J. Math. Anal. Appl. 482 (2020) 123527

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a free boundary problem for finitely extensible bead-spring chain molecules in dilute polymers

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A R T I C L E   I N F O

Article history:
Received 15 April 2019
Available online 24 September 2019
Submitted by D. Wang

Keywords:
FENE model
Suspensions of extensible bead-spring chain molecules
Dilute polymers
Compressible Navier-Stokes equations
Fokker-Planck-type equation
Free boundary problems

A B S T R A C T

We investigate the global existence of weak solutions to a free boundary problem governing the evolution of finitely extensible bead-spring chains in dilute polymers. We construct weak solutions of the two-phase model by performing the asymptotic limit as the adiabatic exponent γ goes to ∞ for a macroscopic model which arises from the kinetic theory of dilute solutions of nonhomogeneous polymeric liquids. In this context the polymeric molecules are idealized as bead-spring chains with finitely extensible nonlinear elastic (FENE) type spring potentials. This class of models involves the unsteady, compressible, isentropic, isothermal Navier-Stokes system in a bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$ coupled with a Fokker-Planck-Smoluchowski-type diffusion equation (cf. Barrett and Süli [3], [4], [7]). The convergence of these solutions, up to a subsequence, to the free-boundary problem is established using weak convergence methods, compactness arguments which rely on the monotonicity properties of certain quantities in the spirit of [12].

1. Introduction

Systems modelling the interaction of fluids and polymeric molecules are of great scientific interest in many branches of applied physics, chemistry, biology and engineering. They are of use in many industrial and medical applications such as food processing and blood flows. Polymeric molecules are very complex objects, and their description and investigation present many challenges. One of the most interesting models is the FENE (Finite Extensible Nonlinear Elastic) dumbbell model. In this model, a polymer is idealized as an elastic dumbbell consisting of beads joined by a spring. We refer the reader to Bird, Amstrong and Hassager [8], [9], Doi and Edwards [11] for some physical introduction to the model, Öttinger [16] for a more mathematical treatment following the stochastic framework and Owens and Phillips [17] for the computational aspects of the problem.

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https://doi.org/10.1016/j.jmaa.2019.123527
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In order to gain some perspective of the complexity of the problem let us recall that one of the starting points in the investigation of polymeric flows is due to Kirkwood and Riseman, who treated the perturbation of the velocity field due to the polymer's presence by steady state hydrodynamics ignoring the dynamical motion of the polymer. Subsequently, Bird, Curtis, Armstrong and Hassager in [10] advanced significantly Kirkwood's early theory introducing a general kinetic theoretical framework for both diluted and concentrated polymeric systems. In that context, the macromolecules are modelled as freely jointed bead-rod or bead-spring chains.

The configurational distribution function, solution of an evolution (diffusion) equation of the Fokker-Planck Smoluchowski-type, is the foundation of polymer dynamics: it is central to the estimation of the components of the stress tensor. The behaviour of the viscoelastic flow in polymeric liquids is affected significantly by the complexity of inter- and intramolecular interactions. At the microscopic level, long chain entanglements are a consequence of chain connectivity and backbone uncrossability due to intermolecular repulsive exclusive volume forces. Macromolecules diffusion (and conformational relaxation) is slowed down due to hydrodynamic drag and Brownian forces.

The microscopic effect due to the interaction between the macroscopic compressible fluid and the polymeric bead-like molecules produces an extra stress term in the momentum equation. This effect is known as micro-macro interaction. Analogously, there is an extra drift term in the Fokker-Planck equation that depends on the spatial gradient of the velocity. This term represents a macro-micro effect. The coupling satisfies the fact that the free-energy dissipates, which is important not only from the physical point of view but also from mathematical considerations, since it allows us to obtain uniform bounds and hence prove global existence of weak solutions.

The resulting system offers a detailed description of the behaviour of the complex mixture of polymer molecules and compressible fluid, and as such, it presents numerous challenges, simultaneously at the level of their derivation, at the level of their numerical simulation, at the level of their physical properties (rheology) and that of their mathematical treatment (see references below).

This paper establishes the existence of global-in-time weak solutions to a free boundary problem governing the evolution of finitely extensible bead-spring chains in dilute polymers. The free boundary problem is defined with the aid of a threshold for the pressure beyond which one has the incompressible Navier-Stokes equations for the fluid and below which one has a compressible model for the gas. We construct weak solutions of the two-phase model by performing the asymptotic limit as the adiabatic exponent γ goes to ∞ for a macroscopic model which arises from the kinetic theory of dilute solutions of nonhomogeneous polymeric liquids. In this context the polymeric molecules are idealized as bead-spring chains with finitely extensible nonlinear elastic (FENE) type spring potentials. This class of models involves the unsteady, compressible, isentropic, isothermal Navier-Stokes system in a bounded domain Ω in $\mathbb{R}^d$, $d = 2$, or 3 coupled with a Fokker-Planck-Smoluchowski-type diffusion equation (cf. Barrett and Süli [3], [4], [7]). The convergence of these solutions, up to a subsequence, to the free-boundary problem is established using techniques in the spirit of Lions and Masmoudi [15].

For related work in the context of polymeric fluids we refer the reader to [12] where the stability and global existence of weak solutions to a free boundary problem governing the evolution of polymeric fluids is investigated. The starting point in the investigation of Donatelli and Trivisa in [12] is a macroscopic model governing the suspensions of rod-like molecules (known as Doi-Model) in compressible fluids. The model under consideration couples a Fokker-Planck-type equation on the sphere for the orientation distribution of the rods to the Navier-Stokes equations, which are now enhanced by additional stresses reflecting the orientation of the rods on the molecular level. The coupled problem is 5-dimensional (three-dimensions in physical space and two degrees of freedom on the sphere) and it describes the interaction between the orientation of rod-like polymer molecules on the microscopic scale and the macroscopic properties of the fluid in which these molecules are contained. The macroscopic flow leads to a change of the orientation and, in the case of flexible particles, to a change in shape of the suspended microstructure. This process, in turn yields the production of a fluid stress. The free boundary problem is defined by a threshold for the pressure beyond which one has the incompressible Navier-Stokes equations for the fluid and below which one has a compressible model for the gas. Regarding the literature on polymeric fluids for compressible flows we refer the reader to the articles by Bae and Trivisa [1,2], Barrett and Süli [5,6], Donatelli and Trivisa [12] and the reference therein.

1.1. Notations

Before formulating the governing equation of the nonlinear system governing our mixture, we fix here some notations we are going to use in the paper.

1.1.1. Notations of macroscopic variables, tensors, forces and coefficients

- ⋆ ρ denotes the density of the fluid.
- ⋆ $\boldsymbol{u}$ represents the velocity field.
- ⋆ ψ denotes the probability distribution function: $\psi = \psi(\boldsymbol{q})$ with $\boldsymbol{q}$ a random conformation vector of $\boldsymbol{q} = (\boldsymbol{q_1}^T, \dots, \boldsymbol{q_K}^T)^T \in \mathbb{R}^{K_d}$ of the chain, with $\boldsymbol{q_i}$ representing the d-component conformation vector of the i-th spring.
- ⋆ $(\rho, \boldsymbol{u}, \psi)$ denote the macroscopic variables which characterise the state of the polymeric fluid.
- ⋆ $M(\boldsymbol{q})$ denotes the total Maxwellian.
- ⋆ $\widehat{\psi} = \psi/M$.
- ⋆ $p = p(\rho)$ denotes the pressure.
- ⋆ $\mathbb{S} = \mathbb{S}[\rho, \boldsymbol{u}]$ denotes the viscous stress tensor.
- ⋆ f denotes a non-dimensional body force.
- ⋆ $\boldsymbol{\tau}$ denotes the elastic extra stress tensor: $\boldsymbol{\tau} = \boldsymbol{\tau}(\psi)$.
- ⋆ $\boldsymbol{\sigma}(\boldsymbol{v}) = \nabla_x \boldsymbol{v}$.
- ⋆ $\zeta(\rho)$ denotes a drag coefficient $\zeta(\rho) \in \mathbb{R},\ \zeta(\rho) > 0$.
- ⋆ D denotes the domain of admissible conformation vectors, $D \subset \mathbb{R}^K$,

$$D = D_1 \times \cdots \times D_K,$$

  D_i bounded open d-dimensional balls centred at the origin.
- ⋆ $\mathcal{O}_i := \big[0, \frac{b_i}{2}\big)$ denotes the image of D_i under $\boldsymbol{q_i} \in D_i \to \frac{1}{2}|\boldsymbol{q}_i|^2$.
- ⋆ U_i denotes the spring potential, $U_i \in C^1(\mathcal{O}_i; \mathbb{R}_{\geq 0}), i = 1, \dots, K$.
- ⋆ $\boldsymbol{A} = (A_{i,j})_{i,j=1}^K$ is the symmetric positive definite *Rouse matrix* or connectivity matrix.
- ⋆ $\eta = \eta(x,t)$ denotes the polymeric number density expressed as

$$\eta(x,t) = \int\limits_D \psi(x,q,t)dq,\ (x,t) \in \Omega \times (0,T]$$

- ⋆ $\mathcal{F}(s) = s(\log s - 1) + 1, \qquad P(s) = \dfrac{s^\gamma}{\gamma - 1}.$

1.1.2. Notations of function spaces

- ⋆ $L^p(0,T;X)$ denotes the Banach set of Bochner measurable functions f from $(0,T)$ to X endowed with either the norm $\left(\int_0^T \|g(\cdot,t)\|_X^p dt\right)^{\frac{1}{p}}$ for $1 \leq p < \infty$ or $\sup\limits_{t>\infty} \|g(\cdot,t)\|_X$ for $p = \infty$. In particular, $f \in L^r(0,T;XY)$ denotes $\left(\int_0^T \big\|\big(\|f(t)\|_{Y_\tau}\big)\big\|_X^p dt\right)^{\frac{1}{p}}$ or $\sup\limits_{t>\infty} \big\|\big(\|f(t)\|_{Y_\tau}\big)\big\|_X$ for $p = \infty$. The notation $L_t^p L_x^q$ will abbreviate the space $L^p(0,T;L^q(\Omega))$.

