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Pricing of contingent claims in large markets

Oleksii Mostovyi[1] · **Pietro Siorpaes**[2]

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Abstract

We consider the problem of pricing in a large market, which arises as a limit of small markets within which there are finitely many traded assets. We show that this framework allows accommodating both marginal-utility-based prices (for stochastic utilities) and arbitrage-free prices. Adopting a stochastic integration theory with respect to a sequence of semimartingales, we introduce the notion of marginal-utility-based prices for the large (post-limit) market and establish their existence, uniqueness and relation to arbitrage-free prices. These results rely on a theorem of independent interest on utility maximisation with a random endowment in a large market that we state and prove first. Further, we provide approximation results for the marginal-utility-based and arbitrage-free prices in the large market by those in small markets. In particular, our framework allows pricing asymptotically replicable claims, where we also show consistency in the pricing methodologies and provide positive examples.

Keywords Infinite-dimensional stochastic control · Large market · Indifference pricing · Fair pricing · Davis pricing · Utility-based pricing · Arbitrage-free pricing · Asymptotic replicability · Duality theory · Semimartingale · Incomplete market

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✉ O. Mostovyi
oleksii.mostovyi@uconn.edu

P. Siorpaes
p.siorpaes@imperial.ac.uk

1 University of Connecticut, Department of Mathematics, Storrs, CT 06269, USA

2 Imperial College London, Department of Mathematics, London SW7 2AZ, UK

1 Introduction

The size and complexity of financial markets have led to the appearance of models with an infinite number of traded securities. Starting from the usual "small" models and supposing that the number of traded stocks is a finite but random number taking values in the set of natural numbers, one directly arrives at the assumption of the availability of countably many tradable assets in the market. Models of this type are considered in Björk and Näslund [3], De Donno et al. [7] and Mostovyi [28], among others. Further, in the context of fixed income derivatives, it is natural to model interest rates with uncountably many traded instruments; see e.g. Carmona and Tehranchi [5], Ringer and Tehranchi [31], De Donno and Pratelli [8] and Ekeland and Taflin [12].

The mathematical foundations of large markets go back to Kabanov and Kramkov [15], where a large market was introduced as a sequence of models with finitely many traded assets. The theory was further developed by Kabanov and Kramkov [16], Klein [18], Klein and Schachermayer [22, 23], Klein [19, 20], Klein et al. [21] and Bálint and Schweizer [2]. Recently, the questions of indifference pricing in such markets have been considered in Anthropelos et al. [1] and Robertson and Spiliopoulos [32]. Relying on ideas of stochastic integration with respect to infinite-dimensional stochastic processes, characterisations of the large market themselves (post-limit) have appeared. The theory for such markets was developed through a series of works including [5, 12, 7, 8, 31, 21] and more recently in Cuchiero et al. [6] and Kardaras [17]. Investigating the post-limit models requires stochastic integration with respect to infinite-dimensional stochastic processes that is less developed than stochastic integration with respect to finite-dimensional semimartingales. Further, *completeness* is a common assumption in traditional interest rate modelling. Thus pricing in the large (post-limits) market models in the context of fixed-income derivatives often inherits certain replicability assumptions; see e.g. [5, Assumption 5.1].

This paper focuses on two *pricing approaches in (fully) incomplete large markets*, without any a priori replicability assumptions, where modelling and establishing results in the large (post-limit) market itself is a significant part of our analysis. In particular, *in stochastic utility settings, we develop marginal-utility-based pricing in the large (post-limit) market and show its consistency with arbitrage-free pricing*. For this, we first establish a theorem on utility maximisation with a random endowment for the large market, a result which is interesting by itself. Further, we provide an approximation result by marginal-utility-based prices in small markets. Finally, we apply our results to asymptotically replicable claims, whose pricing in the large market has a particularly nice structure. We note that in settings of exponential utility, the problem of utility-based pricing has been considered in [1] in the large (pre-limit) market, that is, without considering the limiting markets (associated with the presence of infinitely many traded assets, which in turn requires some notion of stochastic integration with respect to an infinite-dimensional stochastic process).

On the technical level, as there are fewer stochastic analysis tools for studying stochastic integration with respect to an infinite-dimensional semimartingale, we deal with more obstacles. In particular, we do not use the optional decomposition theorem, which was crucial for optimal investment with a random endowment in a small market; see Hugonnier and Kramkov [13]. We note that the optional decomposition theorem for a large market has been recently developed in [17] under continuity of the underlying stock price processes. We could not use it as our formulation deals with semimartingales which might admit jumps. Our approach allows us to include the closures of the domains of the key optimisation problems (crucially for the proofs and to obtain more complete characterisations of the underlying problems) and to circumvent both the non-replicability and asymptotic replicability assumptions often imposed in the literature, even for small markets. An application of such a formulation is the pricing of asymptotically replicable claims as a particular case of our results, which in particular apply to asymptotically complete markets where every claim is replicable or asymptotically replicable.

The remainder of this paper is organised as follows. In Sect. 2, we introduce the model. In Sect. 3, we establish utility maximisation results with a random endowment in a large market. In Sect. 4, we introduce the notion of a marginal-utility-based price in a large market, prove its existence and provide a condition for its uniqueness. In Sect. 5, we prove the convergence of marginal-utility-based prices in small markets to the ones in the large market. In Sect. 6, we show an application of our setting to asymptotically replicable claims, where asymptotically complete markets form a particular case.

2 Model

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and where $\mathcal{F}_0$ is trivial. We suppose that the large market consists of a riskless asset $S^0 \equiv 1$ and a sequence of risky assets $S = (S^n)_{n\in\mathbb{N}}$, where each S^n, $n \in \mathbb{N}$, is a semimartingale that specifies the price of the nth risky asset. We also suppose that there is a non-traded contingent claim with payment process $(F^i)_{i=1}^N$. If $(q_i)_{i=1}^N = q$ is the finite family that specifies the number of such claims, the cumulative payoff is given by

$$qF := (qF_t)_{t\in[0,T]} = \left(\sum_{i=1}^N q_i F_t^i\right)_{t\in[0,T]}.$$

Both processes S and F are given exogenously.

The notion of a trading strategy in the large market is given as follows. For $n \in \mathbb{N}$, an *n-elementary strategy* is an $\mathbb{R}^n$-valued predictable and $(S^k)_{k=1,\dots,n}$-integrable process. An *elementary strategy* is a strategy which is n-elementary for some $n \in \mathbb{N}$. Further, an n-elementary strategy H is *x-admissible* for a given $x \geq 0$ if

$$H \cdot S = \int_0^{\cdot} \sum_{k=1}^n H_t^k dS_t^k \geq -x \qquad \mathbb{P}\text{-a.s.}$$

Let $\mathcal{H}^n$ denote the set of n-elementary strategies which are x-admissible for some $x \geq 0$, and $\mathcal{H}$ the set of admissible elementary strategies. The n-small market is the market where one can trade in $S^0, \dots, S^n$ (and hold q shares of F for some $q \in \mathbb{R}^N$). By a small market, we mean one which is n-small for some $n \in \mathbb{N}$.

To pass to the limit as $n \to \infty$, we follow De Donno and Pratelli [9] and recall that $\mathbb{R}^{\mathbb{N}}$ is the space of real-valued sequences. An *unbounded functional* on $\mathbb{R}^{\mathbb{N}}$ is a linear functional $\bar{H}$ whose domain $\mathrm{Dom}(\bar{H})$ is a subspace of $\mathbb{R}^{\mathbb{N}}$. A *simple integrand* is a finite sum of bounded predictable processes of the form $\sum_{k=1}^n h^k e^k$, where e^k is the Dirac delta at point k and h^k is a one-dimensional bounded and predictable process, $k \in \mathbb{N}$.

A process H with values in the set of unbounded functionals on $\mathbb{R}^{\mathbb{N}}$ is *predictable* if there is a sequence $(H^n)_{n\in\mathbb{N}}$ of simple integrands (as just defined) such that $H = \lim_{n\to\infty} H^n$ pointwise, in the sense that for every $x \in \mathrm{Dom}(H)$, the sequence $(H^n(x))_{n\in\mathbb{N}}$ converges to $H(x)$ as $n \to \infty$.

A predictable process H with values in the set of unbounded functionals on $\mathbb{R}^{\mathbb{N}}$ is *integrable* with respect to S if there is a sequence $(H^n)_{n\in\mathbb{N}}$ of simple integrands such that $(H^n)_{n\in\mathbb{N}}$ converges to H pointwise and the sequence of semimartingales $(H^n \cdot S)_{n\in\mathbb{N}}$ converges to a semimartingale Y in the semimartingale topology. In this case, we set

$$H \cdot S := Y.$$

To put the concept of a stochastic integral as above in the context of optimal investment, we further need to specify the notion of admissibility. Thus for $x \geq 0$, we say that a predictable process with values in the set of unbounded functionals is an *x-admissible generalised strategy* if H is integrable with respect to S and there is an approximating sequence $(H^n)_{n\in\mathbb{N}}$ of x-admissible elementary strategies such that $(H^n \cdot S)_{n\in\mathbb{N}}$ converges to $H \cdot S$ in the semimartingale topology. A predictable process with values in the set of unbounded functionals is an *admissible generalised strategy* if it is x-admissible for some $x \geq 0$.

2.1 Primal problem

Consider a non-traded European contingent claim $f = F_T$ which makes at T payments f^i, $i \in \{1, \dots, N\}$. We define for every $(x, q) \in \mathbb{R} \times \mathbb{R}^{N+1}$ the sets

$$\mathcal{X}^n(x, q) := \{X = x + H \cdot S : \ H \in \mathcal{H}^n \text{ and } X_T + qf \geq 0\}, \qquad n \in \mathbb{N},$$

$$\mathcal{X}(x, q) := \{X = x + H \cdot S : H \text{ is an admissible generalised strategy and } X_T + qf \geq 0\}.$$

We remark that under (2.4) below, $\mathcal{X}(x, 0)$ consists for every $x > 0$ of the wealth processes associated with x-admissible generalised strategies. Thus these wealth processes are nonnegative. Likewise, for every $n \in \mathbb{N}$, $\mathcal{X}^n(x, 0)$ consists of the wealth processes associated with x-admissible n-elementary strategies, and so these wealth processes are nonnegative, too. Moreover, $\mathcal{X}^n(x, q) \subseteq \mathcal{X}^{n+1}(x, q) \subseteq \mathcal{X}(x, q)$ and these sets can be empty for some (x, q). Therefore, we set

$$\mathcal{K}^n := \{(x, q) \in \mathbb{R}^{N+1} : \mathcal{X}^n(x, q) \neq \emptyset\}, \qquad n \in \mathbb{N},$$

$$\mathcal{K} := \{(x, q) \in \mathbb{R}^{N+1} : \mathcal{X}(x, q) \neq \emptyset\}.$$

For the contingent claim, we impose

Assumption 2.1 Every component of f is bounded.

Remark 2.2 In multiple sources, see e.g. Hugonnier and Kramkov [13] and Mostovyi and Sîrbu [29], it is assumed that the contingent claim is bounded by some wealth process in some (typically small) market. Mathematically, this amounts to supposing that $|f| \le C\tilde{X}$ for some maximal $\tilde{X} \in \mathcal{X}^n(1,0)$ for some $n \in \mathbb{N}$. This in particular allows pricing contingent claims unbounded from above and below. In this remark, we show that our settings with stochastic utility are closely related to those with an unbounded contingent claim.

Let us suppose that one starts from a utility $\tilde{U}$ satisfying Assumption 2.3 below (possibly deterministic as in [13], but this assumption does not have to be imposed) and a contingent claim $\tilde{f}$ satisfying

$$|\tilde{f}| \le C\tilde{X} \qquad \text{for some positive maximal } \tilde{X} \in \mathcal{X}^n(1,0) \text{ and } C > 0. \tag{2.1}$$

Then if $\tilde{X}_T > 0$, for any given n-elementary strategy H, there exists a predictable and $S^{n,\tilde{X}} := (\frac{S^0}{\tilde{X}}, \dots, \frac{S^n}{\tilde{X}})$-integrable process H' such that

$$x + H \cdot S_T + q\tilde{f} = \tilde{X}_T\left(x + H' \cdot S_T^{\tilde{X}} + q\frac{\tilde{f}}{\tilde{X}_T}\right).$$

Next, setting

$$f := \frac{\tilde{f}}{\tilde{X}_T} \qquad \text{and} \qquad U(\omega, x) := \tilde{U}\big(\omega, \tilde{X}_T(\omega)x\big), \quad (\omega, x) \in \Omega \times [0,\infty),$$

one can see from (2.1) that $|f| \le C$ and U satisfies Assumption 2.3 below. If the components of $(S^1, \dots, S^n)$ are locally bounded, Delbaen and Schachermayer [10, Theorem 13] implies that $(\frac{S^0}{\tilde{X}}, \dots, \frac{S^n}{\tilde{X}})$ admits an equivalent local martingale measure. This outlines a change of numéraire approach in the context of small markets. For large markets, to the best of the authors' knowledge, a change of numéraire calculus has not yet been developed.

It follows from Assumption 2.1 that

$$(x, 0) \in \operatorname{int} \mathcal{K}^1 \qquad \text{for every } x > 0.$$

To see this, fix $x > 0$. Then for a sufficiently small $\varepsilon > 0$ and every q in a ball in $\mathbb{R}^N$ of radius ε, we have $x + qf \ge 0$ $\mathbb{P}$-a.s.; so a portfolio with x units of the riskless asset S^0 and q shares of the contingent claim (and no risky assets) is admissible. This argument holds for every $x > 0$.

The preferences of an economic agent are given by a utility stochastic field

$$U = U(\omega, x) : \Omega \times [0,\infty) \to \mathbb{R} \cup \{-\infty\}.$$

We suppose that U satisfies the following assumption.

Assumption 2.3 For every $\omega \in \Omega$, the function $x \rightarrow U(\omega, x)$ is strictly concave, strictly increasing, continuously differentiable on $(0, \infty)$ and satisfies the Inada conditions

$$\lim_{x \downarrow 0} U'(\omega, x) = \infty \qquad \text{and} \qquad \lim_{x \uparrow \infty} U'(\omega, x) = 0,$$

where U' denotes the partial derivative with respect to the second argument. At $x = 0$, we set by monotonicity $U(\omega, 0) := \lim_{x \downarrow 0} U(\omega, x)$, which may be $-\infty$. For every $x > 0$, $U(\cdot, x)$ is $\mathcal{F}$-measurable.

