



LIOUVILLE QUANTUM GRAVITY WEIGHTED BY CONFORMAL LOOP ENSEMBLE NESTING STATISTICS

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We study Liouville quantum gravity (LQG) surfaces whose law has been reweighted according to nesting statistics for a conformal loop ensemble (CLE) relative to $n \in \mathbb{N}_0$ marked points $z_1, \dots, z_n$. The idea is to consider a reweighting by $\prod_{B \subseteq \{1, \dots, n\}} e^{\sigma_B N_B}$, where $\sigma_B \in \mathbb{R}$ and N_B is the number of CLE loops surrounding the points z_i for $i \in B$. This is made precise via an approximation procedure where as part of the proof we derive strong spatial independence results for CLE. The reweighting induces logarithmic singularities for the Liouville field at $z_1, \dots, z_n$ with a magnitude depending explicitly on $\sigma_1, \dots, \sigma_n$. We define the partition function of the surface, compute it for $n \in \{0, 1\}$, and derive a recursive formula expressing the $n > 1$ point partition function in terms of lower-order partition functions. The proof of the latter result is based on a continuum peeling process previously studied by Miller, Sheffield and Werner in the case $n = 0$, and we derive an explicit formula for the generator of a boundary length process that can be associated with the exploration for general n . We use the recursive formula to partly characterize for which values of $(\sigma_B : B \subseteq \{1, \dots, n\})$ the partition function is finite. Finally, we give a new proof for the law of the conformal radius of CLE, which was originally established by Schramm, Sheffield, and Wilson.

1. Introduction	101
2. Discrete motivation	115
3. Reweighted Lévy processes: generator and explicit computation	120
4. Some functionals of Lévy excursions	140
5. Background on CLE and LQG	145
6. Law of the CLE conformal radius	161
7. CLE weighted by nesting statistics	165
8. LQG disks weighted by CLE nesting statistics	180
9. Recursive formula and finiteness of partition functions	197
Appendix A. A result on renewal processes	204
Appendix B. Integral computations	207
Acknowledgments	211
References	211

1. Introduction

Liouville quantum gravity (LQG) is a theory of random fractal surfaces with conformal symmetries which arise as the scaling limit of discrete surfaces known as random planar maps. It has its origin in theoretical physics [Polyakov 1981] and has been an active field of study in probability theory for the past fifteen years.

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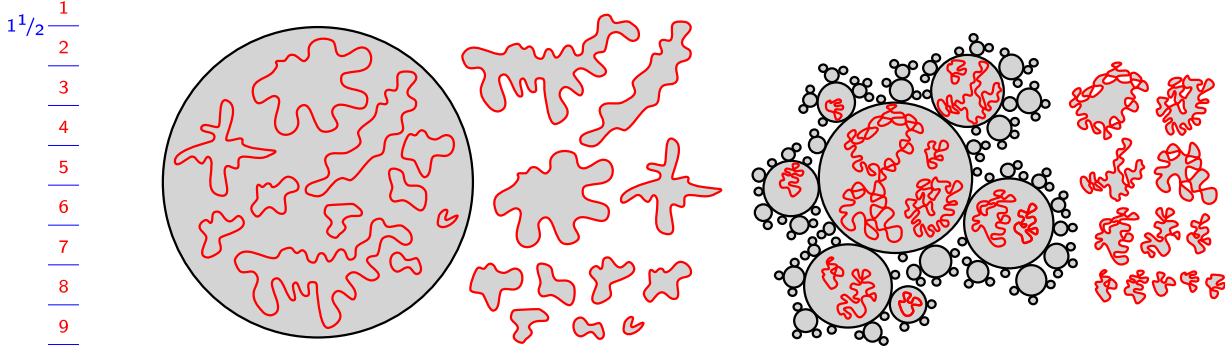


Figure 1. Left: the quantum surfaces obtained by cutting an LQG disk with parameter γ with a (simple) nonnested CLE with parameter $\kappa = \gamma^2 \in (8/3, 4)$ are again independent quantum disks conditionally on all of their boundary lengths. Right: the analogous statement holds when we decorate a generalized quantum disk of parameter γ with a (nonsimple) nonnested CLE of parameter $\kappa' = 16/\gamma^2 \in (4, 8)$.

A particularly fruitful direction in the study of LQG has been the discovery of couplings between LQG surfaces and conformally invariant random fractal curves called Schramm–Loewner evolutions (SLE) [Schramm 2000]. An extensive theory of conformal welding for LQG surfaces, where SLE curves arise as interfaces, has been developed [Sheffield 2016; Duplantier et al. 2021] (see also [Holden and Powell 2021; Ang and Gwynne 2021; Ang et al. 2023a; 2024]). These results have had a number of applications for LQG, SLE, and random planar maps; see [Gwynne et al. 2023] for a survey.

Miller, Sheffield and Werner [Miller et al. 2022; Miller et al. 2021] study closely related couplings between LQG surfaces and so-called conformal loop ensembles (CLE) [Sheffield 2009; Sheffield and Werner 2012], which are loop versions of SLE. They prove that a CLE drawn on top of an independent LQG disk (which is arguably the most natural LQG surface with a disk topology) breaks the disk into independent smaller LQG disks and explicitly describe an exploration of the CLE decorated LQG disk called the continuum percolation interface (CPI), which is the continuum counterpart of the peeling process for random planar maps. They consider both the regular LQG disk, which has disk topology, and the generalized LQG disk, which has pinch points, where by a pinch point we mean a point whose removal disconnects the generalized disk into two disjoint components. See Figure 1. Their approach allows them to obtain several new results about SLE and CLE.

We study a model which generalizes the one in [Miller et al. 2021; 2022]. Our LQG disks will have a finite collection of marked points $z = (z_i : i \in A)$ (for $A \subseteq \mathbb{N}$ finite) where the Liouville field (which is a distribution in the unit disk that describes the distortion of the Euclidean metric) has logarithmic singularities. These surfaces arise in a natural probabilistic construction where the law of an LQG disk with a CLE has been reweighted according to CLE loop nesting statistics around the points z . In the remaining introduction we will first introduce the variants of CLE and LQG disks which will be studied in the paper (Sections 1A to 1E) and then we state our main results (Section 1F), namely exploration results for these reweighted loop decorated LQG disks, and recursive formulas as well as finiteness conditions for their partition functions.

1A. Motivation and parameters. Let us first give some motivation for our model, and let $\gamma \in (0, 2)$. The γ -LQG disk is a particularly natural γ -LQG surface since, for example, it arises as scaling limits of random planar maps with disk topology [Bettinelli and Miermont 2017; Miller and Sheffield 2021]. For concreteness, let us focus on the regular (not generalized) disk. As demonstrated by Miller, Sheffield, and Werner, it is natural to decorate this disk with an independent CLE_κ with $\kappa = \gamma^2$, which is a random countable collection of noncrossing nested loops whose law is conformally invariant. Let h be the random distribution in $\mathbb{D}$ describing the unit boundary length γ -LQG disk and let Γ denote the independent CLE_κ in $\mathbb{D}$. The field h induces an area measure in $\mathbb{D}$ called the γ -LQG area measure which heuristically speaking has density $e^{\gamma h}$ relative to the Lebesgue area measure. It also induces a γ -LQG length measure along the loops of Γ .

For finite and nonempty $A \subseteq \mathbb{N}$, parameters $(\sigma_B \in \mathbb{R} : B \subseteq A)$, and points $\mathbf{z} = (z_i \in \mathbb{D} : i \in A)$ sampled independently according to the γ -LQG area measure, we are interested in reweighting the law of $(h, \mathbf{z}, \Gamma)$ by

$$\prod_{\emptyset \neq B \subseteq A} e^{\sigma_B N_B}, \quad (1-1)$$

where N_B is the number of CLE loops surrounding the points z_i for $i \in B$ and not surrounding the points z_i for $i \notin B$. We often write σ_i, N_i instead of $\sigma_{\{i\}}, N_{\{i\}}$ for $i \in A$ to simplify notation. Since any fixed point is a.s. surrounded by a loop of a nonnested CLE, the iterative construction of a nested CLE from nonnested CLEs implies that $N_i = \infty$ a.s. for any $i \in A$ and therefore this reweighting does not make rigorous sense. When trying to define it rigorously, it is natural to apply an approximation procedure where one only considers the CLE loops of size above a certain threshold δ . When defining the size of a CLE loop one wants to choose an *intrinsic* definition, namely one which does not depend on the embedding h of the γ -LQG disk into $\mathbb{D}$. For example, the Euclidean diameter of the loop would not be an appropriate measure, while the γ -LQG area surrounded by the loop or the γ -LQG length of the loop would be appropriate. Therefore, if $\sigma_i > 0$ (resp. $\sigma_i < 0$) then the reweighting we apply will both favor Γ with a high (resp. low) density of loops around z_i and h which is particularly large (resp. small) near z_i . This explains why the reweighting (1-1) affects the law of *both* Γ and h . One can argue at least heuristically that the effect of the reweighting should factorize in the sense that Γ and h are still independent (given $\mathbf{z}$) after the reweighting. In the rest of this subsection we will give precise definitions of the loop ensemble Γ and field h that we expect to get when applying the reweighting (1-1). Section 1C contains an explanation on how the heuristics above match the definitions made in this paper.

The γ -LQG disk decorated with an independent CLE_κ for $\kappa = \gamma^2 \in (8/3, 4)$ is believed to arise as the scaling limit of random planar maps decorated by a loop $O(n)$ model. Similarly, we conjecture that the variant of the γ -LQG disk obtained via the reweighting (1-1) describes the scaling limit of random planar maps with a loop $O(n)$ model whose law has been reweighted according to nesting statistics relative to $\#A$ marked vertices. We refer to Section 2 for precise conjectures.

Throughout the text, several parameters will have to be tuned in a very particular way for the results to hold. Below, we first associate to each CLE parameter κ the corresponding LQG parameter γ and define how the CLE reweighting parameter σ relates to the strength α_σ^κ of the corresponding logarithmic

¹/₂ singularity of the Liouville field. The parameter ρ_σ^κ appears in the renormalization term in the definition of reweighted CLE. The relationships between the various parameters will be justified later. The reader might wish to skip this definition and refer back to it whenever necessary.

Definition 1.1. Fix $\kappa \in (8/3, 8)$ and let $\gamma = \sqrt{\kappa} \wedge 4/\sqrt{\kappa}$. We will write $Q_\beta = \beta/2 + 2/\beta$ for $\beta > 0$. Let $\sigma \mapsto \alpha_\sigma^\kappa$ be the strictly increasing bijection from $(-\infty, -\log(-\cos(4\pi/\kappa)))$ to $(Q_\gamma - \sqrt{\kappa}/4, Q_\gamma)$ given via

$$e^{-\sigma} = \frac{-\cos(4\pi/\kappa)}{-\cos(4\pi/\kappa - \pi \cdot 2\alpha_\sigma^\kappa/\sqrt{\kappa})}$$

and let $\sigma \mapsto \rho_\sigma^\kappa$ be the strictly increasing bijection from $\mathbb{R}$ to $(-1 + 2/\kappa + 3\kappa/32, \infty)$ given via

$$e^{-\sigma} = \frac{-\cos(4\pi/\kappa)}{\cos(\pi \sqrt{(1 - 4/\kappa)^2 - 8\rho_\sigma^\kappa/\kappa})}.$$

Note that $\rho_\sigma^\kappa = \alpha_\sigma^\kappa(Q_\gamma - Q_{\alpha_\sigma^\kappa})$ when $\sigma < -\log(-\cos(4\pi/\kappa))$. Whenever $A \subseteq \mathbb{N}$ is finite, we let

$$\mathfrak{S}_\kappa^A = \{\sigma \in \mathbb{R}^{\{B: B \subseteq A\}} : \sigma_i \in (-\infty, -\log(-\cos(4\pi/\kappa))) \text{ for } i \in A, \sigma_\emptyset = 0\}.$$

We also write $\mathfrak{S}_\kappa = \bigcup_A \mathfrak{S}_\kappa^A$ where the union is over all finite subsets A of $\mathbb{N}$. To simplify notation, we will frequently write α_i instead of $\alpha_{\sigma_i}^\kappa$ for $i \in A$ when $\sigma \in \mathfrak{S}_\kappa^A$ is clear from the context. For $B \subseteq A$ we let $\sigma|_B = (\sigma_C : C \subseteq B)$.

1B. CLE weighted by nesting statistics. We will now define the variant of CLE that we expect to get when performing the reweighting (1-1). First we state [Theorem 1.2](#), which motivates our definition and is crucial for our weighted CLE to be well-defined. The result should be viewed as making sense of the joint moment generating function of the nesting statistics of a nested CLE, where a nesting statistic counts the number of CLE loops surrounding some points. The difficulty lies in the fact that single points are surrounded by an infinite number of loops, which implies that a regularization procedure is necessary to make sense of this moment generating function.

In the theorem statement and throughout the text, whenever U is a disjoint union of proper simply connected domains and $z \in U$, we write $R(z, U)$ for the conformal radius of z in the connected component of U containing z . Given a closed curve η and a point z , we say that η surrounds z if the index of η around z is ± 1 . We let η^o denote the points surrounded by the curve η .

Theorem 1.2. Let $\kappa \in (8/3, 8)$, $A \subseteq \mathbb{N}$ finite, and $D \subsetneq \mathbb{C}$ be simply connected. Consider distinct points $z_i \in D$ for $i \in A$ and constants $\sigma_B \in \mathbb{R}$ for all nonempty $B \subseteq A$. Let Γ be a nested CLE_κ in D and define $N_B^\epsilon(z)$ to be the number of loops $\eta \in \Gamma$ such that η surrounds z_i if and only if $i \in B$ and such that $R(z_i, \eta^o) \geq \epsilon$ for all $i \in B$. For all $i \in A$, let $\eta_i \in \Gamma$ be the outermost loop surrounding z_i but not z_j for each $j \neq i$. Then

$$\prod_{i \in A} \frac{\epsilon^{\rho_{\sigma_i}^\kappa}}{C^{\kappa_{\sigma_i}}} \mathbb{E} \left(\exp \left(\sum_{\substack{B \subseteq A \\ \#B \geq 1}} \sigma_B N_B^\epsilon(z) \right) \right) \rightarrow \mathbb{E} \left(\prod_{i \in A} e^{\sigma_i R(z_i, \eta_i^o)} \epsilon^{\rho_{\sigma_i}^\kappa} \exp \left(\sum_{\substack{B \subseteq A \\ \#B > 1}} \sigma_B N_B^0(z) \right) \right) < \infty \text{ as } \epsilon \downarrow 0 \quad (1-2)$$

^{11/2}₁ for some (explicit) constants $c_{\sigma_i}^\kappa$ for $i \in A$.¹ We define $\Phi_D^{\sigma, \kappa}(\mathbf{z})$ to be the limit on the right-hand side of the
₂ display above where $\mathbf{z} = (z_i : i \in A)$ and $\sigma = (\sigma_B : B \subseteq A)$ (with $\sigma_\emptyset = 0$). For $\delta > 0$ and $K \subseteq D$ compact,
₃

$$\begin{aligned} & \sup\{\Phi_D^{\sigma, \kappa}(\mathbf{z}) : \mathbf{z} = (z_i : i \in A) \in K^A \text{ and } |z_i - z_j| \geq \delta \ \forall i \neq j\} < \infty, \\ & \inf\{\Phi_D^{\sigma, \kappa}(\mathbf{z}) : \mathbf{z} = (z_i : i \in A) \in K^A \text{ and } |z_i - z_j| \geq \delta \ \forall i \neq j\} > 0. \end{aligned}$$

₄ The proof of the theorem is found in [Section 7](#). It will in particular rely on a strong spatial independence
₅ result for CLE ([Theorem 7.1](#)). Two key properties of the function $\Phi_D^{\sigma, \kappa}$ are its conformal covariance and
₆ a type of recursion relation which is closely related to the Markov property of CLE. These properties are
₇ inherited from the associated properties of CLE_κ , and are stated precisely in [Lemmas 7.5](#) and [7.6](#). We
₈ refer to the end of [Section 1C](#) for a discussion of a possible relationship between $\Phi_D^{\sigma, \kappa}$ and a conformal
₉ field theory (CFT) for CLE.

₁₀ In light of [Theorem 1.2](#), it is natural to expect that we get the following variant of CLE when performing
₁₁ the reweighting (1-1). Before giving the definition, we recall that a loop ensemble is a countable collection
₁₂ of (nonoriented) loops. For $\kappa \in (8/3, 8)$ a CLE_κ is a random loop ensemble which comes in two variants,
₁₃ nested and nonnested, where a nonnested CLE_κ is obtained by sampling a nested CLE_κ and only keeping
₁₄ the loops not surrounded by any other loop, where we use the word surrounded as explained above
₁₅ [Theorem 1.2](#). If Γ is a loop ensemble in a domain $D \subseteq \mathbb{C}$ then we define the complementary components
₁₆ of Γ to be the open connected components of the set $\bigcup_{\eta \in \Gamma} \eta^\circ$. The reader may want to come back to
₁₇ part (iii) of the definition below after reading [Section 5A](#), as CPIs are precisely defined there.

₁₈ **Definition 1.3.** Let $\kappa \in (8/3, 8)$, $D \subsetneq \mathbb{C}$ be simply connected, $A \subseteq \mathbb{N}$ be finite, $\sigma = (\sigma_B : B \subseteq A)$ with
₁₉ $\sigma_\emptyset = 0$, and $\mathbf{z} = (z_i : i \in A)$ be as in [Theorem 1.2](#).

₂₀ (i) A nonnested $\text{CLE}_\kappa^\sigma(\mathbf{z})$ in D is a loop ensemble whose law has the following Radon–Nikodym derivative
₂₁ with respect to a nonnested CLE_κ Γ in D :

$$\frac{1}{\Phi_D^{\sigma, \kappa}(\mathbf{z})} \prod_{\eta \in \Gamma} e^{\sigma_{\mathcal{I}(\eta^\circ)}} \prod_{U \in \mathfrak{U}(\eta^\circ)} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(\mathbf{z}|_{\mathcal{I}(U)}),$$

₂₂ where $\mathfrak{U}(V)$ denotes the connected components of an open set V , $\mathcal{I}(U) := \{i \in A : z_i \in U\}$ for $U \subseteq D$,
₂₃ and $\mathbf{z}|_{\mathcal{I}(U)} = (z_i : i \in \mathcal{I}(U))$.

₂₄ (ii) A nested $\text{CLE}_\kappa^\sigma(\mathbf{z})$ is a loop ensemble with the law of $\bigcup_{k \geq 0} \Gamma_k$ where Γ_k for $k \in \mathbb{N}_0$ are loop ensembles
₂₅ sampled iteratively as follows:

- ₂₆ • Let Γ_0 be a nonnested $\text{CLE}_\kappa^\sigma(\mathbf{z})$.
- ₂₇ • Given $\Gamma_0, \dots, \Gamma_{k-1}$, sample an independent nonnested $\text{CLE}_\kappa^{\sigma|_{\mathcal{I}(U)}}(\mathbf{z}|_{\mathcal{I}(U)})$ in U for each complemen-
₂₈ tary component U of Γ_{k-1} . Let Γ_k be the union of these loop ensembles.

₂₉ (iii) Consider two prime ends w_0 and w_∞ of D , and a parameter $\beta \in [-1, 1]$. We say that λ is a CPI
₃₀ from w_0 to w_∞ with asymmetry parameter β within nested $\text{CLE}_\kappa^\sigma(\mathbf{z})$ Γ if the conditional law of λ given
₃₁

¹The sum in the expression for the limit does not range over singletons, i.e., B with $\#B = 1$, so that N_B^0 is a.s. finite for the
₃₂ considered sets B .

¹/₂ the outermost loops of Γ is the same as that of a CPI from w_0 to w_∞ with asymmetry parameter β within
²/₂ a CLE_κ given the outermost loops of this CLE_κ (see [Section 5A](#)).
³/₂

⁴ If we do not specify where a $\text{CLE}_\kappa^\sigma(z)$ is nested or nonnested we refer to the nested variant. We
⁵ point out that the construction in (iii) is well-defined since the law of nonnested $\text{CLE}_\kappa^\sigma(z)$ is absolutely
⁶ continuous with respect to the law of a nonnested CLE_κ . Note however that the nested $\text{CLE}_\kappa^\sigma(z)$ is singular
⁷ with respect to a nested CLE_κ except if $\sigma_i = 0$ for all $i \in A$, in which case it is absolutely continuous.
⁸ Indeed, in both the case of a CLE and a reweighted CLE, one can see that $\lim_{n \rightarrow \infty} \log(R(z_i, (\eta_i^n)^o))/n$
⁹ exists a.s. and is a deterministic constant if η_n^i denotes the n -th CLE loop surrounding z_i , and these limits
¹⁰ differ when one considers different σ_i values.

¹¹ **1C. Regular LQG disks weighted by CLE nesting statistics.** We will now introduce the LQG surface
¹² we expect to obtain upon applying the reweighting (1-1).
¹³

¹⁴ In fact, when we define the surface here in the introduction we focus on the even smaller parameter
¹⁵ range where $\alpha_i \in (Q_\gamma - \sqrt{\kappa}/4, 2)$ for all $i \in A$ since the formulas can be written in a simpler form in
¹⁶ this case due to the existence of the α_i -LQG area measure. The general case $\alpha_i \in (Q_\gamma - \sqrt{\kappa}/4, Q_\gamma)$
¹⁷ can be treated via a direct generalization (see [Definition 8.1](#) and [Proposition 8.10](#)). For $\Lambda \geq 0$, $\ell > 0$,
¹⁸ $\kappa \in (8/3, 8) \setminus \{4\}$, $A \subseteq \mathbb{N}$ finite, and $\sigma \in \mathfrak{S}_A$ such that $\alpha_i = \alpha_{\sigma_i}^\kappa \in (Q_\gamma - \sqrt{\kappa}/4, 2)$ for all $i \in A$, we define
¹⁹ a measure $M_{\Lambda, \ell}^{\sigma, \kappa}$ on tuples $(h, z) \in H^{-1}(\mathbb{D}) \times \mathbb{D}^A$ by setting

$$M_{\Lambda, \ell}^{\sigma, \kappa}(dh, dz) = e^{-\Lambda \mu_h^\gamma(\mathbb{D})} \Phi_{\mathbb{D}}^{\sigma, \kappa}(z) \prod_{i \in A} \mu_h^{\alpha_i}(dz_i) P(dh), \quad (1-3)$$

²² where P is the probability measure describing the field of an embedding of a γ -LQG disk without marked
²³ points and with boundary length ℓ , and $\mu_h^{\alpha_i}$ is the α_i -LQG measure associated to the field h .
²⁴

²⁵ Let us explain how this definition relates to the motivation (1-1) by performing the following back of
²⁶ the envelope computation. We present it in the singleton case of $A = \{i\}$, writing $z = z_i$ and $\alpha = \alpha_i$, but
²⁷ the generalization to more points is immediate. Recall that we let h be an embedding of a γ -LQG disk
²⁸ and Γ denotes an independent CLE_κ in the unit disk $\mathbb{D}$. Let us fix $\epsilon > 0$ and consider the γ -LQG metric
²⁹ ball with γ -LQG area ϵ centered at the point $z \in \mathbb{D}$. At a heuristic level, we approximate this metric ball
³⁰ by a Euclidean ball $B_r(z)$ with $r > 0$ implicitly given by

$$\epsilon = r^{2+\gamma^2/2} e^{\gamma h_r(z)}.$$

³² Here $h_r(z)$ denotes the circle average of the field on $\partial B_r(z)$ and the right-hand side is an approximation
³³ of $\mu_h^\gamma(B_r(z))$. Let N_ϵ be the number of loops in Γ surrounding $B_r(z)$, which we view as an intrinsic way
³⁴ of regularizing the loop count around the point z as alluded to in [Section 1A](#). In accordance with (1-1) we
³⁵ consider the reweighting
³⁶

$$\epsilon^{\alpha/\gamma-1} e^{\sigma N_\epsilon} \cdot r^{\gamma^2/2} e^{\gamma h_r(z)} = r^{\alpha(Q_\gamma - Q_\alpha)} e^{\sigma N_\epsilon} r^{\alpha^2/2} e^{\alpha h_r(z)},$$

³⁹/₂ where the prefactor involving ϵ is chosen precisely so that the right-hand side has a limit. We now see that
⁴⁰ the expression on the right-hand side matches with (1-3) in the $\epsilon \rightarrow 0$ limit by the definition of α -LQG

¹/₂ measures and by the definition made in [Theorem 1.2](#). Needless to say, this is only a heuristic argument but it explains the origin of the definitions made here.

We are excluding the case $\kappa = 4$ throughout this paper since this corresponds to $\gamma = 2$ and therefore a critical Liouville quantum gravity measure, which we do not consider in this work.

The reason for introducing the exponential term involving the parameter Λ in the definition above is only that we will be able to show that choosing Λ sufficiently large will make the measure above finite. By Girsanov's theorem for the α_i -LQG area measure (see [\[Duplantier and Sheffield 2011, Proposition 3.4\]](#)), it is not hard to see that the measure $M_{\Lambda, \ell}^{\sigma, \kappa}$ is supported on fields (h, z) that have an α_i logarithmic singularity at z_i for each $i \in A$.

A (regular) γ -LQG surface is defined to be an equivalence class of tuples (D, h, z) (with $D \subseteq \mathbb{C}$ simply connected, h a distribution on D , and $z = (z_i : i \in A)$ a tuple of points in D for $A \subseteq \mathbb{N}$ finite) where $(D, h, z) \sim (D', h', z')$ if there is a conformal map $\phi : D \rightarrow D'$ such that

$$h = h' \circ \phi + Q_\gamma \log |\phi'|, \quad \phi(z_i) = z'_i \text{ for all } i \in A. \quad (1-4)$$

We let $[(D, h, z)]$ denote the γ -LQG surface given by the equivalence class of (D, h, z) and call h an embedding of $[(D, h, z)]$ into D . See [Section 5B](#) for further details and motivation.

²⁰/₂ For $\kappa \in (8/3, 8) \setminus \{4\}$ we consider the measure given by taking the pushforward of $M_{\Lambda, \ell}^{\sigma, \kappa}$ by the function which maps (h, z) to $[(\mathbb{D}, h, z)]$. Using the conformal covariance of the function $\Phi_{\mathbb{D}}^{\sigma, \kappa}$ and of the α_i -LQG area measure for all $i \in A$ one can check that this measure on γ -LQG surfaces does not depend on the embedding h that we chose for the γ -LQG disk above. This follows from the change of coordinates formula for $\mu_h^{\alpha_i}$ in combination with [Lemma 7.5](#), and the precise values of α_i appearing in [Definition 1.1](#) are the only ones which make this property true. When $\kappa < 4$ we write $\bar{M}_{\Lambda, \ell}^{\sigma, \kappa}$ for this measure on γ -LQG surfaces.

We now specialize to the case $\kappa \in (8/3, 4)$; the case $\kappa \in (4, 8)$ will be discussed in relation with the generalized LQG disk below. For $\kappa \in (8/3, 4)$ we define the *weight* $W_{\Lambda, \ell}^{\sigma, \kappa}$ and the *partition function* $Z_{\Lambda, \ell}^{\sigma, \kappa}$ of the disk as

$$W_{\Lambda, \ell}^{\sigma, \kappa} = |M_{\Lambda, \ell}^{\sigma, \kappa}| \quad \text{and} \quad Z_{\Lambda, \ell}^{\sigma, \kappa} = \ell^{-1-4/\kappa} W_{\Lambda, \ell}^{\sigma, \kappa}, \quad (1-5)$$

where $|\cdot|$ denotes the total mass of a measure.² Note that $W_{0, \ell}^{-, \kappa} = 1$. The following scaling property is immediate:

$$Z_{\Lambda, \ell}^{\sigma, \kappa} = \ell^{-1-4/\kappa + (2/\sqrt{\kappa}) \sum_{i \in A} \alpha_i} Z_{\Lambda \ell^2, 1}^{\sigma, \kappa}. \quad (1-6)$$

We will see in [Theorem 1.4](#) and [Remark 1.5](#) that $Z_{\Lambda, \ell}^{\sigma, \kappa}$ is the natural definition of the partition function for the disk measure introduced above and we conjecture in [Section 2](#) that $Z_{\Lambda, \ell}^{\sigma, \kappa}$ describes the scaling limit of the associated random planar map partition function.

²The scaling by $\ell^{1+4/\kappa}$ we do when defining the partition function corresponds to having a disk with a marked point on the boundary whose conditional law given the field is that of a point sampled from the γ -LQG boundary measure. This is the natural convention in our setting since the disks we study will come naturally equipped with such a marked boundary point. We mention this since in the random planar map literature one frequently sees a scaling factor $\ell^{4/\kappa}$ which does not take into account the $1/\ell$ normalization factor arising from sampling a boundary point according to the γ -LQG measure.

1D. Relationships with conformal field theory. We will briefly comment on relationships between the objects defined above and conformal field theory (CFT). First we remark that if (h, z) is sampled from $M_{\Lambda, \ell}^{\sigma, \kappa}$ (normalized to be a probability measure, provided this is possible) and we condition on the location of the marked points z , then the conditional law of the field h is the following: it is the field with singularities $((z_i, \alpha_i) : i \in A)$, total boundary length ℓ , and cosmological constant $\Lambda \geq 0$ that appears in Liouville conformal field theory (LCFT) [Huang et al. 2018]; see Section 8A. At the end of Section 8A we explain the relationship between our partition function and correlation functions appearing in LCFT. In particular, we will see that, for $\kappa \in (8/3, 4)$,

$$W_{\Lambda, \ell}^{\sigma, \kappa} = \widehat{C} \ell^{-2+2Q_\gamma/\gamma} \int_{\mathbb{D}^A} \Phi_{\mathbb{D}}^{\sigma, \kappa}(z) Z_{\Lambda, \ell, (\alpha, z), (\beta, w)}^{\text{LCFT}} dz, \quad (1-7)$$

where $\widehat{C} > 0$ is an explicit constant and $Z_{\Lambda, \ell, (\alpha, z), (\beta, w)}^{\text{LCFT}}$ is a so-called fixed boundary length LCFT partition function. The tuple (α, z) indicates the logarithmic singularities within the domain while (β, w) are three logarithmic γ singularities on the boundary of the domain. Choosing the location of the latter three singularities corresponds to fixing the embedding of the LQG surface. These LCFT partition functions are closely related to LCFT correlation functions.

One can hope that (1-7) uniquely characterizes $\Phi_{\mathbb{D}}^{\sigma, \kappa}(z)$, which would provide a potential path for computing this interesting CLE observable if the other unknowns in the formula (namely, $W_{\Lambda, \ell}^{\sigma, \kappa}$ and $Z_{\Lambda, \ell, (\alpha, z), (\beta, w)}^{\text{LCFT}}$) were to be identified. See also Remark 1.10 and Section 8A for more details on this.

The functions $\Phi_{\mathbb{D}}^{\sigma, \kappa}$ are of particular interest due to their potential relevance for the construction of a CFT for CLE. It is expected that one can define a CFT for CLE where the n -point function is related to connection probabilities for n points in discrete loop models converging to CLE [Ikhlef et al. 2016; Picco et al. 2016; Jacobsen and Saleur 2019; He et al. 2020]. Such connection probabilities are further related to functions of the form of the right-hand side of (1-2); see, e.g., [Ang and Sun 2021, Section 1.4]. The latter work considers the variant of $\Phi_{\mathbb{D}}^{\sigma, \kappa}$ on a sphere with three marked points and shows that this function is given by an imaginary counterpart of the three point structure constant in Liouville CFT (known as the DOZZ formula; see [Kupiainen et al. 2020]). In particular, this variant of $\Phi_{\mathbb{D}}^{\sigma, \kappa}$ should define the three point structure constant of the CFT for CLE on a sphere and we expect that $\Phi_{\mathbb{D}}^{\sigma, \kappa}$ with more points is related to the higher order correlation functions of the theory. The field of such a theory is a variant of the CLE nesting field constructed in [Miller et al. 2015].

Finally we remark that we expect $\Phi_{\mathbb{D}}^{\sigma, \kappa}$ to satisfy the following asymptotic property as $z_i \rightarrow z_j$ for $i, j \in A$ distinct:

$$\Phi_{\mathbb{D}}^{\sigma, \kappa}(z) \sim \hat{c}_\sigma^\kappa |z_i - z_j|^{\rho_{\sigma_i}^\kappa + \rho_{\sigma_j}^\kappa - \rho_{\sigma_{i,j}}^\kappa} \Phi_{\mathbb{D}}^{\sigma', \kappa}(z'), \quad z' = (z_k : k \neq i),$$

where $\hat{c}_\sigma^\kappa$ is a (nonexplicit) constant and $\sigma' = (\sigma'_B : B \subseteq A \setminus \{i\})$ is given by $\sigma'_B = \sigma_B$ for $B \not\ni j$ and $\sigma'_B = \sigma_{\{i\} \cup B}$ for $B \ni j$.³ Such a property for a correlation function $\Phi_{\mathbb{D}}^{\sigma, \kappa}$ is related to fusion rules and operator product expansion.

³We do not give a proof of this property but it follows immediately via a heuristic argument from Theorem 1.2 and the Markov property of CLE if we approximate the number of CLE loops $N_{\{i,j\}}^0$ around z_i and z_j by the number of CLE loops around z_j with conformal radius larger than $|z_i - z_j|$.

1E. Generalized LQG disks weighted by CLE nesting statistics. The reader can skip definitions and results related to the generalized LQG disk on a first reading if they wish, but we will still provide a very brief introduction here.

For $\gamma \in (\sqrt{2}, 2)$, the generalized γ -LQG disk can, roughly speaking, be described as a tree of regular γ -LQG disks. See Figure 1, right. The tree structure is encoded by the time reversal of a spectrally positive Lévy excursion $E : [0, \ell] \rightarrow [0, \infty)$ with exponent $4/\gamma^2$ and we call the duration ℓ of the excursion the generalized boundary length of the disk. Each jump of the excursion is associated with an independent regular γ -LQG disk of boundary length equal to the magnitude of the jump. We refer to Section 5C for more details.

In our setting, we will consider a variant of the generalized γ -LQG disk with additional marked points indexed by a finite set $A \subseteq \mathbb{N}$. Fix $\kappa' = 16/\gamma^2 \in (4, 8)$, $\sigma \in \mathfrak{S}_{\kappa'}^A$, $\Lambda \geq 0$, and $\ell > 0$.⁴ The measure on surfaces we consider is defined as

$$\bar{M}_{\Lambda, \ell}^{\sigma, \kappa'} = \sum_{Q \in \Pi(e, A)} \prod_{\substack{t < \ell \\ \Delta e_t \neq 0}} M_{\Lambda, |\Delta e_t|}^{\sigma|_{\Delta Q_t}, \kappa'} P(de), \quad (1-8)$$

where P is the law of E and $\Pi(e, A)$ is the set of nonincreasing (in the sense of inclusion) càdlàg functions $Q : [0, \ell] \rightarrow \mathcal{P}(A)$ such that $Q_0 = A$, $Q_{\ell-} = \emptyset$ and $\Delta Q_t := Q_{t-} \setminus Q_t = \emptyset$ whenever $\Delta e_t := e_t - e_{t-} = 0$. The interpretation of Q_t is the collection of indices of points in the unexplored part of the generalized γ -LQG disk at time t , so in particular ΔQ_t is the collection of indices of points which are in the regular γ -LQG disk with boundary length Δe_t . A surface sampled from $\bar{M}_{\Lambda, \ell}^{\sigma, \kappa'}$ should be viewed as the gluing of surfaces sampled from $M_{\Lambda, |\Delta e_t|}^{\sigma|_{\Delta Q_t}, \kappa'}$ into a tree structure according to the excursion e .

The weight $W_{\Lambda, \ell}^{\sigma, \kappa}$ and the partition function $Z_{\Lambda, \ell}^{\sigma, \kappa}$ of the disk are defined as

$$W_{\Lambda, \ell}^{\sigma, \kappa'} = |\bar{M}_{\Lambda, \ell}^{\sigma, \kappa'}| \quad \text{and} \quad Z_{\Lambda, \ell}^{\sigma, \kappa'} = \ell^{-1-4/\kappa'} W_{\Lambda, \ell}^{\sigma, \kappa'}, \quad (1-9)$$

and (1-6) still holds with $\Lambda \ell^{8/\kappa'}$ instead of $\Lambda \ell^2$ in the subscript on the right-hand side and with κ' instead of κ . The exponent $8/\kappa'$ describes how the LQG area scales with the generalized boundary length of the disk.

1F. Main results on partition functions and the law of the exploration. For $\kappa \in (8/3, 4)$ consider a nonnested CLE_κ Γ in $\mathbb{D}$ and fix distinct points $w_0, w_\infty \in \partial\mathbb{D}$. In [Miller et al. 2017; 2022], Miller, Sheffield, and Werner consider a natural exploration of Γ . The exploration is a continuous curve λ known as the continuum percolation interface (CPI) which starts at w_0 and ends at w_∞ . For a parameter $\beta \in [-1, 1]$, first assign independently to each loop in Γ a counterclockwise or clockwise orientation with probability $(1 + \beta)/2$ and $(1 - \beta)/2$, respectively. Then the CPI λ is an $\text{SLE}_{\kappa'}$ -type curve ($\kappa' = 16/\kappa$) that stays in the carpet and leaves the counterclockwise (resp. clockwise) loops to its right (resp. left). See Section 5A for further details.

We consider an embedding h of a (regular) γ -LQG disk of boundary length $\ell > 0$ into the unit disk $\mathbb{D}$ with the property that a point sampled from the boundary measure of the disk is mapped to w_0 under

⁴We write κ' instead of κ in most of the paper when this parameter takes its value in $(4, 8)$.

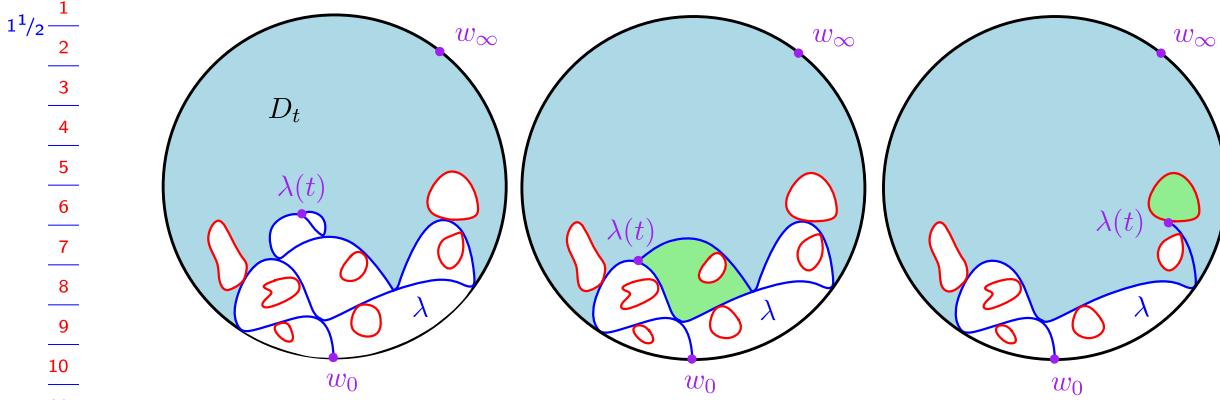


Figure 2. Left: illustration of the CPI λ (blue curve) until some time $t > 0$ and the loops attached to $\lambda|_{[0,t]}$ (red). We have that $\xi(t)$ is the union of the blue curve and the red loops, while D_t is the domain on blue background. Center: we have $\Delta L_t < 0$ since λ hits the left boundary of the domain at time t . The domain which is cut off at time t is shown in green. Right: we have $\Delta R_t > 0$ since λ hits a counterclockwise oriented loop on its right side at time t . The domain which is cut off at time t is shown in green.

the embedding and such that the clockwise (resp. counterclockwise) boundary arc from w_0 to w_∞ has length ℓ_L (resp. ℓ_R) and where $\ell_L, \ell_R > 0$ are such that $\ell_L + \ell_R = \ell$. We suppose that the CLE_κ and CPI have been sampled independently of h . As explained later, by conformal invariance of CLEs and CPIs, we can view the CLE and CPI as being drawn on top of the γ -LQG disk.

There is a natural way to measure the length of SLE-type curves using the γ -LQG measure, which we refer to as γ -LQG or quantum natural lengths (see Section 5B). Here, we suppose λ has been parametrized by its quantum natural length and denote its total duration by ζ .

For $t < \zeta$, let $\xi(t)$ denote the union of $\lambda([0, t])$ and the loops touching $\lambda([0, t])$, namely

$$\xi(t) := \lambda([0, t]) \cup \bigcup_{\substack{\eta \in \Gamma \\ \eta \cap \lambda([0, t]) \neq \emptyset}} \eta. \quad (1-10)$$

Let D_t be the complementary component of $\mathbb{D} \setminus \xi(t)$ which has w_∞ on its boundary. For $t \in [0, \zeta)$, the two points $\lambda(t)$ and w_∞ divide the boundary ∂D_t of D_t into a left boundary arc and a right boundary arc. Denote the quantum boundary lengths of these by L_t and R_t , respectively. We set $L_t = R_t = 0$ for $t \geq \zeta$.

If the CPI λ hits the left boundary of D_{t-} at time t then $\Delta L_t := L_t - L_{t-} < 0$ and the domain which is separated from w_∞ by λ at time t has quantum boundary length equal to $|\Delta L_t|$. If the CPI λ hits a clockwise oriented loop η at time t then $\Delta L_t := L_t - L_{t-} > 0$ and the quantum length of the loop is equal to ΔL_t . The same statements hold with right, R , and counterclockwise instead of left, L , and clockwise, respectively. See Figure 2.

The objects defined in the previous three paragraphs also make sense in the setting of LQG disks and CLE with additional marked points, as defined in Sections 1B and 1C, and we will describe this case in more detail in this paragraph. Assume $\kappa \in (8/3, 4)$, $\sigma \in \mathfrak{S}_\kappa$, $\Lambda \geq 0$, and $\ell > 0$ are such that $W_{\Lambda, \ell}^{\sigma, \kappa} < \infty$

¹/₂ (we address the admissibility question in [Theorem 1.11](#) below). First sample (h, z) according to $M_{\Lambda, \ell}^{\sigma, \kappa}$
² normalized to be a probability measure and then sample $\Gamma \sim \text{CLE}_{\kappa}^{\sigma}(z)$ conditionally independently given z .
³ Let λ be a CPI within Γ as in [Definition 1.3](#), define L , R and ζ as in the previous two paragraphs and for
⁴ $t < \zeta$ let $P_t \subseteq A$ denote the indices of the points contained in D_t (with $P_t = \emptyset$ for $t \geq \zeta$). We summarize
⁵ all these processes in a tuple
⁶

$$Y = (L, R, P).$$

⁷
⁸ Above we considered the case $\kappa \in (8/3, 4)$ but all of the objects can be defined in an analogous way for
⁹ $\kappa' \in (4, 8)$. Indeed, if $\kappa' \in (4, 8)$, we consider a generalized LQG surface sampled from $\bar{M}_{\Lambda, \ell}^{\sigma, \kappa'}$, assuming
¹⁰ the parameters are such that this measure has finite mass so that it can be renormalized to be a probability
¹¹ measure. Decorating the LQG surface with a $\text{CLE}_{\kappa'}^{\sigma}(z)$ and a CPI, we can define $\zeta > 0$ and a process
¹² $Y = (L, R, P)$ as before. We refer to [Section 5D](#) for a precise description of these objects in the case of
¹³ no marked points; the construction is analogous in the case with marked points.

¹⁴ **Theorem 1.4.** *Let $\kappa \in (8/3, 4)$ (resp. $\kappa \in (4, 8)$) and consider the surfaces and processes introduced*
¹⁵ *above. The process Y is a strong Markov process and the rate at which the process jumps from (ℓ_L, ℓ_R, B)*
¹⁶ *to $(\ell_L + s, \ell_R, C)$ and $(\ell_L - s, \ell_R, C)$ for $C \subseteq B$, respectively, is, for a constant $v_{\beta}^{\kappa} > 0$, given by*
¹⁷

$$-v_{\beta}^{\kappa} \cos\left(\frac{4\pi}{\kappa}\right)(1 - \beta) \frac{Z_{\Lambda, \ell_L + \ell_R + s}^{\sigma|C, \kappa} Z_{\Lambda, s}^{\sigma|B \setminus C, \kappa}}{Z_{\Lambda, \ell_L + \ell_R}^{\sigma|B, \kappa}} \quad \text{and} \quad v_{\beta}^{\kappa} \frac{Z_{\Lambda, \ell_L + \ell_R - s}^{\sigma|C, \kappa} Z_{\Lambda, s}^{\sigma|B \setminus C, \kappa}}{Z_{\Lambda, \ell_L + \ell_R}^{\sigma|B, \kappa}}. \quad (1-11)$$

¹⁸
¹⁹
²⁰ ²⁰/₂ *The same holds for jumps from (ℓ_L, ℓ_R, B) to $(\ell_L, \ell_R + s, C)$ and $(\ell_L, \ell_R - s, C)$, respectively, except*
²¹ *that the multiplicative factor of $1 - \beta$ is replaced by $1 + \beta$ in the first formula.*

²² *Conditionally on the process Y the quantum surfaces which are cut out by the CPI λ (corresponding*
²³ *to the negative jumps of L and R) and the quantum surfaces given by the inside of the discovered loops*
²⁴ *(corresponding to the positive jumps of L and R) are independent regular (resp. generalized) quantum*
²⁵ *disks with marked points and boundary length as given by Y .*
²⁶

²⁷ A more precise version of the theorem which also includes an explicit formula for the generator of the
²⁸ process Y is found in [Theorem 9.1](#).
²⁹

³⁰ **Remark 1.5.** The CPI is the continuum counterpart of the peeling process on planar maps, and the jump
³¹ rates appearing in the theorem are reminiscent of those encountered in the peeling process; see [\[Budd](#)
³² [2018, equation \(14\)\]](#) and [Section 2](#). Namely, the probability of a particular event is proportional to the
³³ product of the partition function of the surfaces we encounter on this event; see the numerators in (1-11).

³⁴ **Remark 1.6.** Variants of the above theorem can be proved in a number of related settings. For example,
³⁵ instead of letting the CPI go towards a target point w_{∞} on the boundary of the disk, one can, for instance,
³⁶ explore towards one of the marked points z_i or always go into the component with largest boundary
³⁷ length when the domain of exploration is cut in two. The latter exploration in the setting with no marked
³⁸ points is considered in [\[Miller et al. 2022, Theorem 1.2\]](#). The transition rates in this case are obtained
³⁹ from the transition rates in the boundary-to-boundary exploration, and this proof carries through without
⁴⁰ changes to the setting of marked points.

The special case of the theorem where $n = 0$ was proved in [Miller et al. 2022]. Our proof takes that result as a starting point and obtains the general result by applying a reweighting via Girsanov's theorem (see Proposition 8.11).

From the proof leading to the above theorem one can prove a recursive formula for the partition function of the disk with $\#A \geq 2$ marked points in terms of the partition function of the disks with strictly fewer marked points. We remark that the process (L, R) appearing in the theorem below can be described explicitly (up to a rescaling of time which does not affect the values of the expectations below) via a simple reweighting of a Lévy process; see Section 3. This result arises from decomposing LQG surfaces using CPI curves as explained in the paragraph below the theorem statement.

Theorem 1.7. *Let $\kappa \in (8/3, 8) \setminus \{4\}$, $A \subseteq \mathbb{N}$ finite, $\sigma \in \mathfrak{S}_\kappa^A$, $\Lambda \geq 0$, and $\ell > 0$, and suppose $\#A \geq 2$. Define $X = L + R$, where (L, R) is as in Theorem 1.4 for $A = \emptyset$ and $\Lambda = 0$. Then*

$$W_{\Lambda, \ell}^{\sigma, \kappa} = \sum_{m \in \mathbb{N}_0} V_{\Lambda, \ell}^m, \quad (1-12)$$

where

$$V_{\Lambda, \ell}^0 = \sum_{\emptyset \neq B \subsetneq A} \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{t < \zeta} e^{\sigma_B 1(\Delta X_t > 0)} \prod_{s < t} W_{\Lambda, |\Delta X_s|}^{-, \kappa} W_{\Lambda, |\Delta X_t|}^{\sigma|_{B \setminus A}, \kappa} W_{\Lambda, X_t}^{\sigma|_{B \setminus A}, \kappa} \right),$$

$$V_{\Lambda, \ell}^{m+1} = \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{t < \zeta} \prod_{s < t} W_{\Lambda, |\Delta X_s|}^{-, \kappa} e^{\sigma_A 1(\Delta X_t > 0)} V_{\Lambda, |\Delta X_t|}^m W_{\Lambda, X_t}^{-, \kappa} \right), \quad m \in \mathbb{N}_0.$$

We now explain the proof idea and the intuition underlying the result. When we explore the CLE decorated LQG disk with the CPI as in Theorem 1.4 there is a positive chance that all the marked points are separated from the terminal point w_∞ of the CPI at the same time, i.e., that they lie in the same complementary component of ξ . On the event that this happens, we can do a similar exploration with a new CPI in this complementary component, and we can iterate this procedure until not all the marked points lie in the same complementary component of ξ . Let $m \in \mathbb{N}_0$ denote the number of explorations we need to do before the first exploration where the points are separated. In our definition of $W_{\Lambda, \ell}^{\sigma, \kappa}$ we can split the integral into a sum of integrals according to the value of m , and this gives the decomposition appearing in Theorem 1.7. The proof of the theorem (found in Section 9) builds on this intuition but also needs to rule out certain pathological behaviors of the CPI.

In the case $\#A \in \{0, 1\}$ the partition function has the following explicit closed formula. The function $\bar{K}_\nu$ (with $\nu > 0$) appearing in the statement is given by the formula

$$\bar{K}_\nu(x) := \frac{2(x/2)^\nu}{\Gamma(\nu)} K_\nu(x), \quad x > 0, \quad \bar{K}_\nu(0) = 1, \quad (1-13)$$

where K_ν is the modified Bessel function of the second kind (see the beginning of Section 4).

Theorem 1.8. *Let $\kappa \in (8/3, 8) \setminus \{4\}$, $A = \{i\}$, $\sigma \in \mathfrak{S}_\kappa^A$, $\Lambda \geq 0$, and $\ell > 0$, and set $\alpha = \alpha_{\sigma_i}^\kappa$. Then $W_{\Lambda, \ell}^{-, \kappa}, W_{\Lambda, \ell}^{\sigma, \kappa} < \infty$,*

$$W_{\Lambda, \ell}^{-, \kappa} = \bar{K}_{4/\kappa} \left(2\ell \left(\frac{\Lambda}{4 \sin(\pi \gamma^2/4)} \right)^{(\kappa/8) \vee (1/2)} \right),$$

$1^{1/2}$ and

$$\frac{W_{\Lambda, \ell}^{\sigma, \kappa}}{W_{0,1}^{\sigma, \kappa}} = \ell^{2\alpha/\sqrt{\kappa}} \bar{K}_{2/\sqrt{\kappa} \cdot (Q_\gamma - \alpha)} \left(2\ell \left(\frac{\Lambda}{4 \sin(\pi \gamma^2/4)} \right)^{(\kappa/8) \vee (1/2)} \right).$$

We remark that the formulas in [Theorem 1.8](#) are an almost immediate consequence of [\[Ang et al. 2023b\]](#) in the case $\kappa \in (8/3, 4)$ (indeed, we will relate the weights appearing in the theorem to the partition functions in the cited paper), while they follow by combining the cited paper with an explicit computation for Lévy excursions in the case $\kappa \in (4, 8)$.

Note that the formulas in [Theorem 1.8](#) take the same form for $\kappa \in (8/3, 4)$ and $\kappa \in (4, 8)$ (the exponent $(\kappa/8) \vee (1/2)$ may appear at first to take a different form, but for both parameter ranges it represents how areas scale with boundary lengths). It is interesting that the formulas take the same form since the proofs are quite different for the two parameter ranges. The fact that the formulas in the two parameter ranges coincide supports the statement that the generalized disk should be viewed as the natural extension of the regular disk for $\kappa \in (4, 8)$.

Remark 1.9. When defining the weights $W_{\Lambda, \ell}^{\sigma, \kappa}$ and partition functions $Z_{\Lambda, \ell}^{\sigma, \kappa}$ we had to make some arbitrary choices for multiplicative constants involved in their definition. Conceptually, the partition function should give the *relative* weight of different configurations, so, for example, multiplying all partition functions by a constant leaves the physical model unchanged. The degree of freedom we have lies precisely in the multiplicative constant for the zero-point and one-point partition functions, so if we fix these then the remaining partition functions are given. One can in particular observe that in [Theorem 1.4](#), replacing

$$Z_{\Lambda, \ell}^{\sigma|B, \kappa} \quad \text{by} \quad c \cdot \prod_{i \in B} c_i \cdot Z_{\Lambda, \ell}^{\sigma|B, \kappa}$$

for constants $c, c_i > 0, i \in A$, and each $B \subseteq A$ leaves the jump rates unchanged. Due to the arbitrariness in the multiplicative constant for the one-point partition function, we only consider the *ratio* $W_{\Lambda, \ell}^{\sigma, \kappa} / W_{0,1}^{\sigma, \kappa}$ in [Theorem 1.8](#) but we remark that the denominator $W_{0,1}^{\sigma, \kappa}$ can be computed explicitly using results of [\[Remy and Zhu 2022\]](#). For the zero-point partition function we fixed the multiplicative constant by requiring that $W_{\Lambda, \ell}^{-, \kappa} = 1$.

Remark 1.10. Combining [Theorems 1.7](#) and [1.8](#), one can in principle compute the weights $W_{\Lambda, \ell}^{\sigma, \kappa}$ for an arbitrary number of marked points and any choice of σ . Such recursive computation of partition functions is closely related to topological recursion, enumeration of planar maps and the loop equation in matrix models; see, for example, [\[Eynard 2016; Eynard and Orantin 2009\]](#). On the contrary, our approach is quite different from [\[Belavin et al. 1984; Segal 2004; Guillarmou et al. 2021; ≥ 2024; Wu 2022\]](#), where higher-order correlation functions in LCFT (which, like our weights $W_{\Lambda, \ell}^{\sigma, \kappa}$, are closely related to partition functions) are expressed in terms of lower-order correlation functions via so-called conformal bootstrap; this recursive computation is of a different nature since the modulus is fixed, while we integrate over the modulus.

A key question is when we have admissibility, namely for what parameters the partition function is finite. For $\kappa \in (8/3, 8) \setminus \{4\}$ define the set $\mathcal{A}_\kappa$ of admissible parameters by

$$\mathcal{A}_\kappa = \{(\sigma, \Lambda) \in \mathfrak{S}_\kappa \times [0, \infty) : Z_{\Lambda, 1}^{\sigma, \kappa} < \infty\}.$$

¹₂ We will not entirely identify the set $\mathcal{A}_\kappa$ but the following theorem gives some of its properties. For example, assertion (ii) says that for more than one marked point it is necessary to have a positive cosmological constant $\Lambda > 0$ in order to ensure admissibility. Another feature that is worth emphasizing is assertion (iv), which says that for an arbitrary number of marked points — and, more generally, for arbitrary values of $\sigma_i \in (-\infty, -\log(-\cos(4\pi/\kappa)))$ — we can find $(\sigma, \Lambda) \in \mathcal{A}_\kappa$. Conversely, from assertion (vii) we see that if the number of points is at least two then we can find (σ, Λ) with $\Lambda > 0$ such that $(\sigma, \Lambda) \notin \mathcal{A}_\kappa$; combined with the previous sentence this implies that the set $\mathcal{A}_\kappa$ is nontrivial in the setting of at least two points. Precisely identifying the set $\mathcal{A}_\kappa$ is an interesting open problem.

Theorem 1.11. *Let $A \subseteq \mathbb{N}$ be finite and nonempty, $\sigma \in \mathfrak{S}_\kappa^A$, and $\Lambda \geq 0$.*

- (i) *If $\#A = 1$ then $(\sigma, \Lambda) \in \mathcal{A}_\kappa$.*
- (ii) *If $\#A \geq 2$ and $\Lambda = 0$ then $(\sigma, \Lambda) \notin \mathcal{A}_\kappa$.*
- (iii) *If $(\sigma, \Lambda) \in \mathcal{A}_\kappa$ and $B \subseteq A$ then $(\sigma|_B, \Lambda) \in \mathcal{A}_\kappa$.*
- (iv) *If $\sigma_B < \sigma_i$ whenever $i \in B \subseteq A$ and $\#B \geq 2$ then $(\sigma, \Lambda) \in \mathcal{A}_\kappa$ for all $\Lambda > 0$.*
- (v) *If $(\sigma, \Lambda) \in \mathcal{A}_\kappa$ for $\Lambda > 0$ then $(\sigma, \Lambda') \in \mathcal{A}_\kappa$ for all $\Lambda' > 0$.*
- (vi) *Suppose σ' is such that $\sigma'_i = \sigma_i$ for all $i \in A$ and $\sigma'_B \leq \sigma_B$ for all $B \subseteq A$ such that $\#B \geq 2$. Then if $(\sigma, \Lambda) \in \mathcal{A}_\kappa$ we have $(\sigma', \Lambda) \in \mathcal{A}_\kappa$.*
- (vii) *Suppose σ' and $B' \subseteq A$ are such that $\#B' \geq 2$ and $\sigma'_B = \sigma_B$ for all $B \subseteq A$, $B' \neq B$. For σ'_B , sufficiently large (depending on σ_B for $B \neq B'$) we have $(\sigma', \Lambda) \notin \mathcal{A}_\kappa$.*

We give an alternative proof of the following theorem, which was first proved in [Schramm et al. 2009] via Itô calculus.⁵ The proof is based on ideas from this paper in the $\kappa \neq 4$ case and the level line coupling of CLE₄ with the GFF for $\kappa = 4$. In the case $\kappa \neq 4$ we will use the explicit law of the boundary length process associated with the CPI for the case of $\#A = 0$ marked points (as derived in [Miller et al. 2021; 2022]) along with an exact computation for this process. See Section 6. The theorem plays an important role in our proof of Theorem 1.2 and we want to emphasize that our proof of Theorem 1.12 does not rely on Theorem 1.2.

Theorem 1.12 [Schramm et al. 2009]. *Let Γ be a nonnested CLE _{κ} in $\mathbb{D}$ for $\kappa \in (8/3, 8)$. We write η for the loop surrounding 0 and recall that η° denotes the set of points surrounded by η . Then, for all $\rho > -1 + 2/\kappa + 3\kappa/32$,*

$$\mathbb{E}(R(0, \eta^\circ)^\rho) = \frac{-\cos(4\pi/\kappa)}{\cos(\pi\sqrt{(1-4/\kappa)^2 - 8\rho/\kappa})},$$

while the expectation is infinite for $\rho \leq -1 + 2/\kappa + 3\kappa/32$. The density of $-\log R(0, \eta^\circ)$ on $(0, \infty)$ is

$$\begin{aligned} x &\mapsto \frac{-\cos(4\pi/\kappa)}{4\pi/\kappa} \sum_{n \geq 0} (-1)^n \left(n + \frac{1}{2}\right) \exp\left(\frac{(1-4/\kappa)^2}{8/\kappa} x - \frac{(n+1/2)^2}{8/\kappa} x\right) \\ &= -\frac{4\sqrt{2\pi/\kappa} \cos(4\pi/\kappa)}{x^{3/2}} \sum_{n \geq 0} (-1)^n \left(n + \frac{1}{2}\right) \exp\left(\frac{(1-4/\kappa)^2}{8/\kappa} x - \frac{8\pi^2}{\kappa} \left(n + \frac{1}{2}\right)^2 x^{-1}\right). \end{aligned}$$

⁵The expression for the density in Theorem 1.12 can be found just below equation (2.22) in [Ciesielski and Taylor 1962].

^{11/2} **1G. Outline.** In [Section 2](#) we will describe some discrete counterparts of our results in the setting of random planar maps and formulate scaling limit conjectures. In [Section 3](#) we study certain spatially inhomogeneous jump Markov processes (for which the process Y appearing in [Theorem 1.4](#) is a special case), derive a formula for their generator, and do an explicit computation that is used later in the proof of [Theorem 1.12](#). In [Section 4](#) we do computations for stable Lévy excursions which will be used when computing the partition function of the generalized LQG disk with zero or one marked points. We emphasize that the two latter sections can be read without any knowledge of LQG and CLE and only build on the theory of stochastic processes with values in $\mathbb{R}$ or in $\mathbb{R}^2$.

