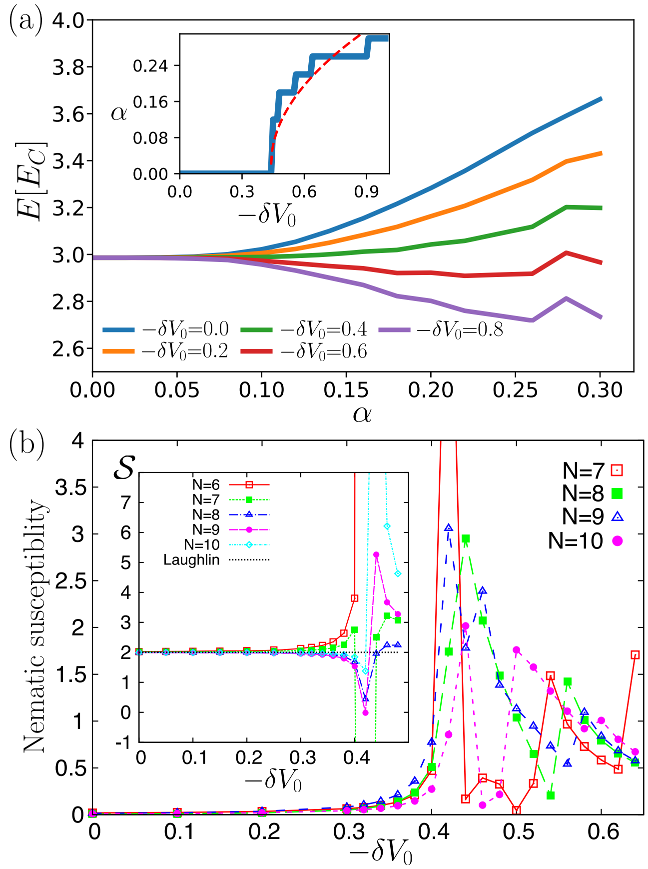

FIG. 4. (a) Energy of the wave functions in Eq. (1) for various δV_0. We discretize the α range in steps of $0.02\ell_B$. Inset shows the value of α which minimizes the total energy at each δV_0. The Coulomb energy and $V_0 = 1$ energy are calculated, respectively, for $N = 15$, 20, 25, 30 and the thermodynamic values are obtained from finite-size extrapolation. The red dashed line is the fitting of the data with the form of $\kappa\sqrt{\delta V_0^{\text{critical}} - \delta V_0}$, where $\kappa = 0.439$ and $\delta V_0^{\text{critical}} = -0.44$. (b) Nematic susceptibility from Ref. [56] for bosons at $\nu_b = 1/2$ on the torus with a square unit cell for several system sizes indicated in the legend. As V_0 is reduced and the critical point is approached, we observe a sharp increase and strong finite-size fluctuations in the nematic susceptibility. Similar behavior is found for the shift $\mathcal{S}$ shown in the inset. We extract the shift from the Hall viscosity η^A on the torus near the square aspect ratio. In the gapped FQH phase, the shift is quantized to the $\mathcal{S} = 2$ value expected in the Laughlin state. Similar to the nematic susceptibility, as the FQHN critical point is approached, we observe large fluctuations in $\mathcal{S}$, which is no longer quantized.

agreement with our previous exact results for the closing of the neutral gap. Upon further reducing V_0, the lowest energy state appears at a nonzero α^* value. The scaling of α^* near criticality is presented in the inset of Fig. 4(a), which is consistent with the mean-field exponent 1/2.

