



Global Regularity of the 2D HVBK Equations

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Abstract

The Hall–Vinen–Bekharevich–Khalatnikov equations are a macroscopic model of superfluidity at nonzero temperatures. For smooth, compactly supported data, we prove the global well-posedness of strong solutions to these equations in $\mathbb{R}^2$, in the incompressible and isothermal case. The proof utilises a contraction mapping argument to establish local well-posedness for high-regularity data, following which we demonstrate global regularity using an analogue of the Beale–Kato–Majda criterion in this context. In the Appendix, we address the sufficient conditions on a 2D vorticity field, in order to have a finite kinetic energy.

Keywords HVBK equations · Superfluids · Global well-posedness · Navier–Stokes equations

1 Introduction

Most substances upon isobaric cooling transition from a gas to liquid, before eventually turning into a solid phase (or in some cases, a variety of them). Helium is an exception—at pressures below 25 bars, liquid Helium-4 transforms into a *superfluid* phase when cooled across the *lambda line* (approximately 2.17 K).¹ This superfluid phase was experimentally discovered (Kapitza 1938; Allen and Misener 1938) over 80 years ago and has since been an important subject of interest to, and investigation by, the physics community. As far as a theoretical explanation of superfluidity goes,

¹ Helium-3 also displays a superfluid phase, albeit at a significantly lower temperature (~ 2 milliKelvin) due to its fermionic nature. Most experimental research has focused on Helium-4.

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there are several pieces to the puzzle that work in limited ranges of validity; however, a single universal theory that explains everything to reasonable satisfaction continues to elude us. The most well-known models currently in use may be classified in many ways (Barenghi et al. 2014b), the one on the basis of length scales being of interest here.

One of the first theories was by Fröhlich (in 1937), proposing a model of order–disorder transition where a fraction of the helium atoms are trapped in a lattice that denotes the ground state, while the remaining atoms are in an excited state. London (1938) modified this theory by suggesting that the ground state could actually be a degenerate Bose gas. Shortly thereafter, Tisza (1938) worked on the details of London’s model, suggesting modifications to match with experimental data. Landau (1941) presented his famous two-fluid model, which would later earn him the Nobel Prize. This was a semi-classical model in which he considered the superfluid to be described by a classical velocity field, and the normal fluid to be the excitations (phonons and rotons) of this underlying superfluid field. The phonons were the excitations corresponding to linear modes (quantised sound waves), while the rotons were those of rotational modes that were identified with a local minimum in the energy–momentum diagram. He emphasised that the two fluids cannot really be compared to a (classical) multiphase flow where each point in spacetime can be *uniquely* identified with a given phase. Landau’s model made experimentally testable and qualitatively accurate predictions, particularly with regards to the critical velocity—when the superfluid is moving below the critical velocity, it is dissipation-free; if it moves faster, then it has enough energy to excite some phonons and rotons, which can dissipate energy via interactions with the walls of the container.

These works, along with the identification of the superfluid phase with a BEC, led to the increasing use of the Gross–Pitaevskii equation (GPE), a nonlinear dispersive PDE. The wavefunction that solves this equation is an order parameter which vanishes at superfluid vortices² (also known as “topological defects”) (Paoletti and Lathrop 2011). The main limitations (Roberts and Berloff 2001) of the GPE are that it is valid only for low-energy scattering of the condensate particles, and also only at absolute zero. The low-energy limit leads to a local potential (Dirac delta) and must therefore be replaced with more general, non-local potentials to describe the bosonic interaction dynamics at higher energies. At $T = 0\text{K}$, the helium is completely condensed into the ground state and is composed only of the inviscid superfluid phase. As previously mentioned, this phase has no viscosity, and at sufficiently low velocities, no excitations either. The nonlinear Schrödinger equation (NLS), as the GPE is known in mathematics literature, has been studied extensively by mathematicians for well-posedness (Colliander et al., webpage) and scattering (see Tao 2006; Dodson 2016 and references therein) results over the last few decades. However, as the system is heated to nonzero temperatures, the presence of (and interactions between) the two fluids must be accounted for. This has led to the emergence of various models (Berloff et al. 2014), each of which work well at certain characteristic length scales (micro-, meso- and macro-scales). The basic idea underlying all these models is that an interaction between the two fluids

² The existence of these vortices was predicted by Onsager (1953) and Feynman (1955) in the mid-50s and was directly observed in 2006 (Bewley et al. 2006) by pinning tracer particles to the vortices to visualise them.

would cause momentum and energy to flow between them, and the normal fluid would continuously dissipate energy through viscosity.

Micro-scale Models Micro-scale models describe the system at a quantum level (length scales $\ll$ inter-vortex distance) and are essentially modifications of the NLS. One way is to simply ignore the normal fluid altogether and represent its dissipative effect on the superfluid by imposing a high-frequency cut-off (Kobayashi and Tsubota 2005), or adding nonlinear damping terms (Antonelli and Sparber 2010), or fractional diffusion operators (Darwich and Molinet 2016; Darwich 2018). Another approach is to assume a given velocity field for the normal fluid and include its effect on the superfluid in the form of a material transport term (material derivative) of the density of the superfluid (Carlson 1996). Finally, in this hierarchy of models, the most sophisticated may be one by Pitaevskii (1959), in which a set of equations coupling (bidirectionally) the superfluid and the normal fluid were derived from first principles.

Meso-scale Models At nonzero temperatures, the superfluid contains some excitations (vortices) that can interact with the normal fluid. To describe the dynamics at length scales comparable to the inter-vortex spacing, Schwarz (1978, 1985, 1988) utilised a vortex model where he treated the superfluid to be incompressible and irrotational, except at the vortices. Each vortex is subject to a drag force from the superfluid velocity field (of the remaining vortices) and the normal fluid velocity field. Apart from this, numerical simulations of the system are designed to include *vortex reconnections*, predicted to occur in superfluids by Feynman (1955). Vortex reconnection also plays an important role in quantum turbulence (Vinen 2006; Paoletti and Lathrop 2011; Barenghi et al. 2014b), since reconnection leads to the formation of Kelvin waves along the vortices, which interact nonlinearly with each other resulting in dissipation through phonon emission.

Macro-scale Models When the length scales of interest are much larger than the inter-vortex spacing, the superfluid can be modelled like a classical inviscid fluid. This prompted an effort to express the manifestation of quantum effects on large scales, discarding the discreteness of the vortex filaments for a continuum viewpoint, thus giving way to the Hall–Vinen–Bekharevich–Khalatnikov (HVBK) model (Barenghi et al. 2001). In this model, the viscous normal fluid is described using the Navier–Stokes equations, while the inviscid superfluid is represented by the Euler equations. This model is known to work well in situations when the vortex filaments are more or less aligned with each other and is thus used to study superfluid Couette flow in rotationally symmetric domains (Barenghi and Jones 1988; Henderson and Barenghi 2004; Peralta et al. 2009). In this paper, we prove the global regularity of strong solutions to the HVBK equations in two dimensions.

Although the original HVBK model treated the superfluid as inviscid, this leads to significant difficulties during numerical simulations since there would not be small-scale dissipation, forcing one to resolve extremely high frequencies. There have been multiple ways reported in the literature to get around this: suppressing high frequencies in the superfluid velocity (Salort et al. 2011; Barenghi et al. 2014a), using subgrid models (Tchoufag and Sagaut 2010) similar to the ones from large Eddy simulation

(LES) and adding an artificial superfluid viscosity (Roche et al. 2009). In this work, we follow the last method, i.e. by treating the superfluid as viscous. We will now state and describe the model in consideration.

1.1 Notation

We use the subscript x on a Banach space to denote the Banach space is defined over $\mathbb{R}^2$. For instance, L_x^p stands for Lebesgue space $L^p(\mathbb{R}^2)$, and similarly for the Sobolev spaces H_x^m . In the same way, spaces over time are denoted with a subscript t , like in L_t^p . Here, the time interval is understood from context and will be explicitly specified otherwise. We also use subscripts on integrals to denote the variable being integrated; for example, $\int_x$ would stand for $\int_{\mathbb{R}^2}$, $\int_t$ for $\int_0^T$ and so on.

Since we are in 2D, the curl operator (acting on a 2D field) is often replaced by the *perpendicular gradient* operator, denoted by:

$$\nabla^\perp := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \nabla$$

We also use the notation $X \lesssim Y$ to imply that there exists a positive constant C such that $X \leq CY$. The dependence of the constant on various parameters (including the initial data) will be denoted using a subscript as $X \lesssim_{k_1, k_2} Y$ or $X \leq C_{k_1, k_2} Y$.

Finally, the subscript i accompanying velocity/vorticity fields (u_i, ω_i) , density (ρ_i) or vorticity (v_i) can take the values $\{n, s\}$ and is used to represent the quantities corresponding to the normal fluid and the superfluid respectively.