In [Section 5](#) we present relevant background material on LQG and CLE, and in particular we give a precise description of the results on CLE explorations of [\[Miller et al. 2021; 2022\]](#) that are reviewed briefly above. We then conclude the proof of [Theorem 1.12](#) in [Section 6](#) and give the proof of [Theorem 1.2](#) in [Section 7](#). The technical bulk of the latter section is devoted to a spatial independence result for CLE.

In [Section 8](#) we give the precise definition of the LQG surface with marked points that appears in [Theorem 1.4](#) and conclude the proof of [Theorem 1.8](#). We also argue that the disk with marked points can be obtained via a limiting argument involving the disk with no marked points and we use the latter result to prove part of [Theorem 1.4](#). Finally, in [Section 9](#) we use inputs from the previous sections to conclude the proofs of [Theorems 1.4, 1.7, and 1.11](#).

The appendices contain various results and computations that are needed for the remainder of the paper. We mention in particular a result on renewal processes of independent interest in [Appendix A](#).

^{201/2} **Notation.** We let $\mathbb{N}$ denote the positive integers $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathbb{N}_0$ denote the nonnegative integers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. For $z \in \mathbb{C}$ and $r > 0$ let $B_r(z) = \{w \in \mathbb{C} : |z - w| < r\}$ denote the open ball of radius r centered at z . For an interval $I \subseteq \mathbb{R}$ we let $D(I)$ denote the set of real-valued càdlàg functions on I . We will often identify a curve with its trace.

For functions $f, g : S \rightarrow (0, \infty)$ and some set S we write $f(x) \lesssim g(x)$ if there is a constant $C \in (0, \infty)$ such that for all $x \in S$ we have $f(x) \leq Cg(x)$. The dependence of C on other parameters will be either clear from the context or stated explicitly.

2. Discrete motivation

We will elaborate on the discrete counterparts of our results in the setting of random planar maps. For concreteness, we will describe all things in the setting of triangulations, but analogous constructions and conjectures can be stated in the setting of other planar maps (quadrangulations, general maps, etc.) as well. We call a planar map a triangulation with boundary if all faces have exactly three edges except for one distinguished face, called the exterior face, with arbitrary degree. The number of edges along the boundary of the exterior face is called the perimeter of the planar map. We allow the boundary of the exterior face to be nonsimple.

Fix parameters

$$A \subseteq \mathbb{N} \text{ finite}, \quad n \in (0, 2), \quad g, h, \Lambda \geq 0 \quad \text{and} \quad \sigma_B \in \mathbb{R} \quad \text{for all } B \subseteq A \text{ with } \sigma_\emptyset = 0.$$

^{391/2} Let $T = (V_T, E_T, F_T, e)$ be a finite planar triangulation with boundary of perimeter ℓ , where e is a distinguished (root) edge on the boundary. We consider a collection Γ of vertex-disjoint (possibly nested)

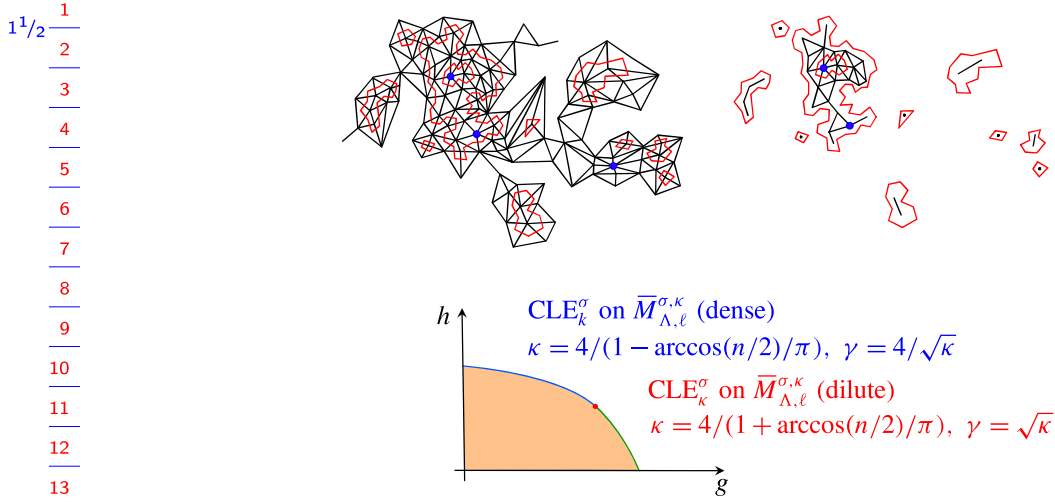


Figure 3. Top: an illustration of the spatial Markov property of the $O(n)$ planar map model weighted by nesting statistics: conditionally on the boundary lengths of the outermost loops and conditionally on the points surrounded by the outermost loops, the planar maps surrounded by these outermost loops are independent and have the law of planar maps with a loop $O(n)$ model and marked points. Bottom: the orange area is the subcritical range where convergence to the continuum random tree is expected. The generic critical range where we conjecture convergence to the $\sqrt{8/3}$ -LQG disk (also known as the Brownian disk) without loops is colored in green. Finally, the expected scaling limits of the blue range and the red point are indicated in the figure. All other parameters are inadmissible.

loops where the vertices of the loops are on the faces of T . Figure 3 illustrates the above definitions. For distinct points $x_i \in V_T$ with $i \in A$ and $\emptyset \neq B \subseteq A$, we let $N_B^\Gamma(x)$ denote the number of loops surrounding x_i for each $i \in B$ but not surrounding the point x_i whenever $i \notin B$. We let $L(\Gamma)$ be the total length of all the loops in Γ . If $A = \emptyset$ we write $\sigma = -$.

To each such triangulation T , configuration of loops Γ on T , and $\mathbf{x} = (x_i : i \in A)$ where $x_i \in V_T$ with $i \in A$ are distinct, we associate the weight

$$m_{\Lambda, \ell}^{\sigma, n, g, h}(T, \Gamma, \mathbf{x}) = g^{\#F_T - L(\Gamma)} h^{L(\Gamma)} n^{\#\Gamma} e^{-\Lambda \#V_T} \cdot \prod_{\emptyset \neq B \subseteq A} e^{\sigma_B N_B^\Gamma(\mathbf{x})}.$$

We say that $(n, g, h, \Lambda, \sigma)$ is admissible if the total mass

$$|m_{\Lambda, \ell}^{\sigma, n, g, h}| := \sum_{(T, \Gamma, \mathbf{x})} m_{\Lambda, \ell}^{\sigma, n, g, h}(T, \Gamma, \mathbf{x})$$

is finite for each $\ell \geq 1$, so that we can define a probability measure

$$p_{\Lambda, \ell}^{\sigma, n, g, h}(T, \Gamma, \mathbf{x}) = \frac{m_{\Lambda, \ell}^{\sigma, n, g, h}(T, \Gamma, \mathbf{x})}{|m_{\Lambda, \ell}^{\sigma, n, g, h}|}. \quad (2-1)$$

When $A = \emptyset$ and $\Lambda = 0$ this model is the classical loop $O(n)$ model on a planar map, which has been extensively studied using combinatorial and probabilistic techniques; see, e.g., [Borot et al. 2012; Budd 2018; Chen et al. 2020]. The model for nonzero σ is much less studied; however the asymptotics of the law of the nesting statistic $N_A^\Gamma(x)$ when $\#A = 1$ have been investigated in [Borot et al. 2023]. The paper [Ikhlef et al. 2016] studies a loop model on a lattice where the weight of a configuration depends on the topological configuration of loops around a collection of marked points. We call the parameter Λ the cosmological constant, as in, e.g., [David et al. 2016; Huang et al. 2018].

The model defined above satisfies a very simple Markov property. Consider

$$P = (T, \Gamma, \mathbf{x}) \sim p_{\Lambda, \ell}^{\sigma, n, g, h}$$

and define Γ_* to be the outermost loops of Γ . For each loop $\eta \in \Gamma_*$ we let P_η be the planar map enclosed by the loop η (see Figure 3) and let $\sigma|_{B_\eta} = (\sigma_C : C \subseteq B_\eta)$ where B_η is the collection of $i \in A$ such that x_i is surrounded by the loop η . We also write ℓ_η for the perimeter of P_η . Then conditionally on $(\ell_\eta : \eta \in \Gamma_*)$ and $(B_\eta : \eta \in \Gamma_*)$, the maps $(P_\eta : \eta \in \Gamma_*)$ are independent and have the laws

$$P_\eta \sim p_{\Lambda, \ell_\eta}^{\sigma|_{B_\eta}, n, g, h} \quad \text{for all } \eta \in \Gamma_*.$$

One can explore the planar map via a peeling process where at each step one peels through an edge, which means the following. Consider the part of the planar map that has been explored already, sample one of its boundary edges in a Markovian way, and consider the unexplored face on the other side of this edge. We then discover one of the following three scenarios:

- (i) a face through which no loop passes,
- (ii) (at once) an outermost loop (i.e., an element of Γ_*) together with all the faces intersecting it and surrounded by it, or
- (iii) a face for which all three vertices lie on the boundary of the undiscovered part of the map.

See [Budd 2018] for a description in the setting of quadrangulations. One can write down the probabilities for each of these scenarios in terms of the partition function of the surfaces we get after doing the peeling step. The continuum results in this paper can be viewed as the continuum analog of this story: the case (ii) will correspond to the discovery of CLE loops by a CPI, the case (iii) to boundary intersections of the CPI and (i) results in the Lévy compensation in the $\kappa < 4$ case (whereas in the $\kappa > 4$ case, the scenario (i) disappears in the scaling limit).

2A. Planar maps without marked points. In [Borot et al. 2012; Budd 2018] an asymptotic expression for the partition function of planar maps with the loop $O(n)$ model (i.e., the case $A = \emptyset$ and $\Lambda = 0$) was established in the setting of quadrangulations when the loops were required to be rigid, i.e., a loop leaves and enters a quadrangle through opposite sides of a face. The scaling exponent they derived took a different form in four different phases: subcritical, generic critical, nongeneric critical dense, and nongeneric critical dilute. We focus only on what corresponds under scaling limit conjectures to the latter two phases, and we write $\mathcal{D} \subseteq (0, \infty)^2$ and $\{(g_*, h_*)\} \subseteq (0, \infty)^2$ for the respective parameter ranges. The

¹/₂ analogous result of [Borot et al. 2012; Budd 2018] in the setting of triangulations would be, for some
² constant $C(n, g, h) > 0$ and with $\kappa = \kappa(n, g, h)$ and $\gamma = \gamma(n, g, h)$ as in Figure 3,
³

$$|m_{\ell,0}^{-,n,g,h}| \sim C(n, g, h) \ell^{-1-4/\kappa} \quad \text{as } \ell \rightarrow \infty. \quad (2-2)$$

⁴
⁵ It is expected that a random planar map sampled according to $(T_\ell, \Gamma_\ell) \sim p_{0,\ell}^{\sigma,n,g,h}$ converges in the
⁶ scaling limit to a γ -LQG surface decorated by a CLE_κ . To be slightly more precise, T_ℓ defines a metric
⁷ measure space by equipping T_ℓ with its graph distance and giving each vertex unit mass, and (T_ℓ, Γ_ℓ)
⁸ defines a loop-decorated metric measure space. It is believed that if we rescale distances and mass
⁹ appropriately as $\ell \rightarrow \infty$ then (T_ℓ, Γ_ℓ) converges in the scaling limit to the γ -LQG disk decorated
¹⁰ by an independent CLE_κ . The convergence here is for some natural generalization of the Gromov–
¹¹ Hausdorff–Prokhorov topology [Abraham et al. 2013] for loop-decorated metric measure spaces, e.g., the
¹² Gromov–Hausdorff–Prokhorov–Loop (GHPL) topology described in [Gwynne et al. 2021]. We formulate
¹³ the following precise conjecture.
¹⁴

¹⁵ **Conjecture 2.1.** *Let $n \in (0, 2)$, $(g, h) \in \mathcal{D} \cup \{(g_*, h_*)\}$ and let $(T_\ell, \Gamma_\ell) \sim p_{0,\ell}^{-,n,g,h}$. Let $\kappa = \kappa(n, g, h)$
¹⁶ and $a(\kappa) := 2 \wedge (8/\kappa)$. Then there exist unique constants $d(\kappa) > 0$ and $c, c' > 0$ such that (T_ℓ, Γ_ℓ) viewed
¹⁷ as a loop-decorated metric measure space, where the metric is given by the graph distance multiplied by
¹⁸ $c\ell^{-2/d(\kappa)}$ and the measure by the counting measure on the vertices multiplied by $c'\ell^{-a(\kappa)}$, converges in
¹⁹ law to $\bar{M}_{0,1}^{-,\kappa}$ (normalized to be a probability measure) as $\ell \rightarrow \infty$ in the GHPL topology. This is a γ -LQG
²⁰ surface with $\gamma = \gamma(n, g, h)$.
²¹*

²² In fact, the constant $d(\kappa)$ should be the almost sure Hausdorff dimension of the γ -LQG disk (see
²³ [Gwynne and Miller 2021; Ding et al. 2020] for further information on this). Conjectures of this kind can
²⁴ be found in much earlier literature; see, for instance, the papers on the loop $O(n)$ model on planar maps
²⁵ referenced above. However, unlike in the cited papers, we construct a candidate limiting surface in the
²⁶ dense regime explicitly as a generalized (rather than regular) γ -LQG disk. The limiting surface should
²⁷ be a generalized (not regular) disk in the dense regime since, as explained above when describing the
²⁸ Markov property, the planar map has the same law as the map surrounded by an $O(n)$ loop conditioned
²⁹ on boundary length, and the latter map converges in the scaling limit to a generalized disk (that is, a
³⁰ disk with pinch points corresponding to the pivotals of the $O(n)$ loop model in the dense regime). One
³¹ can also verify that the scaling properties of the partition function (2-2) in the dilute and dense regimes
³² are consistent with the respective continuum scaling properties of the regular and generalized disks (see
³³ Sections 1C and 1E, in particular (1-6)).

³⁴ We conjectured above that in the dense regime, $(T_\ell, \Gamma_\ell) \sim p_{0,\ell}^{-,n,g,h}$ converges to a generalized (rather
³⁵ than regular) disk in the scaling limit. However, if we condition on the event that the boundary of (T_ℓ, Γ_ℓ)
³⁶ is simple, we believe that we instead get a regular (not generalized) disk. In particular, the limiting
³⁷ behavior of the planar map depends on whether we require the boundary to be simple or not. In private
³⁸ communication, Linxiao Chen has provided evidence for this belief (in the case of quadrangulations with
³⁹ a rigid $O(n)$ decoration) by showing that the partition function for maps with simple boundary (in the
⁴⁰ dense regime) grows like ℓ^{-1-4/γ^2} , namely similarly to the regular γ -LQG disk.

¹₁ The case $n = 1$ and $g = h$ with $(g, h) \in \mathcal{D}$ describes percolation-decorated triangulations with
²₂ monochromatic (vertex) boundary and percolation parameter $1/2$. Indeed, a percolation configuration
³₃ with monochromatic boundary can be mapped to a collection of loops in the bulk of the planar map
⁴₄ by sending each percolation configuration to the collection of interfaces between clusters of opposite
⁵₅ colors. Uniformly sampled triangulations (which can also be seen as percolation-decorated triangulations
⁶₆ with free boundary conditions and percolation parameter $1/2$ after forgetting the percolation decoration)
⁷₇ do not have pinch points in the scaling limit (even when the boundary of the planar map is allowed to
⁸₈ be nonsimple). At first sight this may seem to contradict [Conjecture 2.1](#), where we predict that we get
⁹₉ a generalized disk (i.e., a disk with pinch points) in the case $n = 1$, $g = h$, and $(g, h) \in \mathcal{D}$. However,
¹⁰₁₀ there is no contradiction since the boundary data are different in the two settings (monochromatic versus
¹¹₁₁ free). The crucial point to take away from this discussion is that conditioning the vertex boundary to
¹²₁₂ be monochromatic is expected to result in the appearance of pinch-points in the scaling limit while
¹³₁₃ percolation-decorated random triangulations without this conditioning do not have pinch points in the
¹⁴₁₄ scaling limit.
¹⁵₁₅

¹⁶₁₆ **2B. Planar maps with marked points.** Motivated by [Conjecture 2.1](#) and the results stated in the introduc-
¹⁷₁₇ tion, we formulate the following conjecture. The topology of convergence is for a variant of the GHPL
¹⁸₁₈ topology which accommodates the appearance of marked points. A key takeaway from the conjecture is
¹⁹₁₉ the relationship between the σ_i 's and the $\alpha_{\sigma_i}^\kappa$'s, where the latter parameters describe the magnitude of the
²⁰₂₀ logarithmic singularities of the surface (see [Definition 1.1](#)).
²¹₂₁

²²₂₂ **Conjecture 2.2.** Let $n \in (0, 2)$ and let $(g, h) \in \mathcal{D} \cup \{(g_*, h_*)\}$. Let $\kappa = \kappa(n, g, h)$, $a(\kappa)$, $d(\kappa)$ and $c, c' > 0$
²³₂₃ be as in [Conjecture 2.1](#) and let $(\sigma, \Lambda) \in \mathcal{A}_\kappa$. Then $(n, g, h, \Lambda, \sigma)$ is admissible and we may sample

$$(T_\ell, \Gamma_\ell, \mathbf{x}_\ell) \sim p_{c' \Lambda \ell^{-a(\kappa)}, \ell}^{\sigma, n, g, h}.$$

²⁴₂₄ By rescaling the counting measure by $c' \ell^{-a(\kappa)}$ and the graph metric by $c \ell^{-2/d(\kappa)}$ we obtain a loop-
²⁵₂₅ decorated metric measure space which converges in law to $\bar{M}_{\Lambda, 1}^{\sigma, \kappa}$ (normalized to be a probability measure)
²⁶₂₆ as $\ell \rightarrow \infty$ for the GHPL topology. If $(\sigma, \Lambda) \in \mathcal{A}_\kappa$ then for some function C , we have
²⁷₂₇
²⁸₂₈

$$|m_{c' \Lambda \ell^{-a(\kappa)}, \ell}^{\sigma, n, g, h}| \sim C(n, g, h, \sigma, \Lambda) \ell^{-1-4/\kappa + \sum_{i=1}^n 2\alpha_i/\sqrt{\kappa}} \quad \text{as } \ell \rightarrow \infty \quad (2-3)$$

²⁹₂₉ and
³⁰₃₀

$$\frac{C(n, g, h, \sigma, \Lambda) \cdot C(n, g, h, -, \Lambda)^{\#A-1}}{\prod_{i \in A} C(n, g, h, \sigma_i, \Lambda)} = \frac{Z_{\Lambda, 1}^{\sigma, \kappa} \cdot (Z_{\Lambda, 1}^{-, \kappa})^{\#A-1}}{\prod_{i \in A} Z_{\Lambda, 1}^{\sigma_i, \kappa}}. \quad (2-4)$$

³¹₃₁ The measures $p_{\Lambda, \ell}^{\sigma, n, g, h}$ and $\bar{M}_{\Lambda, \ell}^{\sigma, \kappa}$ are (at least heuristically) obtained from $p_{\Lambda, \ell}^{-, n, g, h}$ and $\bar{M}_{\Lambda, \ell}^{-, \kappa}$, respec-
³²₃₂ tively, by performing a reweighting according to the number of loops surrounding the marked points on
³³₃₃ the surface. Given [Conjecture 2.1](#) this makes the scaling limit result of [Conjecture 2.2](#) plausible, and the
³⁴₃₄ asymptotics of the partition functions in (2-3) are reasonable to expect in analogy with the continuum.
³⁵₃₅

³⁶₃₆ In (2-4) notice that we only claim equality of certain *ratios* of the constants $C(n, g, h, \sigma, \Lambda)$ rather
³⁷₃₇ than statements for the constants $C(n, g, h, \sigma, \Lambda)$ themselves. When defining the partition functions in
³⁸₃₈ both the discrete and the continuum there is some arbitrariness in the normalizing constants. Indeed, the
³⁹₃₉
⁴⁰₄₀

¹/₂ transition probabilities of the peeling process in the discrete and the transition rates of the continuum process Y are all functions of

$$\frac{|m_{\Lambda,\ell}^{\sigma,n,g,h}| \cdot |m_{\Lambda,\ell}^{-,n,g,h}|^{\#A-1}}{\prod_{i \in A} |m_{\Lambda,\ell}^{\sigma_i,n,g,h}|} \quad \text{and} \quad \frac{Z_{\Lambda,1}^{\sigma,\kappa} \cdot (Z_{\Lambda,1}^{-,\kappa})^{\#A-1}}{\prod_{i \in A} Z_{\Lambda,1}^{\sigma_i,\kappa}},$$

respectively, modulo the degree of freedom in the continuum corresponding the speed of the process. See Remark 1.9.

Conjecture 2.2 concerns the case when $(\sigma, \Lambda) \in \mathcal{A}_\kappa$. One can ask about properties of the discrete model for $(\sigma, \Lambda) \notin \mathcal{A}_\kappa$. We will not give a complete picture of this case, but remark that when $\sigma_i \gg 1$ then the model is not admissible since we can make $m_{0,\ell}^{\sigma,n,g,h}(T, \Gamma, \mathbf{x})$ arbitrarily large by considering a map T with a long and thin “tube” with many loops wrapping around the tube, such that the tube separates x_i and the boundary of T ; by increasing the length of the tube and adding more loops (which increases $N_{\{i\}}^\Gamma(\mathbf{x})$), the weight $m_{0,\ell}^{\sigma,n,g,h}(T, \Gamma, \mathbf{x})$ increases. We have nonadmissibility when $\sigma_{ij} \gg 1$ for a similar reason, namely it can be proved by letting x_i and x_j be close and considering a long and thin tube separating x_i and x_j from the boundary of the map T . Even if $(\sigma, \Lambda) \notin \mathcal{A}_\kappa$ it might be possible to understand the asymptotic geometry of the planar map, e.g., by considering infinite measures instead of probability measures and/or allowing infinite planar maps.

2C. Other discrete models. We remark that γ -LQG disks with α singularities are also expected to arise as the scaling limit of planar maps with other statistical physics models than the loop $O(n)$ model. In the more general setting one considers tuples $(M, P, \mathbf{x})$ with $M = (V_M, E_M, F_M)$ a planar map, P a statistical physics model on M , and $\mathbf{x} = (x_i : i \in A)$, $x_i \in V_M$. One associates a weight $w(M, P, \mathbf{x})$ to each such tuple. For certain choices of w we expect that one still gets a surface defined as in (1-3) or (1-8) in the scaling limit, but typically with the function Φ defined differently when $n > 1$. One still needs Φ to satisfy the conformal covariance property of Lemma 7.5 in order for the reweighting one does in the continuum to be intrinsic to the surface (i.e., independent of the chosen embedding), which in particular gives a unique choice for Φ (modulo multiplication by a constant) when $n = 1$. One example of a way to define w is to let it be the indicator function for some rare event for P at the vertices $\mathbf{x}$ (e.g., an arm event). We believe that rigorous scaling limit results could be established in the special case of critical percolation on triangulations with one uniformly chosen vertex conditioned to have a certain configuration of arms to the boundary of the triangulation, building on [Bernardi et al. 2023; Gwynne et al. 2021].

3. Reweighted Lévy processes: generator and explicit computation

We study certain spatially inhomogeneous jump Markov processes. The processes are constructed by applying reweightings to stable Lévy processes. We derive a formula for their generator (Theorem 3.7) and do an explicit computation (Proposition 3.12).

Determining the generator is technically challenging and builds in particular on new techniques that exploit the scale invariance of the process to control its fluctuations close to its starting point, as well as more standard tools like Palm’s formula for Poisson point processes. We expect that variants of our

^{11/2}₁ techniques also work for a large class of other Markov processes that are given by reweighted Lévy
₂ processes. The explicit calculation is done via a martingale argument involving the generator of a simple
₃ variant of the Markov process.
₄

₅ The Markov processes we consider would typically arise in settings similar to our paper, namely in the
₆ setting of growth-fragmentation processes where the jumps represent splitting/fragmentation events, the
₇ rates of the various jumps can be expressed in terms of the respective partition functions, and we keep
₈ track of a fixed number of marked points. In this section we consider jumps rates (or partition functions)
₉ of a more general form than the rest of the paper. We emphasize that the proofs in this section do not
₁₀ build on properties of LQG and CLE but are done via direct analysis of the Markov processes.

₁₁ Let us start by recalling the notion of a generator of a Markov process. Let E be a metric space and let
₁₂ $X = (X_t : t \geq 0)$ be a time-homogeneous càdlàg Markov process on E so X is a random variable in the
₁₃ space $D([0, \infty), E)$ of càdlàg functions. Then for $f : E \rightarrow \mathbb{R}$ let

$$\mathcal{G}f(x) = \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_x(f(X_t) - f(x)) \quad (3-1)$$

₁₆ provided the limit exists for all $x \in E$. We call the operator $\mathcal{G}$ the generator of X . Sometimes, uniform
₁₇ convergence in x is required in (3-1) but here it is more convenient to work with the definition given
₁₈ above. Throughout this section, we will not determine the whole set of functions f for which (3-1) exists
₁₉ but instead show the existence of and give the expression for $\mathcal{G}f$ for certain nice (namely, smooth and
₂₀ compactly supported) functions f .
_{20 1/2}

₂₁ The section is divided into two subsections where we consider the cases with and without, respectively,
₂₂ Lévy compensation. These correspond to the cases where the LQG disks and the CLE are simple,
₂₃ $\kappa \in (8/3, 4)$, and nonsimple, $\kappa' \in (4, 8)$, respectively. At a heuristic level, the Lévy compensation of
₂₄ the boundary length process of a CPI occurs while the CPI is moving in between CLE loops (see also
₂₅ the discrete intuition right before the start of [Section 2A](#)). It is not so surprising that a compensation is
₂₆ needed in the simple case and not in the nonsimple case since in the former case the CLE loops do not
₂₇ touch and the trunk of the CPI “spends more time” in between the loops.
₂₈

₂₉ **3A. With Lévy compensation.** Let $\kappa \in (8/3, 4)$ and $\beta \in [-1, 1]$. We consider independent compensated
₃₀ Lévy processes L and R such that the generator of the process (L, R) is given by

$$\begin{aligned} \mathcal{L}_{\kappa, \beta} f(l, r) = & \int_{\mathbb{R}} \left(f(l+h, r) - f(l, r) - h \frac{\partial f(l, r)}{\partial l} \right) \mu_L(dh) \\ & + \int_{\mathbb{R}} \left(f(l, r+h) - f(l, r) - h \frac{\partial f(l, r)}{\partial r} \right) \mu_R(dh) \end{aligned} \quad (3-2)$$

₃₆ for $f \in C_c^\infty(\mathbb{R}^2)$ where μ_L and μ_R (the jump measures of L and R) are given by

$$\begin{aligned} \frac{\mu_L(dh)}{dh} &= \frac{-\cos(4\pi/\kappa)(1-\beta)/2}{h^{1+4/\kappa}} 1_{(0, \infty)}(h) + \frac{1/2}{|h|^{1+4/\kappa}} 1_{(-\infty, 0)}(h), \\ \frac{\mu_R(dh)}{dh} &= \frac{-\cos(4\pi/\kappa)(1+\beta)/2}{h^{1+4/\kappa}} 1_{(0, \infty)}(h) + \frac{1/2}{|h|^{1+4/\kappa}} 1_{(-\infty, 0)}(h). \end{aligned}$$

¹/₂ See [Kallenberg 2002, Theorem 19.10]. More informally, L is a jump process which transitions from l to $l + h$ at rate $-\cos(4\pi/\kappa)(1 - \beta)/2 \cdot h^{-1-4/\kappa}$ and from l to $l - h$ at rate $(1/2) \cdot h^{-1-4/\kappa}$. This is exactly a stable Lévy process of index $4/\kappa \in (1, 2)$ and therefore we need compensation, which is reflected in the generator expression via the compensation term $-h\partial f(l, r)/\partial l$. The definition is of course chosen completely analogously in the case of R .

The compensation term in (3-2) is so that (L, R) satisfies stable scaling with exponent $4/\kappa$, that is, $((L_{\lambda^{4/\kappa}t}/\lambda, R_{\lambda^{4/\kappa}t}/\lambda) : t \geq 0)$ has the same law as (L, R) for any $\lambda > 0$. From the definition, we see that $\mathcal{L}_{\kappa, \beta} f$ is a bounded function for $f \in C_c^\infty(\mathbb{R}^2)$. Also, as stated in [Kallenberg 2002, Lemma 19.21], for $f \in C_c^\infty(\mathbb{R}^2)$ the process

$$t \mapsto f(L_t, R_t) - f(L_0, R_0) - \int_0^t \mathcal{L}_{\kappa, \beta} f(L_s, R_s) ds \quad (3-3)$$

is a martingale. We also let $X = L + R$.

In [Miller et al. 2022] a certain reweighting is performed which we now explain. We remark that besides the existence of the reweighted process (L', R') considered below, the results of Proposition 3.1 are new. For $l, r > 0$ and $f \in C_c^\infty((0, \infty)^2)$ we let

$$w_\kappa(l, r) = (l + r)^{-1-4/\kappa} 1(l, r > 0) \quad \text{and} \quad \mathcal{G}_{\kappa, \beta} f(l, r) = \frac{1}{w_\kappa(l, r)} \mathcal{L}_{\kappa, \beta}(w_\kappa f)(l, r). \quad (3-4)$$

By convention, $\mathcal{G}_{\kappa, \beta} f(0, 0) = 0$. From the definition, one can check that $\mathcal{G}_{\kappa, \beta} f$ is a bounded function.

Proposition 3.1. *For each $l, r > 0$ there is a strong Markov càdlàg process $((L'_t, R'_t) : t \geq 0)$ starting at (l, r) , taking values in $(0, \infty)^2 \cup \{(0, 0)\}$ and absorbed at $(0, 0)$ (i.e., after the process (L', R') hits $(0, 0)$ it stays at this point) such that*

$$\mathbb{E}_{(l, r)}(g(L'_t, R'_t) : t < \zeta') = \mathbb{E}_{(l, r)}\left(\frac{w_\kappa(L_t, R_t)}{w_\kappa(l, r)} g(L|_{[0, t]}, R|_{[0, t]}) : t < \tau_0\right), \quad (3-5)$$

whenever $g : D([0, t])^2 \rightarrow [0, \infty]$ is measurable on the space of real-valued càdlàg functions, $t \geq 0$, $\zeta' = \inf\{t \geq 0 : L'_t = R'_t = 0\}$ and $\tau_0 = \inf\{t \geq 0 : L_t \leq 0 \text{ or } R_t \leq 0\}$. If we define $X' = L' + R'$, then $X'_t \rightarrow 0$ as $t \rightarrow \infty$ almost surely. Let $f \in C_c^\infty((0, \infty)^2)$. Then

$$t \mapsto f(L'_t, R'_t) - f(L'_0, R'_0) - \int_0^t \mathcal{G}_{\kappa, \beta} f(L'_s, R'_s) ds \quad (3-6)$$

is a (càdlàg) martingale. Finally, $\mathcal{G}_{\kappa, \beta}$ is the generator of (L', R') on $C_c^\infty((0, \infty)^2)$.

Proof. We begin by defining a Markov process (L', R') with values in $(0, \infty)^2 \cup \{(0, 0)\}$ by specifying its transition kernel; for $t \geq 0$ and $l, r > 0$ let

$$\begin{aligned} p'_{(l, r)}(dl', dr') &= \left(1 - \mathbb{E}_{(l, r)}\left(\frac{w_\kappa(L_t, R_t)}{w_\kappa(l, r)}; t < \tau_0\right)\right) \delta_{(0, 0)}(dl', dr') \\ &\quad + \frac{w_\kappa(l', r')}{w_\kappa(l, r)} \mathbb{P}_{(l, r)}(L_t \in dl', R_t \in dr', t < \tau_0), \end{aligned}$$

$$p'_{(0, 0)} = \delta_{(0, 0)}.$$

¹/₂ The semigroup property of these transition kernels is straightforward to verify and to see that $(p^t : t \geq 0)$ is a family of Markov kernels, it therefore suffices to show that the coefficient in front of $\delta_{(0,0)}$ above is nonnegative. To this end, we consider smooth $h_n \in C_c^\infty(\mathbb{R})$ such that $h_n \geq 0$, $h_n|_{(1/n,n)} = 1$ and $h_n|_{(1/(2n),2n)^c} = 0$. Let $\tau_m = \inf\{t \geq 0 : L_t \notin (1/m, m) \text{ or } R_t \notin (1/m, m)\}$. Then we observe that

$$\begin{aligned} & \mathbb{E}_{(l,r)}(h_n(L_{t \wedge \tau_m})h_n(R_{t \wedge \tau_m})w_\kappa(L_{t \wedge \tau_m}, R_{t \wedge \tau_m})) \\ &= w_\kappa(l, r) + \int_0^t \mathbb{E}_{(l,r)}(\mathcal{L}_{\kappa,\beta}((u, v) \mapsto h_n(u)h_n(v)w_\kappa(u, v))(L_s, R_s); s < \tau_m) ds \end{aligned}$$

for all n with $l, r \in (1/n, n)$. The key is that by [Lemma B.3](#), we have $\mathcal{L}_{\kappa,\beta}w_\kappa = 0$. Since $L_s, R_s \in [1/m, m]$ when $s < \tau_m$ it is thus easy to see that the integral term above goes to 0 as $n \rightarrow \infty$ for m fixed. The claim then follows by applying Fatou's lemma by first letting $n \rightarrow \infty$ and then taking $m \rightarrow \infty$.

As mentioned above, $(p_t : t \geq 0)$ is a family of Markov kernels and hence a process (L', R') with these transition kernels can be constructed. To see that (L', R') has a càdlàg version, we will construct several supermartingales and argue that these processes having càdlàg versions implies the existence of a càdlàg version for (L', R') .

Let $\rho_L = \mathbb{P}(L_1 > 0)$ and $\rho_R = \mathbb{P}(R_1 > 0)$ be the positivity parameters of the Lévy processes L and R . We also let $\hat{\rho}_L = 1 - \rho_L$ and $\hat{\rho}_R = 1 - \rho_R$. By the definition of L and R we cannot have $\hat{\rho}_L, \hat{\rho}_R > 0$. We show that

$$(X')^{1+4/\kappa} \quad \text{and} \quad (L')^{\hat{\rho}_L \cdot 4/\kappa} (X')^{1+4/\kappa}$$

are supermartingales and so (by Doob's regularization theorem for martingales) there is a version of (L', R') such that these two processes are càdlàg and we will switch to such a version. Working with this version, we see that X' is càdlàg and hence L' is also càdlàg and, finally, the process $R' = X' - L'$ is càdlàg.

We next establish the supermartingale property. By (3-5) applied with $g(u, v) = (u_t + v_t)^{1+4/\kappa}$ we have

$$\mathbb{E}_{(l,r)}((X'_t)^{1+4/\kappa}) = (l+r)^{1+4/\kappa} \mathbb{P}_{(l,r)}(t < \tau_0) \leq (l+r)^{1+4/\kappa}.$$

Then applying (3-5) applied with $g(u, v) = (u_t)^{\hat{\rho}_L \cdot 4/\kappa} (u_t + v_t)^{1+4/\kappa}$,

$$\begin{aligned} \mathbb{E}_{(l,r)}((L'_t)^{\hat{\rho}_L \cdot 4/\kappa} (X'_t)^{1+4/\kappa}) &\leq (l+r)^{1+4/\kappa} \mathbb{E}_{(l,r)}((L_t)^{\hat{\rho}_L \cdot 4/\kappa}; \inf_{[0,t]} L > 0) \\ &= (l+r)^{1+4/\kappa} l^{\hat{\rho}_L \cdot 4/\kappa}. \end{aligned}$$

The final equality is established in [\[Kyprianou and Pardo 2022, Section 5.4\]](#) (this martingale was first identified in [\[Silverstein 1980, Theorem 2\]](#); see the remarks below (1.15) in the same reference). The Markov property of (L', R') implies the supermartingale property. By supermartingale convergence, we have that $\lim_{t \rightarrow \infty} (X'_t)^{1+4/\kappa}$ exists almost surely and since

$$\mathbb{E}_{(l,r)}((X'_t)^{1+4/\kappa}) = (l+r)^{1+4/\kappa} \mathbb{P}_{(l,r)}(t < \tau_0) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

we in fact have $(X'_t)^{1+4/\kappa} \rightarrow 0$ as $t \rightarrow \infty$ a.s. and thus $X'_t \rightarrow 0$ as $t \rightarrow \infty$ a.s.

The claim (3-5) can now be checked using the definition of the transition kernels by first considering functions g that only depend on finitely many marginals and then using a monotone class argument. To

$1^{1/2}$ get the strong Markov property, note that for $f \in C_c^\infty((0, \infty)^2)$ the map

$$\begin{aligned} (l, r) &\mapsto \mathbb{E}_{(l,r)}(f(L'_t, R'_t)) = \frac{1}{w_\kappa(l, r)} \mathbb{E}_{(l,r)}\left((w_\kappa f)(L_t, R_t); \inf_{[0,t]} L > 0, \inf_{[0,t]} R > 0\right) \\ &= \mathbb{E}_{(0,0)}\left(\frac{(w_\kappa f)(l + t^{\kappa/4}L_1, r + t^{\kappa/4}R_1)}{w_\kappa(l, r)}; \inf_{[0,1]} L > -lt^{-\kappa/4}, \inf_{[0,1]} R > -rt^{-\kappa/4}\right) \end{aligned}$$

from $(0, \infty)^2 \cup \{(0, 0)\}$ to $\mathbb{R}$ is continuous and bounded. The strong Markov property can be deduced from this using the same reasoning that shows that Feller processes are strong Markov processes; see, for instance, [Kallenberg 2002, Theorem 19.17].

Finally, we need to show that the process defined in (3-6) is a martingale. By the Markov property of (L', R') it suffices to prove that

$$\mathbb{E}_{(l,r)}(f(L'_t, R'_t)) = f(l, r) + \int_0^t \mathbb{E}_{(l,r)}(\mathcal{G}_{\kappa,\beta} f(L_s, R_s)) ds \quad (3-7)$$

for all $t \geq 0$. To see this, note that by (3-3)

$$t \mapsto (w_\kappa f)(L_t, R_t) - (w_\kappa f)(l, r) - \int_0^t \mathcal{L}_{\kappa,\beta}(w_\kappa f)(L_s, R_s) ds$$

is a martingale starting from 0 and hence we can use optional stopping at the stopping time $t \wedge \tau_0$ to obtain

$$\mathbb{E}_{(l,r)}((w_\kappa f)(L_{t \wedge \tau_0}, R_{t \wedge \tau_0})) = (w_\kappa f)(l, r) + \mathbb{E}_{(l,r)}\left(\int_0^t \mathcal{L}_{\kappa,\beta}(w_\kappa f)(L_s, R_s) 1(s < \tau_0) ds\right).$$

The claim (3-7) now follows by first interchanging the integral and the expectation and then using the definition of the process (L', R') . To obtain the generator, note that by right-continuity of (L, R) and the fact that $\mathcal{L}_{\kappa,\beta}(w_\kappa f)$ is a bounded continuous function, we get

$$\frac{1}{t} \int_0^t \mathcal{L}_{\kappa,\beta}(w_\kappa f)(L_s, R_s) 1(s < \tau_0) ds \rightarrow \mathcal{L}_{\kappa,\beta}(w_\kappa f)(l, r) \quad \text{as } t \downarrow 0$$

provided (L, R) starts from (l, r) . The generator result now follows from dominated convergence since $\mathcal{L}_{\kappa,\beta}(w_\kappa f)$ is a bounded function and hence the terms in the limit above are bounded by a deterministic constant. $\square$

Remark 3.2. The process $(L')^{\hat{\rho}_L \cdot 4/\kappa} (R')^{\hat{\rho}_R \cdot 4/\kappa} (X')^{1+4/\kappa}$ is a martingale where we define $\hat{\rho}_L = \mathbb{P}(L_1 < 0)$ and $\hat{\rho}_R = \mathbb{P}(R_1 < 0)$. Indeed,

$$\begin{aligned} \mathbb{E}_{(l,r)}((L'_t)^{\hat{\rho}_L \cdot 4/\kappa} (R'_t)^{\hat{\rho}_R \cdot 4/\kappa} (X'_t)^{1+4/\kappa}) &= (l+r)^{1+4/\kappa} \mathbb{E}_{(l,r)}((L_t)^{\hat{\rho}_L \cdot 4/\kappa} (R_t)^{\hat{\rho}_R \cdot 4/\kappa}; \inf_{[0,t]} L > 0, \inf_{[0,t]} R > 0) \\ &= (l+r)^{1+4/\kappa} \mathbb{E}_{(l,r)}((L_t)^{\hat{\rho}_L \cdot 4/\kappa}; \inf_{[0,t]} L > 0) \mathbb{E}_{(l,r)}((R_t)^{\hat{\rho}_R \cdot 4/\kappa}; \inf_{[0,t]} R > 0) \\ &= (l+r)^{1+4/\kappa} l^{\hat{\rho}_L \cdot 4/\kappa} r^{\hat{\rho}_R \cdot 4/\kappa} \end{aligned}$$

and the Markov property of (L', R') yields the martingale property. While we are not making use of this martingale anywhere in this paper, we believe it to be potentially useful in understanding the dependence of some functionals of (L', R') on β .

The main results of this section will be about reweighting the law of (L', R') . We begin by performing a reweighting of the Lévy process (L, R) , where we need the following result on Poisson point processes.

Lemma 3.3. *Let ξ be a Poisson random measure with intensity μ on a measure space $(E, \mathcal{E})$. Also, let $f : E \rightarrow \mathbb{R}$ be measurable such that*

$$\int \mu(dx) |e^{f(x)} - 1| < \infty \quad \text{and let} \quad Z := \exp\left(\int \xi(dx) f(x) - \int \mu(dx) (e^{f(x)} - 1)\right).$$

Then $\mathbb{E}(Z) = 1$. Now reweight the law of ξ by Z . Then under the reweighted measure, ξ is a Poisson random measure with intensity measure $e^f \mu$.

Proof. Let ξ' be a Poisson random measure of intensity $e^f \mu$. For $g : E \rightarrow [0, \infty)$ measurable we have (the second and final equalities follow from Campbell's formula)

$$\begin{aligned} \mathbb{E}(Z \cdot e^{\int \xi(dx) g(x)}) &= \mathbb{E}(e^{\int \xi(dx) (f(x) + g(x))}) e^{-\int \mu(dx) (e^{f(x)} - 1)} \\ &= e^{\int \mu(dx) (e^{f(x) + g(x)} - 1)} e^{-\int \mu(dx) (e^{f(x)} - 1)} \\ &= e^{\int e^{f(x)} \mu(dx) (e^{g(x)} - 1)} = \mathbb{E}(e^{\int \xi'(dx) g(x)}), \end{aligned}$$

which implies the lemma by a monotone class argument. $\square$

The following lemma is now an immediate consequence of [Lemma 3.3](#). We would like to stress that the Lévy compensation term appearing in the generator of (L, R) is unaffected by the reweighting below. We again refer to [\[Kallenberg 2002, Theorem 19.10\]](#) for information on generators of Lévy processes. The assumption $1 - W^\varnothing(h) \lesssim h^2$ ensures the absolute continuity between $(L^\varnothing, R^\varnothing)$ and (L, R) below.

Lemma 3.4. *Suppose $W^\varnothing : [0, \infty) \rightarrow (0, 1]$ is measurable with $1 - W^\varnothing(h) \lesssim h^2$. Let $(L^\varnothing, R^\varnothing)$ be two independent Lévy processes with generator defined for $l, r \in \mathbb{R}$ and $f \in C_c^\infty(\mathbb{R}^2)$ by*

$$\begin{aligned} \mathcal{L}_{\kappa, \beta}^\varnothing f(l, r) &= \int_{\mathbb{R}} \left(W^\varnothing(|h|) (f(l+h, r) - f(l, r)) - h \frac{\partial f(l, r)}{\partial l} \right) \mu_L(dh) \\ &\quad + \int_{\mathbb{R}} \left(W^\varnothing(|h|) (f(l, r+h) - f(l, r)) - h \frac{\partial f(l, r)}{\partial r} \right) \mu_R(dh). \end{aligned}$$

Then whenever $f \in C_c^\infty(\mathbb{R}^2)$, the process

$$t \mapsto f(L_t^\varnothing, R_t^\varnothing) - f(L_0^\varnothing, R_0^\varnothing) - \int_0^t \mathcal{L}_{\kappa, \beta}^\varnothing f(L_s^\varnothing, R_s^\varnothing) ds$$

is a martingale. We let $X^\varnothing = L^\varnothing + R^\varnothing$. Then for all $g : D([0, t])^2 \rightarrow [0, \infty]$ we have

$$\mathbb{E}_{(l, r)}(g(L_{[0, t]}^\varnothing, R_{[0, t]}^\varnothing)) = \frac{\mathbb{E}_{(l, r)}\left(\prod_{s < t} W^\varnothing(|\Delta X_s|) \cdot g(L_{[0, t]}^\varnothing, R_{[0, t]}^\varnothing)\right)}{\exp\left(t \int (\mu_L + \mu_R)(dh) (W^\varnothing(|h|) - 1)\right)}.$$

We define $\tau_0^\varnothing = \inf\{t \geq 0 : L_t^\varnothing \leq 0 \text{ or } R_t^\varnothing \leq 0\}$ and $\alpha = \int (\mu_L + \mu_R)(dh) (W^\varnothing(|h|) - 1)$.

¹/₂ ¹ *Proof.* The martingale property of the process defined in the statement follows from standard results
² on Lévy processes; see, for instance, [Kallenberg 2002, Theorem 19.10 and Lemma 19.21], so it only
³ remains to prove the second part. By definition of the Lévy processes (L, R) and $(L^\varnothing, R^\varnothing)$ we have
⁴

$$\begin{aligned} & \left(t \mapsto \sum_{\substack{s \leq t \\ |\Delta X_s| > \frac{1}{n}}} \Delta L_s - t \mu_L \left(\mathbb{R} \setminus \left[-\frac{1}{n}, \frac{1}{n} \right] \right) \right) \rightarrow L, \\ & \left(t \mapsto \sum_{\substack{s \leq t \\ |\Delta X_s| > \frac{1}{n}}} \Delta R_s - t \mu_R \left(\mathbb{R} \setminus \left[-\frac{1}{n}, \frac{1}{n} \right] \right) \right) \rightarrow R, \\ & \left(t \mapsto \sum_{\substack{s \leq t \\ |\Delta X_s| > \frac{1}{n}}} \Delta L_s^\varnothing - t \mu_L \left(\mathbb{R} \setminus \left[-\frac{1}{n}, \frac{1}{n} \right] \right) \right) \rightarrow L^\varnothing, \\ & \left(t \mapsto \sum_{\substack{s \leq t \\ |\Delta X_s| > \frac{1}{n}}} \Delta R_s^\varnothing - t \mu_R \left(\mathbb{R} \setminus \left[-\frac{1}{n}, \frac{1}{n} \right] \right) \right) \rightarrow R^\varnothing \end{aligned}$$

uniformly on compact sets in probability as $n \rightarrow \infty$, where the processes $L^\varnothing$ and $R^\varnothing$ were defined via the
generator in such a way that the compensation term is the same as for L and R , respectively. We now apply
Lemma 3.3 (individually) to the Poisson point processes $\sum_{t \geq 0: \Delta L_t \neq 0} \delta_{(t, \Delta L_t)}$ and $\sum_{t \geq 0: \Delta R_t \neq 0} \delta_{(t, \Delta R_t)}$. $\square$

²⁰/₂ Throughout this paper, we will make frequent use of Palm's formula for Poisson point processes.
The theory of Palm distributions for general point processes appears, for instance, as a special case in
[Kallenberg 2017, Chapter 6] but let us state the classical case for Poisson point processes (without proof)
here for future reference.

Lemma 3.5. *Let ξ be a Poisson random measure with intensity μ on a measurable space $(E, \mathcal{E})$. Consider
a measurable function $f : E \times \mathcal{M}(E) \rightarrow [0, \infty]$ where $\mathcal{M}(E)$ is the collection of measures on E endowed
with the σ -algebra generated by the evaluation maps $\nu \mapsto \nu(A)$ for $A \in \mathcal{E}$. Then*

$$\mathbb{E} \left(\int f(x, \xi) \xi(dx) \right) = \int \mathbb{E}(f(x, \xi + \delta_x)) \mu(dx).$$

The result extends to f taking values in $\mathbb{R}$ provided either side of the equality above is finite when f is
replaced by $|f|$.

Let us return to the study of the processes (L, R) , (L', R') and $(L^\varnothing, R^\varnothing)$. A key input is the following
Palm's formula type result that allows us to replace the expected sum of the jumps of a process by an
integral over time and over space with respect to some intensity jump measure.

This lemma will be used in the following context: If we consider a random surface and, using a
peeling-type process, cut it into smaller surfaces with boundary lengths given by the jumps of a Markov
process, then we can express the weight of the whole surface in terms of an expectation of a sum over the
jumps of the Markov process. The lemma allows us to rewrite this expectation as an expectation of an
integral with respect to some intensity measure of the jumps.

Lemma 3.6. Consider the setting of [Lemma 3.4](#). Suppose that $f : D([0, \infty))^2 \times [0, \infty) \times \mathbb{R} \rightarrow [0, \infty]$ is measurable with $f(\cdot, \cdot, \cdot, 0) = 0$. Then

$$\begin{aligned} \mathbb{E}_{(l,r)} \left(\sum_{s < t} f(L^\varnothing, R^\varnothing, s, \Delta X_s^\varnothing); t < \tau_0^\varnothing \right) \\ = \mathbb{E}_{(l,r)} \left(\int_0^{t \wedge \tau_0^\varnothing} ds \int W^\varnothing(|h|) \mu_L(dh) a_s^t(L^\varnothing|_{[0,s]}, R^\varnothing|_{[0,s]}, L_s^\varnothing + h, R_s^\varnothing) \right) \\ + \mathbb{E}_{(l,r)} \left(\int_0^{t \wedge \tau_0^\varnothing} ds \int W^\varnothing(|h|) \mu_R(dh) a_s^t(L^\varnothing|_{[0,s]}, R^\varnothing|_{[0,s]}, L_s^\varnothing, R_s^\varnothing + h) \right), \end{aligned}$$

where $a_s^t(u_L, u_R, l', r') = \mathbb{E}_{(l',r')}(f(u_L \oplus L^\varnothing, u_R \oplus R^\varnothing, s, h); t - s < \tau_0^\varnothing) 1(l', r' > 0)$ and where $\oplus$ denotes the concatenation of two functions. Moreover

$$\begin{aligned} \mathbb{E}_{(l,r)} \left(\sum_{s < \zeta'} \prod_{s' < s} W^\varnothing(|\Delta X_{s'}'|) \cdot g(\Delta X_s', L'_{\cdot+s}, R'_{\cdot+s}) \right) \\ = \mathbb{E}_{(l,r)} \left(\int_0^{\zeta'} \prod_{s' < s} W^\varnothing(|\Delta X_{s'}'|) \int \frac{w_\kappa(L'_s + h, R'_s) \mu_L(dh)}{w_\kappa(L'_s, R'_s)} \mathbb{E}_{(L'_s+h, R'_s)}(g(h, L', R')) ds \right) \\ + \mathbb{E}_{(l,r)} \left(\int_0^{\zeta'} \prod_{s' < s} W^\varnothing(|\Delta X_{s'}'|) \int \frac{w_\kappa(L'_s, R'_s + h) \mu_R(dh)}{w_\kappa(L'_s, R'_s)} \mathbb{E}_{(L'_s, R'_s+h)}(g(h, L', R')) ds \right), \end{aligned}$$

whenever $g : \mathbb{R} \times D([0, \infty))^2 \rightarrow [0, \infty]$ is measurable with $g(0, \cdot, \cdot) = 0$.

Proof. Since the jumps of $(L^\varnothing, R^\varnothing)$ form a Poisson point process with an explicit intensity measure, by applying Palm's formula (see [Lemma 3.5](#)), we get

$$\begin{aligned} \mathbb{E}_{(l,r)} \left(\sum_{s < t} f(L^\varnothing, R^\varnothing, s, \Delta X_s^\varnothing); t < \tau_0^\varnothing \right) \\ = \int_0^t ds \int W^\varnothing(|h|) \mu_L(dh) \mathbb{E}_{(l,r)}(f(L^\varnothing + h 1_{[s,\infty)}, R^\varnothing, s, h); B_{h,t,s}^L) \\ + \int_0^t ds \int W^\varnothing(|h|) \mu_R(dh) \mathbb{E}_{(l,r)}(f(L^\varnothing, R^\varnothing + h 1_{[s,\infty)}, s, h); B_{h,t,s}^R), \end{aligned}$$

where

$$\begin{aligned} B_{h,t,s}^L &= \{\inf_{[0,t]} L^\varnothing, \inf_{[0,t]} R^\varnothing, \inf_{[t,s]} L^\varnothing + h, \inf_{[t,s]} R^\varnothing > 0\}, \\ B_{h,t,s}^R &= \{\inf_{[0,t]} L^\varnothing, \inf_{[0,t]} R^\varnothing, \inf_{[t,s]} L^\varnothing, \inf_{[t,s]} R^\varnothing + h > 0\}. \end{aligned}$$

By conditioning $(L^\varnothing, R^\varnothing)$ on its values up to time s and using the Markov property of $(L^\varnothing, R^\varnothing)$ we get

$$\mathbb{E}_{(l,r)}(f(L^\varnothing + h 1_{[s,\infty)}, R^\varnothing, s, h); B_{h,t,s}^L) = \mathbb{E}_{(l,r)}(a_s^t(L_{[0,s]}^\varnothing, R_{[0,s]}^\varnothing, L_s^\varnothing + h, R_s^\varnothing); s < \tau_s^\varnothing)$$

¹/₂ We analogously reexpress the second term to deduce the result. For the second part, we begin by observing that monotone convergence implies

$$\begin{aligned} & \mathbb{E}_{(l,r)} \left(\sum_{s < \zeta'} \prod_{s' < s} W^{\varnothing}(|\Delta X'_{s'}|) \cdot g(\Delta X'_s, L'_{\cdot+s}, R'_{\cdot+s}) \right) \\ &= \lim_{n \rightarrow \infty} \sum_{k \geq 0} \mathbb{E}_{(l,r)} \left(\sum_{s < k2^{-n}} \prod_{s' < s} W^{\varnothing}(|\Delta X'_{s'}|) \cdot g(\Delta X'_s, L'_{\cdot+s}, R'_{\cdot+s}) 1(\zeta' \in (k2^{-n}, (k+1)2^{-n}]) \right). \end{aligned}$$

By the Markov property of (L', R') at time $k2^{-n}$ and the construction of (L', R') we get

$$\begin{aligned} & \mathbb{E}_{(l,r)} \left(\sum_{s < k2^{-n}} \prod_{s' < s} W^{\varnothing}(|\Delta X'_{s'}|) \cdot g(\Delta X'_s, L'_{\cdot+s}, R'_{\cdot+s}) 1(\zeta' \in (k2^{-n}, (k+1)2^{-n}]) \right) \\ &= \mathbb{E}_{(l,r)} \left(\sum_{s < k2^{-n}} \mathbb{E}_{(L'_{k2^{-n}}, R'_{k2^{-n}})} (g(h, u_L \oplus L', u_R \oplus R'); \zeta' \leq 2^{-n}) \mid \begin{array}{l} h = \Delta X'_s \\ u_L = L'_{s+} \cdot \mathbb{I}_{[0, k2^{-n}]} \\ u_R = R'_{s+} \cdot \mathbb{I}_{[0, k2^{-n}]} \end{array} \right. \\ & \quad \left. \cdot \prod_{s' < s} W^{\varnothing}(|\Delta X'_{s'}|) \cdot 1(k2^{-n} < \zeta') \right). \end{aligned}$$

By the construction of (L', R') and the definition of $(L^{\varnothing}, R^{\varnothing})$ the expression above equals

$$\begin{aligned} & e^{\alpha k2^{-n}} \mathbb{E}_{(l,r)} \left(\sum_{s < k2^{-n}} \mathbb{E}_{(L^{\varnothing}_{k2^{-n}}, R^{\varnothing}_{k2^{-n}})} (g(h, u_L \oplus L', u_R \oplus R'); \zeta' \leq 2^{-n}) \mid \begin{array}{l} h = \Delta X^{\varnothing}_s \\ u_L = L^{\varnothing}_{s+} \cdot \mathbb{I}_{[0, k2^{-n}]} \\ u_R = R^{\varnothing}_{s+} \cdot \mathbb{I}_{[0, k2^{-n}]} \end{array} \right. \\ & \quad \left. \prod_{s' \in [s, k2^{-n})} W^{\varnothing}(|\Delta X^{\varnothing}_{s'}|)^{-1} \frac{w_{\kappa}(L^{\varnothing}_{k2^{-n}}, R^{\varnothing}_{k2^{-n}})}{w_{\kappa}(l, r)} 1(k2^{-n} < \tau_0^{\varnothing}) \right). \end{aligned}$$

We now apply the first part of the lemma, rewrite the expression and take the limit as $n \rightarrow \infty$ (again using monotone convergence) to deduce the result. $\square$

In the remainder of this section, we consider a finite subset $A \subseteq \mathbb{N}$ and smooth functions $W^B : (0, \infty) \rightarrow (0, \infty)$ for each $B \subseteq A$ such that

$$\begin{aligned} & W^{\varnothing} \text{ is nonincreasing, } \quad W^{\varnothing}(h) \leq 1, \quad 1 - W^{\varnothing}(h) \lesssim h^2 \quad \text{for } h > 0, \\ & \text{and } \quad W^B(x) \lesssim W^B(y) \quad \text{for all } B \subseteq A \text{ and whenever } 0 < y \leq x \leq 2y. \quad (3-8) \end{aligned}$$

By convention we will write $W^{\varnothing}(0) = 1$ and $W^B(0) = 0$ if $B \neq \varnothing$. We also consider constants $\sigma_B \in \mathbb{R}$ for $B \subseteq A$ such that $\sigma_{\varnothing} = 0$.

For $u : [0, \infty) \rightarrow \mathbb{R}$ càdlàg we let $\Pi(u, B)$ be the set of nonincreasing (in the sense of inclusion) functions $Q : [0, \infty) \rightarrow \mathcal{P}(A)$ such that $Q_0 = B$, $Q_t = \varnothing$ for t sufficiently large and $\Delta Q_t := Q_{t-} \setminus Q_t = \varnothing$ whenever $\Delta u_t := u_t - u_{t-} = 0$.

Theorem 3.7. For each $B \subseteq A$ and $l, r > 0$, we now define a process $\mathbf{Y}^W = (L^W, R^W, P^W)$ taking values in $(0, \infty)^2 \times \mathcal{P}(A)$, starting at (l, r, B) and absorbed at $(0, 0, \emptyset)$ by

$$\mathbb{E}_{(l,r,B)}(g(L^W, R^W, P^W)) = \frac{1}{W^B(l+r)} \mathbb{E}_{(l,r)} \left(\sum_{Q \in \Pi(X', B)} g(L', R', Q) \prod_{\substack{s < \zeta' \\ \Delta X'_s \neq 0}} e^{\sigma_{Q_s - \setminus Q_s} 1(\Delta X'_s > 0)} W^{Q_s - \setminus Q_s}(\Delta X'_s) \right). \quad (3-9)$$

For this to be well-defined, we also assume that the right-hand side of the equation is 1 for $g = 1$ and for all $B \subseteq A$ and $l, r > 0$. We also let $\zeta^W = \inf\{t \geq 0 : \mathbf{Y}_t^W = (0, 0, \emptyset)\}$. Then $\mathbf{Y}^W$ is a strong Markov process and the following is well-defined:

$$\begin{aligned} \mathcal{G}_{\kappa, \beta}^W f(l, r, B) &:= f(l, r, B) \int (\mu_L + \mu_R)(dh) (W^\emptyset(|h|) - 1) \\ &+ \frac{1}{w_\kappa(l, r) W^B(l+r)} \mathcal{L}_{\kappa, \beta}^\emptyset((u, v) \mapsto w_\kappa(u, v) W^B(u+v) f(u, v, B))(l, r) \\ &+ \sum_{C \subsetneq B} \int \mu_L(dh) \frac{W^{B \setminus C}(|h|) w_\kappa(l+h, r) W^C(l+r+h)}{w_\kappa(l, r) W^B(l+r)} e^{\sigma_{B \setminus C} 1(h>0)} f(l+h, r, C) \\ &+ \sum_{C \subsetneq B} \int \mu_R(dh) \frac{W^{B \setminus C}(|h|) w_\kappa(l, r+h) W^C(l+r+h)}{w_\kappa(l, r) W^B(l+r)} e^{\sigma_{B \setminus C} 1(h>0)} f(l, r+h, C) \end{aligned}$$

for $f : (0, \infty)^2 \times \mathcal{P}(A) \rightarrow \mathbb{R}$ with $f(\cdot, B) \in C_c^\infty((0, \infty)^2)$ for all $B \subseteq A$. Furthermore,

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_{(l,r,B)}(f(\mathbf{Y}_t^W) - f(l, r, B)) = \mathcal{G}_{\kappa, \beta}^W f(l, r, B),$$

i.e., $\mathcal{G}_{\kappa, \beta}^W$ is the generator of $\mathbf{Y}^W$ on the set of functions f considered above.