By controlling the investment, the goal of an agent is to maximise the expected utility. The value functions are given by

$$u^n(x, q) = \sup_{X \in \mathcal{X}^n(x,q)} \mathbb{E}[U(X_T + qF_T)], \qquad (x, q) \in \mathcal{K}^n, n \in \mathbb{N}, \tag{2.2}$$

$$u(x, q) = \sup_{X \in \mathcal{X}(x,q)} \mathbb{E}[U(X_T + qF_T)], \qquad (x, q) \in \mathcal{K}. \tag{2.3}$$

Here and below, we use the convention that

$$\text{if } \mathbb{E}[U^-(X_T + qf)] = \infty, \qquad \text{we set } \mathbb{E}[U(X_T + qf)] := -\infty.$$

If will be convenient to extend the definitions of u^n and u to $\mathbb{R}^{N+1}$ by setting

$$u^n(x, q) := -\infty, \qquad (x, q) \in \mathbb{R}^{N+1} \backslash \mathcal{K}^n, n \in \mathbb{N},$$
$$u(x, q) := -\infty, \qquad (x, q) \in \mathbb{R}^{N+1} \backslash \mathcal{K}.$$

To ensure that the utility maximisation problems (2.2) and (2.3) are non-degenerate, we need to impose no-arbitrage conditions. With

$$\mathcal{Z}^n = \{\text{martingale} Z > 0 : Z_0 = 1 \text{ and } ZX \text{ is a local martingale for every } X \in \mathcal{X}^n(1, 0)\}$$

for $n \in \mathbb{N}$ and

$$\mathcal{Z} := \bigcap_{n \geq 1} \mathcal{Z}^n,$$

we suppose that

$$\mathcal{Z} \neq \emptyset. \tag{2.4}$$

Remark 2.4 Condition (2.4) is closely related, but stronger than the existence of an equivalent separating measure in the large market, that is, a probability measure $\mathbb{Q}$ such that $\mathbb{E}_{\mathbb{Q}}[X_T] \leq 1$ for every $X \in \mathcal{X}(1, 0)$. Unlike for small markets, where the existence of an equivalent separating measure implies the existence of an equivalent σ-martingale measure by the results in Delbaen and Schachermayer [11] (in particular, [11, Proposition 4.7 and Theorem 1.1]), a counterexample in Cuchiero et al. [6, Sect. 6] demonstrates that in large markets, the existence of an equivalent separating measure does not imply the existence of a σ-martingale measure. Further, by [6, Theorem 3.3], the existence of an equivalent separating measure in a large market is equivalent to the condition of *no asymptotic free lunch with vanishing risk* (NAFLVR); see [6] for details.

2.2 Dual problem

We begin by setting in small markets

$$\mathcal{L}^n := (-\mathcal{K}^n)^o, \qquad n \in \mathbb{N},$$

that is, the respective polars of $-\mathcal{K}^n$, $n \in \mathbb{N}$, in $\mathbb{R}^{N+1}$. We refer to Rockafellar [33, Sect. 15] for the definition and properties of a polar of a set. Naturally, we extend this definition to the large market by setting

$$\mathcal{L} := -\mathcal{K}^o.$$

We introduce, or rather recall, the classical sets of supermartingale deflators in small markets and define for $y > 0$ and $n \in \mathbb{N}$

$$\mathcal{Y}^n(y) := \{Y \geq 0 : \ Y_0 = y \text{ and } XY \text{ is a supermartingale for every } X \in \mathcal{X}^n(1,0)\}.$$

We set for $(y, r) \in \mathcal{L}^n$,

$$\begin{aligned}\mathcal{Y}^n(y,r) := \{Y \in \mathcal{Y}^n(y) : \mathbb{E}[Y_T(X_T + qf)] \leq xy + qr \\ \text{for every } (x,q) \in \mathcal{K}^n \text{ and } X \in \mathcal{X}^n(x,q)\}.\end{aligned}$$

Similarly, in the large market, we define for $y > 0$ and $(y, r) \in \mathcal{L}$,

$$\begin{aligned}\mathcal{Y}(y) &= \{Y \geq 0 : Y_0 = y \text{ and } XY \text{ is a supermartingale} \\ &\qquad \text{for every } X \in \mathcal{X}(1,0)\}, \qquad (2.5)\\ \mathcal{Y}(y,r) &:= \{Y \in \mathcal{Y}(y) : \mathbb{E}[Y_T(X_T + qf)] \leq xy + qr \\ &\qquad \text{for every } (x,q) \in \mathcal{K} \text{ and } X \in \mathcal{X}(x,q)\}.\end{aligned}$$

Let us set

$$V(\omega, y) := \sup_{x>0} \big(U(\omega, x) - xy\big), \qquad (\omega, y) \in \Omega \times [0, \infty). \qquad (2.6)$$

We note that $-V$ satisfies Assumption 2.3. Now we can state the dual problems for small markets and the large market as

$$v^n(y,r) = \inf_{Y \in \mathcal{Y}^n(y,r)} \mathbb{E}[V(Y_T)], \qquad (y,r) \in \mathcal{L}^n, n \in \mathbb{N}, \qquad (2.7)$$

$$v(y,r) = \inf_{Y \in \mathcal{Y}(y,r)} \mathbb{E}[V(Y_T)], \qquad (y,r) \in \mathcal{L}, \qquad (2.8)$$

where we use the convention that

$$\text{if } \mathbb{E}[V^+(Y_T)] = \infty, \qquad \text{we set } \mathbb{E}[V(Y_T)] := \infty.$$

Further, we extend the definitions of v^n and v to $\mathbb{R}$ by setting

$$\begin{aligned} v^n(y,r) &:= \infty, \qquad (y,r) \in \mathbb{R}^{N+1} \backslash \mathcal{L}^n, n \in \mathbb{N}, \\ v(y,r) &:= \infty, \qquad (y,r) \in \mathbb{R}^{N+1} \backslash \mathcal{L}. \end{aligned}$$

Let us set

$$\tilde{w}(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)], \qquad y > 0, \tag{2.9}$$

and suppose that

$$u(x,0) > -\infty, \quad x > 0, \qquad \text{and} \qquad \tilde{w}(y) < \infty, \quad y > 0. \tag{2.10}$$

3 Utility maximisation with a random endowment in a large market

Theorem 3.1 *Suppose Assumptions 2.3 and 2.1 as well as (2.4) and (2.10) hold. Then*:

(i) *The functions u and v are finite on* $\operatorname{int} \mathcal{K}$ *and* $\operatorname{ri} \mathcal{L}$, *respectively, and satisfy*

$$\begin{aligned} u(x,q) &= \inf_{(y,r) \in \mathcal{L}} \big(v(y,r) + xy + qr\big), \qquad (x,q) \in \mathcal{K}, \\ v(y,r) &= \sup_{(x,q) \in \mathcal{K}} \big(u(x,q) + xy + qr\big), \qquad (y,r) \in \mathcal{L}. \end{aligned}$$

(ii) *The functions u and $-v$ are concave and upper semi-continuous, and $u < \infty$ on $\mathcal{K}$. For every $(x,q) \in \{u > -\infty\}$, there exists a unique maximiser to (2.3). In turn, $v > -\infty$ on $\mathcal{L}$. For every $(y,r) \in \{v < \infty\}$, there exists a unique solution to (2.8).*

(iii) *For every $(x,q) \in \operatorname{int} \mathcal{K}$, the subdifferential of u at (x,q) is a nonempty subset of* $\operatorname{ri} \mathcal{L}$, *and $(y,r) \in \partial u(x,q)$ if and only if the following conditions hold*:

$$\begin{aligned} \widehat{Y}_T(y,r) &= U'\big(\cdot, \widehat{X}_T(x,q) + qf\big) \qquad \mathbb{P}\text{-a.s.}, \\ \mathbb{E}[\widehat{Y}_T(\widehat{X}_T + qf)] &= xy + qr, \\ |v(y,r)| &< \infty. \end{aligned}$$

For every (x,q) and (y,r) in $\mathbb{R}^{N+1}$, let us set

$$\begin{aligned} \mathcal{C}^n(x,q) &:= \{g \in \mathbb{L}^0_+ : g \le X_T + qf \text{ for some } X \in \mathcal{X}^n(x,q)\}, \qquad n \in \mathbb{N}, \\ \mathcal{C}(x,q) &:= \{g \in \mathbb{L}^0_+ : g \le X_T + qf \text{ for some } X \in \mathcal{X}(x,q)\}, \\ \mathcal{D}^n(y,r) &:= \{h \in \mathbb{L}^0_+ : h \le Y_T \text{ for some } Y \in \mathcal{Y}^n(y,r)\}, \qquad n \in \mathbb{N}, \\ \mathcal{D}(y,r) &:= \{h \in \mathbb{L}^0_+ : h \le Y_T \text{ for some } Y \in \mathcal{Y}(y,r)\}. \end{aligned} \tag{3.1}$$

For small markets, we recall Hugonnier and Kramkov [13, Proposition 1] whose intricate proof was based on a delicate parametrisation of the dual domain, the optional decomposition theorem from Kramkov [24] and superreplications results for finite-dimensional models from Delbaen and Schachermayer [11].

Proposition 3.2 *If* (2.4) *and Assumption* 2.1 *hold, then for every* $n \in \mathbb{N}$, *the families* $(\mathcal{C}^n(x,q))_{(x,q)\in\mathcal{K}^n}$ *and* $(\mathcal{D}^n(y,r))_{(y,r)\in\mathcal{L}^n}$ *in* (3.1) *have the following properties*:

1) *For every* $(x,q) \in \operatorname{int}\mathcal{K}^n$, *the set* $\mathcal{C}^n(x,q)$ *contains a strictly positive constant. A nonnegative random variable* g *belongs to* $\mathcal{C}^n(x,q)$ *if and only if*

$$\mathbb{E}[gh] \le xy + qr \qquad \text{for every } (y,r) \in \mathcal{L}^n \text{ and } h \in \mathcal{D}^n(y,r).$$

2) *For every* $(y,r) \in \operatorname{ri}\mathcal{L}^n$, *the set* $\mathcal{D}^n(y,r)$ *contains a strictly positive random variable. A nonnegative function* h *belongs to* $\mathcal{D}^n(y,r)$ *if and only if*

$$\mathbb{E}[gh] \le xy + qr \qquad \text{for every } (x,q) \in \mathcal{K}^n \text{ and } g \in \mathcal{C}^n(x,q).$$

Here is an analogous result, but for the large market.

Proposition 3.3 *If* (2.4) *and Assumption* 2.1 *hold, then the families* $(\mathcal{C}(x,q))_{(x,q)\in\mathcal{K}}$ *and* $(\mathcal{D}(y,r))_{(y,r)\in\mathcal{L}}$ *in* (3.1) *have the following properties*:

(i) *For every* $(x,q) \in \operatorname{int}\mathcal{K}$, *the set* $\mathcal{C}(x,q)$ *contains a strictly positive constant. A nonnegative random variable* g *belongs to* $\mathcal{C}(x,q)$ *if and only if*

$$\mathbb{E}[gh] \le xy + qr \qquad \text{for every } (y,r) \in \mathcal{L} \text{ and } h \in \mathcal{D}(y,r). \tag{3.2}$$

(ii) *For every* $(y,r) \in \operatorname{ri}\mathcal{L}$, *the set* $\mathcal{D}(y,r)$ *contains a strictly positive random variable. A nonnegative function* h *belongs to* $\mathcal{D}(y,r)$ *if and only if*

$$\mathbb{E}[gh] \le xy + qr \qquad \text{for every } (x,q) \in \mathcal{K} \text{ and } g \in \mathcal{C}(x,q). \tag{3.3}$$

We summarise the characterisations of Assumption 2.1 in the following result. **Here and below**, we also use the notations $\mathcal{M}$ and $\mathcal{M}^n$ for the sets of probability measures whose densities are in $\mathcal{Z}$ and $\mathcal{Z}^n$, respectively. Both kinds of notation are so common in the literature that we believe this will cause no confusion.

Lemma 3.4 *If* (2.4) *and Assumption* 2.1 *hold, then we have*

(i) $(x,0) \in \operatorname{int}\mathcal{K}$ *for every* $x > 0$;

(ii) *for every* $q \neq 0$, *there exists* $x > 0$ *such that* $(x,q) \in \operatorname{int}\mathcal{K}$;

(iii) (*trivially*) *there exists a nonnegative wealth process of a generalised strategy such that* $X_T \ge \sum_{i=1}^N |f^i|$;

(iv) (*trivially*) $\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[\sum_{i=1}^N |f^i|] < \infty$.

3.1 Proof of Proposition 3.3 for the large market only

We begin with the following characterisation of the set $\mathcal{K}$.

Lemma 3.5 *If* (2.4) *and Assumption* 2.1 *hold, then $\mathcal{K}$ is closed in $\mathbb{R}^{N+1}$ and thus for every $(x,q) \in \mathrm{cl}\,\mathcal{K}$, we have $\mathcal{X}(x,q) \neq \emptyset$.*

Proof Let $(x^n, q^n)_{n\in\mathbb{N}} \subseteq \mathcal{K}$ be a sequence converging to $(x,q) \in \mathrm{cl}\,\mathcal{K}$, where the closure is taken in $\mathbb{R}^{N+1}$. Consider $X^n \in \mathcal{X}(x^n, q^n)$, $n \in \mathbb{N}$, and set $Z^n := X^n + C$, $n \in \mathbb{N}$, where C is a sufficiently large positive constant such that $Z^n_T \geq 0$ for every $n \in \mathbb{N}$ (e.g. $C = N \max_{n\in\mathbb{N}} \|q^n\|_\infty \max_{i\in\{1,\dots,N\}} \|f_i\|_\infty$). One can show that the Z^n are nonnegative $\mathbb{Q}$-supermartingales for every $\mathbb{Q} \in \mathcal{M}$ (see also De Donno et al. [7, Sect. 2]). By passing to convex combinations, which we do not relabel, and using Fatou-convergence under any such $\mathbb{Q}$, we can obtain a process Z as a Fatou-limit of $(Z^n)_{n\in\mathbb{N}}$ on the set of rational times and in T. By construction, we have

$$Z_T - C + qf = \lim_{n\to\infty} (Z^n_T - C + q^n f) = \lim_{n\to\infty} (X^n_T + q^n f) \geq 0. \tag{3.4}$$

We also have

$$Z_0 \leq \liminf_{n\to\infty} Z^n_0 = x + C.$$

Therefore, as Z is a supermartingale for every $\mathbb{Q} \in \mathcal{M}$, we deduce that

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}[Z_T] \leq Z_0 \leq x + C.$$

Now [7, Theorem 3.1] implies that there exists an admissible generalised strategy H such that

$$x + C + H \cdot S_T \geq Z_T \qquad \mathbb{P}\text{-a.s.} \tag{3.5}$$

Let us set

$$X := x + H \cdot S.$$

Then using (3.4) and (3.5), we have that $X_0 = x$ and

$$X_T + qf = x + C + H \cdot S_T + qf - C \geq Z_T + qf - C \geq 0,$$

where the first inequality uses (3.5) and the second (3.4). We deduce that $X \in \mathcal{X}(x,q)$ and thus $(x,q) \in \mathcal{K}$. We conclude that $\mathcal{K}$ is closed. □

Let us consider for the dual domain the parametrisation

$$\mathcal{M}(\rho) := \{\mathbb{Q} \in \mathcal{M} : \mathbb{E}_{\mathbb{Q}}[f] = \rho\}, \qquad \rho \in \mathbb{R}^N. \tag{3.6}$$

Let us set

$$\mathcal{P}' := \{\rho \in \mathbb{R}^N : \mathcal{M}(\rho) \neq \emptyset\} \qquad \text{and} \qquad \mathcal{P} := \{\rho \in \mathbb{R}^N : (1,\rho) \in \mathrm{ri}\,\mathcal{L}\}. \tag{3.7}$$

For the proofs below, we impose the following non-replicability assumption, which allows us to handle the most difficult case. On the other hand, the cases when some of the components of f are replicable can be handled by reducing the dimensionality of the problem, and if all components of f are replicable, we can analyse (2.3) via the results from optimal investment without a random endowment; see e.g. De Donno et al. [7], Mostovyi [28] and also the discussion in Sect. 6.

Definition 3.6 We say that a bounded random variable g is *replicable in the large market* if there exists an admissible generalised wealth process $X = x + H \cdot S$, where $x \in \mathbb{R}$ and H is an admissible generalised strategy, such that $-X$ is also admissible and $X_T = g$.