1.2 Organisation of the Paper

In Sect. 2, we present and discuss the mathematical model, along with statements of the main results. Sect. 3 contains some apriori estimates, followed by the proof of local well-posedness and pressure regularity in Sect. 4. The global well-posedness for high-regularity data, including the analogue of the Beale–Kato–Majda condition, constitutes Sect. 5. Finally, we consider global regularity for smooth initial data in Sect. 6. The Appendix contains a simple lemma, showing weaker and sufficient conditions on the vorticity in order to have a finite kinetic energy in 2D.

2 Mathematical Model and Main Results

We first consider the general form of the incompressible HVBK equations as given in section 3.3 of Barenghi (2001). Eq. (HVBK) are repeated below for convenience (with slightly altered notation).

$$\begin{aligned}
\nabla \cdot u_n &= 0 \quad , \quad \nabla \cdot u_s = 0 \\
\partial_t u_n + (u_n \cdot \nabla) u_n &= -\frac{1}{\rho} \nabla p_n - \frac{\rho_s}{\rho_n} S \nabla \Theta + v_n \Delta u_n + \frac{\rho_s}{\rho} F \\
\partial_t u_s + (u_s \cdot \nabla) u_s &= -\frac{1}{\rho} \nabla p_s + S \nabla \Theta + T - \frac{\rho_n}{\rho} F \\
\omega_s &= \nabla \times u_s \quad , \quad T = -\lambda_s \omega_s \times (\nabla \times \hat{\omega}_s) \\
\lambda_s &= \frac{\Gamma}{4\pi} \ln \left(\frac{b_0}{a_0} \right) \quad , \quad b_0 = \left(\frac{\Gamma}{2|\omega_s|} \right)^{\frac{1}{2}} \\
F &= \frac{B}{2} \hat{\omega}_s \times [\omega_s \times (u_n - u_s) + T] + \frac{B'}{2} [\omega_s \times (u_n - u_s) + T]
\end{aligned} \tag{HVBK}$$

In the above equations, subscripts n and s are used to denote the normal and superfluid components. u_n and u_s are the fluid velocities, ω_s is the superfluid vorticity, $\hat{\omega}_s$ is the unit vector along the vortex filament, ρ_n and ρ_s are their (constant) densities, $\rho = \rho_n + \rho_s$ is the total density, p is the pressure, S is the entropy, Θ is the temperature, v_n is the normal fluid viscosity, T is the vortex tension term (force associated with bent vortex filaments), F is the “mutual friction” (the coupling between the two fluids), Γ is the quantum of circulation in a superfluid vortex filament, B and B' are temperature-dependent positive friction coefficients, and a_0 is a positive constant that denotes the vortex core radius.

Now, we consider the isothermal case in 2D: the temperature gradient terms drop out from both equations; and the direction of the vorticity is fixed, so vortex lines do not bend, i.e. the vortex tension term is not relevant. As mentioned before, we will also be including a superfluid viscosity. Finally, we absorb ρ in the definition of pressure and arrive at the following form of the equations.

$$\nabla \cdot u_n = 0 \quad , \quad \nabla \cdot u_s = 0 \tag{1}$$

$$\partial_t u_n + (u_n \cdot \nabla) u_n = -\nabla p_n + v_n \Delta u_n + \frac{\rho_s}{\rho} F \tag{2}$$

$$\partial_t u_s + (u_s \cdot \nabla) u_s = -\nabla p_s + v_s \Delta u_s - \frac{\rho_n}{\rho} F \tag{3}$$

$$F = -\frac{B}{2} |\omega_s| (u_n - u_s) + \frac{B'}{2} \omega_s \times (u_n - u_s) \tag{4}$$

This is the system that will be dealt with in this paper and is very similar to those considered in Roche et al. (2009), Tchoufag and Sagaut (2010) and Verma et al. (2019). Note that the first term in the mutual friction indeed acts like a (nonlinear) drag and works to equalise the two velocity fields. The second term is a transport in the direction perpendicular to the relative velocity of the two fluids, so it is not a retarding force in itself. However, that is not concerning since it will be shown that this second term vanishes in the energy estimates, and behaves the same way as the first term as far as the contraction mapping argument goes. Furthermore, it is also evident that

multiplying the normal fluid equation by ρ_n and the superfluid equation by ρ_s , and adding the resulting equations leads to a cancellation of the mutual friction term. This is not surprising, as the friction force is internal to the entire system (of normal fluid and superfluid) and cannot alter the total momentum.

We seek mild solutions to (1)–(4), expressed formally as:

$$u_n(t) = e^{v_n t \Delta} u_n^0 - \int_0^t e^{v_n(t-\tau)\Delta} \mathcal{P} \left[u_n \cdot \nabla u_n - \frac{\rho_s}{\rho} F \right] d\tau \quad (5)$$

$$u_s(t) = e^{v_s t \Delta} u_s^0 - \int_0^t e^{v_s(t-\tau)\Delta} \mathcal{P} \left[u_s \cdot \nabla u_s + \frac{\rho_n}{\rho} F \right] d\tau \quad (6)$$

Here, $\mathcal{P}$ is the Leray projector that gives as output the divergence-free part of a vector field; it is used to eliminate the pressure term.

It is a classical result that the incompressible Navier–Stokes in 2D is globally well-posed. Since the current model has some added retardation effect (with the mutual friction), one can argue heuristically that this cannot hurt the extension of global regularity to this nonlinearly coupled two-fluid system (see remarks following the main results). This is the physical intuition behind the main results stated below.

Theorem 2.1 (Global well-posedness for high-regularity data) *Let $m > \frac{d}{2} + 1 = 2$, and $u_n^0, u_s^0 \in H^m(\mathbb{R}^2)$ be a pair of divergence-free initial velocity fields. Then,*

- (i) *there exist a unique pair of mild solutions $u_n, u_s \in C([0, \infty[; H^m(\mathbb{R}^2)) \cap L^2([0, \infty[; \dot{H}^{m+1}(\mathbb{R}^2))$ to the incompressible, isothermal 2D HVBK equations described by (1)–(4); and*
- (ii) *the solution map is Lipschitz continuous from $H^m(\mathbb{R}^2) \mapsto C([0, \infty[; H^m(\mathbb{R}^2)) \cap L^2([0, \infty[; \dot{H}^{m+1}(\mathbb{R}^2))$.*
- (iii) *In addition, if the Sobolev index is upgraded to $m \geq 4$, then there also exist a unique pair of pressure fields $p_n, p_s \in C([0, \infty[; H^m(\mathbb{R}^2))$;*
- (iv) *The fields are instantaneously smoothed (due to viscosity), so that $u_n, u_s, p_n, p_s \in C^\infty([0, \infty[; H^\infty(\mathbb{R}^2))$.*

The proof of Theorem 2.1 will utilise a contraction mapping argument for local well-posedness. A Beale–Kato–Majda (Beale et al. 1984) analogue will be used to show global well-posedness, and elliptic regularity applied to the pressure–Poisson equation for the pressure field estimates. Once global well-posedness is established for high-regularity data, we can easily extend the result to the following corollary.

Corollary 2.2 (Global well-posedness for smooth, compactly supported data) *Given a pair of divergence-free velocity fields $u_n^0, u_s^0 \in C_c^\infty(\mathbb{R}^2)$, there exist a unique pair of smooth (classical) solutions $u_n, u_s \in C^\infty([0, \infty[; H^\infty(\mathbb{R}^2))$ to the incompressible, isothermal 2D HVBK equations described by (1)–(4). Moreover, the pressure fields are also smooth and unique, i.e. $p_n, p_s \in C^\infty([0, \infty[; H^\infty(\mathbb{R}^2))$.*

At this stage, some remarks are in order, particularly about the mutual friction term and the difficulties it poses:

1. Superfluid He-4 is completely inviscid (and dissipation-free) below its critical velocity. As mentioned in Sect. 1, we consider a viscous fluid to help with the analysis. The inviscid-superfluid version of the HVBK equations has not been studied, either numerically or analytically, to the best of the authors' knowledge.
2. The physical intuition that the addition of a dissipative mutual friction to the 2D equations would not affect their global regularity need not be valid when the superfluid is treated as inviscid. In fact, it is not known whether the system is even locally well-posed in this case. Note that while the advective nonlinearity preserves the L_x^2 norm of the vorticity, the same is not true of the nonlinear mutual friction. This means that the vorticity may not be bounded in $L_t^\infty L_x^2$, possibly leading to a breakdown of the system. This is still an open, and interesting, problem.
3. The ability of the advective term to be written as the gradient of a symmetric tensor $[\nabla \cdot (u \otimes u)]$, also known as the *conservation form*, enables easier manipulation of the nonlinearity. For instance, one of the most useful properties of high-regularity Sobolev spaces $(H_x^m, m > \frac{d}{2})$ is that they form an algebra so that $\|uv\|_{H_x^m} \lesssim \|u\|_{H_x^m} \|v\|_{H_x^m}$. This fact when applied to the advective term in the conservation form gives us the same contribution from each velocity in the nonlinearity, as opposed to one being the norm of the velocity and the other a norm of the velocity gradient. This is particularly useful when deriving the $L_t^\infty L_x^\infty$ controlling norm in the analysis of Navier–Stokes (used to show that the existence time for $C_t^0 H_x^m$ solutions is independent of the index m). The same goes for heat kernel estimates also, where the conservation form allows for the gradient to be accounted for by the heat kernel instead of the velocity fields (albeit at the cost of temporally singular terms). These steps/methods are no longer as effective when it comes to the mutual friction, since it cannot be reduced to conservation form. There is a clear demarcation, heuristically, of the velocity and the velocity gradient (or vorticity) factors in this term. This asymmetry in the nonlinearity, along with the lopsided presence of only the superfluid vorticity, induces difficulties in the treatment of the mutual friction term. This would be especially true when considering the inviscid superfluid—the lack of a conservation form of mutual friction means one cannot get around the analysis of the gradient of the velocity (more precisely, vorticity), a luxury that is not afforded when lacking viscous diffusion.
4. Another problem associated with the mutual friction term is related to the estimates of the pressure. When seeking mild solutions, we eliminate the pressure gradient term from the Navier–Stokes using the Leray projector. This gives us an evolution equation for the velocity field alone. Once a mild solution is established, we can get some estimates on the pressure field by taking the divergence of the Navier–Stokes and using incompressibility. This gives us the *pressure-Poisson equation*.

$$-\Delta p = \nabla \cdot ((u \cdot \nabla)u) = \nabla \cdot \nabla \cdot (u \otimes u) \quad (7)$$

This means $p = (-\Delta)^{-1} \nabla \cdot \nabla \cdot (u \otimes u)$. Note how the derivatives roughly “cancel” out on the RHS. Thus, p is expected to have a regularity similar to $u \otimes u$. This is indeed true (at least in one direction)—since the kernel that maps $u \otimes u$ to p is rotationally symmetric and decays as $|x|^{-d}$, we can utilise the Calderon–

Zygmund inequality to conclude $\|p\|_{L_x^q} \lesssim \|u \otimes u\|_{L_x^q} \leq \|u\|_{L_x^{2q}}^2$ for $1 < q < \infty$. This extension to almost all Lebesgue spaces would not be possible without the conservation form of the advective term. With mutual friction added, there are extra terms on the RHS of (7). Clearly, these terms do not have a conservation form; as a result, they do not admit such a simple estimate for the pressure. There is still some respite though: when we have smooth solutions (as in Corollary 2.2). In this case, the pressure field is also smooth (Sect. 6). Also, as stated in Theorem 2.1, if the assumption on the Sobolev index m is upgraded to $m \geq 4$, then we have unique $p_n, p_s \in C_t^0 H_x^m$. The reason for this upgrade is again due to the non-conservation form of the mutual friction, which means the RHS in the pressure-Poisson equation contains some terms involving the gradient of the vorticity (equivalently, two derivatives of the velocity).

3 Apriori Estimates

In this section, we will derive some apriori estimates for the kinetic energy and enstrophy. Note that these derivations are formal and require $u_i, \omega_i \in L_{[0, T_*]}^\infty L_x^2 \cap L_{[0, T_*]}^2 \dot{H}_x^1$, where $i \in \{n, s\}$ and T_* is the maximal existence time defined in Sect. 4.2. These regularity conditions are indeed satisfied by the mild solutions.

3.1 Energy Estimate

Multiplying the normal fluid equation (2) by $\rho_n u_n$, the superfluid equation (3) by $\rho_s u_s$, adding the resulting equations, and integrating over $\mathbb{R}^2$, we obtain (after integrating by parts with vanishing fields at infinity, and using the incompressibility conditions (1)):

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \rho_n \|u_n\|_{L_x^2}^2 + \frac{1}{2} \rho_s \|u_s\|_{L_x^2}^2 \right) + \rho_n v_n \|\nabla u_n\|_{L_x^2}^2 + \rho_s v_s \|\nabla u_s\|_{L_x^2}^2 \\ & + \frac{\rho_n \rho_s}{\rho} \frac{B}{2} \int_{\mathbb{R}^2} |\omega_s| |u_n - u_s|^2 = 0 \end{aligned} \quad (8)$$

Dropping the last (non-negative) term, and integrating over time, we get:

$$\begin{aligned} & \left(\frac{1}{2} \rho_n \|u_n\|_{L_x^2}^2 + \frac{1}{2} \rho_s \|u_s\|_{L_x^2}^2 \right) (t) + \rho_n v_n \|\nabla u_n\|_{L_{[0,t]}^2 L_x^2}^2 + \rho_s v_s \|\nabla u_s\|_{L_{[0,t]}^2 L_x^2}^2 \\ & \leq \frac{1}{2} \rho_n \|u_n^0\|_{L_x^2}^2 + \frac{1}{2} \rho_s \|u_s^0\|_{L_x^2}^2 \equiv E^0 \end{aligned} \quad (9)$$

where $0 \leq t < T_*$ and E^0 is the initial energy.

Thus, we see that the energy at any time (and dissipation upto any time) is bounded above by the initial energy. Recall that we will choose initial data satisfying the assumptions of Lemma A.1, so that the kinetic energy is finite at $t = 0$.

3.2 Vorticity Equations

Operating on (2) and (3) with the curl operator, we get the evolution equations for the scalar vorticity fields (which are also incompressible).

$$\partial_t \omega_n + (u_n \cdot \nabla) \omega_n = v_n \Delta \omega_n + \frac{\rho_s}{\rho} \mathcal{T} \quad (10)$$

$$\partial_t \omega_s + (u_s \cdot \nabla) \omega_s = v_s \Delta \omega_s - \frac{\rho_n}{\rho} \mathcal{T} \quad (11)$$

$$\mathcal{T} = -\frac{B}{2} \nabla \times [|\omega_s|(u_n - u_s)] + \frac{B'}{2} (u_n - u_s) \cdot \nabla \omega_s \quad (12)$$

3.3 Enstrophy Estimate

Just as in section 8, we multiply (10) by $\rho_n \omega_n$, (11) by $\rho_s \omega_s$, add the resulting equations, and integrate over $\mathbb{R}^2$ to arrive at:

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \rho_n \|\omega_n\|_{L_x^2}^2 + \frac{1}{2} \rho_s \|\omega_s\|_{L_x^2}^2 \right) + \rho_n v_n \|\nabla \omega_n\|_{L_x^2}^2 + \rho_s v_s \|\nabla \omega_s\|_{L_x^2}^2 \\ & + \frac{\rho_n \rho_s}{\rho} \frac{B}{2} \int_{\mathbb{R}^2} |\omega_s| |\omega_n - \omega_s|^2 \\ & = \frac{\rho_n \rho_s}{2\rho} \int_{\mathbb{R}^2} (\omega_n - \omega_s)(u_n - u_s) \cdot (B' \nabla \omega_s - B \nabla^\perp |\omega_s|) \end{aligned} \quad (13)$$

Dropping the last term (non-negative) on the LHS, and using Hölder's inequality,

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \rho_n \|\omega_n\|_{L_x^2}^2 + \frac{1}{2} \rho_s \|\omega_s\|_{L_x^2}^2 \right) + \rho_n v_n \|\nabla \omega_n\|_{L_x^2}^2 + \rho_s v_s \|\nabla \omega_s\|_{L_x^2}^2 \\ & \leq \frac{\rho_n \rho_s}{2\rho} (B + B') \|\omega_n - \omega_s\|_{L_x^4} \|u_n - u_s\|_{L_x^4} \|\nabla \omega_s\|_{L_x^2} \end{aligned}$$

We use Cauchy's inequality $(ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon})$ to extract the $\|\nabla \omega_s\|_{L_x^2}$ term from the RHS and absorb into the LHS. We also use the two-dimensional Ladyzhenskaya inequality $(\|u\|_{L_x^4}^2 \lesssim \|u\|_{L_x^2} \|\nabla u\|_{L_x^2})$ for handling the other terms on the RHS. This leaves us with:

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \rho_n \|\omega_n\|_{L_x^2}^2 + \frac{1}{2} \rho_s \|\omega_s\|_{L_x^2}^2 \right) + \rho_n v_n \|\nabla \omega_n\|_{L_x^2}^2 + \frac{\rho_s v_s}{2} \|\nabla \omega_s\|_{L_x^2}^2 \\ & \lesssim_{\rho_i, v_i, B, B'} \|u_n - u_s\|_{L_x^2} \|\nabla u_n - \nabla u_s\|_{L_x^2} \|\omega_n - \omega_s\|_{L_x^2} \|\nabla \omega_n - \nabla \omega_s\|_{L_x^2} \end{aligned}$$

Using the $L_t^\infty L_x^2$ bound on u_n and u_s from (9), and Cauchy's inequality again, the RHS can be controlled as follows.

$$\text{RHS} \leq C_{\rho_i, v_i} E^0 \|\omega_n - \omega_s\|_{L_x^2}^2 \|\nabla u_n - \nabla u_s\|_{L_x^2}^2 + \frac{\rho_n v_n}{2} \|\nabla \omega_n\|_{L_x^2}^2 + \frac{\rho_s v_n}{4} \|\nabla \omega_s\|_{L_x^2}^2$$

The second and third terms on the RHS can be absorbed into the corresponding terms on the LHS.