- ⋆ The space $L^r_M(\Omega \times D)$ denotes the space of measurable functions f with the norm $\|f\|_{L^r_M(\Omega\times D)} = \left(\iint_{\Omega\times D} M|f|d\boldsymbol{q}dx\right)^{1/r}$. For any $r \in [0,\infty)$ we define $Z_r = \{f \in L^r_M(\Omega \times D) \mid f \geq 0 \text{ a.e. on } \Omega \times D\}$.
- ⋆ $\mathcal{M}((0,T)\times\Omega)$ is the space of bounded measures on $(0,T)\times\Omega$.
- ⋆ $C(T)$ is a function only depending on initial data and T, $C_w([0,T];X)$, is the space of continuous function from $(0,T)$ to X endowed with the weak topology.
- ⋆ $\rightharpoonup$ and $\rightarrow$ denote weak limit and strong limit, respectively.

1.2. Modelling

The main physical assumptions on our model are outlined below:

- ⋆ A macro-molecule is idealized as an "elastic dumbbell" consisting of two "beads" joined by a spring. The "bead-spring chain model" (considered in the present article) consists of $K+1$ beads coupled with K elastic springs representing a polymer chain.
- ⋆ The polymer molecules are described by their density at each time t, position x and probability distribution ψ. This is a kinetic description of the polymer molecules.
- ⋆ The right-hand side of the Navier–Stokes momentum equation includes an elastic extra-stress tensor $\boldsymbol{\tau}$ (produced due to the interaction of the compressible fluid and the polymeric molecules) which is the sum of the classical Kramers expression and a quadratic interaction term. The elastic extra-stress tensor stems from the random movement of the polymer chains and is defined through the associated probability density function that satisfies a Fokker–Planck-type parabolic equation, a crucial feature of which is the presence of a centre-of-mass diffusion term.
- ⋆ The non-Newtonian elastic extra stress tensor $\boldsymbol{\tau}$ (cf. (4) below), depends on the probability density function ψ, which, in addition to time t and space x, also depends on the conformation vector $(q_1^T,...q_K^T)^T \in \mathbb{R}^{3K}$, with q_i representing the 3-component conformation/orientation vector of the i-th spring in the chain.
- ⋆ The Kolmogorov equation satisfied by ψ is a second-order parabolic equation, the Fokker–Planck equation, whose transport coefficients depend on the velocity field $\boldsymbol{u}$, and the hydrodynamic drag coefficient appearing in the Fokker–Planck equation is, generally, a nonlinear function of the density ρ.

1.3. Governing equations

Our starting point is the governing equation of the general non-homogeneous bead-spring chain models with centre of mass diffusion. This class of models is governed by a system of nonlinear partial differential equations that arise from the kinetic theory of dilute polymeric solutions. The bead-spring chain molecules are dispersed in a compressible isentropic, isothermal Newtonian fluid confined to a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d, d = 2$ or 3, with boundary $\partial\Omega$. The governing equations of the system **(P)** read:

$$\partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0 \tag{1}$$

$$\partial_t(\rho\boldsymbol{u}) + \operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla_x p_F = \operatorname{div}\mathbb{S}[\boldsymbol{u},\rho] + \rho f + \operatorname{div}\boldsymbol{\tau} \tag{2}$$

$$\partial_t\psi + \operatorname{div}(\psi\boldsymbol{u}) + \sum_{i=1}^K \nabla_{\boldsymbol{q_i}}\cdot(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{q_i}(\psi)) \tag{3}$$

$$= \epsilon\Delta_x\left(\frac{\psi}{\zeta(\rho)}\right) + \frac{1}{4\lambda}\sum_{i=1}^K\sum_{j=1}^K A_{i,j}\nabla_{\boldsymbol{q_i}}\cdot\left(M\nabla_{\boldsymbol{q_i}}\left(\frac{\psi}{\zeta(\rho)M}\right)\right)$$

The state of the mixture is characterised by the macroscopic variables: the density ρ, the velocity vector field $\boldsymbol{u}$, and the probability distribution function $\boldsymbol{\psi}$. The physical properties of the polymeric fluid are reflected through the constitutive relations. These relations which are stated below express how the fluid pressure p_F, the viscous stress tensor $\mathbb{S} = \mathbb{S}[\rho, \boldsymbol{u}]$, the elastic extra stress tensor $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{\psi})$ depend on the macroscopic variables and in addition specify the physical laws that characterise the behaviour of the probability distribution function $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{q})$ in terms of the conformation vector $\boldsymbol{q}$ as well as other potentials present in the system. The constant parameter ε denotes the centre-of-mass diffusion coefficient, which is strictly positive. The positive parameter λ is called the *Deborah number*; it characterises the elastic relaxation property of the fluid.

1.4. Constitutive relations

⋆ The viscous stress tensor $\mathbb{S} = \mathbb{S}[\rho, \boldsymbol{u}]$ follows Newton's law for viscosity

$$\mathbb{S}[\rho, \boldsymbol{u}] = \mu^S(\rho) \left[D(\boldsymbol{u}) - \frac{1}{d} \operatorname{div} \boldsymbol{u} \mathbb{I} \right] + \mu^B(\rho) \operatorname{div} \boldsymbol{u} \mathbb{I}, \tag{4}$$

where $\mu^S(\rho) > 0, \mu^B(\rho) \geq 0$ denote the shear and bulk viscosities, $\mathbb{I}$ the $d \times d$ identity tensor and $D(\boldsymbol{u})$ is the rate of strain tensor

$$D(\boldsymbol{u}) := \frac{1}{2} \left(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T \right),$$

with $(\nabla_x \boldsymbol{u})(x,t) \in \mathbb{R}^{d \times d}$ and $(\nabla_x \boldsymbol{u})_{ij} = \dfrac{\partial \boldsymbol{u}_i}{\partial x_j}$.

⋆ The elastic extra stress tensor $\boldsymbol{\tau}$ is defined by

$$\boldsymbol{\tau}(\psi)(x,t) := \boldsymbol{\tau_1}(\psi)(x,t) - \xi \left(\int_D \psi d\boldsymbol{q} \right)^2 \mathbb{I} = \boldsymbol{\tau_1}(\psi)(x,t) - \xi \eta^2(x,t) \mathbb{I}, \tag{5}$$

with $\xi > 0$ and

$$\boldsymbol{\tau_1}(\psi) := k \left[\left(\sum_{i=1}^{K} \boldsymbol{C_i}(\psi) \right) - (K+1) \int_D \psi(\boldsymbol{q}) d\boldsymbol{q} \, \mathbb{I} \right], \tag{6}$$

where $k > 0$ and

$$\boldsymbol{C}_i(\psi)(x,t) := \int_d \psi(x, \boldsymbol{q}, t) U_i' \left(\frac{1}{2} |\boldsymbol{q}_i|^2 \right) \boldsymbol{q_i} \boldsymbol{q_i}^T d\boldsymbol{q}.$$

1.5. Additional hypothesis on the potentials

⋆ $\boldsymbol{F}_i$ denotes the elastic spring-force $\boldsymbol{F}_i : D_i \subset \mathbb{R}^d \to \mathbb{R}^d$ of the i-th spring in the chain defined by

$$\boldsymbol{F}_i(\boldsymbol{q_i}) := U_i' \left(\frac{1}{2} |\boldsymbol{q_i}|^2 \right) \boldsymbol{q_i}, \; i = 1, \ldots, K. \tag{7}$$

Using this notation, $\boldsymbol{C}_i(\cdot)$ can be expressed as

$$\boldsymbol{C}_i(\psi)(x,t) := \int_d \psi(x, \boldsymbol{q}, t) \boldsymbol{F}_i(\boldsymbol{q_i}) \boldsymbol{q_i}^T d\boldsymbol{q}.$$

⋆ M_i represents the partial Maxwellian associated with the spring potential U_i defined by

$$M_i(\boldsymbol{q_i}) := \frac{1}{\mathcal{Z}_i} e^{-U_i(\frac{1}{2}|\boldsymbol{q_i}|^2)}, \ \mathcal{Z}_i := \int\limits_{D_i} e^{-U_i(\frac{1}{2}|\boldsymbol{q_i}|^2)} d\boldsymbol{q_i}. \tag{8}$$

The (total) Maxwellian in the model is then

$$M(\boldsymbol{q}) := \prod_{i=1}^{K} M_i(\boldsymbol{q_i}) \ \forall \boldsymbol{q} \in D. \tag{9}$$

⋆ Observe that,

$$\begin{aligned} M(\boldsymbol{q})\nabla_{\boldsymbol{q_i}}[M(\boldsymbol{q})]^{-1} &= -[M(\boldsymbol{q})]^{-1}\nabla_{\boldsymbol{q_i}}[M(\boldsymbol{q})] \\ &= \nabla_{\boldsymbol{q_i}}\left(U_i\left(\frac{1}{2}|\boldsymbol{q_i}|^2\right)\right) = U_i'\left(\frac{1}{2}|\boldsymbol{q_i}|^2\right)\boldsymbol{q_i}, \end{aligned} \tag{10}$$

and by definition

$$\int\limits_{D} M(\boldsymbol{q})d\boldsymbol{q} = 1.$$

⋆ We assume that $D_i = B(0, \sqrt{b_i})$ with $b_i > 0$ is the ball centred at the origin of radius $\sqrt{b_i}$ and that there exist constants $c_{ij} > 0\, j = 1, \ldots, 4$, and $\theta_i > 1$ such that the spring potential $U_i \in C^1[0, \frac{b_i}{2})$ and the associated partial Maxwellian M_i satisfy $\forall \boldsymbol{q_i} \in D_i$

$$c_{i1}[\mathrm{dist}(\boldsymbol{q_i}, \partial D_i)]^{\theta_i} \le M_i(\boldsymbol{q_i}) \le c_{i2}[\mathrm{dist}(\boldsymbol{q_i}, \partial D_i)]^{\theta_i} \tag{11}$$

$$c_{i3} \le \mathrm{dist}(\boldsymbol{q_i}, \partial D_i) U_i'\left(\frac{1}{2}|\boldsymbol{q_i}|^2\right) \le c_{i4}. \tag{12}$$

It follows from (12) that

$$\int\limits_{D_i} \left[1 + \left[U_i\left(\frac{1}{2}|\boldsymbol{q_i}|^2\right)\right]^2\right] M_i(\boldsymbol{q_i})d\boldsymbol{q_i} < \infty. \tag{13}$$

From now on, for simplicity, we will assume that the viscosity coefficients μ^S, μ^B will not depend on the density ρ and we set the drag coefficient $\zeta = 1$.