Assumption 3.7 We suppose that every component of f is non-replicable in the following sense: For every $q \in \mathbb{R}^N \setminus \{0\}$, the random variable qf is not replicable in the large market. We note that this is equivalent to $\mathcal{L}$ being open in $\mathbb{R}^{N+1}$, by a line-by-line adaptation of the argument from Hugonnier and Kramkov [13, Lemma 7].

Lemma 3.8 *If* (2.4) *and Assumption* 2.1 *as well as Assumption* 3.7 *hold*, *then*

$$\mathcal{P}' = \mathcal{P}$$

and

$$\bigcup_{\rho \in \mathcal{P}} \mathcal{M}(\rho) = \mathcal{M}.$$

Proof Fix $q \in \mathbb{R}^N \setminus \{0\}$ and consider qf. One can see that Assumption 3.7 implies that for every constant x with $(x, q) \in \mathcal{K}$, there exists $X \in \mathcal{X}(x, q)$ such that

$$\mathbb{P}[X_T + qf > 0] > 0.$$

Then for $\mathbb{Q} \in \mathcal{M}(\rho)$, using the supermartingale property of X under $\mathbb{Q}$, we have

$$0 < \mathbb{E}_{\mathbb{Q}}[X_T + qf] \le x + q\rho.$$

As (x, q) is arbitrary in $\mathcal{K}$, we conclude that $\rho \in \mathcal{P}$. Therefore we get

$$\mathcal{P}' \subseteq \mathcal{P}. \tag{3.8}$$

On the other hand, for a fixed $q \in \mathbb{R}^N$ and $x := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[qf]$, by [7, Theorem 3.1], there exists an x-admissible generalised strategy H such that

$$qf \le x + H \cdot S_T.$$

This implies that $x + H \cdot S \in \mathcal{X}(x, -q)$ so that $(x, -q) \in \mathcal{K}$. As a result, we have

$$q\rho \le x \qquad \text{for all } \rho \in \mathcal{P}.$$

We deduce that

$$\sup_{\rho \in \mathcal{P}'} q\rho = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[qf] = x \ge \sup_{\rho \in \mathcal{P}} q\rho.$$

As q is arbitrary, we conclude that

$$\mathcal{P}' \supseteq \mathcal{P}. \tag{3.9}$$

Combining (3.8) and (3.9), we deduce that

$$\mathcal{P}' = \mathcal{P}$$

and thus

$$\mathcal{M} = \bigcup_{\rho \in \mathcal{P}'} \mathcal{M}(\rho) = \bigcup_{\rho \in \mathcal{P}} \mathcal{M}(\rho).$$

□

Lemma 3.9 *If* (2.4) *and Assumption* 2.1 *hold, then for every* $(x, q) \in \mathcal{K}$, *we have* $g \in \mathcal{C}(x, q)$ *if and only if*

$$\mathbb{E}_{\mathbb{Q}}[g] \leq x + q\rho \qquad \textit{for every } \rho \in \mathcal{P} \textit{ and } \mathbb{Q} \in \mathcal{M}(\rho). \tag{3.10}$$

Proof Consider a nonnegative random variable g such that (3.10) holds. Denote

$$h := g - qf.$$

Then boundedness of f implies $h \geq -C$ for some constant $C > 0$. Therefore

$$\begin{aligned} \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[h + C] &= \sup_{\rho \in \mathcal{P}} \sup_{\mathbb{Q} \in \mathcal{M}(\rho)} \mathbb{E}_{\mathbb{Q}}[h + C] \\ &= \sup_{\rho \in \mathcal{P}} \sup_{\mathbb{Q} \in \mathcal{M}(\rho)} \mathbb{E}_{\mathbb{Q}}[g - qf + C] \leq x + C. \end{aligned}$$

As $h + C \in \mathbb{L}^0_+$, [7, Theorem 3.1] implies the existence of an $(x + C)$-admissible generalised strategy H such that

$$h + C \leq x + C + H \cdot S_T,$$

and thus

$$0 \leq g \leq x + H \cdot S_T + qf.$$

We deduce that $g \in \mathcal{C}(x, q)$.

Conversely, let $g \in \mathcal{C}(x, q)$. One can see that for every $\rho \in \mathcal{P}$, the density process of $\mathbb{Q} \in \mathcal{M}(\rho)$ belongs to $\mathcal{Y}(1, \rho)$. This implies (3.10). □

Proof of Proposition 3.3 Let $(x, q) \in \operatorname{int} \mathcal{K}$. Then there exists $\varepsilon > 0$ such that $(x - \varepsilon, q) \in \mathcal{K}$. Now pick $X \in \mathcal{X}(x - \varepsilon, q)$; then $X + \varepsilon \in \mathcal{X}(x, q)$ and

$$X_T + \varepsilon + qf \geq \varepsilon > 0.$$

Therefore $\varepsilon \in \mathcal{C}(x, q)$, and thus $\mathcal{C}(x, q)$ contains a positive constant.

If $g \in \mathcal{C}(x, q)$, (3.2) follows from the construction of the sets $\mathcal{D}(y, r)$, $(y, r) \in \mathcal{L}$. Conversely, assume that $g \in \mathbb{L}^0_+$, satisfies (3.2). As for every $\rho \in \mathcal{P}$, the density process of $\mathbb{Q} \in \mathcal{M}(\rho)$ belongs to $\mathcal{Y}(1, \rho)$, we deduce that g satisfies (3.10). So $g \in \mathcal{C}(x, q)$ by Lemma 3.9.

For (ii), it is enough to prove the assertion for $(y, r) = (1, \rho)$ for some $\rho \in \mathcal{P}$ as $c\mathcal{D}(y, r) = \mathcal{D}(cy, cr)$ for every $c > 0$ and $(y, r) \in \mathcal{L}$. By Lemma 3.8, for every $\rho \in \mathcal{P}$, there exists $\mathbb{Q} \in \mathcal{M}(\rho)$. The density process $\mathbb{Q} \in \mathcal{M}(\rho)$ belongs to $\mathcal{Y}(1, \rho)$. As $\mathbb{Q} \approx \mathbb{P}$, $\frac{d\mathbb{Q}}{d\mathbb{P}} > 0$ $\mathbb{P}$-a.s.

If $h \in \mathcal{D}(1, \rho)$, then (3.3) follows from the definition of the set $\mathcal{Y}(1, \rho)$. Conversely, consider $h \in \mathbb{L}^0_+$ satisfying (3.3). Then in particular, we have

$$\mathbb{E}[gh] \leq 1 \qquad \text{for every } g \in \mathcal{C}(1, 0),$$

where $\mathcal{C}(1, 0) \neq \emptyset$ by Lemma 3.4. Therefore, by Mostovyi [28, Lemma 3.4], h is a terminal value of an element of $\mathcal{Y}(1)$ and satisfies (3.3), i.e., $h \in \mathcal{Y}(1, \rho)$. □

3.2 Proving Theorem 3.1 for large markets

The proof of the following result is an adaptation of the proof of Mostovyi [27, Lemma 2.6] and is skipped.

Lemma 3.10 *Under the conditions of Theorem 3.1, we have*

$$u(x, q) > -\infty, \quad (x, q) \in \operatorname{int} \mathcal{K} \qquad \textit{and} \qquad v(y, r) < \infty, \quad (y, r) \in \operatorname{ri} \mathcal{L}.$$

Lemma 3.11 *Under the conditions of Theorem 3.1, we have*

$$u(x, q) \leq v(y, r) + xy + qr \qquad \textit{for every } (x, q) \in \mathcal{K} \textit{ and every } (y, r) \in \mathcal{L}. \tag{3.11}$$

As a consequence, we have

$$u(x, q) < \infty \quad \textit{and} \quad v(y, r) > -\infty \qquad \textit{on } \mathbb{R}^{N+1}. \tag{3.12}$$

Proof Fix $(x, q) \in \mathcal{K}$ and $(y, r) \in \mathcal{L}$. For any $X \in \mathcal{X}(x, q)$ and $Y \in \mathcal{Y}(y, r)$, we have

$$U(X_T + qf) \leq V(Y_T) + (X_T + qf)Y_T \qquad \mathbb{P}\text{-a.s.}$$

Taking expectations and recalling (2.5), we obtain

$$\mathbb{E}[U(X_T + qf)] \leq \mathbb{E}[V(Y_T)] + \mathbb{E}[(X_T + qf)Y_T] \leq \mathbb{E}[V(Y_T)] + xy + qr. \tag{3.13}$$

As X and Y are arbitrary elements of $\mathcal{X}(x, q)$ and $\mathcal{Y}(y, r)$, taking in (3.13) the supremum over $X \in \mathcal{X}(x, q)$ and (then) the infimum over $Y \in \mathcal{Y}(y, r)$ gives

$$u(x, q) \leq v(y, r) + xy + qr,$$

which is precisely (3.11). In turn, (3.12) follows from (3.11) and Lemma 3.10. □

Proof of Theorem 3.1 The proof is an adaptation of the closely related proof of [27, Theorem 2.4]. Therefore we only highlight one point: In order to show that $\partial u(x, q) \subseteq \operatorname{ri} \mathcal{L}$ for $(x, q) \in \operatorname{int} \mathcal{K}$, one can observe that in the fully non-replicable case (as in Assumption 3.7), $0 < U'(\omega, \widehat{X}_T + qf)$ belongs to $\mathcal{D}(y', r')$ for every $(y', r') \in \partial u(x, q)$, and then one can show that $(y', r') \in \operatorname{ri} \mathcal{L}$. □

4 Marginal-utility-based pricing in the large market

We consider the following definition.

Definition 4.1 Let $f^i \in \mathbb{L}^0$, $i \in \{1, \dots, N\}$, and $x > 0$. A vector $\rho \in \mathbb{R}^N$ is a *marginal-utility-based price* for f given the initial capital x if

$$\mathbb{E}[U(X_T + qf)] \le u(x, 0) \qquad \text{for all } q \in \mathbb{R}^N \text{ and } X \in \mathcal{X}(x - q\rho, q). \tag{4.1}$$

We denote the set of marginal-utility-based prices by $\Pi(x)$.

This definition is a natural extension of standard definitions of the utility-based prices in the literature (see e.g. Hugonnier et al. [14, Definition 3.1]) to a stochastic utility and a large market. Let us observe that given our formulation (2.3) of the utility maximisation problem for the large market, marginal-utility-based prices can be characterised by

$$\{\rho : \ u(x - q\rho, q) \le u(x, 0) \text{ for every } q \in \mathbb{R}^N\}. \tag{4.2}$$

We note that the initial wealth is important in both formulations (4.1) and (4.2), and thus the marginal-utility-based prices depend in general on the initial wealth x. This observation has a clear financial interpretation.

Further, (4.2) leads to a natural characterisation of the set of marginal-utility-based prices as

$$\Pi(x) = \left\{ \frac{r}{y} : \ (y, r) \in \partial u(x, 0) \right\}. \tag{4.3}$$

In (2.3), given the concavity of u and in view of Lemma 3.4, we immediately obtain the existence of marginal-utility-based prices for every $x > 0$. If we fix $x > 0$ first and then compute $\Pi(x)$, the question of whether $\Pi(x)$ is a singleton or not becomes important as the uniqueness of marginal-utility-based prices is a necessary condition for the well-posedness (in the sense of Hadamard) of the marginal-utility-based pricing problem. That uniqueness is a desirable feature both from the mathematical and financial viewpoints. If $\Pi(x)$ is a singleton, we get the representation

$$\Pi(x) = \left\{ \frac{u_q(x, 0)}{u_x(x, 0)} \right\}.$$

Below, we provide a sufficient condition for the uniqueness of the marginal-utility-based prices given the initial wealth x.

Theorem 4.2 *Under the conditions of Theorem 3.1, let $x > 0$ be fixed and consider $y := u_x(x, 0)$ and $\widehat{Y}(y)$, the optimiser to (2.9) at y. If $\mathbb{E}[\widehat{Y}_T(y)] = y$, then $\Pi(x)$ is a singleton, and for*

$$\rho := \mathbb{E}\left[\frac{\widehat{Y}_T(y)}{y} f \right] \in \mathcal{P}, \tag{4.4}$$

we have

$$\Pi(x) = \{\rho\}\,.$$

For the proof of Theorem 4.2, we need the following lemma.

Lemma 4.3 *Under the conditions of Theorem 3.1, consider (2.3) for $q = 0$, i.e., $u(x,0) = \sup_{X\in\mathcal{X}(x,0)} \mathbb{E}[U(X_T)]$, $x > 0$, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive numbers converging to $x > 0$. Then for the optimisers to (2.3), we have $\widehat{X}_T(x_n,0) \to \widehat{X}_T(x,0)$ in probability.*

Proof Let us denote

$$g := \widehat{X}_T(x,0) \qquad \text{and} \qquad g^n := \widehat{X}_T(x_n,0), \quad n\in\mathbb{N}.$$

Then if (g^n) does not converge to g in probability, there exists $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}[|g^n - g| > \varepsilon] > \varepsilon.$$

It follows from (2.4) that the set $\{X_T : X \in \mathcal{X}(1,0)\}$ is bounded in $\mathbb{L}^1$ under some probability measure which is equivalent to $\mathbb{P}$. Therefore that set is bounded in $\mathbb{L}^0$ under $\mathbb{P}$, and by passing to a smaller ε if necessary, we get

$$\limsup_{n\to\infty} \mathbb{P}\bigg[|g^n - g| > \varepsilon,\, |g^n + g| < \frac{1}{\varepsilon}\bigg] > \varepsilon. \tag{4.5}$$

From the concavity of $U(\omega,\,\cdot\,)$, $\omega\in\Omega$, we deduce that

$$U\bigg(\frac{g^n+g}{2}\bigg) \geq \frac{1}{2}\big(U(g^n)+U(g)\big),$$

whereas (4.5) and the strict concavity of $U(\omega,\,\cdot\,)$, $\omega\in\Omega$, imply the existence of a random variable $\eta > 0$ and a constant $\delta > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\bigg[U\bigg(\frac{g^n+g}{2}\bigg) \geq \frac{1}{2}\big(U(g^n)+U(g)\big) + \eta\bigg] > \delta.$$

(If U is deterministic, η can be chosen to be a constant.) Because $u(\,\cdot\,,0)$ is concave and finite on $(0,\infty)$, it follows that $u(\,\cdot\,,0)$ is continuous on $(0,\infty)$, and with the sets $A_n := \{U(\frac{g^n+g}{2}) \geq \frac{1}{2}(U(g^n)+U(g))+\eta\}$, $n\in\mathbb{N}$, we have

$$\limsup_{n\to\infty} \mathbb{E}\bigg[U\bigg(\frac{g^n+g}{2}\bigg)\bigg] \geq u(x,0) + \limsup_{n\to\infty} \mathbb{E}[\eta 1_{A_n}] > u(x,0). \tag{4.6}$$

Now we pass to convex combinations $\tilde{g}^n \in \operatorname{conv}(g^n, g^{n+1},\dots)$, $n\in\mathbb{N}$, which converge $\mathbb{P}$-a.s. to some random variable $\tilde{g}$. By the symmetry between the primal and dual value functions and since $-V$ satisfies Assumption 2.3, Mostovyi [26, Lemma 3.5] also implies the uniform integrability of $(U^+(\tilde{g}^n))_{n\in\mathbb{N}}$. Hence we get

$$\limsup_{n\to\infty} \mathbb{E}\left[U\left(\frac{\tilde{g}^n + g}{2}\right)\right] \leq \mathbb{E}\left[U\left(\frac{\tilde{g} + g}{2}\right)\right],$$

and concavity of U gives via (4.6) that

$$\begin{aligned}\mathbb{E}\left[U\left(\frac{\tilde{g} + g}{2}\right)\right] &\geq \limsup_{n\to\infty} \mathbb{E}\left[U\left(\frac{\tilde{g}^n + g}{2}\right)\right] \\ &\geq \limsup_{n\to\infty} \mathbb{E}\left[U\left(\frac{g^n + g}{2}\right)\right] > u(x, 0). \qquad (4.7)\end{aligned}$$

Using De Donno et al. [7, Lemma 3.3], we deduce the existence of $X \in \mathcal{X}(x, 0)$ such that

$$X_T \geq \frac{\tilde{g} + g}{2} \qquad \mathbb{P}\text{-a.s.}$$

Combining the latter inequality with (4.7), we conclude that

$$\mathbb{E}[U(X_T)] > u(x, 0),$$

which is a contradiction. □

Proof of Theorem 4.2 Fix $q \in \mathbb{R}^N$ with $(x - q\rho, q) \in \mathcal{K}$ and consider an arbitrary $X \in \mathcal{X}(x - q\rho, q)$. From the boundedness of f, one can show that $\widehat{Y}(y)X$ is a $\mathbb{P}$-supermartingale. From [28, Theorem 2.2], we have

$$u(x, 0) - xy = \tilde{w}(y).$$

Therefore, using the conjugacy of U and V, we get

$$\begin{aligned}\mathbb{E}[U(X_T + qf)] &\leq \mathbb{E}\left[V\left(\widehat{Y}_T(y)\right) + \widehat{Y}_T(y)(X_T + qf)\right] \\ &= \tilde{w}(y) + \mathbb{E}[\widehat{Y}_T(y)(X_T + qf)] \\ &= u(x, 0) - xy + \mathbb{E}[\widehat{Y}_T(y)(X_T + qf)] \\ &\leq u(x, 0) - xy + y(x - q\rho + q\rho) = u(x, 0). \qquad (4.8)\end{aligned}$$

As q is an arbitrary element of $\mathbb{R}^N$ with $(x - q\rho, q) \in \mathcal{K}$ and X is an arbitrary element of $\mathcal{X}(x - q\rho, q)$, we deduce from (4.8) (comparing (4.8) with (4.1)) that ρ is a marginal-utility-based price for f.