The proof is a quite technical and we will need some preparatory lemmas. The first one given below is the simple ingredient, namely a Palm's formula result.

Lemma 3.8. Let $\mathbf{Y}^W$ be as in [Theorem 3.7](#). Consider $f : (0, \infty)^2 \times \mathcal{P}(A) \rightarrow [0, \infty]$ measurable and fix $M' \in [0, \infty]$. For $B \subseteq A, l, r > 0$ and $t \geq 0$ define

$$\begin{aligned} E_t^{M'}(u, v) &:= \left\{ u([0, t]), v([0, t]) \subseteq \left(\frac{1}{M'}, M' \right) \right\}, \\ a_{(l,r)}^t(B, C, C_0, \dots, C_{k-1}) &:= \mathbb{E}_{(l,r)} \left(\prod_{s < t} W^\emptyset(|\Delta X'_s|) \sum_{t_0 < \dots < t_{k-1} < t} \prod_{i=0}^{k-1} e^{\sigma_{C_i} 1(\Delta X'_{t_i} > 0)} \frac{W^{C_i}(|\Delta X'_{t_i}|)}{W^\emptyset(|\Delta X'_{t_i}|)} \right. \\ &\quad \left. \cdot W^C(X'_t) f(L'_t, R'_t, C); E_t^{M'}(L', R') \right), \end{aligned}$$

$$\mu_{(l,r)}^U(dx, dy) := e^{\sigma_U 1(x+y>0)} \frac{W^U(|x+y|)}{W^\emptyset(|x+y|)} (\mu_L(dx) \delta_0(dy) + \delta_0(dx) \mu_R(dy)),$$

$$\alpha = \int (\mu_L + \mu_R)(dh) (W^\emptyset(|h|) - 1),$$

¹/₂ when $u, v \in D([0, \infty))$, $C \subseteq B$ and $B \setminus C = C_0 \cup \dots \cup C_{k-1}$ is a union of disjoint sets. With these definitions we can write

$$\mathbb{E}_{(l,r,B)}(f(Y_t^W); E_t^{M'}(L^W, R^W)) = \frac{1}{W^B(l+r)} \sum a'_{(l,r)}(B, C, C_0, \dots, C_{k-1}),$$

where the sum ranges over all $C \subseteq B$ and all (ordered) partitions $B \setminus C = C_0 \cup \dots \cup C_{k-1}$ into disjoint nonempty sets. Moreover $a'_{(l,r)}(B, C, C_0, \dots, C_{k-1})$ equals

$$\begin{aligned} & \frac{e^{\alpha t}}{w_\kappa(l, r)} \int_{0 < t_0 < \dots < t_{k-1} < t} dt_0 \dots dt_{k-1} \mathbb{E}_{(l,r)} \left(\int_{(\mathbb{R}^2)^k} \prod_{i=1}^{k-1} \mu_{(L_{t_i}^\varnothing + x_0 + \dots + x_{i-1}, R_{t_i}^\varnothing + y_0 + \dots + y_{i-1})}^{C_i} (dx_i, dy_i) \right. \\ & \quad \cdot w_\kappa(L_t^\varnothing + x_0 + \dots + x_{k-1}, R_t^\varnothing + y_0 + \dots + y_{k-1}) \\ & \quad \cdot f(L_t^\varnothing + x_0 + \dots + x_{k-1}, R_t^\varnothing + y_0 + \dots + y_{k-1}, C) \\ & \quad \cdot W^C(L_t^\varnothing + x_0 + \dots + x_{k-1} + R_t^\varnothing + y_0 + \dots + y_{k-1}) \\ & \quad \cdot \left. 1 \left(E_t^{M'} \left(L^\varnothing + \sum_{i=0}^{k-1} x_i 1_{[t_i, \infty)}, R^\varnothing + \sum_{i=0}^{k-1} y_i 1_{[t_i, \infty)} \right) \right) \right). \quad (3-10) \end{aligned}$$

Proof. The first claim is easy to see from the definitions and the second claim follows by applying Lemma 3.4 and then the first part of Lemma 3.6 several times. $\square$

Lemma 3.9. Let Y^W be as in Theorem 3.7. Then Y^W satisfies the strong Markov property.

Proof. The weak Markov property of Y^W is a simple consequence of the definition and the weak Markov property of (L', R') . Consider now

$$f : (0, \infty)^2 \times \mathcal{P}(A) \rightarrow \mathbb{R}$$

with $f(\cdot, B) \in C_c^\infty((0, \infty)^2)$ for all $B \subseteq A$. Using the first part of Lemma 3.8 with $M' = \infty$, the definition of (L', R') and stable scaling of (L, R) it is not difficult to see that

$$(l, r) \mapsto \mathbb{E}_{(l,r,B)}(f(Y_t^W))$$

is continuous from $(0, \infty)^2$ to $\mathbb{R}$ for each $B \subseteq A$ (here one needs to make use of the continuity of W^C and the fact that $f(\cdot, \cdot, C)$ has compact support for each $C \subseteq B$). Using standard ideas from the theory of Feller processes (see, e.g., [Kallenberg 2002, Theorem 19.17]) it is then classical to deduce that the process Y^W has the strong Markov property. $\square$

A key insight which allows analytical tools to be applied in the proof of Theorem 3.7 without having detailed information about the functions W^B near 0 and ∞ for $B \subseteq A$ is established with the following lemma. The main idea of the argument applies to general Markov chains and requires only some continuity properties of the process in the starting point; in our case we get these via stable scaling and the properties (3-8) of the functions W^B .

Lemma 3.10. *Let Y^W be as in Theorem 3.7. Fix $l, r > 0$, $B \subseteq A$, $\epsilon > 0$ and $M \in (0, \infty)$. Then there exists $\infty > M' > M$ such that*

$$\mathbb{P}_{(l,r,B)}\left(L^W([0, t]) \not\subseteq \left(\frac{1}{M'}, M'\right), (L^W, R^W)_t \in \left(\frac{1}{M}, M\right)^2\right) \leq \epsilon \cdot t \quad \text{for all } t \geq 0. \quad (3-11)$$

By symmetry the analogous expression with L^W and R^W interchanged also holds.

Proof. Fix $\epsilon_0 \in (0, 1)$ sufficiently small and M_0 sufficiently large such that $l/(1 + \epsilon_0), l, r, r(1 + \epsilon_0) \in (1/M_0, M_0)$ and

$$\left[\frac{1}{M}, M\right] \subseteq \left(\frac{(1 + \epsilon_0)^{-1}}{M_0}, r(1 - (1 + \epsilon_0)^{-2} + M_0)\right).$$

Let $l_- = l/(1 + \epsilon_0)$ and $r_+ = r(1 + \epsilon_0)$ and fix some $l_+ \in (l_-, l)$ and $r_- \in (r, r_+)$. The significance of these conditions will become apparent towards the end of the proof.

Note that for any $n \geq 1$, $c \in (0, 1)$ and any $M' > M_0$ we have the lower bound (where, throughout, we set $M'' = 2M'$)

$$\mathbb{P}_{(l_+, r_-, B)}\left(L^W([0, 1]) \not\subseteq \left(\frac{1}{M''}, M''\right), 1 < \zeta^W\right) \geq \sum_{i=0}^{n-1} \mathbb{P}_{(l_+, r_-, B)}(A_i) \quad (3-12)$$

where

$$A_i = \left\{ L^W\left(\left[0, \frac{ci}{n}\right]\right) \subseteq \left(\frac{1}{M''}, M''\right), L^W\left(\left[\frac{ci}{n}, \frac{c(i+1)}{n}\right]\right) \not\subseteq \left(\frac{1}{M''}, M''\right), \right. \\ \left. (L^W, R^W)_{c(i+1)/n} \in \left(\frac{1}{M_0}, M_0\right)^2, 1 < \zeta^W \right\}.$$

The idea of the proof is to show that each of the probabilities $\mathbb{P}_{(l_+, r_-, B)}(A_i)$ on the right-hand side of (3-12) are (modulo multiplication by a constant) at least as large as the left-hand side of (3-11) with $t = c/n$. We will prove this by using the Markov property of (L', R') and that the properties of (L', R') do not depend too strongly on its starting point. From this we get that the left-hand side of (3-11) is (modulo multiplication by a constant) bounded above by the left-hand side of (3-12) times t . Observing that the left-hand side of (3-12) converges to 0 as $M'' \rightarrow \infty$ we conclude the proof. In order to carry out this strategy, we first observe that we have the inclusion $A'_i \subseteq A_i$ where

$$A'_i = \left\{ L^W\left(\left[0, \frac{ci}{n}\right]\right) \subseteq (l_-, l), R^W\left(\left[0, \frac{ci}{n}\right]\right) \subseteq (r, r_+), P_{ci/n}^W = B, \right.$$

$$\left. L^W\left(\left[\frac{ci}{n}, \frac{c(i+1)}{n}\right]\right) \not\subseteq \left(\frac{1}{M''}, M''\right), (L^W, R^W)_{c(i+1)/n} \in \left(\frac{1}{M_0}, M_0\right)^2, 1 < \zeta^W \right\}.$$

¹/₂ The Markov property of Y^W as stated in [Lemma 3.9](#) at times i/n and $(i+1)/n$ implies that

$$\begin{aligned} \mathbb{P}_{(l_+, r_-, B)}(A'_i) &\geq \mathbb{P}_{(l_+, r_-, B)}\left(L^W\left(\left[0, \frac{ci}{n}\right]\right) \subseteq (l_-, l), R^W\left(\left[0, \frac{ci}{n}\right]\right) \subseteq (r, r_+), P_{ci/n}^W = B\right) \\ &\quad \cdot \inf_{\substack{l' \in (l_-, l) \\ r' \in (r, r_+)}} \mathbb{P}_{(l', r', B)}\left(L^W\left(\left[0, \frac{c}{n}\right]\right) \not\subseteq \left(\frac{1}{M''}, M''\right), (L^W, R^W)_{c/n} \in \left(\frac{1}{M_0}, M_0\right)^2\right) \\ &\quad \cdot \inf_{\substack{l', r' \in (1/M_0, M_0) \\ C \subseteq B}} \mathbb{P}_{(l', r', C)}\left(1 - \frac{c(i+1)}{n} < \zeta^W\right). \end{aligned} \quad (3-13)$$

From the definitions we can see that the first and third terms in the product above are lower bounded by a positive constant which does not depend on M'' , c , i or n . Combining this with (3-12), (3-13), and the fact that $A'_i \subseteq A_i$ yields that

$$\begin{aligned} \frac{1}{n} \mathbb{P}_{(l_+, r_-, B)}\left(L^W([0, 1]) \not\subseteq \left(\frac{1}{M''}, M''\right), 1 < \zeta^W\right) \\ \gtrsim \inf_{\substack{l' \in (l_-, l) \\ r' \in (r, r_+)}} \mathbb{P}_{(l', r', B)}\left(L^W\left(\left[0, \frac{c}{n}\right]\right) \not\subseteq \left(\frac{1}{M''}, M''\right), (L^W, R^W)_{c/n} \in \left(\frac{1}{M_0}, M_0\right)^2\right) \end{aligned} \quad (3-14)$$

²⁰/₂ for all $n \geq 1$ and all $M' > M_0$. It remains to understand the dependence of the terms in the infimum on the right-hand side on (l', r') ; more precisely we must show that (modulo multiplication by a constant) it is lower bounded by the considered probability with $(l', r') = (l, r)$ and the considered intervals slightly changed.

For $l' \in (l_-, l]$, $r' \in [r, r_+)$ and $x' = l' + r'$ by [Lemma 3.8](#) and by unpacking the definitions we get

$$\begin{aligned} \mathbb{P}_{(l', r', B)}\left(L^W\left(\left[0, \frac{c}{n}\right]\right) \not\subseteq \left(\frac{1}{M''}, M''\right), (L^W, R^W)_{c/n} \in \left(\frac{1}{M_0}, M_0\right)^2\right) \\ = \frac{1}{W^B(x')(x')^{-1-4/\kappa}} \\ \cdot \sum \mathbb{E}_{(0,0)}\left(W^C(x' + X_{c/n})(x' + X_{c/n})^{-1-4/\kappa} 1\left(l' + L_{c/n}, r' + R_{c/n} \in \left(\frac{1}{M_0}, M_0\right)\right)\right. \\ \cdot \sum_{0 < t_0 < \dots < t_{k-1} < c/n} \prod_{i=0}^{k-1} e^{\sigma_{C_i} 1(\Delta X_{t_i} > 0)} W^{C_i}(|\Delta X_{t_i}|) \prod_{s \in (0, c/n) \setminus \{t_0, \dots, t_{k-1}\}} W^{\emptyset}(|\Delta X_s|) \\ \cdot 1\left(l' + \inf_{[0, c/n]} L > 0, r' + \inf_{[0, c/n]} R > 0, l' + L\left(\left[0, \frac{c}{n}\right]\right) \not\subseteq \left(\frac{1}{M'}, M'\right)\right)\Bigg), \end{aligned} \quad (3-15)$$

where the first sum ranges over all $C \subseteq B$ and all (ordered) partitions $B \setminus C = C_0 \cup \dots \cup C_{k-1}$ into disjoint nonempty sets. We now analyze the dependence on (l', r') for each of the product terms in the expectation. The integrand in the expectation is not monotonic in l' but is (up to multiplicative factors)

$1^{1/2}$ $\frac{1}{2}$ monotonic in r' . To deal with the l' dependence, we observe that, by stable scaling when $L_0 = R_0 = 0$,

$$\frac{3}{4} \quad (L, R) \stackrel{d}{=} \frac{l'}{l} \cdot ((L, R)_{(l/l')^{4/\kappa}} : t \geq 0) =: (\tilde{L}, \tilde{R}). \quad (3-16)$$

$\frac{6}{7}$ Also let $\tilde{X} = \tilde{L} + \tilde{R}$ and write $x = l + r$. Since $(L, R) \stackrel{d}{=} (\tilde{L}, \tilde{R})$, the right-hand side of (3-15) is unchanged if we replace (L, R) by $(\tilde{L}, \tilde{R})$ and we will do this when bounding it from below. For the first term in the expectation in the display (3-15), we note that

$$\begin{aligned} \frac{10}{11} & \left\{ l' + \tilde{L}_{c/n} \in \left(\frac{1}{M_0}, M_0 \right), r' + \tilde{R}_{c/n} \in \left(\frac{1}{M_0}, M_0 \right) \right\} \\ \frac{12}{13} & \supseteq \left\{ l + L_{c(l/l')^{4/\kappa}/n} \in \frac{l}{l'} \left(\frac{1}{M_0}, M_0 \right), r + R_{c(l/l')^{4/\kappa}/n} \in r - \frac{r'l'}{l} + \frac{l}{l'} \left(\frac{1}{M_0}, M_0 \right) \right\} \\ \frac{14}{15} & \supseteq \left\{ l + L_{c(l/l')^{4/\kappa}/n} \in \left(\frac{(1+\epsilon_0)^{-1}}{M_0}, M_0 \right), r + R_{c(l/l')^{4/\kappa}/n} \in \left(\frac{(1+\epsilon_0)^{-1}}{M_0}, r(1 - (1+\epsilon_0)^2) + M_0 \right) \right\} \end{aligned}$$

$\frac{17}{18}$ and remark that $x \mapsto W^C(x)x^{-1-4/\kappa}$ is upper and lower bounded by finite and positive constants on each compact set contained in $(0, \infty)$. For the second term in the expectation in (3-15), we use (3-8), to lower bound

$$\begin{aligned} \frac{20^{1/2}}{21} & \sum_{0 < t_0 < \dots < t_{k-1} < c/n, i=0}^{k-1} \prod e^{\sigma_{C_i} 1(\Delta \tilde{X}_{t_i} > 0)} W^{C_i}(|\Delta \tilde{X}_{t_i}|) \prod_{s \in (0, c/n) \setminus \{t_0, \dots, t_{k-1}\}} W^{\varnothing}(|\Delta \tilde{X}_s|) \\ \frac{22}{23} & \gtrsim \sum_{0 < t_0 < \dots < t_{k-1} < c(l/l')^{4/\kappa}/n, i=0}^{k-1} \prod e^{\sigma_{C_i} 1(\Delta X_{t_i} > 0)} W^{C_i}(|\Delta X_{t_i}|) \prod_{s \in (0, c(l/l')^{4/\kappa}/n) \setminus \{t_0, \dots, t_{k-1}\}} W^{\varnothing}(|\Delta X_s|). \end{aligned}$$

$\frac{27}{28}$ Note that in the display above, only times where the respective process in question jumps contribute to the sums and products. Finally, for the third term in the expectation in (3-15),

$$\begin{aligned} \frac{29}{30} & \left\{ l' + \inf_{[0, c/n]} \tilde{L} > 0, r' + \inf_{[0, c/n]} \tilde{R} > 0, l' + \tilde{L} \left(\left[0, \frac{c}{n} \right] \right) \not\subseteq (1/M'', M'') \right\} \\ \frac{31}{32} & \supseteq \left\{ l + \inf_{[0, c(l/l')^{4/\kappa}/n]} L > 0, r + \inf_{[0, c(l/l')^{4/\kappa}/n]} R > 0, l + L \left(\left[0, \frac{c}{n} \right] \right) \not\subseteq (1/M'', (1+\epsilon_0)M'') \right\}. \end{aligned}$$

$\frac{34}{35}$ We now use the three last displays to bound the right-hand side of (3-15) (with $(\tilde{L}, \tilde{R})$ instead of (L, R)) and get

$$\begin{aligned} \frac{37}{38} & \mathbb{P}_{(l', r', B)} \left(L^W \left(\left[0, \frac{c}{n} \right] \right) \not\subseteq \left(\frac{1}{M''}, M'' \right), (L^W, R^W)_{c/n} \in \left(\frac{1}{M_0}, M_0 \right)^2 \right) \\ \frac{39^{1/2}}{40} & \gtrsim \mathbb{P}_{(l, r, B)} \left(L^W \left(\left[0, \frac{c}{n} \right] \right) \not\subseteq \left(\frac{1}{M''}, (1+\epsilon_0)M'' \right), (L^W, R^W)_{c(l/l')^{4/\kappa}/n} \in (a, b)^2 \right) \end{aligned}$$

¹/₂ for all $l' \in (l_-, l)$, $r' \in (r, r_+)$ where $(a, b) = ((1 + \epsilon_0)^{-1}/M_0, r(1 - (1 + \epsilon_0)^{-2}) + M_0)$. We can further lower bound this using the Markov property to obtain

$$\begin{aligned} & \mathbb{P}_{(l', r', B)} \left(L^W \left(\left[0, \frac{c}{n} \right] \right) \not\subseteq \left(\frac{1}{M''}, M'' \right), (L^W, R^W)_{c/n} \in \left(\frac{1}{M_0}, M_0 \right)^2 \right) \\ & \gtrsim \mathbb{P}_{(l, r, B)} \left(L^W \left(\left[0, \frac{c}{n} \right] \right) \not\subseteq \left(\frac{1}{M''}, (1 + \epsilon_0)M'' \right), (L^W, R^W)_{c/n} \in \left(\frac{1}{M}, M \right)^2 \right) \\ & \quad \cdot \inf_{\substack{l'', r'' \in (1/M, M) \\ C \subseteq B}} \mathbb{P}_{(l'', r'', C)} ((L^W, R^W)_{c(l/l')^{4/\kappa}/n - c/n} \in (a, b)^2) \end{aligned}$$

By construction, $[1/M, M] \subseteq (a, b)$ and one can see from the definitions that the infimum in the display above is bounded below by a positive constant uniformly in $l' \in (l_-, l)$. Combining this with (3-14) yields

$$\begin{aligned} & \mathbb{P}_{(l, r, B)} \left(L^W \left(\left[0, \frac{c}{n} \right] \right) \not\subseteq \left(\frac{1}{M''}, (1 + \epsilon_0)M'' \right), (L^W, R^W)_{c/n} \in \left(\frac{1}{M}, M \right)^2 \right) \\ & \lesssim \frac{1}{n} \mathbb{P}_{(l_+, r_-, B)} \left(L^W([0, 1]) \not\subseteq \left(\frac{1}{M''}, M'' \right), 1 < \zeta^W \right) \end{aligned}$$

for all $n \geq 1$ and $c \in (0, 1)$. Using that $\epsilon_0 \in (0, 1)$ this further gives that, for some constant $C_0 > 0$ and all $t \in (0, 1)$ and $M' > M_0$ (recalling that $M'' = 2M'$),

$$\begin{aligned} & \mathbb{P}_{(l, r, B)} \left(L^W([0, t]) \not\subseteq \left(\frac{1}{M'}, M' \right), (L^W, R^W)_t \in \left(\frac{1}{M}, M \right)^2 \right) \\ & \leq C_0 t \mathbb{P}_{(l_+, r_-, B)} \left(L^W([0, 1]) \not\subseteq \left(\frac{1}{M''}, M'' \right), 1 < \zeta^W \right). \end{aligned}$$

The lemma now follows from this inequality and

$$\mathbb{P}_{(l_+, r_-, B)} \left(L^W([0, 1]) \not\subseteq \left(\frac{1}{M''}, M'' \right), 1 < \zeta^W \right) \rightarrow 0$$

as $M'' \rightarrow \infty$. □

Lemma 3.11. Consider the setting as in Theorem 3.7. Then $\int_0^1 W^B(x) dx < \infty$ for all $B \subseteq A$.

Proof. The case $B = \emptyset$ is clear by (3-8), so assume that $B \neq \emptyset$ and consider $l = r = 1/2$. By the assumptions of Theorem 3.7 we get that

$$\begin{aligned} \infty & > W^B(1) \geq \mathbb{E}_{(l, r)} \left(\sum_{t < \zeta'} e^{\sigma_B 1(\Delta X'_t > 0)} \prod_{s \neq t} W^\emptyset(|\Delta X'_s|) \cdot W^B(|\Delta X'_t|) \right) \\ & = \mathbb{E}_{(l, r)} \left(\int_0^{\zeta'} \prod_{s < t} W^\emptyset(|\Delta X'_s|) \int \frac{w_\kappa(L'_s + h, R'_s) W^B(|h|) e^{\sigma_B 1(h > 0)} \mu_L(dh)}{w_\kappa(L'_t, R'_t)} W^\emptyset(X'_t + h) dt \right. \\ & \quad \left. + \int_0^{\zeta'} \prod_{s < t} W^\emptyset(|\Delta X'_s|) \int \frac{w_\kappa(L'_s, R'_s + h) W^B(|h|) e^{\sigma_B 1(h > 0)} \mu_R(dh)}{w_\kappa(L'_t, R'_t)} W^\emptyset(X'_t + h) dt \right), \end{aligned}$$

where to go to the second line we use the second statement in Lemma 3.6 and that, by assumption, the right-hand side of (3-9) is equal to 1 for $g = 1$. If we had that $\int_0^1 W^B(x) dx = \infty$ then the integrand in the final expectation above would be almost surely infinite resulting in a contradiction. $\square$

Proof of Theorem 3.7. The strong Markov property appears as Lemma 3.9. We will now prove the main claim of the theorem. Let us remark that all expressions in the definition of $\mathcal{G}_{\kappa,\beta}^W f$ are well-defined by Lemma 3.11. By Lemma 3.8 with $M' = \infty$ and applied to the function $(u, v, C) \mapsto f(u, v, B)1(B = C)$, and by the expression of the generator of $(L^\varnothing, R^\varnothing)$ as given in Lemma 3.4, we get by the definitions and by Lemma 3.4 that

$$\mathbb{E}_{(l,r,B)}(f(Y_t^W); P_t^W = B) = \frac{e^{\alpha t}}{w_\kappa(l, r) W^B(l+r)} \mathbb{E}_{(l,r)}(f(L_{t \wedge \tau_0^\varnothing}^\varnothing, R_{t \wedge \tau_0^\varnothing}^\varnothing, B) W^B(X_{t \wedge \tau_0^\varnothing}^\varnothing) w_\kappa(L_{t \wedge \tau_0^\varnothing}^\varnothing, R_{t \wedge \tau_0^\varnothing}^\varnothing))$$

Now by the martingale statement in Lemma 3.4 and optional stopping at time $t \wedge \tau_0^\varnothing$ we obtain that

$$\begin{aligned} \mathbb{E}_{(l,r)}(f(L_{t \wedge \tau_0^\varnothing}^\varnothing, R_{t \wedge \tau_0^\varnothing}^\varnothing, B) W^B(X_{t \wedge \tau_0^\varnothing}^\varnothing) w_\kappa(L_{t \wedge \tau_0^\varnothing}^\varnothing, R_{t \wedge \tau_0^\varnothing}^\varnothing)) \\ = f(l, r, B) W^B(l+r) w_\kappa(l, r) + \mathbb{E}_{(l,r)} \left(\int_0^t \mathcal{L}_{\kappa,\beta}^\varnothing g(L_s^\varnothing, R_s^\varnothing) 1(s < \tau_0^\varnothing) ds \right), \end{aligned}$$

where $g(l', r') := f(l', r', B) W^B(l' + r') w_\kappa(l', r')$. Since $\mathcal{L}_{\kappa,\beta}^\varnothing g$ is a bounded function and $(L^\varnothing, R^\varnothing)$ is right-continuous we obtain (as in Proposition 3.1) by dominated convergence that

$$\frac{1}{t} \mathbb{E}_{(l,r)} \left(\int_0^t \mathcal{L}_{\kappa,\beta}^\varnothing g(L_s^\varnothing, R_s^\varnothing) 1(s < \tau_0^\varnothing) ds \right) \rightarrow \mathcal{L}_{\kappa,\beta}^\varnothing g(l, r)$$

as $t \downarrow 0$. Combining the above observations yields that

$$\begin{aligned} \lim_{t \downarrow 0} \frac{1}{t} (\mathbb{E}_{(l,r,B)}(f(Y_t^W); P_t^W = B) - f(l, r, B)) \\ = f(l, r, B) \int (\mu_L + \mu_R)(dh) (W^\varnothing(|h|) - 1) \\ + \frac{1}{w_\kappa(l, r) W^B(l+r)} \mathcal{L}_{\kappa,\beta}^\varnothing((u, v) \mapsto w_\kappa(u, v) W^B(u+v) f(u, v, B))(l, r). \end{aligned}$$

This in particular implies the result when $B = \varnothing$.

Let M be such that $f(\cdot, \cdot, C)$ has support contained in $(1/M, M)^2$ for all $C \subseteq A$. We will now show that

$$\begin{aligned} \lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_{(l,r,B)}(f(Y_t^W); E_t^{M'}(L^W, R^W), P_t^W \neq B) \\ = \sum_{\varnothing \neq C_0 \subseteq B} \int \mu_L(dh) \frac{W^{C_0}(|h|) w_\kappa(l+h, r) W^{B \setminus C_0}(l+r+h)}{w_\kappa(l, r) W^B(l+r)} e^{\sigma_{B \setminus C} 1(h>0)} f(l+h, r, B \setminus C_0) \\ + \sum_{\varnothing \neq C_0 \subseteq B} \int \mu_R(dh) \frac{W^{C_0}(|h|) w_\kappa(l, r+h) W^{B \setminus C_0}(l+r+h)}{w_\kappa(l, r) W^B(l+r)} e^{\sigma_{B \setminus C} 1(h>0)} f(l, r+h, B \setminus C_0) \end{aligned}$$

for all $M' > M$ where $E_t^{M'}(u, v) = \{u([0, t]), v([0, t]) \subseteq (1/M', M')\}$ whenever $u, v \in D([0, \infty))$.

Lemma 3.10 (with the constant M appearing in the statement of this lemma having been defined above) then

¹/₂ implies the theorem. The display above is an immediate consequence of (3-10) within Lemma 3.8; the key is that the indicator in the final line of (3-10) yields deterministic bounds on the other quantities in the integral and we then see that if $k > 1$ then the integral is $O(t^2)$ and if $k = 1$ we obtain exactly the limit above. $\square$

For the computation of the law of the conformal radius of a simple conformal loop ensemble, we will need to determine a particular functional associated to the process (L', R') . Thus we conclude this section with the following explicit computation. The proof starts by expressing $A_{\pm}$ (as defined in the proposition below) in terms of an integral over time, using the Palm's formula type result of Lemma 3.6. Then the key trick is to observe that the integrand can be expressed in terms of $\mathcal{G}_{\kappa, \beta} f$ for an appropriate function f , which allows us to use a martingale argument with (3-6) to compute the integral. The fact that we can find such an f explicitly is very particular to the setting considered here and does not generalize to arbitrary functionals.

Proposition 3.12. *Suppose that $l, r > 0$ such that $l + r = 1$ and let*

$$A_+ = \mathbb{E}_{(l, r)} \left(\sum_{\substack{t < \zeta' \\ \Delta X'_t > 0}} |\Delta X'_t|^\theta \right) \quad \text{and} \quad A_- = \mathbb{E}_{(l, r)} \left(\sum_{\substack{t < \zeta' \\ \Delta X'_t < 0}} |\Delta X'_t|^\theta \right).$$

Then $A_+ < \infty$, $A_- < 1$ and

$$\frac{A_+}{1 - A_-} = \frac{\cos(4\pi/\kappa)}{\cos(4\pi/\kappa - \pi\theta)},$$

²⁰/₂ whenever we have $\theta \in (4/\kappa + 1/2, 4/\kappa + 3/2)$. Also, if $\theta \notin (4/\kappa, 1 + 8/\kappa)$ then $A_+ = \infty$. Recall that the subscript (l, r) in the display above indicates that (L', R') starts from (l, r) .

Proof. By the second part of Lemma 3.6 with $W^\varnothing \equiv 1$ and $g(x, l, r) = x^\theta 1(x > 0)$ we get

$$A_+ = \int_0^\infty \mathbb{E}_{(l, r)} \left(\int_0^\infty \mu_L(dh) \frac{h^\theta w_\kappa(L'_t + h, R'_t)}{w_\kappa(L'_t, R'_t)} + \int_0^\infty \mu_R(dh) \frac{h^\theta w_\kappa(L'_t, R'_t + h)}{w_\kappa(L'_t, R'_t)}; t < \zeta' \right) dt.$$

Note that if $\theta \notin (4/\kappa, 1 + 8/\kappa)$ then

$$\begin{aligned} & \int_0^\infty \mu_L(dh) \frac{h^\theta w_\kappa(L'_t + h, R'_t)}{w_\kappa(L'_t, R'_t)} + \int_0^\infty \mu_R(dh) \frac{h^\theta w_\kappa(L'_t, R'_t + h)}{w_\kappa(L'_t, R'_t)} \\ &= -\cos\left(\frac{4\pi}{\kappa}\right) \int_0^\infty \frac{h^\theta dh}{h^{1+4/\kappa}} \frac{(L'_t + R'_t)^{1+4/\kappa}}{(L'_t + R'_t + h)^{1+4/\kappa}} = \infty \quad \text{for all } t < \zeta' \text{ a.s.} \end{aligned}$$

This in particular implies the final assertion of the proposition.

Suppose henceforth that $\theta \in (4/\kappa + 1/2, 4/\kappa + 3/2)$ and define σ by $e^{-\sigma} = \cos(4\pi/\kappa) / \cos(4\pi/\kappa - \pi\theta)$. Applying Lemma 3.6 to A_- , similarly as it was applied to A_+ above, we obtain

$$e^\sigma A_+ + A_- = \int_0^\infty \mathbb{E}_{(l, r)} (g(L'_t, R'_t); t < \zeta') dt,$$

where

$$g(l', r') = \int_{-\infty}^\infty \mu_L(dh) \frac{e^{\sigma 1(h>0)} |h|^\theta w_\kappa(l' + h, r')}{w_\kappa(l', r')} + \int_{-\infty}^\infty \mu_R(dh) \frac{e^{\sigma 1(h>0)} |h|^\theta w_\kappa(l', r' + h)}{w_\kappa(l', r')}.$$

³⁹/₂ Note that g is well-defined and finite since $\theta \in (4/\kappa, 1 + 8/\kappa)$.

The key is now that if we let $f(l', r') = (l' + r')^\theta 1(l', r' > 0)$ then $\mathcal{G}_{\kappa, \beta} f(l', r') = -g(l', r')$ for all $l', r' > 0$, where we recall the definition of the generator $\mathcal{G}_{\kappa, \beta}$ appearing in (3-4). To see this, by scaling, it suffices to consider $l' + r' = 1$. Let us write $\theta' = \theta - 1 - 4/\kappa$. Then

$$\begin{aligned} \mathcal{G}_{\kappa, \beta} f(l', r') + g(l', r') &= \frac{1}{w_\kappa(l', r')} \mathcal{L}_{\kappa, \beta}(f w_\kappa)(l', r') + g(l', r') \\ &= -\cos\left(\frac{4\pi}{\kappa}\right) \int_0^\infty \frac{dh}{h^{1+4/\kappa}} ((1+h)^{\theta'} - 1 - \theta' h) \\ &\quad + \int_0^\infty \frac{dh}{2h^{1+4/\kappa}} ((1-h)^{\theta'} 1(h < l') - 1 + \theta' h) + \int_0^{l'} \frac{h^\theta dh}{2(1-h)^{1+4/\kappa}} \\ &\quad + \int_0^\infty \frac{dh}{2h^{1+4/\kappa}} ((1-h)^{\theta'} 1(h < r') - 1 + \theta' h) + \int_0^{r'} \frac{h^\theta dh}{2(1-h)^{1+4/\kappa}} \\ &\quad - \cos\left(\frac{4\pi}{\kappa} - \pi\theta\right) \int_0^\infty \frac{h^{\theta'} dh}{(1+h)^{1+4/\kappa}} \\ &= 0 \end{aligned}$$

by Lemmas and B.2 and B.3. Putting the results obtained so far together, we deduce that

$$e^\sigma A_+ + A_- = - \int_0^\infty \mathbb{E}_{(l,r)}(\mathcal{G}_{\kappa, \beta} f(L'_t, R'_t); t < \zeta') dt. \quad (3-17)$$

In order to conclude the proof it is sufficient to show that the right-hand side of (3-17) is equal to 1.

We consider functions $h_n \in C_c^\infty(\mathbb{R})$ with $h_n \geq 0$, $h_n|_{(2^{-n}, 2^n)} = 1$ and $h_n|_{(2^{-n-1}, 2^{n+1})^c} = 0$. Also let $\zeta'_m = \inf\{t \geq 0 : L'_t \notin (2^{-m}, 2^m) \text{ or } R'_t \notin (2^{-m}, 2^m)\}$. By Proposition 3.1 (in particular, (3-6)) and the optional stopping theorem at time $t \wedge \zeta'_m$, for $t \geq 0$,

$$\mathbb{E}_{(l,r)}(u_n(L'_{t \wedge \zeta'_m}, R'_{t \wedge \zeta'_m})) = 1 + \int_0^t \mathbb{E}_{(l,r)}(\mathcal{G}_{\kappa, \beta} u_n(L'_s, R'_s); s < \zeta'_m) ds, \quad (3-18)$$

where $u_n(l', r') = h_n(l')h_n(r')f(l', r')$ and provided that $l, r \in (2^{-n}, 2^n)$. We now consider the limits $n \rightarrow \infty$, $m \rightarrow \infty$ and then $t \rightarrow \infty$ in (3-18) on both sides.

We begin with the right-hand side. For each $m \geq 1$ there exists a constant C_m such that $|\mathcal{G}_{\kappa, \beta} u_n| \leq C_m$ on $[2^{-m}, 2^m]^2$ and for all $n \geq 1$. Thus for the $n \rightarrow \infty$ limit, we can use dominated convergence to get

$$\int_0^t \mathbb{E}_{(l,r)}(\mathcal{G}_{\kappa, \beta} u_n(L'_s, R'_s); s < \zeta'_m) ds \rightarrow \int_0^t \mathbb{E}_{(l,r)}(\mathcal{G}_{\kappa, \beta} f(L'_s, R'_s); s < \zeta'_m) ds \leq 0 \quad (3-19)$$

as $n \rightarrow \infty$ for all $m \geq 1$ and $t > 0$. Observe now that since $\mathcal{G}_{\kappa, \beta} f = -g \leq 0$ we can use monotone convergence for the $m \rightarrow \infty$ and $t \rightarrow \infty$ limit to get

$$\lim_{t \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^t \mathbb{E}_{(l,r)}(\mathcal{G}_{\kappa, \beta} f(L'_s, R'_s); s < \zeta'_m) ds = \int_0^\infty \mathbb{E}_{(l,r)}(\mathcal{G}_{\kappa, \beta} f(L'_s, R'_s); s < \zeta') ds.$$

Therefore, in order to conclude the proof it is sufficient to show that the left-hand side of (3-18) goes to zero as we send $n \rightarrow \infty$, then $m \rightarrow \infty$, and finally $t \rightarrow \infty$.

¹/₂ Fatou's lemma and (3-18) together with (3-19) yield

$$\mathbb{E}_{(l,r)}((X'_{t \wedge \zeta'_m})^\theta) \leq \liminf_{n \rightarrow \infty} \mathbb{E}_{(l,r)}(u_n(L'_{t \wedge \zeta'_m}, R'_{t \wedge \zeta'_m})) \leq 1.$$

Let $\theta' > \theta$ such that $\theta' \in (4/\kappa + 1/2, 4/\kappa + 3/2)$. Then by the same argument the above display also holds with θ replaced by θ' and therefore the family $((X'_{t \wedge \zeta'_m})^{\theta'} : m \geq 1, t > 0)$ is uniformly integrable. Consequently since $X'_{t \wedge \zeta'_m} \rightarrow X'_t$ a.s. as $m \rightarrow \infty$ and $X'_t \rightarrow 0$ a.s. as $t \rightarrow \infty$ we deduce that (using monotone convergence for the first equality)

$$\lim_{t \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E}_{(l,r)}(u_n(L'_{t \wedge \zeta'_m}, R'_{t \wedge \zeta'_m})) = \lim_{t \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbb{E}_{(l,r)}((X'_{t \wedge \zeta'_m})^{\theta'}) = 0, \quad (3-20)$$

which concludes the proof. $\square$

3B. Without compensation. Let $\kappa' \in (4, 8)$ and $\beta \in [-1, 1]$. Consider two independent Lévy processes L and R (this time without compensation) where this time the generator of (L, R) is

$$\mathcal{L}_{\kappa', \beta} f(l, r) = \int_{\mathbb{R}} (f(l+h, r) - f(l, r)) \mu_L(dh) + \int_{\mathbb{R}} (f(l, r+h) - f(l, r)) \mu_R(dh)$$

for $f \in C_c^\infty(\mathbb{R}^2)$ and where the jump measures μ_L and μ_R of L and R are given by

$$\begin{aligned} \frac{\mu_L(dh)}{dh} &= \frac{-\cos(4\pi/\kappa')(1-\beta)/2}{h^{1+4/\kappa'}} 1_{(0,\infty)}(h) + \frac{1/2}{|h|^{1+4/\kappa'}} 1_{(-\infty,0)}(h), \\ \frac{\mu_R(dh)}{dh} &= \frac{-\cos(4\pi/\kappa')(1+\beta)/2}{h^{1+4/\kappa'}} 1_{(0,\infty)}(h) + \frac{1/2}{|h|^{1+4/\kappa'}} 1_{(-\infty,0)}(h). \end{aligned}$$

By definition, $\mathcal{L}_{\kappa', \beta} f$ is a bounded function whenever $f \in C_c^\infty(\mathbb{R}^2)$ and in this case (see again, for instance, [Kallenberg 2002, Theorem 19.10 and Lemma 19.21]), the process

$$t \mapsto f(L_t, R_t) - f(L_0, R_0) - \int_0^t \mathcal{L}_{\kappa', \beta} f(L_s, R_s) ds$$

is a martingale.

Let $X = L + R$. Just like in the previous section, we now want to consider a reweighting construction. For $l, r > 0$ and $f \in C_c^\infty(\mathbb{R}^2)$ we let

$$w_{\kappa'}(l, r) = (l+r)^{-1-4/\kappa'} 1_{(l, r > 0)} \quad \text{and} \quad \mathcal{G}_{\kappa', \beta} f(l, r) = \frac{1}{w_{\kappa'}(l, r)} \mathcal{L}_{\kappa', \beta}(w_{\kappa'} f)(l, r).$$

By convention, we write $\mathcal{G}_{\kappa', \beta} f(0, 0) = 0$ and one can check that $\mathcal{G}_{\kappa', \beta} f$ is a bounded function. Let us remark that from Lemma B.4 we get that, for $f \in C_c^\infty((0, \infty)^2)$ and $l, r > 0$,

$$\mathcal{G}_{\kappa', \beta} f(l, r) = \int_{\mathbb{R}} (f(l+h, r) - f(l, r)) \frac{w_{\kappa'}(l+h, r) \mu_L(dh)}{w_{\kappa'}(l, r)} + \int_{\mathbb{R}} (f(l, r+h) - f(l, r)) \frac{w_{\kappa'}(l, r+h) \mu_R(dh)}{w_{\kappa'}(l, r)}.$$

Throughout this section we will not give the proofs since they are all essentially identical to the ones in the previous section but only highlight the small differences (the intermediate lemmas needed for some of the proofs are also exactly analogous).

The only differing ingredient in the proof of the following proposition is that [Lemma B.4](#) is used instead of [B.3](#) as was the case in the proof of [Proposition 3.1](#).

Proposition 3.13. *For each $l, r > 0$ there is a strong Markov càdlàg process (L', R') starting at (l, r) , taking values in $(0, \infty)^2 \cup \{(0, 0)\}$ and absorbed at $(0, 0)$ such that*

$$\mathbb{E}_{(l,r)}(g(L'|_{[0,t]}, R'|_{[0,t]}; t < \zeta') = \mathbb{E}_{(l,r)}\left(\frac{w_{\kappa'}(L_t, R_t)}{w_{\kappa'}(l, r)} g(L|_{[0,t]}, R|_{[0,t]}; t < \tau_0\right),$$

whenever the function $g : D([0, t])^2 \rightarrow [0, \infty]$ is measurable on the space of càdlàg functions, $t \geq 0$, $\zeta' = \inf\{t \geq 0 : L'_t = R'_t = 0\}$ and where $\tau_0 = \inf\{t \geq 0 : L_t \leq 0 \text{ or } R_t \leq 0\}$. Moreover, $X'_t \rightarrow 0$ as $t \rightarrow \infty$ almost surely where $X' = L' + R'$. For $f \in C_c^\infty((0, \infty)^2)$ the process

$$t \mapsto f(L'_t, R'_t) - \int_0^{t \wedge \zeta'} \mathcal{G}_{\kappa', \beta} f(L'_s, R'_s) ds$$

is a (càdlàg) martingale. Finally, $\mathcal{G}_{\kappa', \beta}$ is the generator of (L', R') on $C_c^\infty((0, \infty)^2)$.

Using [Lemma 3.3](#) we immediately obtain the following result.

Lemma 3.14. *Suppose $W^\varnothing : [0, \infty) \rightarrow (0, 1]$ is measurable with $1 - W^\varnothing(h) \lesssim h^{8/\kappa'}$. Let $(L^\varnothing, R^\varnothing)$ be two independent Lévy processes with generator defined for $l, r > 0$ by*

$$\mathcal{L}_{\kappa', \beta}^\varnothing f(l, r) = \int_{\mathbb{R}} (f(l+h, r) - f(l, r)) W^\varnothing(h) \mu_L(dh) + \int_{\mathbb{R}} (f(l, r+h) - f(l, r)) W^\varnothing(h) \mu_R(dh).$$

In fact, whenever $f \in C_c^\infty(\mathbb{R}^2)$, the process

$$t \mapsto f(L_t^\varnothing, R_t^\varnothing) - f(L_0^\varnothing, R_0^\varnothing) - \int_0^t \mathcal{L}_{\kappa', \beta}^\varnothing f(L_s^\varnothing, R_s^\varnothing) ds$$

is a martingale. We let $X^\varnothing = L^\varnothing + R^\varnothing$. Then for all $g : D([0, t])^2 \rightarrow [0, \infty]$ we have

$$\mathbb{E}_{(l,r)}(g(L^\varnothing|_{[0,t]}, R^\varnothing|_{[0,t]})) = \frac{\mathbb{E}_{(l,r)}\left(\prod_{s \leq t} W^\varnothing(|\Delta X_s|) \cdot g(L|_{[0,t]}, R|_{[0,t]})\right)}{\exp\left(t \int (\mu_L + \mu_R)(dh) (W^\varnothing(|h|) - 1)\right)}.$$

We define $\tau_0^\varnothing = \inf\{t \geq 0 : L_t^\varnothing \leq 0 \text{ or } R_t^\varnothing \leq 0\}$ and $\alpha = \int (\mu_L + \mu_R)(dh) (W^\varnothing(|h|) - 1)$.

As in the previous section, we consider a finite subset $A \subseteq \mathbb{N}$ and smooth functions $W^B : (0, \infty) \rightarrow (0, \infty)$ for each $B \subseteq A$ such that

$W^\varnothing$ is nonincreasing, $W^\varnothing(h) \leq 1$, $1 - W^\varnothing(h) \lesssim h^{8/\kappa'}$ for $h > 0$

and $W^B(x) \lesssim W^B(y)$ for all $B \subseteq A$ and whenever $0 < y \leq x \leq 2y$. (3-21)

By convention we will write $W^\varnothing(0) = 1$ and $W^B(0) = 0$ if $B \neq \varnothing$. We also consider constants $\sigma_B \in \mathbb{R}$ for $B \subseteq A$ with $\sigma_\varnothing = 0$.

Recall also the following definition: for $u : [0, \infty) \rightarrow \mathbb{R}$ càdlàg we let $\Pi(u, B)$ be the set of càdlàg nonincreasing (in the sense of inclusion) functions $Q : [0, \infty) \rightarrow \mathcal{P}(A)$ such that $Q_0 = B$, $Q_t = \varnothing$ for t sufficiently large and $\Delta Q_t = Q_{t-} \setminus Q_t = \varnothing$ whenever $\Delta u_t = 0$.

Theorem 3.15. For each $B \subseteq A$ and $l, r > 0$, we now define a new process $\mathbf{Y}^W = (L^W, R^W, P^W)$ taking values in $(0, \infty)^2 \times \mathcal{P}(A)$, starting at (l, r, B) and absorbed at $(0, 0, \emptyset)$ by

$$\mathbb{E}_{(l,r,B)}(g(L^W, R^W, P^W)) = \frac{1}{W^B(l+r)} \mathbb{E}_{(l,r)} \left(\sum_{Q \in \Pi(X', B)} g(L', R', Q) \prod_{\substack{s < \zeta' \\ \Delta X'_s \neq 0}} e^{\sigma_{Q_s - \setminus Q_s} 1(\Delta X'_s > 0)} W^{Q_s - \setminus Q_s}(\Delta X'_s) \right).$$

For this to make sense, we also assume that the right-hand side of the equation is 1 for $g = 1$ and for all $B \subseteq A$ and $l, r > 0$. We let $\zeta^W = \inf\{t \geq 0 : \mathbf{Y}^W = (0, 0, \emptyset)\}$. Then $\mathbf{Y}^W$ is a strong Markov process and the following expression is well-defined:

$$\begin{aligned} \mathcal{G}_{\kappa', \beta}^W f(l, r, B) &:= f(l, r, B) \int (\mu_L + \mu_R)(dh) (W^\emptyset(h) - 1) \\ &+ \frac{1}{w_{\kappa'}(l, r) W^B(l+r)} \mathcal{L}_{\kappa', \beta}^\emptyset((u, v) \mapsto w_{\kappa'}(u, v) W^B(u+v) f(u, v, B))(l, r) \\ &+ \sum_{C \subsetneq B} \int \mu_L(dh) \frac{W^{B \setminus C}(|h|) w_{\kappa'}(l+h, r) W^C(l+r+h)}{w_{\kappa'}(l, r) W^B(l+r)} e^{\sigma_{B \setminus C} 1(h > 0)} f(l+h, r, C) \\ &+ \sum_{C \subsetneq B} \int \mu_R(dh) \frac{W^{B \setminus C}(|h|) w_{\kappa'}(l, r+h) W^C(l+r+h)}{w_{\kappa'}(l, r) W^B(l+r)} e^{\sigma_{B \setminus C} 1(h > 0)} f(l, r+h, C) \end{aligned}$$

for $f : (0, \infty)^2 \times \mathcal{P}(A) \rightarrow \mathbb{R}$ with $f(\cdot, B) \in C_c^\infty((0, \infty)^2)$ for all $B \subseteq A$. Then

$$\lim_{t \downarrow 0} \frac{1}{t} \mathbb{E}_{(l,r,B)}(f(\mathbf{Y}_t^W) - f(l, r, B)) = \mathcal{G}_{\kappa', \beta}^W f(l, r, B),$$

i.e., $\mathcal{G}_{\kappa', \beta}^W$ is the generator of $\mathbf{Y}^W$ on the set of functions f considered above.

The proof of the following proposition is exactly analogous to the one of [Proposition 3.12](#) except that [Lemma B.4](#) is used instead of [Lemma B.3](#) in the computations.

Proposition 3.16. Suppose that $l, r > 0$ such that $l + r = 1$ and let

$$A_+ = \mathbb{E}_{(l,r)} \left(\sum_{\substack{t < \zeta' \\ \Delta X'_t > 0}} |\Delta X'_t|^\theta \right) \quad \text{and} \quad A_- = \mathbb{E}_{(l,r)} \left(\sum_{\substack{t < \zeta' \\ \Delta X'_t < 0}} |\Delta X'_t|^\theta \right).$$

Then $A_+ < \infty$, $A_- < 1$ and $A_+/(1 - A_-) = \cos(4\pi/\kappa')/\cos(4\pi/\kappa' - \pi\theta)$ whenever we have $\theta \in (4/\kappa' + 1/2, 4/\kappa' + 3/2)$. Also, if $\theta \notin (4/\kappa', 1 + 8/\kappa')$ then $A_+ = \infty$.

4. Some functionals of Lévy excursions

We will carry out computations for stable Lévy excursions. These will be used when finding the partition function for the generalized LQG disk with zero marked points or one marked point in [Section 8A](#).

Suppose that $\nu \in (1, 2)$. Throughout this section, let $(B_t)_{t \geq 0}$ be a stable Lévy process with exponent ν and no negative jumps started at 0. We fix the normalization as in [Curien and Kortchemski 2014], that is, by requiring that

$$\mathbb{E}(e^{-\lambda B_t}) = e^{-t\lambda^\nu} \quad \text{for } \lambda > 0 \text{ and } t \geq 0.$$

The jump measure of the Lévy process B is $\pi(dh) = \Gamma(-\nu)^{-1} h^{-1-\nu} 1_{(0, \infty)}(h) dh$. Let us define

$$\tau_y = \inf\{t \geq 0 : B_t < -y\} \quad \text{and} \quad \tau_{y-} = \inf\{t \geq 0 : B_t \leq -y\}$$

for $y \geq 0$. Note that we have $\tau_y \neq \tau_{y-}$ precisely if $B|_{[\tau_{y-}, \tau_y]}$ is an excursion from $-y$ to $-y$ staying strictly above level $-y$ on the interval (τ_{y-}, τ_y) .

Then by excursion theory (see, for instance, [Kallenberg 2002, Chapter 22]) there exists a random nonnegative process (b_t) starting at 0 and absorbed again at 0 at time 1 with the following property. Let $\tilde{n}_\ell$ be the law of $(\ell^{1/\nu} b_{t/\ell} : t \geq 0)$, i.e., $\tilde{n}_\ell$ is a probability measure on excursions of duration ℓ which is obtained by rescaling b appropriately. Define

$$\tilde{n} = \frac{1}{\nu \cdot \Gamma(1 - 1/\nu)} \int_0^\infty d\ell \ell^{-1-1/\nu} \tilde{n}_\ell,$$

so that $\tilde{n}$ is an infinite measure on excursions of nonfixed duration obtained by combining the measures $\tilde{n}_\ell$ (called the excursion measure of the Lévy process). Then

$$\xi := \sum_{\substack{y \geq 0 \\ \tau_y > \tau_{y-}}} \delta_{(y, B_{(\tau_{y-} + \cdot) \wedge \tau_y} - B_{\tau_{y-}})} \quad \text{is a PPP with intensity } \lambda|_{(0, \infty)} \otimes \tilde{n}$$

on $(0, \infty) \times D([0, \infty))$ where λ is the Lebesgue measure on $\mathbb{R}$ and $D([0, \infty))$ is the space of càdlàg functions. In other words, the excursions made by B above its running infimum indexed by the value of the infimum have law of a Poisson point process (PPP) with intensity $\lambda|_{(0, \infty)} \otimes \tilde{n}$. The process $(\ell^{1/\nu} b_{t/\ell} : t \geq 0)$ is called the spectrally positive stable Lévy excursion with exponent ν and length ℓ . We refer to [Curien and Kortchemski 2014, Section 3.1.1] for more information on these constructions.

We will now compute some explicit functionals associated to such Lévy excursions. Recall the definition of the modified Bessel functions of the second kind:

$$K_\nu(x) = \frac{1}{2(x/2)^\nu} \int_0^\infty e^{-(x/2)^2 y - 1/y} \frac{dy}{y^{1+\nu}}.$$

We will now prove the following theorem; as the proof will reveal, it is quite remarkable that the expectations of the functionals below have a completely explicit form.

Theorem 4.1. *Consider $\nu \in (1, 2)$, $c > 0$, $\theta \in (1, 1 + 1/\nu)$ and let*

$$g(x) = \frac{2(cx/2)^\nu}{\Gamma(\nu)} K_\nu(cx) \quad \text{for } x > 0 \text{ and } g(0) = 1,$$

$$f(x) = x^{1+\nu} K_{1+\nu-\theta\nu}(cx) \quad \text{for } x > 0 \text{ and } f(0) = 0.$$

¹/₂ Let $c' = c^\nu 2^{1-\nu}$. Then, for all $\ell > 0$,

$$\begin{aligned} \mathbb{E}\left(\prod_{t \leq 1} g(\ell^{1/\nu} \Delta b_t)\right) &= \frac{2(c'\ell/2)^{1/\nu}}{\Gamma(1/\nu)} K_{1/\nu}(c'\ell), \\ \mathbb{E}\left(\sum_{t \leq 1} f(\ell^{1/\nu} \Delta b_t) \prod_{s \neq t} g(\ell^{1/\nu} \Delta b_s)\right) &= \frac{\sqrt{1/\nu} \Gamma(-1/\nu) \sin(\pi(1 + 1/\nu - \theta))}{\sqrt{\nu} \Gamma(-\nu) \sin(\pi(1 + \nu - \theta\nu))} \ell^{1+1/\nu} K_{1+1/\nu-\theta}(c'\ell). \end{aligned}$$

To prove the theorem we need two lemmas, one on Lévy processes and the other on the computation of some integrals involving the modified Bessel functions of the second kind. The latter is [Lemma B.1](#).

Lemma 4.2. Consider $\nu \in (1, 2)$. Let $G : [0, \infty) \rightarrow [0, \infty)$ be twice continuously differentiable with $G(0) = G'(0) = 0$. Suppose that $\lambda > 0$ and $\rho_\lambda > 0$ are such that

$$\lambda = \frac{1}{\Gamma(-\nu)} \int_0^\infty \frac{dh}{h^{1+\nu}} (e^{-\rho_\lambda h - G(h)} - 1 + \rho_\lambda h). \quad (4-1)$$

The integral in the display above is finite for any $\rho_\lambda > 0$ because of the assumption $G(0) = G'(0) = 0$. Then, with B as above,

$$-\rho_\lambda = \log \mathbb{E}(e^{-\lambda \tau_1 - \sum_{t < \tau_1} G(\Delta B_t)}) \quad (4-2)$$

$$= \frac{1}{\nu \Gamma(1 - 1/\nu)} \int_0^\infty \frac{d\ell}{\ell^{1+1/\nu}} (e^{-\lambda \ell} \mathbb{E}(e^{-\sum_{t \leq 1} G(\ell^{1/\nu} \Delta b_t)}) - 1). \quad (4-3)$$

Let $F : [0, \infty) \rightarrow [0, \infty)$ be twice continuously differentiable with $F(0) = F'(0) = 0$. Then

$$\begin{aligned} -\frac{d}{d\lambda} (e^{-\rho_\lambda}) \frac{1}{\Gamma(-\nu)} \int_0^\infty \frac{dh}{h^{1+\nu}} e^{-\rho_\lambda h - G(h)} F(h) \\ = \mathbb{E}\left(\sum_{t < \tau_1} F(\Delta B_t) e^{-\lambda \tau_1 - \sum_{s < \tau_1} G(\Delta B_s)}\right) \\ = \frac{e^{-\rho_\lambda}}{\nu \Gamma(1 - 1/\nu)} \int_0^\infty \frac{d\ell}{\ell^{1+1/\nu}} \mathbb{E}\left(\sum_{t \leq 1} F(\ell^{1/\nu} \Delta b_t) e^{-\lambda \ell - \sum_{s \leq 1} G(\ell^{1/\nu} \Delta b_s)}\right), \end{aligned} \quad (4-4)$$

where now we need to assume that a $\rho_\lambda > 0$ satisfying (4-1) is defined on some open set of $\lambda > 0$ values in order for the derivative to be well-defined.

Proof. Let M be the càdlàg process

$$M_T = e^{-\rho_\lambda B_T - \lambda T - \sum_{t \leq T} G(\Delta B_t)}, \quad T \geq 0.$$

We will first argue that $\mathbb{E}(M_T) = 1$ for any $T \geq 0$. Since B has stationary and independent increments, it will follow that M is a martingale. By the construction of the compensated Lévy process B as a compensated sum over its jumps (see, e.g., [Bertoin 1996](#), Chapter I), we have

$$M_T^n := e^{-\rho_\lambda (\sum_{t \leq T: \Delta B_t > 1/n} \Delta B_t - T \Gamma(-\nu)^{-1} \int_{1/n}^\infty h \cdot h^{-1-\nu} dh) - \sum_{t \leq T: \Delta B_t > 1/n} G(\Delta B_t) - \lambda T} \rightarrow M_T$$

$1^{1/2}$ in probability as $n \rightarrow \infty$. Campbell's formula and the definition of λ imply that

$$\log \mathbb{E}(M_T^n) = -\lambda T + \frac{T}{\Gamma(-\nu)} \int_{1/n}^{\infty} (e^{-\rho_\lambda h - G(h)} - 1 + \rho_\lambda h) \frac{dh}{h^{1+\nu}} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4-5)$$

A similar computation shows that $(M_T^n : n \geq 1)$ is bounded in L^2 and in particular uniformly integrable. Therefore $\mathbb{E}(M_T^n) \rightarrow \mathbb{E}(M_T)$ as $n \rightarrow \infty$. Using this, (4-1), and (4-5), we get that $\mathbb{E}(M_T) = 1$ as required. The fact that M is a martingale and optional stopping now implies that, for any $T \geq 0$,

$$1 = \mathbb{E}(M_{T \wedge \tau_1}) = \mathbb{E}(e^{-\rho_\lambda B_{T \wedge \tau_1} - \lambda(T \wedge \tau_1) - \sum_{t \leq T \wedge \tau_1} G(\Delta B_t)}).$$

Sending $T \rightarrow \infty$ and using dominated convergence with dominating function e^{ρ_λ} we get the first equality of (4-2). For the second equality of (4-2), we observe that

$$\lambda \tau_1 + \sum_{t < \tau_1} G(\Delta B_t) = \int_{(0,1) \times D([0,\infty))} \xi(dy, de) \left(\lambda \zeta_e + \sum_{t < \zeta_e} G(\Delta e_t) \right) \quad \text{a.s.}, \quad (4-6)$$

where $\zeta_e = \inf\{t > 0 : e_t = 0\}$. Hence since n is the intensity measure of the Poisson point process ξ , we obtain

$$\log \mathbb{E}(e^{-\lambda \tau_1 - \sum_{t < \tau_1} G(\Delta B_t)}) = \int_{(0,1) \times D([0,\infty))} n(dy, de) (e^{\lambda \zeta_e + \sum_{t < \zeta_e} G(\Delta e_t)} - 1).$$

The result follows from the definition of n .