1.5.1. Further examples

In the classical FENE dumbbell model, $K = 1$ and the spring force is given by

$$\boldsymbol{F}(\boldsymbol{q}) = \left(1 - \frac{|\boldsymbol{q}|^2}{b}\right)^{-1} \boldsymbol{q}, \boldsymbol{q} \in D = B(0, b^{\frac{1}{2}}),$$

corresponding to

$$U(s) = -\frac{b}{2}\log\left(1 - \frac{2s}{b}\right) \in O = [0, b/2), b > 2.$$

More generally, in the FENE bead-spring chain model, one considers $K + 1$ beads linearly coupled with K springs, each with a FENE spring potential. Direct calculations show that the partial Maxwellian M_i and the elastic potentials $U_i, i = 1, ..., K$, of the FENE bead spring chain satisfy the conditions (11) and (12) with $\theta_i := \frac{b_i}{2}$, provided that $b_i > 2, i = 1, ..., K$. Thus, (13) also holds when $b_i > 2, i = 1, ..., K$.

1.6. The free boundary problem

In this article we are concerned with the free-boundary problem for the system (1)-(10). In particular we consider a compressible model which includes the incompressible case only beyond a certain threshold for the pressure. Indeed the model the free boundary problem ($\boldsymbol{P_F}$) is defined by the following equations in $(0,T)\times\Omega$:

$$\partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0 \tag{14}$$

$$0 \le \rho \le 1$$

$$\partial_t(\rho\boldsymbol{u}) + \operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla_x[p_F + \xi\eta^2] = \operatorname{div}\mathbb{S}[\boldsymbol{u},\rho] + \rho f + \operatorname{div}\boldsymbol{\tau_1} \tag{15}$$

$$\partial_t\psi + \operatorname{div}(\psi\boldsymbol{u}) + \sum_{i=1}^{K}\nabla_{\boldsymbol{q_i}}\cdot(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{q_i}(\psi)) \tag{16}$$

$$= \epsilon\Delta_x\left(\frac{\psi}{\zeta(\rho)}\right) + \frac{1}{4\lambda}\sum_{i=1}^{K}\sum_{j=1}^{K}A_{i,j}\nabla_{\boldsymbol{q_i}}\cdot\left(M\nabla_{\boldsymbol{q_i}}\left(\frac{\psi}{\zeta M}\right)\right)$$

$$\frac{\partial\eta}{\partial t} + \nabla_x\cdot(\boldsymbol{u}\eta) = \varepsilon\Delta_x\left(\frac{\eta}{\zeta(\rho)}\right) \tag{17}$$

and the free boundary conditions

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{a.e. on } \{\rho = 1\} \tag{18}$$

$$p_F \ge 0 \quad \text{a.e. in } \{\rho = 1\} \tag{19}$$

$$p_F = 0 \quad \text{a.e. in } \{\rho < 1\} \tag{20}$$

The unknowns for our problem are the density ρ, the velocity vector field $\boldsymbol{u}$, the pressure p_F, which is Lagrange multiplier associated with the incompressibility constraint (18) $\operatorname{div}\boldsymbol{u} = 0$ a.e. in $\{\rho = 1\}$, the probability distribution ψ and the polymeric number density η. It is important to observe that the pressure p_F appears only in what we call the congested regions $\{\rho = 1\}$ and that the conditions (19), (20), can be formulated as one constraint

$$\rho p_F = p_F \ge 0. \tag{21}$$

1.6.1. Boundary conditions

Let $\partial\bar{D}_i := D_1\times\cdots\times D_{i-1}\times\partial D_i\times D_{i+1}\times\cdots\times D_K$. We consider the problem ($\mathbf{P_F}$) on a bounded domain with the following boundary conditions.

$$\boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega. \tag{22}$$

$$\left[\frac{1}{4\lambda}\sum_{j=1}^{K}A_{ij}M\nabla_{\boldsymbol{q_j}}\left(\frac{\psi}{\zeta(\rho)M}\right) - \sigma(\boldsymbol{u})\boldsymbol{q_i}\psi\right]\cdot\frac{\boldsymbol{q_i}}{|\boldsymbol{q_i}|} = 0, \tag{23}$$

$$\text{on } \Omega\times\partial\bar{D}_i\times(0,T], \text{ for } i = 1,\dots,K,$$

$$\varepsilon\nabla_x\left(\frac{\psi}{\zeta(\rho)}\right)\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\times D\times(0,T], \tag{24}$$

noting that $\boldsymbol{q_j}$ is normal to ∂D_i, as D_i is a bounded ball centred at the origin, and $\boldsymbol{n}$ is normal to $\partial\Omega$.

1.6.2. Initial data

The system ($\mathbf{P_F}$) must be complemented with initial conditions, namely

$$\rho(x,0) = \rho_0(x),\ \rho\boldsymbol{u}(x,0) = (\rho_0\boldsymbol{u}_0)(x),\ x \in \Omega, \tag{25}$$

$$\psi(\cdot,\cdot,0) = \psi_0(\cdot,\cdot) \geq 0,\ \text{on } \Omega \times D. \tag{26}$$

$$\eta(x,0) = \int_D \psi_0(x,\boldsymbol{q})d\boldsymbol{q},\ \text{for } x \in \Omega. \tag{27}$$

1.7. Outline and overall strategy of the proof

The outline of this article and overall strategy of the proof are as follows: Section 1 presents the main motivation for the upcoming investigation, the modelling aspects of the problem: the physical setting, constitutive relations, the free-boundary problem, the statement of the problem and. Section 2 introduces the main result, namely the global existence of the weak solutions to the free-boundary problem. This is achieved by rigorously showing that these solutions can be obtained as the limit of weak solutions to the Doi model for compressible fluids as the adiabatic exponent $\gamma_n \to \infty$. The approximating scheme and an outline of the proof of the global existence of approximate solutions are presented in Section 3. The proof follows the line of argument introduced by Barrett and Süli [7] which is based on the use of the rather special quantity

$$\frac{\psi}{M}, \quad M \text{ the Maxwellian,}$$

which if bounded appropriately results to a new formulation of the equation verified by the probability distribution function and an additional partial differential equation. Section 4 presents the proof of the main theorem. The global existence of weak solutions to the free boundary problem is obtained by (a) showing the convergence of $(\rho_n - 1)_+ \to 0$; (b) establishing the L^1 uniform bound for the approximate pressure ρ_n^γ; (c) establishing the convergence of the approximating sequence $(\rho_n, \boldsymbol{u}_n, \psi_n, \eta_n)$ through the proof of compactness for the solution sequence by using monotonicity properties of certain crucial quantities that depend on the macroscopic variables. Section 5 presents further extensions and related models.

2. Main result

The goal of this paper is to prove the existence of weak solutions to the free-boundary problem (15)-(20), so we introduce the notion of weak solutions we are going to use throughout the paper.

2.1. Notion of weak solution to problem (P_F)

Definition 2.1. **[Weak solution of the problem ($\boldsymbol{P_F}$)]** A vector $(\rho, \boldsymbol{u}, \psi, \eta)$ is called a weak solution to (14)-(20) with boundary data (22)-(24) and initial data (25)-(27) if the equations

$$\partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = 0 \tag{28}$$

$$\partial_t(\rho\boldsymbol{u}) + \operatorname{div}(\rho\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x(p_F + \xi\eta^2) = \operatorname{div}\mathbb{S}[\boldsymbol{u},\rho] + \rho f + \operatorname{div}\boldsymbol{\tau_1} \tag{29}$$

$$\partial_t\psi + \operatorname{div}(\psi\boldsymbol{u}) + \sum_{i=1}^K \nabla_{\boldsymbol{q_i}} \cdot (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{q_i}(\psi)) \tag{30}$$

$$= \epsilon\Delta_x\left(\frac{\psi}{\zeta(\rho)}\right)\frac{1}{4\lambda} + \sum_{i=1}^K\sum_{j=1}^K A_{i,j}\nabla_{\boldsymbol{q_i}} \cdot \left(M\nabla_{\boldsymbol{q_i}}\left(\frac{\psi}{\zeta(\rho)M}\right)\right)$$

$$\partial_t \eta + \operatorname{div}(\eta \boldsymbol{u}) - \varepsilon \Delta_x \left(\frac{\eta}{\zeta(\rho)} \right) = 0 \tag{31}$$

are satisfied in the sense of distributions, the divergence free condition $\operatorname{div} \boldsymbol{u} = 0$ is satisfied a.e. in $\{\rho = 1\}$, the constraint $0 \leq \rho \leq 1$ is satisfied a.e. in $(0,T) \times \Omega$ and the following regularity properties hold

$$\begin{gathered} \rho \in C([0,T]; L^p(\Omega)),\ 1 \leq p < \infty, \\ \boldsymbol{u} \in L^2(0,T; (W_0^{1,2}(\Omega))),\ \rho|\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega)), \\ p_F \in \mathcal{M}((0,T) \times \Omega) \\ \eta \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; \dot{H}^1(\Omega)) \\ \widehat{\psi} \in L^p(0,T; Z_1) \cap H^1(0,T; M^{-1}(H^s(\Omega \times D))'),\ 1 \leq p < \infty. \end{gathered}$$

Moreover p_F is so regular that the condition

$$p_F(\rho - 1) = 0,$$

is satisfied in the sense of distributions. In this work we will prove the existence of weak solutions to the free boundary problem (14)-(20) by showing rigorously that they can be obtained as a limit of the weak solutions to the macroscopic fluid-particle problem

$$\partial_t \rho_n + \operatorname{div}(\rho_n \boldsymbol{u}_n) = 0 \tag{32}$$

$$\partial_t(\rho_n \boldsymbol{u}_n) + \operatorname{div}(\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n) + \nabla_x(p_{Fn} + \xi \eta_n^2) = \operatorname{div} \mathbb{S}[\boldsymbol{u_n}, \rho_n] + \rho f + \operatorname{div} \boldsymbol{\tau_{1n}} \tag{33}$$

$$\partial_t \psi_n + \operatorname{div}(\psi_n \boldsymbol{u}_n) + \sum_{i=1}^K \nabla_{\boldsymbol{q_i}} \cdot (\boldsymbol{\sigma(u_n) q_i}(\psi_n)) \tag{34}$$

$$= \epsilon \Delta_x \left(\frac{\psi_n}{\zeta(\rho_n)} \right) + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{i,j} \nabla_{\boldsymbol{q_i}} \cdot \left(M \nabla_{\boldsymbol{q_i}} \left(\frac{\psi_n}{\zeta(\rho_n) M} \right) \right)$$

$$\partial_t \eta_n + \nabla \cdot (\eta_n \boldsymbol{u}_n) - \varepsilon \Delta_x \left(\frac{\eta_n}{\zeta(\rho_n)} \right) = 0, \tag{35}$$

where $\boldsymbol{\tau_{1n}} = \boldsymbol{\tau_1}(\psi_n)$ and

$$p_{Fn} = (\rho_n)^{\gamma_n},\ \gamma_n \to \infty,\ \text{as } n \to \infty.$$

Now we are ready to state the main existence results for our problem.