To show the uniqueness of ρ, consider some $\pi \in \mathbb{R}^N$ with $\pi \neq \rho$. First suppose that $\pi_i < \rho_i$ for some $i \in \{1, \ldots, N\}$. For $C_k := \|f_k\|_\infty$ and $\vec{c} := (C_1, \ldots, C_N)$ and with e_i being the ith unit vector in $\mathbb{R}^N$, consider a sequence $(s_n)_{n\in\mathbb{N}}$ of positive numbers with $s_n \to 0$ and such that $s_n e_i(\vec{c} + \pi) < x$, $n \in \mathbb{N}$, and set

$$q_n := s_n e_i, \quad X^n := \widehat{X}\big(x - q_n(\vec{c} + \pi), 0\big) + q_n\vec{c}, \qquad n \in \mathbb{N}.$$

Then we have

$$X_0^n = x - q_n\pi \quad \text{and} \quad X^n \in \mathcal{X}(x - q_n\pi, q_n), \qquad n \in \mathbb{N}.$$

We deduce that

$$\begin{aligned} u(x - q_n\pi, q_n) &\geq \mathbb{E}[U(X_T^n + q_n f)] \\ &\geq \mathbb{E}\Big[U\Big(\widehat{X}_T\big(x - q_n(\vec{c} + \pi), 0\big)\Big)\Big] + \mathbb{E}[q_n(\vec{c} + f)U'(X_T^n + q_n f)] \\ &= u\big(x - q_n(\vec{c} - \pi), 0\big) + \mathbb{E}[q_n(\vec{c} + f)U'(X_T^n + q_n f)]. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} &\liminf_{n\to\infty} \frac{u(x - q_n\pi, q_n) - u(x - q_n(\vec{c} - \pi), 0)}{s_n} \\ &\geq \liminf_{n\to\infty} \mathbb{E}[e_i(\vec{c} + f)U'(X_T^n + q_n f)] \\ &\geq \mathbb{E}\big[e_i(\vec{c} + f)U'\big(\widehat{X}_T(x, 0)\big)\big] \\ &= \mathbb{E}[e_i(\vec{c} + f)\widehat{Y}_T(y, 0)] \\ &= e_i(\vec{c} + \rho)y, \end{aligned}$$

where in the second inequality we have used Fatou's lemma and the assertion of Lemma 4.3. We deduce that

$$\begin{aligned} \liminf_{n\to\infty} \frac{u(x - q_n\pi, q_n) - u(x, 0)}{s_n} &= \liminf_{n\to\infty} \frac{u(x - q_n\pi, q_n) - u(x - q_n(\vec{c} + \pi), 0)}{s_n} \\ &\quad + \liminf_{n\to\infty} \frac{u(x - q_n(\vec{c} + \pi), 0) - u(x, 0)}{s_n} \\ &\geq e_i(\vec{c} + \rho)y - e_i u_x(x, 0)(\vec{c} + \pi) \\ &= e_i(\rho - \pi)y > 0. \end{aligned}$$

As $s_n > 0$, $n \in \mathbb{N}$, we deduce that π is not a marginal-utility-based price as π does not satisfy (4.2). As π_i was an arbitrary number smaller than ρ_i, we deduce that every π with $\pi_i < \rho_i$ for some i is not a marginal-utility-based price for f. Denoting $\tilde{f} = -f$, we can apply the argument above to show that every $\tilde{\pi}$ such that $\tilde{\pi}_i < -\rho_i$ for some i is not a marginal-utility-based price for $-f$, and thus every π such that $\pi_i > \rho_i$ for some i is not a marginal-utility-based price for f. As $i \in \{1, \dots, N\}$ was arbitrary, we deduce that every π such that $\pi_i \neq \rho_i$ for some $i \in \{1, \dots, N\}$ is not a marginal-utility-based price. In other words, under our assumptions, the marginal-utility-based price ρ given by (4.4) is unique.

Finally, to show that $\rho \in \mathcal{P}$, we observe that since $\mathbb{E}[\widehat{Y}_T(y)] = y$, we deduce that $\frac{\hat{Y}}{y}$ is the density process of an element of $\mathcal{M}$. Therefore $\rho \in \mathcal{P}$ by Lemma 3.8. □

5 The marginal-utility-based prices in a large market are a limit of marginal-utility-based prices in small markets

For the convergence results, we need to strengthen (2.10) to ensure that the value functions in small markets are finite-valued, too.

Assumption 5.1 There exists $\tilde{n} \in \mathbb{N}$ such that

$$u^{\tilde{n}}(x, 0) > -\infty, \quad x > 0, \qquad \text{and} \qquad \tilde{w}(y) < \infty, \quad y > 0.$$

Remark 5.2 If we impose Assumptions 2.3, 2.1, 5.1 and (2.4), an application of Mostovyi [27, Lemma 2.6] implies (2.10) and, for all $n \geq \tilde{n}$,

$$u^n(x, q) > -\infty, \quad (x, q) \in \operatorname{int} \mathcal{K}^n \qquad \text{and} \qquad v^n(y, r) < \infty, \quad (y, r) \in \operatorname{ri} \mathcal{L}^n.$$

Under Assumption 5.1, the marginal-utility-based prices $\Pi^n(x)$, $n \in \mathbb{N}$, in small markets can be characterised similarly to Theorem 4.2.

5.1 The case when the dual minimisers in small and large markets are true martingales

Recall that the dual problem in a small market without an endowment is

$$\tilde{w}^n(y) := \inf_{Y \in \mathcal{Y}^n(y)} \mathbb{E}[V(Y_T)], \qquad y > 0, n \in \mathbb{N}. \tag{5.1}$$

It follows from Theorem A.1 below that under Assumptions 2.3, 5.1 and (2.4), (5.1) admits a unique minimiser for every $(y, n) \in (0, \infty) \times \mathbb{N}$.

The following result gives convergence of the marginal-utility-based prices in small markets to the one in the large market under the assumption that the dual minimisers are true martingales.

Lemma 5.3 *Suppose that Assumptions 2.3, 2.1, 5.1 and (2.4) hold and consider a sequence $(x^n)_{n \in \mathbb{N}}$ of strictly positive numbers converging to $x > 0$. For $\tilde{n}$ given in Assumption 2.3, the quantities $y^n := u^n_x(x^n, 0)$, $n \geq \tilde{n}$, and $y := u_x(x, 0)$ are then well defined and we have*

$$\lim_{n \to \infty} y^n = y > 0.$$

Moreover, if both the minimiser $\widehat{Y}(y)$ to (2.9) and the minimisers $\widehat{Y}^n(y^n)$, $n \geq \tilde{n}$, to (5.1) are martingales, we have

$$\lim_{n \to \infty} \rho^n(x) = \lim_{n \to \infty} \mathbb{E}\left[\frac{1}{y^n} \widehat{Y}^n_T(y^n, 0) f\right] = \mathbb{E}\left[\frac{1}{y} \widehat{Y}_T(y, 0) f\right] = \rho(x), \tag{5.2}$$

that is, the marginal-utility-based prices are singletons, have representations as in (5.2) and converge.

Proof By Mostovyi [28, Lemma 2.4], we have $u^n(\cdot, 0) \to u(\cdot, 0)$ (in the case of deterministic utility, this follows from De Donno et al. [7, Proposition 4.3]). By [28, Theorem 2.2], $u(\cdot, 0)$ is continuously differentiable, and by Theorem A.1 below, so are $u^n(\cdot, 0), n \in \mathbb{N}$. Therefore $y^n, n \geq \tilde{n}$, and y are well defined, and Rockafellar [33, Theorem 24.5] gives $y^n \to y$.

Next, similarly to Lemma 4.3, one can show that $\widehat{Y}^n_T(y^n) \to \widehat{Y}_T(y)$ in probability. As $\widehat{Y}^n(y^n), n \geq \tilde{n}$, and $\widehat{Y}(y)$ are martingales and $y^n \to y$, we deduce via Scheffé's lemma that $\widehat{Y}^n_T(y^n)f \to \widehat{Y}_T(y)f$ in $\mathbb{L}^1(\mathbb{P})$. Finally, using Theorem 4.2, where a similar argument can be used to obtain the representation in (5.2) of the marginal-utility-based prices in small markets, we deduce that (5.2) holds. □

The following example illustrates the assertions of Lemma 5.3. It also demonstrates that superreplication prices in small markets do not converge to that (those) in the large market in general. Nevertheless, the martingale property of the dual minimisers, as in Lemma 5.3, ensures the convergence of the marginal-utility-based prices.

Example 5.4 Consider a one period-setting where $\Omega = \{\omega_n : n \in \mathbb{N}_0\}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}_1$ is the power set of Ω. We suppose that $\mathbb{P}[\{\omega_n\}] > 0, n \geq 0$, and that the asset prices are given by

$$S^0 \equiv 1, \qquad S^n_0 = s^n, \qquad S^n_1(\omega_n) = 1, \qquad S^n_1(\omega_k) = 0, \quad k \neq n,$$

where s^n are strictly positive numbers such that $\sum_{n=1}^{\infty} s^n < 1$.

One can see that for every $n \in \mathbb{N}$, the market with traded securities $S^0, \dots, S^n$ is incomplete. An example of a non-replicable claim in every such small market is

$$f := 1 - 1_{\{\omega_0\}} = \sum_{k=1}^{\infty} S^k_1. \tag{5.3}$$

The superreplication price for f is $\pi^n = 1, n \in \mathbb{N}$. This can also be obtained via (5.9) below as

$$\pi^n = \sup_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[1 - 1_{\{\omega_0\}}] = 1 - \inf_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{Q}[1_{\{\omega_0\}}] = 1, \qquad n \in \mathbb{N}.$$

Here, the elements of $\mathcal{M}^n$ can be identified with sequences $(q_0, q_1, \dots)$ of strictly positive numbers adding up to 1 and such that $q_j = s^j, j = 1, \dots, n$. For a given $\mathbb{Q} \in \mathcal{M}^n$, q_k equals $\mathbb{Q}[\{\omega_k\}], k \in \{0, 1, \dots\}$.

On the other hand, the large market is complete as every $1_{\{\omega_n\}}, n \in \mathbb{N}$, can be replicated by one share of S^n with the initial cost s^n, and $1_{\{\omega_0\}} = S^0_1 - \sum_{k=1}^{\infty} S^k_1$ so that $1_{\{\omega_0\}}$ can be replicated with the initial cost $1 - \sum_{k=1}^{\infty} s^k$. In particular, f in (5.3) can be replicated with the initial cost $\pi = \sum_{k=1}^{\infty} s^k < 1$.

To recapitulate, we do not have convergence of superreplication prices for f as

$$\lim_{n\to\infty} \pi^n = 1 > \sum_{k=1}^{\infty} s^k = \pi.$$

Let us fix a *deterministic* utility function U satisfying Assumption 2.3. In the large market, the marginal-utility-based price is equal to $\pi = \sum_{k=1}^{\infty} s^k$ by Theorem 4.2, and since the unique martingale measure in this market has the density

$$Z_1 = \frac{1 - \sum_{j=1}^{\infty} s^j}{\mathbb{P}[\{\omega_0\}]} 1_{\{\omega_0\}} + \sum_{i=1}^{\infty} \frac{s^i}{\mathbb{P}[\{\omega_i\}]} 1_{\{\omega_i\}}, \tag{5.4}$$

the minimiser to (2.9) is $\widehat{Y}_1(y) = yZ_1$ for every $y > 0$.