The proof of (4-4) will make use of Palm's formula in several places (see Lemma 3.5). Let us start with the excursion theory perspective as this is somewhat simpler. Indeed, we write

$$\begin{aligned} X &:= \sum_{t < \tau_1} F(\Delta B_t) e^{-\lambda \tau_1 - \sum_{s < \tau_1} G(\Delta B_s)} \\ &= \int_{(0,1) \times D([0,\infty))} \xi(dy, de) \left(\sum_{t < \zeta_e} F(\Delta e_t) \right) e^{-\int_{(0,1) \times D([0,\infty))} \xi(dy', de') (\lambda \zeta_{e'} + \sum_{s < \zeta_{e'}} G(\Delta e'_s))}. \end{aligned}$$

By Palm's formula we obtain

$$\begin{aligned} \mathbb{E}(X) &= \int_{(0,1) \times D([0,\infty))} n(dy, de) \left(\sum_{t < \zeta_e} F(\Delta e_t) \right) \\ &\quad \cdot \mathbb{E}(e^{-\int_{(0,1) \times D([0,\infty))} (\xi + \delta_{(y,e)})(dy', de') (\lambda \zeta_{e'} + \sum_{s < \zeta_{e'}} G(\Delta e'_s))}) \\ &= \int_{(0,1) \times D([0,\infty))} n(dy, de) \left(\sum_{t < \zeta_e} F(\Delta e_t) e^{-\lambda \zeta_e - \sum_{s < \zeta_e} G(\Delta e_s)} \right) \\ &\quad \cdot \mathbb{E}(e^{-\int_{(0,1) \times D([0,\infty))} \xi(dy', de') (\lambda \zeta_{e'} + \sum_{s < \zeta_{e'}} G(\Delta e'_s))}). \end{aligned}$$

Note that by (4-6), the final expectation in the display above equals $e^{-\rho_\lambda}$ and the second equality of (4-4) follows again from the definition of n .

Lastly, we compute $\mathbb{E}(X)$ using the Poisson point process structure of the jumps of B . We will need the notation

$$B^{(t,h)} = B + h \cdot 1_{[t,\infty)} \quad \text{and} \quad \tau_1^{(t,h)} = \inf\{s \geq 0 : B_s^{(t,h)} < -1\}.$$

Again by Palm's formula we get

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}\left(\sum_{t \geq 0} F(\Delta B_t) 1(t < \tau_1) e^{-\lambda \tau_1 - \sum_{s < \tau_1} G(\Delta B_s)}\right) \\ &= \frac{1}{\Gamma(-\nu)} \int_0^\infty dt \int_0^\infty \frac{dh}{h^{1+\nu}} F(h) \mathbb{E}(e^{-\lambda \tau_1^{(t,h)} - \sum_{s < \tau_1} G(\Delta B_s^{(t,h)})}; t < \tau_1^{(t,h)}). \end{aligned}$$

Note that $\{t < \tau_1^{(t,h)}\} = \{t < \tau_1\}$ a.s. and by using the strong Markov property of B at time τ_1 we get

$$\begin{aligned} \mathbb{E}(e^{-\lambda \tau_1^{(t,h)} - \sum_{s < \tau_1} G(\Delta B_s^{(t,h)})}; t < \tau_1^{(t,h)}) &= \mathbb{E}(e^{-\lambda \tau_1^{(t,h)} - \sum_{s < \tau_1} G(\Delta B_s^{(t,h)})}; t < \tau_1) \\ &= e^{-G(h)} \mathbb{E}(e^{-\lambda \tau_1 - \sum_{s < \tau_1} G(\Delta B_s)}; t < \tau_1) \mathbb{E}(e^{-\lambda \tau_h - \sum_{s < \tau_h} G(\Delta B_s)}) \\ &= e^{-G(h)} \mathbb{E}(e^{-\lambda \tau_1 - \sum_{s < \tau_1} G(\Delta B_s)}; t < \tau_1) e^{-\rho_\lambda h}. \end{aligned}$$

Plugging this into the expression for $\mathbb{E}(X)$ derived above yields

$$\mathbb{E}(X) = \frac{1}{\Gamma(-\nu)} \int_0^\infty \frac{dh}{h^{1+\nu}} F(h) e^{-G(h) - \rho_\lambda h} \mathbb{E}(e^{-\lambda \tau_1 - \sum_{s < \tau_1} G(\Delta B_s)} \tau_1).$$

The first equality of (4-4) now follows by using that the expectation on the right-hand side of the last display is equal to $d(e^{-\rho_\lambda})/d\lambda$. $\square$

Proof of Theorem 4.1. Let $F(\ell) = f(\ell)/g(\ell)$ and $G(\ell) = -\log g(\ell)$. Let us first compute ρ_λ for $\lambda > 0$ in the setting of Lemma 4.2. We have

$$\begin{aligned} \lambda &= \frac{1}{\Gamma(-\nu)} \int_0^\infty \frac{dh}{h^{1+\nu}} \left(e^{-\rho_\lambda h} \frac{2(ch/2)^\nu}{\Gamma(\nu)} K_\nu(ch) - 1 + \rho_\lambda h \right) \\ &= c^\nu 2^{1-\nu} \begin{cases} \cos \nu \theta_\lambda & \text{if } \rho_\lambda \in (0, c], \text{ where } \theta_\lambda \in [0, \pi/2) \text{ is defined by } \rho_\lambda = c \cos \theta_\lambda, \\ \cosh \nu \theta_\lambda & \text{if } \rho_\lambda > c, \text{ where } \theta_\lambda > 0 \text{ is defined by } \rho_\lambda = c \cosh \theta_\lambda, \end{cases} \end{aligned}$$

by Lemma B.1 and Euler's reflection formula. This can be inverted:

$$\rho_\lambda = c \begin{cases} \cos(\psi_\lambda/\nu) & \text{if } \lambda \in (0, c^\nu 2^{1-\nu}], \text{ where } \psi_\lambda \in [0, \pi/2) \text{ and } \lambda = c^\nu 2^{1-\nu} \cos \psi_\lambda, \\ \cosh(\psi_\lambda/\nu) & \text{if } \lambda > c^\nu 2^{1-\nu}, \text{ where } \psi_\lambda > 0 \text{ and } \lambda = c^\nu 2^{1-\nu} \cosh \psi_\lambda. \end{cases}$$

Moreover, again by Lemma B.1 and Euler's reflection formula,

$$\begin{aligned} &\frac{1}{\nu \Gamma(1-1/\nu)} \int_0^\infty \frac{d\ell}{\ell^{1+1/\nu}} \left(e^{-\lambda \ell} \frac{2(c'\ell/2)^{1/\nu}}{\Gamma(1/\nu)} K_{1/\nu}(c'\ell) - 1 \right) \\ &= -(c')^{1/\nu} 2^{1-1/\nu} \cdot \begin{cases} \cos(\psi_\lambda/\nu) & \text{if } \lambda \in (0, c'], \text{ where } \psi_\lambda \in [0, \pi/2) \text{ and } \lambda = c' \cos \psi_\lambda, \\ \cosh(\psi_\lambda/\nu) & \text{if } \lambda > c', \text{ where } \psi_\lambda > 0 \text{ and } \lambda = c' \cosh \psi_\lambda. \end{cases} \\ &= -\rho_\lambda \end{aligned}$$

$1^{1/2}$ by the definition of c' . The first part of the theorem then follows from the injectivity of the Laplace transform and the first part of [Lemma 4.2](#). For the second part of the theorem, again by a Laplace inversion argument and the now the second part of [Lemma 4.2](#), it suffices to show that

$$-\frac{d(e^{-\rho_\lambda})/d\lambda}{\Gamma(-v)} \int_0^\infty \frac{e^{-\rho_\lambda h} dh}{h^{1+v}} f(h) = \frac{e^{-\rho_\lambda}}{v\Gamma(1-1/v)} \int_0^\infty \frac{e^{-\lambda \ell} d\ell}{\ell^{1+1/v}} C \ell^{1+1/v} K_{1+1/v-\theta}(c'\ell)$$

where

$$C = \frac{\sqrt{1/v} \Gamma(-1/v) \sin(\pi(1+1/v-\theta))}{\sqrt{v} \Gamma(-v) \sin(\pi(1+v-\theta v))}.$$

This can be verified directly via [Lemma B.1](#). In the computation it is useful to note that $\psi_\lambda = v\theta_\lambda$. $\square$

5. Background on CLE and LQG

We present various background materials which are needed in the remainder of the paper and in order to give precise statements of our main results. In [Section 5A](#) we review results on CLE explorations from [\[Miller et al. 2017; 2021; 2022\]](#). In [Section 5B](#) we introduce the Gaussian free field and the notions of regular and generalized LQG surfaces. In [Section 5C](#) we define regular and generalized LQG disks. In the case of the regular disk we allow the disk to have a given number of marked points at given locations. Our definition coincides with the one given in [\[Huang et al. 2018\]](#) and we will consider this disk in both the regular and the generalized cases in [Section 8](#). Finally, in [Section 5D](#) we review results from [\[Miller et al. 2021; 2022\]](#) on explorations of CLE decorated LQG disks and explain some consequences of these results.

5A. CLE explorations. We assume basic familiarity with SLE [\[Werner 2004; Lawler 2005\]](#) and CLE [\[Sheffield 2009; Sheffield and Werner 2012\]](#). We will however present one of the outcomes from [\[Miller et al. 2017\]](#) that will be used in this paper. The ideas below are illustrated in [Figure 4](#).

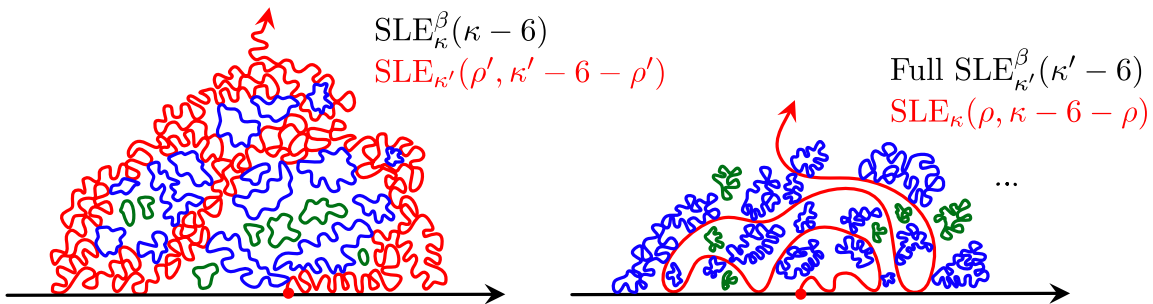


Figure 4. Left: a CPI (red) in a CLE_κ with $\kappa \in (8/3, 4)$ and with parameter β has $SLE_{\kappa'}(\rho', \kappa' - 6 - \rho')$ as its marginal law and the exploration path is a $SLE_\kappa^\beta(\kappa - 6)$ curve. Right: when $\kappa \in (4, 8)$, then the marginal law of a CPI (red) in a $CLE_{\kappa'}$ and with parameter β is a $SLE_\kappa(\rho, \kappa - 6 - \rho)$ curve and the obtained exploration path is then a full $SLE_{\kappa'}^\beta(\kappa' - 6)$ curve. Here κ and κ' are related by $\kappa' = 16/\kappa$. The blue loops are those discovered by the CPI while the green ones are not discovered by it.

The concept that is important is that of a continuum percolation interface (CPI). We begin with the simple case $\kappa < 4$ (the $\kappa = 4$ case is slightly different [Miller et al. 2017; Lehmkuehler 2023] and will not be needed here). Fix a parameter $\beta \in [-1, 1]$ and let Γ be a nonnested CLE_κ in $\mathbb{H}$. Conditionally on Γ , we independently assign to each loop a counterclockwise orientation with probability $(1 + \beta)/2$ and a clockwise orientation with probability $(1 - \beta)/2$. This way, we obtain a collection Γ_β of oriented loops. A CPI λ from 0 to ∞ in Γ_β is a (random) continuous curve in $\overline{\mathbb{H}}$ that is generated by a Loewner chain, does not intersect the interior of any loops in Γ_β and obeys the following properties:

- (i) The law of (Γ_β, λ) satisfies scale invariance.
- (ii) All counterclockwise (resp. clockwise) loops in Γ_β intersecting λ are on the right (resp. on the left) of the curve λ .
- (iii) Fix $t \geq 0$ and let K_t be the union of $\lambda([0, t])$ and the filling of all loops in Γ intersecting $\lambda([0, t])$. We condition on $\lambda|_{[0, t]}$ and all the loops in Γ_β intersecting $\lambda([0, t])$. Given this information, the conditional law of (Γ_β, λ) is given by sampling independent nonnested CLE_κ (with i.i.d. orientations as before) in the bounded complementary components of K_t and an independent copy of (Γ_β, λ) mapped conformally into the unbounded complementary component of K_t where we concatenate $\lambda|_{[0, t]}$ with the curve in the mapped in domain.

Details of this definition can be found in [Miller et al. 2017, Section 2.3]. By [Miller et al. 2017, Section 4] this uniquely characterizes the law of (Γ_β, λ) , however almost surely the conditional law of λ given Γ_β is nonatomic as established in [Miller et al. 2020]. Informally this means that almost surely, we cannot deterministically obtain λ from Γ_β . By [Miller et al. 2022, Theorem 1.6], the law of λ is a $\text{SLE}_{\kappa'}(\rho', \kappa' - 6 - \rho')$ curve where $\kappa' = 16/\kappa$ and

$$\rho' = \frac{2}{\pi} \arctan \frac{\sin(\pi \kappa'/2)}{1 + \cos(\pi \kappa'/2) - 2/(1 - \beta)} \in [\kappa' - 6, 0],$$

where the branch of $\arctan(\cdot)$ is chosen for each value β such that the value on the right lies in the specified interval. The curve obtained by following λ and tracing each loop that λ intersects according to its orientation and in chronological order is a so-called $\text{SLE}_\kappa^\beta(\kappa - 6)$ curve (see [Sheffield 2009] and [Werner and Wu 2013]).

When $\kappa' \in (4, 8)$ there is an analogous concept of a CLE percolation interface which we will also abbreviate by CPI. Again, we fix $\beta \in [-1, 1]$, let Γ be a nonnested $\text{CLE}_{\kappa'}$ in $\mathbb{H}$ and obtain a CLE with orientations Γ_β in precisely the same way as in the simple case. It turns out that a.s. there exists a unique simple curve λ from 0 to ∞ in $\overline{\mathbb{H}}$ which does not intersect the interior of any loop in Γ_β and which has the property that each counterclockwise (resp. clockwise) loop in Γ_β that intersects λ is right (resp. left) of λ . When $\beta = 1$ (resp. $\beta = -1$), λ simply traces the negative (resp. positive) real line. If $\beta \in (-1, 1)$ then λ has the law of a $\text{SLE}_\kappa(\rho, \kappa - 6 - \rho)$ curve where $\kappa = 16/\kappa'$ and

$$\rho = \frac{2}{\pi} \arctan \left(\frac{\sin(\pi \kappa/2)}{1 + \cos(\pi \kappa/2) - 2/(1 - \beta)} \right) - 2 \in (-2, \kappa - 4)$$

¹/₂ as established in [Miller et al. 2021, Theorem 1.3]. As in the simple case, the conditional law of Γ_β given λ and all loops in Γ_β that λ intersects is obtained by sampling independent nonnested $\text{CLE}_{\kappa'}$ (with i.i.d. orientations as before) within each complementary component of the union of λ and the filling of all the loops it touches (see [Miller et al. 2017, Section 6]). Lastly, one can obtain a curve by moving along λ and tracing a loop whenever it is first hit by λ according to its orientation. This can be defined rigorously and the curve so obtained is a full $\text{SLE}_{\kappa'}^\beta(\kappa' - 6)$; see [Miller et al. 2017, Section 6] for definitions and details on these results.

In the above discussion, we explained the background on CPIs in the upper half plane starting at 0 and ending at ∞ . If we consider an arbitrary simply connected domain with two distinguished prime ends, then by mapping the CPI and CLE to this domain along a conformal transformation which sends 0 (resp. ∞) to the first (resp. second) distinguished prime end, then this allows us to generalize the above definition to the setting of arbitrary simply connected domains. We call the tuple (Γ, λ) a CLE coupled with a CPI with asymmetry parameter β .

5B. The Gaussian free field and Liouville quantum gravity. For any domain $D \subseteq \mathbb{C}$, let $C_c^\infty(D)'$ denote the space of distributions (also known as generalized functions) on the domain D with its usual topology; the σ -algebra is generated by the evaluation maps $h \rightarrow h(f)$ for $f \in C_c^\infty(D)$. Note that this space is not metrizable and hence not Polish. For technical reasons, it will therefore be advantageous to also work with a subspace of $C_c^\infty(D)'$ on some occasions (the disadvantage of this subspace is that it is not invariant under precompositions with conformal transformations). If D has harmonically nontrivial boundary, we let $H_0^1(D)$ be the Sobolev space which is the Hilbert space closure of $C_c^\infty(D)$ with respect to the inner product

$$(f, g)_\nabla = \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) dz$$

for $f, g \in C_c^\infty(D)$ (the 2π factor makes the explicit Green's function expressions below simpler). Its Hilbert space dual $H^{-1}(D) := H_0^1(D)'$ embeds continuously into $C_c^\infty(D)'$. The space $H^{-1}(D)$ is a metric space and it is also separable and hence Polish.

Let $G_D : \mathbb{D} \times \mathbb{D} \rightarrow (0, \infty]$ be the Dirichlet Green's function on $\mathbb{D}$:

$$G_D(z, w) = -\log(|z - w|) + \log(|1 - z\bar{w}|).$$

The Dirichlet Gaussian free field on $\mathbb{D}$ (see [Sheffield 2007; Werner and Powell 2021]) is the random element h_D of $H^{-1}(\mathbb{D})$ such that $h_D(f)$ is a centered Gaussian random variable for each $f \in C_c^\infty(\mathbb{D})$ and such that

$$\mathbb{E}(h_D(f)h_D(g)) = \int_{\mathbb{D} \times \mathbb{D}} G_D(z, w) f(z)g(w) dz dw.$$

One can deterministically associate to h_D its circle average approximation which is a collection of random variables $((h_D)_\epsilon(z)) : \epsilon > 0, z \in (1 - \epsilon)\mathbb{D}$ which is continuous in its parameters; see [Duplantier and Sheffield 2011]. Then

$$\text{Var}((h_D)_\epsilon(z)) = \log\left(\frac{1}{\epsilon}\right) + R(z, \mathbb{D}) + o(1) \quad \text{as } \epsilon \rightarrow 0 \text{ for all } z \in \mathbb{D}. \quad (5-1)$$

¹/₂ If $\gamma \in (0, 2)$ then there is a measure $\mu_{h_D}^\gamma$ (called the γ -LQG area measure) supported in $\mathbb{D}$ such that
² whenever $f \in C_b(\mathbb{C})$ (i.e., f is continuous and bounded), we have
³

$$\int_{(1-\epsilon)\mathbb{D}} f(z) \epsilon^{\gamma/2} e^{\gamma(h_D)_\epsilon(z)} dz \rightarrow \int_{\mathbb{D}} f(z) \mu_{h_D}^\gamma(dz)$$

⁴ as $\epsilon \rightarrow 0$ in L^1 and
⁵

$$\mathbb{E} \left(\int_{\mathbb{D}} f(z) \mu_{h_D}^\gamma(dz) \right) = \int_{\mathbb{D}} f(z) R(z, \mathbb{D})^{\gamma/2} dz. \quad (5-2)$$

⁶ This type of result is classical and due to Kahane [1985], see also [Duplantier and Sheffield 2011] and
⁷ [Berestycki 2017] (the conformal radius factor appears in order to make this definition agree with the one
⁸ involving circle averages). Note that if h is a random function inducing a distribution in $H^{-1}(\mathbb{D})$, then
⁹ (in view of the circle average definition) it is natural to define

$$\mu_{h_D+h}^\gamma(dz) := e^{\gamma h(z)} \mu_{h_D}^\gamma(dz). \quad (5-3)$$

¹⁰ There is an analogous concept of an LQG boundary measure which we introduce in the particular case
¹¹ of a Neumann GFF on the unit disk. Throughout this paper, $G : \mathbb{D} \times \mathbb{D} \rightarrow (0, \infty]$ denotes the Green's
¹² function for the Neumann GFF in $\mathbb{D}$ with mean zero on $\partial\mathbb{D}$, namely

$$G(z, w) = -\log(|z - w| |1 - z\bar{w}|).$$

¹³ An important property of G and G_D which underlies conformal covariance properties of several measures
¹⁴ appearing in this paper is that for any Möbius transformation $f : \mathbb{D} \rightarrow \mathbb{D}$ we have
¹⁵

$$G(f(z), f(w)) = G(z, w) - \log |f'(z)| - \log |f'(w)| \quad \text{and} \quad G_D(f(z), f(w)) = G_D(z, w).$$

¹⁶ The Neumann GFF h^0 on $\mathbb{D}$ with mean zero on $\partial\mathbb{D}$ is then the random element of $H^{-1}(\mathbb{D})$ such that
¹⁷ $h^0(f)$ is a centered Gaussian random variable for all $f \in C_c^\infty(\mathbb{D})$ and
¹⁸

$$\mathbb{E}(h^0(f)h^0(g)) = \int_{\mathbb{D} \times \mathbb{D}} G(z, w) f(z) g(w) dz dw.$$

¹⁹ One key property of h^0 is the following spatial Markov property that it satisfies: Consider disjoint balls
²⁰ $B_{r_i}(z_i) \subseteq \mathbb{D}$ for $i = 1, \dots, n$. Then there are random harmonic functions h_i on $B_{r_i}(z_i)$ and i.i.d. Dirichlet
²¹ Gaussian free fields h_i in $\mathbb{D}$ for $i \leq n$ such that $(h_i : i \leq n)$ and $(h_i : i \leq n)$ are independent, and
²²

$$h^0|_{\bigcup_i B_{r_i}(z_i)} = \sum_{i=1}^n h_i \left(\frac{\cdot - z_i}{r_i} \right) 1_{B_{r_i}(z_i)} + \sum_{i=1}^n h_i 1_{B_{r_i}(z_i)}.$$

²³ We call the first summand the projection of h^0 onto $H^{-1}(\bigcup_i B_{r_i}(z_i))$ and the second summand the
²⁴ harmonic extension of h^0 restricted to the complement of $\bigcup_i B_{r_i}(z_i)$. Note that the same is true for any
²⁵ continuous function g on $\mathbb{D}$: We can uniquely decompose g into a function supported on $\bigcup_i B_{r_i}(z_i)$
²⁶ and a continuous function which agrees with g on $\mathbb{D} \setminus \bigcup_i B_{r_i}(z_i)$ and is harmonic on $\bigcup_i B_{r_i}(z_i)$. Thus
²⁷ this projection decomposition extends to the case when h^0 is replaced by a field which is the sum of a
²⁸ continuous function and an independent field which is absolutely continuous with respect to h^0 .
²⁹

To understand the LQG area and boundary measure associated to h^0 , let us observe that

$$G(z, w) = G_D(z, w) - 2 \log(|1 - z\bar{w}|). \quad (5-4)$$

By the Markov property explained above, we can construct a random continuous function $\mathfrak{h}$ on $\mathbb{D}$ (extended to be 0 on $\mathbb{C} \setminus \mathbb{D}$) independent of a Dirichlet GFF h_D on $\mathbb{D}$ such that $\mathfrak{h}$ is a centered Gaussian process with

$$\mathbb{E}(\mathfrak{h}(z)\mathfrak{h}(w)) = -2 \log(|1 - z\bar{w}|) \quad \text{for all } z, w \in \mathbb{D}.$$

By (5-4) $h^0 = h_D + \mathfrak{h}$ is then a Neumann GFF on $\mathbb{D}$ with mean zero on the boundary. Thus $\mu_{h^0}^\gamma$ is defined via (5-3) and it follows that, for each $f \in C_b(\mathbb{C})$ and all $\gamma \in (0, 2)$,

$$\int_{(1-\epsilon)\mathbb{D}} f(z) \epsilon^{\gamma^2/2} e^{\gamma h_\epsilon^0(z)} dz \rightarrow \int_{\mathbb{D}} f(z) \mu_{h^0}^\gamma(dz) \quad \text{in } L^1 \text{ as } \epsilon \rightarrow 0.$$

We can now define the semicircle approximations

$$(h_\epsilon^0(z) : \epsilon \in (0, 2), z \in \partial\mathbb{D}) \text{ to } h^0 \quad \text{and} \quad (\mathfrak{h}_\epsilon(z) : \epsilon \in (0, 2), z \in \partial\mathbb{D}) \text{ to } \mathfrak{h};$$

see again [Duplantier and Sheffield 2011]. The γ -LQG boundary measure associated to h^0 when $\gamma \in (0, 2)$ is the measure $\nu_{h^0}^\gamma$ supported on $\partial\mathbb{D}$ such that, for all $f \in C_b(\partial\mathbb{D})$,

$$\int_{\partial\mathbb{D}} f(w) \epsilon^{\gamma^2/4} e^{\gamma h_\epsilon^0(w)/2} dw \rightarrow \int_{\partial\mathbb{D}} f(w) \nu_{h^0}^\gamma(dw) \quad \text{in } L^1 \text{ as } \epsilon \rightarrow 0. \quad (5-5)$$

Note that $\text{Var}(h_\epsilon^0(z)) = 2 \log(1/\epsilon) + o(1)$ as $\epsilon \rightarrow 0$ and hence the expected total mass of $\nu_{h^0}^\gamma$ is 2π . It is not hard to deduce from this, that the above convergence also holds when h_ϵ^0 is replaced by $\mathfrak{h}_\epsilon$ and it follows that $\nu_{h^0}^\gamma$ has a version which is measurable with respect to $\mathfrak{h}$. Again, when $\mathfrak{g}$ is a random function inducing a distribution in $H^{-1}(D)$, we set

$$\nu_{h^0+\mathfrak{g}}^\gamma(dw) = e^{\gamma \mathfrak{g}(w)/2} \nu_{h^0}^\gamma(dw).$$

This construction can be generalized by replacing the Lebesgue measure (as the background measure) by more general measures, in particular those induced by the Minkowski content of (random) curves. This way one can associate to segments of SLE-type curves a γ -LQG length. We refer the reader to [Benoist 2018] for details on this.

Let us recall the definition of a decorated γ -LQG surface (the labeling of the marked points by arbitrary index sets will simplify the notation at a later stage in the paper). Below ∂D is the set of prime ends of a domain D and $\bar{D} = D \cup \partial D$. In the definition below, we will also use the notation $\phi^{-1}(\mathbf{z}) = (\phi^{-1}(z_i) : i \in A)$, $\phi^{-1}(\mathbf{w}) = (\phi^{-1}(w_j) : j \in B)$ and $\phi^{-1} \circ \Gamma = \{\phi^{-1} \circ \eta' : \eta' \in \Gamma\}$.

Definition 5.1. For $\gamma \in (0, 2)$ we define an equivalence relation on tuples $(D, h, \mathbf{z}, \mathbf{w}, \eta, \Gamma)$, where $D \subseteq \mathbb{C}$ is simply connected, $h \in C_c^\infty(D)'$, $\mathbf{z} = (z_i : i \in A)$ with $z_i \in D$ for $i \in A \subseteq \mathbb{N}$, $\mathbf{w} = (w_j : j \in B)$ with $w_j \in \partial D$ for $j \in B \subseteq \mathbb{N}$, η is either a placeholder value — or a continuous curve in $\bar{D}$ viewed up to reparametrization, and Γ is a collection of loops (each loop viewed up to reparametrization). We let

$$(D, h, \mathbf{z}, \mathbf{w}, \eta, \Gamma) \sim_\gamma (D', h \circ \phi + Q_\gamma \log |\phi'|, \phi^{-1}(\mathbf{z}), \phi^{-1}(\mathbf{w}), \phi^{-1} \circ \eta, \phi^{-1} \circ \Gamma)$$

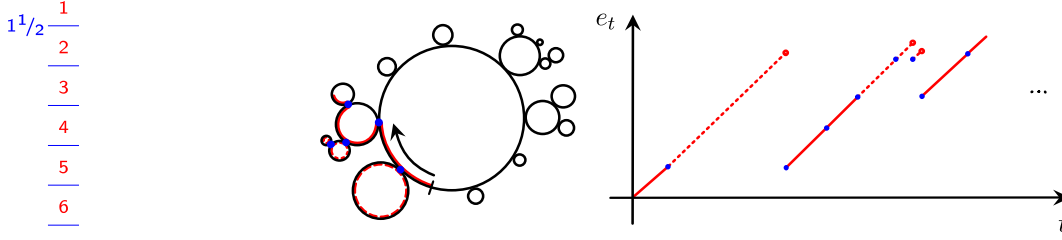


Figure 5. We illustrate the encoding of a looptree with a distinguished point (marked with a line segment) when the total length of all the loops in the looptree is finite. We trace the loops in clockwise order starting at the distinguished point. In this simple example, the process e increases linearly with slope 1 except when a loop is completed in which case it jumps down by the height of the loop. The dotted lines represent the completed loops at this point of the exploration and the blue points correspond to the start of tracing a new loop.

if $\phi : D' \rightarrow D$ is a conformal map. A regular decorated γ -LQG surface is an equivalence class of tuples $(D, h, z, \mathbf{w}, \eta, \Gamma)$. We let $[(D, h, z, \mathbf{w}, \eta, \Gamma)]$ denote the equivalence class associated to $(D, h, z, \mathbf{w}, \eta, \Gamma)$. We write $-$ for the empty tuple and the empty collection and use the shorthand notation $(D, h, z, \mathbf{w}, -, -) = (D, h, z, \mathbf{w})$. We call $[(D, h, z, \mathbf{w})]$ a regular γ -LQG surface.

The motivation behind the above definition (in particular, for the transformation rule for the field h) is that the γ -LQG area measure and γ -LQG boundary measure are invariant under the map ϕ in the sense

$$\mu_h^\gamma = \phi_* \mu_{h \circ \phi + Q_\gamma \log |\phi'|}^\gamma, \quad \nu_h^\gamma = \phi_* \nu_{h \circ \phi + Q_\gamma \log |\phi'|}^\gamma. \quad (5-6)$$

In fact, above we defined ν_h^γ only for the unit disk, and by using that (5-6) holds for the unit disk we get a natural definition of ν_h^γ for a general simply connected domain D by defining

$$\nu_h^\gamma := \phi_* \nu_{h \circ \phi + Q_\gamma \log |\phi'|}^\gamma,$$

where $\phi : \mathbb{D} \rightarrow D$ is conformal. See [Sheffield and Wang 2016] for a proof that (5-6) holds simultaneously for all conformal transformations ϕ .

What the definition of γ -LQG surface above excludes is the possibility of pinch points that the surface might have. The definition of such generalized γ -LQG surfaces relies on the notion of looptrees as introduced in [Curien and Kortchemski 2014] (see also [Duplantier et al. 2021, Section 10] where they appear in the context of forested lines). We will recall the motivation for the following definition below.

Definition 5.2. A generalized (decorated) γ -LQG surface is a pair

$$S = (e, \{(t, S_t) : t \geq 0, \Delta e_t \neq 0\}),$$

where $e : [0, \infty) \rightarrow [0, \infty)$ is a nonconstant càdlàg function with only negative jumps such that $e_0 = 0$ and $e_t = 0$ for all $t \geq \zeta_e = \inf\{s > 0 : e_s = 0\}$, and S_t are regular (decorated) γ -LQG surfaces whenever $\Delta e_t = e_t - e_{t-} \neq 0$. We call ζ_e the boundary length of S .

We will now explain how e encodes a looptree. The construction is illustrated in Figure 5. We define an equivalence relation $\sim_e$ on the interval $[0, \zeta_e]$ by specifying

$$t \sim_e \sup\{s < t : e_s \leq e_t\} \quad \text{when } \Delta e_t \neq 0.$$

The quotient topological space $[0, \zeta_e] / \sim_e$ is still a metric space (i.e., the quotient pseudometric is a metric) and is called the looptree associated to the function e . This topological space has a distinguished point, namely the equivalence class $[0]$. Each time $t \in [0, \zeta_e]$ with $\Delta e_t \neq 0$ corresponds to one particular loop in the looptree and each of these loops has a distinguished point $[t]$ on it. The intuitive picture of S is now that we ‘glue’ each S_t when $\Delta e_t \neq 0$ into the loop corresponding to time t according to the boundary length of the quantum surface; to fix this gluing uniquely, it is performed so that the first marked boundary point of each quantum surface is glued to the distinguished point on the respective loop (needless to say, this assumes that each quantum surface has at least one marked boundary point). If some of the decorated quantum surfaces have a continuous curve as part of its decoration, the picture is that we concatenate these continuous curve segments so that the curve starts at $[0]$ (provided this is possible).

5C. Regular and generalized Liouville quantum gravity disks. The Liouville quantum gravity (LQG) disk is an LQG surface of particular importance since it arises as the scaling limit of random planar maps with disk topology; see Section 2. As we will see, when this disk is coupled to an independent conformal loop ensemble (CLE) and reweighted according to the CLE nesting statistics around n marked points, the field of the disk will get logarithmic singularities at the marked points. It turns out that the latter field is precisely equal to a field which arises in Liouville conformal field theory (LCFT) on the disk as defined by Huang, Rhodes, and Vargas [Huang et al. 2018]. In Definition 5.3 right below we give their definition of the boundary length ℓ Liouville field with marked points.

Definition 5.3. Suppose that $\alpha_i \in \mathbb{R}$ for $i \in A$ and $\beta_j \in \mathbb{R}$ for $j \in B$ for some finite possibly empty index sets $A, B \subseteq \mathbb{N}$ satisfy

$$\alpha_i < Q_\gamma \text{ for all } i, \quad \beta_j < Q_\gamma \text{ for all } j, \quad Q_\gamma - \sum_{i \in A} \alpha_i - \frac{1}{2} \sum_{j \in B} \beta_j < \frac{2}{\gamma} \wedge \min\{Q_\gamma - \beta_j : j \in B\}. \quad (5-7)$$

Let h^0 denote a Neumann GFF in $\mathbb{D}$ with mean zero on $\partial\mathbb{D}$. For distinct points $z_i \in \mathbb{D}$ where $i \in A$ and $w_j \in \partial\mathbb{D}$ for $j \in B$ define

$$h = h^0 + \sum_{i \in A} \alpha_i G(\cdot, z_i) + \sum_{j \in B} \frac{\beta_j}{2} G(\cdot, w_j), \quad h_* = h - \frac{2}{\gamma} \log v_h^\gamma(\partial\mathbb{D}) + \frac{2}{\gamma} \log \ell.$$

For $\Lambda \geq 0$ and $\ell > 0$ we define a probability measure by

$$P_{\Lambda, \ell}^{(\alpha, z), (\beta, w)}(X) := \frac{C_{(\beta, w)}^{(\alpha, z)} \mathbb{E}(1_X(h_*) e^{-\Lambda \mu_{h_*}^\gamma(\mathbb{D})} v_h^\gamma(\partial\mathbb{D})^{2Q_\gamma/\gamma - \sum_{i \in A} 2\alpha_i/\gamma - \sum_{j \in B} \beta_j/\gamma})}{Z_{\Lambda, \ell}^{(\alpha, z), (\beta, w)}} \quad (5-8)$$

¹/₂ for $X \subseteq C_c^\infty(D)'$ measurable where

$$C_{(\beta, w)}^{(\alpha, z)} = \prod_{i \in A} R(z_i, \mathbb{D})^{-\alpha_i^2/2} \prod_{\substack{i, i' \in A \\ i < i'}} e^{\alpha_i \alpha_{i'} G(z_i, z_{i'})} \prod_{\substack{j, j' \in B \\ j < j'}} e^{\beta_j/2 \cdot \beta_{j'}/2 \cdot G(w_j, w_{j'})} \prod_{\substack{i \in A \\ j \in B}} e^{\alpha_i \beta_j/2 \cdot G(z_i, w_j)}$$

and where the denominator in (5-8) is the normalizing constant which ensures that (5-8) defines a probability measure. The measure is called the Liouville (disk) field with marked points $((z_i, \alpha_i) : i \in A)$, $((w_j, \beta_j) : j \in B)$, cosmological constant Λ and boundary length ℓ .

Remark 5.4. It follows from the proof of [Huang et al. 2018, Corollary 3.10] that if (5-7) holds, then the boundary length $\nu_h(\partial\mathbb{D})$ has finite moments of sufficiently high order so that (5-8) is well-defined as a probability measure.⁶ The cited corollary is for the case $\Lambda = 0$ but having $\Lambda > 0$ is only making the expectation smaller; however we require $\alpha_i < Q_\gamma$ such that $\mu_{h*}^\gamma(\mathbb{D}) < \infty$. In the setting of no bulk points ($A = \emptyset$) and $\beta_j = \gamma$ for all $j \in B$ we need to have $\#B \geq 3$ in order for (5-7) to be satisfied.

One natural question is whether there are relations between disks with different singularities. The proposition below resolves this question in the special case where we show how to construct a Liouville field with one α bulk singularity and one γ boundary singularity by starting with a Liouville field with no bulk singularities and three γ boundary singularities, forgetting two of the boundary marked points and sampling a bulk singularity essentially from the α -LQG measure. In Sections 6 and 8A, this proposition will be key as it allows us to explicitly compute some partition functions and in particular establish their finiteness.

Proposition 5.5. *Consider distinct and counterclockwise oriented points $w_a, w_b, w_c \in \partial\mathbb{D}$. Whenever $w_a = \tilde{w}_a, \tilde{w}_b, \tilde{w}_c$ are distinct, let $\psi_{\tilde{w}} : \mathbb{D} \rightarrow \mathbb{D}$ be the unique conformal or anticonformal transformation mapping $\tilde{w} = (\tilde{w}_a, \tilde{w}_b, \tilde{w}_c)$ to $w = (w_a, w_b, w_c)$. Then*

$$\begin{aligned} Z_{\Lambda, \ell}^{(\alpha, 0), (\gamma, 1)} \int P_{\Lambda, \ell}^{(\alpha, 0), (\gamma, 1)}(dk) \int_{\partial\mathbb{D}} v_k^\gamma(d\tilde{w}_b) \int_{\partial\mathbb{D}} v_k^\gamma(d\tilde{w}_c) f(k \circ \psi_{\tilde{w}}^{-1} + Q_\gamma \log |(\psi_{\tilde{w}}^{-1})'|, \psi_{\tilde{w}}(0)) \\ = 2c_w \ell^2 \int_{\mathbb{D}} dz R(z, \mathbb{D})^{\alpha(Q_\gamma - Q_\alpha)} Z_{\Lambda, \ell}^{(\alpha, z), ((\gamma, \gamma, \gamma), w)} \int P_{\Lambda, \ell}^{(\alpha, z), ((\gamma, \gamma, \gamma), w)}(dk) f(k, z), \end{aligned}$$

where $c_w = \exp(-(G(w_a, w_b) + G(w_b, w_c) + G(w_c, w_a))/2)$. If $\alpha < 2$ we have

$$\begin{aligned} \int_{\mathbb{D}} dz R(z, \mathbb{D})^{\alpha(Q_\gamma - Q_\alpha)} Z_{\Lambda, \ell}^{(\alpha, z), ((\gamma, \gamma, \gamma), w)} \int P_{\Lambda, \ell}^{(\alpha, z), ((\gamma, \gamma, \gamma), w)}(dk) f(k, z) \\ = Z_{\Lambda, \ell}^{(-, -), ((\gamma, \gamma, \gamma), w)} \int P_{\Lambda, \ell}^{(-, -), ((\gamma, \gamma, \gamma), w)}(dk) \int_{\mathbb{D}} \mu_k^\alpha(dz) R(z, \mathbb{D})^{\alpha(Q_\gamma - Q_\alpha)} f(k, z). \end{aligned}$$

The key in the proof of the proposition above is the following elementary change of variables result. It will allow us in the proof of the proposition to switch from an integral over an interior point to an integral over two boundary points.

⁶There is a typo in the statement of [Huang et al. 2018, Proposition 3.2] where there is an incorrect minus sign in the exponent of the Poincaré metric at an interior point; in our setting the correct definition to make things consistent with the remainder of [Huang et al. 2018] is given above where the exponent on the conformal radius is negative.

Lemma 5.6. *Let $g : \mathbb{D} \rightarrow [0, \infty]$ be measurable. Let I be the set of distinct $(u, v) \in (\partial\mathbb{D} \setminus \{1\})^2$. Fix $(w_-, w_+) \in I$ such that $(1, w_-, w_+)$ is ordered counterclockwise. Then*

$$\int_I g(\psi_{\tilde{w}_\pm}(0)) |\psi'_{\tilde{w}_\pm}(1) \psi'_{\tilde{w}_\pm}(\tilde{w}_b) \psi'_{\tilde{w}_\pm}(\tilde{w}_c)| \cdot |\psi'_{\tilde{w}_\pm}(0)|^2 d\tilde{w}_- d\tilde{w}_+ = 2e^{-(G(1, w_-) + G(1, w_+) + G(w_-, w_+))/2} \int_{\mathbb{D}} g(z) dz,$$

where $\psi_{\tilde{w}_\pm} : \mathbb{D} \rightarrow \mathbb{D}$ is the unique conformal or anticonformal transformation mapping $(1, \tilde{w}_-, \tilde{w}_+)$ to $(1, w_-, w_+)$ whenever $(\tilde{w}_-, \tilde{w}_+) \in I$.

Proof. Let $f : \mathbb{H} \rightarrow \mathbb{D}$ given by $f(u) = (u - i)/(u + i)$ be the Möbius transformation mapping (∞, i) to $(1, 0)$ with inverse $f^{-1}(z) = i(1 + z)/(1 - z)$. By performing a change of coordinates, for any measurable function $\tilde{g} : \mathbb{H} \rightarrow [0, \infty]$ and $x_- < x_+$ we have

$$\int_{\mathbb{H}} \tilde{g}(u) du = \int_{\mathbb{R}^2} \tilde{g}(\psi_{\tilde{x}_\pm}^*(i)) \frac{(x_+ - x_-)^2}{(\tilde{x}_+ - \tilde{x}_-)^3} 1(\tilde{x}_- < \tilde{x}_+) d\tilde{x}_- d\tilde{x}_+, \quad (5-9)$$

where

$$\psi_{\tilde{x}_\pm}^* = \frac{(x_+ - x_-)(\cdot) + x_- \tilde{x}_+ - x_+ \tilde{x}_-}{\tilde{x}_+ - \tilde{x}_-}.$$

Let $x_\pm = f^{-1}(w_\pm)$. If $\tilde{x}_\pm = f^{-1}(\tilde{w}_\pm)$ then $f \circ \psi_{\tilde{x}_\pm}^* = \psi_{\tilde{w}_\pm} \circ f$ and hence

$$\begin{aligned} \psi'_{\tilde{w}_\pm}(\tilde{w}_+) &= \frac{f'(x_+)}{f'(\tilde{x}_+)} \frac{x_+ - x_-}{\tilde{x}_+ - \tilde{x}_-}, & \psi'_{\tilde{w}_\pm}(\tilde{w}_-) &= \frac{f'(x_-)}{f'(\tilde{x}_-)} \frac{x_+ - x_-}{\tilde{x}_+ - \tilde{x}_-}, \\ \psi'_{\tilde{w}_\pm}(0) &= \frac{f'(\psi_{\tilde{x}_\pm}^*(i))}{f'(i)} \frac{x_+ - x_-}{\tilde{x}_+ - \tilde{x}_-}, & \psi'_{\tilde{w}_\pm}(1) &= \frac{\tilde{x}_+ - \tilde{x}_-}{x_+ - x_-}. \end{aligned} \quad (5-10)$$

We now consider $\tilde{g} = (g \circ f) \cdot |f'|^2$ and let I' be the set of $(u, v) \in I$ such that $(1, u, v)$ is counterclockwise ordered. Then by two changes of coordinates and (5-9) we get

$$\begin{aligned} \int_{\mathbb{D}} g(z) dz &= \int_{\mathbb{H}} g(f(u)) |f'(u)|^2 du \\ &= \int_{\mathbb{R}^2} g((f \circ \psi_{\tilde{x}_\pm}^*)(i)) |f'(\psi_{\tilde{x}_\pm}^*(i))|^2 \frac{(x_+ - x_-)^2}{(\tilde{x}_+ - \tilde{x}_-)^3} 1(\tilde{x}_- < \tilde{x}_+) d\tilde{x}_- d\tilde{x}_+ \\ &= \int_{I'} g((f \circ \psi_{f^{-1}(\tilde{w}_\pm)}^*)(i)) |f'(\psi_{f^{-1}(\tilde{w}_\pm)}^*(i))|^2 \frac{(x_+ - x_-)^2 |(f^{-1})'(\tilde{w}_-)| \cdot |(f^{-1})'(\tilde{w}_+)|}{(f^{-1}(\tilde{w}_+) - f^{-1}(\tilde{w}_-))^3} d\tilde{w}_- d\tilde{w}_+. \end{aligned}$$

Note that $(f \circ \psi_{f^{-1}(\tilde{w}_\pm)}^*)(i) = \psi_{\tilde{w}_\pm}(0)$ and if $\tilde{x}_\pm = f^{-1}(\tilde{w}_\pm)$ then, by (5-10),

$$\begin{aligned} &|f'(\psi_{\tilde{x}_\pm}^*(i))|^2 \frac{(x_+ - x_-)^2 |(f^{-1})'(\tilde{w}_-)| \cdot |(f^{-1})'(\tilde{w}_+)|}{(\tilde{x}_+ - \tilde{x}_-)^3} \\ &= |\psi'_{\tilde{w}_\pm}(\tilde{w}_+)| \cdot |\psi'_{\tilde{w}_\pm}(\tilde{w}_-)| \cdot |\psi'_{\tilde{w}_\pm}(1)| \cdot |\psi'_{\tilde{w}_\pm}(0)|^2 \frac{|f'(i)|}{|f'(x_-)| \cdot |f'(x_+)| (x_+ - x_-)}. \end{aligned}$$

¹/₂ Finally

$$\begin{aligned} \frac{|f'(i)|}{|f'(x_-)| \cdot |f'(x_+)| (x_+ - x_-)} &= \frac{|(f^{-1})'(w_-)| \cdot |(f^{-1})'(w_+)|}{2(f^{-1}(w_+) - f^{-1}(w_-))} = \frac{1}{|1 - w_-| \cdot |1 - w_+| \cdot |w_- - w_+|} \\ &= e^{(G(1, w_-) + G(1, w_+) + G(w_-, w_+))/2}. \end{aligned}$$

Putting everything together yields the claim. Note that the extra factor of 2 in the statement arises since on the right-hand side of (5-9) we only consider tuples $(\tilde{x}_-, \tilde{x}_+)$ that give rise to a conformal (rather than anticonformal) transformation $\psi_{\tilde{x}_\pm}^*$. $\square$

Proof of Proposition 5.5. Without loss of generality we may assume that $\Lambda = 0$ and $\ell = 1$; the general case follows by scaling and the fact that $\mu_k^\gamma(\mathbb{D})$ is a measurable function of the field k . Also, without loss of generality $w_a = 1$. In the notation of Definition 5.3 and by Girsanov's theorem we get

$$\begin{aligned} &\mathbb{E} \left(\int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} v_{h^0}^\gamma(d\tilde{w}_b) v_{h^0}^\gamma(d\tilde{w}_c) g(\tilde{w}_b, \tilde{w}_c, h^0) \right) \\ &= \int_{\partial\mathbb{D}} \int_{\partial\mathbb{D}} e^{(\gamma/2)^2 G(\tilde{w}_b, \tilde{w}_c)} \mathbb{E} \left(g \left(\tilde{w}_b, \tilde{w}_c, h + \frac{\gamma}{2} \cdot G(\cdot, \tilde{w}_b) + \frac{\gamma}{2} \cdot G(\cdot, \tilde{w}_c) \right) \right) d\tilde{w}_b d\tilde{w}_c \end{aligned}$$

for any measurable function $g : (\partial\mathbb{D})^2 \times C_c^\infty(D)' \rightarrow [0, \infty]$; this in particular yields

$$\begin{aligned} &Z_{0,1}^{(\alpha,0),(\gamma,1)} \int P_{0,1}^{(\alpha,0),(\gamma,1)}(dk) \int_{\partial\mathbb{D}} v_k^\gamma(d\tilde{w}_b) \int_{\partial\mathbb{D}} v_k^\gamma(d\tilde{w}_c) f(k \circ \psi_{\tilde{w}}^{-1} + Q_\gamma \log |(\psi_{\tilde{w}}^{-1})'|, \psi_{\tilde{w}}(0)) \\ &= \int_{(\partial\mathbb{D} \setminus \{1\})^2} d\tilde{w}_b d\tilde{w}_c Z_{0,1}^{(\alpha,0),((\gamma,\gamma,\gamma),\tilde{w})} \int P_{0,1}^{(\alpha,0),((\gamma,\gamma,\gamma),\tilde{w})}(dk) f(k \circ \psi_{\tilde{w}}^{-1} + Q_\gamma \log |(\psi_{\tilde{w}}^{-1})'|, \psi_{\tilde{w}}(0)) \\ &= \int_{(\partial\mathbb{D} \setminus \{1\})^2} d\tilde{w}_b d\tilde{w}_c \prod_i |\psi'_{\tilde{w}}(\tilde{w}_i)| \cdot |\psi'_{\tilde{w}}(0)|^{\alpha(Q_\gamma - \alpha/2)} \\ &\quad \cdot Z_{0,1}^{(\alpha, \psi_{\tilde{w}}(0)), ((\gamma,\gamma,\gamma), \mathbf{w})} \int P_{0,1}^{(\alpha, \psi_{\tilde{w}}(0)), ((\gamma,\gamma,\gamma), \mathbf{w})}(dk) f(k, \psi_{\tilde{w}}(0)), \end{aligned}$$

where the second equality follows from the change of coordinates formula [Huang et al. 2018, Theorem 3.5].

Note that

$$|\psi'_{\tilde{w}}(0)|^{\alpha(Q_\gamma - \alpha/2)} = R(\psi_{\tilde{w}}(0), \mathbb{D})^{\alpha(Q_\gamma - Q_\alpha)} |\psi'_{\tilde{w}}(0)|^2.$$

The first claim now follows from Lemma 5.6. The second claim follows from another application of Girsanov's theorem. $\square$

Definition 5.7. Let k be the Liouville field with $\#A = 0$, $\#B = 3$, $\beta_j = \gamma$ for all $j \in B$, where the singularities are located at distinct points $w_j \in \partial\mathbb{D}$ for $j \in B$ and we have boundary length ℓ and cosmological constant 0. Then the γ -LQG surface $[(\mathbb{D}, k, \mathbf{w})]$ is called the (regular) γ -LQG disk with boundary length ℓ and three marked boundary points. We get the (regular) γ -LQG disk with boundary length ℓ and zero (resp. one, two) marked points by keeping zero (resp. one, two) of the marked points.

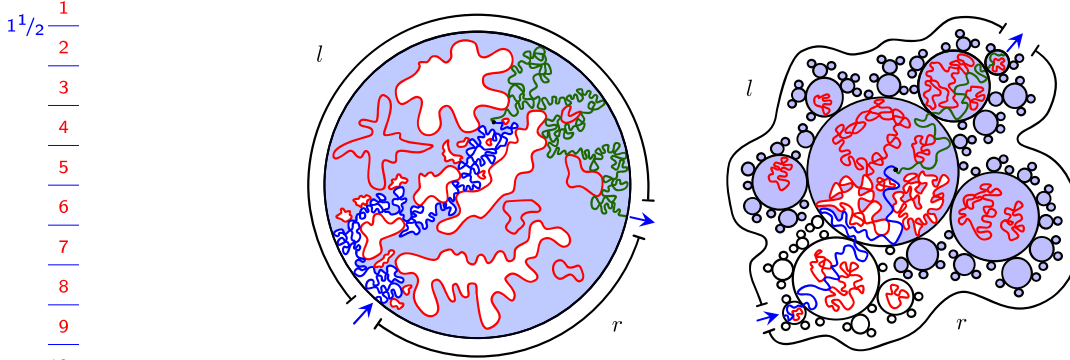


Figure 6. Left: we consider a regular γ -LQG disk decorated by a CLE_κ (red loops) and a CPI with asymmetry parameter β ; the CPI up to some time t is colored in blue and the remaining part of the CPI is colored in green. The future part of the disk at time t (which is the surface in which the exploration continues) is shown in light blue. Note that the disks colored in white are precisely the disks that touch the CPI up to time t . Right: the figure illustrates a generalized γ -LQG disk decorated by a $\text{CLE}_{\kappa'}$ and a CPI with asymmetry parameter β with the same coloring convention as in the simple case.

It follows from [Huang et al. 2018, Theorem 3.5] that the law of $[(\mathbb{D}, k, \mathbf{w})]$ does not depend on the locations of the marked points $(w_j : j \in B)$. The paper [Cercle 2021] shows that this definition agrees with the one given in [Duplantier et al. 2021, Definition 4.21].

We will now define the generalized γ -LQG disk with (generalized) boundary length $\ell > 0$. This will be a generalized γ -LQG surface in the sense of Definition 5.2. Recall the definition of spectrally positive stable Lévy excursions from Section 4.

Definition 5.8. Suppose that $\sqrt{2} < \gamma < 2$ and $\ell > 0$. Let $(E_t : t \in [0, \ell])$ be the time reversal of a spectrally positive stable Lévy excursion with exponent $4/\gamma^2$ and duration ℓ . Conditionally on E , let S_t be independent γ -LQG disks with one marked point sampled from the γ -LQG boundary measure and with boundary lengths $|\Delta E_t|$ whenever $t \in (0, \ell)$ is such that $\Delta E_t \neq 0$. Then

$$(E, \{(t, S_t) : t < \ell, \Delta E_t \neq 0\})$$

is called the generalized γ -LQG disk with boundary length ℓ .

We emphasize the time reversal of the excursion in the definition. Indeed, a positive excursion with only positive jumps is easier to construct, but the process with only negative jumps is more natural from the perspective of the looptree.

5D. Exploring CLE decorated LQG disks. We briefly review the results by Miller, Sheffield and Werner on exploring LQG surfaces decorated by independent CLE. The case where the quantum disk is decorated by simple CLE appears in [Miller et al. 2022] and the case where the generalized quantum disk is decorated by a nonsimple CLE appears in [Miller et al. 2021].

We begin by explaining the simple case. See the left-hand side of Figure 6 for an illustration of the ideas below. We consider parameters

$$\sqrt{8/3} < \gamma < 2, \quad \kappa = \gamma^2, \quad \beta \in [-1, 1] \quad \text{and} \quad \ell_L, \ell_R > 0.$$

Let $(\mathbb{D}, h)$ be any embedding of a regular γ -LQG disk $S = [(\mathbb{D}, h)]$ with boundary length $\ell = \ell_L + \ell_R$ into $\mathbb{D}$. Let $w_0 \in \partial\mathbb{D}$ be chosen uniformly according to the γ -LQG boundary measure and let $w_\infty \in \partial D$ be such that the clockwise boundary arc from w_0 to w_∞ has length ℓ_L . Then the counterclockwise arc from w_0 to w_∞ has length ℓ_R .

Conditionally on h and on (w_0, w_∞) , let (Γ, λ) be a nonnested CLE_κ in $\mathbb{D}$ coupled with a CPI from w_0 to w_∞ with asymmetry parameter β (see Section 5A). By the conformal invariance of the law of (Γ, λ) , the law of the decorated regular γ -LQG surface $[(\mathbb{D}, h, w_0, w_\infty, \lambda, \Gamma)]$ does not depend on the embedding $(\mathbb{D}, h)$ of the quantum disk. We may assume that λ is parametrized according to γ -LQG length and call its total duration ζ . For each $t < \zeta$ we write $\xi(t)$ for the union of $\lambda([0, t])$ and all the loops in Γ touching it (see (1-10)). Denote by D_t the complementary component of $\mathbb{D} \setminus \xi(t)$ containing the boundary point w_∞ .

Let L_t (resp. R_t) denote the γ -LQG length of the clockwise (resp. counterclockwise) boundary arc of D_t from λ_t to b . By convention, we set $L_t = R_t = 0$ for $t \geq \zeta$. It turns out that this process has a càdlàg version $((L_t, R_t) : t \geq 0)$ which takes values in $(0, \infty)^2 \cup \{(0, 0)\}$. Let $X = L + R$. To help the reader compare the construction here to the one in the nonsimple case presented below, let us also define the regular γ -LQG surface $S_t := [(D_t, h|_{D_t})]$.

When $\Delta X_t := X_t - X_{t-} \neq 0$ we let $\Delta D_t = (\bigcap_{t' < t} D_{t'}) \setminus D_t$ and we call the quantum surface $\Delta S_t := [(\Delta D_t, h|_{\Delta D_t})]$ the cut out quantum surface at time t . The surface ΔS_t has boundary length $|\Delta X_t|$. Note that L (resp. R) has a positive jump at time t if λ hits a clockwise (resp. counterclockwise) oriented loop η for the first time at time t , and we have $\Delta D_t = \eta^o$ and hence $\Delta S_t = [(\eta^o, h|_{\eta^o})]$ in this case. On the other hand, L (resp. R) has a negative jump at time t if λ hits the boundary of its past hull at time t on the left (resp. right) side. In the latter case, we call ΔD_t a trunk component.

Theorem 5.9 [Miller et al. 2022]. *There exists a constant $v_\kappa^\beta \in (0, \infty)$ such that*

$$(L, R) \stackrel{d}{=} ((L'_{v_\kappa^\beta t}, R'_{v_\kappa^\beta t}) : t \geq 0),$$

where (L', R') is defined in Proposition 3.1. Conditionally on (L, R) the cut out quantum surfaces which correspond to jump times $t \geq 0$ with $\Delta X_t \neq 0$ are independent regular γ -LQG disks with boundary lengths $|\Delta X_t|$.

Remark 5.10. In Section 3 the variables L and R were used to denote two independent Lévy processes. In all other parts of the paper, including this section, L and R denote boundary length processes within LQG disks.

In order to use LQG tools to understand CLE, it is useful to split a regular γ -LQG disk decorated by a CLE into the components obtained by restricting the surface to all the individual CLE loops. The following corollary explains how one can evaluate functionals depending on the restrictions to the CLE loops.

Corollary 5.11. *Let Ψ be a measurable function on the space of regular γ -LQG surfaces with values in $[0, \infty]$. Consider a regular γ -LQG disk S with boundary length $\ell > 0$ and decorated with an independent nonnested CLE_κ . Let $(S_n : n \geq 1)$ be the γ -LQG surfaces cut out by the CLE loops with some arbitrary ordering. Then*

$$\mathbb{E}_\ell \left(\sum_{n \geq 1} \Psi(S_n) \right) = \sum_{k \geq 0} f_k(\ell)$$

where

$$f_0(\ell) := \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{t \in (0, \zeta) \\ \Delta X_t > 0}} \Psi(\Delta S_t) \right) \quad \text{and} \quad f_k(\ell) := \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{t \in (0, \zeta) \\ \Delta X_t < 0}} f_{k-1}(|\Delta X_t|) \right) \quad \text{for } k \geq 1.$$

The subscript ℓ on the left-hand side of the statement indicates the total boundary length of the γ -LQG disk S and a subscript of the form (ℓ_L, ℓ_R) indicates that the CPI exploring the LQG disk is started with boundary lengths (ℓ_L, ℓ_R) .

Proof. The proof is based on an exploration procedure for iteratively discovering all the γ -LQG surfaces cut out by the CLE loops on S .