Theorem 2.2. *Assume that the boundary conditions (22)-(24) and the initial conditions (25)-(27) are satisfied. Then, there exists a weak solution (in the sense of Definition 2.1) of the problem (14)-(20).*

The main Theorem 2.2 will be obtained as a consequence of the following stability result.

Theorem 2.3. *For each $n \in \mathbb{N}$ be fixed, then there exists a global weak solution $(\rho_n, \boldsymbol{u}_n, \psi_n, \eta_n)$ to (32)-(35) such that, as $n \to \infty$*

$$(\rho_n - 1)_+ \to 0 \qquad \text{in } L^\infty(0,T; L^p),\ \text{for any } 1 \leq p < +\infty. \tag{36}$$

Moreover,

$$(\rho_n)^{\gamma_n} \qquad \textit{is bounded in } L^1, \textit{ for } n \textit{ such that } \gamma_n \geq 3, \tag{37}$$

and up to a subsequence there exists $p_F \in \mathcal{M}((0,T)\times\Omega)$ *such that*

$$(\rho_n)^{\gamma_n} \rightharpoonup p_F \qquad \textit{as } n \to \infty. \tag{38}$$

If in addition $\rho_{n0} = \rho_n(x,0) \to \rho_0$ *in* L^1, *then the following convergence holds:*

$$\begin{gathered}
\rho_n \rightharpoonup \rho \textit{ weakly in } L^p((0,T)\times\Omega)\ 1 \leq p < +\infty, \\
\rho_n \boldsymbol{u}_n \rightharpoonup \rho\boldsymbol{u} \textit{ weakly in } L^p((0,T;L^r(\Omega)),\ 1 \leq p < +\infty,\ 1 \leq r < 2, \\
\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n \rightharpoonup \rho\boldsymbol{u}\otimes\boldsymbol{u} \textit{ weakly in } L^p((0,T;L^1(\Omega)), 1 \leq p < +\infty, \\
\psi_n \to \psi \textit{ strongly in } L^p((0,T;L^1(\Omega\times D)),\ 1 \leq p < +\infty, \\
\eta_n \to \eta \textit{ strongly in } L^2(0,T;L^2(\Omega)),
\end{gathered}$$

$0 \leq \rho \leq 1$ *and* $(\rho, \boldsymbol{u}, \psi, \eta)$ *is a weak solution to the problem* (14)-(20) *in the sense of Definition 2.1.*

Finally, we want to point out that the solution we are going to construct satisfies the following energy inequality

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left[\frac{1}{2}\rho|\boldsymbol{u}|^2 + \xi\eta^2 + k\int_D M\mathcal{F}\left(\frac{\psi}{M}\right)d\boldsymbol{q}\right]dx \\
&+\mu^S\int_\Omega |D(\boldsymbol{u}) - \frac{1}{d}(\operatorname{div}_x \boldsymbol{u})\mathbb{I}|^2 dx + \mu^B\int_\Omega |\operatorname{div}_x \boldsymbol{u}|^2 dx \\
&+2\varepsilon\xi\int_\Omega |\nabla_x\eta|^2 dx + +4k\varepsilon\int_{\Omega\times D} M|\nabla_x\sqrt{\frac{\psi}{M}}|^2 d\boldsymbol{q}dx \\
&+\frac{k}{\lambda}\sum_{i=1}^K\sum_{j=1}^K\int_{\Omega\times D} M\nabla_{\boldsymbol{q}_j}\sqrt{\frac{\psi}{M}}\nabla_{\boldsymbol{q}_i}\sqrt{\frac{\psi}{M}}d\boldsymbol{q}dx = \int_\Omega \rho f\cdot\boldsymbol{u}dx.
\end{aligned} \tag{39}$$

The rest of the paper is devoted to the proof of the Theorems 2.2 and 2.3.

3. Formulation of the approximating problem

3.1. The approximating scheme

As already mentioned the solutions of the problem $(\mathbf{P_F})$ will be obtained by means of an approximating procedure. In this section we will set up the approximating scheme we are going to use.

Let be γ_n a sequence of real numbers such that $\gamma_n > \frac{3}{2}$, for any $n \in \mathbb{N}$ and $\gamma_n \to \infty$ as $n \to \infty$, we define $\{\rho_n, \boldsymbol{u}_n, \psi_n, \eta_n\}$ as solutions of the following system denoted as $(\mathbf{P_n})$.

$$\partial_t\rho_n + \operatorname{div}(\rho_n\boldsymbol{u}_n) = 0 \tag{40}$$

$$\partial_t(\rho_n\boldsymbol{u}_n) + \operatorname{div}(\rho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n) + \nabla_x(p_{Fn} + \xi\eta_n^2) = \operatorname{div}\mathbb{S}[\boldsymbol{u_n}, \rho_n] + \rho f + \operatorname{div}\boldsymbol{\tau_{1n}} \tag{41}$$

$$\partial_t\psi_n + \operatorname{div}(\psi_n\boldsymbol{u}_n) + \sum_{i=1}^K \nabla_{\boldsymbol{q_i}}\cdot(\boldsymbol{\sigma}(\boldsymbol{u_n})\boldsymbol{q_i}(\psi_n)) \tag{42}$$

$$= \epsilon \Delta_x \left(\frac{\psi_n}{\zeta(\rho_n)} \right) + \frac{1}{4\lambda} \sum_{i=1}^{K} \sum_{j=1}^{K} A_{i,j} \nabla_{\boldsymbol{q_i}} \cdot \left(M \nabla_{\boldsymbol{q_i}} \left(\frac{\psi_n}{\zeta(\rho_n) M} \right) \right)$$

$$\partial_t \eta_n + \nabla \cdot (\eta_n \boldsymbol{u}_n) - \varepsilon \Delta_x \left(\frac{\eta_n}{\zeta(\rho_n} \right) = 0, \tag{43}$$

where $\boldsymbol{\tau_{1n}} = \boldsymbol{\tau_1}(\psi_n)$ and

$$p_{Fn} = (\rho_n)^{\gamma_n}, \ \gamma_n \to \infty, \ \text{as } n \to \infty.$$

The approximating system must be complemented with boundary and initial data as follows.

3.1.1. Boundary data

$$\boldsymbol{u}_n = 0 \quad \text{on } \partial\Omega. \tag{44}$$

$$\left[\frac{1}{4\lambda} \sum_{j=1}^{K} A_{ij} M \nabla_{\boldsymbol{q_j}} \left(\frac{\psi_n}{\zeta(\rho_n) M} \right) - \sigma(\boldsymbol{u}_n) \boldsymbol{q_i} \psi \right] \cdot \frac{\boldsymbol{q_i}}{|\boldsymbol{q_i}|} = 0, \tag{45}$$

$$\text{on } \Omega \times \partial \bar{D}_i \times (0,T], \ \text{for } i = 1, \dots, K,$$

$$\varepsilon \nabla_x \left(\frac{\psi_n}{\zeta(\rho_n)} \right) \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega \times D \times (0,T]. \tag{46}$$

3.1.2. Initial data

$$\rho_n|_{t=0} = \rho_{n_0}, \quad \rho_n \boldsymbol{u}_n|_{t=0} = m_{n_0}, \quad \eta_n|_{t=0} = \eta_{n_0}, \quad \psi_n|_{t=0} = \psi_{n_0} \tag{47}$$

where

$$\begin{aligned} &\rho_{n_0} \geq 0 \quad \text{a.e.,} \quad \rho_{n_0} \in L^1(\Omega) \cap L^{\gamma_n}(\Omega), \\ &\int (\rho_{n_0})^{\gamma_n} dx \leq c\gamma_n \text{ for some } c, \\ &\qquad m_{n_0} \in L^{\frac{2\gamma_n}{\gamma_n+1}}(\Omega), \\ &\rho_{n_0} |\boldsymbol{u}_{n_0}|^2 \text{ is bounded in } L^1(\Omega), \\ &\qquad \boldsymbol{u}_{n_0} = \frac{m_{n_0}}{\rho_{n_0}} \text{ on } \{\rho_{n_0} > 0\}, \\ &\qquad \boldsymbol{u}_{n_0} = 0 \text{ on } \{\rho_{n_0} = 0\}, \\ &\qquad \psi_{n_0} \in L^1(\Omega \times S^2) \\ &\qquad \eta_{n_0} \in L^2(\Omega \times S^2). \end{aligned} \tag{48}$$

Furthermore we assume that

$$V_n = \fint_{\Omega} \rho_{n_0}, \quad 0 < V_n < V \leq 1, \quad V_n \to V, \tag{49}$$

and

$$\begin{aligned} \rho_{n_0} \boldsymbol{u}_n &\rightharpoonup m_0 \quad \text{weakly in } L^2(\Omega), \\ \rho_{n_0} &\rightharpoonup \rho_0 \quad \text{weakly in } L^1(\Omega). \end{aligned} \tag{50}$$

3.2. Existence of approximate solutions

For any fixed $n \in \mathbb{N}$, the existence of weak solutions for the system (40)-(43) has been proved by Barrett and Süli (2016) [7] (we refer the reader also to a series of earlier works on related models [3–5]).

We can summarize their existence result as follows.