Consider the market where only $S^0, \dots, S^n$ are traded and denote by h^i the proportion of wealth invested in S^i, $i = 0, \dots, n$. Then the corresponding wealth at time 1 is

$$X_1^n(x) = x\left(h^0 + \sum_{i=1}^{n} \frac{h^i}{s^i} 1_{\{\omega_i\}}\right).$$

Consider the auxiliary optimisation problem

$$\begin{aligned} &\max_{h^0,\dots,h^n} \quad \left(1 - \sum_{i=1}^{n} \mathbb{P}[\{\omega_i\}]\right) U(xh^0) + \sum_{i=1}^{n} \mathbb{P}[\{\omega_i\}] U\left(x\left(h^0 + \frac{h^i}{s^i}\right)\right) \\ &\text{subject to} \quad \sum_{i=0}^{n} h^i = 1. \end{aligned} \tag{5.5}$$

This formulation does not include any admissibility condition. However, the computations below show that the solution to this problem gives a positive wealth process and thus the optimiser to the utility maximisation problem without a random endowment. Introducing the Lagrangian $L(h, \lambda)$ for $(h, \lambda) \in \mathbb{R}^{n+2}$ by

$$L(h, \lambda) := \left(1 - \sum_{i=1}^{n} \mathbb{P}[\{\omega_i\}]\right) U(xh^0) + \sum_{i=1}^{n} \mathbb{P}[\{\omega_i\}] U\left(x\left(h^0 + \frac{h^i}{s^i}\right)\right) + \lambda\left(1 - \sum_{i=0}^{n} h^i\right),$$

we get from the optimality conditions that

$$\begin{aligned} 0 &= x\left(1 - \sum_{k=1}^{n} \mathbb{P}[\{\omega_k\}]\right) U'(xh^0) + x \sum_{j=1}^{n} \mathbb{P}[\{\omega_j\}] U'\left(x\left(h^0 + \frac{h^j}{s^j}\right)\right) - \lambda, \\ 0 &= \mathbb{P}[\{\omega_i\}] U'\left(x\left(h^0 + \frac{h^i}{s^i}\right)\right) \frac{x}{s^i} - \lambda, \qquad i = 1, \dots, n, \end{aligned}$$

which leads to

$$\begin{aligned} 0 &= x\left(1 - \sum_{k=1}^{n} \mathbb{P}[\{\omega_k\}]\right) U'(xh^0) - \lambda\left(1 - \sum_{j=1}^{n} s^j\right), \\ 0 &= \mathbb{P}[\{\omega_i\}] U'\left(x\left(h^0 + \frac{h^i}{s^i}\right)\right) \frac{x}{s^i} - \lambda, \qquad i = 1, \dots, n. \end{aligned}$$

Thus the optimal $h^0, \dots, h^n$ are given by

$$h^0 = \frac{1}{x}(U')^{-1}\left(\frac{\lambda}{x}\frac{1-\sum_{j=1}^n s^j}{1-\sum_{k=1}^n \mathbb{P}[\{\omega_k\}]}\right),$$
$$h^i = s^i \frac{1}{x}(U')^{-1}\left(\frac{\lambda}{x}\frac{s^i}{\mathbb{P}[\{\omega_i\}]}\right) - h^0 s^i,$$

where λ is the unique solution to

$$\frac{1-\sum_{j=1}^n s^j}{x}(U')^{-1}\left(\frac{\lambda}{x}\frac{1-\sum_{j=1}^n s^j}{1-\sum_{k=1}^n \mathbb{P}[\{\omega_k\}]}\right) + \sum_{i=1}^n \frac{s^i}{x}(U')^{-1}\left(\frac{\lambda}{x}\frac{s^i}{\mathbb{P}[\{\omega_i\}]}\right) = 1.$$

The existence and uniqueness of λ follows from the strict monotonicity of U' and the Inada conditions. Therefore the optimal wealth at time 1 is

$$\begin{aligned}\widehat{X}_1^n(x)(\omega) = (U')^{-1}&\left(\frac{\lambda}{x}\frac{1-\sum_{j=1}^n s^j}{1-\sum_{k=1}^n \mathbb{P}[\{\omega_k\}]}\right)\left(1-\sum_{\ell=1}^n 1_{\{\omega_\ell\}}\right)\\ &+\sum_{i=1}^n (U')^{-1}\left(\frac{\lambda}{x}\frac{s^i}{\mathbb{P}[\{\omega_i\}]}\right)1_{\{\omega_i\}}.\end{aligned} \tag{5.6}$$

From (5.6), one can see that $\widehat{X}^n(x)(\omega) > 0$ for all $\omega \in \Omega$. Therefore, considering (5.5) allowed us to find a candidate solution which satisfies the admissibility constraint(s) and thus is the optimiser for $u^n(x, 0)$, $x > 0$. Consequently, we obtain from (5.6) that the density of the martingale measure for $(S^0, \dots, S^n)$ given by

$$Z_1^n(\omega) = \frac{1-\sum_{j=1}^n s^j}{1-\sum_{k=1}^n \mathbb{P}[\{\omega_k\}]}\left(1-\sum_{\ell=1}^n 1_{\{\omega_\ell\}}\right) + \sum_{i=1}^n \frac{s^i}{\mathbb{P}[\{\omega_i\}]}1_{\{\omega_i\}} \tag{5.7}$$

is up to a multiplicative constant the (dual) minimiser to (5.1) for every deterministic utility in the market with traded securities $S^0, \dots, S^n$. Therefore, one can see that for every bounded contingent claim $\tilde{f}$, the set of marginal-utility-based prices in that market is

$$\Pi^n(x) = \{\mathbb{E}[Z_1^n \tilde{f}]\}, \qquad n \in \mathbb{N}, x > 0.$$

Next, from (5.7), we get

$$\begin{aligned}\lim_{n\to\infty} Z_1^n(\omega) &= \frac{1-\sum_{j=1}^\infty s^j}{1-\sum_{k=1}^\infty \mathbb{P}[\{\omega_k\}]}1_{\{\omega_0\}} + \sum_{i=1}^\infty \frac{s^i}{\mathbb{P}[\{\omega_i\}]}1_{\{\omega_i\}}\\ &= Z_1(\omega) \qquad \mathbb{P}\text{-a.s.},\end{aligned} \tag{5.8}$$

where Z_1 given by (5.4) is the density of the unique martingale measure in the large market. As $\mathbb{E}[Z_1^n] = 1$, $n \in \mathbb{N}$, and $\mathbb{E}[Z_1] = 1$, Scheffé's lemma implies that the convergence in (5.8) also takes place in $\mathbb{L}^1(\mathbb{P})$. Therefore, for every bounded contingent claim $\tilde{f}$, the corresponding sequence of marginal-utility-based prices in a sequence of small markets converges to the one in the large market. The argument above can be extended with some modifications to stochastic utilities satisfying Assumptions 2.3 and 5.1.

5.2 The case of no martingale assumption

The main result of this subsection is Theorem 5.7, stated without assuming that the marginal-utility-based prices in large or small markets are singletons, but under an additional Assumption 5.6. The key role in the proof is played by the auxiliary minimisation problems (5.29) below. In turn, their domains are given by the polars in $\mathbb{L}^0(\mathbb{P})$ to $\mathcal{C}(x, q)$ defined in (3.1) for $(x, q) \in \operatorname{int}\mathcal{K}$. Here a special role is played by the sets $\Theta(x, q)$ defined in (5.10) below. Lemma 5.5 below establishes properties of $\Theta(x, q)$, and Lemma 5.10 below shows that they generate the polars to $\mathcal{C}(x, q)$. Further, we need the convergence of the domains of the value functions from (2.3); this is established in Lemma 5.14 below. In turn, this will allow us to show the convergence of the value functions in (5.29) below.

We recall that in a small market, the superreplication price of a contingent claim $\tilde{f} \in \mathbb{L}^0_+$ is defined as

$$\pi_n(\tilde{f}) := \inf\{x \in \mathbb{R} : x + H \cdot S_T \geq \tilde{f}\ \mathbb{P}\text{-a.s. for some } H \in \mathcal{H}^n\}. \tag{5.9}$$

Delbaen and Schachermayer [11, Theorem 5.12] for example characterises the superreplication price as

$$\pi_n(\tilde{f}) = \sup_{\mathbb{Q} \in \mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[\tilde{f}], \qquad n \in \mathbb{N}.$$

One can see that $(\pi_n(\tilde{f}))_{n\in\mathbb{N}}$ is decreasing. By setting

$$\pi(\tilde{f}) := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[\tilde{f}],$$

we are trying to build an analogue of the arbitrage-free prices in small markets. From the definition of π_n and π, we immediately get

$$\lim_{n\to\infty} \pi_n(\tilde{f}) = \inf_{n\in\mathbb{N}} \pi_n(\tilde{f}) \geq \pi(\tilde{f}).$$

As pointed out in Example 5.4 above, the inequality can be strict. In this case, we might say that the superreplication prices in small markets do not converge to that in the large market. We might have situations when the domains of the optimisation problems do not converge in the set-theoretic sense, that is,

$$\mathcal{L} \neq \bigcap_{n\in\mathbb{N}} \mathcal{L}^n \qquad \text{and therefore} \qquad \mathcal{K} \neq \bigcup_{n\in\mathbb{N}} \mathcal{K}^n.$$

If this happens, the model in the large market is not a limit of the small models and thus is not as interesting.

Recall that $\mathcal{M}(\rho)$ and $\mathcal{P}$ are defined in (3.6) and (3.7), respectively. Analogously, we can specify $\mathcal{M}^n(\rho)$ and $\mathcal{P}^n$ for the market with traded $(S^0, \dots, S^n)$ and set

$$\Theta^n(x,q) := \bigcup_{\rho\in\mathcal{P}^n} \bigcup_{\mathbb{Q}\in\mathcal{M}^n(\rho)} \frac{1}{x+q\rho}\frac{d\mathbb{Q}}{d\mathbb{P}}, \qquad n \in \mathbb{N},$$

$$\Theta(x,q) := \bigcup_{\rho\in\mathcal{P}} \bigcup_{\mathbb{Q}\in\mathcal{M}(\rho)} \frac{1}{x+q\rho}\frac{d\mathbb{Q}}{d\mathbb{P}}. \tag{5.10}$$

We recall that for $(x,q) \in \operatorname{int}\mathcal{K}$, $\rho \in \mathcal{P}$ and $\mathbb{Q} \in \mathcal{M}(\rho)$, we have $x + q\rho > 0$. The latter inequality also holds for $(x,q) \in \operatorname{int}\mathcal{K}^n$, $\rho \in \mathcal{P}^n$ and $\mathbb{Q} \in \mathcal{M}^n(\rho)$.

Lemma 5.5 *If (2.4) and Assumption 2.1 hold, then for every $(x,q) \in \operatorname{int}\mathcal{K}$, the set $\Theta(x,q)$ is convex, closed under countable convex combinations, and we have*

$$\sup_{h\in\Theta(x,q)} \mathbb{E}[gh] = \sup_{h\in\tilde{\Theta}(x,q)} \mathbb{E}[gh], \qquad g \in \mathcal{C}(x,q). \tag{5.11}$$

Proof Since $(x,q) \in \operatorname{int}\mathcal{K}$, there exists $\delta > 0$ with $(x-\delta, q) \in \operatorname{int}\mathcal{K}$. Therefore we have for every $\rho \in \mathcal{P}$ that

$$x + q\rho \geq \delta > 0.$$

Next, we show that $\Theta(x,q)$ is closed under countable convex combinations. Take $\rho^i \in \mathcal{P}$, $\mathbb{Q}^i \in \mathcal{M}(\rho^i)$ and $\alpha^i = \frac{1}{x+q\rho^i}$, $i \in \mathbb{N}$. As $(x,q) \in \operatorname{int}\mathcal{K}$, one can see that the α^i are uniformly bounded from above. Then $h^i = \alpha^i \frac{d\mathbb{Q}^i}{d\mathbb{P}} \in \Theta(x,q)$. For convex weights $\lambda^i \in [0,1]$ with $\sum_{i=1}^\infty \lambda^i = 1$, we want to show that

$$h := \sum_{i=1}^\infty \lambda^i h^i \in \Theta(x,q).$$

Let us define

$$\alpha := \sum_{j=1}^\infty \lambda^j \alpha^j \qquad \text{and} \qquad \mu^i := \frac{\lambda^i\alpha^i}{\sum_{j=1}^\infty \lambda^j\alpha^j} = \frac{\lambda^i\alpha^i}{\alpha} \in [0,1], \quad i \in \mathbb{N},$$

a probability measure $\mathbb{Q}$ by its density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \sum_{i=1}^\infty \mu^i \frac{d\mathbb{Q}^i}{d\mathbb{P}}, \qquad \text{and} \qquad \rho := \sum_{i=1}^\infty \mu^i\rho^i. \tag{5.12}$$

Then $\mathbb{Q} \in \mathcal{M}$, as an application of the monotone convergence theorem shows that $\mathcal{M}$ is closed under countable convex combinations (see e.g. Mostovyi [28, proof of

Lemma 3.5]) and moreover, one can see that $\mathbb{Q} \in \mathcal{M}(\rho)$. Then we have

$$\begin{aligned} h &= \sum_{i=1}^{\infty} \lambda^i h^i = \sum_{i=1}^{\infty} \lambda^i \alpha^i \frac{d\mathbb{Q}^i}{d\mathbb{P}} \\ &= \left(\sum_{k=1}^{\infty} \lambda^k \alpha^k \right) \sum_{i=1}^{\infty} \frac{\lambda^i \alpha^i}{\sum_{j=1}^{\infty} \lambda^j \alpha^j} \frac{d\mathbb{Q}^i}{d\mathbb{P}} = \alpha \sum_{i=1}^{\infty} \mu^i \frac{d\mathbb{Q}^i}{d\mathbb{P}} = \alpha \frac{d\mathbb{Q}}{d\mathbb{P}}. \end{aligned} \tag{5.13}$$

Next, observe that $\alpha^i (x + q\rho^i) = 1$ yields

$$\begin{aligned} 1 = \sum_{i=1}^{\infty} \lambda^i &= \sum_{i=1}^{\infty} \lambda^i \alpha^i (x + q\rho^i) \\ &= x \sum_{i=1}^{\infty} \lambda^i \alpha^i + q \sum_{j=1}^{\infty} \lambda^j \alpha^j \rho^j \\ &= \left(x + q \sum_{j=1}^{\infty} \frac{\lambda^j \alpha^j}{\sum_{k=1}^{\infty} \lambda^k \alpha^k} \rho^j \right) \left(\sum_{i=1}^{\infty} \lambda^i \alpha^i \right) \\ &= \left(x + q \sum_{j=1}^{\infty} \mu^j \rho^j \right) \left(\sum_{i=1}^{\infty} \lambda^i \alpha^i \right). \end{aligned}$$

That is, we have

$$\left(\sum_{i=1}^{\infty} \lambda^i \alpha^i \right) \left(x + q \sum_{j=1}^{\infty} \mu^j \rho^j \right) = 1,$$

and thus, recalling the definitions of α and ρ, we conclude that

$$\alpha(x + q\rho) = 1. \tag{5.14}$$

To recapitulate, (5.12)–(5.14) imply that

$$h = \sum_{i=1}^{\infty} \lambda^i h^i = \alpha \frac{d\mathbb{Q}}{d\mathbb{P}}, \qquad \text{where } \mathbb{Q} \in \mathcal{M}(\rho) \text{ and } \alpha = \frac{1}{x + q\rho},$$

so that $h \in \Theta(x, q)$. Therefore $\Theta(x, q)$ is closed under countable convex combinations. In turn, (5.11) follows from the respective constructions of $\Theta(x, q)$ and $\tilde{\Theta}(x, q)$ and Fatou's lemma. □

We impose the following assumption, and see how it holds in the examples in Sect. 6. If the optimal Z for the large market are elements of $\mathcal{Z}$, one can typically have a natural candidate for the approximating sequence as in the examples in Sect. 6.

Assumption 5.6 For every $(x, q) \in \operatorname{int}\mathcal{K}$ and $z > 0$, there exists a uniformly integrable sequence $h^n \in \Theta^n(x, q)$, $n \in \mathbb{N}$, such that

$$\sup_{n\in\mathbb{N}} \inf_{h\in\Theta^n(x,q)} \mathbb{E}[V(zh)] = \lim_{n\to\infty} \mathbb{E}[V(zh^n)],$$

and for every $q \in \mathbb{R}^N$, there exists a uniformly integrable sequence $(\tilde{h}_n)_{n\in\mathbb{N}}$, where each $\tilde{h}^n$ is the terminal value of an element of $\mathcal{Z}^n$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \sup_{h\in\mathcal{Z}^n} \mathbb{E}[hqf] = \lim_{n\to\infty} \mathbb{E}[\tilde{h}^n qf].$$

We note that Assumption 5.6 holds if $\{Z_T : Z \in \mathcal{Z}^n\}$ is uniformly integrable for some $n \in \mathbb{N}$, which is much stronger. The primary result of this section is the following theorem. We recall that $\Pi(x)$ is given in (4.3), and $\Pi^n(x)$ is specified entirely similarly for the market with n risky assets.

Theorem 5.7 *Suppose that Assumptions 2.3, 2.1, 5.1, 5.6 and (2.4) hold. Then for every sequence $(x^n)_{n\in\mathbb{N}}$ of strictly positive numbers converging to $x > 0$, we have*

$$\Pi^n(x^n) \longrightarrow \Pi(x)$$

in the sense that for every $\varepsilon > 0$, there exists $n' \in \mathbb{N}$ such that

$$\Pi^n(x^n) \subseteq \Pi(x) + \varepsilon B, \qquad \text{for every } n \geq n', \tag{5.15}$$

where B is the Euclidean unit ball of $\mathbb{R}^N$.