We first pick a boundary point from S uniformly according to the γ -LQG boundary measure (independently of the CLE) and consider the unique point at distance $\ell/2$ along the boundary away from it. We then sample a CPI with asymmetry parameter β within the CLE and let (L, R) , X , and ΔS_t be defined as in the discussion above. Then the γ -LQG surfaces ΔS_t in the case $\Delta X_t > 0$ correspond to the case of CLE loops intersecting the CPI being cut out.

Within each of the trunk components ΔS_t (i.e., $\Delta X_t < 0$) we now independently sample a boundary point uniformly from its γ -LQG boundary measure and subsequently also consider the unique point at distance $|\Delta X_t|/2$ (along the boundary of the trunk component) away from this point. We can now again consider a CPI with asymmetry parameter β from the first to the second of these boundary points in each of the trunk components. Again, the CLE loops intersecting the newly constructed CPI cut out a collection of γ -LQG surfaces and we also obtain a collection of trunk components for each of the CPIs.

By iterating this procedure within each of the trunk components, we obtain a collection of γ -LQG surfaces $(S'_n : n \geq 1)$. Moreover, by construction and [Theorem 5.9](#) we have

$$\mathbb{E}_\ell \left(\sum_{n \geq 1} \Psi(S'_n) \right) = \sum_{k \geq 1} f_k(\ell). \quad (5-11)$$

Clearly, $(S'_n : n \geq 1)$ forms a subcollection of $(S_n : n \geq 1)$, so to establish the corollary, it suffices to show that these collections are identical. To prove this, it is sufficient to establish the identity

$$\mathbb{E}_\ell \left(\sum_{n \geq 1} A(S'_n) \right) = \mathbb{E}_\ell \left(\sum_{n \geq 1} A(S_n) \right), \quad (5-12)$$

where $A([(D, h)]) = \mu_h^\gamma(D)$ denotes the total area of $[(D, h)]$. We will now verify this identity. The right-hand side of (5-12) equals $\mathbb{E}_\ell(A(S)) = \mathbb{E}_1(A(S)) \ell^2$ since (modulo boundary curves with zero LQG area) S can be viewed as a disjoint union of the surfaces S_n . To prove that the left-hand side of (5-12) is

$1^{1/2}$ also equal to $\mathbb{E}_1(A(S)) \ell^2$ we will use the explicit formula (5-11) and Proposition 3.12 with $\theta = 2$. Indeed, in the notation of the proposition just mentioned we have $f_0(\ell) = A_+ \mathbb{E}_1(A(S)) \ell^2$ and by induction we see that $f_k(\ell) = A_+ A_-^k \mathbb{E}_1(A(S)) \ell^2$. Therefore by Proposition 3.12 we obtain

$$\mathbb{E}_\ell \left(\sum_{n \geq 1} \Psi(S'_n) \right) = \sum_{k \geq 1} f_k(\ell) = \frac{A_+}{1 - A_-} \mathbb{E}_1(A(S)) \ell^2 = \mathbb{E}_1(A(S)) \ell^2.$$

The claim follows. $\square$

We will also need a more general version of the previous corollary in order to prove Theorem 1.7. In the following, recall the definition of $\Pi(X, B)$ just before the statement of Theorem 3.7. The next result will be used in the context where $\Psi'(S)$ is the weight of a γ -LQG surface with no marked points and $\Psi_i(S)$ is the weight of a γ -LQG disk with one marked point, indexed by $i \in A$ (see (1-5)). Applied in this context, the corollary says that the weight $f^B(\ell)$ of an LQG disk with the marked points contained in distinct outermost CLE loops can be computed using an iterative CPI exploration. The requirement that the marked points are in distinct outermost CLE loops corresponds to the fact that in the definition of $f^B(\ell)$ we only sum over $m \in \mathbb{N}_0^B$ with distinct components.

Corollary 5.12. *Let $(\Psi_i : i \in A)$ and Ψ' be measurable functions on the space of regular γ -LQG surfaces with values in $[0, \infty]$. Consider the same setup as in Corollary 5.11. We assume that*

$$\mathbb{E}_\ell(\Psi'(S)) = \mathbb{E}_\ell \left(\prod_{n \geq 1} \Psi'(S_n) \right). \quad (5-13)$$

For $B \subseteq A$ let

$$f^B(\ell) = \mathbb{E}_\ell \left(\sum_{m \in \mathbb{N}_0^B} \prod_{i \in B} \Psi_i(S_{m_i}) \prod_{n' \notin \{m_i : i \in B\}} \Psi'(S_{n'}) \right),$$

where the sum is only over tuples with distinct components. Then $f^B(\ell) = \sum_{k \geq 0} f_k^B(\ell)$ where

$$f_0^B(\ell) = \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{Q \in \Pi^+(X, B)} \prod_{\substack{t \in (0, \zeta) \\ \Delta X_t < 0 \\ \Delta Q_t \neq \emptyset}} f^{\Delta Q_t}(|\Delta X_t|) \cdot \prod_{\substack{t \in (0, \zeta) \\ \Delta Q_t = \emptyset}} \Psi'(\Delta S_t) \prod_{\substack{t \in (0, \zeta) \\ \Delta X_t > 0 \\ \Delta Q_t = \{i\}}} \Psi_i(\Delta S_t) \right),$$

$$f_{k+1}^B(\ell) = \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{t < \zeta \\ \Delta X_t < 0}} f_k^B(|\Delta X_t|) \prod_{s \in (0, \zeta) \setminus \{t\}} \Psi'(\Delta S_s) \right) \text{ for } k \geq 0,$$

where we write $\Pi^+(X, B)$ for the set of $Q \in \Pi(X, B)$ such that $\#\Delta Q_t \leq 1$ whenever $\Delta X_t > 0$ and we require that $\Delta Q_t \neq B$ whenever $\Delta X_t < 0$.

Proof. Let us condition on both the CLE and the LQG surface, and fix $m \in \mathbb{N}_0^B$ with $(m_i : i \in B)$ distinct. The idea is to define an iterative CPI exploration until we have discovered all loops η_i corresponding to the surfaces (S_{m_i}) , i.e., each such loop η_i intersects at least one of the CPIs. Suppose we are exploring the CLE decorated LQG disk with a CPI. If all the η_i are in the same cut out component (and none intersect

^{11/2}₁ the CPI), we do another CPI exploration in the one cut out component containing all the η_i . We repeat
₂ this until not all of the η_i are in the same cut out component of the CPI. This procedure terminates by the
₃ argument in the previous corollary showing that the iterative CPI exploration discovers all CLE loops.
₄ The corollary follows since f_k^B is equal to the expectation defining f^B restricted to the event that we
₅ need exactly $k + 1$ such chordal explorations in order for not all η_i to be in the same complementary
₆ component of the CPI. Indeed, the terms in the sum in the definition of $f_0^B(\ell)$ correspond precisely to
₇ the event that not all η_i are in the same complementary component in CPI exploration. The iterative
₈ definition for $f_{k+1}^B(\ell)$ precisely captures repeating the construction in one of the cut out components. $\square$
₉

₁₀ The situation is analogous in the setting of a generalized γ -LQG disk decorated by a nonsimple CLE.
₁₁ See now the right side of [Figure 6](#) for a graphical depiction of the following definitions and results. We
₁₂ fix parameters

$$\sqrt{2} < \gamma < 2, \quad \kappa = \frac{16}{\gamma^2}, \quad \beta \in [-1, 1] \quad \text{and} \quad \ell_L, \ell_R > 0.$$

₁₅ The detailed definitions are a bit more involved than in the simple case considered above and we refer to
₁₆ [\[Miller et al. 2021\]](#) for more details. Consider a generalized γ -LQG disk with boundary length $\ell = \ell_L + \ell_R$,
₁₇ that is,

$$S = (E, \{(s, [(\mathbb{D}, h_s, w_0^s)]) : s < l + r, \Delta E_s \neq 0\}),$$

₂₀ where E is the time reversal of a spectrally positive stable Lévy excursion with exponent $4/\gamma^2$ and
₂₁ duration ℓ , and conditionally on E , we have that $(\mathbb{D}, h_s)$ is an embedding of an independent regular
₂₂ γ -LQG disk of boundary lengths $|\Delta E_s|$ and w_0^s is independently picked according to the γ -LQG boundary
₂₃ measures for each $s < \ell$ with $\Delta E_s \neq 0$. As explained in [Section 5B](#) we view the LQG surfaces as “glued
₂₄ into” the loops of the looptree defined by E , and we do this so that the point w_0^s is identified with the
₂₅ point on the loop that is closest to the root of the looptree in the metric of the looptree.

₂₆ The times 0 and ℓ_L (in the parametrization of the excursion E) correspond to two points on the looptree
₂₇ associated to E (see [Section 5B](#)) and there exists a unique curve of minimal length from the first to the
₂₈ second of the points above with the property that it only traces loops of the looptree in clockwise order—
₂₉ we call this curve a clockwise geodesic. Recall that each $s < \ell$ for which $\Delta E_s \neq 0$ is associated with
₃₀ a loop in the looptree and let $I(E, \ell_L)$ denote the collection of such times corresponding to loops that
₃₁ intersect the clockwise geodesic mentioned above. For each $s \in I(E, \ell_L)$, we let $L(E, \ell_L)_s$ be the length
₃₂ of the intersection of the clockwise geodesic with the loop corresponding to the time s .

₃₃ Now, for each regular γ -LQG disk of the generalized γ -LQG disk we sample an independent $\text{CLE}_{\kappa'}$
₃₄ (on $\mathbb{D}$). After sampling i.i.d. orientations for all the $\text{CLE}_{\kappa'}$ loops (clockwise and counterclockwise with
₃₅ probability $(1 - \beta)/2$ and $(1 + \beta)/2$, respectively), we deterministically associate to each $s \in I(E, \ell_L)$ a
₃₆ CPI with asymmetry parameter β from w_0^s to w_∞^s where w_∞^s is chosen so that the clockwise boundary
₃₇ segment from w_0^s to w_∞^s has γ -LQG length $L(E, \ell_L)_s$ with respect to the field h_s ; formally, this yields
₃₈ a decorated generalized γ -LQG surface where the surfaces corresponding to the loops connecting the
₃₉ (generalized) boundary points corresponding to 0 and ℓ_L are decorated by a collection of loops together
₄₀ with a curve and all other surfaces are only decorated by a collection of loops. We parametrize the CPI

¹/₂ by the γ -LQG length of the CPI in the generalized quantum disk, i.e., it is the sum of all the γ -LQG lengths of the CPIs in the regular γ -LQG disks. We let ζ be the total γ -LQG length of the CPI.

For $t \in [0, \zeta)$, we (informally) define a generalized γ -LQG surface S_t as follows (see again [Miller et al. 2021] for details). Let us consider the CPI stopped at time t . For $s < \ell$ with $\Delta E_s \neq 0$, let A_t^s be the set of points in $\mathbb{D}$ such that the corresponding point in the $[(\mathbb{D}, h_s)]$ (which is a part of S) can be connected within S to the generalized boundary point associated to l along a path which does not intersect the CPI stopped at time t and which does not intersect the interior of any of the CLE loops intersecting the CPI stopped at time t . This is illustrated in the right-hand side of Figure 6, where the purple (resp. white) points are contained in A_t^s (resp. $\mathbb{D} \setminus A_t^s$).

It can be argued that the collection of sets $(A_t^s : s < \ell, \Delta E_s \neq 0)$ defines a generalized γ -LQG disk S_t with two marked points; the point visited by the CPI at time t and the terminal point of the CPI (the latter of which is equal to the boundary point of the original generalized disk corresponding to time ℓ_L for E). For $t < \zeta$, the clockwise (resp. counterclockwise) generalized γ -LQG length from the first to the second one of these generalized boundary points is called L_t (resp. R_t). If $t \geq \zeta$, we simply set $L_t = R_t = 0$. As in the simple case, it turns out that (L, R) has càdlàg version $((L, R) : t \geq 0)$ which takes values in $(0, \infty)^2 \cup \{(0, 0)\}$ and we also make the definition $X = L + R$.

Whenever $t \in (0, \zeta)$ is such that $\Delta X_t \neq 0$, the collection of sets $(\Delta A_t^s : s \in (0, l+r), \Delta E_s \neq 0)$ with $\Delta A_t^s = (\bigcap_{t' < t} A_{t'}^s) \setminus A_t^s$ also defines a generalized γ -LQG surface ΔS_t (called the cut out generalized quantum surfaces at time t). As in the simple case, if $\Delta X_t > 0$ then the CPI discovers a CLE loop at time t and ΔS_t is the generalized γ -LQG surface given by the interior of the CLE loop. If $\Delta X_t < 0$, this corresponds to the CPI hitting its past hull and cutting out what we call a trunk component.

Theorem 5.13 [Miller et al. 2021]. *There exists a constant $v_k^\beta \in (0, \infty)$ such that*

$$(L, R) \stackrel{d}{=} ((L'_{v_k^\beta t}, R'_{v_k^\beta t}) : t \geq 0),$$

where (L', R') is defined in Proposition 3.13. Conditionally on (L, R) the cut out (generalized) quantum surfaces which correspond to jump times $t \geq 0$ with $\Delta X_t \neq 0$ are independent generalized γ -LQG disks with boundary length $|\Delta X_t|$.

Corollary 5.14. *Let Ψ be a measurable function on the space of generalized γ -LQG surfaces with values in $[0, \infty]$. Consider a generalized γ -LQG disk S with (generalized) boundary length $\ell > 0$ with each of its (regular) components decorated by an independent nonnested $\text{CLE}_{\kappa'}$. Let $(S_n : n \geq 1)$ be the (generalized) γ -LQG surfaces cut out by the CLE loops. Then*

$$\mathbb{E}_\ell \left(\sum_{n \geq 1} \Psi(S_n) \right) = \sum_{k \geq 0} f_k(\ell),$$

where

$$f_0(\ell) := \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{t \in (0, \zeta) \\ \Delta X_t > 0}} \Psi(\Delta S_t) \right) \quad \text{and} \quad f_k(\ell) := \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{t \in (0, \zeta) \\ \Delta X_t < 0}} f_{k-1}(|\Delta X_t|) \right) \quad \text{for } k \geq 1.$$

¹_{1^{1/2}} The subscript ℓ on the left-hand side of the statement indicates the total boundary length of the γ -LQG disk S and a subscript of the form (ℓ_L, ℓ_R) indicates that the CPI exploring the LQG disk is started with boundary lengths (ℓ_L, ℓ_R) .

⁵₆ *Proof.* The proof is entirely analogous to that of [Corollary 5.11](#) except that we now use [Proposition 3.16](#) with $\theta = \gamma^2/2$ instead of [Proposition 3.12](#) with $\theta = 2$. $\square$

⁷₈ **Remark 5.15.** [Corollary 5.12](#) also has an analogous version for the theory of generalized γ -LQG disks discussed above (with only the word “simple” replaced by “generalized” and [Corollary 5.14](#) in place of [Corollary 5.11](#)).

6. Law of the CLE conformal radius

¹³₁₄ We will give a new proof of [Theorem 1.12](#) when $\kappa \neq 4$ using techniques from Liouville quantum gravity. We also present a new proof using similar ideas in the $\kappa = 4$ case. The cases of simple and nonsimple CLEs will be considered separately although the proofs are essentially identical and will rely on [\[Miller et al. 2022\]](#) and [\[Miller et al. 2021\]](#), respectively (see [Section 5D](#) for a review).

¹⁸₁₉ **Theorem 6.1.** Let $\kappa \in (8/3, 4)$ and $\alpha \in (2/\gamma + \gamma/4, 2)$ where $\gamma = \sqrt{\kappa}$. Let Γ be a nonnested CLE_κ in $\mathbb{D}$ and let η_0 be the loop in Γ that surrounds 0. Then

$$c := \mathbb{E}(R(0, \eta_0^\circ)^{\alpha(Q_\gamma - Q_\alpha)}) = \frac{\cos(4\pi/\kappa)}{\cos(4\pi/\kappa - \pi \cdot 2\alpha/\gamma)}.$$

²²₂₃ *Proof.* The key idea of the proof is the following. To a regular γ -LQG surface $[(D, k)]$ we associate a value

$$\Psi([(D, k)]) = \int_D R(z, D)^{\alpha(Q_\gamma - Q_\alpha)} \mu_k^\alpha(dz) \in [0, \infty]$$

²⁶₂₇ provided the measure μ_k^α is defined — and $\Psi([(D, k)]) = 0$ otherwise, say. It is not hard to check that Ψ is well-defined (i.e., does not depend on the representative (D, k) of its equivalence class) by using the definition of a regular γ -LQG surface ([Definition 5.1](#)) and how μ_k^α transforms under the application of a conformal map (see [\(5-6\)](#)).

³⁰₃₁ Let $(\mathbb{D}, h)$ be an embedding of a γ -LQG disk with boundary length 1 into the unit disk $\mathbb{D}$ which is independent of Γ , i.e., h and Γ are independent. Then

$$\mathbb{E}(\Psi([\mathbb{D}, h)]) = \mathbb{E}\left(\int_{\mathbb{D}} R(z, \mathbb{D})^{\alpha(Q_\gamma - Q_\alpha)} \mu_h^\alpha(dz)\right) < \infty,$$

³⁵₃₆ where the finiteness is a consequence of the definition of h as a γ -LQG disk and the second part of [Proposition 5.5](#) with $f = 1$ and $\Lambda = 0$ (note that $\alpha > \gamma/2$ so that the Liouville partition functions appearing on both sides of the second equation in the proposition are finite; see [Remark 5.4](#)). Moreover, $[(\mathbb{D}, h + 2/\gamma \cdot \log \ell)]$ is a regular γ -LQG disk with boundary length ℓ by [Definition 5.7](#) and

$$\mathbb{E}(\Psi([\mathbb{D}, h + 2/\gamma \cdot \log \ell])) = \ell^{2\alpha/\gamma} \mathbb{E}(\Psi([\mathbb{D}, h)]). \quad (6-1)$$

¹/₂ For any $z \in \mathbb{D}$, let η_z be the loop around z (if no such loop exists, which is a probability zero event for any fixed z , we take η_z to be the curve tracing $\partial\mathbb{D}$, say). Recalling that $c = \mathbb{E}(R(0, \eta_0^o)^{\alpha(Q_\alpha - Q_\gamma)}) \in (0, \infty]$ and using conformal invariance of CLE, we get

$$\mathbb{E}(R(z, \eta_z^o)^{\alpha(Q_\alpha - Q_\gamma)}) = c \cdot R(z, \mathbb{D})^{\alpha(Q_\alpha - Q_\gamma)} \quad \text{for all } z \in \mathbb{D}.$$

We now first use the previous equation, exchange integration with expectation and rewrite the expression within the expectation to obtain

$$\begin{aligned} c \cdot \mathbb{E}(\Psi([\mathbb{D}, h])) &= \mathbb{E}\left(\int_{\mathbb{D}} \mathbb{E}(R(z, \eta_z^o)^{\alpha(Q_\gamma - Q_\alpha)}) \mu_h^\alpha(dz)\right) = \mathbb{E}\left(\int_{\mathbb{D}} R(z, \eta_z^o)^{\alpha(Q_\gamma - Q_\alpha)} \mu_h^\alpha(dz)\right) \\ &= \mathbb{E}\left(\sum_{\eta \in \Gamma} \int_{\eta^o} R(z, \eta^o)^{\alpha(Q_\gamma - Q_\alpha)} \mu_h^\alpha(dz)\right) = \mathbb{E}\left(\sum_{n \geq 1} \Psi(S_n)\right), \end{aligned}$$

where (S_n) denotes the quantum surfaces cut out of $[(\mathbb{D}, h)]$ by Γ , noting that we are taking the expectation over both the LQG field and the CLE.

Let (L', R') and $X' = L' + R'$ be as in [Proposition 3.1](#) and conditionally on (L', R') let S'_t be independent regular γ -LQG surfaces with boundary length $|\Delta X'_t|$ whenever $\Delta X'_t \neq 0$. By [Corollary 5.11](#) and [Theorem 5.9](#) we have

$$\mathbb{E}\left(\sum_{n \geq 1} \Psi(S_n)\right) = \sum_{k \geq 0} f_k(1)$$

where

$$\begin{aligned} f_0(\ell) &= \mathbb{E}_{(\ell/2, \ell/2)}\left(\sum_{\substack{t \in (0, \zeta') \\ \Delta X'_t > 0}} \Psi(S'_t)\right) = \mathbb{E}(\Psi([\mathbb{D}, h])) \ell^{2\alpha/\gamma} \mathbb{E}_{(1/2, 1/2)}\left(\sum_{\substack{t \in (0, \zeta') \\ \Delta X'_t > 0}} (\Delta X'_t)^{2\alpha/\gamma}\right), \\ f_k(\ell) &= \mathbb{E}_{(\ell/2, \ell/2)}\left(\sum_{\substack{t \in (0, \zeta) \\ \Delta X'_t < 0}} f_{k-1}(|\Delta X'_t|)\right) \quad \text{for } k \geq 1, \end{aligned}$$

where we used (6-1) to get the second equality. Note that in the first of the previous two displays, we are taking an expectation over the random variable (S_n) while in the second one we are taking one over $(L', R', X', (S'_t))$. By [Proposition 3.12](#) with $\theta = \alpha\gamma/2$ we see by induction (and using the notation of the aforementioned proposition) that for all $k \geq 1$,

$$f_k(\ell) = \mathbb{E}(\Psi([\mathbb{D}, h])) \ell^{2\alpha/\gamma} A_+ A_-^k.$$

Using this, we obtain that

$$\sum_{k \geq 1} f_k(1) = \frac{\cos(4\pi/\kappa)}{\cos(4\pi/\kappa - \pi \cdot 2\alpha/\gamma)} \mathbb{E}(\Psi([\mathbb{D}, h]))$$

and the claim follows. $\square$

Theorem 6.2. Let $\kappa' \in (4, 8)$ and $\alpha \in (1/\gamma + \gamma/2, 2)$ with $\gamma = 4/\sqrt{\kappa'}$. Let Γ be a nonnested $\text{CLE}_{\kappa'}$ in $\mathbb{D}$ and let η_0 be the loop in Γ that surrounds 0. Then

$$c := \mathbb{E}(R(0, \eta_0^{\alpha(Q_\gamma - Q_\alpha)}) = \frac{\cos(4\pi/\kappa')}{\cos(4\pi/\kappa' - \pi \cdot \alpha\gamma/2)}.$$

Proof. The proof is very similar to the proof of [Theorem 6.1](#). We define the following functional on the space of generalized γ -LQG surfaces:

$$\Psi(e, \{(t, [(D_t, k_t)]) : t \geq 0, \Delta E_t \neq 0\}) = \sum_{\substack{t \geq 0 \\ \Delta E_t \neq 0}} \int_{D_t} R(z, D_t)^{\alpha(Q_\gamma - Q_\alpha)} \mu_{k_t}^\alpha(dz) \in [0, \infty]$$

provided the measure $\mu_{k_t}^\alpha$ is defined for all t and define the functional to be 0 otherwise. Again, the integrals do not depend on the particular equivalence class of the γ -LQG surfaces $[(D_t, k_t)]$. Let $(E_t : t \in [0, 1])$ be the time reversal of a spectrally positive stable Lévy excursion with exponent $4/\gamma^2$ and duration 1 and conditionally on E , consider independent regular γ -LQG quantum disks $[(\mathbb{D}, h_t)]$ for any $t \in [0, 1]$ with $\Delta E_t \neq 0$. Then by stable scaling

$$S_\ell = (\ell^{\gamma^2/4} E_{\cdot/\ell}, \{(t, [(\mathbb{D}, h_t + 2/\gamma \cdot \log(\ell^{\gamma^2/4} |\Delta E_{t/\ell}|)]) : t < \ell, \Delta E_{t/\ell} \neq 0\})$$

is a generalized γ -LQG disk with boundary length ℓ . By definition, we therefore obtain

$$\begin{aligned} \mathbb{E}(\Psi(S_\ell)) &= \mathbb{E}\left(\sum_{\substack{t < \ell \\ \Delta E_{t/\ell} \neq 0}} (\ell^{\gamma^2/4} |\Delta E_{t/\ell}|)^{2\alpha/\gamma}\right) \mathbb{E}\left(\int_{\mathbb{D}} R(z, \mathbb{D})^{\alpha(Q_\gamma - Q_\alpha)} \mu_h^\alpha(dz)\right) \\ &= \ell^{\alpha\gamma/2} \mathbb{E}\left(\sum_{\substack{t < 1 \\ \Delta E_t \neq 0}} |\Delta E_t|^{2\alpha/\gamma}\right) \mathbb{E}\left(\int_{\mathbb{D}} R(z, \mathbb{D})^{\alpha(Q_\gamma - Q_\alpha)} \mu_h^\alpha(dz)\right) \end{aligned}$$

noting that the two expectations on the right-hand side are finite—for the first expectation, this is a consequence of [Theorem 4.1](#) and for the second expectation this follows from [Proposition 5.5](#) with $f = 1$. If S is a generalized γ -LQG disk with boundary length 1 then we can decorate this surface with an independent $\text{CLE}_{\kappa'}$ on each of its regular γ -LQG disks components. By a similar argument as in the proof of [Theorem 6.1](#) we get

$$c \cdot \mathbb{E}(\Psi(S)) = \mathbb{E}\left(\sum_{n \geq 1} \Psi(S_n)\right),$$

where (S_n) denotes the collection of generalized γ -LQG surfaces cut out by the $\text{CLE}_{\kappa'}$ decoration. The conclusion now follows similarly as in the proof of [Theorem 6.1](#), now using [Corollary 5.14](#), [Theorem 5.13](#) and [Proposition 3.16](#). $\square$

The $\kappa = 4$ case is actually the one which is understood best due to the level-line coupling with the Gaussian free field by Miller and Sheffield (see, e.g., [\[Aru et al. 2019\]](#) for a self-contained presentation).

For the sake of completeness, we also give a proof of the computation of the law of the conformal radius in a similar spirit as in the $\kappa \neq 4$ case. Note however that the CLE decoration is not independent of

the field but instead determined by it. While the proof is interesting, let us remark that the law of the conformal radius can also be read off more directly in the proof of the level-line coupling (see again [Aru et al. 2019]) or one could attempt to take the limit $\kappa \rightarrow 4$ in the theorems above.

Proposition 6.3. *Consider $\Gamma \sim \text{CLE}_4$ (nonnested) in $\mathbb{D}$ and let η_0 be the loop in Γ that surrounds 0. Then, for any $\gamma \in (0, 2)$,*

$$\mathbb{E}(R(0, \eta_0^o)^{\gamma^2/2}) = \frac{1}{\cosh(\pi\gamma)}.$$

Proof. Let us recall the level-line coupling of a CLE_4 with a Dirichlet GFF due to Miller and Sheffield: Let h be a Dirichlet GFF in $\mathbb{D}$ as introduced in Section 5B, let us write $\Gamma = \{\tilde{\eta}_i : i \geq 1\}$ and conditionally on Γ , let $(h_i : i \geq 1)$ be i.i.d. Dirichlet GFFs in $\tilde{\eta}_i^o$ and $(\sigma_i : i \geq 1)$ be independent i.i.d. and uniform on $\{\pm 1\}$; then

$$h \stackrel{d}{=} \sum_{i \geq 1} (\pi \sigma_i + h_i) 1_{\tilde{\eta}_i^o}.$$

It follows that, for $\gamma \in (0, 2)$,

$$\mathbb{E}\left(\int_{\mathbb{D}} \mu_h^\gamma(dz)\right) = \mathbb{E}\left(\sum_{i \geq 1} \int_{\tilde{\eta}_i^o} \mu_{\pi \sigma_i + h_i}^\gamma(dz)\right) = \mathbb{E}\left(\sum_{i \geq 1} \mathbb{E}\left(\int_{\tilde{\eta}_i^o} \mu_{\pi \sigma_i + h_i}^\gamma(dz) \mid \Gamma\right)\right). \quad (6-2)$$

By (5-2) we get

$$\begin{aligned} \mathbb{E}\left(\int_{\mathbb{D}} \mu_h^\gamma(dz)\right) &= \int_{\mathbb{D}} R(z, \mathbb{D})^{\gamma^2/2} dz, \\ \mathbb{E}\left(\int_{\tilde{\eta}_i^o} \mu_{\pi \sigma_i + h_i}^\gamma(dz) \mid \Gamma\right) &= \cosh(\gamma\pi) \int_{\tilde{\eta}_i^o} R(z, \tilde{\eta}_i^o)^{\gamma^2/2} dz \quad \text{a.s.}, \end{aligned}$$

where we use for the second identity that $\mathbb{E}(e^{\gamma\pi\sigma_i}) = \cosh(\gamma\pi)$. For $z \in \mathbb{D}$, let η_z be the a.s. unique loop in Γ surrounding the point z taking η_z to be the curve tracing $\partial\mathbb{D}$ if no such loop exists. Then $\mathbb{E}(R(z, \eta_z^o)^{\gamma^2/2}) = R(z, \mathbb{D})^{\gamma^2/2} \mathbb{E}(R(0, \eta_0)^{\gamma^2/2})$ for $z \in \mathbb{D}$ by conformal invariance of Γ and thus, by (6-2),

$$\begin{aligned} \cosh(\gamma\pi)^{-1} \int_{\mathbb{D}} R(z, \mathbb{D})^{\gamma^2/2} dz &= \mathbb{E}\left(\sum_{i \geq 1} \int_{\tilde{\eta}_i^o} R(z, \tilde{\eta}_i^o)^{\gamma^2/2} dz\right) = \mathbb{E}\left(\int_{\mathbb{D}} R(z, \eta_z^o)^{\gamma^2/2} dz\right) \\ &= \int_{\mathbb{D}} \mathbb{E}(R(z, \eta_z^o)^{\gamma^2/2}) dz = \mathbb{E}(R(0, \eta_0^o)^{\gamma^2/2}) \int_{\mathbb{D}} R(z, \mathbb{D})^{\gamma^2/2} dz. \end{aligned}$$

The claim follows. $\square$

Combining the results above we conclude the proof of Theorem 1.12.

Proof of Theorem 1.12. Let us first consider $\kappa \in (8/3, 4)$ and $\gamma = \sqrt{\kappa}$. If $\alpha \in (2/\gamma + \gamma/4, 2)$, let $\rho = \alpha(Q_\gamma - Q_\alpha) \in (-1 + 2/\kappa + 3\kappa/32, \sqrt{\kappa} + 4/\sqrt{\kappa} - 4)$. Since

$$\sqrt{(1 - 4/\kappa)^2 - 8\rho/\kappa} = 4/\kappa - 2\alpha/\kappa + 1$$

$1^{1/2}$ $\frac{1}{2}$ by [Theorem 6.1](#) we get

$$\mathbb{E}(R(0, \eta_0^\rho)^\rho) = \mathbb{E}(R(0, \eta_0^\rho)^{\alpha(Q_\gamma - Q_\alpha)}) = \frac{\cos(4\pi/\kappa)}{\cos(4\pi/\kappa - \pi \cdot 2\alpha/\gamma)} = \frac{-\cos(4\pi/\kappa)}{\cos(\pi \sqrt{(1 - 4/\kappa)^2 - 8\rho/\kappa})},$$

$\frac{5}{6}$ when $\rho \in (-1 + 2/\kappa + 3\kappa/32, \sqrt{\kappa} + 4/\sqrt{\kappa} - 4)$. In particular, the two functions

$$\rho \mapsto \mathbb{E}(R(o, \eta_0^\rho)^\rho) \quad \text{and} \quad \rho \mapsto \frac{-\cos(4\pi/\kappa)}{\cos(\pi \sqrt{(1 - 4/\kappa)^2 - 8\rho/\kappa})}$$

$\frac{9}{10}$ are both analytic functions on $\{\rho \in \mathbb{C} : \Re \rho > -1 + 2/\kappa + 3\kappa/32\}$ (for the second function this statement is obvious) and agree on a set containing an accumulation point. The two functions therefore are identical on $\{\rho \in \mathbb{C} : \Re \rho > -1 + 2/\kappa + 3\kappa/32\}$ as required.

$\frac{12}{13}$ In the $\kappa \in (4, 8)$ case we let $\gamma = 4/\sqrt{\kappa}$, consider $\alpha \in (1/\gamma + \gamma/2, 2)$ and define $\rho = \alpha(Q_\gamma - Q_\alpha) \in (-1 + 2/\kappa + 3\kappa/32, \sqrt{\kappa} + 4/\sqrt{\kappa} - 4)$. Then we have

$$\sqrt{(1 - 4/\kappa)^2 - 8\rho/\kappa} = 4/\kappa - \alpha\gamma/2 + 1$$

$\frac{16}{17}$ and the result follows for $\rho \in (-1 + 2/\kappa + 3\kappa/32, \sqrt{\kappa} + 4/\sqrt{\kappa} - 4)$ by [Theorem 6.2](#) and the same reasoning as in the $\kappa < 4$ case. We again get the full range of ρ values by analytic continuation.

$\frac{18}{19}$ Finally, we consider the $\kappa = 4$ case. [Proposition 6.3](#) yields the claim for $\rho \in (0, 2)$. The moment generating function of the random variable $-\log R(0, \eta_0^\rho)$ characterizes its law and we see that the moment generating function of $-\log R(0, \eta_0^\rho)$ agrees with that of the random variable $\tau_{\pm 1} = \inf\{t \geq 0 : |B_t| = \pi\}$ where B is a standard Brownian motion. The moment generating function of $\tau_{\pm 1}$ is explicit on the entire range where it is finite. $\square$

7. CLE weighted by nesting statistics

$\frac{25}{26}$ We prove [Theorem 1.2](#), which justifies that the CLE weighting considered in the introduction is natural, since, given a tuple of points, it corresponds in some sense to reweighting the law of a CLE coupled to an LQG surface by the number of CLE loops surrounding any subset of these points. See [Section 1C](#) for a description of this weighting in the case of a regular LQG disk. A key input is [Theorem 7.1](#), which is a spatial independence property for CLE that is necessary in order to establish finiteness of the (claimed) limit in [Theorem 1.2](#). [Theorem 7.1](#) will be proved in [Section 7C](#) for $\kappa \in (8/3, 4]$ and in [Section 7D](#) for $\kappa \in (4, 8)$.

$\frac{33}{34}$ **7A. Weighting CLE by nesting statistics: Proof of [Theorem 1.2](#).** As discussed in the introduction, we want to mimic in the continuum the definition of the $O(n)$ model weighted by the number of loops surrounding points. Due to the fact that the number of loops in a nested CLE surrounding a single point is almost surely infinite, a renormalization is needed to weight a CLE by the exponential of the number of loops surrounding points. Making sense of this rigorously will be the main achievement of this section.

$\frac{38}{39}$ **Theorem 7.1.** *Let $\delta > 0$ and let $z_1, \dots, z_n \in \mathbb{D}$ be such that*

$$|z_i - z_j| > \delta \quad \text{and} \quad |z_i| < 1 - \delta \quad \text{for all } i, j \leq n, i \neq j.$$

¹_{1/2} Let Γ be a nested CLE_κ in $\mathbb{D}$ for $\kappa \in (8/3, 8)$ and define η_i to be the outermost loop surrounding z_i which does not surround z_j for all $j \neq i$. Suppose $\rho_i > -1 + 2/\kappa + 3\kappa/32$ for all $i = 1, \dots, n$. Then there is a constant $C > 0$ depending only on $\delta, \rho_1, \dots, \rho_n$ such that

$$\mathbb{E} \left(\prod_{i=1}^n R(z_i, \eta_i^o)^{\rho_i} \right) \leq C.$$

The proof of the theorem is deferred to Sections 7C and 7D in the simple and nonsimple case, respectively. The proof of this result will make strong use of a quasi-independence property of the CLE: the conformal radii of the outermost loops only surrounding one particular point do not correlate strongly with each other; indeed the proofs in Sections 7C and 7D will exploit two different constructions of CLE from which this quasi-independence can be deduced. Before we can establish Theorem 1.2 we will also need two lemmas on CLE. Related results also appear in [Miller et al. 2015], however, both the finiteness and convergence of exponential moments of nesting statistics presented here is new.

Lemma 7.2. *Let Γ be a nested CLE_κ in $\mathbb{D}$ where $\kappa \in (8/3, 8)$. Whenever $\epsilon > 0$ we write N_ϵ for the number of loops $\eta \in \Gamma$ surrounding 0 and with $R(0, \eta^o) > \epsilon$. Then*

$$\mathbb{E}(e^{\sigma N_\epsilon}) < \infty \quad \text{for all } \epsilon > 0, \sigma \in \mathbb{R} \quad \text{and} \quad \epsilon^{\rho_\sigma^\kappa} \mathbb{E}(e^{\sigma N_\epsilon}) \rightarrow c_\sigma^\kappa \quad \text{as } \epsilon \downarrow 0$$

for some explicit constant $c_\sigma^\kappa \in (0, \infty)$.

²⁰_{1/2} *Proof.* Let $(\eta_i : i \geq 0)$ be the loops in Γ surrounding 0 ordered according to their nesting structure (i.e., η_i surrounds η_{i+1} for $i \geq 0$) with the convention that η_0 parametrizes $\partial\mathbb{D}$. Then by the spatial Markov property of CLE_κ

$$X_i = -\log R(0, \eta_i^o) + \log R(0, \eta_{i-1}^o), \quad i \geq 1,$$

are i.i.d. and their density f and Laplace transform $\mathcal{L}f$ are given in Theorem 1.12. Also note that $N_\epsilon = \sup\{n \geq 0 : X_1 + \dots + X_n \leq \log(1/\epsilon)\}$ for $\epsilon > 0$. Also by Definition 1.1 we have $\sigma = -\log \mathcal{L}f(\rho_\sigma^\kappa)$. The claim now follows from Proposition A.1 where the conditions can be verified using Theorem 1.12. $\square$

Lemma 7.3. *Let $z \in \mathbb{D} \setminus \{0\}$ and let Γ be a nested CLE_κ in $\mathbb{D}$ with $\kappa \in (8/3, 8)$. We also let N_{0z} be the number of loops in Γ surrounding both 0 and z . Then for all $\delta \in (0, 1)$ and $\theta \geq 0$ there is a constant $C > 0$ such that $\mathbb{E}(\exp(\theta N_{0z})) \leq C$ for $|z| \geq \delta$.*

Proof. Let η_i be the $2i$ -th loop surrounding 0 (counted from the boundary of $\mathbb{D}$) with the convention that η_0 parametrizes $\partial\mathbb{D}$ and let us write A_w for the event that η_1 surrounds w . Then $\mathbb{P}(A_w) \rightarrow 0$ as $|w| \rightarrow 1$; indeed, if this was not the case then there would exist (w_n) with $|w_n| \rightarrow 1$ and $\limsup_{n \rightarrow \infty} \mathbb{P}(A_{w_n}) > 0$ and hence $\mathbb{P}(A_{w_n} \text{ i.o.}) > 0$ which is impossible since $\eta_1 \subseteq \mathbb{D}$. Let N'_{0z} be the largest i such that η_i surrounds z so that in particular

$$N'_{0z} \leq 2N_{0z} + 1. \tag{7-1}$$

³⁹_{1/2} Let $\phi_i : \mathbb{D} \rightarrow C_i$ be conformal transformations with $\phi_i(0) = 0$ and $\phi_i'(0) > 0$ where C_i is the connected component of η_i^o containing 0. Consider $\psi > \theta$, let $\epsilon > 0$ be such that $\mathbb{P}(A_w) \leq e^{-\psi}$ for all $|w| \geq 1 - \epsilon$,

$1^{1/2}$ and define $\tau_z = \inf\{i \geq 0 : |\phi_i^{-1}(z)| \geq 1 - \epsilon\}$ where by convention $\phi_i^{-1}(z) = 1$ if $z \notin C_i$. By the Markov property of nested CLE_κ ,

$$\mathbb{E}(e^{\theta N'_{0z}}) = \mathbb{E}(e^{\theta \tau_z} \mathbb{E}(e^{\theta N'_{0w}}) |_{w=\phi_{\tau_z}^{-1}(z)}) \leq \mathbb{E}(e^{\theta \tau_z}) \sup_{|w| \geq 1-\epsilon} \mathbb{E}(e^{\theta N'_{0w}}).$$

Let us treat the two factors separately. First of all, by the growth theorem for univalent functions [Lawler 2005, Theorem 3.21] applied to $\varphi_i = \phi_i / \phi'_i(0)$ at the value $\phi^{-1}(z)$ whenever $z \in C_i$,

$$|\varphi_i(\phi_i^{-1}(z))| \leq \frac{|\phi_i^{-1}(z)|}{(1 - |\phi_i^{-1}(z)|)^2} \leq \frac{1}{(1 - |\phi_i^{-1}(z)|)^2}.$$

Since $|\varphi_i(\phi_i^{-1}(z))| = |z|/R(0, \eta_i^\circ)$, we obtain that

$$|\phi_i^{-1}(z)| \geq 1 - \sqrt{\frac{R(0, \eta_i^\circ)}{|z|}} \quad \text{whenever } z \in C_i.$$

Let M_z be the number of $i \geq 1$ such that $R(0, \eta_i^\circ) \geq |z|\epsilon^2$. Then by the above estimate, $\tau_z \leq M_z + 1$ and therefore by Lemma 7.2, $\mathbb{E}(e^{\theta \tau_z}) \leq e^\theta \mathbb{E}(e^{\theta M_z}) \leq C$ for some constant $C < \infty$ only depending on δ . Let us now consider the second factor. For $n \geq 0$ and $|w| \geq 1 - \epsilon$, by the Markov property for nested CLE_κ ,

$$\mathbb{P}(N'_{0w} \geq n + 1) = \mathbb{E}(\mathbb{P}(A_v) |_{v=\phi_n^{-1}(w)}; N'_{0w} \geq n, z \in C_n).$$

By the Schwarz lemma, $|\phi_n^{-1}(w)| \geq |w|$ and thus $\mathbb{P}(N'_{0w} \geq n + 1) \leq e^{-\psi} \mathbb{P}(N_{0w} \geq n)$ for $|w| \geq 1 - \epsilon$. Therefore $\mathbb{P}(N'_{0w} \geq n) \leq \exp(-\psi n)$ for all $n \geq 0$ and $|w| \geq 1 - \epsilon$. The claim follows by combining everything and using (7-1). $\square$

We can now combine the above to deduce the main result.

Proof of Theorem 1.2. Without loss of generality $A = [n] = \{1, \dots, n\}$. Let C_i be the connected component of η_i° containing z_i for $i \leq n$ (recall that η_i denotes the outermost loop in Γ surrounding z_i but not z_j for $j \neq i$). We explore the nested CLE Γ from the outside until we discover the loops $(\eta_1, \dots, \eta_n)$. By the strong Markov property of the CLE and since $R(z_i, \eta_i^\circ) = R(z_i, C_i)$, we get

$$\prod_{i=1}^n \epsilon^{\rho_{\sigma_i}^\kappa} \mathbb{E}(e^{\sum_{B \subseteq [n]: \#B \geq 1} \sigma_B N_B^\epsilon(z)}) = \mathbb{E}\left(\prod_{i=1}^n e^{\sigma_i 1(R(z_i, C_i) \geq \epsilon)} \epsilon^{\rho_{\sigma_i}^\kappa} \mathbb{E}(e^{\sigma_i N^{\epsilon_i}}) |_{\epsilon_i = \epsilon/R(z_i, C_i)} \cdot e^{\sum_{B \subseteq [n]: \#B > 1} \sigma_B N_B^\epsilon(z)}\right),$$

where N^ϵ is the number of loops η in a CLE_κ $\tilde{\Gamma}$ in $\mathbb{D}$ surrounding 0 with $R(0, \eta^\circ) \geq \epsilon$. From Lemma 7.2 it follows that

$$\prod_{i=1}^n e^{\sigma_i 1(R(z_i, \eta_i^\circ) \geq \epsilon)} \epsilon^{\rho_{\sigma_i}^\kappa} \mathbb{E}(e^{\sigma_i N^{\epsilon_i}}) |_{\epsilon_i = \epsilon/R(z_i, \eta_i^\circ)} \cdot e^{\sum_{B \subseteq [n]: \#B > 1} \sigma_B N_B^\epsilon(z)}$$

$$\rightarrow \prod_{i=1}^n e^{\sigma_i c_{\sigma_i}^\kappa R(z_i, \eta_i^\circ)^{\rho_{\sigma_i}^\kappa}} \cdot e^{\sum_{B \subseteq [n]: \#B > 1} \sigma_B N_B^0(z)} \quad \text{a.s. as } \epsilon \downarrow 0. \quad (7-2)$$

¹/₂ Also by [Lemma 7.2](#), there exists a constant $c > 1$ such that, for all $\epsilon \leq 1$,

$$\begin{aligned} \epsilon^{\rho_{\sigma_i}^{\kappa}} \mathbb{E}(e^{\sigma_i N^{\epsilon_i}}) |_{\epsilon_i = \epsilon / R(z_i, \eta_i^o)} &\leq c R(z_i, \eta_i^o)^{\rho_{\sigma_i}^{\kappa}} 1(\epsilon \leq R(z_i, \eta_i^o)) + \epsilon^{\rho_{\sigma_i}^{\kappa}} 1(\epsilon > R(z_i, \eta_i^o)) \\ &\leq 2c R(z_i, \eta_i^o)^{\rho_{\sigma_i}^{\kappa} \wedge 0} \quad \text{for all } i \leq n. \end{aligned}$$

In the first line of the above display, we split into the cases $\epsilon \leq R(z_i, \eta_i^o)$ (where [Lemma 7.2](#) is applicable) and $\epsilon > R(z_i, \eta_i^o)$ where the expectation on the left-hand side equals 1. Thus we can deduce the first part of the theorem using dominated convergence for the sequence in (7-2) and using the dominating function

$$\prod_{i=1}^n 2c e^{\sigma_i \vee 0} R(z_i, \eta_i^o)^{\rho_i \wedge 0} \cdot \prod_{\substack{B \subseteq [n] \\ \#B > 1}} e^{\sigma_B N_B^{\epsilon}(z)}.$$

The fact that this expression has finite expectation follows from [Theorem 7.1](#) and [Lemma 7.3](#) after using Hölder's inequality with exponents p and $(q_B : B \subseteq [n], \#B > 1)$, taking the exponent p sufficiently close to 1 so that $(\rho_i \wedge 0)p > -1 + 2/\kappa + 3\kappa/32$ for all $i \leq n$ (noting that $-1 + 2/\kappa + 3\kappa/32 < 0$).

For the universal upper bound at the end of the statement of the theorem we use Hölder's inequality with the above exponents to get

$$\mathbb{E}\left(\prod_{i \in [n]} R(z_i, \eta_i^o)^{\rho_{\sigma_i}^{\kappa}} \exp\left(\sum_{\substack{B \subseteq [n] \\ \#B > 1}} \sigma_B N_B^0(z)\right)\right) \leq \mathbb{E}\left(\prod_{i \in [n]} e^{\sigma_i} R(z_i, \eta_i^o)^{\rho_{\sigma_i}^{\kappa}}\right)^{1/p} \prod_{\substack{B \subseteq [n] \\ \#B > 1}} \mathbb{E}(e^{q_B \sigma_B N_B^0(z)})^{1/q_B}.$$

The result now follows from [Theorem 7.1](#) and [Lemma 7.3](#).

It remains to establish the universal lower bound. Let $\tilde{\eta}_{\epsilon}$ be the outermost loop in $\tilde{\Gamma}$ surrounding 0 and contained in $\epsilon \mathbb{D}$. For each i , let $\phi_i : \mathbb{D} \rightarrow D$ be a conformal map with $\phi_i(0) = z_i$. Then by compactness of K and since $\delta > 0$ there exists $\epsilon_0 = \epsilon_0(D, K, \delta) \in (0, 1)$ such that $\phi_i(\epsilon_0 \mathbb{D}) \cap \{z_1, \dots, z_n\} = \{z_i\}$ for all i . Thus by conformal invariance of CLE, for any $c > 0$,

$$\mathbb{P}(R(z_i, \eta_i^o) > c) \geq \mathbb{P}(|\phi_i'(0)| R(0, \tilde{\eta}_{\epsilon_0}) > c).$$

Moreover, $|\phi'(0)| \geq \delta_0$ for some $\delta_0 = \delta_0(D, K, \delta)$ (again, this follows from the compactness of K and the fact that $\delta > 0$). Hence, by a union bound,

$$\mathbb{P}(R(z_i, \eta_i^o) > c \forall i) \geq 1 - n \cdot \mathbb{P}\left(R(0, \tilde{\eta}_{\epsilon_0}) \leq \frac{c}{\delta_0}\right) \geq \frac{1}{2}$$

for $c = c(D, K, \delta, n) > 0$ sufficiently small and by decreasing it further, we can also ensure that $R(z_i, D) < 1/c$ for all i . By Hölder's inequality,

$$\begin{aligned} \frac{1}{2} \prod_{i \in [n]} c^{\rho_i/2} &\leq \mathbb{E}\left(\prod_{i \in [n]} R(z_i, \eta_i^o)^{\rho_{\sigma_i}^{\kappa}/2}\right) \\ &\leq \mathbb{E}\left(\prod_{i \in [n]} R(z_i, \eta_i^o)^{\rho_{\sigma_i}^{\kappa}} \prod_{\substack{B \subseteq [n] \\ \#B > 1}} e^{\sigma_B N_B^0(z)}\right)^{1/2} \mathbb{E}\left(\prod_{\substack{B \subseteq [n] \\ \#B > 1}} e^{-\sigma_B N_B^0(z)}\right)^{1/2}. \end{aligned}$$

¹₁¹₂ The claim now follows since the second term on the right-hand side can be bounded from above by a constant that only depends on D, K, δ, n and σ . $\square$

³₄ The next lemma will not be used until [Section 7D](#). However, since the flavor of the proof is similar to other proofs given in this section, we state and prove it here.

⁶**Lemma 7.4.** *Let Γ be a CLE_κ in $\mathbb{D}$ with $\kappa \in (8/3, 8)$ and let $r \in (0, 1)$. If $\tilde{\eta}_r$ is the outermost loop surrounding 0 that is contained in $r\mathbb{D}$, then $\mathbb{E}(R(0, \tilde{\eta}_r^\circ)^\rho) < \infty$ if and only if $\rho > -1 + 2/\kappa + 3\kappa/32$. If $\tilde{\eta}_r$ is the outermost loop surrounding 0 that is surrounded by $\tilde{\eta}_r$ then we have $\mathbb{E}(R(0, \tilde{\eta}_r^\circ)^\rho) < \infty$.*

⁹*Proof.* The second assertion of the lemma is an immediate consequence of the first assertion by the Markov property of CLE_κ and [Theorem 1.12](#). When $\rho \leq -1 + 2/\kappa + 3\kappa/32$ or $\rho \geq 0$, the claim also follows directly from [Theorem 1.12](#), so suppose that $\rho \in (-1 + 2/\kappa + 3\kappa/32, 0)$. We first prove the result for a single $r \in (0, 1)$ (sufficiently close to 1) and will at the end deduce the general case. Let η_i be the $2i$ -th loop surrounding 0 (counted from the boundary of $\mathbb{D}$) with the convention that η_0 parametrizes $\partial\mathbb{D}$. Since $\eta_1 \subseteq \mathbb{D}$ a.s. (i.e., the loop η_1 a.s. does not touch the boundary of the domain),

$$p_r := \mathbb{P}(\eta_1 \subseteq r\mathbb{D}) \rightarrow 1 \quad \text{as } r \uparrow 1 \text{ for } n \geq 0. \quad (7-3)$$

¹⁷Let C_i be the connected component of η_i° containing 0 and let $\phi_i : \mathbb{D} \rightarrow C_i$ be the unique conformal transformation with $\phi_i(0) = 0$ and $\phi_i'(0) > 0$. For each $n \in \mathbb{N}$ define an event A_n as

$$A_n = \{\phi_i(\eta_{i+1}) \not\subseteq r\mathbb{D} \text{ for } i = 0, \dots, n-1 \text{ and } \phi_n(\eta_{n+1}) \subseteq r\mathbb{D}\}.$$

²⁰¹₂ By the spatial Markov property of CLE_κ , $\mathbb{P}(A_n) = (1 - p_r)^n p_r$. In particular, almost surely A_n occurs for some $n \geq 0$ when $p_r > 0$. Moreover, by the Schwarz lemma, on the event A_n we have the inclusion $\tilde{\eta}_r^\circ \supseteq \eta_{n+1}^\circ$. Hence, when $1/p + 1/q = 1$,

$$\begin{aligned} \mathbb{E}(R(0, \tilde{\eta}_r^\circ)^\rho) &\leq \sum_{n \geq 0} \mathbb{E}(R(0, \eta_{n+1}^\circ)^\rho; A_n) \leq \sum_{n \geq 0} \mathbb{E}(R(0, \eta_{n+1})^{\rho p})^{1/p} \mathbb{P}(A_n)^{1/q} \\ &= \sum_{n \geq 0} \mathbb{E}(R(0, \eta_1)^{\rho p})^{(n+1)/p} (1 - p_r)^{n/q} p_r^{1/q}. \end{aligned}$$

²⁹By [Theorem 1.12](#), we can take $p > 1$ sufficiently close to 1 such that $\mathbb{E}(R(0, \eta_1)^{\rho p}) < \infty$ and then by [\(7-3\)](#) we take $r \in (0, 1)$ sufficiently close to 1 such that

$$\mathbb{E}(R(0, \eta_1)^{\rho p})^{1/p} (1 - p_r)^{1/q} < 1.$$

³³Thus $\mathbb{E}(R(0, \tilde{\eta}_r^\circ)^\rho) < \infty$ for r sufficiently close to 1. We now prove that $\mathbb{E}(R(0, \tilde{\eta}_{r^k}^\circ)^\rho) \leq \mathbb{E}(R(0, \tilde{\eta}_r^\circ)^\rho)^k$ for $k \geq 1$ by induction from which the general statement directly follows. The $k = 1$ case is trivial. For the induction step, let C_r be the connected component of $\tilde{\eta}_r$ containing 0 and let $\psi_r : \mathbb{D} \rightarrow C_r$ be the unique conformal transformation with $\psi_r(0) = 0$ and $\psi_r'(0) > 0$. By the Schwarz lemma $|\psi(z)| \leq r|z|$ for $z \in \mathbb{D}$ so that $\psi_r(r^{k-1}\mathbb{D}) \subseteq r^k\mathbb{D}$ and hence, by the spatial Markov property of CLE_κ ,

$$\mathbb{E}(R(0, \tilde{\eta}_{r^k}^\circ)^\rho) \leq \mathbb{E}(R(0, \tilde{\eta}_{r^{k-1}}^\circ)^\rho) \mathbb{E}(R(0, \tilde{\eta}_r^\circ)^\rho)$$

³⁹¹₂ since $\rho \leq 0$. The claim follows. $\square$

7B. Properties of the nesting statistic. We list some key properties that the function $\Phi_D^{\sigma, \kappa}$ introduced in Theorem 1.2 satisfies. In the later sections on LQG, these properties (rather than the precise definition) of the function $\Phi_D^{\sigma, \kappa}$ will be used. We will make use of the terminology from Section 5A, in particular the notion of a CPI. There are completely analogous statements for the two lemmas in the $\kappa = 4$ case which we omit since they will not be used in this paper and the description of the CPIs is slightly different in that case.

Lemma 7.5. Fix $\kappa \in (8/3, 8)$, $\sigma = (\sigma_B : \emptyset \neq B \subseteq A)$ and consider a conformal transformation $\psi : D \rightarrow D'$ for D and D' simply connected strict subsets of $\mathbb{C}$. Let Γ be a nested CLE_κ in D and write Γ_0 for the collection of its outermost loops. For $\mathbf{z} \in D^A$ (with $z_i \neq z_j$ for $i \neq j$) and $U \subseteq D$, we write $\mathcal{I}(U) = \{i \in A : z_i \in U\}$. Then

$$\begin{aligned} \Phi_{D'}^{\sigma, \kappa}(\psi(z_i) : i \in A) &= \prod_{i \in A} |\psi'(z_i)|^{\rho_{\sigma_i}^\kappa} \cdot \Phi_D^{\sigma, \kappa}(\mathbf{z}), \\ \Phi_D^{\sigma, \kappa}(\mathbf{z}) &= \mathbb{E} \left(\prod_{\eta \in \Gamma_0} e^{\sigma_{\mathcal{I}(\eta^o)}} \prod_{U \in \mathfrak{U}(\eta^o)} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(\mathbf{z}|_{\mathcal{I}(U)}) \right), \end{aligned} \quad (7-4)$$

where $\mathfrak{U}(V)$ denotes the set of connected components of an open set V .

Now suppose that $\kappa \neq 4$ and consider a CPI λ (with some asymmetry parameter) in Γ from one prime end w_0 to another prime end w_∞ of D . Let L_λ be the collection of all loops in Γ_0 that λ intersects and let C_λ be the collection of all open complementary components of the union of λ with all loops in L_λ . Then

$$\Phi_D^{\sigma, \kappa}(\mathbf{z}) = \mathbb{E} \left(\prod_{\eta \in L_\lambda} e^{\sigma_{\mathcal{I}(\eta^o)}} \cdot \prod_{U \in C_\lambda} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(\mathbf{z}|_{\mathcal{I}(U)}) \right). \quad (7-5)$$

Proof. The first equality of (7-4) is satisfied by conformal invariance of CLE_κ and since we have $R(z_i, U)|\psi'(z_i)| = R(\psi(z_i), \psi(U))$ for any set $U \subseteq D$ which is a disjoint union of simply connected open sets (this follows directly from the definition of conformal radii). The second equality will follow by conditioning on the outermost loops Γ_0 of the nested CLE_κ . In the following, we use the notation appearing in Theorem 1.2 and we will split the outermost CLE loops into those surrounding exactly one point and those surrounding at least two points. We obtain

$$\begin{aligned} &\mathbb{E} \left(\prod_{i \in A} e^{\sigma_i} R(z_i, \eta_i^o)^{\rho_{\sigma_i}^\kappa} \prod_{\substack{B \subseteq A \\ \#B > 1}} e^{\sigma_B N_B^0(\mathbf{z})} \mid \Gamma_0 \right) \\ &= \prod_{\substack{i \in A \\ \eta_i \in \Gamma_0}} e^{\sigma_i} R(z_i, \eta_i^o)^{\rho_{\sigma_i}^\kappa} \cdot \left(\prod_{\eta \in \Gamma_0 \setminus \{\eta_i : i \in A\}} e^{\sigma_{\mathcal{I}(\eta^o)}} \prod_{U \in \mathfrak{U}(\eta^o)} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(\mathbf{z}|_{\mathcal{I}(U)}) \right) \\ &= \prod_{\eta \in \Gamma_0} e^{\sigma_{\mathcal{I}(\eta^o)}} \prod_{U \in \mathfrak{U}(\eta^o)} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(\mathbf{z}|_{\mathcal{I}(U)}), \end{aligned} \quad (7-6)$$

where the second equality follows from the definitions. The second equality in (7-4) thus follows by taking expectations on both sides and by using the definition of $\Phi_D^{\sigma, \kappa}(\mathbf{z})$.