Theorem 3.1. *The triple $(\rho_n, \boldsymbol{u}_n, \widehat{\psi}_n)$, is a global weak solution to problem $(\mathbf{P_n})$ in the sense that the following relations hold true*

$$\int_0^T \Big\langle \frac{\partial \rho_n}{\partial t}, \chi \Big\rangle_{W^{1,6}(\Omega)} dt - \int_0^T \int_\Omega \rho_n \boldsymbol{u}_n \cdot \nabla_x \chi dx dt = 0, \tag{51}$$

for any $\chi \in L^2(0,T;W^{1,6}(\Omega))$ with $\rho_n(\cdot,0) = \rho_{n,0}(\cdot)$,

$$\begin{aligned}\int_0^T \Big\langle \frac{\partial(\rho_n \boldsymbol{u}_n)}{\partial t}, \boldsymbol{w} \Big\rangle_{W_0^{1,r}(\Omega)} dt + \int_0^T \int_\Omega [\mathbb{S}(\boldsymbol{u}_n) - \rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n - c_p \rho_n^{\gamma_n} \mathbb{I}] : \nabla_x \boldsymbol{w} dx dt \\ = \int_0^T \int_\Omega [\rho_n \boldsymbol{f} \cdot \boldsymbol{w} - (\boldsymbol{\tau_1}(M\widehat{\psi_n}) - \xi \eta_n^2 \mathbb{I}) : \nabla_x \boldsymbol{w}] dx dt \\ \text{for all } \boldsymbol{w} \in L^{\frac{\gamma_n+\vartheta}{\vartheta}}(0,T;W_0^{1,r}(\Omega))\end{aligned} \tag{52}$$

with $(\rho_n \boldsymbol{u_n})(\cdot,0) = (\rho_{n,0}\boldsymbol{u_{n,0}})(\cdot)$ and $\vartheta(\gamma)$ is defined as

$$\vartheta(\gamma) := \frac{\gamma}{v(\gamma)} = \begin{cases} \frac{2\gamma-3}{3} \text{ for } \frac{3}{2} < \gamma \le 4, \\ \frac{5}{12}\gamma \text{ for } 4 \le \gamma \end{cases}$$

and $r = \max\left\{4, \frac{6\gamma}{2\gamma-3}\right\}$.

$$\begin{aligned}\int_0^T \Big\langle M\frac{\partial \widehat{\psi}_n}{\partial t}, \varphi \Big\rangle_{H^s(\Omega\times D)} dt + \frac{1}{4\lambda}\sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_0^T \int_{\Omega\times D} M \nabla_{\boldsymbol{q_j}} \widehat{\psi}_n \cdot \nabla_{\boldsymbol{q_j}} \varphi d\boldsymbol{q} dx dt \\ + \int_0^T \int_{\Omega\times D} M[\varepsilon \nabla_x \widehat{\psi}_n - \boldsymbol{u}_n \widehat{\psi}_n] \cdot \nabla_x \varphi d\boldsymbol{q} dx dt \\ + \int_0^T \int_{\Omega\times D} M \sum_{i=1}^K [\sigma(\boldsymbol{u}_n)\boldsymbol{q_i}] \widehat{\psi}_n \cdot \nabla_{\boldsymbol{q_i}} \varphi d\boldsymbol{q} dx dt = 0, \\ \text{for all } \varphi \in L^2(0,T;H^s(\Omega\times D)).\end{aligned} \tag{53}$$

In addition, the weak solution $(\rho_n, \boldsymbol{u}_n, \widehat{\psi_n})$ satisfies the inequality

$$\frac{1}{2}\int_\Omega \rho_n(t')|\boldsymbol{u}_n(t')|^2 dx + \int_\Omega \frac{(\rho_n(t'))^{\gamma_n}}{\gamma_n - 1} dx + k \int_{\Omega\times D} M\mathcal{F}(\widehat{\psi}_n(t')) d\boldsymbol{q} dx$$

$$
\begin{aligned}
&+\mu^S c_0 \int_0^{t'} \|\boldsymbol{u}_n\|^2_{H^1(\Omega)} dt \\
&+k\int_0^{t'} \int_{\Omega\times D} M\left[\frac{a_0}{2\lambda}|\nabla_{\boldsymbol{q}}|\sqrt{\widehat{\psi_n}}|^2 + 2\varepsilon|\nabla_x\sqrt{\widehat{\psi_n}}|^2\right] d\boldsymbol{q}dxdt \\
&+\xi\|\eta_n(t')\|^2_{L^2(\Omega)} + 2\xi\varepsilon\int_0^{t'} \|\nabla_x\eta_n\|_{L^2(\Omega)}dt \\
&\leq e^{t'}\left[\frac{1}{2}\rho_0|\boldsymbol{u_{n0}}|^2dx + \int_\Omega \frac{(\rho_{n_0})^{\gamma_n}}{\gamma_n - 1}dx + k\int_{\Omega\times D} M\mathcal{F}(\widehat{\psi}_{n0})d\boldsymbol{q}dx\right. \\
&\left.+\xi\int_\Omega\left(\int_D M\widehat{\psi}_{n0}d\boldsymbol{q}\right)^2 dx + \frac{1}{2}\int_0^{t'}\|f\|^2_{L^\infty(\Omega)}dt\int_\Omega \rho_{n0}dx\right] = E_{n_0}.
\end{aligned} \tag{54}
$$

Proof. For the sake of completeness we present now an outline of the proof. For the details we refer the reader to Barrett and Süli [7]. The proof relies on two key observations:

- If $\dfrac{\psi}{M}$ is bounded above then, for $L \in \mathbb{R}_+$ sufficiently large, the third term in (3) is equal to

$$
\sum_{i=1}^K \nabla_{\boldsymbol{q_i}} \cdot \left(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{q_i}M\beta^L\left(\frac{\psi}{M}\right)\right),
$$

where $\zeta = 1$ and $\beta^L \in C(\mathbb{R})$ denotes a cut-off function such as $\beta^L(s) := \min\{s, L\}$. It follows that for sufficiently large L any solution of (3) such as $\dfrac{\psi}{M}$ is bounded above by L also satisfies

$$
\begin{aligned}
&\partial_t\psi + \operatorname{div}(\psi\boldsymbol{u}) + \sum_{i=1}^K \nabla_{\boldsymbol{q_i}} \cdot \left(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{q_i}M\beta^L\left(\frac{\psi}{M}\right)\right) \\
&= \epsilon\Delta_x\left(\frac{\psi}{\zeta(\rho)}\right)\frac{1}{4\lambda}\sum_{i=1}^K\sum_{j=1}^K A_{i,j}\nabla_{\boldsymbol{q_i}} \cdot \left(M\nabla_{\boldsymbol{q_i}}\left(\frac{\psi}{M}\right)\right)
\end{aligned} \tag{55}
$$

in $\Omega \times D \times (0, T]$ supplemented with the following boundary conditions:

$$
\left[\frac{1}{4\lambda}\sum_{j=1}^K A_{i,j}M\nabla_{\boldsymbol{q_j}}\left(\frac{\psi}{M}\right) - \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{q_i}M\beta^L\left(\frac{\psi}{M}\right)\right] \cdot \frac{\boldsymbol{q_i}}{|\boldsymbol{q_i}|} = 0 \tag{56}
$$

on $\Omega \times \partial\bar{D}_i \times (0, T]$ for $i = 1, \ldots, K$.

$$
\varepsilon\nabla_x\psi \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \times D \times (0, T] \tag{57}
$$

and the initial conditions:

$$
\psi(\cdot,\cdot,0) = M(\cdot)\beta^L\left(\frac{\psi_0(\cdot,\cdot)}{M(\cdot)}\right) \geq 0, \ \text{on}\,\Omega \times D. \tag{58}
$$

The model with cut-off parameter $L > 1$ is further regularized, by introducing into the continuity equation a dissipation term of the form $\alpha\Delta\rho$, with $\alpha > 0$ and supplementing the resulting parabolic equation with a homogeneous Neumann boundary condition on $\partial\Omega \times (0, T]$. Moreover, the equation of state (1.3) is replaced by a regularized equation of state,

$$p_\kappa(\rho) = p(\rho) + \kappa(\rho^4 + \rho^\Gamma), \text{ with } \kappa \in \mathbb{R}_+ \ \Gamma = \max\{\gamma, 8\}.$$

- The second key element of the proof is that instead of taking the limits $\kappa \to 0_+, \alpha \to 0_+, L \to +\infty$ to deduce the existence of solutions to $(\boldsymbol{P_n})$ the authors (semi)discretise the problem with respect to t, with step size Δt. The existence of solutions to this problem is established by employing Schauder's fixed point theorem. Next one derives bounds on the sequence of solutions to the time discretised problem uniform in the time step Δt and the cut-off parameter L, and thus permit the extraction of weakly convergent subsequences, as $L \to \infty$ and $\Delta t \to 0_+$, with $\Delta t = o(L - 1)$. The weakly convergent subsequences are then shown to converge strongly in suitable norms. This allows to the passing to the limit as $L \to +\infty$, with $\Delta t = o(L - 1)$. The result follows by passing to the limit as $\alpha, \kappa \to 0_+$. □

3.3. A priori estimates and compactness for the approximating sequences

We start this section by collecting all the a priori estimates that can be deduced by the energy estimate (54). In particular for any fixed $\gamma_n > 3/2$ we have

$$\begin{gathered}
\rho_n \in L^\infty(0,T; L^{\gamma_n}(\Omega)), \quad \nabla \boldsymbol{u}_n \in L^2(0;T;L^2(\Omega)),\\
\sqrt{\rho_n}\boldsymbol{u}_n \in L^\infty(0,T;L^2(\Omega)), \quad \rho_n\boldsymbol{u}_n \in C_w([0,T]; L^{\frac{2\gamma_n}{\gamma_n+1}}(\Omega)),\\
\eta_n \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\dot{H}^1(\Omega))\\
\mathcal{F}(\widehat{\psi}_n) \in L^\infty(0,T;L^1(\Omega \times D))\\
M^{1/2}\nabla_x\sqrt{\widehat{\psi}_n} \in L^2(0,T;L^2(\Omega \times D)), \quad M^{1/2}\nabla_q\sqrt{\widehat{\psi}_n} \in L^2(0,T;L^2(\Omega \times D)).
\end{gathered}$$

By using (42) and (54) we get also that

$$M\frac{\partial\widehat{\psi}_n}{\partial t} \in L^2(0,T;H^s(\Omega \times D)')$$

Moreover, by applying well established techniques (see for example [7] or [12]) we are able to show the following uniform bound for the density

$$\rho_n \in L^\infty(0,T;L^1 \cap L^\Gamma(\Omega)), \qquad \text{for any } \Gamma \geq 8.$$

With the preceding bounds we can get some further estimates on the elastic stress tensor

$$\boldsymbol{\tau}(\boldsymbol{\psi})(x,t) := \boldsymbol{\tau_1}(\boldsymbol{\psi})(x,t) - \xi\left(\int_D \psi d\boldsymbol{q}\right)^2 \mathbb{I},$$

where

$$\boldsymbol{\tau_1}(\psi) := k\left[\left(\sum_{i=1}^K \boldsymbol{C_i}(\psi)\right) - (K+1)\int_D \psi(\boldsymbol{q})d\boldsymbol{q}\,\mathbb{I}\right].$$

In fact by following the same lines of arguments as in [7] it is possible to prove that

$$\|\boldsymbol{C_i}(M\widehat{\psi}_n)\|_{L^2(0,T;L^{\frac{4}{3}}(\Omega))} + \|\boldsymbol{C_i}(M\widehat{\psi}_n)\|_{L^{\frac{4(d+2)}{3d+4}}(0,T;\Omega)} \leq C$$

from which it is straightforward to deduce that

$$\|\boldsymbol{\tau_1}(M\widehat{\psi}_n)\|_{L^2(0,T;L^{\frac{4}{3}}(\Omega))} + \|\boldsymbol{\tau_1}(M\widehat{\psi}_n)\|_{L^{\frac{4(d+2)}{3d+4}}(0,T;\Omega)} \leq C \tag{59}$$