Remark 5.8 The assertions of Theorem 5.7 hold without any assumption on whether any of the sets $\Pi^n(x)$ or $\Pi(x)$ are singletons or not.

Lemma 5.9 *If (2.4) and Assumption 2.1 hold, then*

$$\overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n} \subseteq \mathcal{K},$$

where the closure is taken in $\mathbb{R}^{N+1}$.

Proof Let $(x^n, q^n) \in \mathcal{K}^n$, $n \in \mathbb{N}$, be a sequence converging to $(x, q) \in \overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n}$. Then for every $n \in \mathbb{N}$, we have for some $x^n + H^n \cdot S \in \mathcal{X}^n(x^n, q^n)$ that

$$x^n + H^n \cdot S_T + q^n f \geq 0 \qquad \mathbb{P}\text{-a.s.}$$

For an appropriate $C \in \mathbb{R}^N$, let us rewrite the latter inequalities as

$$\frac{H^n}{|x^n| + q^n C} \cdot S_T \geq \frac{-x^n - q^n f}{|x^n| + q^n C} \geq -1 \qquad \mathbb{P}\text{-a.s.}, n \in \mathbb{N}.$$

Therefore, De Donno et al. [7, Lemma 3.3] applies, and we deduce from this lemma that there exists a 1-admissible generalised strategy $\tilde{H}$ such that

$$\bar{H} \cdot S_T \geq \frac{-x - qf}{|x| + qC}.$$

Then $H := (|x| + qC)\bar{H}$ is a generalised admissible strategy that satisfies

$$x + H \cdot S_T + qf \geq 0 \qquad \mathbb{P}\text{-a.s.}$$

In particular, we deduce that $x + H \cdot S_T \in \mathcal{X}(x, q) \neq \emptyset$ and thus $(x, q) \in \mathcal{K}$. □

For every $(x, q) \in \operatorname{int} \mathcal{K}$, we define the sets

$$\begin{aligned} \mathcal{B}^n(x, q) &:= \{(y, r) \in \mathcal{L}^n : \ xy + qr \leq 1\}, \qquad n \in \mathbb{N}, \\ \mathcal{B}(x, q) &:= \{(y, r) \in \mathcal{L} : \ xy + qr \leq 1\}, \\ \tilde{\mathcal{D}}^n(x, q) &:= \bigcup_{(y,r) \in \mathcal{B}^n(x,q)} \mathcal{D}^n(y, r), \qquad n \in \mathbb{N}, \\ \tilde{\mathcal{D}}(x, q) &:= \bigcup_{(y,r) \in \mathcal{B}(x,q)} \mathcal{D}(y, r). \end{aligned} \tag{5.16}$$

Using Propositions 3.2 and 3.3, one can show that the sets $\tilde{\mathcal{D}}^n(x, q)$ and $\tilde{\mathcal{D}}(x, q)$ are the polars to $\mathcal{C}^n(x, q)$ and $\mathcal{C}(x, q)$, respectively, and for every $\mathbb{Q} \in \mathcal{M}(\rho)$, $\rho \in \mathcal{P}$, there exists $\alpha = \alpha(\rho) = \frac{1}{x+q\rho}$ with $\alpha \frac{d\mathbb{Q}}{d\mathbb{P}} \in \tilde{\mathcal{D}}(x, q)$ and $(\alpha + \delta)\frac{d\mathbb{Q}}{d\mathbb{P}} \notin \tilde{\mathcal{D}}(x, q)$ for every $\delta > 0$.

Lemma 5.10 *If* (2.4) *and Assumption* 2.1 *hold, then for every* $(x, q) \in \operatorname{int} \mathcal{K}$, *we have*

$$\tilde{\Theta}(x, q) = \tilde{\mathcal{D}}(x, q). \tag{5.17}$$

Proof Fix $(x, q) \in \operatorname{int} \mathcal{K}$ and consider an arbitrary $g \in \mathcal{C}(x, q)$. Then we have

$$g \leq X_T + qf \qquad \mathbb{P}\text{-a.s.}$$

for some $X \in \mathcal{X}(x, q)$. Therefore for every $\rho \in \mathcal{P}$ and $\mathbb{Q} \in \mathcal{M}(\rho)$, the supermartingale property of X under $\mathbb{Q}$ gives

$$\mathbb{E}_{\mathbb{Q}}[g] \leq \mathbb{E}_{\mathbb{Q}}[X_T + qf] \leq x + q\rho.$$

This implies that for $h := \frac{1}{x+q\rho} \frac{d\mathbb{Q}}{d\mathbb{P}} \in \Theta(x, q)$, we have

$$\mathbb{E}[hg] \leq 1. \tag{5.18}$$

Next, for every $\tilde{h} \in \tilde{\Theta}(x, q)$, by the respective definitions of $\tilde{\Theta}(x, q)$ and $\Theta(x, q)$, there exists a sequence $(h^n)_{n \in \mathbb{N}} \subseteq \Theta(x, q)$ such that $\lim_{n\to\infty} h^n \geq \tilde{h}$ and (5.18) holds for every h^n. Fatou's lemma implies that $\mathbb{E}[\tilde{h}g] \le 1$, and thus $\tilde{h} \in \tilde{\mathcal{D}}(x,q)$ by Proposition 3.3. As $g \in \mathcal{C}(x,q)$ and $\tilde{h} \in \tilde{\Theta}(x,q)$ are arbitrary, this shows that

$$\tilde{\Theta}(x,q) \subseteq \tilde{\mathcal{D}}(x,q). \tag{5.19}$$

Conversely, consider $g \in \mathbb{L}^0_+$ with

$$\mathbb{E}[gZ] \le 1 \qquad \text{for every } Z \in \Theta(x,q). \tag{5.20}$$

We want to show that

$$g \le X_T + qf$$

for some $X \in \mathcal{X}(x,q)$. Now (5.20) implies that for every $\rho \in \mathcal{P}$ and $\mathbb{Q} \in \mathcal{M}(\rho)$, we have

$$\mathbb{E}_{\mathbb{Q}}\bigg[g\frac{1}{x+q\rho}\bigg] \le 1.$$

Then we get

$$\mathbb{E}_{\mathbb{Q}}\bigg[(g-qf)\frac{1}{x+q\rho}\bigg] \le 1 - \frac{q\rho}{x+q\rho} = \frac{x}{x+q\rho}.$$

Therefore we obtain

$$\mathbb{E}_{\mathbb{Q}}[g-qf] \le x. \tag{5.21}$$

Let $C \in \mathbb{R}^N$ be such that $g - q(f-C) \in \mathbb{L}^0_+$. Then from (5.21), we have

$$\mathbb{E}_{\mathbb{Q}}\big[\big(g-q(f-C)\big)\big] \le x + qC.$$

The latter inequality holds for every $\rho \in \mathcal{P}$ and $\mathbb{Q} \in \mathcal{M}(\rho)$, where the right-hand side does not depend on ρ. Consequently, from Lemma 3.8, we deduce that

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\big[\big(g-q(f-C)\big)\big] \le x + qC.$$

Now De Donno et al. [7, Theorem 3.1] asserts that there exists an $(x+qC)$-admissible generalised strategy H such that

$$g - q(f-C) \le x + qC + H \cdot S_T,$$

and thus $X := x + H \cdot S \in \mathcal{X}(x,q)$ and X superreplicates $g - qf$. In turn, this implies that $g \in \mathcal{C}(x,q)$ and so $\mathcal{C}(x,q) \supseteq (\tilde{\Theta}(x,q))^o$. Also, from the construction of $\tilde{\mathcal{D}}(x,q)$ in (5.16) and Proposition 3.3, it follows that $\tilde{\mathcal{D}}(x,q) = (\mathcal{C}(x,q))^o$. As a result, we obtain

$$\tilde{\mathcal{D}}(x,q) = \big(\mathcal{C}(x,q)\big)^o \subseteq \big(\tilde{\Theta}(x,q)\big)^{oo} = \tilde{\Theta}(x,q), \tag{5.22}$$

where the last equality uses the bipolar theorem of Brannath and Schachermayer [4] and we note that $\tilde{\Theta}(x,q)$ is convex, solid and closed in $\mathbb{L}^0$ by construction.

Finally, (5.19) and (5.22) imply the assertion (5.17) of the lemma. □

Lemma 5.11 *Under the conditions of Theorem 5.7, let $(x,q) \in \operatorname{int}\mathcal{K}$ be fixed and $\bar{h}^n \in \Theta^n(x,q)$, $n \in \mathbb{N}$, a uniformly integrable sequence with $\lim_{n\to\infty} \bar{h}^n = h$ $\mathbb{P}$-a.s. Then $h \in \tilde{\mathcal{D}}(x,q)$.*

Proof Consider an arbitrary $g \in \mathcal{C}(x,q)$. Then there exists $X \in \mathcal{X}(x,q)$ such that

$$g \leq X_T + qf.$$

For a sufficiently large constant $C > 0$ such that $X + C \in \mathcal{X}(x+C,0)$, consider an approximating sequence $\tilde{X}^n \in \mathcal{X}^n(x+C,0)$, $n \in \mathbb{N}$, and set $X^n := \tilde{X}^n - C$, $n \in \mathbb{N}$. By passing if necessary to subsequences, which we do not relabel, we have

$$\begin{aligned}\mathbb{E}[h(g+C)] &\leq \mathbb{E}[h(X_T + qf + C)]\\ &= \mathbb{E}\Big[\liminf_{n\to\infty} \bar{h}^n(X^n_T + qf + C)\Big]\\ &\leq \liminf_{n\to\infty} \mathbb{E}[\bar{h}^n(X^n_T + qf + C)]\\ &\leq 1 + C\liminf_{n\to\infty}\mathbb{E}[\bar{h}^n],\end{aligned}$$

where the second inequality uses Fatou's lemma and the last one the definition of the sets $\Theta^n(x,q)$ and the uniform integrability of $\bar{h}^n$, $n \in \mathbb{N}$. Now one can see that $\mathbb{E}[hg] \leq 1$. As $g \in \mathcal{C}(x,q)$ was arbitrary, we deduce that $h \in (\mathcal{C}(x,q))^o = \tilde{\mathcal{D}}(x,q)$. □

Remark 5.12 Assumption 5.6 implies that

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[qf] = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[qf] \qquad \text{for every } q \in \mathbb{R}^N. \tag{5.23}$$

To prove this, fix $q \in \mathbb{R}^N$. By Assumption 5.6, there exist $Z^n \in \mathcal{Z}^n$, $n \in \mathbb{N}$, such that $\mathbb{Q}^n$, $n \in \mathbb{N}$, with $\frac{d\mathbb{Q}^n}{d\mathbb{P}} = Z^n_T$ is a maximising sequence for (5.23), where $\{Z^n_T : n \in \mathbb{N}\}$ is uniformly integrable. By passing to convex combinations, we obtain a sequence, still denoted by $Z^n \in \mathcal{Z}^n$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} Z^n_T = h$ $\mathbb{P}$-a.s. for some nonnegative random variable h. Then we have

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[qf] = \lim_{n\to\infty} \mathbb{E}[Z^n_T qf] = \mathbb{E}[hqf]. \tag{5.24}$$

Lemma 5.11 implies that $h \in \tilde{\mathcal{D}}(1,0)$. Using Lemmas 5.5 and 5.10 (note that since qf is bounded, it is in $\mathcal{C}(x,0)$ for a sufficiently large x; see Proposition 3.3), we obtain

$$\sup_{Z\in\Theta(1,0)} \mathbb{E}[Z_T qf] = \sup_{h\in\mathcal{D}(1,0)} \mathbb{E}[hqf] \geq \mathbb{E}[hqf]. \tag{5.25}$$

Combining (5.24) and (5.25), we deduce (5.23).

Remark 5.13 When $N = 1$, (5.23) is equivalent to assuming that

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f],$$

$$\lim_{n\to\infty} \inf_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = \inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f], \tag{5.26}$$

which is the convergence of super- and subreplication prices to those in the large market, respectively. Without passing to the limit, that is, without considering $\inf_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f]$ and $\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[f]$ in the limiting market, (5.26) is closely related to Anthropelos et al. [1, Assumption 4.1].

For the proof of Theorem 5.7, we need the following result. With Assumption 5.6, we can strengthen Lemma 5.9 as follows.

Lemma 5.14 *Under the assumptions of Theorem 5.7, we have*

$$\overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n} = \mathcal{K}. \tag{5.27}$$

Proof Recall that by Lemma 3.5, $\mathcal{K}$ is closed. Fix $(x,q) \in \mathcal{K}$ and take a sequence $((x^k,q^k))_{k\in\mathbb{N}} \subseteq \operatorname{int}\mathcal{K}$ converging to (x,q). As $(x,0) \in \mathcal{K}^n$ for every $n \in \mathbb{N}$, it is enough to consider $q \neq 0$, and thus it is enough to consider $q^k \neq 0$, $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$. As $(x^k,q^k) \in \operatorname{int}\mathcal{K}$, there exists $\delta^k > 0$ such that

$$(x^k - \delta^k, q^k) \in \mathcal{K}.$$

From the definition of $\mathcal{K}$, it follows that there exists $X \in \mathcal{X}(x^k - \delta^k, q^k)$ such that

$$X + q^k f \geq 0 \qquad \mathbb{P}\text{-a.s.}$$

Using the supermartingale property of X under every $\mathbb{Q} \in \mathcal{M}$, we deduce that

$$x^k - \delta^k + \mathbb{E}_{\mathbb{Q}}[q^k f] \geq \mathbb{E}_{\mathbb{Q}}[X_T + q^k f] \geq 0, \qquad \mathbb{Q} \in \mathcal{M},$$

and thus

$$\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[-q^k f] \leq x^k - \delta^k.$$

Then using Assumption 5.6, we can rewrite this inequality as

$$x^k - \delta^k \geq \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}[-q^k f] = \lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[-q^k f],$$

and thus we obtain

$$\lim_{n\to\infty} \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[-q^k f] \leq x^k - \delta^k.$$

Fix $\varepsilon_n > 0$. Then the above inequality implies the existence of $n = n(\varepsilon_n) \in \mathbb{N}$ with

$$\sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[-q^k f] \leq x^k - \delta^k + \varepsilon_n. \tag{5.28}$$

By the superreplication results for small markets, see e.g. Delbaen and Schachermayer [11, Theorem 5.12], (5.28) implies that there exists an $(x^k-\delta^k+\varepsilon_n)$-admissible n-elementary strategy H^n such that

$$x^k - \delta^k + \varepsilon_n + H^n \cdot S_T + q^k f \geq 0 \qquad \mathbb{P}\text{-a.s.}$$

Therefore $X := x^k - \delta^k + \varepsilon_n + H^n \cdot S \in \mathcal{X}^n(x^k - \delta^k + \varepsilon_n, q^k)$, and in particular $(x^k - \delta^k + \varepsilon_n, q^k) \in \mathcal{K}^n$. We deduce that

$$(x^k - \delta^k + \varepsilon_n, q^k) \in \bigcup_{n\in\mathbb{N}} \mathcal{K}^n.$$

As ε_n was arbitrary, by picking for example $\varepsilon_n = \delta^k$, we deduce that

$$\big((x^k, q^k)\big)_{k\in\mathbb{N}} \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{K}^n,$$

and therefore, since $((x^k, q^k))_{k\in\mathbb{N}} \subseteq \operatorname{int}\mathcal{K}$ is convergent to (x, q), we deduce that $(x, q) \in \overline{\bigcup_{n\in\mathbb{N}} \mathcal{K}^n}$. Therefore (5.27) holds. □