The statement (7-5) is obtained as follows. First of all, note that Γ and λ are conditionally independent given Γ_0 . Therefore

$$\mathbb{E} \left(\prod_{i \in A} e^{\sigma_i} R(z_i, \eta_i^o)^{\rho_{\sigma_i}^\kappa} \prod_{\substack{B \subseteq A \\ \#B > 1}} e^{\sigma_B N_B^0(z)} \mid (\lambda, \Gamma_0) \right) = \prod_{\eta \in \Gamma_0} e^{\sigma_{\mathcal{I}(\eta^o)}} \prod_{U \in \mathfrak{U}(\eta^o)} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(z|_{\mathcal{I}(U)})$$

by (7-6). By the tower property we get

$$\mathbb{E} \left(\prod_{i \in A} e^{\sigma_i} R(z_i, \eta_i^o)^{\rho_{\sigma_i}^\kappa} \prod_{\substack{B \subseteq A \\ \#B > 1}} e^{\sigma_B N_B^0(z)} \mid (\lambda, L_\lambda) \right) = \mathbb{E} \left(\prod_{\eta \in \Gamma_0} e^{\sigma_{\mathcal{I}(\eta^o)}} \prod_{U \in \mathfrak{U}(\eta^o)} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(z|_{\mathcal{I}(U)}) \mid (\lambda, L_\lambda) \right).$$

By Section 5A, the conditional law of Γ_0 given (λ, L_λ) is obtained by taking the union of L_λ together with an independent nonnested CLE_κ Γ_U within each $U \in C'_\lambda := C_\lambda \setminus \bigcup_{\eta \in L_\lambda} \mathfrak{U}(\eta^o)$. We deduce that

$$\begin{aligned} & \mathbb{E} \left(\prod_{i \in A} e^{\sigma_i} R(z_i, \eta_i^o)^{\rho_{\sigma_i}^\kappa} \prod_{\substack{B \subseteq A \\ \#B > 1}} e^{\sigma_B N_B^0(z)} \mid (\lambda, L_\lambda) \right) \\ &= \prod_{\eta \in L_\lambda} e^{\sigma_{\mathcal{I}(\eta^o)}} \prod_{U \in \mathfrak{U}(\eta^o)} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(z|_{\mathcal{I}(U)}) \cdot \prod_{U \in C'_\lambda} \mathbb{E} \left(\prod_{\eta \in \Gamma_U} e^{\sigma_{\mathcal{I}(\eta^o)}} \prod_{V \in \mathfrak{U}(\eta^o)} \Phi_V^{\sigma|_{\mathcal{I}(V)}, \kappa}(z|_{\mathcal{I}(V)}) \mid \Gamma_U \right) \\ &= \prod_{\eta \in L_\lambda} e^{\sigma_{\mathcal{I}(\eta^o)}} \cdot \prod_{U \in C_\lambda} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(z|_{\mathcal{I}(U)}), \end{aligned}$$

where the final equality follows from the second part of (7-4) applied to $\Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(z|_{\mathcal{I}(U)})$. The claim (7-5) now follows by taking the expectation on both sides. $\square$

Lemma 7.6. Consider $\kappa \in (8/3, 8) \setminus \{4\}$, $\beta \in [-1, 1]$ and a simply connected domain $D \subsetneq \mathbb{C}$. We also fix two distinct prime ends w_0 and w_∞ . Let (Γ, λ) be a coupling of a CLE_κ in D and a CPI with asymmetry parameter β from w_0 to w_∞ in D .

Similarly, we let (Γ', λ') be the coupling of a $\text{CLE}_\kappa^\sigma(z)$ with a corresponding CPI from w_0 to w_∞ in D (see Definition 1.3). Recall the notation of (L_λ, C_λ) from Lemma 7.5 and define $(L_{\lambda'}, C_{\lambda'})$ analogously. Also recall the other notation from this lemma. Then the Radon–Nikodym derivative of $(\lambda', L_{\lambda'})$ with respect to (λ, L_λ) is given by

$$\frac{1}{\Phi_D^{\sigma, \kappa}(z)} \prod_{\eta \in L_\lambda} e^{\sigma_{\mathcal{I}(\eta^o)}} \cdot \prod_{U \in C_\lambda} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(z|_{\mathcal{I}(U)}).$$

The conditional law of Γ' given $(\lambda', L_{\lambda'})$ is given by taking an independent $\text{CLE}_\kappa^{\sigma|_{\mathcal{I}(U)}}(z|_{\mathcal{I}(U)})$ for each $U \in C_{\lambda'}$ together with the loops $L_{\lambda'}$.

Proof. By definition, the Radon–Nikodym derivative of Γ'_0 with respect to Γ_0 is given by

$$\frac{1}{\Phi_D^{\sigma, \kappa}(z)} \prod_{\eta \in \Gamma_0} e^{\sigma_{\mathcal{I}(\eta^o)}} \prod_{U \in \mathfrak{U}(\eta^o)} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(z|_{\mathcal{I}(U)}).$$

11¹/₂ Therefore the Radon–Nikodym derivative of $(\lambda', L_{\lambda'})$ with respect to (λ, L_{λ}) equals

$$\begin{aligned} & \frac{1}{\Phi_D^{\sigma, \kappa}(z)} \mathbb{E} \left(\prod_{\eta \in \Gamma_0} e^{\sigma_{\mathcal{I}(\eta^o)}} \prod_{U \in \mathcal{U}(\eta^o)} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(z|_{\mathcal{I}(U)}) \mid (\lambda, L_{\lambda}) \right) \\ &= \frac{1}{\Phi_D^{\sigma, \kappa}(z)} \prod_{\eta \in L_{\lambda}} e^{\sigma_{\mathcal{I}(\eta^o)}} \cdot \prod_{U \in C_{\lambda}} \Phi_U^{\sigma|_{\mathcal{I}(U)}, \kappa}(z|_{\mathcal{I}(U)}), \quad (7-7) \end{aligned}$$

where the equality follows from [Section 5A](#) as in [Lemma 7.5](#). To see the final part of the lemma, by the definition of a nested $\text{CLE}_{\kappa}^{\sigma}(z)$, it suffices to describe the conditional law of Γ_0' given the process $(\lambda', L_{\lambda'})$. The statement in this case is a consequence again of reweighting the conditional law of Γ_0 given (λ, L_{λ}) described in [Section 5A](#). $\square$

7C. Spatial independence for simple CLE. The aim of this section will be to establish [Theorem 7.1](#) in the case $\kappa \in (8/3, 4]$. For this, we rely on the construction of nested CLE_{κ} using the Brownian loop soup. We thus begin by quickly reviewing this construction. Consider $D \subseteq \mathbb{C}$ and define the following infinite and nonatomic measure on $C([0, \infty), \mathbb{C})$ by

$$\mu_D = \int_D dz \int_0^{\infty} dt \frac{p_t(z, z)}{t} P_z^t(\cdot; \omega([0, t]) \subseteq D).$$

Here $p_t(z, w) = \exp(-|z - w|^2/(2t))/(2\pi t)$ is the heat kernel, P_z^t is the law of a Brownian bridge in $\mathbb{C}$ from z to z of duration t , i.e., it is the law of $z + B_{\cdot \wedge t} - (1 \wedge (\cdot/t)) \cdot B_t$ where B is a two-dimensional standard Brownian motion and ω is the coordinate process on $C([0, \infty), \mathbb{C})$. The measure μ_D is called the *Brownian loop measure* and was introduced in [\[Lawler and Werner 2004\]](#).

Two key properties of the Brownian loop measure are its conformal invariance and its restriction property. To state them properly, let $\sim$ be an equivalence relation on $C(\partial\mathbb{D}, \mathbb{C})$ such that $\eta \sim \tilde{\eta}$ if there is an orientation-preserving homeomorphism $\psi : \partial\mathbb{D} \rightarrow \partial\mathbb{D}$ such that $\eta(s) = \tilde{\eta}(\psi(s))$ for all $s \in \partial\mathbb{D}$. Then $C(\partial\mathbb{D}, \mathbb{C})/\sim$ is the set of loops on $\mathbb{C}$ viewed modulo reparametrization of time. The measure μ_D induces a measure $\tilde{\mu}_D$ on $C(\partial\mathbb{D}, \mathbb{C})/\sim$. Conformal invariance means that $([\ell] \mapsto [\phi \circ \ell])_* \tilde{\mu}_D = \tilde{\mu}_{D'}$ for every conformal transformation $\phi : D \rightarrow D'$, and the restriction property means that $\tilde{\mu}_D$ restricted to the loops that are contained in D' is the measure $\tilde{\mu}_{D'}$ whenever $D' \subseteq D \subseteq \mathbb{C}$. The conformal invariance can be found in [\[Lawler and Werner 2004\]](#) while the restriction property is immediate from the definition.

Let $\mathcal{L}_c$ be a Poisson point process with intensity $c\mu_D$ (called the Brownian loop soup of intensity c in D). $\mathcal{L}_c$ is a.s. an infinite collection of loops and we say that two loops $\ell, \ell' \in \mathcal{L}_c$ are in the same cluster if there exist loops $\ell_0, \dots, \ell_n \in \mathcal{L}_c$ such that $\ell_0 = \ell$, $\ell_n = \ell'$, and ℓ_i intersects ℓ_{i+1} for $i = 0, \dots, n-1$. For each cluster, we can consider the closure of the union of all the loops within it; whenever the outer boundary of a cluster is a simple curve, we add it to a set $\mathcal{C}_c$ (all curves are viewed up to reparametrization). In fact, it turns out that almost surely, all clusters have a simple outer boundary. The remarkable result due to Sheffield and Werner [\[2012, Theorem 1.6\]](#) is that $\mathcal{C}_c$ is a CLE.

Theorem 7.7 [\[Sheffield and Werner 2012\]](#). *When $c \in (0, 1]$, $\mathcal{C}_c$ is almost surely an infinite collection of disjoint simple loops and has the law of a nonnested CLE_{κ} in D with $\kappa \in (8/3, 4]$ where κ and c are related via $c = (3\kappa - 8)(6 - \kappa)/(2\kappa)$.*

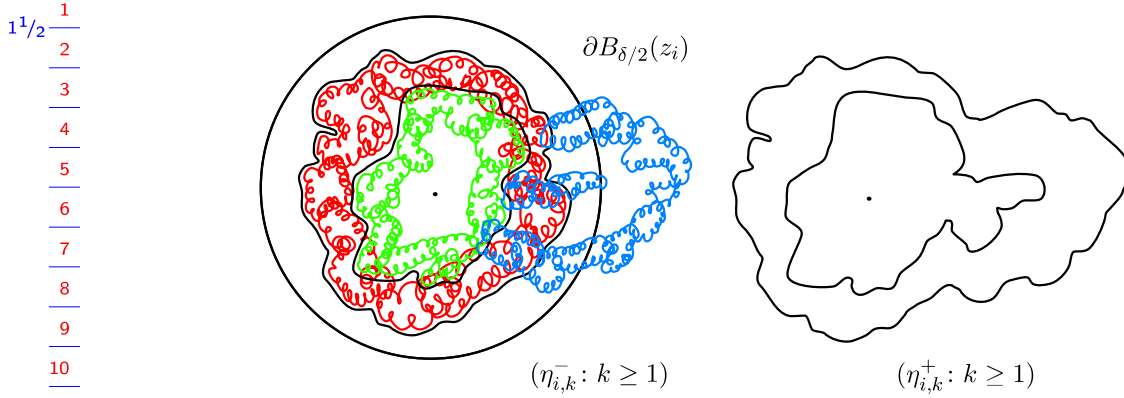


Figure 7. Illustration of the proof of [Theorem 7.1](#) for $\kappa \in (8/3, 4]$. Left: the clusters of the loop soup restricted to the ball $B_{\delta/2}(z_i)$ that touch the outermost and second outermost cluster are drawn in red and green, respectively. The blue loops are not contained in $B_{\delta/2}(z_i)$ and we therefore obtain larger clusters. Right: these are the (nested) clusters of the whole CLE. Note that some of the red loops on the left no longer touch the boundary of the new outermost cluster and hence now contribute to the boundary of the second outermost cluster.

It is also true that $\mathcal{L}_c$ encodes an entire nested CLE_κ . Indeed, let $\mathcal{C}_c^1 = \mathcal{C}_c$ and inductively define $\mathcal{C}_c^{n+1}$ from $\mathcal{C}_c^n$ and $\mathcal{L}_c$ as follows: Let $\mathcal{L}_c^{n+1}$ be all the loops in $\mathcal{L}_c$ that are surrounded by a loop in $\mathcal{C}_c^n$ but do not intersect any loop in $\mathcal{C}_c^n$. The collection $\mathcal{C}_c^{n+1}$ is then obtained from $\mathcal{L}_c^{n+1}$ as its set of outer cluster boundaries (analogous to the construction of $\mathcal{C}_c$ from $\mathcal{L}_c$). Finally, let $\mathcal{C}_c^\infty = \bigcup_{n \geq 1} \mathcal{C}_c^n$.

Proposition 7.8. When $c \in (0, 1]$, $\mathcal{C}_c^\infty$ is a nested CLE_κ in D with $\kappa \in (8/3, 4]$ where κ and c are related by $c = (3\kappa - 8)(6 - \kappa)/(2\kappa)$.

Proof. This is an immediate consequence of [\[Qian and Werner 2019, Theorem 1\]](#) and [Theorem 7.7](#). $\square$

We can now establish [Theorem 7.1](#) in the case where the CLE is simple.

Proof of Theorem 7.1 for $\kappa \in (8/3, 4]$. Without loss of generality $D = \mathbb{D}$ and $\rho_1, \dots, \rho_n \leq 0$. The proof will exploit the coupling in [Proposition 7.8](#) of a nested CLE_κ $\mathcal{C}_c^\infty$ with a Brownian loop soup $\mathcal{L}_c$. Note that the balls $B_{\delta/2}(z_i) \subseteq \mathbb{D}$ for all $i = 1, \dots, n$ are pairwise disjoint. For each $i \leq n$, if $\mathcal{L}_c(i)$ denotes the set of loops in $\mathcal{L}_c$ that are contained in $B_{\delta/2}(z_i)$ then $\mathcal{L}_c(i)$ has the law of a Brownian loop soup in $B_{\delta/2}(z_i)$ and hence defines a nested CLE_κ $\mathcal{C}_c^\infty(i)$ in $B_{\delta/2}(z_i)$ via the outer boundaries of its clusters. For $i \leq n$, let

$$(\eta_{i,k}^- : k \geq 1) \quad \text{and} \quad (\eta_{i,k}^+ : k \geq 1)$$

denote the loops surrounding z_i in $\mathcal{C}_c^\infty(i)$ and $\mathcal{C}_c^\infty$, respectively, such that both collections of loops are ordered according to their nesting structure.

Fix $i \leq n$. The key property we need is that $(\eta_{i,k}^-)^o \subseteq (\eta_{i,k}^+)^o$ for all $k \geq 1$. We prove this key property by induction on k . See [Figure 7](#). The induction hypothesis $k = 1$ is clear. For the induction step, we assume that the claim holds for k and we wish to deduce it for $k + 1$. This follows from the simple observation that a Brownian loop that is strictly contained in $(\eta_{i,k}^-)^o$ is also strictly contained in $(\eta_{i,k}^+)^o$.

In the notation of the theorem, let N be the minimal integer such that

$$(\eta_{i,N}^+)^o \subseteq \eta_i^o \quad \text{for all } i \leq n.$$

Combining this with the inclusion $(\eta_{i,k}^-)^o \subseteq (\eta_{i,k}^+)^o$ whenever $i, k \geq 1$ and applying Hölder's inequality with exponents $p, q > 1$ such that $1/p + 1/q = 1$ yields

$$\begin{aligned} \mathbb{E} \left(\prod_{i=1}^n R(z_i, \eta_i^o)^{\rho_i} \right) &\leq \sum_{k \geq 1} \mathbb{E} \left(\prod_{i=1}^n R(z_i, (\eta_{i,k}^-)^o)^{\rho_i}; N = k \right) \\ &\leq \sum_{k \geq 1} \mathbb{E} \left(\prod_{i=1}^n R(z_i, (\eta_{i,k}^-)^o)^{\rho_i p} \right)^{1/p} \mathbb{P}(N = k)^{1/q} \\ &= \sum_{k \geq 1} \prod_{i=1}^n \mathbb{E} \left(R(z_i, (\eta_{i,k}^-)^o)^{\rho_i p} \right)^{1/p} \mathbb{P}(N = k)^{1/q} \\ &= \sum_{k \geq 1} \prod_{i=1}^n \left(\frac{\delta}{2} \right)^{\rho_i} \mathbb{E} \left(R(0, \tilde{\eta}^o)^{\rho_i p} \right)^{k/p} \mathbb{P}(N = k)^{1/q}, \end{aligned}$$

where $\tilde{\eta}$ is the outermost loop in $\mathcal{C}_c^\infty$ surrounding 0, and we used independence of the CLEs $(\mathcal{C}_c^\infty(i) : i \leq n)$ and the conformal invariance of CLE. We take $p > 1$ sufficiently close to 1 such that $\rho_i p > -1 + 2/\kappa + 3\kappa/32$ for all $i \leq n$. Observe that

$$N \leq \max\{N_{ij} : 1 \leq i < j \leq n\} \leq \sum_{1 \leq i < j \leq n} N_{ij},$$

where N_{ij} is the number of loops in $\mathcal{C}_c^\infty$ surrounding both z_i and z_j . Now take $\theta > 0$ such that

$$\prod_{i=1}^n \mathbb{E} \left(R(0, \tilde{\eta}^o)^{\rho_i p} \right)^{1/p} < e^{\theta/q}.$$

By Markov's inequality, Hölder's inequality and [Lemma 7.3](#),

$$\mathbb{P}(N = k) \leq e^{-\theta k} \mathbb{E}(e^{\theta N}) \leq e^{-\theta k} \mathbb{E} \left(\prod_{1 \leq i < j \leq n} e^{\theta N_{ij}} \right) \leq e^{-\theta k} \prod_{1 \leq i < j \leq n} \mathbb{E}(e^{n(n-1)/2 \cdot \theta N_{ij}})^{2/(n(n-1))} \leq C \cdot e^{-\theta k}$$

for some constant $C > 0$ where C only depends on δ, θ and the number of points n . The result is now immediate. $\square$

7D. Spatial independence for nonsimple CLE. We will prove [Theorem 7.1](#) for $\kappa' \in (4, 8)$. The idea of the proof is to use a coupling between $\text{CLE}_{\kappa'}$ and a GFF h based on imaginary geometry and then use spatial independence properties of the GFF. Throughout the section we set $\kappa = 16/\kappa'$.

We say that h is a GFF with *clockwise (resp. counterclockwise) boundary data* on $(\mathbb{D}, -i)$ if it is a Dirichlet GFF in $\mathbb{D}$ with boundary data given by

$$-\frac{\pi}{\sqrt{\kappa'}} + \frac{\chi\pi}{2} + \chi \arg(z) \quad \text{and} \quad \frac{\pi}{\sqrt{\kappa'}} - \frac{3\chi\pi}{2} + \chi \arg(z),$$

respectively, where $\chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2$ and $\arg(z) \in [-\pi/2, 3\pi/2)$. For D an arbitrary simply connected domain with a distinguished prime end z_0 we say that h is a GFF on (D, z_0) with clockwise (resp. counterclockwise) boundary data if $h \circ \phi - \chi \arg(\phi')$ is a GFF with clockwise (resp. counterclockwise) boundary data in $\mathbb{D}$ for a conformal map $\phi : \mathbb{D} \rightarrow D$ satisfying $\phi(-i) = z_0$. We use the name clockwise (resp. counterclockwise) since in the coupling between the GFF and the branching $\text{SLE}_{\kappa'}$ described in (i) and (ii) right below, if the branching $\text{SLE}_{\kappa'}$ encloses a domain D by tracing its boundary in clockwise (resp. counterclockwise) direction, then $h|_D$ will have clockwise (resp. counterclockwise) boundary data.

Lemma 7.9. *Let $D \subseteq \mathbb{C}$ be a domain, let $\delta, p > 0$, let $z \in D$ be such that $\text{dist}(z, \mathbb{C} \setminus D) > \delta$, and let h be a GFF in D with clockwise or counterclockwise boundary data. Let $\mathfrak{h}$ denote the harmonic extension of $h|_{D \setminus B}$ to B where $B = B_{\delta/2}(z)$. Then there is a constant $c = c(D, \delta, p) \in (0, \delta/4)$ such that for all $r < c$ we have*

$$\mathbb{E}(\exp(p \|\mathfrak{h}|_{B_r(z)}\|_{\nabla}^2)) < c^{-1}.$$

Proof. Using, for instance, [Evans 2010, Section 2.2, Theorem 7] (and the harmonicity of the function $\mathfrak{h}$) we can see that there is a universal constant $c_0 > 0$ such that, for all $w \in B_{\delta/2}(z)$,

$$|\nabla \mathfrak{h}(w)| \leq \frac{c_0 \text{ave}(|\mathfrak{h}|; \partial B_{\delta}(z))}{\delta},$$

where $\text{ave}(|\mathfrak{h}|; \partial B_{\delta}(z))$ denotes the average of $|\mathfrak{h}|$ over $\partial B_{\delta}(z)$. Therefore, for $r \in (0, \delta/2)$ and by Jensen's inequality,

$$\|\mathfrak{h}|_{B_r(z)}\|_{\nabla}^2 \leq \pi c_0^2 \left(\frac{r}{\delta}\right)^2 \text{ave}(\mathfrak{h}^2; \partial B_{\delta}(z)). \quad (7-8)$$

Again using Jensen's inequality implies

$$\exp\left(\pi p c_0^2 \left(\frac{r}{\delta}\right)^2 \text{ave}(\mathfrak{h}^2; \partial B_{\delta}(z))\right) \leq \text{ave}(e^{\pi p c_0^2 (r/\delta)^2 \mathfrak{h}^2}; \partial B_{\delta}(z)). \quad (7-9)$$

By combining (7-8) and (7-9) we get

$$\begin{aligned} \mathbb{E}(\exp(p \|\mathfrak{h}|_{B_r(z)}\|_{\nabla}^2)) &\leq \mathbb{E}\left(\exp\left(\pi p c_0^2 \left(\frac{r}{\delta}\right)^2 \text{ave}(\mathfrak{h}^2; \partial B_{\delta}(z))\right)\right) \\ &\leq \text{ave}(\mathbb{E}(e^{\pi p c_0^2 (r/\delta)^2 \mathfrak{h}^2}); \partial B_{\delta}(z)). \end{aligned}$$

The lemma now follows since the random variables $\mathfrak{h}(w)$ for $w \in \partial B_{\delta}(z)$ are Gaussian with variance and expectation bounded above by a constant depending only on D and δ . $\square$

We will now recall that there exists a coupling between the following three random objects, such that each object determines the other three objects:

- (i) a GFF h on $(\mathbb{D}, -i)$ with clockwise boundary data,
- (ii) a branching $\text{SLE}_{\kappa'}$ $\mathcal{L} = (\lambda_z : z \in \mathcal{Q})$ in $\mathbb{D}$ started from $-i$ with force point at $(-i)^-$ (where $\mathcal{Q}$ is the set of rational points in $\mathbb{D}$),
- (iii) a $\text{CLE}_{\kappa'}$ Γ in $\mathbb{D}$.

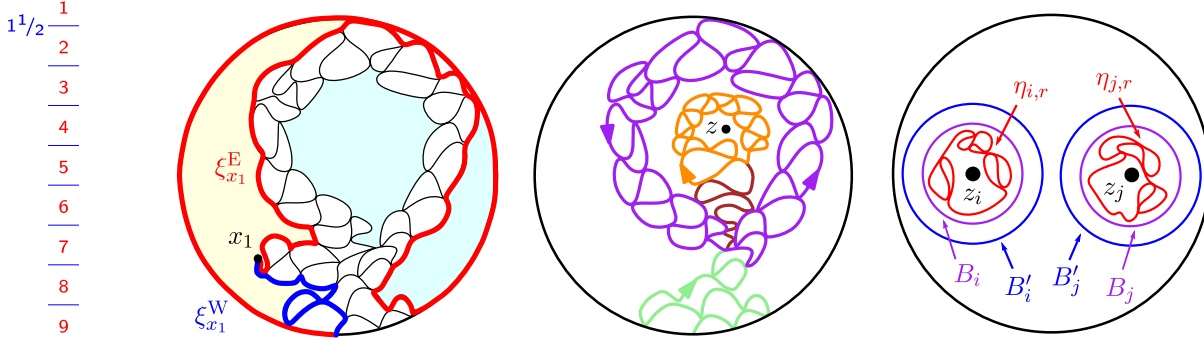


Figure 8. Left: illustration of the coupling between (i) and (ii) and the definition of a pocket (see Definition 7.10). The figure shows λ_{x_1} for some $x_1 \in \mathcal{Q}$ (black), $\xi_{x_1}^W$ (blue), and $\xi_{x_1}^E$ (red). The two domains in light blue are pockets while the domain in light yellow is not a pocket since its red boundary arc is ordered clockwise and its blue boundary arc is ordered counterclockwise. Center: illustration of the coupling between (ii) and (iii). The path obtained by concatenating the green, purple, brown, and orange path segments is $\lambda_z|_{[0, \tau_2]}$, while the path segments $\lambda_z|_{[\sigma_1, \tau_1]}$ (purple) and $\lambda_z|_{[\sigma_2, \tau_2]}$ (orange) are contained in the outermost and in the second outermost CLE loop surrounding z , respectively. Right: illustration of the proof of Theorem 7.1 for $\kappa' \in (4, 8)$. The relative size of the various circles is not to scale.

There is also a coupling with counterclockwise instead of clockwise boundary data in (i) and with $(-i)^+$ instead of $(-i)^-$ in (ii). We will focus on the case of clockwise boundary data and force point at $(-i)^-$ and only point out the cases where the coupling is different for counterclockwise boundary data and force point at $(-i)^+$. We refer to Figure 8 for an illustration of the coupling.

A branching $\text{SLE}_{\kappa'}$ in $\mathbb{D}$ started from $-i$ with force point at $(-i)^-$ is a set of curves $\mathcal{L} = (\lambda_z : z \in \mathcal{Q})$ such that for each $z \in \mathcal{Q}$, the curve λ_z has the law of a radial $\text{SLE}_{\kappa'}(\kappa' - 6)$ from $-i$ to z with the following force point behavior: The force point is initially located infinitesimally to the left (i.e., clockwise) of the starting point $-i$; the force point flips to the other side whenever the curve completes a clockwise (resp. counterclockwise) loop around z and before the flip, the force point lies right (resp. left) of the tip of the curve. For any $z, w \in \mathcal{Q}$, λ_z and λ_w are identical until z and w lie in different complementary components of the curve, and afterwards the two curves evolve independently.

The coupling between the objects in (i) and (ii) is based on the theory of imaginary geometry [Miller and Sheffield 2016; 2017]. Recall from [Miller and Sheffield 2017] that for each $z \in \mathcal{Q}$, one can define the following curves:

- an east-going flow line $\xi_z^E : [0, \infty) \rightarrow \overline{\mathbb{D}}$ for h started at z which by convention merges into the curve $\partial\mathbb{D}$ upon hitting it and traces a segment of $\partial\mathbb{D}$ in counterclockwise direction until hitting $-i$,
- a west-going flow line $\xi_z^W : [0, \infty) \rightarrow \overline{\mathbb{D}}$ for h started at z which terminates at $-i$ but does not merge into $\partial\mathbb{D}$ before that.

The flow lines are determined locally by h in the sense that if $z \in U \subseteq \mathbb{D}$ for some fixed domain U then the curves ξ_z^E, ξ_z^W stopped when they first leave U are a.s. determined by $h|_U$. If an east-going (resp.

^{11/2}₁ west-going) flow line hits another east-going (resp. west-going) flow line, then these two flow lines merge,
₂ while if an east-going (resp. west-going) flow line hits a west-going (resp. east-going) flow line, then it
₃ will bounce off without crossing the west-going (resp. east-going) flow line.
₄

₅ We next review some properties of the coupling between (i) and (ii) here and refer the reader to [Miller
₆ and Sheffield 2017, Section 4] for a complete description. For any $z \in \mathcal{Q}$ the curve λ_z can be constructed
₇ by first sampling its left and right boundaries (viewed as curves in the universal cover of $\mathbb{D} \setminus \{z\}$), which
₈ are given by ξ_z^W and ξ_z^E , respectively. Suppose $V \subseteq \mathbb{D}$ is a complementary component of $\mathbb{D} \setminus (\xi_z^W \cup \xi_z^E)$
₉ which lies between ξ_z^W and ξ_z^E , and let $x \in \partial V$ (resp. $y \in \partial V$) denote the point of intersection between
₁₀ ξ_z^W and ξ_z^E on ∂V which is ordered last (resp. first).

₁₁ We obtain λ_z restricted to V as follows: To the field $h|_V$ we associate the counterflow line in V
₁₂ from x to y (see [Miller and Sheffield 2016, Theorem 1.1]), which has the marginal law of a chordal
₁₃ $\text{SLE}_{\kappa'}(\kappa'/2 - 4; \kappa'/2 - 4)$ in V (from x to y); this is a boundary filling SLE with force points. If the GFF
₁₄ has clockwise boundary data and the force point of $\mathcal{L}$ is at $(-i)^+$ (i.e., counterclockwise or to the right
₁₅ of $-i$) then the coupling described above is identical.

₁₆ For $z \in \mathcal{Q}$ let V be a complementary component of λ_z , i.e., V is a connected component of the set
₁₇ $\mathbb{D} \setminus \lambda_z$. For a connected set $I \subseteq \partial V \cap \lambda_z$ we say that I is oriented clockwise (resp. counterclockwise) if
₁₈ the points of I are visited in clockwise (resp. counterclockwise) order by λ_z . In particular, this definition
₁₉ can be applied to the full boundary ∂V (i.e., the case $I = \partial V$) when $\partial V \subseteq \lambda_z$. If $\partial V \cap \partial \mathbb{D} \neq \emptyset$ then
₂₀ we say that ∂V is oriented clockwise (resp. counterclockwise) if $\partial V \cap \lambda_z$ is oriented clockwise (resp.
₂₁ ^{201/2} counterclockwise) and the boundary data of h are clockwise (resp. counterclockwise). The crucial property
₂₂ of the coupling with h is the following: if ∂V is ordered clockwise (resp. counterclockwise) then the
₂₃ conditional law of $h|_V$ given λ_z is that of a GFF with clockwise (resp. counterclockwise) boundary data
₂₄ and the fields in the different complementary components of λ_z are conditionally independent given λ_z .

₂₅ We define a function $\Upsilon_{\text{GFF} \rightarrow \text{SLE}}$ which takes as input a distribution on some simply connected domain D
₂₆ and outputs a collection of curves on that domain, such that in the coupling above we have $\mathcal{L} =$
₂₇ $\Upsilon_{\text{GFF} \rightarrow \text{SLE}}(h)$ almost surely.

₂₈ **Definition 7.10.** Consider a coupling of the objects in (i) and (ii) above. A domain $V \subseteq \mathbb{D}$ is called a
₂₉ pocket if it is a complementary component of λ_z for some $z \in \mathcal{Q}$ and if ∂V is ordered either clockwise or
₃₀ counterclockwise. For $U \subseteq \mathbb{D}$ we say that V is a pocket in U if V is a pocket and the closure of V is
₃₁ contained in U .

₃₂ If $z, w \in \mathcal{Q}$, V is a complementary component of λ_z , and $w \in V$, then λ_w can be written as the
₃₃ concatenation of λ_z until this curve encloses V and a curve λ' contained in $\bar{V}$. Let $\mathcal{L}|_V$ denote the
₃₄ collection of paths consisting of these curves λ' for all $w \in \mathcal{Q} \cap V$.

₃₅ **Lemma 7.11.** *The set of pockets V in U is a measurable function of $h|_U$, and for any pocket $V \subseteq U$ it*
₃₆ *holds a.s. that $\Upsilon_{\text{GFF} \rightarrow \text{SLE}}(h|_V)$ is well-defined and equal to $\mathcal{L}|_V$.*

₃₇ *Proof.* We encourage the reader to look at Figure 9 while reading the proof. We will argue that with
₃₈ ^{391/2} probability 1, a domain $V \subseteq \mathbb{D}$ with simple boundary is a pocket if and only if we can find a sequence of

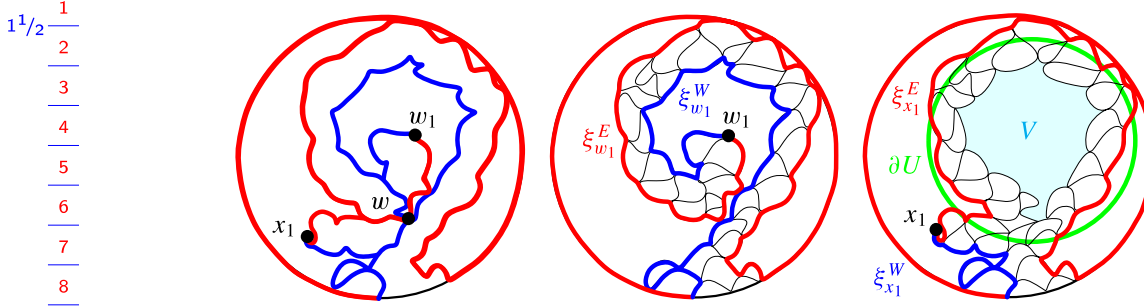


Figure 9. Illustration of the proof of Lemma 7.11. Left: $\hat{\xi}_w^W, \xi_{w_1}^W, \xi_{x_1}^W$ (blue), $\hat{\xi}_w^E, \xi_{w_1}^E$, and $\xi_{x_1}^E$ (red). Center: λ_{w_1} (black), $\xi_{w_1}^W$ (blue), and $\xi_{w_1}^E$ (red). Right: λ_{x_1} (black), $\xi_{x_1}^W$ (blue), $\xi_{x_1}^E$ (red), and a pocket V (light blue). The pocket V is a pocket in U , where U is the disk with boundary ∂U (light green).

points $(w_j) \subseteq V \cap \mathcal{Q}$ converging to some $w \in \mathbb{D}$ and curves $\hat{\xi}_w^W, \hat{\xi}_w^E$ starting at w and ending at $-i$ such that the following holds (or the following with the role of left/right and E/W swapped):

- The boundary of V is equal to an initial segment of $\hat{\xi}_w^W$.
- The curves $\xi_{w_j}^W, \xi_{w_j}^E$ merge into $\hat{\xi}_w^W, \hat{\xi}_w^E$, respectively, such that $\xi_{w_j}^W \Delta \hat{\xi}_w^W$ and $\xi_{w_j}^E \Delta \hat{\xi}_w^E$ have diameters converging to 0 as $j \rightarrow \infty$.
- The curve $\xi_{w_j}^E$ contains $\hat{\xi}_w^E, \hat{\xi}_w^W \cap V = \emptyset$ and $\hat{\xi}_w^E \cap V = \emptyset$.

The first assertion of the lemma then follows since flow lines are locally determined by the field to which they are coupled.

If V is a pocket, pick $x_1 \in \mathcal{Q}$ such that V is a complementary component of λ_{x_1} and let τ denote the time at which λ_{x_1} encloses V (that is, the time at which λ_{x_1} finishes tracing ∂V). Existence of appropriate points (w_j) is now immediate since we can let $(w_j) \subseteq V \cap \mathcal{Q}$ be points converging to $\lambda_{x_1}(\tau)$ and we can let $\hat{\xi}_w^W$ and $\hat{\xi}_w^E$ be the left and right, respectively, boundary of λ_{x_1} infinitesimally before time τ .

Conversely, suppose we are given points (w_j) and curves $\hat{\xi}_w^W, \hat{\xi}_w^E$ satisfying the given properties and assume without loss of generality that ∂V is equal to an initial segment of $\hat{\xi}_w^W$ and not of $\hat{\xi}_w^E$. We claim that $\hat{\xi}_w^W$ and $\hat{\xi}_w^E$ are the left and right, respectively, boundaries of λ_{z_1} infinitesimally before this curve encloses V . If the claim was not true then λ_{z_1} would enter V at some point $\hat{z} \in \partial V \setminus \{w\}$ before finishing to trace ∂V , so (by definition of $\mathcal{L}$) all the curves λ_{z_j} would enter V at $\hat{z}$. This contradicts the assumption that the left and right boundaries of λ_{z_j} converge to the curves $\hat{\xi}_w^W$ and $\hat{\xi}_w^E$ so the claim is correct. Furthermore, by letting $x_1 \in V^c \cap \mathcal{Q}$ be a point which is *not* lying between $\xi_{z_1}^E$ and $\xi_{z_1}^W$ and which is in the same complementary component of λ_{z_1} infinitesimally before this curve finishes tracing ∂V , we get that $\lambda_{x_1} \cap \lambda_{z_1}$ is equal to λ_{x_1} stopped upon enclosing V and that V is a complementary component of x_1 . Also observe that ∂V is oriented counterclockwise, while it would have been oriented clockwise if we had assumed that ∂V is equal to an initial segment of $\hat{\xi}_w^E$ instead of $\hat{\xi}_w^W$.

We have that $\Upsilon_{\text{GFF} \rightarrow \text{SLE}}(h|_V)$ is a.s. well-defined for all pockets V since $h|_V$ has the law of a GFF with clockwise boundary data (if ∂V is oriented clockwise) or counterclockwise boundary data (if ∂V is

^{11/2}₁ oriented counterclockwise). We get further that $\Upsilon_{\text{GFF} \rightarrow \text{SLE}}(h|_V) = \mathcal{L}|_V$ a.s. since by the construction of
₂ $\mathcal{L}$ in terms of h we have that $\mathcal{L}|_V$ is determined a.s. by the flow lines of the field $h|_V$. $\square$
₃

₄ The coupling between the objects in (ii) and (iii) is from [Sheffield 2009] and is also reviewed in,
₅ e.g., [Miller et al. 2014] and [Gwynne et al. 2023, Section 3.6.3 and Remark 3.28]. We will not give a
₆ complete description of the construction but recall some key properties.

₇ We say that λ_z *completes a counterclockwise turn around* z at time τ if λ_z hits its left boundary at
₈ time τ . We say that λ_z makes a counterclockwise turn around z during $[\sigma, \tau]$ if it completes a clockwise
₉ turn around z at time τ and $\sigma = \sup\{t \in (0, \tau) : \lambda_z(t) = \lambda_z(\tau)\}$. Define clockwise turns in the same manner
₁₀ with right instead of left. Let $[\sigma_1, \tau_1]$ be the first interval during which λ_z makes a counterclockwise
₁₁ turn around z . Iteratively, for each even (resp. odd) $k \in \mathbb{N}$ let $[\sigma_k, \tau_k]$ denote the first interval after
₁₂ $[\sigma_{k-1}, \tau_{k-1}]$ during which λ_z makes a clockwise (resp. counterclockwise) turn around z . The coupling
₁₃ between the objects in (ii) and (iii) is such that $\lambda_z([\sigma_k, \tau_k]) \subseteq \eta_k$, where η_k is the k -th outermost loop
₁₄ of Γ surrounding z . We can find a sequence of $z_j \in \mathcal{Q}$ converging to $\lambda_z(\sigma_k^z)$ such that η_k is the limit of
₁₅ $\lambda_{z_j}([\sigma_k^{z_j}, \tau_k^{z_j}])$ for (say) the Hausdorff topology as $j \rightarrow \infty$.

₁₆ If the GFF in (i) has counterclockwise boundary data instead of clockwise boundary data so that $\mathcal{L}$
₁₇ has a force point at $(-i)^+$ instead, then the exact same holds, except that the role of clockwise and
₁₈ counterclockwise loops is swapped. Let $\Upsilon_{\text{SLE} \rightarrow \text{CLE}}$ be a function such that $\Gamma = \Upsilon_{\text{SLE} \rightarrow \text{CLE}}(\mathcal{L})$ a.s.

₁₉ For a pocket V , let $\Gamma|_V := \{\eta \in \Gamma : \eta \subseteq \bar{V}\}$ denote the loops of Γ which are contained in $\bar{V}$. A key
₂₀ property of the coupling between (ii) and (iii) is that if V is a pocket then $\Gamma|_V = \Upsilon_{\text{SLE} \rightarrow \text{CLE}}(\mathcal{L}|_V)$ almost
₂₁ surely. This can be seen by observing that if ∂V is oriented clockwise (resp. counterclockwise) then the
₂₂ outermost loop in V around a point $z \in V \cap \mathcal{Q}$ corresponds to the first counterclockwise (resp. clockwise)
₂₃ turn made by λ_z around z after entering V .

₂₄ For $U \subseteq D$ let Γ^U denote the collection of loops η for which there exists a pocket V in U such that
₂₅ $\eta \subseteq V$.

₂₆ **Lemma 7.12.** *For any fixed domain $U \subseteq \mathbb{D}$ the collection of loops Γ^U is a measurable function of $h|_U$. If*
₂₇ *$\eta, \eta' \in \Gamma|_U$ and η' is surrounded by η then $\eta' \in \Gamma^U$.*

₂₈ *Proof.* The first assertion is immediate by the last assertion of Lemma 7.11 and since, as stated above the
₂₉ lemma, $\Gamma|_V = \Upsilon_{\text{SLE} \rightarrow \text{CLE}}(\mathcal{L}|_V)$ for each pocket V . We get the second assertion by observing that the
₃₀ complementary component of η which contains η' is a pocket. $\square$
₃₁

₃₂ Note that $\Gamma^U \subseteq \{\eta \in \Gamma : \eta \in U\}$ and that Lemma 7.12 does not hold with the set on the right-hand side
₃₃ instead of Γ^U , i.e., the set of loops contained in U is not a measurable function of $h|_U$.

₃₄ *Proof of Theorem 7.1 for $\kappa' \in (4, 8)$.* It is sufficient to consider the case when $\rho_i < 0$ for all i since this
₃₅ implies the general case. Pick $p, q > 1$ such that $p\rho_i > -1 + 2/\kappa' + 3\kappa'/32$ and $1/p + 1/q = 1$. Then in
₃₆ the notation of Lemma 7.9, pick $r \in (0, c(\mathbb{D}, \delta, n(q-1)/2))$.

₃₇ Consider the coupling between a Dirichlet GFF h with clockwise boundary conditions and a $\text{CLE}_{\kappa'}$
₃₈ in $\mathbb{D}$ described in (i), (ii) and (iii) above. Define $B'_i := B_r(z_i)$. Let $\eta_{i,r}$ denote the outermost loop in
₃₉ $\Gamma^{B'_i}$ containing z_i and define $I_i = R(z_i, \eta_{i,r})^{\rho_i}$. Note that $I_i \geq R(z_i, \eta_i^o)^{\rho_i}$ since $\eta_{i,r}$ is either equal to or
₄₀ surrounded by η_i . Define $B_i := B_{\delta/3}(z_i)$.

Let $\mathcal{F}$ denote the σ -algebra generated by h restricted to $\mathbb{D} \setminus \bigcup_i B_i$, let $\tilde{h}_i$ be a GFF in B_i with clockwise boundary conditions, define $\tilde{I}_i$ just as I_i but with the field $\tilde{h}_i$ instead of h , and let X_i denote the Radon–Nikodym derivative of $h|_{B'_i}$ given $\mathcal{F}$ with respect to $\tilde{h}_i|_{B'_i}$. We have

$$\begin{aligned} \mathbb{E}\left(\prod_i I_i\right) &= \mathbb{E}\left(\mathbb{E}\left(\prod_i I_i \mid \mathcal{F}\right)\right) = \mathbb{E}\left(\prod_i \mathbb{E}(I_i \mid \mathcal{F})\right) = \mathbb{E}\left(\prod_i \mathbb{E}(\tilde{I}_i X_i \mid \mathcal{F})\right) \\ &\leq \mathbb{E}\left(\prod_i \mathbb{E}(\tilde{I}_i^p \mid \mathcal{F})^{1/p} \mathbb{E}(X_i^q \mid \mathcal{F})^{1/q}\right) = \prod_i \mathbb{E}(\tilde{I}_i^p)^{1/p} \mathbb{E}\left(\prod_i \mathbb{E}(X_i^q \mid \mathcal{F})^{1/q}\right) \\ &\leq \prod_i \mathbb{E}(\tilde{I}_i^p)^{1/p} \mathbb{E}(\mathbb{E}(X_i^q \mid \mathcal{F})^{n/q})^{1/n}. \end{aligned} \quad (7-10)$$

By the last assertion of [Lemma 7.12](#), the second outermost loop in B'_i surrounding z_i is either equal to or surrounded by $\eta_{i,r}$. Using this and [Lemma 7.4](#) we get

$$\prod_i \mathbb{E}(\tilde{I}_i^p) < \infty.$$

For fixed i let $\mathfrak{h}$ denote the harmonic extension of h from ∂B_i to B_i , minus the harmonic extension of the boundary data of $\tilde{h}_i$ in B_i . We have

$$X_i = \exp\left((\tilde{h}_i|_{B'_i}, \mathfrak{h}|_{B'_i})_{\nabla} - \frac{\|\mathfrak{h}|_{B'_i}\|_{\nabla}^2}{2}\right)$$

and further

$$\begin{aligned} \mathbb{E}(\mathbb{E}(X_i^q \mid \mathcal{F})^{n/q}) &= \mathbb{E}\left(\mathbb{E}\left(\exp\left(q(\tilde{h}_i|_{B'_i}, \mathfrak{h}|_{B'_i})_{\nabla} - \frac{q\|\mathfrak{h}|_{B'_i}\|_{\nabla}^2}{2}\right) \mid \mathcal{F}\right)^{n/q}\right) \\ &= \mathbb{E}\left(\exp\left(\frac{n}{q} \cdot \frac{(q^2 - q)\|\mathfrak{h}|_{B'_i}\|_{\nabla}^2}{2}\right)\right). \end{aligned}$$

This is finite by [Lemma 7.9](#), which implies that the right-hand side of (7-10) is bounded by a finite constant depending only on δ . $\square$

8. LQG disks weighted by CLE nesting statistics

8A. Definition of the LQG disk weighted by CLE nesting statistics. We introduce the LQG surfaces that are studied in the paper. We proceed in two steps: First, we construct a measure on fields on the unit disk ([Definition 8.1](#)) and then use this to build the weighted LQG surface and weighted generalized LQG surface in the simple and nonsimple cases (that is, without and with pinch points), respectively, ([Definitions 8.5](#) and [8.7](#)). Then we derive explicit formulas for the partition functions of the disks in the case of zero or one marked bulk points, and finally we explain the relationship between the partition functions and LCFT correlation functions.

In the following definition we use the notation from [Sections 5B](#) and [5C](#) and [Definition 1.1](#). Let us point out that the measure defined below is not a probability measure.

Definition 8.1. Suppose that $A, B \subseteq \mathbb{N}$ are finite with $\#B = 3$ and that $w_j \in \partial\mathbb{D}$ for $j \in B$ are distinct points. Also, we consider $\kappa \in (8/3, 8) \setminus \{4\}$, $\sigma \in \mathfrak{S}_\kappa^A$, $\ell > 0$ and $\Lambda \geq 0$. Let $\alpha_i = \alpha_{\sigma_i}^\kappa$ for $i \in A$ and $\gamma = \sqrt{\kappa} \wedge 4/\sqrt{\kappa}$. We can now define a measure $M_{\Lambda, \ell}^{\sigma, \kappa}$ on the space $H^{-1}(D) \times \mathbb{D}^A$ by

$$M_{\Lambda, \ell}^{\sigma, \kappa}(X) = \frac{\ell^{2/\gamma \sum \alpha_i}}{Z_{0, \ell}^{(-, -), ((\gamma, \gamma, \gamma), \mathbf{w})}} \int_{\mathbb{D}^A} d\mathbf{z} \Phi_{\mathbb{D}}^{\sigma, \kappa}(\mathbf{z}) Z_{\Lambda, \ell}^{(\alpha, \mathbf{z}), ((\gamma, \gamma, \gamma), \mathbf{w})} \int P_{\Lambda, \ell}^{(\alpha, \mathbf{z}), ((\gamma, \gamma, \gamma), \mathbf{w})}(dk) 1_X(k, \mathbf{z})$$

for $X \subseteq H^{-1}(\mathbb{D}) \times \mathbb{D}^A$ measurable. We write $M_{\mathbf{w}, \Lambda, \ell}^{\sigma, \kappa}$ when we want to emphasize the dependence on $\mathbf{w}$ and we write $-$ for the empty tuple.

Remark 8.2. By [Definition 1.1](#), when $\kappa < 4$, the condition $\sigma_i < -\log \cos(4\pi/\kappa)$ translates to $\alpha_{\sigma_i}^\kappa > 2/\gamma + \gamma/4$ and when $\kappa > 4$ it translates to $\alpha_{\sigma_i}^\kappa > 1/\gamma + \gamma/2$. These constraints appear also in [Section 6](#) for the same reason. Indeed, in both cases this threshold arises because we need to assume finiteness of CLE conformal radius moments.

Remark 8.3. The constant prefactor in the definition of the measure is chosen such that $|M_{0, \ell}^{-, \kappa}| = 1$ and $|M_{0, \ell}^{\sigma, \kappa}| = \ell^{2/\gamma \sum \alpha_i} |M_{0, 1}^{\sigma, \kappa}|$ for $\ell > 0$. These conventions are the most convenient ones when stating results such as [Proposition 8.11](#).

The following change of coordinates result implies that the law of the γ -LQG surface associated to the field in the definition above does not depend on $\mathbf{w}$.

Lemma 8.4. Let us consider the setting of [Definition 8.1](#) with $B = \{a, b, c\}$ and consider another tuple (w'_a, w'_b, w'_c) of distinct points. Let $\phi : \mathbb{D} \rightarrow \mathbb{D}$ be the unique conformal or anticonformal transformation mapping (w_a, w_b, w_c) to (w'_a, w'_b, w'_c) . Then

$$\int M_{\mathbf{w}', \Lambda, \ell}^{\sigma, \kappa}(dk', dz') f(k' \circ \phi + Q_\gamma \log |\phi'|, \phi^{-1}(z')) = \int M_{\mathbf{w}, \Lambda, \ell}^{\sigma, \kappa}(dk, dz) f(k, z).$$

Proof. By unpacking the definitions and subsequently using the change of coordinates formula [\[Huang et al. 2018, Theorem 3.5\]](#) (see also [Section 5B](#) where the relevant Liouville conformal field theory notions were introduced) we get

$$\begin{aligned} & \int M_{\mathbf{w}', \Lambda, \ell}^{\sigma, \kappa}(dk', dz') f(k' \circ \phi + Q_\gamma \log |\phi'|, \phi^{-1}(z')) \\ &= \frac{\ell^{2/\gamma \sum \alpha_i}}{Z_{0, \ell}^{(-, -), ((\gamma, \gamma, \gamma), \mathbf{w}')}} \int_{\mathbb{D}^A} dz' \Phi_{\mathbb{D}}^{\sigma, \kappa}(z') \\ & \quad \cdot Z_{\Lambda, \ell}^{(\alpha, z'), ((\gamma, \gamma, \gamma), \mathbf{w}')} \int P_{\Lambda, \ell}^{(\alpha, z'), ((\gamma, \gamma, \gamma), \mathbf{w}')} (dk') f(k' \circ \phi + Q_\gamma \log |\phi'|, \phi^{-1}(z')) \\ &= \frac{\ell^{2/\gamma \sum \alpha_i}}{Z_{0, \ell}^{(-, -), ((\gamma, \gamma, \gamma), \mathbf{w}')}} \int_{\mathbb{D}^A} dz' \Phi_{\mathbb{D}}^{\sigma, \kappa}(z') \cdot \prod_{i \in A} |(\phi^{-1})'(z'_i)|^{\alpha_{\sigma_i}^\kappa (Q_\gamma - \alpha_{\sigma_i}^\kappa / 2)} \prod_{j \in B} |(\phi^{-1})'(w'_j)| \\ & \quad \cdot Z_{\Lambda, \ell}^{(\alpha, \phi^{-1}(z')), ((\gamma, \gamma, \gamma), \mathbf{w})} \int P_{\Lambda, \ell}^{(\alpha, \phi^{-1}(z')), ((\gamma, \gamma, \gamma), \mathbf{w})} (dk) f(k, \phi^{-1}(z')). \end{aligned}$$

Moreover, again by [Huang et al. 2018, Theorem 3.5],

$$Z_{0,\ell}^{(-,-),((\gamma,\gamma,\gamma),w')} = Z_{0,\ell}^{(-,-),((\gamma,\gamma,\gamma),w)} \prod_{j \in B} |(\phi^{-1})'(w'_j)|,$$

and the claim is then immediate after making the change of coordinates $z' = \phi(z)$ and using the behavior of $\Phi_{\mathbb{D}}^{\sigma,\kappa}$ under conformal transformations as stated in Lemma 7.5. $\square$

The previous lemma implies that we can define a γ -LQG surface with marked points in the following way.

Definition 8.5. Let $\kappa \in (8/3, 4)$, $\ell > 0$, $\Lambda \geq 0$ and $\sigma \in \mathfrak{S}_{\kappa}^A$ for finite $A \subseteq \mathbb{N}$. Then we define a measure $\bar{M}_{\Lambda,\ell}^{\sigma,\kappa}$ on the space of regular γ -LQG surfaces by

$$\bar{M}_{\Lambda,\ell}^{\sigma,\kappa}(X) := \int M_{w,\Lambda,\ell}^{\sigma,\kappa}(dk, dz) 1([(\mathbb{D}, k, z)] \in X)$$

for each measurable set X . We also let

$$W_{\Lambda,\ell}^{\sigma,\kappa} := |\bar{M}_{\Lambda,\ell}^{\sigma,\kappa}| = |M_{\Lambda,\ell}^{\sigma,\kappa}|.$$

If $W_{\Lambda,\ell}^{\sigma,\kappa} < \infty$ we say that $(\sigma, \kappa, \Lambda, \ell)$ is admissible and call $\bar{M}_{\Lambda,\ell}^{\sigma,\kappa} / W_{\Lambda,\ell}^{\sigma,\kappa}$ the regular γ -LQG disk with boundary length ℓ , cosmological constant Λ and weights σ .

Remark 8.6. We consider fields with three marked boundary points since this guarantees that (5-7) is satisfied; see Remark 5.4. Having three marked boundary points gives a natural way to fix the embedding of the disk since we can map the three marked boundary points to three fixed points on the boundary of the disk.

Next, we will make an analogous definition in the case when $\kappa' \in (4, 8)$. In the following definition, recall the definition of a Lévy excursion from Section 5C and the definition of generalized disk without marked points from Definition 5.8. In the definition below and throughout the text we write $\sigma|_B = (\sigma_C : C \subseteq B)$ whenever $B \subseteq A$.

Definition 8.7. Let $\kappa' \in (4, 8)$, $A \subseteq \mathbb{N}$ finite, $\sigma \in \mathfrak{S}_{\kappa'}^A$, $\Lambda \geq 0$ and $\ell > 0$. We now define $\bar{M}_{\Lambda,\ell}^{\sigma,\kappa'}$ to be a measure on the space of generalized γ -LQG surfaces as follows. Let E be the time reversal of a length ℓ spectrally positive Lévy excursion with exponent $4/\gamma^2$. Then we set

$$\bar{M}_{\Lambda,\ell}^{\sigma,\kappa'}(X) := \mathbb{E} \left(\sum_{Q \in \Pi(E,A)} \int \prod_{\substack{t \leq \ell \\ \Delta E_t \neq 0}} M_{w,\Lambda,|\Delta E_t|}^{\sigma|_{Q_t \setminus Q_{t-}, \kappa'}}(dk_t, dz_t) 1((E, \{(t, [(\mathbb{D}, k_t, z_t, w_a)]) : t < \ell, \Delta E_t \neq 0\}) \in X) \right)$$

for each measurable set X where $\Pi(E, A)$ denotes the set of nonincreasing càdlàg functions $Q : [0, \ell] \rightarrow \mathcal{P}(A)$ such that $Q_0 = A$, $Q_{\ell-} = \emptyset$ and $\Delta Q_t := Q_{t-} \setminus Q_t = \emptyset$ whenever $\Delta E_t = 0$. We also let

$$W_{\Lambda,\ell}^{\sigma,\kappa'} := |\bar{M}_{\Lambda,\ell}^{\sigma,\kappa'}|.$$

If $W_{\Lambda,\ell}^{\sigma,\kappa'} < \infty$ we say that $(\sigma, \kappa', \Lambda, \ell)$ is admissible and call $\bar{M}_{\Lambda,\ell}^{\sigma,\kappa'} / W_{\Lambda,\ell}^{\sigma,\kappa'}$ the generalized γ -LQG disk with boundary length ℓ , cosmological constant Λ and weights σ .

¹
^{1¹/₂} We do not include $\mathbf{w}$ in the notation $\bar{M}_{\Lambda,\ell}^{\sigma,\kappa'}$ since this measure does not depend on the choice of $\mathbf{w}$ (by Lemma 8.4).

A key question is when admissibility holds and what the values for the partition functions are. Here, we will answer this question in the case of a single bulk singularity, i.e., $\#A = 1$ or no bulk singularities, i.e., $\#A = 0$ both in the simple and the nonsimple case. The case of more than one singularity will be studied in Section 9. By making use of Proposition 5.5 the proof will be quite short. For the following results, recall the definition of modified Bessel functions of the second kind (see the beginning of Section 4) and define $\bar{K}_\nu$ as in (1-13).

Lemma 8.8. *Suppose that $\kappa \in (8/3, 8) \setminus \{4\}$ and let $\gamma = \sqrt{\kappa} \wedge 4 / \sqrt{\kappa}$. Assume that $A = \{i\}$ and let $\alpha = \alpha_{\sigma_i}^\kappa$. Then $|M_{\Lambda,\ell}^{-,\kappa}|, |M_{\Lambda,\ell}^{\sigma,\kappa}| < \infty$ for all $\ell > 0$ and $\Lambda \geq 0$, and*

$$|M_{\Lambda,\ell}^{-,\kappa}| = \bar{K}_{4/\gamma^2} \left(\ell \cdot \sqrt{\frac{\Lambda}{\sin(\pi\gamma^2/4)}} \right), \quad \frac{|M_{\Lambda,\ell}^{\sigma,\kappa}|}{|M_{0,1}^{\sigma,\kappa}|} = \ell^{2\alpha/\gamma} \bar{K}_{2/\gamma \cdot (Q_\gamma - \alpha)} \left(\ell \cdot \sqrt{\frac{\Lambda}{\sin(\pi\gamma^2/4)}} \right).$$

Proof. By Definition 8.1 (with $\mathbf{w}$ as in that definition) we get

$$|M_{\Lambda,\ell}^{-,\kappa}| = \frac{Z_{\Lambda,\ell}^{(-,-),((\gamma,\gamma,\gamma),\mathbf{w})}}{Z_{0,\ell}^{(-,-),((\gamma,\gamma,\gamma),\mathbf{w})}} = \int e^{-\Lambda\mu_k^\gamma(\mathbb{D})} P_{0,\ell}^{(-,-),((\gamma,\gamma,\gamma),\mathbf{w})}(dk).$$

²⁰
^{20¹/₂} The two results [Ang and Gwynne 2021, Theorem 1.2] and [Ang et al. 2023b, Theorem 1.3] show that the law of the area of the unit boundary length γ -LQG disk as defined in Definition 5.7 is an inverse Gamma distribution with shape parameter $4/\gamma^2$ and scale parameter $1/(4 \sin(\pi\gamma^2/4))$. Thus

$$\begin{aligned} \int e^{-\Lambda\mu_k^\gamma(\mathbb{D})} P_{0,\ell}^{(-,-),((\gamma,\gamma,\gamma),\mathbf{w})}(dk) &= \frac{1}{\Gamma(4/\gamma^2)(4 \sin(\pi\gamma^2/4))^{4/\gamma^2}} \int_0^\infty e^{-\Lambda\ell^2 t} \frac{e^{-1/(4 \sin(\pi\gamma^2/4)t)} dt}{t^{1+4/\gamma^2}} \\ &= \frac{1}{\Gamma(4/\gamma^2)} \int_0^\infty e^{-\Lambda\ell^2 x/(4 \sin(\pi\gamma^2/4)) - 1/x} \frac{dx}{x^{1+4/\gamma^2}}, \end{aligned}$$

and the first claim of the lemma follows from the definition of the modified Bessel functions as given at the beginning of Section 4.