From the previous a priori estimates, extracting a subsequence, we can deduce the following convergence results

$$\begin{aligned}
&\rho_n \rightharpoonup \rho \quad \text{weakly in } \ L^\infty(0,T;L^p(\Omega)),\ \rho \in L^\infty(0,T;L^1\cap L^p(\Omega)),\ 1\leq p<+\infty,\\
&\sqrt{\rho_n}\boldsymbol{u}_n \rightharpoonup \sqrt{\rho}\boldsymbol{u} \quad \text{weakly in } \ L^2(0,T;L^2(\Omega)),\\
&\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \ \text{ weakly in } \ L^2(0,T;H^1(\Omega)),\\
&\rho_n\boldsymbol{u}_n \rightharpoonup \rho\boldsymbol{u} \ \text{ weakly in } \ L^2(0,T;L^{\frac{6\gamma_n}{\gamma_n+6}}(\Omega)),\\
&\rho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n \rightharpoonup \rho\boldsymbol{u}\otimes\boldsymbol{u} \quad \text{in weakly } \ L^2(0,T;L^{\frac{6\gamma_n}{4\gamma_n+3}}(\Omega)),\\
&\eta_n \rightharpoonup \eta \ \text{ weakly in } \ L^2(0,T;H^1(\Omega)),\ \ \eta\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)),\\
&M^{1/2}\nabla_x\sqrt{\widehat{\psi}_n} \rightharpoonup M^{1/2}\nabla_x\sqrt{\widehat{\psi}} \ \text{ weakly in } \ L^2(0,T;L^2(\Omega\times D)),\\
&M^{1/2}\nabla_q\sqrt{\widehat{\psi}_n} \rightharpoonup M^{1/2}\nabla_q\sqrt{\widehat{\psi}} \ \text{ weakly in } \ L^2(0,T;L^2(\Omega\times D)),\\
&M\frac{\partial\widehat{\psi}_n}{\partial t} \rightharpoonup M\frac{\partial\widehat{\psi}}{\partial t} \ \text{ weakly in } \ L^2(0,T;H^s(\Omega\times D)'),\\
&\psi_n \to \psi \quad \text{strongly in } \ L^p(0,T;L^1(\Omega\times D)),\ \ p\geq 1,\\
&\boldsymbol{\tau}(\widehat{\psi}_n) \to \boldsymbol{\tau}(\widehat{\psi}) \ \text{ strongly in } \ L^r(0,T;\Omega)
\end{aligned} \tag{60}$$

4. Proof of the Main Theorem 2.2

In this section we are going to prove the existence of a global weak solution for the problem $(\mathbf{P_F})$, the Main Theorem 2.2. We start by showing a stability result for the approximating sequences, Theorem 2.3.

4.1. Proof of the Theorem 2.3

One of the main issues in the proof is to get a uniform L^1 bound in $n\in\mathbb{N}$ for $\rho_n^{\gamma_n}$. Indeed from (54) we only have $\int \rho_n^{\gamma_n} \leq C(\gamma_n - 1)$.

For simplicity we divide the proof in different steps.

Step 1: Convergence of $(\rho_{\mathbf{n}} - \mathbf{1})_+$ to 0.
The energy inequality (54) with the initial condition (48) gives

$$\int_\Omega (\rho_n)^{\gamma_n}dx \leq (\gamma_n - 1)E_{n_0} + \int_\Omega (\rho_{n_0})^{\gamma_n}dx \leq (\gamma_n-1)E_{n_0} + c\gamma_n \leq c\gamma_n. \tag{61}$$

Since $\gamma_n\to\infty$, for any $1<p<+\infty$ there exists $n\in\mathbb{N}$ such that $\gamma_n>p$. Then by Hölder inequality we get

$$\|\rho_n\|_{L^\infty_t L^p_x} \leq \|\rho_n\|^{\theta_n}_{L^\infty_t L^1_x}\|\rho_n\|^{1-\theta_n}_{L^\infty_t L^1_x} \leq V_n^{\theta_n}(c\gamma_n)^{\frac{1-\theta_n}{\gamma_n}},$$

where V_n is defined in (49) and θ_n is so that $\frac{1}{p} = \theta_n + \frac{1-\theta_n}{\gamma_n}$. We have that $\theta_n \to \frac{1}{p}$, as $n \to \infty$ and we end up with

$$\|\rho_n\|_{L_t^\infty L_x^p} \le \liminf_{n\to\infty} \|\rho_n\|_{L_t^\infty L_x^p} \le V^{1/p}.$$

We define the function ϕ_n as follows

$$\phi_n = (\rho_n - 1)_+,$$

by using again the energy inequality (54) we can compute

$$\int_\Omega (1+\phi_n)^{\gamma_n} \mathbf{1}_{\{\phi_n>0\}} dx \le \int_\Omega \rho^{\gamma_n} dx \le c\gamma_n. \tag{62}$$

We apply the inequality

$$(1+x)^k \ge 1 + c_p k^p x^p, \quad p>1, k \text{ large}, \ x>0$$

with $k=\gamma_n$, $x=\phi_n$ to the right hand side of (62), so we obtain

$$c_p\gamma_n^p \int_\Omega \phi_n^p dx \le |\Omega| + c_p\gamma_n^p \int_\Omega \phi_n^p dx \le \int_\Omega (1+\phi_n)^\gamma \mathbf{1}_{\{\phi_n>0\}} dx \le c\gamma_n.$$

Therefore we have

$$\int_\Omega \phi_n^p dx \le \frac{c}{c_p \gamma_n^{p-1}},$$

and, as $n\to\infty$ we get

$$(\rho_n - 1)_+ \to 0 \qquad \text{in } L^\infty(0,T;L^p(\Omega)),\ 1 \le p < +\infty.$$

Step 2: $\mathbf{L^1}$ uniform bound of $(\rho_\mathbf{n})^{\gamma_\mathbf{n}}$.
In order to prove a uniform bound for the pressure p_F, we start by assuming that we know

$$(\rho_n)^{\gamma_n+1} \quad \text{is uniformly bounded in } L^1(0,T;L^1(\Omega)), \tag{63}$$

hence we have

$$\begin{aligned}\int_0^T\int_\Omega (\rho_n)^{\gamma_n} dxdt &= \int_0^T \left(\int_{\Omega\cap\{\rho_n>1\}} (\rho_n)^{\gamma_n} dx + \int_{\Omega\cap\{\rho_n\le 1\}} (\rho_n)^{\gamma_n} dx \right) dt \\ &\le \int_0^T \left(\int_\Omega \left((\rho_n)^{\gamma_n+1} + \rho_n\right) dx \right) dt.\end{aligned} \tag{64}$$

Since $\rho_n \in L^\infty(0,T;L^1(\Omega))$ and (63) holds, from (64) it follows the uniform L^1 bound for $(\rho_n)^{\gamma_n}$.

The only thing we need to prove is (63). We recall that for ρ_n we don't have L^∞ bounds, but on the other hand, because of (61) there exists a constant $\tilde{c}$ such that for any $n \in \mathbb{N}$ the following estimate holds

$$\|\rho_n\|_{L_t^\infty L_x^{\gamma_n}} \leq \tilde{c}, \tag{65}$$

where $\tilde{c} = \sup_{\gamma>0}(c\gamma)^{1/\gamma}$.

Let us define, now, the operator $\mathcal{B}$ as the inverse of the divergence operator. We denote the solution v of

$$\operatorname{div} v = g \ \text{ in } \ \Omega, \quad v = 0 \ \text{ on } \ \partial\Omega$$

by $v = \mathcal{B}g$. The operator $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ enjoys the following properties

$$\mathcal{B} : \Big\{g \in L^p; \int_\Omega g dx = 0\Big\} \to W_0^{1,p}(\Omega),$$
$$\|\mathcal{B}(g)\|_{W^{1,p}(\Omega)} \leq C\|g\|_{L^p(\Omega)}.$$

If g can be written as $g = \operatorname{div} h$ for a certain $h \in L^r$ with $h \cdot \hat{n} = 0$ on $\partial\Omega$, then

$$\|\mathcal{B}(g)\|_{L^r(\Omega)} \leq C\|h\|_{L^r(\Omega)}.$$

With same lines of arguments as in [12] it is possible to prove that our approximating sequences satisfy also the equation (40) in the sense of renormalized solutions, namely

$$\partial_t b(\rho_n)_\epsilon + \operatorname{div}(b(\rho_n)_\epsilon u) + \Big(\big[b'(\rho_n)\rho_n - b(\rho_n)\big] \operatorname{div} \boldsymbol{u}_n\Big)_\epsilon = r_\epsilon, \tag{66}$$

where as proved in Lions [14], $r_\epsilon \to 0$ in $L^2((0,T) \times \mathbb{R}^3)$. Let us take a test function of the form

$$\phi_i = \chi(t)\mathcal{B}_i\Big[b(\rho_n)_\epsilon - \oint_\Omega b(\rho_n)_\epsilon dy\Big],$$

where

$$\oint_\Omega b(\rho_n)_\epsilon dy = \frac{1}{|\Omega|}\int_\Omega b(\rho_n)_\epsilon dy, \quad \chi \in \mathcal{D}(0,T)$$

and test it against (41). Then, with the aid of (66) and by setting for simplicity all the constants equal to 1 we can compute