Corollary 5.15 *Under the assumptions of Theorem 5.7, we have*

$$\mathcal{L} = \bigcap_{n\in\mathbb{N}} \mathcal{L}^n.$$

Corollary 5.16 *Under the assumptions of Theorem 5.7, the sets of closures of arbitrage-free prices in small markets converge to the closure of the set of arbitrage-free prices in the large market.*

Now for every $(x, q) \in \operatorname{int}\mathcal{K}$, let us define

$$\begin{aligned} \tilde{v}^n(z) &:= \inf_{h\in\Theta^n(x,q)} \mathbb{E}[V(zh)], \qquad z > 0, n \geq 1, \\ \tilde{v}(z) &:= \inf_{h\in\Theta(x,q)} \mathbb{E}[V(zh)], \qquad z > 0. \end{aligned} \tag{5.29}$$

Lemma 5.17 *Under the assumptions of Theorem 5.7, for every $(x, q) \in \operatorname{int}\mathcal{K}$, there exists $n_0 \in \mathbb{N}$ such that*

$$\begin{aligned} \tilde{v}^n(z) &= \inf_{h\in\tilde{\mathcal{D}}^n(x,q)} \mathbb{E}[V(zh)], \qquad z > 0, n \geq n_0, \\ \tilde{v}(z) &= \inf_{h\in\tilde{\mathcal{D}}(x,q)} \mathbb{E}[V(zh)], \qquad z > 0. \end{aligned}$$

Proof First, we observe that $(x, q) \in \operatorname{int}\mathcal{K}$ and Lemma 5.14 imply that there exists $n_0 \in \mathbb{N}$ such that $(x, q) \in \mathcal{K}^n$ for every $n \geq n_0$. Entirely similarly to the proofs of Lemmas 5.5 and 5.10, we can obtain similar assertions (convexity, closedness under countable convex combinations, (5.11) and (5.17)) for $\tilde{\Theta}^n(x, q)$, $n \geq n_0$, like for $\tilde{\Theta}(x, q)$. Now the assertion of the lemma follows from Theorem A.2 below. □

Proof of Theorem 5.7 Fix $(x, q) \in \operatorname{int}\mathcal{K}$. One can see (e.g. using Lemma 5.17) the monotonicity of $\tilde{v}^n$: as $\mathcal{M}^n$ is decreasing in n, $\tilde{v}^n$ is increasing in n for $n \geq n_0$, where n_0 is given by Lemma 5.17, and

$$\sup_{n\geq 1} \tilde{v}^n(z) = \lim_{n\to\infty} \tilde{v}^n(z) \leq \tilde{v}(z) < \infty, \qquad z > 0. \tag{5.30}$$

Fix $z > 0$. Assumption 5.6 implies the existence of a uniformly integrable sequence $h^n \in \Theta^n(x, q)$, $n \geq n_0$, such that

$$\liminf_{n\to\infty} \tilde{v}^n(z) = \lim_{n\to\infty} \mathbb{E}[V(zh^n)]. \tag{5.31}$$

By passing to convex combinations $\tilde{h}^n \in \operatorname{conv}(h^n, h^{n+1}, \dots)$, $n \in \mathbb{N}$, we can obtain a sequence such that $\tilde{h}^n \in \Theta^n(x, q)$, where the convexity of $\Theta^n(x, q)$ can be shown similarly to Lemma 5.5, and such that $(\tilde{h}^n)$ converges to some limit $\tilde{h}$ $\mathbb{P}$-a.s. By Lemma 5.11, we deduce that $\tilde{h} \in \tilde{\mathcal{D}}(x, q)$.

As $\tilde{h} \in \bigcap_{n\in\mathbb{N}} \mathcal{Y}^n(y^n) \subseteq \mathcal{Y}^1(\bar{y})$ for some $\bar{y} \in (0, \infty)$ and also $(\tilde{h}^n)_{n\geq\tilde{n}} \subseteq \mathcal{Y}^{\tilde{n}}(\bar{y})$, where $\tilde{n}$ is given by Assumption 5.1, we conclude via Mostovyi [26, Lemma 3.5] that the sequence $(V^-(z\tilde{h}^n))_{n\in\mathbb{N}}$ is uniformly integrable. Therefore, using the convexity of $V(\,\cdot\,, \omega)$, $\omega \in \Omega$, we obtain

$$\begin{aligned}\tilde{v}(z) \leq \mathbb{E}[V(z\tilde{h})] &\leq \liminf_{n\to\infty} \mathbb{E}[V(z\tilde{h}^n)] \\ &\leq \liminf_{n\to\infty} \mathbb{E}[V(zh^n)] = \liminf_{n\to\infty} \tilde{v}^n(z),\end{aligned} \tag{5.32}$$

where the last equality uses (5.31). Combining (5.30) and (5.32), we obtain

$$\tilde{v}(z) = \lim_{n\to\infty} \tilde{v}^n(z), \qquad z > 0. \tag{5.33}$$

Recall that by the construction of $\tilde{\mathcal{D}}(x, q)$ in (5.16) and Proposition 3.3, it follows that $\tilde{\mathcal{D}}(x, q)$ and $\mathcal{C}(x, q)$ satisfy the assumptions of Theorem A.1 below. Further, by Assumption 5.1 (see also Remark 5.2), the finiteness of $\tilde{v}(z)$, $z > 0$, and $u(sx, sq)$, $s > 0$, holds. Therefore Theorem A.1 yields

$$\begin{aligned}\tilde{u}(s) := u(sx, sq) &= \sup_{g\in\mathcal{C}(x,q)} \mathbb{E}[U(sg)] \\ &= \inf_{z>0}\Big(\inf_{h\in\tilde{\mathcal{D}}(x,q)} \mathbb{E}[V(zh)] + zs\Big) \\ &= \inf_{z>0}\big(\tilde{v}(z) + sz\big), \qquad s > 0,\end{aligned} \tag{5.34}$$

where the last equality uses Lemma 5.17. By construction, both $-\tilde{u}$ and v as well as $\liminf_{n\to\infty} \tilde{v}^n$ are convex and finite-valued. From (5.34) and (5.33), we get

$$\tilde{u}(s) = \inf_{z>0} \left(\tilde{v}(z) + sz\right) = \inf_{z>0} \left(\lim_{n\to\infty} \tilde{v}^n(z) + sz\right), \qquad s > 0. \tag{5.35}$$

A similar construction gives that for every $n \geq n_0$, we have

$$\tilde{u}^n(s) := u^n(sx, sq) = \inf_{z>0} \left(\tilde{v}^n(z) + sz\right), \qquad s > 0. \tag{5.36}$$

This shows that $\tilde{u}^n$, $n \geq n_0$, is a monotone sequence and

$$-\infty < \tilde{u}^n(s) \leq \tilde{u}(s), \qquad s > 0, n \geq n_0.$$

Therefore we have

$$\tilde{u}^\infty(s) := \lim_{n\to\infty} \tilde{u}^n(s) \leq \tilde{u}(s), \qquad s > 0. \tag{5.37}$$

Further, combining (5.37) with (5.35) and (5.36) and using the monotonicity of $\tilde{v}^n$ for $n \geq n_0$, we get

$$\inf_{z>0} \left(\sup_{k\geq n_0} \tilde{v}^k(z) + sz\right) = \tilde{u}(s) \geq \inf_{z>0} \left(\tilde{v}^n(z) + sz\right), \qquad n \geq n_0.$$

By the conjugacy of $\tilde{u}^n$ and $\tilde{v}^n$ and from the monotonicity of $\tilde{u}^n$ for $n \geq n_0$, we obtain

$$\tilde{v}^n(z) = \sup_{s>0} \left(\tilde{u}^n(s) - sz\right) \leq \sup_{s>0} \left(\tilde{u}^\infty(s) - sz\right), \qquad z > 0, n \geq n_0.$$

Therefore, using (5.33), we obtain

$$\tilde{v}(z) \leq \sup_{s>0} \left(\tilde{u}^\infty(s) - sz\right), \qquad z > 0. \tag{5.38}$$

One can see that $\tilde{u}^\infty$ is a concave function as a pointwise limit of concave functions, and further that $\tilde{u}^\infty$ is finite-valued. Let $\widehat{v}$ denote its convex conjugate; then (5.38) implies that

$$\widehat{v}(z) = \sup_{s>0} \left(\tilde{u}^\infty(s) - sz\right) \geq \tilde{v}(z), \qquad z > 0. \tag{5.39}$$

Therefore, the biconjugation characterisation and (5.39) imply that

$$\tilde{u}^\infty(s) = \inf_{z>0} \left(\widehat{v}(z) + zs\right) \geq \inf_{z>0} \left(\tilde{v}(z) + zs\right) = \tilde{u}(s), \qquad s > 0, \tag{5.40}$$

where the last equality uses (5.35). As a result, combining (5.37) and (5.40), we get

$$\tilde{u}^\infty(s) = \tilde{u}(s), \qquad s > 0.$$

In particular, recalling (5.34) and (5.36), we conclude that

$$u(x, q) = \lim_{n\to\infty} u^n(x, q).$$

As $(x,q) \in \operatorname{int}\mathcal{K}$ was arbitrary, we deduce that

$$u(x,q) = \lim_{n\to\infty} u^n(x,q), \qquad (x,q) \in \operatorname{int}\mathcal{K},$$

which implies (5.15) via Rockafellar [33, Theorem 24.5]. Note that division by y in (4.3) leads to no issues as $\partial u(x,q) \subseteq \operatorname{ri}\mathcal{L}$ by item (iii) of Theorem 3.1 and thus, as the subdifferential is closed, the set $\{y : (y,r) \in \partial u(x,q)\}$ is bounded away from 0. □

Remark 5.18 A close look at the proof of Theorem 5.7 shows that under its conditions, it shows the convergence of utility-based prices, and not only marginal-utility-based prices.

6 Pricing of asymptotically replicable claims

An asymptotically replicable claim is one that is replicable in the large market, but possibly not in any small market. We give below examples of such claims and markets admitting such claims. Intuitively, the sets of arbitrage-free prices for small markets should converge to singletons.

Contingent claims which are replicable in some small market are well studied in the literature. To develop a theory of arbitrage-free or marginal-utility-based pricing for such claims, one needs to analyse large markets as the arbitrage-free and marginal-utility-based prices in large markets will match those in the smallest small markets in which these contingent claims are replicable. Below, we focus on claims which are not replicable in the small markets, but are replicable in the large one. Examples below show such markets and claims. The definition of asymptotic replicability can be stated as follows.

Definition 6.1 Under (2.4), a componentwise bounded contingent claim f is *asymptotically replicable* if it is not replicable in any small market, but is replicable in the large market, that is, every component of f^i is replicable in the sense of Definition 3.6. We say that such a contingent claim f is *asymptotically replicable at* x if x is the initial value of the admissible generalised wealth processes appearing in Definition 3.6.

We denote by AFP^n, $n \in \mathbb{N}$, and AFP the sets of arbitrage-free prices for f in small and large markets, respectively. Formally, as in Siorpaes [34], we define

$$\begin{aligned} \mathrm{AFP}^n &:= \{p \in \mathbb{R}^N : q \in \mathbb{R}^N \text{ and } X \in \mathcal{X}^n(-pq, q) \text{ imply } X_T = -qf\}, \qquad n \in \mathbb{N}, \\ \mathrm{AFP} &:= \{p \in \mathbb{R}^N : q \in \mathbb{R}^N \text{ and } X \in \mathcal{X}(-pq, q) \text{ imply } X_T = -qf\}. \end{aligned} \tag{6.1}$$

The following result shows the consistency of various pricing methodologies for asymptotically replicable claims.

Lemma 6.2 *Under the assumptions of Theorem 5.7, suppose that f is asymptotically replicable at $\pi \in \mathbb{R}^N$. Then we have*

$$\mathcal{P} = \Pi(x) = \mathrm{AFP} = \{\pi\}, \qquad x > 0, \tag{6.2}$$

and for every $\varepsilon > 0$, there exists $n' \in \mathbb{N}$ such that for every $n \geq n'$, we have

$$\Pi^n \subseteq \mathcal{P}^n \subseteq \mathcal{P} + \varepsilon B, \qquad x > 0, \tag{6.3}$$

and

$$\mathrm{AFP}^n \subseteq \mathrm{AFP} + \varepsilon B, \tag{6.4}$$

where B is the unit ball of $\mathbb{R}^N$.

Proof If one can replicate f in the large market with an initial price $\pi \in \mathbb{R}^N$, then the associated utility maximisation problem degenerates to that without f, as follows. Using the admissibility of the generalised wealth process replicating f in the sense of Definition 3.6, one can show that

$$u(x, q) = u(x + q\pi, 0), \qquad (x, q) \in \mathbb{R}^{N+1},$$

where the boundedness of f and (2.4) ensure that no admissibility issues arise by passing from $u(x, q)$ to $u(x + q\pi, 0)$ and back. Then for $\rho = \pi$ and every $x > 0$, we have

$$u(x - q\rho, q) = u(x + q\pi - q\rho, 0) = u(x, 0), \qquad q \in \mathbb{R}^N,$$

and thus (4.2) holds, i.e., π is the marginal-utility-based price at x, for every $x > 0$, i.e., $\pi \in \Pi(x)$, $x > 0$.

If $\rho \in \mathbb{R}^N$ with $\rho^i \neq \pi^i$ for some $i \in \{1, \dots, N\}$, then for $q = \mathrm{sign}(\pi^i - \rho^i)e^i$, where e^i is the ith unit vector in $\mathbb{R}^N$, we have

$$u(x - q\rho, q) = u(x + q\pi - q\rho, 0) > u(x, 0)$$

as $u(\cdot\,, 0)$ is strictly increasing (see e.g. [28, Theorem 2.2]). Thus ρ is not a marginal-utility-based price, for every $x > 0$.

Further, by observing that $\mathcal{K}$ contains straight lines passing through the origin and using Rockafellar [33, Theorem 14.6], one can see that $\mathcal{L} = \{y(1, \pi) : \; y \geq 0\}$ and $\pi \in \mathcal{P}$. It follows from the definition of arbitrage-free prices that $\mathrm{AFP} = \{\pi\}$, and so we deduce (6.2). In turn, analogously to the proof of Theorem 3.1, we can use $\partial u^n \subseteq \mathrm{ri}\,\mathcal{L}^n$ to obtain $\Pi^n \subseteq \mathcal{P}^n$ and $\mathrm{AFP}^n = \mathcal{P}^n$. Now (6.3) and (6.4) follow from Corollary 5.15. □

6.1 Examples

The following (positive) Examples 6.3 and 6.5 illustrate the results of Sects. 4–6, in particular Theorem 4.2 and Lemmas 5.3 and 6.2. Note that the assumptions of Lemma 5.3 are particularly convenient for the characterisation of asymptotically replicable claims (and in particular asymptotically complete markets).

Example 6.3 Let $S_0^n = \exp(-\frac{1}{2^n}) - \frac{1}{2} \in (0,1)$, $n \in \mathbb{N}$, and let S_1^n be i.i.d. Bernoulli random variables so that $\mathbb{P}[S_1^n = 0] = \mathbb{P}[S_1^n = 1] = \frac{1}{2}$. Suppose that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_1 = \sigma(S_1^n, n \in \mathbb{N})$. Consider an asymptotically replicable $f = \sum_{k=1}^{\infty} \frac{1}{2^k} S_1^k$ and fix a deterministic utility function U satisfying Assumption 2.3 and such that Assumption 5.1 holds.