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\rho_n}{2} \|\omega_n\|_{L_x^2}^2 + \frac{\rho_s}{2} \|\omega_s\|_{L_x^2}^2 \right) + \frac{\rho_n \nu_n}{2} \|\nabla \omega_n\|_{L_x^2}^2 + \frac{\rho_s \nu_s}{4} \|\nabla \omega_s\|_{L_x^2}^2 \\ & \lesssim_{\rho_i, \nu_i} E^0 \|\omega_n - \omega_s\|_{L_x^2}^2 \|\nabla u_n - \nabla u_s\|_{L_x^2}^2 \\ & \lesssim_{\rho_i, \nu_i} \left(\frac{\rho_n}{2} \|\omega_n\|_{L_x^2}^2 + \frac{\rho_s}{2} \|\omega_s\|_{L_x^2}^2 \right) E^0 \left[\|\nabla u_n\|_{L_x^2}^2 + \|\nabla u_s\|_{L_x^2}^2 \right] \end{aligned} \quad (14)$$

We then drop non-negative terms on the LHS and use Grönwall's inequality to arrive at:

$$\frac{\rho_n}{2} \|\omega_n\|_{L_x^2}^2(t) + \frac{\rho_s}{2} \|\omega_s\|_{L_x^2}^2(t) \leq \Omega^0 e^{C E^0 \left[\|\nabla u_n\|_{L_{[0,t]}^2 L_x^2}^2 + \|\nabla u_s\|_{L_{[0,t]}^2 L_x^2}^2 \right]} \leq \Omega^0 e^{C(E^0)^2} \quad (15)$$

where $\Omega^0 = \frac{\rho_n}{2} \|\omega_n^0\|_{L_x^2}^2 + \frac{\rho_s}{2} \|\omega_s^0\|_{L_x^2}^2$ is the initial enstrophy, and the constant C in the exponent depends only on the densities, the viscosities, and the coupling constants (B, B') . The second inequality in (15) is a result of the bound in (9), and thus, we observe that ω_n and ω_s are bounded in $L_t^\infty L_x^2$. Substituting this upper bound on the vorticities back into (14), we integrate over time to calculate upper bounds for $\|\nabla \omega_n\|_{L_{[0,T_*]}^2 L_x^2}$ and $\|\nabla \omega_s\|_{L_{[0,T_*]}^2 L_x^2}$ as well. In summary, for $i \in \{n, s\}$,

$$\|\omega_i\|_{L_{[0,T_*]}^\infty L_x^2}, \|\nabla \omega_i\|_{L_{[0,T_*]}^2 L_x^2} \leq C_{\Omega^0, E^0, \rho_i, \nu_i, B, B'} \quad (16)$$

The apriori bounds in (9) and (16) will be repeatedly used in the proof of global well-posedness (Sect. 5).

4 Local Well-Posedness

In this section, we will establish the local well-posedness of the system (1)–(4) for high-regularity data. We will use the properties of the heat kernel to set up a contraction mapping that allows us to show the existence of a unique (mild) solution. Since we have a two-fluid system, the contraction map will be “alternating”, in the sense that for a given u_s , we contract the normal fluid map, and for a given u_n , we contract the superfluid map.

The appropriate space for the contraction is denoted by X_i , and the corresponding norm is defined as:

$$\|u_i\|_{X_i} := \|u_i\|_{C_t^0 H_x^m} + \nu_i^{\frac{1}{2}} \|\nabla u_i\|_{L_t^2 H_x^m} \quad (17)$$

where $i \in \{n, s\}$. In what follows, the spacetime subscripts denote the domains $t \in [0, T[$ (where T is yet to be determined) and $x \in \mathbb{R}^2$.

4.1 Mild Solution and Bound on the Norm

Starting with divergence-free initial data (in accordance with Lemma A.1) $u_n^0, u_s^0 \in H^m(\mathbb{R}^2)$ for $m > \frac{d}{2} + 1 = 2$, we write the required contraction maps in the form of mild solutions corresponding to (2) and (3).

$$\Phi(t) = e^{v_n t \Delta} u_n^0 - \int_0^t e^{v_n(t-\tau)\Delta} \mathcal{P} [u_n \cdot \nabla u_n + \beta_n |\omega_s| (u_n - u_s) - \beta'_n \omega_s \times (u_n - u_s)] \, d\tau \quad (18)$$

$$\Psi(t) = e^{v_s t \Delta} u_s^0 - \int_0^t e^{v_s(t-\tau)\Delta} \mathcal{P} [u_s \cdot \nabla u_s - \beta_s |\omega_s| (u_n - u_s) + \beta'_s \omega_s \times (u_n - u_s)] \, d\tau \quad (19)$$

Here, $u_n, u_s \in H^m(\mathbb{R}^2)$ as well, and the (positive) constants β 's are defined below.³

$$\begin{aligned} \beta_n &= \frac{\rho_s}{\rho} \frac{B}{2}, & \beta'_n &= \frac{\rho_s}{\rho} \frac{B'}{2} \\ \beta_s &= \frac{\rho_n}{\rho} \frac{B}{2}, & \beta'_s &= \frac{\rho_n}{\rho} \frac{B'}{2} \end{aligned}$$

From (18), we have almost everywhere (a.e.) in time:

$$\frac{\partial \Phi}{\partial t} = v_n \Delta \Phi + F_n$$

where $F_n = -\mathcal{P} [u_n \cdot \nabla u_n + \beta_n |\omega_s| (u_n - u_s) - \beta'_n \omega_s \times (u_n - u_s)]$.

For each multi-index α with $0 \leq |\alpha| \leq m$, we can derive a higher-order energy inequality for the contraction maps in (18) and (19).

$$\frac{\partial}{\partial t} D^\alpha \Phi = v_n \Delta D^\alpha \Phi + D^\alpha F_n$$

Multiplying the above equation by $D^\alpha \Phi$, integrating over $\mathbb{R}^2$, using the Holder's inequality, and summing over all $0 \leq |\alpha| \leq m$ leads to:

$$\frac{d}{dt} \|\Phi\|_{H_x^m}^2 + v_n \|\nabla \Phi\|_{H_x^m}^2 \leq \|\Phi\|_{H_x^m} \|F_n\|_{H_x^m}$$

Integrating over time, from 0 to T , and using the fact that $\Phi(0) = u_n^0$:

$$\|\Phi\|_{C_t^0 H_x^m}^2 + v_n \|\nabla \Phi\|_{L_t^2 H_x^m}^2 \leq \|u_n^0\|_{H_x^m}^2 + \|\Phi\|_{C_t^0 H_x^m} \|F_n\|_{L_t^1 H_x^m}$$

³ As was pointed out by one of the referees, the subscripts in the definition of the β 's may seem counter-intuitive at first glance. This choice is dictated by the "swapping" of the densities in the mutual friction term. This, in turn, is to ensure that the coupling conserves overall momentum of both fluids.

Finally, we use the inequality $ab \leq \frac{a^2+b^2}{2}$ to simplify the second term on the RHS and obtain:

$$\|\Phi\|_{X_n} \lesssim \|u_n^0\|_{H_x^m} + \|F_n\|_{L_t^1 H_x^m} \quad (20)$$

$$\|\Psi\|_{X_s} \lesssim \|u_s^0\|_{H_x^m} + \|F_s\|_{L_t^1 H_x^m} \quad (21)$$

Equation (21) is derived in the same way as (20).