If $\#A = \{i\}$ then $\Phi_{\mathbb{D}}^{\sigma,\kappa}(\mathbf{z}) = R(z_i, \mathbb{D})^\rho$ where $\rho := \rho_{\sigma_i}^\kappa = \alpha(Q_\gamma - Q_\alpha)$. By Proposition 5.5 therefore

$$|M_{\Lambda,\ell}^{\sigma,\kappa}| = \frac{\ell^{2\alpha/\gamma}}{Z_{0,\ell}^{(-,-),((\gamma,\gamma,\gamma),\mathbf{w})}} \frac{1}{2} e^{(G(w_a, w_b) + G(w_b, w_c) + G(w_c, w_a))/2} Z_{\Lambda,\ell}^{(\alpha,0),(\gamma,1)} < \infty,$$

where the finiteness of the partition functions on the right-hand side is recalled in Definition 5.3. Therefore,

$$\ell^{-2\alpha/\gamma} \frac{|M_{\Lambda,\ell}^{\sigma,\kappa}|}{|M_{0,1}^{\sigma,\kappa}|} = \frac{Z_{\Lambda,\ell}^{(\alpha,0),(\gamma,1)}}{Z_{0,\ell}^{(\alpha,0),(\gamma,1)}} = \frac{Z_{\Lambda\ell^2,1}^{(\alpha,0),(\gamma,1)}}{Z_{0,1}^{(\alpha,0),(\gamma,1)}} = \int e^{-\Lambda\ell^2\mu_k^\gamma(\mathbb{D})} P_{0,1}^{(\alpha,0),(\gamma,1)}(dk).$$

³⁹
^{39¹/₂} As stated in [Ang et al. 2023b, Theorem 1.2], the law of $\mu_k^\gamma(\mathbb{D})$ when $k \sim P_{0,1}^{(\alpha,0),(\gamma,1)}$ is an inverse Gamma distribution with shape parameter $2/\gamma \cdot (Q_\gamma - \alpha)$ and scale parameter $1/(4 \sin(\pi\gamma^2/4))$; note that in their setup the boundary γ singularity is mapped to a point that is uniformly sampled from $\partial\mathbb{D}$ which of course

¹₂ does not affect the law of the total area. The second claim now follows from a computation similar to the one for the first claim. $\square$

⁴ We now deduce the corresponding results for the total masses of the measures on the space of (generalized) LQG surfaces and conclude the proof of [Theorem 1.8](#). One thing worth remarking is that in the definition of a Lévy excursion we introduced an arbitrary convention: if b is a unit length Lévy excursion constructed from a Lévy process B (see [Section 4](#)) and $c > 0$ a constant then $c \cdot b$ is the unit length Lévy excursion associated to the Lévy process $c \cdot B$. This arbitrary choice in multiplicative normalization is reflected in the appearance of the $\sin(\pi\gamma^2/4)$ term below; by changing the normalization of the Lévy excursion in the $\kappa > 4$ case, we could also replace it by another arbitrary constant depending on γ and κ , for instance, $\sin(\pi\kappa/4)$.

¹² *Proof of Theorem 1.8.* The case $\kappa \in (8/3, 4)$ is immediate from [Lemma 8.8](#) and [Definition 8.5](#). For the $\kappa \in (4, 8)$ case let

$$c = \sqrt{\frac{\Lambda}{\sin(\pi\gamma^2/4)}}, \quad v = 4/\gamma^2, \quad \theta = \alpha\gamma/2, \quad c' = c^\nu 2^{1-\nu} = 2 \left(\frac{\Lambda}{4 \sin(\pi\gamma^2/4)} \right)^{2/\gamma^2}.$$

¹⁷ We only consider the $\Lambda > 0$ case; the $\Lambda = 0$ case then follows by monotone convergence. By [Definition 8.7](#) we have

$$W_{\Lambda,\ell}^{-,\kappa} = \mathbb{E} \left(\prod_{\substack{t < \ell \\ \Delta E_t \neq 0}} |M_{\Lambda,|\Delta E_t|}^{-,\kappa}| \right), \quad W_{\Lambda,\ell}^{\sigma,\kappa} = \mathbb{E} \left(\sum_{\substack{t < \ell \\ \Delta E_t \neq 0}} |M_{\Lambda,|\Delta E_t|}^{\sigma,\kappa}| \prod_{\substack{s \neq t \\ \Delta E_s \neq 0}} |M_{\Lambda,|\Delta E_s|}^{-,\kappa}| \right),$$

²² where E is a spectrally positive Lévy excursion of exponent $4/\gamma^2$ and duration ℓ (since we are only using the sequence of jump heights, we do not have to consider the time reversal). Thus by [Theorem 4.1](#) and [Lemma 8.8](#), we get

$$W_{\Lambda,\ell}^{-,\kappa} = \mathbb{E} \left(\prod_{\substack{t < \ell \\ \Delta E_t \neq 0}} \bar{K}_v(c\Delta E_t) \right) = \bar{K}_{1/v}(c'\ell)$$

²⁹ from which the first claim follows. Also

$$\begin{aligned} \frac{W_{\Lambda,\ell}^{\sigma,\kappa}}{|M_{0,1}^{\sigma,\kappa}|} &= \frac{2(c/2)^{1+\nu-\theta\nu}}{\Gamma(1+\nu-\theta\nu)} \mathbb{E} \left(\sum_{\substack{t < \ell \\ \Delta E_t \neq 0}} (\Delta E_t)^{1+\nu} K_{1+\nu-\theta\nu}(c\Delta E_t) \prod_{\substack{s \neq t \\ \Delta E_s \neq 0}} \bar{K}_v(c\Delta E_s) \right) \\ &= \frac{2(c/2)^{1+\nu-\theta\nu}}{\Gamma(1+\nu-\theta\nu)} \cdot \frac{\sqrt{1/\nu} \Gamma(-1/\nu) \sin(\pi(1+1/\nu-\theta))}{\sqrt{\nu} \Gamma(-\nu) \sin(\pi(1+\nu-\theta\nu))} \ell^{1+1/\nu} K_{1+1/\nu-\theta}(c'\ell) \\ &= c'' \ell^{\alpha\gamma/2} \bar{K}_{\gamma/2 \cdot (Q_\gamma - \alpha)} \left(2\ell \left(\frac{\Lambda}{4 \sin(\pi\gamma^2/4)} \right)^{2/\gamma^2} \right) \end{aligned}$$

³⁸ for a constant $c'' > 0$ which does not depend on Λ and ℓ . The claim follows. $\square$

³⁹₂ For future reference, let us also record the following corollary which is a direct consequence of properties of modified Bessel functions and will be applied in [Section 9](#).

Corollary 8.9. Suppose that $\kappa \in (8/3, 8) \setminus \{4\}$, $A = \{i\}$ and $\Lambda > \Lambda' > 0$. Then there exists a constant $C = C(\sigma_i, \kappa, \Lambda, \Lambda')$ such that $W_{\Lambda, \ell}^{\sigma, \kappa} \leq C W_{\Lambda', \ell}^{\sigma, \kappa}$ for all $\ell > 0$. There is $C_\kappa > 0$ such that, for all $\Lambda \geq 0$ and $\ell > 0$,

$$1 - W_{\Lambda, \ell}^{\sigma, \kappa} \leq C_\kappa \Lambda \ell^{2 \wedge (8/\kappa)}.$$

Proof. Let us first recall the following asymptotic for modified Bessel functions of the second kind (see Equations (9.6.2) and (9.6.10) in [Abramowitz and Stegun 1964]):

$$K_\nu(x) = \frac{\pi}{2 \sin(\pi \nu)} \left(\frac{1}{\Gamma(-\nu+1)} \left(\frac{x}{2}\right)^{-\nu} + \frac{1}{\Gamma(-\nu+2)} \left(\frac{x}{2}\right)^{-\nu+2} + \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \right) + O(x^{-\nu+4} + x^{\nu+2}) \quad \text{as } x \rightarrow 0$$

for all $\nu > 0$. Therefore, by Euler's reflection formula and the definition of $\bar{K}_\nu$,

$$1 - \bar{K}_\nu(x) = \frac{1}{\nu-1} \left(\frac{x}{2}\right)^2 + \frac{\pi}{\sin(-\pi \nu) \Gamma(\nu) \Gamma(\nu+1)} \left(\frac{x}{2}\right)^{2\nu} + O(x^4 + x^{2\nu+2}) \quad \text{as } x \rightarrow 0. \quad (8-1)$$

In particular we obtain $\bar{K}_\nu(x) \rightarrow 1$ as $x \rightarrow 0$ and this combined with the Bessel function asymptotic (see Equation (9.7.1) in [Abramowitz and Stegun 1964])

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \quad \text{as } x \rightarrow \infty$$

and Theorem 1.8 readily yields the first claim. For the second claim, we observe that (8-1) implies that for $\nu \in (0, 2)$ there is $C'_\nu > 0$ such that

$$1 - \bar{K}_\nu(x) \leq C'_\nu x^{2 \wedge (2\nu)} \quad \text{for all } x > 0.$$

The claim then follows again directly from Theorem 1.8. $\square$

In the $\kappa \in (8/3, 4)$ case, by considering the total mass of the measure in Definition 8.1 we get

$$W_{\Lambda, \ell}^{\sigma, \kappa} = \frac{\ell^{2/\gamma} \sum \alpha_i}{Z_{0, \ell}^{(-, -), ((\gamma, \gamma, \gamma), w)}} \int_{\mathbb{D}^A} dz \Phi_{\mathbb{D}}^{\sigma, \kappa}(z) Z_{\Lambda, \ell}^{(\alpha, z), ((\gamma, \gamma, \gamma), w)}. \quad (8-2)$$

Let us now consider the setting of Definition 5.3 and define $s := \sum_i \alpha_i + 1/2 \sum_j \beta_j - Q_\gamma$. In the context of Liouville conformal field theory, the constants

$$Z_{\Lambda, \ell, (\alpha, z), (\beta, w)}^{\text{LCFT}} := \frac{2\ell^{2s/\gamma-1}}{\gamma} e^{-\sum_j \beta_j^2/8} Z_{\Lambda, \ell}^{(\alpha, z), (\beta, w)}$$

are the fixed boundary length LCFT partition functions (with certain boundary and bulk singularities)—the multiplicative prefactor is an arbitrary convention which we include to make our presentation consistent with [Huang et al. 2018]. We believe the relation (8-2) to be possibly useful for the following reason.

This work explains how the expressions $W_{\Lambda, \ell}^{\sigma, \kappa}$ can in principle be iteratively computed (see Remark 1.10) so if the fixed boundary length LCFT partition functions could also be determined as a function of the

¹/₂ position of the singularities and the cosmological constant, then this could be used to obtain information about the CLE observable $\Phi_{\mathbb{D}}^{\sigma, \kappa}$.

We remark that the constant $Z_{0, \ell}^{(-, -), ((\gamma, \gamma, \gamma), \mathbf{w})}$ appearing in (8-2) can be determined explicitly using results of [Remy and Zhu 2022]. We also remark that the fixed boundary length LCFT partition functions are closely related to the LCFT correlation functions; indeed, if h is as in Definition 5.3 then the LCFT correlation function is given by

$$Z_{\Lambda, (\alpha, z), (\beta, \mathbf{w})}^{\text{LCFT}} = e^{-\sum_j \beta_j^2/8} C_{(\beta, \mathbf{w})}^{(\alpha, z)} \int \mathbb{E}(e^{-\Lambda \mu_{h+c}(\mathbb{D})}) e^{cs} dc = \int Z_{\Lambda, \ell, (\alpha, z), (\beta, \mathbf{w})}^{\text{LCFT}} d\ell,$$

where the second equality is obtained by making the change of coordinates $c = 2/\gamma \cdot (\log \ell - \log \nu_h(\partial \mathbb{D}))$. Note however that the LCFT correlation functions (in contrast to the fixed boundary length LCFT partition functions) will be insufficient to extract information about the CLE observable $\Phi_{\mathbb{D}}^{\sigma, \kappa}$ since the LCFT correlation function is a polynomial in the cosmological constant Λ ; this can be seen by making a change of coordinates $\ell' = \ell \sqrt{\Lambda}$ in the above display concerning LCFT correlation functions.

8B. The LQG disk weighted by CLE nesting statistics as a limit. This section is devoted to the proof of the following proposition. It says that the field describing a γ -LQG disk (or a single component of such a disk in the generalized case) with zero interior marked points (see Section 5C) converges to the field of a disk with singularities $(\alpha_i : i \in A)$ (see Section 8A), upon a reweighting by the expression

$$\prod_{i \in A} (\epsilon_i^{\alpha_i^2/2} e^{\alpha_i h_\epsilon(z_i)}),$$

where the z_i points are sampled independently from the Lebesgue measure and $h_\epsilon(z_i)$ is an appropriate approximation to h evaluated at z_i . In other words, we are showing how to obtain the measures in Section 8A using a limiting procedure which is going to be crucial in the proofs in Section 8C in order to transfer results from Section 5D, i.e., the case without marked points, to our setting with marked points. In Section 8C we will be in a scenario where the mollifier we use to define $h_\epsilon(z_i)$ is not entirely independent of the field h (because these mollifiers will depend on mapping out functions defined via the field h) and for this reason we prove the result below in this generality. The reason for including the space E in the result below is that we want to allow for sampling other geometric objects conditionally on the field h , for example we will sample a CPI where its target point is sampled from the LQG boundary measure associated to h .

Proposition 8.10. *Let $\kappa \in (8/3, 8) \setminus \{4\}$, $\ell > 0$, $r \in (0, 1/4)$, $A \subseteq \mathbb{N}$ finite, $\Lambda \geq 0$, and define $K \subseteq \mathbb{D}^A$ by*

$$K = \{z = (z_i : i \in A) \in B_{1-2r}(0)^A : B_{2r}(z_i) \cap B_{2r}(z_j) = \emptyset \forall i \neq j\}.$$

Consider some measurable space E . Let $B = \{a, b, c\}$, $(\beta_a, \beta_b, \beta_c) = (\gamma, \gamma, \gamma)$, and let $\mathbf{w} = (w_a, w_b, w_c)$ be three distinct boundary points of $\partial \mathbb{D}$ ordered counterclockwise. We consider a probability kernel μ from $H^{-1}(\mathbb{D})$ to $C(\mathbb{D} \times (0, r/2) \times \mathbb{D}) \times E$ (so μ_k is a probability measure on $C(\mathbb{D} \times (0, r/2) \times \mathbb{D}) \times E$ for each $k \in H^{-1}(\mathbb{D})$) where we denote elements θ of $C(\mathbb{D} \times (0, r/2) \times \mathbb{D})$ by $(z, \epsilon, w) \mapsto \theta_\epsilon^z(w)$. Let $G_{B_r(z)}$ denote the Dirichlet Green's function in $B_r(z)$. We assume that for all $k \in H^{-1}(\mathbb{D})$, $((\theta, x) \mapsto \theta)_ \mu_k$ is*

¹/₂ supported on functions θ with the property that for all $z \in B_{1-2r}(0)$ and $\epsilon \in (0, r/2)$ we have

$$\begin{aligned} \int_{\mathbb{D}} \theta_{\epsilon}^z(w) dw &= 1 + o_{\epsilon}(1), \quad \text{supp}(\theta_{\epsilon}^z) \subseteq B_{2\epsilon}(z), \\ \int_{\mathbb{D}^2} G(w, w') \theta_{\epsilon}^z(w) \theta_{\epsilon}^z(w') dw dw' &= \log \frac{1}{R(z, \mathbb{D})\epsilon} + o_{\epsilon}(1), \\ \int_{B_r(z)^2} G_{B_r(z)}(w, w') \theta_{\epsilon}^z(w) \theta_{\epsilon}^z(w') dw dw' &= \log \frac{r}{\epsilon} + o_{\epsilon}(1) \end{aligned} \quad (8-3)$$

as $\epsilon \rightarrow 0$ where the $o_{\epsilon}(1)$ is uniform in $z \in B_{1-2r}(0)$, in θ and in k .⁷ Lastly, we suppose that $f : H^{-1}(\mathbb{D}) \times \mathbb{D}^A \times E \rightarrow [0, \infty)$ is measurable, bounded and satisfies $f(k, z, x) = 0$ for $z \notin K$. Almost every k sampled from $M_{w,0,\ell}^{-,\kappa}$ satisfies $\lim_{n \rightarrow \infty} f(k + g_n^z, z, x) = f(k, z, x)$ if (g_n^z) is a sequence of continuous functions with compact support in $\bigcup_i B_{2r}(z_i)$ which converge uniformly to 0 away from the points of z such that $\lim_{n \rightarrow \infty} \|g_n^z\|_{H^{-1}(\mathbb{D})} = 0$. Then

$$\begin{aligned} \int f(k, z, x) \Phi_{\mathbb{D}}^{\sigma, \kappa}(z) \prod_{i \in A} (\epsilon^{\alpha_i^2/2} e^{\alpha_i k(\theta_{\epsilon}^{z_i})}) e^{-\Lambda \mu_k^{\gamma}(\mathbb{D})} \mu_{k^{\perp z}}(d\theta, dx) dz M_{w,0,\ell}^{-,\kappa}(dk) \\ \rightarrow \int f(k, z, x) \mu_{k^{\perp z}}(d\theta, dx) M_{w,\Lambda,\ell}^{\sigma, \kappa}(dk, dz) \end{aligned} \quad (8-4)$$

²⁰/₂ as $\epsilon \rightarrow 0$ where dz denotes Lebesgue measure on tuples $z = (z_i \in \mathbb{D} : i \in A)$ and $k^{\perp z}$ denotes the harmonic extension of k restricted to the complement of $\bigcup_i B_r(z_i)$ (see [Section 5B](#)).

Proof. For each $z \in K^A$, we decompose h^0 (a Neumann GFF in $\mathbb{D}$) as $h^0 = h^z + \mathfrak{h}^z$ into its projection onto $H^{-1}(\bigcup_i B_r(z_i))$ (called h^z) and the harmonic extension of h^0 restricted to the complement of $\bigcup_i B_r(z_i)$ (called $\mathfrak{h}^z$). Crucially, h^z and $\mathfrak{h}^z$ are independent and h^z has the law of an independent Dirichlet GFF on each ball $B_r(z_i)$ for $i \in A$. Let $G^z : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{R}$ (resp. $\mathcal{G}^z : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{R}$) denote the function describing the covariance of h^z (resp. $\mathfrak{h}^z$), and note that $G = G^z + \mathcal{G}^z$. Recall that (using that $h^0 = \mathfrak{h}^z + h^z$)

$$h = \mathfrak{h}^z + h^z + \sum_{j \in B} \frac{\gamma}{2} G(\cdot, w_j), \quad h_* = h - \frac{2}{\gamma} \log v_h^{\gamma}(\partial \mathbb{D}) + \frac{2}{\gamma} \log \ell.$$

As pointed out in [Section 5B](#) we have that $v_h^{\gamma}(\partial \mathbb{D}) = v_{h-h^z}^{\gamma}(\partial \mathbb{D})$ and we see that this random variable is measurable with respect to $\mathfrak{h}^z$. Furthermore,

$$h_*^{\perp z} = \mathfrak{h}^z + \sum_{j \in B} \frac{\gamma}{2} G(\cdot, w_j) - \frac{2}{\gamma} \log v_h^{\gamma}(\partial \mathbb{D}) + \frac{2}{\gamma} \log \ell,$$

which is hence also measurable with respect to $\mathfrak{h}^z$ (this is the crucial observation which will momentarily allow us to apply Girsanov's theorem). By unpacking the definitions and using the notation (with no

⁷When $\theta_{\epsilon}^z(w) dw$ is replaced by the probability measure σ_{ϵ}^z with uniform mass on the circle $\partial B_{\epsilon}(z)$, (8-3) is also satisfied.

One can make sense of the function $(z, \epsilon) \mapsto k(\sigma_{\epsilon}^z)$ in the setting of the proposition and the random process is called the circle average process associated to k . However, for technical reasons, we work with smooth bump functions instead.

marked points in the bulk, $\mathbf{w}$ as the marked points on the boundary and boundary length ℓ) appearing in Definition 5.3, we obtain that the left-hand side of (8-4) equals $c_0 \cdot \int_K A_\epsilon(\mathbf{z}) d\mathbf{z}$ where

$$A_\epsilon(\mathbf{z}) := \mathbb{E} \left(\int v_h^\gamma(\partial\mathbb{D})^{4/\gamma^2-2} e^{-\Lambda \mu_{h_*}(\mathbb{D})} f(h_*, \mathbf{z}, x) \Phi_{\mathbb{D}}^{\sigma, \kappa}(\mathbf{z}) \prod_{i \in A} \epsilon^{\alpha_i^2/2} e^{\alpha_i h_*(\theta_\epsilon^{z_i})} \mu_{h_*^\perp \mathbf{z}}(d\theta, dx) \right)$$

and $c_0 = \mathbb{E}(v_h^\gamma(\partial\mathbb{D})^{4/\gamma^2-2})^{-1}$. For $\mathbf{z} \in K$ we now condition on $\mathfrak{h}^{\mathbf{z}}$, fix a realization of θ and apply Girsanov's theorem in the setting obtained after conditioning on the field $\mathfrak{h}^{\mathbf{z}}$ by considering the reweighting

$$\begin{aligned} \prod_{i \in A} \epsilon^{\alpha_i^2/2} e^{\alpha_i h_*(\theta_\epsilon^{z_i})} &= e^{h^{\mathbf{z}}(\phi) - \frac{1}{2} \int \phi(w) \phi(w') G^{\mathbf{z}}(w, w') dw dw'} \\ &\cdot \prod_{i \in A} (\epsilon^{\alpha_i^2/2} e^{\alpha_i^2/2 \cdot \int_{B_r(z_i)} G^{\mathbf{z}}(w, w') \theta_\epsilon^{z_i}(w) \theta_\epsilon^{z_i}(w') dw dw'}) \\ &\cdot \prod_{i \in A} (e^{\alpha_i (\mathfrak{h}^{\mathbf{z}} + \gamma/2 \sum_{j \in B} G(\cdot, w_j))(\theta_\epsilon^{z_i})} v_h(\partial\mathbb{D})^{-2\alpha_i/\gamma} \ell^{2\alpha_i/\gamma}), \end{aligned}$$

where $\phi = \sum_{i \in A} \alpha_i \theta_\epsilon^{z_i}$ and we used that $G^{\mathbf{z}}(w, w') = 0$ if $w \in B_r(z_i)$ and $w' \in B_r(z_j)$ for $i \neq j$. Note that we have written the expression in the display above so that the first line on the right-hand side has expectation 1 and is the term we are reweighting by when applying Girsanov's theorem. Indeed, the Girsanov weighting

$$\exp \left(h^{\mathbf{z}}(\phi) - \frac{1}{2} \int \phi(w) \phi(w') G^{\mathbf{z}}(w, w') dw dw' \right)$$

induces a shift of the field $h^{\mathbf{z}}$ by

$$\sum_{i \in A} \alpha_i \int G^{\mathbf{z}}(\cdot, y) \theta_\epsilon^{z_i}(y) dy.$$

Formally, for $\mathbf{z} \in K$ the field $h_{\mathbf{z}, \epsilon}$ has the same law as that of h weighted by

$$\exp(h^{\mathbf{z}}(\phi) - \frac{1}{2} \int \phi(w) \phi(w') G^{\mathbf{z}}(w, w') dw dw'),$$

where we make the definitions

$$\begin{aligned} G_\epsilon^{\mathbf{z}}(z, w) &= \int G^{\mathbf{z}}(z, y) \theta_\epsilon^w(y) dy, \\ h_{\mathbf{z}, \epsilon} &= \mathfrak{h}^{\mathbf{z}} + h^{\mathbf{z}} + \sum_{i \in A} \alpha_i G_\epsilon^{\mathbf{z}}(\cdot, z_i) + \sum_{j \in B} \frac{\gamma}{2} G(\cdot, w_j), \\ h_{*\mathbf{z}, \epsilon} &= h_{\mathbf{z}, \epsilon} - \frac{2}{\gamma} \log v_{h_{\mathbf{z}, \epsilon}}^\gamma(\partial\mathbb{D}) + \frac{2}{\gamma} \log \ell. \end{aligned}$$

For later reference we also let $(h_{\mathbf{z}, 0}, h_{*\mathbf{z}, 0})$ and $(h_{\mathbf{z}}, h_{*\mathbf{z}})$ be defined analogously, only with $G^{\mathbf{z}}$ and G , respectively, in place of $G_\epsilon^{\mathbf{z}}$ (compare the latter definition with Definition 5.3). Girsanov's theorem yields

$$A_\epsilon(\mathbf{z}) = \ell^{2/\gamma \sum_{i \in A} \alpha_i} \mathbb{E} \left(\int B_\epsilon(\mathbf{z}, \theta, x) \mu_{h_*^\perp \mathbf{z}}(d\theta, dx) \right)$$

¹/₂ where

$$B_\epsilon(\mathbf{z}, \theta, x) := v_{h_{\mathbf{z}, \epsilon}}^\gamma (\partial \mathbb{D})^{4/\gamma^2 - 2 - 2/\gamma \sum_{i \in A} \alpha_i} e^{-\Lambda \mu_{h_{\mathbf{z}, \epsilon}}(\mathbb{D})} f(h_{\mathbf{z}, \epsilon}, \mathbf{z}, x) \Phi_{\mathbb{D}}^{\sigma, \kappa}(\mathbf{z}) \\ \cdot \prod_{i \in A} (\epsilon^{\alpha_i^2/2} e^{\alpha_i^2/2 \cdot \int_{B_r(z_i)} G^z(w, w') \theta_\epsilon^{z_i}(w) \theta_\epsilon^{z_i}(w') dw dw'} e^{\alpha_i(\mathfrak{h}^z + \gamma/2 \sum_{j \in B} G(\cdot, w_j))(\theta_\epsilon^{z_i})}).$$

By our assumptions (noting that $G^z = G_{B_r(z_i)}$ on $B_r(z_i)^2$ by definition), we see that

$$B_\epsilon(\mathbf{z}, \theta, x) \rightarrow v_{h_{\mathbf{z}, 0}}^\gamma (\partial \mathbb{D})^{4/\gamma^2 - 2 - 2/\gamma \sum_{i \in A} \alpha_i} e^{-\Lambda \mu_{h_{\mathbf{z}, 0}}(\mathbb{D})} f(h_{\mathbf{z}, 0}, \mathbf{z}, x) \Phi_{\mathbb{D}}^{\sigma, \kappa}(\mathbf{z}) \\ \cdot \prod_{i \in A} (r^{\alpha_i^2/2} e^{\alpha_i(\mathfrak{h}^z(z_i) + \gamma/2 \sum_{j \in B} G(z_i, w_j))}).$$

as $\epsilon \rightarrow 0$. Note that the right-hand side does not depend on θ . Moreover, since $4/\gamma^2 - 2 - 2/\gamma \sum_{i \in A} \alpha_i < 0$ and using our assumptions, we may bound

$$B_\epsilon(\mathbf{z}, \theta, x) \lesssim v_{\mathfrak{h}^z + h^z}^\gamma (\partial \mathbb{D})^{4/\gamma^2 - 2 - 2/\gamma \sum_{i \in A} \alpha_i} \prod_{i \in A} e^{\alpha_i \sup_{B_{2r/3}(z_i)} \mathfrak{h}^z} =: C_\epsilon(\mathbf{z}),$$

where the implicit constant does not depend on $\mathbf{z} \in K$ and also not on θ ; to upper bound $\Phi_{\mathbb{D}}^{\sigma, \kappa}(\mathbf{z})$ by a deterministic constant for $\mathbf{z} \in K$ we used [Theorem 1.2](#). By the mean value property of harmonic functions, we see that $\sup_{B_{2r/3}(z_i)} \mathfrak{h}^z \leq C \mathfrak{h}^z(z_i)$ uniformly in $\mathbf{z} \in K$ for some constant $C < \infty$. Using Hölder's inequality and the fact that the total mass $v_{\mathfrak{h}^z + h^z}^\gamma (\partial \mathbb{D})$ has all negative moments and that these are bounded uniformly in $\mathbf{z} \in K$ (see [\[Rhodes and Vargas 2014, Theorem 2.12\]](#)) we see that $\sup_{\mathbf{z} \in K} \mathbb{E}(C_\epsilon(\mathbf{z})) < \infty$.

Thus by dominated convergence, we obtain

$$\int_K A_\epsilon(\mathbf{z}) d\mathbf{z} \rightarrow \ell^{2/\gamma \sum_{i \in A} \alpha_i} \int \mathbb{E} \left(v_{h_{\mathbf{z}, 0}}^\gamma (\partial \mathbb{D})^{4/\gamma^2 - 2 - 2/\gamma \sum_{i \in A} \alpha_i} e^{-\Lambda \mu_{h_{\mathbf{z}, 0}}(\mathbb{D})} f(h_{\mathbf{z}, 0}, \mathbf{z}, x) \right. \\ \left. \Phi_{\mathbb{D}}^{\sigma, \kappa}(\mathbf{z}) \cdot \prod_{i \in A} (r^{\alpha_i^2/2} e^{\alpha_i(\mathfrak{h}^z(z_i) + \gamma/2 \sum_{j \in B} G(z_i, w_j))}) \right) \mu_{k \perp z}(d\theta, dx) d\mathbf{z}$$

as $\epsilon \rightarrow 0$. The right-hand side can be rewritten as follows. Note that since $\mathcal{G}^z = G - G^z$ and by the continuity of $\mathcal{G}^z$ we have $\mathcal{G}^z(z_i, z_i) = -\log R(z_i, \mathbb{D}) - \log(r)$ and furthermore $\mathcal{G}^z(z_i, z_j) = G(z_i, z_j)$ for $i \neq j$. We perform a Girsanov shift by

$$e^{\sum_{i \in A} \alpha_i \mathfrak{h}^z(z_i)} = e^{\sum_{i \in A} \alpha_i \mathfrak{h}^z(z_i) - 1/2 \sum_{i, j \in A} \alpha_i \alpha_j \mathcal{G}^z(z_i, z_j)} \prod_{i \in A} (r R(z_i, \mathbb{D}))^{-\alpha_i^2/2} \prod_{\substack{i, i' \in A \\ i < i'}} e^{\alpha_i \alpha_{i'} G(z_i, z_j)},$$

which shifts the field $\mathfrak{h}^z$ up by $\sum_{i \in A} \alpha_i \mathcal{G}^z(\cdot, z_i)$. Rearranging the resulting expression (which now involves the fields h_z and $h_{\mathbf{z}}$) yields the claim. $\square$

8C. Chordal exploration of LQG disk weighted by CLE nesting statistics. This section is devoted to the proof of [Proposition 8.11](#) below. The proposition says that if we consider an LQG disk with an independent CLE, weighted by the CLE nesting statistics (as in [Definitions 8.5, 8.7, and 1.3](#)), then the boundary length process of the CPI has a law which is reweighted in an explicit way as compared to the

^{11/2}₁ case with no marked points, and the loop-decorated LQG surfaces in the complementary components of
₂ the CPI have the law of independent CLE decorated LQG disks reweighted by CLE nesting statistics,
₃ conditioned on their boundary lengths and marked points.
₄

₅ The idea of the proof is to start with the analogous result for disks without marked points (proved in
₆ [Miller et al. 2021; 2022]) and then do a particular reweighting of the measures. The reweighted measure
₇ gives us the surface and loop ensemble in Definition 8.5 (or, in the generalized case, Definition 8.7) and
₈ Definition 1.3, respectively, via an application of the Girsanov theorem. A closely related strategy was
₉ used to prove a conformal welding result for a surface with an α singularity on its boundary in [Ang et al.
₁₀ 2024]. However, the case considered here is substantially more technically challenging, e.g., since there
₁₁ are multiple marked points the locations of which are random and the considered mollifiers and curves
₁₂ are not always independent of the field (even if we condition on the location of the marked points).

₁₃ We borrow the following terminology from probability theory in the setting of (finite or infinite)
₁₄ nonprobability measures, i.e., measures whose total mass is different from 1. Suppose that E , E' and
₁₅ E'' are measurable spaces, μ is measure on E , $f : E \times E' \rightarrow E''$ is measurable, and ν is a probability
₁₆ kernel from E to E' , i.e., $\nu = (\nu_e : e \in E)$ is a collection of probability measures on E' such that the map
₁₇ $e \mapsto \nu_e(A)$ from E to $\mathbb{R}$ is measurable for any measurable $A \subseteq E'$. Let $(\mu \otimes \nu)(de, de') = \mu(de)\nu_e(de')$
₁₈ and define $\zeta = f_*(\mu \otimes \nu)$ to be the pushforward of $\mu \otimes \nu$ along f . In words we will describe the
₁₉ construction of ζ as follows: sample $e \sim \mu$, conditionally on e sample e' according to ν_e , and let ζ denote
₂₀ the law of $f(e, e')$. It is often more convenient to introduce ζ in words as in the previous sentence since
₂₁ the precise formula for f is rather involved.

₂₂ In order to state Proposition 8.11 precisely, we first need to introduce some more notation and definitions.
₂₃ For $\kappa \in (8/3, 8) \setminus \{4\}$ and a finite and nonempty set $A \subseteq \mathbb{N}$ let $\sigma \in \mathfrak{S}_\kappa^A$. Also let $\gamma = \sqrt{\kappa} \wedge 4/\sqrt{\kappa}$, $\Lambda \geq 0$,
₂₄ $\ell_L, \ell_R > 0$, and $\beta \in [-1, 1]$. Let $\ell = \ell_L + \ell_R$.

₂₅ We define three measures $\overline{\text{ML}}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}$, $\overline{\text{ML}}_{\Lambda, \ell}^{\sigma, \kappa}$ and $\overline{\text{MC}}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}$ on the space of decorated regular γ -LQG
₂₆ surfaces. We first consider the $\kappa \in (8/3, 4)$ case.

- ₂₇ • Let $(h, z) \sim M_{\Lambda, \ell_L + \ell_R}^{\sigma, \kappa}$ (see Definition 8.1 with w given by $w_0 = -i$ and $w_{\pm 1} = \pm 1$). Let $w_\infty \in \partial \mathbb{D}$ be
₂₈ the point such that the clockwise (resp. counterclockwise) arc from w_0 to w_∞ has length ℓ_L (resp. ℓ_R) from
₂₉ w_0 to w_∞ . Let $\Gamma \sim \text{CLE}_\kappa^\sigma(z)$ and then sample an associated CPI λ with parameter β (see Definition 1.3
₃₀ and Section 5A). We obtain a decorated γ -LQG surface $[(\mathbb{D}, h, z, \Gamma, \lambda)]$ and we call its law $\overline{\text{ML}}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}$.
- ₃₁ • If $[(\mathbb{D}, h, z, \Gamma, \lambda)] \sim \overline{\text{ML}}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}$ we write $\overline{\text{ML}}_{\Lambda, \ell}^{\sigma, \kappa}$ for the law of $[(\mathbb{D}, h, z, \Gamma)]$.
- ₃₂ • Let Γ_0 denote the loops in Γ intersecting λ . If $[(\mathbb{D}, h, z, \Gamma, \lambda)] \sim \overline{\text{ML}}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}$ then $\overline{\text{MC}}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}$ denotes
₃₃ the law of $[(h, z, \Gamma_0, \lambda)]$.

₃₄ As in Section 5D, we can associate to $S = [(\mathbb{D}, h, z, \Gamma, \lambda)]$ the following objects: the γ -LQG length ζ of the
₃₅ CPI, for each $t < \zeta$ the future part of the disk S_t (this is now a loop decorated γ -LQG surface with marked
₃₆ points) and boundary length processes L and R (writing $X = L + R$). Let $J(X) := \{t < \zeta : \Delta X_t \neq 0\}$. Then
₃₇ for each $t \in J(X)$, the cut out γ -LQG surfaces ΔS_t is now a loop decorated γ -LQG surface with marked
₃₈ points, more precisely $\Delta S_t = [(\Delta D_t, h|_{\Delta D_t}, z|_{\mathcal{I}_z(\Delta D_t)}, \{\eta \in \Gamma : \eta^o \subsetneq \Delta D_t\})]$ where $\mathcal{I}_z(U) := \{i \in A : z_i \in U\}$
₃₉ whenever U is open. Here, ΔD_t is one of the complementary components that the CPI with attached CLE
₄₀

¹/₂ loops cuts out. We write $P_t = \mathcal{I}_z(D_t)$ for the indices of the points which are in the future of part S_t of the disk. Then $\Delta P_t = P_{t-} \setminus P_t$ for $t \in J(X)$ are the indices of the marked points on ΔS_t . The process P is a set-valued càdlàg process and we summarize the processes as $Y = (L, R, P)$.

If $S = [(\mathbb{D}, h, z, \Gamma_0, \lambda)]$, we can again associate to it $\zeta, L, R, X, S_t, \Delta S_t, P$ and Y , the only difference being that S_t and ΔS_t are now γ -LQG disks with marked points but without a loop decoration. As in Section 5D, we can write the cut out γ -LQG surface as $\Delta S_t = [(\Delta D_t, h|_{\Delta D_t}, z|_{\mathcal{I}_z(\Delta D_t)})]$.

In the case $\kappa \in (4, 8)$ we define these measures analogously. We refer the reader to Section 5D for information on the relevant constructions in this case.

In the following, we will leave the dependence of the boundary length processes and cut out quantum surfaces on the underlying decorated γ -LQG surface implicit. For example, the process Y on the left-hand side of the first indented equation below depends on the integration variable S and we leave this dependence implicit in the notation.

Proposition 8.11. *In the setting above, for any measurable function f with values in $[0, 1]$,*

$$\begin{aligned} & \int f(Y, \{(t, \Delta S_t) : t \in J(X)\}) \overline{\text{ML}}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}(dS) \\ &= \mathbb{E} \left(\sum_{Q \in \Pi(X^0, A)} \int f((L^0, R^0, Q), \{(t, S'_t) : t \in J(X^0)\}) \prod_{t \in J(X^0)} e^{\sigma \Delta Q_t \mathbf{1}(\Delta X_t^0 > 0)} \overline{\text{ML}}_{\Lambda, |\Delta X_t^0|}^{\sigma \mathbf{1}_{\Delta Q_t}, \kappa}(dS'_t) \right), \end{aligned}$$

²⁰/₂ where the expectation on the right-hand side is taken with respect to the random process (L^0, R^0) from Section 5D (corresponding to $A = \emptyset$) with $X^0 = L^0 + R^0$. In particular, the case $f \equiv 1$ yields

$$W_{\Lambda, \ell}^{\sigma, \kappa} = \mathbb{E} \left(\sum_{Q \in \Pi(X^0, A)} \int \prod_{t \in J(X^0)} e^{\sigma \Delta Q_t \mathbf{1}(\Delta X_t^0 > 0)} W_{\Lambda, |\Delta X_t^0|}^{\sigma \mathbf{1}_{\Delta Q_t}, \kappa} \right).$$

We will also use a version of the proposition in the case where we decorate a γ -LQG disk with marked points with a (nested) weighted CLE and consider the collection of components which are cut out by the outermost CLE loops.

Proposition 8.12. *Let $(S_n : n \geq 1)$ and $(\ell_n : n \geq 1)$ be the γ -LQG surfaces cut out by the collection of outermost loops on S and the associated boundary lengths (both sequences ordered according to decreasing boundary length). Let $(\ell_n^0 : n \geq 1)$ be the sequence of boundary lengths of a nonnested CLE_κ decorating a (regular) γ -LQG disk (also ordered according to decreasing boundary length). Then, for any measurable function f with values in $[0, 1]$,*

$$\int f(\{(\ell_n, S_n) : n \geq 1\}) \overline{\text{ML}}_{\Lambda, \ell}^{\sigma, \kappa}(dS) = \mathbb{E} \left(\sum_{\bigcup_n B_n = A} \int f(\{(\ell_n^0, S'_n) : n \geq 1\}) \prod_{n \geq 1} e^{\sigma B_n} \overline{\text{ML}}_{\Lambda, \ell_n^0}^{\sigma \mathbf{1}_{B_n}, \kappa}(dS'_n) \right),$$

where the sum on the right-hand side is over all partitions of A into disjoint sets.

³⁹/₂ *Proof.* The proof is omitted since it is identical to that of Proposition 8.11. In fact, the proof is slightly easier than that of Proposition 8.11 since we do not need to consider the CPI. $\square$

The next lemma is a variant of [Proposition 8.11](#) where we consider surfaces decorated by (Γ_0, λ) instead of (Γ, λ) . The statement therefore only differs by the fact that we are not considering the decoration by loops inside the cut out surfaces.

Lemma 8.13. *In the setting above, for any measurable function f with values in $[0, 1]$,*

$$\begin{aligned} & \int f(Y, \{(t, \Delta S_t) : t \in J(X)\}) \overline{\text{MC}}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}(dS) \\ &= \mathbb{E} \left(\sum_{Q \in \Pi(X^0, A)} \int f((L^0, R^0, Q), \{(t, S'_t) : t \in J(X^0)\}) \prod_{t \in J(X^0)} e^{\sigma_{\Delta Q_t} 1(\Delta X_t^0 > 0)} \bar{M}_{\Lambda, |\Delta X_t^0|}^{\sigma |_{\Delta Q_t}, \kappa}(dS'_t) \right), \end{aligned}$$

where the expectation on the right-hand side is taken with respect to the random process (L^0, R^0) from [Section 5D](#) (corresponding to $A = \emptyset$) with $X^0 = L^0 + R^0$.

Proof of Proposition 8.11 given Lemma 8.13. We only need to prove the first statement since the second one is immediate. The difference between the proposition and the lemma is that in the latter case we are only considering the CLE loops that are intersecting the CPI while in the former case we are considering a full nested CLE. The result is then a consequence of the definitions and the final part of [Lemma 7.6](#). $\square$

It remains to prove [Lemma 8.13](#). We will first do the proof for $\kappa \in (8/3, 4)$.

Proof of Lemma 8.13 for $\kappa \in (8/3, 4)$. We start by introducing some notation. Sample the tuple $(h, z) \sim M_{0, \ell}^{\sigma, \kappa}$ ([Definition 8.1](#)), and define $w_\infty, \Gamma, \lambda, \Gamma_0, Y = (L, R, P), X, J(X), \mathcal{I}_z(U)$, and $(\Delta D_t)_{t \in J(X)}$ as above.

Let w_t^0 denote the marked point on $\partial \Delta D_t$ which is equal to the point on $\partial \Delta D_t$ that is hit first by the CPI λ . Then let w_t^1 and w_t^{-1} be sampled independently according to the (normalized) γ -LQG boundary length measure with respect to h on $\partial \Delta D_t$ conditioned on $\mathbf{w}_t := (w_t^0, w_t^1, w_t^{-1})$ being ordered counterclockwise. Let $\psi_t : \mathbb{D} \rightarrow \Delta D_t$ be the unique conformal map which is sending $\mathbf{w}_t$ to $(-i, 1, -1)$ and define

$$\tilde{h}_t = h \circ \psi_t + Q_\gamma \log |\psi'_t|, \quad \tilde{z}_i^t = \psi_t^{-1}(z_i) \text{ for } t \in J(X) \text{ and } i \in \Delta P_t, \quad \tilde{z}^t = (\tilde{z}_i^t : t \in J(X), i \in \Delta P_t) \quad (8-5)$$

whenever $z \in \mathbb{D}^A$. We also let

$$\tilde{M}_{L, R, (\tilde{h}_t)_t} \text{ be the conditional law of } (\psi_t)_{t \in J(X)} \text{ given } L, R, (\tilde{h}_t)_{t \in J(X)}$$

in the setting of $A = \emptyset$.⁸ We denote the law of $(h, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)})$ by $\text{MC}_{0, \ell_L, \ell_R}^{-, \kappa}$. For any nonnegative measurable function f , by [\[Miller et al. 2022\]](#) (see [Section 5D](#) for a review),

$$\begin{aligned} & \int f(L, R, (\tilde{h}_t, \psi_t)_t) d\text{MC}_{0, \ell_L, \ell_R}^{-, \kappa}(h, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) \\ &= \mathbb{E} \left(\int f(L^0, R^0, (\tilde{h}_t, \psi_t)_t) \tilde{M}_{L^0, R^0, (\tilde{h}_t)_{t \in J(X^0)}}(d(\psi_t)_{t \in J(X^0)}) \prod_{t \in J(X^0)} M_{0, |\Delta X_t^0|}^{-, \kappa}(d\tilde{h}_t) \right). \quad (8-6) \end{aligned}$$

⁸In fact, it is stated in [\[Miller et al. 2022, Remark 3.6\]](#) that $(\psi_t)_{t \in J(X)}$ is almost determined by $L, R, (\tilde{h}_t)_{t \in J(X)}$ in the sense that $L, R, (\tilde{h}_t)_{t \in J(X)}$ determine a collection of conformal transformations $(\tilde{\psi}_t)_{t \in J(X)}$ such $\tilde{\psi}_t = \phi \circ \psi_t$ for some conformal map $\phi : \mathbb{D} \rightarrow \mathbb{D}$. We do not rely on this result.

^{11/2}₁ Note that we are abusing notation slightly since $\tilde{h}_t$ and ψ_t are functions of the integration variables on the left-hand side while they are integration variables on the right-hand side.

₃ Given a tuple $\mathbf{z} \in \mathbb{D}^A$ we write $P_t = \{i \in A : z_i \in D_t\}$, leaving the dependence on $\mathbf{z}$ implicit. In the following, we will write λ_U for the Lebesgue measure on U where $U \subseteq \mathbb{C}$ is open. The product measure on U^B for $B \subseteq A$ is denoted by $\lambda_U^{\otimes B}$ with the convention that $\lambda_U^{\otimes \emptyset}$ is a Dirac delta of mass 1 on the empty tuple.

₇ For any $t \in J(X)$, the pushforward of $\lambda_{\mathbb{D}}^{\otimes B}$ along ψ_t^{-1} is given by $(|\psi_t' \circ \psi_t^{-1}|^{-2} \lambda_{\Delta D_t})^{\otimes B}$. Using (8-6), we obtain that, for f nonnegative and measurable,

$$\begin{aligned} & \int f((L, R, P), (\tilde{h}_t, \tilde{\mathbf{z}}^t, \psi_t)_t) d\text{MC}_{0, \ell_L, \ell_R}^{-, \kappa}(h, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) \prod_{t \in J(X)} \prod_{i \in \Delta P_t} |\psi_t'(\tilde{z}_i^t)|^{-2} \lambda_{\mathbb{D}}^{\otimes A}(d\mathbf{z}) \\ &= \mathbb{E} \left(\int \sum_{Q \in \Pi(X, A)} f((L^0, R^0, Q), (\tilde{h}_t, \tilde{\mathbf{z}}^t, \psi_t)_t) \tilde{M}_{L^0, R^0, (\tilde{h}_t)_t}(d(\psi_t)_{t \in J(X)}) \right. \\ & \quad \left. \prod_{t \in J(X^0)} M_{0, |\Delta X_t^0|}^{-, \kappa}(d\tilde{h}_t) \lambda_{\mathbb{D}}^{\otimes \Delta Q_t}(d\tilde{\mathbf{z}}^t) \right). \quad (8-7) \end{aligned}$$

₁₈ By the same type of abuse of notation as before $\tilde{\mathbf{z}}^t$ has different meanings on the right- and left-hand side of the equation, namely, on the left-hand side it is defined by (8-5) while on the right-hand side it is the integration variable. Let $\theta : \mathbb{C} \rightarrow [0, \infty)$ be a smooth radially symmetric bump function satisfying

$$\int_{\mathbb{C}} \theta = 1, \quad \text{supp}(\theta) \subseteq \mathbb{D}, \quad \iint \theta(z) \theta(z') \log(|z| \vee |z'|)^{-1} dz dz' = 0.$$

₂₅ For $\epsilon \in (0, 1)$ and $z \in \mathbb{D}$ set $\tilde{\theta}_\epsilon^z(\tilde{z}) = \epsilon^{-2} \theta(\epsilon^{-1}(\tilde{z} - z))$ and for a distribution $\tilde{h}$ define $\tilde{h}_\epsilon(z) := \tilde{h}(\tilde{\theta}_\epsilon^z)$. Fix $r \in (0, 1/16)$. Consider a function $f = f_\epsilon$ of the form

$$\begin{aligned} f_\epsilon(Y, (\tilde{h}_t, \tilde{\mathbf{z}}^t, \psi_t)_t) &= \tilde{f}(Y, (\tilde{h}_{t_1}, \tilde{\mathbf{z}}^{t_1}, \psi_{t_1}), \dots, (\tilde{h}_{t_N}, \tilde{\mathbf{z}}^{t_N}, \psi_{t_N})) \\ & \prod_{t \in J(X)} \Phi_{\mathbb{D}}^{\sigma |\Delta P_t|, \kappa}(\tilde{\mathbf{z}}^t) e^{\sigma \Delta P_t 1(\Delta X_t > 0)} e^{-\Lambda \mu_{\tilde{h}_t}^\gamma(\mathbb{D})} \prod_{i \in \Delta P_t} \epsilon^{\alpha_i^2/2} e^{\alpha_i(\tilde{h}_t)_\epsilon(\tilde{z}_i^t)}, \quad (8-8) \end{aligned}$$

₃₁ where $t_1 < \dots < t_N$ are all the times satisfying $|\Delta X_{t_i}| > r$ and where $\tilde{f}$ is assumed to be measurable, take values in $[0, 1]$ and have the following properties:

- ₃₄ • The function $\tilde{f}$ is continuous in each of the fields with respect to the topology of $H^{-1}(\mathbb{D})$.
- ₃₅ • If $n \leq N$ and $\tilde{\mathbf{z}}^{t_n} = (\tilde{z}_i^{t_n} : i \in B)$ then $\tilde{f}$ takes the value 0 whenever

$$\min_{i \in B} (1 - |\tilde{z}_i^{t_n}|) \wedge (1 - |\psi_{t_n}(\tilde{z}_i^{t_n})|) \wedge |\psi_{t_n}'(\tilde{z}_i^{t_n})| \wedge \min_{\substack{i, j \in B \\ i \neq j}} |\tilde{z}_i^{t_n} - \tilde{z}_j^{t_n}| \wedge |\psi_{t_n}(\tilde{z}_i^{t_n}) - \psi_{t_n}(\tilde{z}_j^{t_n})| \leq 4\sqrt{r}.$$

- ₃₉_{1/2} • The function $\tilde{f}$ takes the value 0 if $\Delta P_{t_1} \cup \dots \cup \Delta P_{t_N} \neq A$.

Inserting (8-8) into (8-7), we will show that the left-hand side of (8-7) converges, as $\epsilon \rightarrow 0$, to

$$\int \tilde{f}(Y, (\tilde{h}_{t_n}, \tilde{z}^{t_n}, \psi_{t_n})_n) d\text{MC}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}(h, z, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) \quad (8-9)$$

and that the right-hand side of (8-7) converges, as $\epsilon \rightarrow 0$, to

$$\mathbb{E} \left(\int \sum_{Q \in \Pi(X^0, A)} \tilde{f}((L^0, R^0, Q), (\tilde{h}_{t_n}, \tilde{z}^{t_n}, \psi_{t_n})_n) \prod_{t \in J(X^0)} e^{\sigma_{\Delta Q_t} 1(\Delta X_t^0 > 0)} \right. \\ \left. \tilde{M}_{L^0, R^0, (\tilde{h}_t)_t}^{\sigma, \kappa}(d(\psi_t)_{t \in J(X^0)}) \prod_{t \in J(X^0)} M_{\Lambda, |\Delta X_t^0|}^{\sigma_{\Delta Q_t}, \kappa}(d\tilde{h}_t, d\tilde{z}^t) \right) \quad (8-10)$$

so (8-9) is equal to (8-10). Before proving the convergence results, we claim that Lemma 8.13 follows from the fact that (8-9) and (8-10) are equal. The identification of (8-9) and (8-10) gives by a monotone class argument and using the fact that the constant r was arbitrary that for any nonnegative, bounded, and measurable function $\tilde{f}$,

$$\int \tilde{f}(Y, (\tilde{h}_t, \tilde{z}^t)_t) d\text{MC}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}(h, z, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) \\ = \mathbb{E} \left(\int \sum_{Q \in \Pi(X^0, A)} \tilde{f}((L^0, R^0, Q), (\tilde{h}_t, \tilde{z}^t)_t) \prod_{t \in J(X^0)} e^{\sigma_{\Delta Q_t} 1(\Delta X_t^0 > 0)} M_{\Lambda, |\Delta X_t^0|}^{\sigma_{\Delta Q_t}, \kappa}(d\tilde{h}_t, d\tilde{z}^t) \right). \quad (8-11)$$

The lemma therefore follows by combining the following two observations (both of which follow directly from the definitions):

- If we sample $(h, z, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) \sim \text{MC}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}$ then the law of $[(\mathbb{D}, h, z, \Gamma_0, \lambda)]$ is $\overline{\text{MC}}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa}$.
- If $(\tilde{h}_t, \tilde{z}^t) \sim M_{\Lambda, |\Delta X_t|}^{\sigma_{\Delta Q_t}, \kappa}$ then the law of $[(\mathbb{D}, \tilde{h}_t, \tilde{z}^t)]$ is $\tilde{M}_{\Lambda, |\Delta X_t|}^{\sigma_{\Delta Q_t}, \kappa}$.

It remains to prove that the left- and right-hand sides of (8-7) converge to (8-9) and (8-10), respectively. Convergence of the right-hand side of (8-7) to (8-10) is immediate by Proposition 8.10 since we can check that the functions $(\tilde{\theta}_\epsilon^z)_{z, \epsilon}$ satisfy (8-3) with 0 instead of $o_\epsilon(1)$ and $\tilde{\theta}_\epsilon^z$ instead of θ_ϵ^z . It therefore remains to prove the following:

$$\text{the left-hand side of (8-7) converges to (8-9) as } \epsilon \rightarrow 0. \quad (8-12)$$

We will now prove (8-12). To this end, we suppose that $\tilde{f}(Y, (\tilde{h}_{t_n}, \tilde{z}^{t_n}, \psi_{t_n})_n) > 0$. For each $i \in A$ with $i \in \Delta P_t$, let

$$\delta_i = \epsilon |\psi'_t(\tilde{z}_i^t)|, \quad \theta_{\delta_i}^{z_i} = |\psi'_t \circ \psi_t^{-1}|^{-2} (\tilde{\theta}_\epsilon^{\tilde{z}_i^t} \circ \psi_t^{-1}).$$

Varying ϵ this defines $\theta_\delta^{z_i}$ for $\delta \in (0, r/2)$. Extend θ_δ^z for $z \in \mathbb{D} \setminus \{z_i : i \in A\}$ such that they satisfy (8-3) (with δ instead of ϵ) and set $h_\delta(z) = h(\theta_\delta^z)$.

We first argue that θ_δ^z satisfies the condition (8-3) (with δ instead of ϵ) of Proposition 8.10. Indeed, this follows from [Lawler 2005, Theorem 3.20], which implies that

$$\frac{|(\psi_t^{-1})'(z) - (\psi_t^{-1})'(\tilde{z}_i^t)|}{|(\psi_t^{-1})'(\tilde{z}_i^t)|} = o_{\delta_i}(1) \quad \text{for } |z - \tilde{z}_i^t| < \delta_i$$

as $\delta_i \rightarrow 0$ with the $o_{\delta_i}(1)$ deterministic. The first identity of Lemma 7.5, $\tilde{h}_t = h \circ \psi_t + Q_\gamma \log |\psi_t'|$, and the fact that $\log |\psi_t'|$ is harmonic yields that

$$\prod_{t \in J(X)} \Phi_{\mathbb{D}}^{\sigma|_{\Delta P_t}, \kappa}(\tilde{\mathbf{z}}^t) \prod_{i \in \Delta P_t} \epsilon^{\alpha_i^2/2} e^{\alpha_i(\tilde{h}_t)(\tilde{z}_i^t)} |\psi_t'(\tilde{z}_i^t)|^{-2} = \prod_{t \in J(X)} \Phi_{\Delta D_t}^{\sigma|_{\Delta P_t}, \kappa}(\mathbf{z}|_{\Delta P_t}) \prod_{i \in \Delta P_t} \delta_i^{\alpha_i^2/2} e^{\alpha_i h_{\delta_i}(z_i)}.$$

Let M_h denote the law of $(\Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)})$ given h if $(h, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) \sim \text{MC}_{\Lambda, \ell_L, \ell_R}^{-, \kappa}$. Note that M_h can be made sense of as soon as the γ -LQG boundary measure on the boundary of the domain and on the boundary of each of the components ΔD_t is well-defined. We will use Proposition 8.10 to argue that, as $\epsilon \rightarrow 0$,

$$\begin{aligned} & \int f_\epsilon(\mathbf{Y}, (\tilde{h}_{t_n}, \tilde{\mathbf{z}}^{t_n}, \psi_{t_n})_n) dM_h(\Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) M_{\Lambda, \ell}^{-, \kappa}(dh) dz \\ & \rightarrow \int \tilde{f}(\mathbf{Y}, (\tilde{h}_{t_n}, \tilde{\mathbf{z}}^{t_n}, \psi_{t_n})_n) \frac{\prod_{t \in J(X)} \Phi_{D_t}^{\sigma|_{\Delta P_t}, \kappa}(\mathbf{z}|_{\Delta P_t}) e^{\sigma_{\Delta P_t} 1(\Delta X_t > 0)}}{\Phi_{\mathbb{D}}^{\sigma, \kappa}(\mathbf{z})} \\ & dM_h(\Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) M_{\Lambda, \ell}^{\sigma, \kappa}(dh, d\mathbf{z}), \quad (8-13) \end{aligned}$$

which is sufficient to conclude the proof of (8-12) since the left-hand side of (8-13) is equal to the left-hand side of (8-7) and the right-hand side of (8-13) is equal to (8-9) by Lemma 7.6. To this end, define F by

$$F(h, \mathbf{z}, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) = \tilde{f}(\mathbf{Y}, (\tilde{h}_{t_n}, \tilde{\mathbf{z}}^{t_n}, \psi_{t_n})_n) \frac{\prod_{t \in J(X)} \Phi_{D_t}^{\sigma|_{\Delta P_t}, \kappa}(\mathbf{z}|_{\Delta P_t}) e^{\sigma_{\Delta P_t} 1(\Delta X_t > 0)}}{\Phi_{\mathbb{D}}^{\sigma, \kappa}(\mathbf{z})}.$$

We now want to apply Proposition 8.10 to this functional. In the proof above, we described the probability kernel used to sample $(\Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)})$ and (θ_δ^z) in (8-13) conditionally on $(h, \mathbf{z})$. The key observation is that this probability kernel only depends on $(h, \mathbf{z})$ through $\mathbf{z}$ and the restriction of h to the complement of $\bigcup_i B_r(z_i)$. This is indeed true since the target point of the CPI w_∞ is sampled from the γ -LQG boundary measure with respect to h which is measurable with respect to the restriction of h to the complement of $\bigcup_i B_r(z_i)$. Similarly, the points $(\mathbf{w}_t)_{t \in J(X)}$ are sampled from the γ -LQG boundary measures of the cut out components which are also measurable with respect to the restriction of h to the complement of $\bigcup_i B_r(z_i)$.

In order to prove (8-13) it is therefore sufficient to argue that the function F satisfies the assumptions appearing in Proposition 8.10.

Measurability of F is immediate, and boundedness follows from boundedness of $\tilde{f}$ and the upper and lower bounds at the end of Theorem 1.2. To argue that F satisfies the required continuity property, we consider a sequence (g_m^z) as in Proposition 8.10. By the support condition on (g_m^z) , adding the functions

to the field does not affect the process Y and the boundary lengths of the cut out components. Thus

$$F(h + g_m^z, z, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)}) = \tilde{f}(Y, (\tilde{h}_{t_n} + g_m^z \circ \psi_{t_n}, \tilde{z}^{t_n}, \psi_{t_n})_n) \cdot \frac{\prod_{t \in J(X)} \Phi_{D_t}^{\sigma|_{\Delta P_t}, \kappa}(z|_{\Delta P_t}) e^{\sigma_{\Delta P_t} 1(\Delta X_t > 0)}}{\Phi_{\mathbb{D}}^{\sigma, \kappa}(z)} \\ \rightarrow F(h, z, \Gamma_0, \lambda, (\mathbf{w}_t)_{t \in J(X)})$$

as $m \rightarrow \infty$ as required in [Proposition 8.10](#). $\square$

Proof of [Lemma 8.13](#) for $\kappa \in (4, 8)$. The proof for generalized disks is identical to the case of regular disks, except that a surface is represented by a countable collection of fields rather than a single field due to the difference in topology between the disks. We will therefore just state the counterparts of [\(8-6\)](#) and [\(8-11\)](#) in the setting of generalized disks and leave the details of the proof to the reader. In the following, we will use the index u to denote the time parameter of the excursion that encodes the generalized γ -LQG disk on which we have a CPI, we will use t to denote the time parameter of the exploration process, and we will use s to denote the time parameter of the excursions encoding the cut-out generalized γ -LQG disks. We write κ' instead of κ below.

We start by defining a measure $\text{MC}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa'}$ on tuples $(e, (h_u, z_u, \Gamma_u^0, \lambda_u), (\mathbf{w}_s^t)_{t,s})$.

- First sample $(e, (h_u, z_u)_{u \in J(e)})$ as in [Definition 8.7](#). Note that [Definition 8.7](#) gives a measure on (generalized) γ -LQG surfaces rather than fields but it can be immediately adapted to give a measure on fields.
- Then sample an independent copy Γ_u of $\text{CLE}_{\kappa'}^{\sigma|_B}(z_u)$ for each $u \in J(e)$ if z_u is indexed by the set B . Consider the CPI and write λ_u for the segment of the CPI if it goes through the component of the generalized γ -LQG disk corresponding to the index u (and $\lambda_u = -$ otherwise). Let Γ_u^0 denote the loops in Γ_u intersecting λ_u .
- Define $Y = (L, R, Q)$ and $X = L + R$ as in [Section 5D](#). For $t \in J(X)$, let e^t be the excursion encoding the cut out generalized γ -LQG disk, sample boundary points $\mathbf{w}_s^t$ and consider the mapping out functions ψ_s^t as in the simple case whenever $t \in J(X)$ and $s \in J(e^t)$. Define $\tilde{h}_s^t$ and $\tilde{z}_s^t$ analogously to the simple case.