To characterize the geometric nature of the FQHN transition, we have computed the nematic susceptibility introduced in Ref. [56], see Fig. 4(b). This calculation is performed in the torus geometry [95] by varying the aspect ratio. Figure 4(a) shows that deep in the Laughlin phase the nematic susceptibility is zero while close to the transition point, $\delta V_0 = -0.43$, it jumps to a nonzero value and thereafter varies erratically with system size. Moreover, we have computed the Hall viscosity η^A on the torus [96–98] as the system is tuned towards the FQHN critical point [63]. For gapped FQH states, the Hall viscosity is quantized by the shift, $\eta^A = (\hbar\rho/4)\mathcal{S}$ [99], where ρ is the fluid density. The extracted value of $\mathcal{S}$ is plotted in the inset of Fig. 4(b) as a function of δV_0. Deep in the gapped phase, $\delta V_0 \gtrsim -0.3$, the shift is robustly quantized to the Laughlin value $\mathcal{S} = 2$. Near the transition but still in the gapped phase, the smaller systems depart from the quantized value, which is attributed to an increase in the correlation length. Finally, as we hit the FQHN critical point, the quantization of η^A breaks down completely. In the Supplemental Material [63], we show that the onset of the transition is also signaled by the dynamical response of the system to a geometric quench.

Experimental implications.—The key ingredient of our model—the softening of the V_0 pseudopotential of the Coulomb interaction—can realistically arise due to the finite width of the 2DEG or screening by electrostatic gates. LL mixing, in particular, leads to a large reduction of V_0 [100–102]. Moreover, the node in the single-particle wave functions in higher LLs tends to expose the magnetoroton mode in the long-wavelength limit [103], thus providing a natural setting for FQHNs [104]. One of the main experimental challenges is distinguishing the FQHN from a CDW. Resonant inelastic Raman scattering [36,54,105] or surface acoustic waves [38] can map out the magnetoroton mode and confirm whether it closes in the long-wavelength limit. Recent advances in the scanning tunneling microscopy of FQH states [106–109] could provide further insights into the FQHN formation, generalizing the previous observation of nematicity of free electrons in bismuth [31].

Beyond solid-state systems, interactions in synthetic systems of alkaline atoms can be tuned by coupling to the highly excited Rydberg levels. The Rydberg blockade radius that simulates the V_0 interaction can be tuned in these platforms and both FQH and crystalline phases have been shown to arise in these models [110,111]. The long-range interactions in Rydberg-dressed atoms as well as dipolar gases [112] make them a promising platform for FQH physics [113]. Other platforms where bosonic FQH states can be stabilized are polaritons where an artificial magnetic field and LLs can be generated by rotating the medium through which light propagates [114]. A suitably chosen medium could potentially allow to engineer a desired interaction [115,116].

Conclusions.—In summary, we have presented a microscopic model that exhibits key features of the FQHN transition. Intriguingly, the model shows clearer FQHN signatures compared to short-range models, such as the one in Ref. [56]. We speculate that this is due to the tendency of the Coulomb interaction to "pull down" the magnetoroton mode below the spectral continuum in the long-wavelength limit, as seen in Fig. 2. By contrast, in short-range models, the long-wavelength limit of the magnetoroton mode is clearly inside the continuum [117]. The necessary conditions for the gaplessness of the $L = 2$ neutral excitation in short-range models have been derived in Ref. [57], and it would be interesting to generalize those results to long-range interactions.

While we have provided multiple pieces of evidence for the FQHN critical point, the nature of the phase *past* criticality is not fully understood. At large negative δV_0, we expect a CDW to become the ground state. On the sphere, the study of CDW phases is hindered by frustration effects, requiring very large systems to observe the expected closing of the charge gap. On a torus, by varying the aspect ratio, we have indeed found a proximate phase with a manifold of ground states consistent with CDW ordering [63]. Thus, the FQHN phase found above could be proximate to a CDW.

One important question is, how general are the results above? In the Supplemental Material [63], we show that fermions at $\nu = 1/3$ behave similarly to bosons at $\nu_b = 1/2$ considered above. Moreover, other incompressible states, in particular $\nu = 2/5$ and $3/7$ Jain states, also have collective modes "exposed" below the continuum of the spectrum [118,119]. We leave their study to future work. We note, however, that the FQHN presented above could potentially be realized even beyond noninteracting CF states. For example, we have checked that bosons at $\nu_b = 1$, which realize the Moore-Read state [40,120–123], support a similar phenomenology. Beyond the nematic instability of incompressible FQH states, it would be interesting to explore a *compressible* nematic which could arise in third and higher Landau levels [18,124–126], e.g., due to the quantum (or thermal) melting of a stripe state [26,104,127] or a Pomeranchuk instability of the compressible $\nu = 1/2$ state [48,128].

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