4.2 The Contraction

Since Sobolev spaces with $m > \frac{d}{2}$ form an algebra, i.e. $\|uv\|_{H_x^m} \lesssim \|u\|_{H_x^m} \|v\|_{H_x^m}$,

$$\|F_n\|_{H_x^m} \lesssim \|u_n\|_{H_x^m} \|\nabla u_n\|_{H_x^m} + (\beta_n + \beta'_n) \|\omega_s\|_{H_x^m} \|u_n - u_s\|_{H_x^m}$$

From the Calderon–Zygmund inequality, we have $\|\omega_s\|_{H_x^m} \lesssim \|\nabla u_s\|_{H_x^m}$. Integrating over the time interval $[0, T_n]$,

$$\begin{aligned} \|F_n\|_{L_t^1 H_x^m} &\lesssim \|u_n\|_{C_t^0 H_x^m} \|\nabla u_n\|_{L_t^2 H_x^m} T_n^{\frac{1}{2}} + (\beta_n + \beta'_n) \|\nabla u_s\|_{L_t^2 H_x^m} \|u_n - u_s\|_{C_t^0 H_x^m} T_n^{\frac{1}{2}} \\ &\lesssim v_n^{-\frac{1}{2}} T_n^{\frac{1}{2}} \|u_n\|_{X_n}^2 + (\beta_n + \beta'_n) v_s^{-\frac{1}{2}} T_n^{\frac{1}{2}} \|u_s\|_{X_s} (\|u_n\|_{X_n} + \|u_s\|_{X_s}) \end{aligned} \quad (22)$$

From (22) and (20), and repeating the same procedure for Ψ , we obtain:

$$\begin{aligned} \|\Phi\|_{X_n} &\lesssim \|u_n^0\|_{H_x^m} + T_n^{\frac{1}{2}} \left[v_n^{-\frac{1}{2}} \|u_n\|_{X_n}^2 + v_s^{-\frac{1}{2}} (\beta_n + \beta'_n) \|u_s\|_{X_s} (\|u_n\|_{X_n} + \|u_s\|_{X_s}) \right] \\ \|\Psi\|_{X_s} &\lesssim \|u_s^0\|_{H_x^m} + T_s^{\frac{1}{2}} \left[v_s^{-\frac{1}{2}} \|u_s\|_{X_s}^2 + v_s^{-\frac{1}{2}} (\beta_s + \beta'_s) \|u_s\|_{X_s} (\|u_n\|_{X_n} + \|u_s\|_{X_s}) \right] \end{aligned}$$

Let $B_X(R)$ denote a ball (centred at the origin) of radius R in the space X . Consider $u_n \in B_{X_n}(R_n)$ and $u_s \in B_{X_s}(R_s)$. Thus, the above equations therefore simplify to:

$$\|\Phi\|_{X_n} \lesssim \|u_n^0\|_{H_x^m} + T_n^{\frac{1}{2}} \left[v_n^{-\frac{1}{2}} R_n^2 + v_s^{-\frac{1}{2}} (\beta_n + \beta'_n) R_s (R_n + R_s) \right] \quad (23)$$

$$\|\Psi\|_{X_s} \lesssim \|u_s^0\|_{H_x^m} + T_s^{\frac{1}{2}} \left[v_s^{-\frac{1}{2}} R_s^2 + v_s^{-\frac{1}{2}} (\beta_s + \beta'_s) R_s (R_n + R_s) \right] \quad (24)$$

Now, we will fix the superfluid velocity field $u_s \in B_{X_s}(R_s)$ and the initial normal fluid velocity u_n^0 , and consider two different normal fluid velocity fields $u_n^1, u_n^2 \in B_{X_n}(R_n)$. Then, the difference of the Φ map for the two different velocities is:

$$\Phi(u_n^1) - \Phi(u_n^2) = \int_0^t e^{v_n(t-\tau)\Delta} (F_n(u_n^1) - F_n(u_n^2)) d\tau$$

Using the estimate from (20),

$$\begin{aligned} \Rightarrow \|\Phi(u_n^1) - \Phi(u_n^2)\|_{X_n} &\lesssim \|F_n^1 - F_n^2\|_{L_t^1 H_x^m} \\ &\lesssim \|u_n^1 \cdot \nabla u_n^1 - u_n^2 \cdot \nabla u_n^2 - \beta_n |\omega_s| (u_n^1 - u_n^2) + \beta'_n \omega_s \times (u_n^1 - u_n^2)\|_{L_t^1 H_x^m} \\ &\lesssim T_n^{\frac{1}{2}} \left[v_n^{-\frac{1}{2}} (\|u_n^1\|_{X_n} + \|u_n^2\|_{X_n}) + (\beta_n + \beta'_n) v_s^{-\frac{1}{2}} \|u_s\|_{X_s} \right] \|u_n^1 - u_n^2\|_{X_n} \end{aligned}$$

A similar procedure can be repeated for the Ψ map, starting with the initial data u_s^0 , fixing the normal fluid velocity field $u_n \in B_{X_n}(R_n)$, and considering two different superfluid velocity⁴ fields $u_s^1, u_s^2 \in B_{X_s}(R_s)$. Thus,

$$\|\Phi(u_n^1) - \Phi(u_n^2)\|_{X_n} \lesssim T_n^{\frac{1}{2}} \left[v_n^{-\frac{1}{2}} R_n + (\beta_n + \beta'_n) v_s^{-\frac{1}{2}} R_s \right] \|u_n^1 - u_n^2\|_{X_n} \quad (25)$$

$$\begin{aligned} \|\Psi(u_s^1) - \Psi(u_s^2)\|_{X_s} &\lesssim T_s^{\frac{1}{2}} \left[v_s^{-\frac{1}{2}} (1 + \beta_s + \beta'_s) R_s + (\beta_s + \beta'_s) v_s^{-\frac{1}{2}} R_n \right] \\ &\quad \|u_s^1 - u_s^2\|_{X_n} \end{aligned} \quad (26)$$

From (23), (24), (25) and (26), we see that if the following (sufficient) conditions are satisfied, the maps Φ and Ψ will be contractions. (c_1, c_2, c_3 and c_4 are positive constants that were suppressed in all the inequalities so far.)

$$c_1 \left[\|u_n^0\|_{H_x^m} + T_n^{\frac{1}{2}} \left(v_n^{-\frac{1}{2}} R_n^2 + v_s^{-\frac{1}{2}} (\beta_n + \beta'_n) R_s (R_n + R_s) \right) \right] \leq R_n \quad (27)$$

$$c_2 \left[T_n^{\frac{1}{2}} \left(v_n^{-\frac{1}{2}} R_n + (\beta_n + \beta'_n) v_s^{-\frac{1}{2}} R_s \right) \right] < 1 \quad (28)$$

$$c_3 \left[\|u_s^0\|_{H_x^m} + T_s^{\frac{1}{2}} \left(v_s^{-\frac{1}{2}} R_s^2 + v_s^{-\frac{1}{2}} (\beta_s + \beta'_s) R_s (R_n + R_s) \right) \right] \leq R_s \quad (29)$$

$$c_4 \left[T_s^{\frac{1}{2}} \left(v_s^{-\frac{1}{2}} (1 + \beta_s + \beta'_s) R_s + (\beta_s + \beta'_s) v_s^{-\frac{1}{2}} R_n \right) \right] < 1 \quad (30)$$

Define $\delta = 1 + \beta_n + \beta'_n + \beta_s + \beta'_s$ and $v^{-\frac{1}{2}} = v_n^{-\frac{1}{2}} + v_s^{-\frac{1}{2}}$. Then, the conditions in (27)–(30) are automatically satisfied if the following more conservative inequalities are obeyed (for some sufficiently large $N_n, N_s \in \mathbb{N}$).

$$c_1 \delta \left[\|u_n^0\|_{H_x^m} + T_n^{\frac{1}{2}} v^{-\frac{1}{2}} (R_n + R_s)^2 \right] \leq R_n \quad (31)$$

$$c_2 \delta T_n^{\frac{1}{2}} v^{-\frac{1}{2}} (R_n + R_s) \leq \frac{1}{N_n} \quad (32)$$

⁴ Note that in this case, ω_s is dependent on which of the superfluid velocity fields u_s^1 or u_s^2 is in consideration in that term.

$$c_3 \delta \left[\|u_s^0\|_{H_x^m} + T_s^{\frac{1}{2}} \nu^{-\frac{1}{2}} (R_n + R_s)^2 \right] \leq R_s \quad (33)$$

$$c_4 \delta T_s^{\frac{1}{2}} \nu^{-\frac{1}{2}} (R_n + R_s) \leq \frac{1}{N_s} \quad (34)$$

Upper bounds for T_n and T_s from (32) and (34) are substituted in (31) and (33) to give:

$$c_1 \delta \|u_n^0\|_{H_x^m} + c_3 \delta \|u_s^0\|_{H_x^m} + \left(\frac{c_1}{N_n c_2} + \frac{c_3}{N_s c_4} \right) (R_n + R_s) \leq (R_n + R_s)$$

This is easily satisfied by choosing $R_n = 2c_1 \delta \|u_n^0\|_{H_x^m}$ and $R_s = 2c_3 \delta \|u_s^0\|_{H_x^m}$, and sufficiently large N_n and N_s . This choice also leads to the conclusion that:

$$T_n \leq \frac{\nu}{(2c_2 \delta^2 N_n)^2} \frac{1}{(c_1 \|u_n^0\|_{H_x^m} + c_3 \|u_s^0\|_{H_x^m})^2} \quad (35)$$

$$T_s \leq \frac{\nu}{(2c_4 \delta^2 N_s)^2} \frac{1}{(c_1 \|u_n^0\|_{H_x^m} + c_3 \|u_s^0\|_{H_x^m})^2} \quad (36)$$

From (32)–(34), we observe that the upper bounds on the existence times are inversely proportional to $R_n + R_s$. This allows us to use a bootstrapping argument to establish that there exists a *unique maximal time* $T_* \in (0, \infty]$ such that $\|u_n\|_{X_n}(t) \|u_s\|_{X_s}(t) \rightarrow \infty$ as $t \rightarrow T_*$. In other words, at least one of the two velocity fields blows up in the respective X_i norm as the maximal time is approached.