In the case $A = \emptyset$ we write $(L^0, R^0, X^0, (E^t))$ instead of $(L, R, X, (e^t))$. The counterpart of [\(8-6\)](#) in the case of generalized disks, which follows from [\[Miller et al. 2021\]](#) (see again [Section 5D](#)), is then

$$\int f(L, R, (e^t)_t, (\tilde{h}_s^t, \psi_s^t)_{t,s}) d\text{MC}_{\Lambda, \ell_L, \ell_R}^{-, \kappa'}(e, (h_u, \Gamma_u, \lambda_u)_u, (\mathbf{w}_s^t)_{t,s}) \\ = \mathbb{E} \left(\int f(L^0, R^0, (E^t)_t, (\tilde{h}_s^t, \psi_s^t)_{t,s}) \tilde{M}_{L^0, R^0, (\tilde{h}_s^t)_{t,s}}(d(\psi_s^t)_{t,s}) \prod_{t,s} M_{\Lambda, |\Delta E_s^t|}^{-, \kappa'}(d\tilde{h}_s^t) \right),$$

where, every time we index over t, s, u , we mean $t \in J(X), s \in J(e^t), u \in J(e)$ on the left-hand side and $t \in J(X^0), s \in J(E^t)$ on the right-hand side.

Adding marked points via a limiting procedure as in the case of regular disks we get the following, which is the counterpart of (8-11) in the generalized case:

$$\int f(Y, (e^t)_t, (\tilde{h}_s^t, \tilde{z}_s^t)) d\text{MC}_{\Lambda, \ell_L, \ell_R}^{\sigma, \kappa'}(e, (h_u, z_u, \Gamma_u, \lambda_u)_u, (w_s^t)_{t,s}) \\ = \mathbb{E} \left(\int \sum_{Q, (Q^t)} f((L^0, R^0, Q), (E^t)_t, (\tilde{h}_s^t, \tilde{z}_s^t)_{t,s}) \tilde{M}_{L^0, R^0, (\tilde{h}_s^t)_{t,s}}(d(\psi_s^t)_{t,s}) \prod_{t,s} M_{\Lambda, |\Delta E_s^t|}^{\sigma|_{\Delta Q_s^t}, \kappa'}(d\tilde{h}_s^t, d\tilde{z}_s^t) \right),$$

where the sum is over $Q \in \Pi(X^0, A)$ and $Q^t \in \Pi(E^t, \Delta Q_t)$ for all t . By Definition 8.7 we deduce the result. $\square$

9. Recursive formula and finiteness of partition functions

We will put the result together from previous sections and conclude the proofs of Theorems 1.4, 1.7, and 1.11. Recall the definition of the weight functions in Definitions 8.5 and 8.7; see also (1-5) and (1-9). One simple property that we will use frequently is the following scaling property of the weight functions for marked disks. For $\kappa \in (8/3, 8) \setminus \{4\}$, $\Lambda \geq 0$, $\ell > 0$ and $\sigma \in \mathfrak{S}_\kappa^A$ we have

$$W_{\Lambda, \ell}^{\sigma, \kappa} = \ell^{(2/\sqrt{\kappa}) \sum_{i \in A} \alpha_i} W_{\Lambda \ell^{2 \wedge (8/\kappa)}, 1}^{\sigma|_{B}, \kappa}. \quad (9-1)$$

Throughout the following sections, to simplify notation we will introduce the shorthand notation

$$W_\Lambda^B(\ell) = W_{\Lambda, \ell}^{\sigma|_B, \kappa} \quad (9-2)$$

for $B \subseteq A$ when σ and κ are clear from the context.

9A. Recursive partition function formula and law of boundary length process. We conclude the proofs of Theorems 1.4 and 1.7. The following is an extended version of Theorem 1.4.

Theorem 9.1. *The assertions of Theorem 1.4 hold. The generator of Y is given by $\mathcal{G}_{\kappa, \beta}^W$ from Theorem 3.7 in the $\kappa < 4$ case and Theorem 3.15 in the $\kappa > 4$ case, where the weights $(W^B(\ell) : B \subseteq A, \ell > 0)$ of the latter theorems are given by $(W_{\Lambda, \ell}^{\sigma, \kappa} : B \subseteq A, \ell > 0)$.*

Proof. We first treat the case $\kappa \in (8/3, 4)$ and start by verifying that the weights $W_\Lambda^B(\ell)$ satisfy (3-8). Let us first mention that we have a monotonicity property, namely that $\Lambda \mapsto W_\Lambda^B(\ell)$ is decreasing. This and (9-1) immediately imply the first assertion of (3-8) (monotonicity). The second assertion of (3-8) (boundedness by 1) follows from the monotonicity property and since $W_0^\emptyset(\ell) = 1$. The third assertion of (3-8) is given as the second part of Corollary 8.9. The final assertion of (3-8) follows from (9-1) and the monotonicity property

$$W_\Lambda^B(x) = x^{(2/\sqrt{\kappa}) \sum_{i \in A} \alpha_i} W_{\Lambda x^2}^B(1) \leq x^{(2/\sqrt{\kappa}) \sum_{i \in A} \alpha_i} W_{\Lambda y^2}^B(1) \\ \lesssim y^{(2/\sqrt{\kappa}) \sum_{i \in A} \alpha_i} W_{\Lambda y^2}^B(1) = W_\Lambda^B(y) \quad \text{for } 0 < y \leq x \leq 2y.$$

By Proposition 8.11 and Theorem 5.9 we know that the process Y in Theorem 1.4 is equal to the process Y^W in Theorem 3.7 since it satisfies the defining property (3-9) of Y^W . It is therefore immediate

¹/₂ by the latter theorem and (3-8) that the generator of Y is given by $\mathcal{G}_{\kappa,\beta}^W$. The assertion about the jump rates of Y in Theorem 3.7 is now immediate from the definition of $\mathcal{G}_{\kappa,\beta}^W$. The final assertion of Theorem 1.4 follows from Proposition 8.11.

The proof in the case $\kappa \in (4, 8)$ is the same, except that we consider (3-21), Theorem 3.15 and Theorem 5.13 instead of (3-8), Theorem 3.7 and Theorem 5.9. When verifying (3-21) we again use (9-1) and the fact that the function $\Lambda \mapsto W_\Lambda^B(\ell)$ is decreasing, but the asymptotics of $W_\Lambda^\varnothing(\ell)$ as $\ell \rightarrow 0$ is different and given as part of Corollary 8.9. $\square$

We will now prove Theorem 1.7. The intuition behind the result is explained below the statement of the theorem in the introduction. One technical obstacle with turning this intuition into a proof is that one needs to exclude the possibility that the scenario where all the marked points lie in the same complementary component of the CPI in all the infinitely many iteration steps has positive mass. If we however consider a nested weighted CLE and discover the nested loops layer by layer, then it is clear that the points will be split after finitely many iterations. The key of the proof below is relating this CLE exploration back to the CPI exploration.

Proof of Theorem 1.7. We prove the result in the case $\kappa \in (8/3, 4)$. The nonsimple case where $\kappa \in (4, 8)$ is exactly the same. Let $(\ell_n : n \geq 1)$ denote the γ -LQG lengths of a nonnested CLE_κ which decorates a (regular) γ -LQG disk and let (L, R) be the boundary length process associated to a CPI in a γ -LQG disk.

For notational convenience, let us define an operator

$$\mathcal{A}_\pm h(\ell) := \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{t \in (0, \zeta) \\ \pm \Delta X_t > 0}} h(|\Delta X_t|) \prod_{\substack{s \in (0, \zeta) \setminus \{t\} \\ \Delta X_s \neq 0}} W_\Lambda^\varnothing(|\Delta X_s|) \right).$$

By Proposition 8.11 we have that, for any $B \subseteq A$,

$$W_\Lambda^B(\ell) = \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{Q \in \Pi(X, B)} \prod_{t < \zeta} e^{\sigma_{\Delta Q_t} 1(\Delta X_t > 0)} W_\Lambda^{\Delta Q_t}(|\Delta X_t|) \right). \quad (9-3)$$

Moreover, by Proposition 8.12 we get

$$W_\Lambda^B(\ell) = \mathbb{E}_\ell \left(\sum_{m \geq 1} \sum_{B_1 \cup \dots \cup B_m = B} \sum_{n_1 < \dots < n_m} \prod_{i=1}^m e^{\sigma_{B_i}} W_\Lambda^{B_i}(\ell_{n_i}) \prod_{n \notin \{n_1, \dots, n_m\}} W_\Lambda^\varnothing(\ell_n) \right), \quad (9-4)$$

where the second sum in the display ranges over all ordered partitions of B into m nonempty sets (this includes the case where $m = 1$ and there is just one element in the partition).

We now consider the (possibly infinite) measure $\bar{M}_{\Lambda, \ell}^{\sigma, \kappa}$ and decorate it with an independent nested CLE_κ^σ . Since this measure of γ -LQG disks with marked points carries zero mass on configurations where not all of the marked points are distinct, we can consider the first nesting level $j + 1$ for some $j \geq 0$ in the nested CLE_κ^σ where not all of the marked points lie in the same CLE loop. Splitting into all of the

$1^{1/2}$ possible values for j and then applying [Proposition 8.12](#) yields

$$\begin{aligned} W_{\Lambda}^A(\ell) &= \sum_{j \geq 0} g_j(\ell), \\ g_0(\ell) &= \mathbb{E}_{\ell} \left(\sum_{m \geq 1} \sum_{B_1 \cup \dots \cup B_m = A} \sum_{n_1 < \dots < n_m} \prod_{i=1}^m e^{\sigma_{B_i}} W_{\Lambda}^{B_i}(\ell_{n_i}) \prod_{n \notin \{n_1, \dots, n_m\}} W_{\Lambda}^{\emptyset}(\ell_n) \right), \\ g_j(\ell) &= \mathbb{E}_{\ell} \left(\sum_{n \geq 0} e^{\sigma_A} g_{j-1}(\ell_n) \prod_{n' \neq n} W_{\Lambda}^{\emptyset}(\ell_{n'}) \right). \end{aligned}$$

Note that here the sum over partitions does not include summing over the trivial partition which contains only one element. By [Corollary 5.12](#) we have for each $j \geq 1$ and $\ell > 0$ that

$$g_j(\ell) = \sum_{k_j \geq 0} e^{\sigma_A} \mathcal{A}_{-}^{k_j} \mathcal{A}_{+} g_{j-1}(\ell).$$

By using [Corollary 5.12](#) together with (9-4) we obtain

$$g_0(\ell) = \sum_{k_0 \geq 0} \mathcal{A}_{-}^{k_0} \tilde{g}_0(\ell), \quad \text{where } \tilde{g}_0(\ell) = \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{Q \in \Pi(X, A) \\ \Delta Q_t \subsetneq A \forall t < \zeta}} \prod_{t < \zeta} e^{\sigma_{\Delta Q_t} 1(\Delta X_t > 0)} W_{\Lambda}^{\Delta Q_t}(|\Delta X_t|) \right),$$

$20^{1/2}$ where k_0 represents the number of CPI iterations one needs to do before not all marked points lie in the same complementary component. Putting everything together yields

$$\begin{aligned} W_{\Lambda}^A(\ell) &= \sum_{j \geq 0} \sum_{k_0, \dots, k_j \geq 0} e^{\sigma_A j} \mathcal{A}_{-}^{k_j} \mathcal{A}_{+} \dots \mathcal{A}_{-}^{k_1} \mathcal{A}_{+} \mathcal{A}_{-}^{k_0} \tilde{g}_0(\ell) \\ &= \sum_{m \geq 0} \sum_{j \geq 0} \sum_{k_0 + \dots + k_j + j = m} e^{\sigma_A j} \mathcal{A}_{-}^{k_j} \mathcal{A}_{+} \dots \mathcal{A}_{-}^{k_1} \mathcal{A}_{+} \mathcal{A}_{-}^{k_0} \tilde{g}_0(\ell) = \sum_{m \geq 0} (\mathcal{A}_{-} + e^{\sigma_A} \mathcal{A}_{+})^m \tilde{g}_0(\ell). \end{aligned} \quad (9-5)$$

We will now show that $\tilde{g}_0(\ell) = V_{\Lambda, \ell}^0$ for all $\ell > 0$. To see this, we first sum over all possibilities for the first jump time t and the associated jump $\Delta Q_t = B$ and then combine this with the Markov property of (L, R) together with (9-3):

$$\begin{aligned} \tilde{g}_0(\ell) &= \sum_{\emptyset \neq B \subsetneq A} \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{t < \zeta \\ \Delta X_t \neq 0}} \prod_{s < t} W_{\Lambda}^{\emptyset}(|\Delta X_s|) e^{\sigma_B 1(\Delta X_t > 0)} W_{\Lambda}^B(|\Delta X_t|) \right. \\ &\quad \cdot \left. \sum_{\substack{Q \in \Pi(X, A \setminus B) \\ Q_t = A \setminus B}} \prod_{s \in (t, \zeta)} e^{\sigma_{\Delta Q_s} 1(\Delta X_s > 0)} W_{\Lambda}^{\Delta Q_s}(|\Delta X_s|) \right) \\ &= \sum_{\emptyset \neq B \subsetneq A} \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{t < \zeta \\ \Delta X_t \neq 0}} \prod_{s < t} W_{\Lambda}^{\emptyset}(|\Delta X_s|) e^{\sigma_B 1(\Delta X_t > 0)} W_{\Lambda}^B(|\Delta X_t|) W_{\Lambda}^{A \setminus B}(X_t) \right) \\ &= V_{\Lambda, \ell}^0. \end{aligned}$$

Let us comment in slightly more detail on the second equality above. One first turns the sum over the t values into an integral with the use of the final statement in [Lemma 3.6](#), then one applies (9-3) and finally, one again uses [Lemma 3.6](#) to turn the integration back into a sum over jump times. At an intuitive level, we are applying the Markov property at the time t , however we need to reason slightly differently since the time t appears in a sum within the expectation. We leave the details to the reader.

A similar reasoning as the one in the previous paragraph yields that

$$\mathcal{A}_{\pm} h(\ell) := \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{\substack{t \in (0, \zeta) \\ \pm \Delta X_t > 0}} h(|\Delta X_t|) \prod_{\substack{s < t \\ \Delta X_s \neq 0}} W_{\Lambda}^{\varnothing}(|\Delta X_s|) \cdot W_{\Lambda}^{\varnothing}(X_t) \right)$$

and the statement then follows from (9-5) and the result that $\tilde{g}_0(\ell) = V_{\Lambda, \ell}^0$ for all $\ell > 0$. $\square$

9B. Finiteness of the partition function: Proof of [Theorem 1.11](#). The goal of this subsection is to prove [Theorem 1.11](#), which gives a partial characterization of which parameters are admissible. The theorem is split up into the Lemmas [9.3](#) and [9.4](#) below.

Our proof is based on the recursive expression for the partition function in [Theorem 1.7](#), along with basic monotonicity, scaling and continuity properties of the partition function. We also use Propositions [3.12](#) and [3.16](#) in the proof of (ii). The proof of (iv) in [Lemma 9.4](#) will be particularly involved; see the paragraph before it for a brief outline. For the proof of [Lemma 9.3](#) the following direct consequence of [Theorem 1.7](#) will suffice.

Lemma 9.2. *Suppose that $\#A > 1$. For any $(\sigma, \Lambda) \in \mathfrak{S}_{\kappa}^A$,*

$$W_{\Lambda}^A(\ell) = \sum_{\varnothing \neq B \subseteq A} \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{t < \zeta} e^{\sigma_B 1(\Delta X_t > 0)} \prod_{s < t} W_{\Lambda}^{\varnothing}(|\Delta X_s|) W_{\Lambda}^B(|\Delta X_t|) W_{\Lambda}^{B \setminus A}(X_t) \right). \quad (9-6)$$

In particular, each term on the right-hand side is strictly smaller than the left-hand side.

Proof. This identity follows immediately by using the expression for $W_{\Lambda}^A(\ell)$ in [Theorem 1.7](#). To be more precise, note that it is sufficient to argue that the term on the right-hand side of (9-6) corresponding to $B = A$ is equal to $\sum_{m=1}^{\infty} V_{\Lambda}^m(|\Delta X_t|)$, and we get this by replacing $W_{\Lambda}^A(|\Delta X_t|)$ in the former expression by $\sum_{m=0}^{\infty} V_{\Lambda}^m(|\Delta X_t|)$ and then using the recursive definition of $V_{\Lambda}^m(|\Delta X_t|)$ in [Theorem 1.7](#). $\square$

The following lemma gives [Theorem 1.11](#) except assertion (iv).

Lemma 9.3. *Assertions (i), (ii), (iii), (v), (vi) and (vii) of [Theorem 1.11](#) hold.*

Proof. Assertion (i) of the proposition is immediate from [Theorem 1.8](#) and assertion (vi) is immediate from [Definition 8.1](#) since $W_{\Lambda, \ell}^{\sigma, \kappa}$ is monotone in σ_B for $\#B \geq 2$.

We continue by proving (ii), i.e., if $\#A \geq 2$ and $\Lambda = 0$ then $(\sigma, \Lambda) \notin \mathcal{A}_{\kappa}$. By the final assertion of [Lemma 9.2](#), the fact that $W_0^{\varnothing}(\ell) = 1$ and

$$W_0^A(\ell) = \ell^{(2/\sqrt{\kappa}) \sum_{i \in A} \alpha_i} W_0^A(1)$$

$1^{1/2}$ we have that

$$W_{\Lambda}^A(1) > e^{\sigma_A} W_{\Lambda}^A(1) \mathbb{E}_{(1/2, 1/2)} \left(\sum_{\substack{t < \zeta \\ \Delta X_t > 0}} |\Delta X_t|^{(2/\sqrt{\kappa}) \sum_i \alpha_i} \right). \quad (9-7)$$

By Definition 1.1, $\alpha_i \cdot 2/\sqrt{\kappa} > 4/\kappa + 1/2$, so $(2/\sqrt{\kappa}) \cdot \sum_i \alpha_i > 1 + 8/\kappa$. This implies via an application of Proposition 3.12 (when $\kappa \in (8/3, 4)$) and of Proposition 3.16 (when $\kappa \in (4, 8)$) that the right-hand side of (9-7) is infinite, which gives (ii).

Next we prove (v), i.e., if $(\sigma, \Lambda) \in \mathcal{A}_{\kappa}$ for $\Lambda > 0$ then $(\sigma, \Lambda') \in \mathcal{A}_{\kappa}$ for any $\Lambda' > 0$. It is immediate from Definition 8.1 that $W_{\Lambda}^A(1)$ is decreasing in Λ , so if (v) was not true then there would be some $\Lambda_0 \in (0, \infty)$ such that $W_{\Lambda}^A(1) < \infty$ for all $\Lambda > \Lambda_0$ and $W_{\Lambda}^A(1) = \infty$ for all $\Lambda < \Lambda_0$. We suppose such a Λ_0 exists and want to derive a contradiction. Suppose $\Lambda > \Lambda_0$. Then the right-hand side of (9-6) is infinite since it holds with probability 1 that there is a $t \in (0, \zeta)$ for which $\Lambda |\Delta X_t|^{2 \wedge (8/\kappa)} < \Lambda_0$ and since we have

$$W_{\Lambda}^A(|\Delta X_t|) = |\Delta X_t|^{(2/\sqrt{\kappa}) \sum_i \alpha_i} W_{\Lambda |\Delta X_t|^{2 \wedge (8/\kappa)}}^A(1) = \infty$$

for such $t \in (0, \zeta)$. Therefore $W_{\Lambda}^A(1) = \infty$, which is a contradiction to $\Lambda > \Lambda_0$, and we conclude (v).

Now we prove (iii), i.e., if $(\sigma, \Lambda) \in \mathcal{A}_{\kappa}$ and $B \subseteq A$ then $(\sigma|_B, \Lambda) \in \mathcal{A}_{\kappa}$. We will prove the contrapositive, namely that $(\sigma|_B, \Lambda) \notin \mathcal{A}_{\kappa}$ implies $(\sigma, \Lambda) \notin \mathcal{A}_{\kappa}$. By (v) and the scaling property we have that, for each $\ell > 0$,

$$W_{\Lambda}^B(\ell) < \infty \quad \text{if and only if} \quad (\sigma|_B, \Lambda) \in \mathcal{A}_{\kappa}.$$

Therefore the right-hand side of (9-6) is infinite if $(\sigma|_B, \Lambda) \notin \mathcal{A}_{\kappa}$, so (9-6) gives that $(\sigma, \Lambda) \notin \mathcal{A}_{\kappa}$. This concludes the proof of (iii).

Finally, we prove (vii), i.e., if σ' and $B' \subseteq A$ are such that $\#B' \geq 2$ and $\sigma'_B = \sigma_B$ for all $B \subseteq A$, $B \neq B'$, then by choosing $\sigma'_{B'}$ sufficiently large we have $(\sigma', \Lambda) \notin \mathcal{A}_{\kappa}$. In the following we will not use the simplified notation for the weight functions from (9-2) since the dependence on σ is important. By (iii) it is sufficient to consider the case $B' = A$. The case $(\sigma, \Lambda) \notin \mathcal{A}_{\kappa}$ is immediate since $W_{\Lambda, \ell}^{\sigma', \kappa}$ is increasing in σ'_B , so suppose $(\sigma, \Lambda) \in \mathcal{A}_{\kappa}$. Since $W_{\Lambda, 1}^{\sigma', \kappa}$ increases as Λ decreases, for all $\ell \in (1/2, 1)$,

$$W_{\Lambda, \ell}^{\sigma', \kappa} = \ell^{(2/\sqrt{\kappa}) \sum_i \alpha_i} W_{\Lambda \ell^{2 \wedge (8/\kappa)}, 1}^{\sigma', \kappa} > 2^{-(2/\sqrt{\kappa}) \sum_i \alpha_i} W_{\Lambda, 1}^{\sigma', \kappa}.$$

Applying Lemma 9.2 and this estimate we get

$$\begin{aligned} W_{\Lambda, 1}^{\sigma', \kappa} &> e^{\sigma'_A} \mathbb{E}_{(1/2, 1/2)} \left(\sum_{\substack{t < \zeta \\ \Delta X_t \in (1/2, 1)}} \prod_{s < t} W_{\Lambda, |\Delta X_s|}^{-, \kappa} W_{\Lambda, |\Delta X_t|}^{\sigma', \kappa} W_{\Lambda, X_t}^{-, \kappa} \right) \\ &> 2^{-(2/\sqrt{\kappa}) \sum_i \alpha_i} W_{\Lambda, 1}^{\sigma', \kappa} e^{\sigma'_A} \mathbb{E}_{(1/2, 1/2)} \left(\sum_{\substack{t < \zeta \\ \Delta X_t \in (1/2, 1)}} \prod_{s < t} W_{\Lambda, |\Delta X_s|}^{-, \kappa} W_{\Lambda, X_t}^{-, \kappa} \right). \end{aligned}$$

1^{1/2} It follows that $(\sigma', \Lambda) \notin \mathcal{A}_k$ if

$$2^{-(2/\sqrt{\kappa}) \sum_i \alpha_i} e^{\sigma'_A} \mathbb{E}_{(1/2, 1/2)} \left(\sum_{\substack{t < \zeta \\ \Delta X_t \in (1/2, 1)}} \prod_{s < t} W_{\Lambda, |\Delta X_s|}^{-, \kappa} W_{\Lambda, X_t}^{-, \kappa} \right) > 1,$$

which we can obtain by choosing σ'_A sufficiently large since the expectation on the left-hand side is positive and does not depend on σ'_A . This implies (vii). $\square$

It remains to prove assertion (iv) of [Theorem 1.11](#), which will be done in [Lemma 9.4](#). The idea of the proof is to split the γ -LQG disk with marked points into components containing just one marked point each (by decorating it with a weighted CLE) using [Theorem 1.7](#) and then decoupling the individual disks at the cost of decreasing the cosmological constant Λ .

Lemma 9.4. *Assertion (iv) of [Theorem 1.11](#) holds, that is, if $\sigma_B < \sigma_i$ whenever $i \in B \subseteq A$ and $\#B > 1$ then $(\sigma, \Lambda) \in \mathcal{A}_k$ for all $\Lambda > 0$.*

Proof. The proof will proceed via induction on $\#B$. We will show that there are constants $\rho_{B, \Lambda} < \infty$ and $0 < \Lambda_{B, \Lambda} < \Lambda$ such that for all $\ell > 0$, we have

$$W_{\Lambda}^B(\ell) \leq \rho_{B, \Lambda} \sum_{i \in B} W_{\Lambda_{B, \Lambda}}^{\{i\}}(\ell) \quad \text{and} \quad W_{\Lambda}^B(\ell) \leq \rho_{B, \Lambda} W_{\Lambda_{B, \Lambda}}^{\varnothing}(\ell). \quad (9-8)$$

This implies the lemma since the right-hand sides are finite by assertion (i) of [Theorem 1.11](#).

We start by verifying the induction hypothesis $\#B = 1$. The first identity of (9-8) is trivial in this case and the second one appears as the first statement of [Corollary 8.9](#).

For the induction step, we use the induction hypothesis to get, for $\Lambda_{B, \Lambda} = \min\{\Lambda_{C, \Lambda} : \varnothing \neq C \subsetneq B\}$ and $V_{\Lambda, m}^B(\ell)$ just as $V_{\Lambda, \ell}^m$ in [Theorem 1.7](#) but with A replaced by $B \subseteq A$,

$$\begin{aligned} V_{\Lambda, 0}^B(\ell) &\leq e^{\sigma^*} \sum_{\varnothing \neq C \subsetneq B} \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{t < \zeta} \prod_{s < t} W_{\Lambda}^{\varnothing}(|\Delta X_s|) W_{\Lambda}^C(|\Delta X_t|) W_{\Lambda}^{B \setminus C}(X_t) \right) \\ &\leq e^{\sigma^*} \sum_{\varnothing \neq C \subsetneq B} \rho_{B \setminus C, \Lambda} \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{t < \zeta} \prod_{s < t} W_{\Lambda}^{\varnothing}(|\Delta X_s|) W_{\Lambda}^C(|\Delta X_t|) W_{\Lambda_{B \setminus C, \Lambda}}^{\varnothing}(X_t) \right) \\ &\leq e^{\sigma^*} \sum_{\varnothing \neq C \subsetneq B} \rho_{B \setminus C, \Lambda} e^{\sigma^*} W_{\Lambda_{B, \Lambda}}^C(\ell), \end{aligned}$$

where $\sigma^* = \max\{|\sigma_C| : C \subseteq A\}$. By the induction hypothesis (9-8), we therefore get that there exists $\rho'_{B, \Lambda} < \infty$ such that

$$V_{\Lambda, 0}^B(\ell) \leq \rho'_{B, \Lambda} \sum_{i \in B} W_{\Lambda_{B, \Lambda}}^{\{i\}}(\ell). \quad (9-9)$$

For $i \in B$ we now define

$$U_0^{B, i}(\ell) := W_0^{\{i\}}(\ell), \quad U_{n+1}^{B, i}(\ell) := \mathbb{E}_{(\ell/2, \ell/2)} \left(\sum_{t < \zeta} e^{\sigma_B 1(\Delta X_t > 0)} U_n^{B, i}(|\Delta X_t|) \right).$$

¹/₂ Since $\sigma_B < \sigma_i$ for $i \in B$ with $\#B > 1$ we get

$$U_1^{B,i}(1) < \mathbb{E}_{(1/2,1/2)} \left(\sum_{t < \zeta} e^{\sigma_i 1(\Delta X_t > 0)} W_0^{[i]}(|\Delta X_t|) \right) = W_0^{[i]}(1).$$

We can therefore define $\alpha_{B,i} := W_0^{[i]}(1)/U_1^{B,i}(1) \in (0, 1)$. By scaling, for $\ell > 0$,

$$\mathbb{E}_{(\ell/2,\ell/2)} \left(\sum_{t < \zeta} e^{\sigma_B 1(\Delta X_t > 0)} W_0^{[i]}(|\Delta X_t|) \right) = U_1^{B,i}(\ell) = \alpha_{B,i} W_0^{[i]}(\ell).$$

To proceed, we will argue two statements, each of which is established using induction on n . Note that the induction on $\#B$ above is not complete, and the induction step of that proof requires the further inductions on n that we do right below.

First we argue by induction on n that

$$U_n^{B,i}(\ell) = (\alpha_{B,i})^n W_0^{[i]}(\ell) \quad (9-10)$$

for all $\ell > 0$ and $n \geq 0$. The cases $n = 0, 1$ are immediate, and assuming the assertion is true for $n - 1$ (with $n \geq 2$) we get that it also holds for n since by first applying the definition of $U_n^{B,i}(\ell)$, then using the induction hypothesis, and then using (9-12)

$$U_n^{B,i}(\ell) = \mathbb{E}_{(\ell/2,\ell/2)} \left(\sum_{t < \zeta} e^{\sigma_B 1(\Delta X_t > 0)} U_{n-1}^{B,i}(|\Delta X_t|) \right) \quad (9-11)$$

$$= (\alpha_{B,i})^{n-1} \mathbb{E}_{(\ell/2,\ell/2)} \left(\sum_{t < \zeta} e^{\sigma_i 1(\Delta X_t > 0)} W_0^{[i]}(|\Delta X_t|) \right) = (\alpha_{B,i})^n W_0^{[i]}(\ell), \quad (9-12)$$

which concludes the proof of (9-10) by induction.

Next we argue via yet another induction on n that

$$V_{\Lambda,n}^B(\ell) \leq \rho'_{B,\Lambda} \sum_{i \in B} U_n^{B,i}(\ell) \quad (9-13)$$

for $n \geq 0, \ell > 0$. The case $n = 0$ is (9-9). Assuming the induction hypothesis has been proved for n we see that it also holds for $n + 1$ since

$$\begin{aligned} V_{\Lambda,n+1}^B(\ell) &= \mathbb{E}_{(\ell/2,\ell/2)} \left(\sum_{t < \zeta} \prod_{s \neq t} W_{\Lambda}^{\varnothing}(|\Delta X_s|) e^{\sigma_B 1(\Delta X_t > 0)} V_{\Lambda,n}^B(|\Delta X_t|) \right) \\ &\leq \sum_{i \in B} \rho'_{B,\Lambda} \mathbb{E}_{(\ell/2,\ell/2)} \left(\sum_{t < \zeta} e^{\sigma_B 1(\Delta X_t > 0)} U_n^{B,i}(|\Delta X_t|) \right) = \rho'_{B,\Lambda} \sum_{i \in B} U_{n+1}^{B,i}(\ell), \end{aligned}$$

where we used $W_{\Lambda}^{\varnothing}(|\Delta X_s|) \leq 1$. This completes the proof of (9-13) by induction.

Combining the above, we get

$$W_{\Lambda}^{\sigma|_B}(\ell) = \sum_{n \geq 0} V_{\Lambda,n}^B(\ell) \leq \rho'_{B,\Lambda} \sum_{i \in B} \sum_{n \geq 0} U_n^{B,i}(\ell) = \rho'_{B,\Lambda} \sum_{i \in B} \sum_{n \geq 0} (\alpha_{B,i})^n W_0^{[i]}(\ell) < \infty$$

¹/₂ and the induction step of the main induction (where we prove (9-8)) is done by setting

$$\rho_{B,\Lambda} = \rho'_{B,\Lambda} \sum_{i \in B} \frac{1}{1 - \alpha_{B,i}}. \quad \square$$

Appendix A: A result on renewal processes

This section establishes an asymptotic result on exponentials of renewal processes, namely [Proposition A.1](#) right below. While we think that this addresses a very natural question which has likely appeared earlier in other contexts, we were not able to locate similar statements in existing literature. For the statement below, recall the definition

$$\mathcal{L}f(\lambda) = \int_0^\infty e^{-\lambda t} f(t) dt$$

of the Laplace transform of a function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$.

Proposition A.1. *Let (X_i) be an i.i.d. sequence of random variables in $(0, \infty)$ with density f and cumulative distribution function F and suppose that*

$$\sup_{t \geq 0} \frac{f(t)}{1 - F(t)} < \infty \quad \text{and} \quad \inf_{t \geq 0} \frac{F(t+c) - F(t)}{1 - F(t)} > 0 \quad \text{for some } c > 0.$$

²⁰/₂ Suppose that $\mathcal{L}f(\rho_0)$ exists for some $\rho_0 \in \mathbb{R}$. Assume $\rho > \rho_0$ with $\rho \neq 0$ is such that $\mathcal{L}f(\rho + iy) \neq \mathcal{L}f(\rho)$ for all $y \in \mathbb{R} \setminus \{0\}$. Let $\theta = -\log \mathcal{L}f(\rho) \in \mathbb{R}$ and define N to be the renewal process associated to X , that is,

$$N_t = \sup\{n \geq 0 : X_1 + \cdots + X_n \leq t\} \quad \text{for } t \geq 0.$$

Then

$$\beta(t) := \mathbb{E}(e^{\theta N_t - \rho t}) \rightarrow \frac{e^{-\theta}(1 - e^{-\theta})/\rho}{-(\mathcal{L}f)'(\rho)} \quad \text{as } t \rightarrow \infty$$

and $\beta(t) < \infty$ for all $t \geq 0$.

The proof of the proposition uses as an essential ingredient a Tauberian result which relates the asymptotic behavior of a function to the asymptotic behavior of its Laplace transform. A proof of the theorem below is given in [\[Korevaar 2004, Chapter III, Theorem 7.1\]](#).

Theorem A.2 [\[Korevaar 2004\]](#). *Let $\alpha : [0, \infty) \rightarrow \mathbb{R}$ be measurable and bounded so that the Laplace transform $\mathcal{L}\alpha$ exists on $\{z \in \mathbb{D} : \Re z > 0\}$ and suppose that $\mathcal{L}\alpha$ has an analytic extension to a neighborhood of each point on the imaginary axis which we denote by $\mathcal{L}\alpha$ too. Then*

$$\int_0^t \alpha(s) ds \rightarrow \mathcal{L}\alpha(0) \quad \text{as } t \rightarrow \infty.$$

³⁹/₂ *Proof of Proposition A.1.* The idea of the proof will be to observe the process N at a random independent exponentially distributed time $T_\lambda \sim \text{Exp}(\lambda)$ for $\lambda > 0$ and deduce the result using the Tauberian theorem

$1^{1/2}$ stated in [Theorem A.2](#). We begin by observing that for $\lambda + \rho \neq 0$ we have

$$\begin{aligned}
 \mathbb{E}(e^{\theta N_{T_\lambda} - \rho T_\lambda}) &= \sum_{n \geq 0} e^{\theta n} \mathbb{E}(e^{-\rho T_\lambda}; T_\lambda \in [X_1 + \dots + X_n, X_1 + \dots + X_{n+1})) \\
 &= \frac{\lambda}{\lambda + \rho} \sum_{n \geq 0} e^{\theta n} \mathbb{E}(e^{-(\lambda + \rho)(X_1 + \dots + X_n)} - e^{-(\lambda + \rho)(X_1 + \dots + X_{n+1})}) \\
 &= \frac{\lambda}{\lambda + \rho} \sum_{n \geq 0} (1 - \mathcal{L}f(\lambda + \rho))(e^\theta \mathcal{L}f(\lambda + \rho))^n \\
 &= \frac{\lambda}{\lambda + \rho} \frac{1 - \mathcal{L}f(\lambda + \rho)}{1 - e^\theta \mathcal{L}f(\lambda + \rho)} \rightarrow \frac{e^{-\theta}(1 - e^{-\theta})/\rho}{-(\mathcal{L}f)'(\rho)} \quad \text{as } \lambda \rightarrow 0, \tag{A-1}
 \end{aligned}$$

noting that $\mathcal{L}f(\lambda + \rho) < \mathcal{L}f(\rho) = e^{-\theta}$. If $\lambda = -\rho$ we get by an analogous computation that

$$\mathbb{E}(e^{\theta N_{T_\lambda} - \rho T_\lambda}) = \frac{-\lambda(\mathcal{L}f)'(0)}{1 - e^\theta}.$$

Our goal is to show that the limit in [\(A-1\)](#) (as $\lambda \rightarrow 0$) is the same as the limit of $\beta(t)$ as $t \rightarrow \infty$; this is intuitive since $T_\lambda \rightarrow \infty$ in distribution as $\lambda \rightarrow 0$. In the last step, we will apply [Theorem A.2](#) with $\alpha = \beta'$ to conclude the proof.

The technical bulk of the proof will be establishing the boundedness of β and its derivative β' . Fix $t \geq 0$. By the memoryless property of T_λ , we can bound

$$\begin{aligned}
 \mathbb{E}(e^{\theta N_{T_\lambda} - \rho T_\lambda}) &\geq \mathbb{E}(e^{\theta N_{T_\lambda} - \rho T_\lambda}; T_\lambda \geq t + c) \\
 &= \mathbb{E}(e^{\theta N_{T_\lambda + (t+c)} - \rho(T_\lambda + (t+c))}) \mathbb{P}(T_\lambda \geq t + c). \tag{A-2}
 \end{aligned}$$

To simplify notation, let us write $J_n = X_1 + \dots + X_n$ for the jump times. The key observation which follows directly from the definition is that the conditional law of J_{N_t+1} given (N_t, J_{N_t}) has density $f(\cdot - J_{N_t})1_{(t, \infty)} / (1 - F(t - J_{N_t}))$, so by the renewal property of N at time J_{N_t+1} , for any $\tau \geq 0$,

$$\begin{aligned}
 \mathbb{E}(e^{\theta N_{\tau+(t+c)}}) &\geq \mathbb{E}\left(\frac{e^{\theta N_t}}{1 - F(t - J_{N_t})} \int_t^{\tau+t+c} f(s - J_{N_t}) e^\theta \mathbb{E}(e^{\theta N_{\tau+t+c-s}}) ds\right) \\
 &\geq \mathbb{E}\left(\frac{e^{\theta N_t}}{1 - F(t - J_{N_t})} \int_t^{t+c} f(s - J_{N_t}) e^\theta \mathbb{E}(e^{\theta N_{\tau+t+c-s}}) ds\right) \\
 &\geq e^\theta \mathbb{E}(e^{\theta N_\tau}) \mathbb{E}\left(e^{\theta N_t} \frac{F(t+c - J_{N_t}) - F(t - J_{N_t})}{1 - F(t - J_{N_t})}\right) \\
 &\geq C \cdot \mathbb{E}(e^{\theta N_\tau}) \mathbb{E}(e^{\theta N_t})
 \end{aligned}$$

using the lower bound from the assumptions where C is a constant depending only on θ , f and c .

Therefore [\(A-2\)](#) and the above imply that

$$\begin{aligned}
 \mathbb{E}(e^{\theta N_{T_\lambda} - \rho T_\lambda}) &\geq \mathbb{E}(\mathbb{E}(e^{\theta N_{\tau+(t+c)}})|_{\tau=T_\lambda} e^{-\rho(T_\lambda + (t+c))}) \mathbb{P}(T_\lambda \geq t + c) \\
 &\geq C e^{-\rho c} \mathbb{E}(e^{\theta N_{T_\lambda} - \rho T_\lambda}) \mathbb{E}(e^{\theta N_t - \rho t}) \mathbb{P}(T_\lambda \geq t + c).
 \end{aligned}$$

¹/₂ Since $\mathbb{P}(T_\lambda \geq t + c) = e^{-1}$ for $\lambda = 1/(t + c)$ and the left-hand side of the above estimate is finite, we can cancel the two terms to obtain the required uniform bound on β . To establish the existence and find a uniform bound for β' , let us first write

$$\mathbb{E}(e^{\theta N_t}) = \sum_{n \geq 0} e^{\theta n} \mathbb{P}(t \in [J_n, J_{n+1})) = \sum_{n \geq 1} e^{\theta n} \int_t^\infty (1 - F(t - s)) f^{*n}(s) ds + 1 - F(t),$$

where f^{*n} is the density of J_n . Note that each of the summands is differentiable, indeed a straightforward computation shows that

$$\frac{d}{dt} \int_t^\infty (1 - F(t - s)) f^{*n}(s) ds = f^{*(n+1)}(t) - f^{*n}(t).$$

Moreover, by assumption there is $C' < \infty$ such that

$$\begin{aligned} \sum_{n \geq 0} e^{\theta n} f^{*(n+1)}(t) &= \sum_{n \geq 0} e^{\theta n} \mathbb{E}(f(t - J_n); J_n \geq t) = \sum_{n \geq 0} e^{\theta n} \mathbb{E}\left(\frac{f(t - J_n)}{1 - F(t - J_n)}; t \in [J_n, J_{n+1})\right) \\ &\leq C' \sum_{n \geq 0} e^{\theta n} \mathbb{P}(t \in [J_n, J_{n+1})) = C' \mathbb{E}(e^{\theta N_t}). \end{aligned}$$

From this, it follows directly that β is differentiable and $|\beta'|$ is bounded.

We therefore infer that, for any $\lambda > 0$,

$$\begin{aligned} \mathcal{L}(\beta')(\lambda) &= \int_0^\infty e^{-\lambda t} \beta'(t) dt = \int_0^\infty \lambda e^{-\lambda t} \beta(t) dt - 1 = \mathbb{E}(e^{\theta N_{T_\lambda}} - \rho T_\lambda) - 1 \\ &= -1 + \frac{\lambda}{1 - e^\theta \mathcal{L}f(\lambda + \rho)} \cdot \begin{cases} \frac{1 - \mathcal{L}f(\lambda + \rho)}{\lambda + \rho} & \lambda + \rho \neq 0, \\ -(\mathcal{L}f)'(0) & \lambda + \rho = 0. \end{cases} \end{aligned}$$

We will show that the right-hand side extends to an analytic function on a neighborhood of $\{\lambda \in \mathbb{C} : \Re \lambda \geq 0\}$ by defining the extension at 0 to be

$$\mathcal{L}(\beta')(0) = -1 + \frac{e^{-\theta}(1 - e^{-\theta})/\rho}{-(\mathcal{L}f)'(\rho)}.$$

Our strategy is to show that the right-hand side of the identity above defines an analytic function on a set of the form $U \setminus \{0\}$ where U is an open neighborhood of $\{\lambda \in \mathbb{C} : \Re \lambda \geq 0\}$. This will yield the claim since by (A-1) the function has a removable singularity at 0.

To this end, note that since $\rho > \rho_0$, the function $\mathcal{L}f(\cdot + \rho)$ is an analytic function on a neighborhood of $\{\lambda \in \mathbb{C} : \Re \lambda \geq 0\}$. In order to conclude, it is now sufficient to argue that $e^\theta \mathcal{L}f(\lambda + \rho) \neq 1$ for all $\lambda \in U$ for some set U as above. This property is satisfied since for all $\lambda \in \mathbb{C}$ with $\Re \lambda > 0$ we have $|\mathcal{L}f(\lambda + \rho)| < \mathcal{L}f(\rho) = e^{-\theta}$ as f is a probability density function from $(0, \infty)$ to $[0, \infty)$ and moreover we may use the assumption that $\mathcal{L}f(\rho + \lambda) \neq \mathcal{L}f(\rho)$ for each $\lambda \neq 0$ with $\Re \lambda = 0$.

The result now follows since

$$\beta(t) = \int_0^t \beta'(s) ds + 1 \rightarrow \mathcal{L}(\beta')(0) + 1 = \frac{e^{-\theta}(1 - e^{-\theta})/\rho}{-(\mathcal{L}f)'(\rho)} \quad \text{as } t \rightarrow \infty,$$

¹/₂ where the convergence part follows from [Theorem A.2](#) applied with $\alpha = \beta'$ and where $\mathcal{L}(\beta')(0)$ denotes the analytic continuation of $\mathcal{L}(\beta')$ to 0. $\square$

Appendix B: Integral computations

This paper, and in particular the study of Lévy processes here, relies on the explicit computations of various complicated integrals. These computations could not be located by us in the literature and have therefore been included in this appendix. [Lemma B.1](#) is the key computation that will be used in [Section 4](#) to explicitly compute functionals of Lévy excursions and [Lemmas B.3](#) and [B.4](#) are used in [Sections 3A](#) and [3B](#), respectively, to evaluate functionals associated to the spatially inhomogeneous Markov processes discussed in these sections.

The integrals are rather involved and it is not clear if there exist explicit closed-form antiderivatives of the integrands; the computations rely on the fact that we consider indefinite integrals. Several computations involve the integral of sums of terms which are not integrable individually. A key trick we will use multiple times is to evaluate such integrals via integration by parts. In [Lemma B.1](#) we further rely on Cauchy's residue theorem.

We start with the computation of certain Bessel integrals. Recall the definition of the modified Bessel functions of the second kind:

$$K_\nu(x) = \int_0^\infty e^{-x \cosh(t)} \cosh(\nu t) dt = \frac{1}{2(x/2)^\nu} \int_0^\infty e^{-(x/2)^2 y - 1/y} \frac{dy}{y^{1+\nu}}. \quad (\text{B-1})$$

It is easy to see that $2(x/2)^\nu K_\nu(x) \rightarrow \Gamma(\nu)$ as $x \downarrow 0$. Also, the derivative of $(x/2)^\nu K_\nu(x)$ (a smooth function) with respect to x is $-(x/2)^\nu K_{\nu-1}(x)$.

Lemma B.1. *Suppose that $\alpha \in (0, 1)$, $\beta \in (0, 2) \setminus \{1\}$ and $\rho > -1$. We write $\rho = \cos \theta$ for $\theta \in [0, \pi)$ when $\rho \in (-1, 1]$ and $\rho = \cosh \theta$ for $\theta > 0$ when $\rho > 1$. Then*

$$\int_0^\infty \frac{\cosh(\alpha t) dt}{\cosh(t) + \rho} = \frac{\pi}{\sin(\pi \alpha)} \cdot \begin{cases} \sin \alpha \theta / \sin \theta & \rho \in (-1, 1), \\ \alpha & \rho = 1, \\ \sinh \alpha \theta / \sinh \theta & \rho > 1, \end{cases}$$

$$\int_0^\infty \frac{\cosh(\beta t) dt}{(\cosh(t) + \rho)^2} = \frac{\pi}{\sin(\pi \beta)} \begin{cases} (\beta \cos \beta \theta - \cot \theta \sin \beta \theta) / \sin^2 \theta & \rho \in (-1, 1), \\ \beta(1 - \beta^2)/3 & \rho = 1, \\ (\coth \theta \sinh \beta \theta - \beta \cosh \beta \theta) / \sinh^2 \theta & \rho > 1. \end{cases}$$

For $\nu \in (1, 2)$, $\nu' \in (0, 1)$ we have

$$\int_0^\infty \frac{dh}{h^{1+\nu}} \left(e^{-\rho h} \frac{2(h/2)^\nu}{\Gamma(\nu)} K_\nu(h) - 1 + \rho h \right) = \frac{2^{1-\nu} \pi / \sin(-\pi \nu)}{\Gamma(1+\nu)} \cdot \begin{cases} \cos \nu \theta & \rho \in (-1, 1], \\ \cosh \nu \theta & \rho > 1, \end{cases}$$

$$\int_0^\infty \frac{dh}{h^{1+\nu'}} \left(e^{-\rho h} \frac{2(h/2)^{\nu'}}{\Gamma(\nu')} K_{\nu'}(h) - 1 \right) = \frac{-2^{1-\nu'} \pi / \sin(\pi \nu')}{\Gamma(1+\nu')} \cdot \begin{cases} \cos \nu' \theta & \rho \in (-1, 1], \\ \cosh \nu' \theta & \rho > 1, \end{cases}$$

$$\int_0^\infty dh e^{-\rho h} K_{\nu'}(h) = \int_0^\infty \frac{\cosh(\nu' t) dt}{\cosh(t) + \rho}.$$

¹/₂ *Proof.* The first two integrals are applications of Cauchy's residue theorem. In particular, we carefully choose a rectangular contour such that the asymptotic contribution of the left and right segments vanish and the limit of the top and bottom segments can be related to the integral in the lemma. We only explain the first computation, the second one is analogous. Let us consider the contour integral of $u(z) = e^{\alpha z} (\cosh(z) + \rho)^{-1}$ along the boundary of $(-R, R) + i(0, 2\pi)$ (oriented counterclockwise) and send $R \rightarrow \infty$ to obtain the identity

$$\begin{aligned} 2\pi i \sum_{z \in \mathbb{R} + i(0, 2\pi)} \text{Res}(u, z) &= \int_{\mathbb{R}} \frac{e^{\alpha t}}{\cosh(t) + \rho} dt - \int_{\mathbb{R}} \frac{e^{\alpha(t+2\pi i)}}{\cosh(t+2\pi i) + \rho} dt. \\ &= 2(1 - e^{2\pi i \alpha}) \int_0^\infty \frac{\cosh(\alpha t)}{\cosh(t) + \rho} dt. \end{aligned}$$

Computing the residues yields the result. Let us consider the integrals involving Bessel functions. For the third integral in the statement, we write $h^{-\nu-1}$ as the second derivative of $h^{1-\nu}/(\nu(\nu-1))$ and integrate by parts twice to get

$$\begin{aligned} &\int_0^\infty \frac{dh}{h^{1+\nu}} \left(e^{-\rho h} \frac{2(h/2)^\nu}{\Gamma(\nu)} K_\nu(h) - 1 + \rho h \right) \\ &= \frac{2^{1-\nu}}{\nu(\nu-1)\Gamma(\nu)} \int_0^\infty e^{-\rho h} ((1+\rho^2)h K_{\nu-2}(h) + (2\rho h + (2\rho^2(\nu-1) - 1))K_{\nu-1}(h)) dh \\ &= \frac{2^{1-\nu}}{\nu(\nu-1)\Gamma(\nu)} \int_0^\infty \frac{(1+\rho^2) \cosh((2-\nu)t) + 2\rho \cosh((\nu-1)t)}{(\cosh(t) + \rho)^2} dt \\ &\quad + \frac{2^{1-\nu}(2\rho^2(\nu-1) - 1)}{\nu(\nu-1)\Gamma(\nu)} \int_0^\infty \frac{\cosh((\nu-1)t)}{\cosh(t) + \rho} dt \end{aligned}$$

by the definition (B-1). Using the first part of the lemma and applying trigonometric identities yields the result. The fourth integral can be computed similarly, except that only one integration by parts is needed. For the final integral, (B-1) can be used directly. $\square$

In the proof of Lemmas B.3 and B.4 we will need the Beta function and the incomplete Beta function, which are defined by the respective formulas

$$\begin{aligned} B(a, b) &= \int_0^1 t^{a-1} (1-t)^{b-1} dt = \int_0^\infty t^{a-1} (1+t)^{-a-b} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{for } a, b > 0, \\ B_x(a, b) &= \int_0^x t^{a-1} (1-t)^{b-1} dt \quad \text{for } a > 0, b \in \mathbb{R}, x \in [0, 1). \end{aligned}$$

The next lemma recalls some basic properties of these functions.

Lemma B.2. For $a, b > 0$ and $x \in (0, 1)$ we have $B_x(a, b) + B_{1-x}(b, a) = B(a, b)$ and, for $a > 0, b \in \mathbb{R}, x \in [0, 1)$,

$$B_x(a, b) = \frac{a+b}{b} B_x(a, b+1) - \frac{x^a(1-x)^b}{b}.$$

Moreover, $\Gamma(a)\Gamma(1-a) = \pi / \sin \pi a$ for $a \notin \mathbb{Z}$.

¹/₂ ¹ *Proof.* The first claim is trivial. The second one follows since it holds at $x = 0$ and the derivatives of both sides agree for all $x > 0$. The third claim is Euler's reflection formula. $\square$

⁴ **Lemma B.3.** Let $\kappa \in (8/3, 4)$. For $\theta < 4/\kappa$ and $u \in (0, 1)$ define

$$A_u^\theta = -\cos\left(\frac{4\pi}{\kappa}\right) \int_0^\infty \frac{dh}{h^{1+4/\kappa}} ((1+h)^\theta - 1 - \theta h) + \int_0^\infty \frac{dh}{2h^{1+4/\kappa}} ((1-h)^\theta 1(h < u) - 1 + \theta h) \\ + \int_0^\infty \frac{dh}{2h^{1+4/\kappa}} ((1-h)^\theta 1(h < 1-u) - 1 + \theta h).$$

¹⁰ Then $A_u^{-1-4/\kappa} = 0$ for all $u \in (0, 1)$ and, for all $\theta \in (-1, 4/\kappa)$ and $u \in (0, 1)$,

$$A_u^\theta + \int_0^u \frac{h^\theta dh}{2(1-h)^{1+4/\kappa}} + \int_0^{1-u} \frac{h^\theta dh}{2(1-h)^{1+4/\kappa}} = -\cos(\pi\theta) B\left(\theta + 1, \frac{4}{\kappa} - \theta\right).$$

¹³ *Proof.* We write

$$\frac{1}{h^{1+4/\kappa}} = \frac{d^2}{dh^2} \left(\frac{h^{1-4/\kappa}}{4/\kappa \cdot (4/\kappa - 1)} \right) \quad \text{and} \quad \frac{1}{h^{4/\kappa}} = \frac{d}{dh} \left(\frac{h^{1-4/\kappa}}{1 - 4/\kappa} \right).$$

¹⁷ Applying integration by parts multiple times thus allows us to establish the following integral identities.

¹⁸ For $u \in (0, 1)$ and $\theta < 4/\kappa$ we have

$$\int_0^\infty \frac{dh}{h^{1+4/\kappa}} ((1+h)^\theta - 1 - \theta h) = \frac{\theta(\theta - 1)}{4/\kappa \cdot (4/\kappa - 1)} B\left(2 - \frac{4}{\kappa}, \frac{4}{\kappa} - \theta\right), \\ \int_0^\infty \frac{dh}{h^{1+4/\kappa}} ((1-h)^\theta 1(h < u) - 1 + \theta h) = \frac{\theta(\theta - 1)}{4/\kappa \cdot (4/\kappa - 1)} B_u\left(2 - \frac{4}{\kappa}, \theta - 1\right) \\ + \frac{\theta u^{1-4/\kappa} (1-u)^{\theta-1}}{4/\kappa \cdot (4/\kappa - 1)} - \frac{u^{-4/\kappa} (1-u)^\theta}{4/\kappa}.$$

²⁶ For the $\theta = -1 - 4/\kappa$ case, applying the indented equation in [Lemma B.2](#) four times allows us to express $B_u(2 - 4/\kappa, -2 - 4/\kappa)$ in terms of an expression involving $B_u(2 - 4/\kappa, 2 - 4/\kappa)$. One then verifies that indeed $A_u^\theta = 0$ for $\theta = -1 - 4/\kappa$.

²⁹ Suppose now that $\theta \in (-1, 4/\kappa)$. Then

$$\int_0^\infty \frac{dh}{h^{1+4/\kappa}} ((1-h)^\theta 1(h < u) - 1 + \theta h) + \int_0^{1-u} \frac{h^\theta dh}{(1-h)^{1+4/\kappa}} \\ = \int_0^\infty \frac{dh}{h^{1+4/\kappa}} ((1-h)^\theta 1(h < 1-u) - 1 + \theta h) + \int_0^u \frac{h^\theta dh}{(1-h)^{1+4/\kappa}} \\ = \int_0^\infty \frac{dh}{h^{1+4/\kappa}} ((1-h)^\theta 1(h < 1) - 1 + \theta h) \\ = \frac{(1 - 4/\kappa + \theta)(2 - 4/\kappa + \theta)}{4/\kappa \cdot (4/\kappa - 1)} B\left(2 - \frac{4}{\kappa}, \theta + 1\right),$$

³⁹ where the second equality is obtained by writing it as the $u \uparrow 1$ limit in the second equality of the second display above, and then applying the indented equation of [Lemma B.2](#) twice to determine the limit.

¹/₂ Replacing u by $1 - u$ we also get

$$\int_0^\infty \frac{dh}{h^{1+4/\kappa}} ((1-h)^\theta 1(h < 1-u) - 1 + \theta h) + \int_0^u \frac{h^\theta dh}{(1-h)^{1+4/\kappa}} = \frac{(1-4/\kappa+\theta)(2-4/\kappa+\theta)}{4/\kappa \cdot (4/\kappa-1)} B\left(2-\frac{4}{\kappa}, \theta+1\right),$$

Combining the previous displays,

$$\begin{aligned} A_u^\theta + \int_0^u \frac{h^\theta dh}{2(1-h)^{1+4/\kappa}} + \int_0^{1-u} \frac{h^\theta dh}{2(1-h)^{1+4/\kappa}} \\ = -\cos(4\pi/\kappa) \frac{\theta(\theta-1)}{4/\kappa \cdot (4/\kappa-1)} B\left(2-\frac{4}{\kappa}, \frac{4}{\kappa}-\theta\right) + \frac{(1-4/\kappa+\theta)(2-4/\kappa+\theta)}{4/\kappa \cdot (4/\kappa-1)} B\left(2-\frac{4}{\kappa}, \theta+1\right) \\ = -\cos(\pi\theta) B\left(\theta+1, \frac{4}{\kappa}-\theta\right), \end{aligned}$$

where the final equality follows by writing the Beta functions in terms of Gamma functions and using Euler's reflection formula. $\square$

We now establish a similar result which will appear in the context of Lévy processes without compensation. Note that the actual formula is formally the same as in the previous lemma.

Lemma B.4. Suppose that $\kappa' \in (4, 8)$ and define A_u^θ to be

$$\begin{aligned} -\cos\left(\frac{4\pi}{\kappa'}\right) \int_0^\infty \frac{dh}{h^{1+4/\kappa'}} ((1+h)^\theta - 1) + \int_0^\infty \frac{dh}{2h^{1+4/\kappa'}} ((1-h)^\theta 1(h < u) - 1) \\ + \int_0^\infty \frac{dh}{2h^{1+4/\kappa'}} ((1-h)^\theta 1(h < 1-u) - 1), \end{aligned}$$

whenever $\theta < 4/\kappa'$ and $u \in (0, 1)$. Then $A_u^{-1-4/\kappa'} = 0$ for all $u \in (0, 1)$ and

$$A_u^\theta + \int_0^u \frac{h^\theta dh}{2(1-h)^{1+4/\kappa'}} + \int_0^{1-u} \frac{h^\theta dh}{2(1-h)^{1+4/\kappa'}} = -\cos(\pi\theta) B\left(\theta+1, \frac{4}{\kappa'}-\theta\right).$$

for all $\theta \in (-1, 4/\kappa')$ and $u \in (0, 1)$.

Proof. The proof of this result is completely analogous to the proof of Lemma B.3. By integration by parts,

$$\begin{aligned} \int_0^\infty \frac{dh}{h^{1+4/\kappa'}} ((1+h)^\theta - 1) &= \frac{\theta}{4/\kappa'} B\left(1-\frac{4}{\kappa'}, \frac{4}{\kappa'}-\theta\right), \\ \int_0^\infty \frac{dh}{h^{1+4/\kappa'}} ((1-h)^\theta 1(h < u) - 1) &= -\frac{\theta}{4/\kappa'} B_u\left(1-\frac{4}{\kappa'}, \theta\right) - \frac{u^{-4/\kappa'}(1-u)^\theta}{4/\kappa'}. \end{aligned}$$

The result that $A_u^{-1-4/\kappa'} = 0$ for $u \in (0, 1)$ then follows from a lengthy calculation — see Lemma B.3 for some more detail on this. Suppose now that $\theta \in (-1, 4/\kappa')$. Then

$$\begin{aligned} \int_0^\infty \frac{dh}{h^{1+4/\kappa'}} ((1-h)^\theta 1(h < u) - 1) + \int_0^{1-u} \frac{h^\theta dh}{(1-h)^{1+4/\kappa}} &= \int_0^\infty \frac{dh}{h^{1+4/\kappa'}} ((1-h)^\theta 1(h < 1) - 1) \\ &= -\frac{1-4/\kappa'+\theta}{4/\kappa'} B\left(1-\frac{4}{\kappa'}, \theta+1\right). \end{aligned}$$

¹/₂ The results now follows by putting everything together and simplifying the expressions, making use of
² Euler’s reflection formula. □
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