$$\begin{aligned}
\int_0^T\int_\Omega \chi\rho_n^{\gamma_n} b(\rho_n)_\epsilon dxdt &= \int_0^T\int_\Omega \chi\rho_n^{\gamma_n}\Big[\oint_\Omega b(\rho_n)_\epsilon dy\Big]dxdt \\
&\quad - \int_0^T\int_\Omega \chi_t\rho_n\boldsymbol{u}_n \cdot \mathcal{B}\Big[b(\rho_n)_\epsilon - \oint_\Omega b(\rho_n)_\epsilon dy\Big]dxdt \\
&\quad + \int_0^T\int_\Omega \chi\rho_n\boldsymbol{u}_n \cdot \mathcal{B}\Big[\big((b'(\rho_n)\rho_n - b(\rho_n))\operatorname{div} \boldsymbol{u}_n\big)_\epsilon \\
&\quad - \oint_\Omega \big((b'(\rho_n)\rho_n - b(\rho_n))\operatorname{div} \boldsymbol{u}_n\big)_\epsilon dy\Big]dxdt
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^T\int_\Omega \chi\rho_n \boldsymbol{u}_n\cdot\mathcal{B}\Big[r_\epsilon-\oint_\Omega r_\epsilon dy\Big]dxdt\\
&+\int_0^T\int_\Omega \chi\rho_n \boldsymbol{u}_n\cdot\mathcal{B}\Big[\nabla\cdot\big(b(\rho_n)_\epsilon \boldsymbol{u}_n\big)\Big]dxdt\\
&-\int_0^T\int_\Omega \chi\rho_n \boldsymbol{u}_{ni}\boldsymbol{u}_{nj}\partial_i\mathcal{B}_j\Big[b(\rho_n)_\epsilon-\oint_\Omega b(\rho_n)_\epsilon dy\Big]dxdt\\
&+\int_0^T\int_\Omega \chi\partial_i\boldsymbol{u}_{nj}\partial_i\mathcal{B}_j\Big[b(\rho_n)_\epsilon-\oint_\Omega b(\rho_n)_\epsilon dy\Big]dxdt\\
&+\int_0^T\int_\Omega \chi\,\mathrm{div}\,\boldsymbol{u}_n\Big[b(\rho_n)_\epsilon-\oint_\Omega b(\rho_n)_\epsilon dy\Big]dxdt\\
&-\int_0^T\int_\Omega \chi\eta_n^2\Big[b(\rho_n)_\epsilon-\oint_\Omega b(\rho_n)_\epsilon dy\Big]dxdt\\
&+\int_0^T\int_\Omega \chi\boldsymbol{\tau}_{\mathbf{1}ij}(\psi_n)\partial_i\mathcal{B}_j\Big[b(\rho_n)_\epsilon-\oint_\Omega b(\rho_n)_\epsilon dy\Big]dxdt\\
&=I_1+\cdots+I_{11}.
\end{aligned}
$$

By using the properties of the operator $\mathcal{B}$, the a priori bounds of the Section 3.3 and (65) we estimate each one of the terms $I_1,\cdots,I_{11}$. For details see [13], [12].

For I_1 we have

$$I_1\lesssim C(T).$$

Concerning I_2 we get

$$
\begin{aligned}
I_2&\lesssim \|\rho_n\boldsymbol{u}_n\|_{L^\infty(0,T;L^{\frac{2\gamma_n}{\gamma_n+1}}(\Omega))}\|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^{\frac{6\gamma_n}{5\gamma_n-3}}(\Omega))}\\
&\le C(T)\|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^{\frac{6\gamma_n}{5\gamma_n-3}}(\Omega))}.
\end{aligned}
$$

For I_3, I_4 and I_5 we have,

$$
\begin{aligned}
I_3+I_4&\lesssim \|\rho_n\|_{L^\infty(0,T;L^\gamma(\Omega))}\|\nabla\boldsymbol{u}_n\|^2_{L^2(\Omega\times(0,T))}\|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma_n}{2\gamma_n-3}}(\Omega))}\\
&\le C(T)\|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma_n}{2\gamma_n-3}}(\Omega))},\\
I_5&\lesssim \|\rho_n\boldsymbol{u}_n\|_{L^\infty(0,T;L^{\frac{2\gamma_n}{\gamma_n+1}}(\Omega))}\|r_\epsilon\|_{L^2(\Omega\times(0,T))}\le C(T)\|r_\epsilon\|_{L^2(\Omega\times(0,T))}.
\end{aligned}
$$

We estimate now I_6+I_7,

$$
\begin{aligned}
I_6+I_7&\lesssim \|\rho_n\|_{L^\infty(0,T;L^{\gamma_n}(\Omega))}\|\nabla\boldsymbol{u}_n\|^2_{L^2(\Omega\times(0,T))}\|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma_n}{2\gamma_n-3}}(\Omega))}\\
&\le C(T)\|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma_n}{2\gamma_n-3}}(\Omega))}.
\end{aligned}
$$

For the terms I_8, I_9 we have

$$I_8 + I_9 \lesssim \|\nabla \boldsymbol{u}_n\|_{L^2(\Omega\times(0,T))}\|b(\rho_n)_\epsilon\|_{L^2(\Omega\times(0,T))} \le C(T)\|b(\rho_n)_\epsilon\|_{L^2(\Omega\times(0,T))},$$

and finally the last two terms can be estimates as follows

$$\begin{aligned} I_{10} + I_{11} &\lesssim \|\eta_n\|^2_{L^2(0,T;L^6(\Omega))}\|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^{\frac{3}{2}}(\Omega))} \\ &\quad + \|\boldsymbol{\tau_1}\psi_n\|_{L^2(0,T;L^{4/3}(\Omega))}\|b(\rho_n)_\epsilon\|_{L^2(0,T;L^4(\Omega))} \\ &\le C(T)\|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^4(\Omega))} \end{aligned}$$

In sum,

$$\begin{aligned} &\int_0^T\int_\Omega \chi\rho_n^{\gamma_n}(b(\rho_n))_\epsilon dxdt \\ &\le C(T) + \|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^{\frac{6\gamma_n}{5\gamma_n-3}}(\Omega))} + \|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^{\frac{3\gamma_n}{2\gamma_n-3}}(\Omega))} \\ &+ \|b(\rho_n)_\epsilon\|_{L^\infty(0,T;L^4(\Omega))} + \|b(\rho_n)_\epsilon\|_{L^2(\Omega\times(0,T))} + \|r_\epsilon\|_{L^2(\Omega\times(0,T))}. \end{aligned}$$

By taking the limit $\epsilon \to 0$,

$$\begin{aligned} &\int_0^T\int_\Omega \chi\rho_n^{\gamma_n}b(\rho_n)dxdt \\ &\le C(T) + \|b(\rho_n)\|_{L^\infty(0,T;L^{\frac{6\gamma_n}{5\gamma_n-3}}(\Omega))} + \|b(\rho_n)\|_{L^\infty(0,T;L^{\frac{3\gamma_n}{2\gamma_n-3}}(\Omega))} \\ &+ \|b(\rho_n)\|_{L^\infty(0,T;L^4(\Omega))} + \|b(\rho_n)\|_{L^2(\Omega\times(0,T))}. \end{aligned}$$

We approximate $z \mapsto z$ by a sequence of $\{b_n\}$ in (66), and approximate χ to the identity function of $(0, T)$. Then,

$$\begin{aligned} \int_0^T\int_\Omega \rho_n^{\gamma_n+1}dxdt &\le C(T) + \|\rho_n\|_{L^\infty(0,T;L^{\frac{6\gamma_n}{5\gamma_n-3}}(\Omega))} + \|\rho_n\|_{L^\infty(0,T;L^{\frac{3\gamma_n}{2\gamma_n-3}}(\Omega))} \\ &\quad + \|\rho_n\|_{L^\infty(0,T;L^4(\Omega))} + \|\rho_n\|_{L^2(\Omega\times(0,T))}. \end{aligned} \tag{67}$$

Since $\gamma_n \to \infty$ we can always assume that $\gamma_n \ge N = 3$, hence by taking into account that $\rho_n \in L^\infty(0,T;L^1(\Omega))$ and (65) we have that the right hand side of (67) is uniformly bounded and we can conclude that

$$\int_0^T\int_\Omega \rho_n^{\gamma_n+1}dxdt \le C(T)$$

which completes the proof of (63).

Step 3: Convergence of the approximating scheme.
The compactness properties of the approximating sequence $\{\rho_n, \boldsymbol{u}_n, \eta_n, \psi_n\}$ stated in Section 3.3 and the bounds of the Step 1 and Step 2 entail

$$\begin{aligned}
&\rho_n \boldsymbol{u}_n \to \rho \boldsymbol{u} \ \text{ in } \ L^p(0,T;L^r(\Omega)) \ \text{ for all } \ 1 \le p < \infty, \quad 1 \le r < 2,\\
&\boldsymbol{u}_n \to \boldsymbol{u} \ \text{ in } \ L^p(\Omega \times (0,T)) \cap \{\rho_n > 0\} \ \text{ for all } \ 1 \le p < 2,\\
&\boldsymbol{u}_n \to \boldsymbol{u} \ \text{ in } \ L^2(\Omega \times (0,T)) \cap \{\rho_n \ge \delta\} \ \text{ for all } \ \delta > 0,\\
&\rho_n \boldsymbol{u}_{ni} \boldsymbol{u}_{nj} \to \rho \boldsymbol{u}_i \boldsymbol{u}_j \ \text{ in } \ L^p(0,T;L^1(\Omega)) \ \text{ for all } \ 1 \le p < \infty,\\
&(\rho_n)^{\gamma_n} \rightharpoonup \pi, \ \text{ where } \pi \in \mathcal{M}((0,T) \times \Omega).
\end{aligned}$$

With the above convergence result and those one obtained in the Section 3.3 we can pass into the weak limit in the system (40)-(43), and we get that $\rho, \boldsymbol{u}, \eta, \psi$ is a weak solution of the problem ($\mathbf{P_F}$) provided we prove the conditions (18)-(20). This is equivalent to the proof of

$$\rho\pi = \pi. \tag{68}$$

Setting $s_n = \rho_n \log \rho_n$ and $\bar{s} = \overline{\rho \log \rho}$ and using (40) we get

$$(\rho_n \log \rho_n)_t + \nabla \cdot (\rho_n \log \rho_n \boldsymbol{u}_n) + (\nabla \cdot \boldsymbol{u}_n)\rho_n = 0.$$

Then we apply $(-\Delta)^{-1}\nabla\cdot$ to (41),

$$\frac{d}{dt}\Big[(-\Delta)^{-1}\nabla \cdot (\rho_n \boldsymbol{u}_n)\Big] + (-\Delta)^{-1}\partial_i\partial_j(\rho_n \boldsymbol{u}_{ni}\boldsymbol{u}_{nj}) + 2\nabla \cdot \boldsymbol{u}_n - \rho_n^{\gamma_n} - \eta_n^2 = (-\Delta)^{-1}\nabla \cdot (\nabla \cdot \boldsymbol{\tau_{1n}}).$$