4.3 Lipschitz Continuous Dependence on Initial Data

So far, we have shown the local existence and uniqueness of (mild) solutions. We will now establish Lipschitz continuous dependence of the solutions on initial data, to complete the well-posedness proof. Consider two initial data u_n^0, v_n^0 with corresponding mild solutions u_n, v_n (for a fixed u_s). Then,

$$\begin{aligned} \|u_n - v_n\|_{X_n} &\lesssim \|u_n^0 - v_n^0\|_{H_x^m} + \|u_n \cdot \nabla u_n - v_n \cdot \nabla v_n - \beta_n |\omega_s| (u_n - v_n) \\ &\quad + \beta_n' \omega_s \times (u_n - v_n)\|_{L_t^1 H_x^m} \end{aligned}$$

Just as done in Sect. 4.2, we can show:

$$\begin{aligned} \|u_n - v_n\|_{X_n} &\lesssim \|u_n^0 - v_n^0\|_{H_x^m} \\ &\quad + T_*^{\frac{1}{2}} \left[\nu_n^{-\frac{1}{2}} (\|u_n\|_{X_n} + \|v_n\|_{X_n}) + \nu_s^{-\frac{1}{2}} \|u_s\|_{X_s} \right] \|u_n - v_n\|_{X_n} \end{aligned}$$

From (25), (28) and (32), we can choose T_* sufficiently small so that the second term on the RHS may be absorbed into the LHS. This gives the required result for the normal fluid. The same procedure can be repeated for the superfluid as well.

4.4 Regularity of the Pressure Fields

We will now briefly comment on the regularity of the pressure fields. Applying the divergence operator on (2) and (3), and using incompressibility, we get the Poisson equations that govern the evolution of the pressure fields.

$$\begin{aligned} -\Delta p_n &= \nabla \cdot \nabla \cdot (u_n \otimes u_n) + \beta_n(u_n - u_s) \cdot \nabla |\omega_s| - \beta'_n \nabla \cdot (\omega_s \times (u_n - u_s)) \\ -\Delta p_s &= \nabla \cdot \nabla \cdot (u_s \otimes u_s) - \beta_s(u_n - u_s) \cdot \nabla |\omega_s| + \beta'_s \nabla \cdot (\omega_s \times (u_n - u_s)) \end{aligned} \quad (37)$$

The solutions p_n, p_s of these equations must be unique. At first look, it appears that an arbitrary harmonic function could be added to the pressure fields, breaking uniqueness. But since we seek bounded pressures, Liouville's theorem (see chapter 2 of Evans 2010) guarantees that such a harmonic function is in fact a constant. This constant can be forced to be zero, by demanding that the pressure vanishes at infinity.

Now, for unique solutions of the Poisson equation $-\Delta p = f$, with $p = 0$ on the boundary, elliptic regularity theory estimates (see chapter 6 of Evans 2010) show that $\|p\|_{H_x^{m+2}} \lesssim \|f\|_{H_x^m}$. The RHS of each equation in (37) consists of terms like $(\nabla u)(\nabla u)$, and $(u)(\nabla \omega) \sim (u)(D^2 u)$. If we upgrade our Sobolev index to $m \geq 4$, then we have $m - 2 > \frac{d}{2}$, allowing us to use the algebra property of H_x^{m-2} . Thus,

$$\begin{aligned} \|\text{RHS}\|_{H_x^{m-2}} &\lesssim \|(\nabla u)(\nabla u)\|_{H_x^{m-2}} + \|(u)(D^2 u)\|_{H_x^{m-2}} \\ &\lesssim \|(\nabla u)\|_{H_x^{m-2}}^2 + \|u\|_{H_x^{m-2}} \|D^2 u\|_{H_x^{m-2}} \\ &\lesssim \|u\|_{H_x^m}^2 < \infty \end{aligned}$$

Therefore, this shows that the RHS of (37) $\in C_t^0 H_x^{m-2} \Rightarrow p_n, p_s \in C_t^0 H_x^m$.

4.5 Instantaneous Smoothing

Viscosity-driven momentum diffusion leads to the well-known instantaneous smoothing of the solution. This standard result can be argued as follows, where u can mean either the superfluid or the normal fluid velocity. Since we have established that $u \in C^0([0, T_*]; \dot{H}_x^m) \cap L^2([0, T_*]; \dot{H}_x^{m+1})$, we can conclude that $\|\nabla u\|_{H_x^m}(t) < \infty$ for almost every $t \in [0, T_*]$. In other words, $\|u\|_{H_x^{m+1}}(t) < \infty$ for almost every $t \in [0, T_*]$.

So, for an arbitrary $0 < \delta < T_*$, we can find a time $0 < t_0 < \delta$ such that $\|u\|_{H_x^{m+1}}(t_0) < \infty$. Denoting this velocity field as u^{t_0} and considering it as an H_x^{m+1} datum, we can evolve the system (as shown thus far) to find a local $C_t^0 H_x^{m+1}$ solution in some time interval $[t_0, T'_*]$, where $T'_* - t_0$ is the maximal existence time corresponding to this H_x^{m+1} datum. The choice of t_0 can be made so that $\delta < T'_*$. By the uniqueness of solutions on the interval $[t_0, \delta]$, we may conclude that indeed $u \in C^0([t_0, \delta]; H_x^{m+1})$. Since δ is arbitrary, this implies $u \in C^0([0, T_*]; H_x^{m+1})$. Iterating this argument shows that $u \in C^0([0, T_*]; H_x^\infty)$.

The mild solutions given by (5) and (6) satisfy a.e. in time:

$$\begin{aligned}\partial_t u_n &= v_n \Delta u_n - \mathcal{P}(u_n \cdot \nabla u_n + \beta_n |\omega_s| (u_n - u_s) - \beta'_n \omega_s \times (u_n - u_s)) \\ \partial_t u_s &= v_s \Delta u_s - \mathcal{P}(u_s \cdot \nabla u_s - \beta_s |\omega_s| (u_n - u_s) + \beta'_s \omega_s \times (u_n - u_s))\end{aligned}$$

Since $u_n, u_s \in C^0([0, \infty[; H^\infty(\mathbb{R}^2))$, each term on the RHS of these equations is in $C^0([0, \infty[; H^\infty(\mathbb{R}^2))$. This means the LHS (the time-derivative term) is in $C^0([0, \infty[; H^\infty(\mathbb{R}^2))$, i.e. $u_n, u_s \in C^1([0, \infty[; H^\infty(\mathbb{R}^2))$. This argument can be iterated to show that indeed $u_n, u_s \in C^\infty([0, \infty[; H^\infty(\mathbb{R}^2))$. In other words, the solutions are smooth in time (for $t > 0$), and H^∞ -smooth in space. Repeating the steps of Sect. 4.4 proves the uniqueness and spacetime smoothness of pressure fields as well (for $t > 0$). $\square$

5 Global Well-Posedness

5.1 Higher-Order Energy Estimate

We will now establish global well-posedness for high-regularity data using the method of Beale–Kato–Majda (Beale et al. 1984). We begin by deriving a higher-order energy estimate like in Sect. 4.1. Acting on (2) with a higher-order derivative operator D^α (for $0 \leq |\alpha| \leq m$, multiplying by $D^\alpha u_n$, and summing over all α , we get:

$$\begin{aligned}\frac{d}{dt} \frac{1}{2} \|u_n\|_{H_x^m}^2 + v_n \|\nabla u_n\|_{H_x^m}^2 &= - \sum_{\alpha} \langle D^\alpha u_n, \mathcal{P} D^\alpha [u_n \cdot \nabla u_n \\ &\quad + \beta_n |\omega_s| (u_n - u_s) - \beta'_n \omega_s \times (u_n - u_s)] \rangle\end{aligned}\quad (38)$$

where $\langle, \rangle$ denotes the L_x^2 inner product. The Leray projector is self-adjoint and commutes with derivatives, so it can be dropped. By incompressibility, $\langle D^\alpha u_n, u_n \cdot \nabla D^\alpha u_n \rangle = 0$. So, we add this vanishing quantity to the RHS.

