An Inquiry-Oriented Approach to Determinants

Abstract: In introductory linear algebra classrooms, determinants are often taught in a formulaic way that obscures their rich connections to graphical interpretations of linear transformations. As an alternative, in this paper we introduce an innovative task sequence exploring the determinant conceptualized as a measure of the distortion of space. The task sequence is part of the Inquiry-Oriented Linear Algebra (IOLA) curricular materials, which build from experientially real tasks that allow for active student engagement in the guided reinvention of key mathematical ideas through student and instructor inquiry. The task sequence begins with students developing a conceptualization of the 2×2 matrix determinant as a ratio between the (signed) area of an image object and the area of its pre-image object transformed by the associated linear transformation. They then use this conceptualization to construct for themselves the 2×2 determinant formula and the determinant of a matrix inverse. Finally, students explore 2D and 3D GeoGebra applets to reinvent and explain core theorems related to the determinant involving linear dependence and switching/scaling rows and columns. Student work examples are provided throughout the paper.

Keywords: Determinant, Linear Algebra, Digital Geometry Software, Inquiry-Oriented

1 INTRODUCTION

According to Axler [2, p. 139], "determinants are difficult, non-intuitive, and often defined without motivation". Indeed, matrix determinants are often taught procedurally, with little connection to other core concepts in linear algebra. A solely procedural approach to determinants would be a missed opportunity for students to explore the geometric connections between matrix determinants and linear transformations. Introductory linear algebra courses in the United States are excellent settings to learn the foundational aspects of linear algebra through exploring $\mathbb{R}^n$. In particular, doing so can connect with students' embodied experiences in 2- and 3-dimensional space, as well as with their geometric and graphical experiences in $\mathbb{R}^2$ and $\mathbb{R}^3$. Thus, as part of the Inquiry-Oriented Linear Algebra (IOLA) project, our research team designed a task sequence that builds from students' knowledge of matrices as representations of linear transformations to create a conceptualization of the determinant of a 2×2 or 3×3 matrix as a measure of the (signed) multiplicative change in area or volume, respectively. Such an approach for the task sequence is aligned with current recommendations for a first course in linear algebra: "Determinant of a matrix as the area/volume scaling factor of the linear map described by the matrix" (updated Linear Algebra Curriculum Study Group (LACSG 2.0) Recommendations [22]). One aspect in which the IOLA task sequence stands out as an innovative approach to teaching this topic, however, is that it puts students in a mathematical situation in which they can themselves, with guidance from the instructor, reinvent not only the 2×2 determinant formula but also various determinant properties.

2 LITERATURE REVIEW

There are relatively few research articles that investigate student understanding of the determinant concept. Some of that research is oriented in terms of student misconceptions. For example, Aygor and Ozdag [3] framed their study results in terms of student misconceptions that contributed to conflating determinant operations with matrix operations, such as the matrix operation of switching two rows negating the matrix rather than negating the determinant. Further, Kazunga and Bansilal [15] interpreted their study results as student misconceptions related to incorrect procedures for calculating determinants, overgeneralizing procedures, and conflating a matrix inverse or transpose and its determinant. This might be because of the inherent complexity in relating the numeric, algebraic, and geometric aspects of the determinant [7] or the reliance of the determinant on multi-linearity [6]. One way to support students is to foster their ways of reasoning about determinants geometrically that anticipate the effect matrix operations might have on the determinant. Research shows that Dynamic Geometry Software (DGS) applets allow students to investigate, visualize, make predictions, calculate, simulate, and generalize certain situations, all of which are critical practices for inquiry (e.g., [11]; [10]; [13]; [14]; [19]). In linear algebra, DGS has been used to explore key topics such as linear combinations [18], linear transformations [23], eigentheory ([11], [10]), and determinants ([5],[6]). For example, Donevska-Todorova [5] used DGS to support students' exploration of determinants in a teaching experiment setting and found that DGS can help build meaningful connections between the determinant and conceptions of area in 2D. The author attributes the DGS's ability to simultaneously display numeric and geometric feedback as important for connecting to students' prior geometric understanding.