In each small market $n \in \mathbb{N}$, the superreplication price of f is given by

$$\sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = \sup_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}\bigg[\sum_{k=1}^{n} \frac{1}{2^k} S_1^k + \sum_{j=n+1}^{\infty} \frac{1}{2^j} S_1^j\bigg]$$
$$= \sum_{k=1}^{n} \frac{1}{2^k}\Big(\exp\Big(-\frac{1}{2^k}\Big) - \frac{1}{2}\Big) + \sum_{j=n+1}^{\infty} \frac{1}{2^j}.$$

By similar computations, the subreplication price is given by

$$\inf_{\mathbb{Q}\in\mathcal{M}^n} \mathbb{E}_{\mathbb{Q}}[f] = \sum_{k=1}^{n} \frac{1}{2^k}\Big(\exp\Big(-\frac{1}{2^k}\Big) - \frac{1}{2}\Big).$$

We see that the set of arbitrage-free prices in each small market is given by

$$\mathrm{AFP}^n = \bigg(\sum_{k=1}^{n} \frac{1}{2^k}\Big(\exp\Big(-\frac{1}{2^k}\Big) - \frac{1}{2}\Big), \sum_{k=1}^{n} \frac{1}{2^k}\Big(\exp\Big(-\frac{1}{2^k}\Big) - \frac{1}{2}\Big) + \sum_{j=n+1}^{\infty} \frac{1}{2^j}\bigg),$$

which converges to

$$\bigg\{\sum_{k=1}^{\infty} \frac{1}{2^k}\Big(\exp\Big(-\frac{1}{2^k}\Big) - \frac{1}{2}\Big)\bigg\} = \mathrm{AFP},$$

where the equality follows from the definition (6.1) of AFP.

In the large market, consider the unique element $\hat{\mathbb{Q}}$ of $\mathcal{M}$. Then $\hat{\mathbb{Q}}$ is the dual minimiser in the large market. By Theorem 4.2, whose assumptions are satisfied here, the unique marginal-utility-based price in the large market is given by

$$\mathbb{E}_{\widehat{\mathbb{Q}}}[f] = \sum_{k=1}^{\infty} \frac{1}{2^k}\Big(\exp\Big(-\frac{1}{2^k}\Big) - \frac{1}{2}\Big). \tag{6.5}$$

In every small market, we set

$$\zeta^k := 2S_0^k 1_{\{S_1^k=1\}} + 2(1 - S_0^k)1_{\{S_1^k=0\}}, \qquad k \in \{1,\dots,n\},$$
$$\tilde{\zeta}^j(\alpha^j) := 2\alpha^j 1_{\{S^j=1\}} + 2(1 - \alpha^j)1_{\{S^j=0\}}, \qquad \alpha^j \in (0,1),\ j \in \{n+1,\dots\},$$
$$\mathcal{A}^n := \big\{(\alpha^j)_{j\in\{n+1,\dots\}} :\ \alpha^j \in (0,1),\ j \in \{n+1,\dots\}\big\}.$$

Let V be the convex conjugate of U. In the setting of this example, the parametrisation of the dual domains in the small markets can be given in terms of the elements of $\mathcal{A}^n$, which allows representing the dual value functions as

$$v^n(y) = \inf_{(\alpha^j)_{j\in\{n+1,\dots\}}\in\mathcal{A}^n} \mathbb{E}\Big[V\Big(y\prod_{k=1}^{n}\zeta^k \prod_{j=n+1}^{\infty}\tilde{\zeta}^j(\alpha^j)\Big)\Big], \qquad y>0. \tag{6.6}$$

By conditioning on $\sigma(S_1^1,\dots,S_1^n)$, one can show that the density of the minimal martingale measure $\widehat{\mathbb{Q}}^n$ is the minimiser to (6.6). It is given by

$$\frac{d\widehat{\mathbb{Q}}^n}{d\mathbb{P}} = \prod_{k=1}^{n}\zeta^k \tag{6.7}$$

and corresponds to $\alpha^j = \frac{1}{2}$ for every $j\in\{n+1,\dots\}$. As in the large market case, the dual minimiser does not depend on $y>0$, and therefore the unique marginal-utility-based price does not depend on the initial wealth. We remark that Assumption 5.6 from Sect. 5 holds.

Finally, as $\mathbb{E}[\frac{d\widehat{\mathbb{Q}}^n}{d\mathbb{P}}]=1$, $n\in\mathbb{N}$, Theorem 4.2, whose proof also applies to small markets, implies that the unique marginal-utility-based price in the market with n stocks is given by

$$\begin{aligned}\rho^n = \mathbb{E}_{\widehat{\mathbb{Q}}^n}[f] &= \mathbb{E}_{\widehat{\mathbb{Q}}^n}\Big[\sum_{k=1}^{n}\frac{1}{2^k}S_1^k + \sum_{j=n+1}^{\infty}\frac{1}{2^j}S_1^j\Big]\\ &= \sum_{k=1}^{n}\frac{1}{2^k}\Big(\exp\Big(-\frac{1}{2^n}\Big)-\frac{1}{2}\Big) + \sum_{j=n+1}^{\infty}\frac{1}{2^j}\frac{1}{2}, \qquad n\in\mathbb{N}.\end{aligned}$$

Therefore we obtain

$$\lim_{n\to\infty}\rho^n = \sum_{k=1}^{\infty}\frac{1}{2^k}\Big(\exp\Big(-\frac{1}{2^k}\Big)-\frac{1}{2}\Big),$$

which is by (6.5) the unique marginal-utility-based price for f in the large market.

Remark 6.4 It is striking that in Examples 6.3 and 6.5, the marginal-utility-based prices do not depend on the utility function U. This is not a coincidence, but a detailed explanation is beyond the scope of the current paper. For related results, see Mostovyi et al. [30].

Example 6.5 Let us consider a model, which is not asymptotically complete, where $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ is a complete stochastic basis supporting a countable set of one-dimensional independent Brownian motions W^n, $n\in\mathbb{N}$, $\mathcal{F}_0$ is trivial and $\mathcal{F}=\mathcal{F}_T$ is generated by W^n, $n\in\mathbb{N}$, and some other finite-dimensional Brownian motion independent of $W^n, n\in\mathbb{N}$. Take the riskless asset $S^0\equiv 1$, whereas the dynamics of the risky assets is given by

$$dS_t^n = S_t^n(\mu^n dt + \sigma^n dW_t^n), \qquad n\in\mathbb{N},$$

where S_0^n is deterministic and strictly positive and the constants μ^n and $\sigma^n > 0$, $n \in \mathbb{N}$, are such that the market prices of risk $\lambda^n := \frac{\mu^n}{\sigma^n}$, $n \in \mathbb{N}$, satisfy

$$\sum_{n=1}^{\infty} (\lambda^n)^2 < \infty.$$

Consider the minimal martingale measure for the n-stock model; its density process is given by

$$Z^n = \prod_{k=1}^{n} \mathcal{E}(-\lambda^k \cdot W^k), \qquad n \in \mathbb{N}.$$

One can show that the family $\{Z_T^n : n \in \mathbb{N}\}$ is uniformly integrable and $(Z_T^n)_{n \in \mathbb{N}}$ converges $\mathbb{P}$-a.s. and in $\mathbb{L}^1(\mathbb{P})$ to a random variable Z_T which admits the representation

$$Z_T = \exp\Big(- \sum_{k=1}^{\infty} \lambda^k W_T^k - \frac{T}{2} \sum_{k=1}^{\infty} (\lambda^k)^2 \Big),$$

where $\sum_{k=1}^{\infty} \lambda^k W_T^k$ is also a limit of the sequence $(\sum_{k=1}^{n} \lambda^k W_T^k)_{n \in \mathbb{N}}$ of terminal values which is a uniformly integrable family.

One can see that (2.4) is satisfied. Let us introduce $\mathcal{H}_t^n := \mathcal{F}_t \vee \mathcal{F}_T^{W^1,\dots,W^n}$, $t \in [0,T]$, $n \in \mathbb{N}$. Then for this model and a deterministic utility U satisfying Assumption 2.3 and such that Assumption 5.1 holds and with conjugate V, we have for every $\tilde{Z}^n \in \mathcal{Z}^n$, $n \in \mathbb{N}$, that

$$\begin{aligned} \mathbb{E}[V(y\tilde{Z}_T^n)] &= \mathbb{E}\big[\mathbb{E}[V(y\tilde{Z}_T^n)|\mathcal{H}_T^n]\big] \\ &\geq \mathbb{E}\big[V(y\mathbb{E}[\tilde{Z}_T^n|\mathcal{H}_T^n])\big] \geq \mathbb{E}[V(yZ_T^n)], \qquad n \in \mathbb{N}, \end{aligned} \tag{6.8}$$

where the last inequality can be established similarly to computations in Kramkov and Sîrbu [25, Sect. 7]. Let

$$f = \sum_{k=1}^{\infty} h^k(S_T^k),$$

where the h^k are smooth functions such that $\sum_{k=1}^{\infty} \|h^k\|_\infty < \infty$. Then f is asymptotically replicable and

$$\lim_{n\to\infty} \mathbb{E}[Z_T^n f] = \mathbb{E}[Z_T f]. \tag{6.9}$$

Using (6.8) and (6.9), one can see that Assumption 5.6 holds.

Since additionally $\mathbb{E}[Z_T^n] = 1$, using the argument in Remark 6.4, one can show along the lines of Theorem 4.2 that the set of marginal-utility-based prices for f in the market with the first n risky assets is given by

$$\Pi^n(x) = \{\mathbb{E}[Z_T^n f]\}, \qquad n \in \mathbb{N}, x > 0.$$

In the large market, Lemma 6.2 and Theorem 4.2 imply that

$$\Pi(x) = \{\mathbb{E}[Z_T f]\}, \qquad x > 0,$$

is the set of marginal-utility-based prices in the large market. In view of (6.9), we have $\lim_{n\to\infty} \Pi^n(x_n) = \Pi(x)$ for every sequence $(x_n)_{n\in\mathbb{N}}$ of strictly positive numbers converging to $x > 0$, where the convergence is in the sense of Lemma 6.2, which in the present setting reduces to convergence of singletons.

Appendix

Below we state Mostovyi [26, Theorem 3.2 and Theorem 3.3] which are used above. Their proofs are contained in [26]. Let μ be a finite and positive measure on a measurable space $(\Omega, \mathcal{F})$. Denote by $\mathbb{L}^0 = \mathbb{L}^0(\Omega, \mathcal{F}, \mu)$ the vector space of (equivalence classes of) real-valued measurable functions on $(\Omega, \mathcal{F}, \mu)$ topologised by convergence in measure for μ. Let $\mathbb{L}^0_+$ denote its positive orthant, i.e.,

$$\mathbb{L}^0_+ = \{\xi \in \mathbb{L}^0(\Omega, \mathcal{F}, \mu) : \xi \geq 0\}.$$

For any ξ and η in $\mathbb{L}^0$, we write

$$\langle \xi, \eta \rangle := \int_\Omega \xi \eta d\mu.$$

If ξ and η are both nonnegative, the integral is well defined in $[0, \infty]$. Let $\mathcal{C}$, $\mathcal{D}$ be subsets of $\mathbb{L}^0_+$ that satisfy

(a) we have

$$\begin{aligned} \xi \in \mathcal{C} \quad &\Longleftrightarrow \quad \langle \xi, \eta \rangle \leq 1 \text{ for all } \eta \in \mathcal{D}, \\ \eta \in \mathcal{D} \quad &\Longleftrightarrow \quad \langle \xi, \eta \rangle \leq 1 \text{ for all } \xi \in \mathcal{C}; \end{aligned} \tag{A.1}$$

(b) $\mathcal{C}$ and $\mathcal{D}$ contain at least one strictly positive element, i.e.,

$$\text{there are } \xi^* \in \mathcal{C}, \eta^* \in \mathcal{D} \text{ such that } \min(\xi^*, \eta^*) > 0 \ \mu\text{-a.e.} \tag{A.2}$$

Consider a *stochastic utility function* $U\colon \Omega \times [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ which satisfies Assumption 2.3. Define the *conjugate function* V to U as in (2.6). Now we consider the optimisation problems

$$u(x) = \sup_{\xi \in \mathcal{C}} \int_\Omega U(x\xi) d\mu, \qquad x > 0, \tag{A.3}$$

$$v(y) = \inf_{\eta \in \mathcal{D}} \int_\Omega V(y\eta) d\mu, \qquad y > 0. \tag{A.4}$$

The following result is [26, Theorem 3.2].

Theorem A.1 *Assume that $\mathcal{C}$ and $\mathcal{D}$ satisfy conditions (A.1) and (A.2). Let Assumption 2.3 hold and suppose that*

$$v(y) < \infty \quad \text{for all } y > 0 \qquad \text{and} \qquad u(x) > -\infty \quad \text{for all } x > 0.$$

Then we have:

1) $u(x) < \infty$ *for all* $x > 0$, *and* $v(y) > -\infty$ *for all* $y > 0$. *The functions* u *and* v *satisfy the biconjugacy relations*, *i.e.*,

$$v(y) = \sup_{x>0} \big(u(x) - xy\big), \qquad y > 0,$$

$$u(x) = \inf_{y>0} \big(v(y) + xy\big), \qquad x > 0.$$

The functions u *and* $-v$ *are continuously differentiable on* $(0, \infty)$, *strictly increasing*, *strictly concave and satisfy the Inada conditions*

$$u'(0) := \lim_{x\downarrow 0} u'(x) = \infty, \qquad -v'(0) := \lim_{y\downarrow 0} -v'(y) = \infty,$$

$$u'(\infty) := \lim_{x\to\infty} u'(x) = 0, \qquad -v'(\infty) := \lim_{y\to\infty} -v'(y) = 0.$$

2) *For every* $x > 0$, *the solution* $\hat{\xi}(x)$ *to* (A.3) *exists and is unique. For every* $y > 0$, *the solution* $\hat{\eta}(y)$ *to* (A.4) *exists and is unique. If* $y = u'(x)$, *we have the dual relations*

$$\hat{\eta}(y) = U'\big(\hat{\xi}(x)\big) \qquad \mu\text{-a.e.}$$

and

$$\langle \hat{\xi}(x), \hat{\eta}(y) \rangle = xy.$$

Let $\tilde{\mathcal{D}}$ be a subset of $\mathcal{D}$ such that
(i) $\tilde{\mathcal{D}}$ is closed under countable convex combinations;
(ii) for every $\xi \in \mathcal{C}$, we have

$$\sup_{\eta\in\mathcal{D}} \langle \xi, \eta \rangle = \sup_{\eta\in\tilde{\mathcal{D}}} \langle \xi, \eta \rangle.$$

Likewise, define $\tilde{\mathcal{C}}$ to be a subset of $\mathcal{C}$ such that
(iii) $\tilde{\mathcal{C}}$ is closed under countable convex combinations;
(iv) for every $\eta \in \mathcal{D}$, we have

$$\sup_{\xi\in\mathcal{C}} \langle \xi, \eta \rangle = \sup_{\xi\in\tilde{\mathcal{C}}} \langle \xi, \eta \rangle.$$

The following result is [26, Theorem 3.3].

Theorem A.2 *Under the conditions of Theorem A.1, we have*

$$v(y) = \inf_{\eta \in \tilde{\mathcal{D}}} \int_\Omega V(y\eta)\, d\mu, \qquad y > 0,$$

$$u(x) = \sup_{\xi \in \tilde{\mathcal{C}}} \int_\Omega U(x\xi)\, d\mu, \qquad x > 0.$$

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Declarations

Competing Interests The authors declare no competing interests.

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