$$\begin{aligned}RHS &= - \sum_{\alpha} \langle D^\alpha u_n, D^\alpha [u_n \cdot \nabla u_n] - u_n \cdot \nabla D^\alpha u_n \rangle \\ &\quad - \sum_{\alpha} \langle D^\alpha u_n, D^\alpha [\beta_n |\omega_s| (u_n - u_s) - \beta'_n \omega_s \times (u_n - u_s)] \rangle \\ &\lesssim \|u_n\|_{H_x^m} [\|\nabla u_n\|_{L_x^\infty} \|u_n\|_{H_x^m} + \|\omega_s\|_{L_x^\infty} \|u_n - u_s\|_{H_x^m}] \\ &\lesssim \|u_n\|_{H_x^m} [\|\nabla u_n\|_{L_x^\infty} \|u_n\|_{H_x^m} + \|u_n - u_s\|_{L_x^\infty} \|\omega_s\|_{H_x^m} + \|\omega_s\|_{L_x^\infty} \|u_n - u_s\|_{H_x^m}]\end{aligned}$$

where we used Hölder's inequality and calculus inequalities for Sobolev spaces [Eqns. (3.31) and (3.32) from Majda and Bertozzi (2002)].

Using the above estimate of the RHS in (38),

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \|u_n\|_{H_x^m}^2 + \nu_n \|\nabla u_n\|_{H_x^m}^2 \\ & \lesssim \|\nabla u_n\|_{L_x^\infty} \|u_n\|_{H_x^m}^2 + \|u_n - u_s\|_{L_x^\infty} \|u_n\|_{H_x^m} \|\omega_s\|_{H_x^m} \\ & \quad + \|\omega_s\|_{L_x^\infty} \|u_n - u_s\|_{H_x^m} \|u_n\|_{H_x^m} \end{aligned} \quad (39)$$

A similar calculation for the superfluid equation (3) can be performed.

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \|u_s\|_{H_x^m}^2 + \nu_s \|\nabla u_s\|_{H_x^m}^2 \\ & \lesssim \|\nabla u_s\|_{L_x^\infty} \|u_s\|_{H_x^m}^2 + \|u_n - u_s\|_{L_x^\infty} \|u_s\|_{H_x^m} \|\omega_s\|_{H_x^m} \\ & \quad + \|\omega_s\|_{L_x^\infty} \|u_n - u_s\|_{H_x^m} \|u_s\|_{H_x^m} \end{aligned} \quad (40)$$

Once again, using Cauchy's inequality, we can extract out a $\frac{\nu_s}{2} \|\omega_s\|_{H_x^m}^2$ from the second terms on the RHS of (39) and (40). Since $\|\omega_s\|_{H_x^m} \leq \|\nabla u_s\|_{H_x^m}$, we may add the equations and these extracted terms can be used to cancel out the $\nu_s \|\nabla u_s\|_{H_x^m}^2$ on the LHS. Finally, after dropping $\nu_n \|\nabla u_n\|_{H_x^m}^2$ on the LHS, the following inequality results.

$$\begin{aligned} & \frac{d}{dt} \left(\|u_n\|_{H_x^m}^2 + \|u_s\|_{H_x^m}^2 \right) \lesssim \left(\|u_n\|_{H_x^m}^2 + \|u_s\|_{H_x^m}^2 \right) \\ & \quad \times \left[\|\nabla u_n\|_{L_x^\infty} + \|\nabla u_s\|_{L_x^\infty} + \|\omega_s\|_{L_x^\infty} + \|u_n - u_s\|_{L_x^\infty}^2 \right] \end{aligned} \quad (41)$$

For any time interval $[0, T]$ where $T < \infty$, if we can show that the $L_t^\infty H_x^m$ norms of u_n and u_s are bounded, we may substitute these bounds back into (39) and (40) and derive upper bounds for $\nu_n \|\nabla u_n\|_{H_x^m}^2$ and $\nu_s \|\nabla u_s\|_{H_x^m}^2$, just as was done to arrive at the bounds in (16). This will show that the X_n and X_s norms of the respective velocity fields are bounded over any time interval $[0, T]$ for every $T < \infty$, allowing us to conclude global solutions of high-regularity.

In this regard, we recall a crucial potential theory estimate from Beale et al. (1984).

Lemma 5.1 (BKM potential theory estimate) *For $m > \frac{d}{2} + 1 = 2$,*

$$\|\nabla u\|_{L_x^\infty} \lesssim \|\omega\|_{L_x^2} + \|\omega\|_{L_x^\infty} (1 + \log(1 + \|u\|_{H_x^m})) \quad (42)$$

Thus, the quantity in the square brackets in (41) can be bounded above by:

$$\begin{aligned} [\dots] & \lesssim (\|\omega_n\|_{L_x^2} + \|\omega_s\|_{L_x^2}) + \|u_n - u_s\|_{L_x^\infty}^2 \\ & \quad + (\|\omega_n\|_{L_x^\infty} + \|\omega_s\|_{L_x^\infty}) (1 + \log(1 + \|u_n\|_{H_x^m} + \|u_s\|_{H_x^m})) \end{aligned}$$

From (16), we know that $\|\omega\|_{L_x^2}$ is bounded above for both fluids. So, we can absorb these upper bounds into the implied multiplicative constant in the upper bound. We

can also adjust the argument of the logarithm to make the entire expression in (41) more amenable to the Grönwall's inequality. For this, we note that for $x \geq 0$,

$$\log(1+x) \leq \log(1+x^2+2x) \leq \log\left(2(1+x^2)\right) \lesssim 1 + \log(1+x^2)$$

From the above arguments,

$$[\dots] \lesssim_{\Omega^0, E^0} (1 + \|u_n - u_s\|_{L_x^\infty}^2 + \|\omega_n\|_{L_x^\infty} + \|\omega_s\|_{L_x^\infty}) \left(1 + \log(1 + \|u_n\|_{H_x^m}^2 + \|u_s\|_{H_x^m}^2)\right) \quad (43)$$

Denoting $X = 1 + \|u_n\|_{H_x^m}^2 + \|u_s\|_{H_x^m}^2$, we see that (41) simplifies to

$$\frac{dX}{dt} \lesssim_{\Omega^0, E^0} X(1 + \log X) \left[1 + \|u_n - u_s\|_{L_x^\infty}^2 + \|\omega_n\|_{L_x^\infty} + \|\omega_s\|_{L_x^\infty}\right] \quad (44)$$

At this stage, one can easily apply Grönwall's inequality to draw the conclusion that the $C_t^0 H_x^m$ norm is bounded for any finite time T , if the quantity in the square brackets in (44) is integrable in time over $[0, T]$. If that is the case, the required upper bound for X is given by

$$\sup_{0 \leq t \leq T} X(t) \leq e^{(1+\log X(0))e^{\int_0^T [\dots] dt}} \quad (45)$$

5.2 Analogue of the BKM Condition

We will now prove the time-integrability condition required to proceed from (44) to (45), which will lead us to global solutions from high-regularity data. The claim to be verified is:

Lemma 5.2 (BKM analogue) *For every $0 < T < \infty$,*

$$\int_0^T \left[1 + \|u_n - u_s\|_{L_x^\infty}^2 + \|\omega_n\|_{L_x^\infty} + \|\omega_s\|_{L_x^\infty}\right] dt < \infty \quad (46)$$

Proof 1. The first term in the integral is obviously finite for every finite T .
2. For the second term, we use the Gagliardo-Nirenberg interpolation inequality, followed by the Calderon-Zygmund inequality, and the energy boundedness from (9):

$$\begin{aligned} \|u_n - u_s\|_{L_x^\infty}^2 &\lesssim \|u_n - u_s\|_{L_x^2} \|D^2 u_n - D^2 u_s\|_{L_x^2} \\ &\lesssim \left(\|u_n\|_{L_x^2} + \|u_s\|_{L_x^2}\right) \left(\|\nabla \omega_n\|_{L_x^2} + \|\nabla \omega_s\|_{L_x^2}\right) \\ &\lesssim_{E^0, \rho_i, v_i} \left(\|\nabla \omega_n\|_{L_x^2} + \|\nabla \omega_s\|_{L_x^2}\right) \end{aligned}$$

Thus, integrating over $[0, T]$,

$$\|u_n - u_s\|_{L^2_{[0,T]} L^\infty_x}^2 \lesssim_{E^0, \rho_i, v_i} \left(\|\nabla \omega_n\|_{L^2_{[0,T]} L^2_x} + \|\nabla \omega_s\|_{L^2_{[0,T]} L^2_x} \right) T^{\frac{1}{2}} \quad (47)$$

This is clearly finite, from the bounds in (16).