3 THEORETICAL FRAMEWORK

The IOLA materials [25] are a collection of instructional units, each focused on core ideas from introductory linear algebra. Each unit consists of a task sequence and various instructor support materials for colleagues interested in incorporating inquiry into their pedagogical approach. Broadly speaking, inquiry-based mathematics education is characterized by "student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice" [17, p. 129]. At a more specific level, the IOLA project is aligned with inquiry-oriented instruction, which has four key components: generating student reasoning, building on student reasoning, developing a shared understanding, and formalizing language and notation [16]. The defining characteristic of inquiry-oriented instruction, however, is the reliance on Realistic Mathematics Education([9]; [12]) as an instructional design theory. Realistic Mathematics Education(RME) has three main guiding heuristics. Briefly stated, task sequences for the mathematical phenomena to be organized by students should be based on experientially real starting points (didactical phenomenology); classroom activity should support student development of models-of their mathematical activity that can be leveraged as models-for subsequent mathematical activity (emergent models); and student activity, with instructor guidance, should evolve toward the reinvention of formal notions and ways of reasoning (guided reinvention). In [12], the models-of/models-for development is detailed through four levels of activity: situational, referential, general, and formal. Inquiry and RME work synergistically in the IOLA materials: inquiry-oriented task sequences are first theorized according to RME, further refined according to the nature of students' mathematical reasoning about the tasks and how they may engage meaning fully in the reinvention of important mathematical ideas, which allows for and facilitates the foundational role of inquiry within the classroom [26].

The IOLA materials are developed using design-based research, methodologically following the design-based research spiral [26]. For the Determinants unit, we initially examined the research into student understanding of determinants, the history of determinants, and textbook problems relating to determinants. From this, we developed our own tasks, based on the aforementioned RME heuristics, and conducted a paired-teaching experiment [21] with two students. After analyzing these data and revising the tasks, we implemented the tasks in a classroom teaching experiment [4]. We analyzed data from the classroom teaching experiment and revised the tasks again for implementation with a group of instructors in an online working group [8]. Comments and feedback from the instructors helped to develop the final version of the tasks that we share in this article. The entire unit, with additional detail such as more examples of student reasoning, can be found at `iola.math.vt.edu`.

The Determinants unit is fourth in the IOLA materials and is intended to follow Unit 3: *Italicizing N*. In Unit 1: *The Magic Carpet Ride* [24], students conceptualize linear combinations as travel vectors and use the travel metaphor to make sense of span and linear independence. Unit 2: *Meal Plans* [20] introduces students to systems of equations by looking at school meal plans and helps students understand solution sets by progressively introducing constraints to a system. Unit 3: *Italicizing N* [1] develops the notion of a linear transformation for students starting with concept of italicizing the letter N. By the end of Unit 3, students will have written the matrix for a linear transformation, looked at how the linear transformation affects the entirety of $\mathbb{R}^2$, explored the composition of linear transformations, and the inverse of a linear transformation.

4 THE DETERMINANTS TASK SEQUENCE

The main goal of the Determinants task sequence is to build from students' knowledge of matrices as representations of linear transformations towards conceptualizing the determinant of a 2×2 or 3×3 matrix as a measure of (signed) multiplicative change in area or volume, respectively. Task 1 engages students in experientially real, situational activity by introducing them students to the scenario of developing a measure of a distortion of space. It is designed to support students in suggesting change in area as a way to quantify distortion that objects undergo from specific matrix transformations. In Task 2, students engage in referential activity by utilizing their conception of determinant as a ratio of areas to reinvent the 2×2 determinant formula and the term determinant is introduced. In Task 3, students reinvent $det(A^{-1}) = \frac{1}{det(A)}$ for invertible matrices by coordinating their knowledge of invertible linear transformations with determinants as a measure of change in area. Tasks 1 and 3 provide space for students to use their understanding of composition of functions to reinvent $det(AB) = det(A)det(B)$ for matrices A and B. In Task 4, students further explore the geometric interpretation of matrix transformations and their determinants via GeoGebra applets for 2×2 and 3×3 matrices; the goal of Task 4 is to have students reinvent some of the key properties of the determinant by engaging in generalizing activity across multiple examples created from exploring the applets.

4.1 Task 1: Quantifying the Distortion

The IOLA Determinants unit is intended to naturally extend students' experiences with the Italicizing N IOLA unit. In that unit, students will have explored matrices as linear transformations that distort space by using matrix multiplication to map vectors. Specifically, students will have explored: matrix-times-vector multiplication as a function that maps input vectors to output vectors, composition of such mappings via matrix multiplication, and inverses of matrix transformations. Accordingly, the Determinants unit begins by presenting students with three pre-images and their respective distortions under a sequence of matrix transformations (Figure 1). The left-most column shows three "original" pre-images. Then, the same matrix transforms all pre-images from one column into their images in the next columns, which is shown with labeled arrows between the columns. This is consistent with the notation and representations in the prior IOLA unit, so students should recognize this figure as representing a composition of matrix transformations. On the left side of the slide, there is a brief explanation and the question "What are some things you notice?"