By following the same procedure as in [12] and by taking the limit as $n \to \infty$ we end up with relation,

$$\begin{aligned}
&2\Big[\bar{s}_t + \nabla \cdot (\boldsymbol{u}\bar{s})\Big] + \overline{\rho^{\gamma+1}}\\
&= -\rho\eta^2 - \rho\Big[(-\Delta)^{-1}\nabla \cdot (\nabla \cdot \boldsymbol{\tau} - \nabla\eta)\Big] + \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla \cdot (\rho\boldsymbol{u})\Big]\\
&+ \nabla \cdot \Big[\rho\boldsymbol{u}(-\Delta)^{-1}\nabla \cdot (\rho\boldsymbol{u})\Big]\\
&+ \rho\Big[(-\Delta)^{-1}\partial_i\partial_j(\rho\boldsymbol{u}_i u_j) - u \cdot \nabla(-\Delta)^{-1}\nabla \cdot (\rho\boldsymbol{u})\Big].
\end{aligned}$$

Let $s = \rho \log \rho$, exactly as before (for the details of the proof we refer to [12]), we obtain that

$$\begin{aligned}
&2\Big[s_t + \nabla \cdot (\boldsymbol{u}s)\Big] + \rho\overline{\rho^{\gamma}}\\
&= -\rho\eta^2 - \rho\Big[(-\Delta)^{-1}\nabla \cdot (\nabla \cdot \sigma - \nabla\eta)\Big] + \frac{d}{dt}\Big[\rho(-\Delta)^{-1}\nabla \cdot (\rho\boldsymbol{u})\Big]\\
&+ \nabla \cdot \Big[\rho\boldsymbol{u}(-\Delta)^{-1}\nabla \cdot (\rho\boldsymbol{u})\Big]\\
&+ \rho\Big[(-\Delta)^{-1}\partial_i\partial_j(\rho\boldsymbol{u}_i\boldsymbol{u}_j) - u \cdot \nabla(-\Delta)^{-1}\nabla \cdot (\rho\boldsymbol{u})\Big].
\end{aligned}$$

Comparing the last two relations, we have

$$\partial_t(\bar{s} - s) + \operatorname{div}((\bar{s} - s)\boldsymbol{u}_n) = -\overline{\rho \operatorname{div} \boldsymbol{u}} + \rho_n \operatorname{div} \boldsymbol{u}_n \tag{69}$$

and

$$\partial_t(\bar{s} - s) + \operatorname{div}((\bar{s} - s)\boldsymbol{u}_n) = \frac{1}{2}\left(\rho\pi - \overline{(\rho_n)^{\gamma_n+1}}\right). \tag{70}$$

Now, using that

$$(\rho)^{\gamma_n} \to \mathbf{1}_{\{\rho=1\}}, \quad \text{a.e. in } L^p((0,T)\times\Omega),$$

which yields

$$(\rho)^{\gamma_n}(\rho_n - \rho) \rightharpoonup 0,$$

we obtain

$$\overline{(\rho_n)^{\gamma_n+1}} - \rho\overline{(\rho_n)^{\gamma_n}} = \overline{(\rho_n)^{\gamma_n}(\rho_n-\rho)} = \overline{((\rho_n)^{\gamma_n} - \rho^{\gamma_n})(\rho_n-\rho)} \geq 0. \tag{71}$$

From (71) we obtain,

$$\rho\pi = \rho\overline{(\rho_n)^{\gamma_n}} \leq \overline{(\rho_n)^{\gamma_n+1}}.$$

We integrate (70) in space to get

$$\partial_t \int_\Omega (\bar{s} - s)dx \leq 0.$$

Now, since $(\bar{s}-s)|_{t=0} = 0$ and by the convexity of s we have $s \leq \bar{s}$ and $s = \bar{s}$. Therefore, from (70) we obtain

$$\rho\pi = \overline{(\rho_n)^{\gamma_n+1}} \tag{72}$$

Moreover we have

$$(\rho_n)^{\gamma_n+1} \geq (\rho_n)^{\gamma_n} - \varepsilon. \tag{73}$$

Indeed it is sufficient to use the property $x^{\gamma_n+1} \geq x^{\gamma_n} - \varepsilon$, $\varepsilon > 0$ and any $x \geq 0$ in the case $x = \rho$. By using (72) and by passing to the weak limit in (73) we end up with

$$\rho\pi \geq \pi - \varepsilon,$$

and, as $\varepsilon \to 0$ we conclude with

$$\rho\pi \geq \pi. \tag{74}$$

The last issue to be proved is $\rho\pi \leq \pi$. Since $\rho\pi$ is not defined almost everywhere, in order to give a meaning to the inequality we want to prove, we define by ω_k a smoothing sequence in the space and time variables as follows

$$\omega_k = k^4\omega(k\cdot),$$
$$\omega \in C^\infty(\mathbb{R}^4),\ \omega \geq 0, \quad \fint_{\mathbb{R}^4} \omega dx dt = 1, \quad spt\omega \in B_1(\mathbb{R}^4).$$

We denote by ρ_k and π_k a sequence of smooth functions defined as

$$\rho_k = \rho * \omega, \qquad \pi_k = \pi * \omega$$

and we have that

$$\rho_k \to \rho \quad \text{in } C([0,T];L^p) \cap C([0,T];H^{-1}),$$
$$\pi_k \to \pi \quad \text{in } W^{-1,2} \cap L^1(L^q),$$

for any p, q such that $1/p + 1/q = 1$. Hence we can rewrite $(\rho - 1)\pi$ as

$$(\rho - 1)\pi = (\rho_k - 1)\pi_k + (\rho - \rho_k)\pi_k + (\rho - 1)(\pi - \pi_k) \tag{75}$$

Since $\rho_k \leq 1$, as $k \to \infty$ in (75) we obtain

$$\rho\pi - \pi \leq 0 \tag{76}$$

Considering together and (74) and (76) we have (68) and we conclude the proof of the Theorem 2.3.

4.2. Proof of the Theorem 2.2

The proof of the Theorem 2.2 is a consequence of the Theorem 2.3. The only thing we have to check is that the condition (18) holds in the sense of distribution. This last issue is a consequence of the following lemma (for the proof we refer to [15], Lemma 2.1).

Lemma 4.1. *Let $\boldsymbol{u} \in L^2(0,T;H^1_{loc}(\Omega))$ and $\rho \in L^2_{loc}((0,T)\times\Omega)$ satisfying*

$$\partial_t \rho_n + \operatorname{div}(\rho_n \boldsymbol{u}_n) = 0, \quad \text{in } (0,T)\times\Omega,$$
$$\rho(0) = \rho_0,$$

then the following two assertions are equivalent

(i) $\operatorname{div}\boldsymbol{u} = 0$, *a.e. on* $\{\rho \geq 1\}$ *and* $0 \leq \rho_0 \leq 1$.
(ii) $0 \leq \rho \leq 1$.

The final step is to obtain the energy inequality (39) that we require our global weak solutions have to satisfy. By applying the convergence results proved in the Theorem 2.3 we can pass into the weak limit in the energy inequality (54) and we get

$$\int_\Omega \left[\frac{1}{2}\rho|\boldsymbol{u}|^2 + \xi\eta^2 + k\int_D M\mathcal{F}\left(\frac{\psi}{M}\right)d\boldsymbol{q}\right]dx$$
$$+ \mu^S \int_0^t\int_\Omega |D(\boldsymbol{u}) - \frac{1}{d}(\operatorname{div}_x \boldsymbol{u})\mathbb{I}|^2 dx + \mu^B \int_0^t\int_\Omega |\operatorname{div}_x \boldsymbol{u}|^2 dx$$
$$+ 2\varepsilon\xi \int_0^t\int_\Omega |\nabla_x \eta|^2 dx dx + 4k\varepsilon \int_0^t\int_{\Omega\times D} M|\nabla_x \sqrt{\frac{\psi}{M}}|^2 d\boldsymbol{q}dx$$
$$+ \frac{k}{\lambda}\sum_{i=1}^K\sum_{j=1}^K \int_0^t\int_{\Omega\times D} M\nabla_{\boldsymbol{q}_j}\sqrt{\frac{\psi}{M}}\nabla_{\boldsymbol{q}_i}\sqrt{\frac{\psi}{M}}d\boldsymbol{q}dx$$

$$\leq \frac{1}{2}\int\limits_{\Omega} \rho_0|\boldsymbol{u_0}|^2 dx + k \int\limits_{\Omega\times D} M\mathcal{F}(\widehat{\psi}_0) d\boldsymbol{q} dx + \liminf_{n\to\infty} \int\limits_{\Omega} dx \frac{(\rho_{n0})^{\gamma_n}}{\gamma_n} + \int\limits_{\Omega} \rho f \cdot \boldsymbol{u} dx,$$

a.e. in t. Now, if we take, for any $n > 2$, $\rho_{n0} = \rho_0$, $m_{n0} = m_0$ and since $0 \leq \rho_0 \leq 1$, then

$$\liminf_{n\to\infty} \int\limits_{\Omega} \frac{(\rho_{n0})^{\gamma_n}}{\gamma_n} dx = 0$$

and we end up with the energy inequality (39).

5. Related models

We conclude this paper by mentioning two related models for the problem ($\mathbf{P_F}$), where the conditions (19)-(20) can be generalized.

5.1. *General pressure law fluid*

The free-boundary conditions (19)-(20) can be extended to include the case with a general fluid pressure, namely

$$p_F \geq p(1) \quad \text{a.e. in } \{\rho = 1\} \tag{77}$$

$$p_F = p(\rho) \quad \text{a.e. in } \{\rho < 1\} \tag{78}$$

The polymer behaviour in this case is that of a barotropic fluid in the region $\{\rho < 1\}$ and the condition (21) becomes

$$\rho(p_F - p(\rho)) = p_F - p(\rho).$$

This generalization requires only some technical changes in the energy estimates which can be treated in a similar manner.

5.2. *Congestion constraints*

Our analysis can accommodate non-homogeneous congestions constraints, i.e. a non homogeneous threshold for the pressure. In this case (19)-(20) have the form

$$p_F \geq 0 \quad \text{a.e. in } \{\rho = \rho^*(x)\} \tag{79}$$

$$p_F = 0 \quad \text{a.e. in } \{\rho < \rho^*(x)\}. \tag{80}$$

This can be achieved by introducing in the approximating system (40)-(43) an approximating pressure of the form

$$p_{F\,n} = \left(\frac{\rho_n}{\rho^*}\right)^{\gamma_n}.$$

Acknowledgments

The research of D.D. leading to these results was supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No. 642768 (Project Name: ModCompShock). K.T. gratefully acknowledges the support in part by the National Science Foundation under the grant DMS-1614964 and by the Simons Foundation under the Simons Fellows in Mathematics Award 267399. Part of this research was performed during the visit of K.T. at University of L'Aquila which was supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie Grant Agreement No. 642768 (Project Name: ModCompShock).

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