3. For the third term, consider the vorticity equation (10), reproduced here for convenience.

$$\partial_t \omega_n + u_n \cdot \nabla \omega_n = \nu_n \Delta \omega_n + \frac{\rho_s}{\rho} \mathcal{T}$$

Along the characteristics of the flow (denoted by $x_n^\alpha(t)$, where α is the initial point), the Duhamel solution to this equation can be written as:

$$\omega_n(t, x_n^\alpha(t)) = e^{\nu_n t \Delta} \omega_n^0(\alpha) + \frac{\rho_s}{\rho} \int_0^t e^{\nu_n(t-t') \Delta} \mathcal{T}(t') dt' \quad (48)$$

Now, we make use of an interesting result, the proof of which can be found in Proposition 44 of Tao (2018).

Lemma 5.3 *In two spatial dimensions, if a field evolves according to the forced heat equation $\partial_t u = \nu \Delta u + F$ (with initial condition u^0), then its solution satisfies for any $0 < T < \infty$*

$$\|u\|_{L^2_{[0,T]} L^\infty_x} \lesssim_\nu \|u^0\|_{L^2_x} + \|F\|_{L^1_{[0,T]} L^2_x} \quad (49)$$

Applying this lemma to (48), and absorbing the viscosity factor into the implied constant,

$$\begin{aligned} \|\omega_n\|_{L^2_{[0,T]} L^\infty_x} &\lesssim_{\nu_n} \|\omega_n^0\|_{L^2_x} + \| |\omega_s| (\omega_n - \omega_s) \|_{L^1_{[0,T]} L^2_x} \\ &\quad + \| (u_n - u_s) \cdot (B \nabla^\perp |\omega_s| - B' \nabla \omega_s) \|_{L^1_{[0,T]} L^2_x} \\ &\lesssim_{\nu_n, B, B'} \|\omega_n^0\|_{L^2_x} + \| |\omega_s| (\omega_n - \omega_s) \|_{L^1_{[0,T]} L^2_x} \\ &\quad + \| u_n - u_s \|_{L^2_{[0,T]} L^\infty_x} \|\nabla \omega_s\|_{L^2_{[0,T]} L^2_x} \end{aligned} \quad (50)$$

In the above equation, the first term is finite by the high-regularity assumption on the data, and the finiteness of the last term is inferred from the previous analysis combined with the bound from (16). For the middle term, from Hölder's and Cauchy's inequalities, it is easy to see that

$$\| |\omega_s| (\omega_n - \omega_s) \|_{L^1_{[0,T]} L^2_x} \lesssim \|\omega_n\|_{L^2_{[0,T]} L^4_x}^2 + \|\omega_s\|_{L^2_{[0,T]} L^4_x}^2 \quad (51)$$

Both the terms on the RHS of (51) are handled in the same way, as shown below for the first of the two. From the Ladyzhenskaya inequality in 2D, followed by the Calderon–Zygmund inequality,

$$\begin{aligned} \|\omega_n\|_{L_x^4}^2 &\lesssim \|\omega_n\|_{L_x^2} \|\nabla \omega_n\|_{L_x^2} \lesssim \|\nabla u_n\|_{L_x^2} \|\nabla \omega_n\|_{L_x^2} \\ \Rightarrow \|\omega_n\|_{L_{[0,T]}^2 L_x^4}^2 &\lesssim \|\nabla u_n\|_{L_{[0,T]}^2 L_x^2} \|\nabla \omega_n\|_{L_{[0,T]}^2 L_x^2} < \infty \end{aligned}$$

Once again, from (9) and (16), we have that the quantities in the RHS are bounded above. Since the third term in Lemma 5.2 can be bounded by $\|\omega_n\|_{L_{[0,T]}^2 L_x^\infty} T^{\frac{1}{2}}$, we see that its contribution to the integral is finite.

4. That the fourth (and last) term in Lemma 5.2 is finite is proven exactly as the third term above.

□

This completes the proof of the BKM analogue, showing global well-posedness for high-regularity initial data.

6 Proof of Corollary 2.2

In the previous sections, we showed that the HVBK equations are globally well-posed for high-regularity data. Now, we seek to show that starting from C_c^∞ data, the solutions are globally well-posed and are smooth in space and time.

Since a C_c^∞ function belongs to H_x^m for every m , and data in H_x^m implies global well-posedness, it is easy to see that C_c^∞ data means there is a unique global solution in every H_x^m , which means it is spatially H^∞ -smooth, i.e. $u_n, u_s \in C^0([0, \infty[; H^\infty(\mathbb{R}^2))$. Time regularity can be shown just as in Sect. 4.5. Thus, $u_n, u_s \in C^\infty([0, \infty[; H^\infty(\mathbb{R}^2))$, implying smoothness of the pressure fields (from elliptic regularity). The uniqueness of the pressure fields can be argued as done in Sect. 4.4. □

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Appendix A: Conditions on the Vorticity Field for Finite Kinetic Energy in 2D Incompressible Fluids

It is well known that an incompressible fluid in $\mathbb{R}^2$ with compactly supported vorticity has finite kinetic energy if and only if the integral of the vorticity vanishes [see Prop. 3.3 in Majda and Bertozzi (2002)]. This is due to the slow decay of the Biot–Savart kernel in two dimensions.

$$\begin{aligned} \nabla \times u &= \omega \Rightarrow -\Delta u = \nabla \times \omega \\ \therefore u(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy \end{aligned} \quad (52)$$

However, it turns out that a compactly supported vorticity is not the only way to tackle this problem. In this appendix, we wish to replace this assumption with two weaker conditions: finite enstrophy and finite L^1 norm of the first moment of the vorticity.

Lemma A.1 *For an incompressible fluid in 2D, let there be a vorticity field $\omega : \mathbb{R}^2 \mapsto \mathbb{R}$ such that $\int_{\mathbb{R}^2} \omega \, dx = 0$, and $\|\omega\|_{L^2(\mathbb{R}^2)}$ and $\int_{\mathbb{R}^2} |x| |\omega| \, dx$ are both finite. The associated velocity field $u : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is defined by (52). Then, the kinetic energy of the fluid is finite, i.e. $\|u\|_{L^2(\mathbb{R}^2)} < \infty$. In particular, $\|u\|_{L^2(\mathbb{R}^2)} < \|\omega\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|x\omega\|_{L^1(\mathbb{R}^2)}^{\frac{1}{2}}$.*

Proof Writing (52) in Fourier space,

$$\hat{u}(k) = \frac{ik^\perp}{|k|^2} \hat{\omega}(k)$$

From Plancherel's theorem, we have

$$\|u\|_{L_x^2} \lesssim \|\hat{u}\|_{L_k^2} \lesssim \left(\int \frac{|\hat{\omega}(k)|^2}{|k|^2} dk \right)^{\frac{1}{2}}$$

We split the integral into one over low frequencies and another over high frequencies. The (to-be-determined) cut-off is denoted K .

$$\|u\|_{L_x^2} \lesssim \left(\int_{|k| < K} \frac{|\hat{\omega}(k)|^2}{|k|^2} dk \right)^{\frac{1}{2}} + \left(\int_{|k| \geq K} \frac{|\hat{\omega}(k)|^2}{|k|^2} dk \right)^{\frac{1}{2}}$$

The high-frequency component is easily seen to be bounded by $\frac{1}{K} \|\omega\|_{L_x^2}$. The vanishing integral of the vorticity translates to $\hat{\omega}(0) = 0$. Expanding in a Taylor series about $k = 0$, for some $0 \leq \theta \leq 1$:

$$\hat{\omega}(k) = \hat{\omega}(0) + D\hat{\omega}(\theta k) \cdot k \Rightarrow |\hat{\omega}(k)| \leq |D\hat{\omega}(\theta k)| |k|$$

Also,

$$D\hat{\omega}(k_0) = \int i x \omega(x) e^{ik_0 x} dx \Rightarrow \|D\hat{\omega}\|_{L_k^\infty} \leq \|x\omega\|_{L_x^1}$$

We can thus bound the low-frequency component by $K \|x\omega\|_{L_x^1}$. This gives us:

$$\|u\|_{L_x^2} \lesssim K \|x\omega\|_{L_x^1} + \frac{1}{K} \|\omega\|_{L_x^2}$$

Selecting $K = \|x\omega\|_{L_x^1}^{-\frac{1}{2}} \|\omega\|_{L_x^2}^{\frac{1}{2}}$ gives the required result. $\square$

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