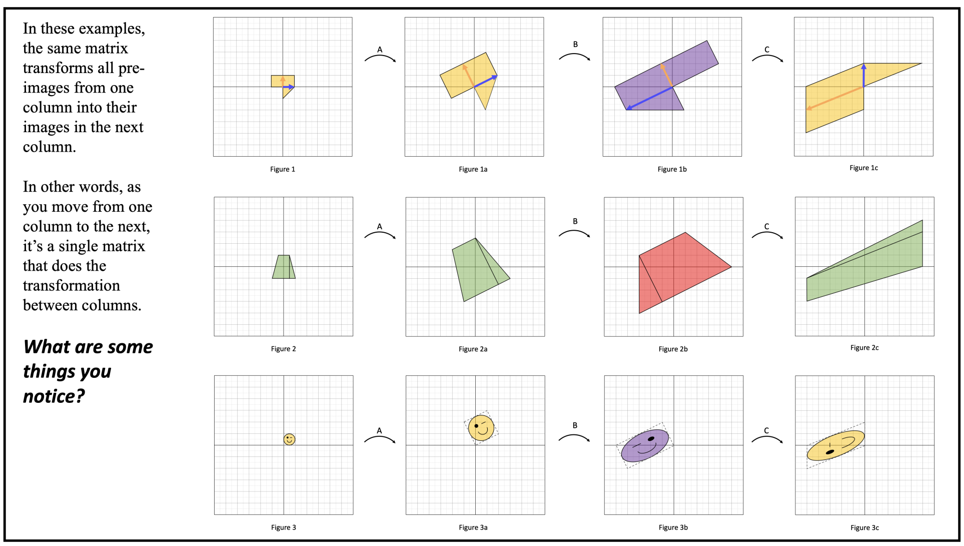


Figure 1. A slide that shows pre-images and their distortions under linear transformations in an array, which helps constitute Task 1.

We designed these objects to allow students to attend to several geometric properties of pre-images and images and to provide a set of linear transformations with a variety of properties. The objects were created to have areas that could be calculated by decomposing into polygons, circles, and ellipses and that would be accessible to students. For instance, the first transformation rotates and dilates each object about the origin, preserving length ratios and angles within each object. The second matrix transformation reflects each object across the line $y = -2x$ and scales them by a factor of 2 in the direction along the line $y = 0.5x$. Finally, the third transformation reflects, rotates, and shears to produce an image that is even more distorted from the original. We included chirality in each object's design (and a color change) to support students' recognition of when a transformation reflects an object. We chose these transformations carefully to exhibit specific characteristics of linear transformations and allow students to contrast and compare across them. For example, we intentionally chose the first transformation to generate a similar (in the geometric sense), but larger, shape as the pre-image. We also intentionally chose a second transformation to reflect and stretch the shape, but not shear it. Finally, we chose a transformation that dramatically changed the characteristics of the shape.

Our goal in this portion of the lesson is to have students attend to and recognize which geometric properties might change under these transformations and recognize at least some of the geometric distortions that the objects undergo with each transformation. We intend for instructors to discuss these geometric properties and distortions, either as observations or questions to explore, with the students before moving to the next slide in the materials, which present students with a prompt to generate "a single real number" to "measure how each transformation changes the size of an object in space" (Figure 2).

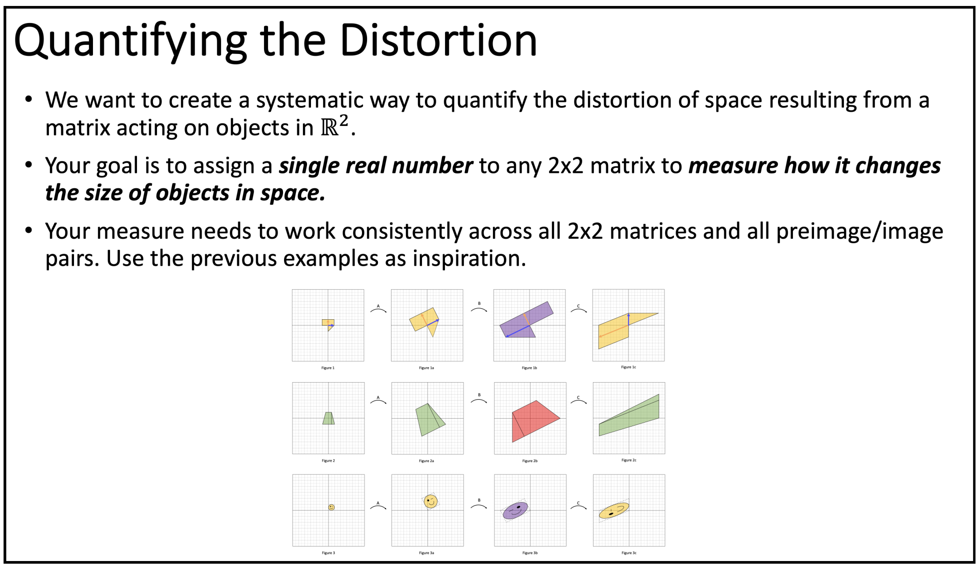


Figure 2. Task 1 of the determinants unit.

Once provided with this prompt, students work in groups to identify which geometric properties from the array can and should be included in quantifying the distortion. By asking students to quantify how each matrix transformation changes the size of each object, we are intending to focus students' attention on areas of objects. In our initial development of these materials, we found that students would often attend to angles, orientation, and other geometric aspects of the matrix transformations that were not independent of the pre-image/images shapes nor the vectors students chose to measure change under the transformation. For instance, angle is not (generally) preserved under linear transformations, but students often attended to easily recognizable angles in a given image and observed the measure of the corresponding part of the figure in the image of that transformation.

While the students explore the figures and work to develop a number for quantifying the distortion, we suggest that the instructor visit with each group to identify which geometric properties the students are attending to and how they are choosing to generate a real number. Some students might try to identify the transformation's corresponding matrix and find some way to encode the entries of the matrix into a single number. Although this is a creative approach, we encourage instructors to identify instances of this happening and direct students' attention back to characterizing and quantifying the geometric distortions of the figures under the transformation. In service of that focus, instructors might push back on what geometric characteristics the encoded number would provide insight into. The central motivator in this task should be the geometry, not the matrix itself, as the later tasks in this unit will connect the entries of the matrix to the geometry of the distortion.

As mentioned, students typically attend to angles between vectors or sides of a polygon, lengths of vectors or line segments, areas of shapes, rotations, reflections, and whether an image is skewed/distorted or a similar figure after undergoing a transformation. We intentionally designed the task to provide multiple instantiations of each of these aspects in the presented figures. Once a student identifies a type of geometric distortion that they want to characterize, it is important that they measure multiple instances because this will allow them to identify whether a measure is dependent on their choice of a vector, angle, or shape in the preimage. For instance, a student might focus on the distortion of angles under each transformation. Under A and B, all angles will remain the same, though they are rotated under A and reflected under B. Initially, students might interpret this as typical of transformations because they might not have attended to angles with prior transformations. However, under transformation C, angles are suddenly distorted in very different ways.

For each geometric property that students focus on, we suggest that instructors inquire into student reasoning to gain a sense of how the students view whether that property satisfies the prompt. This understanding should build toward a richer whole-class discussion once the students have explored the task in small groups. During whole-class discussion, the instructor should guide students toward focusing on the area of pre-images and images. Typically, at least one group in a given class will focus on the areas of geometric objects changing under each of the transformations. If multiple groups have focused on measuring area, an instructor could ask the whole class to compare the groups' work and verify their calculations, as well as encourage a discussion of which unit(s) is/are being used to calculate the areas of each figure (in the task, there are light gray and dark gray grid lines, either of which might serve as a unit). This is important because measuring with different units creates different areas for the pre-image and image under the same linear transformation. Further, students will often focus on areas of the whole large shapes in the figures or on the smaller shapes that compose them. Measuring the areas of the smaller shapes provides additional data points for the whole-class discussion.

Typically, once students focus on the areas of figures, they compare from pre-image to image either additively or multiplicatively. The variety of measurements described in the previous paragraph provides a way to address the additive comparison. For example, instructors could point out that an additive comparison is dependent on both the area of the pre-image object as well as the unit used to measure length in the domain and co-domain, but that the multiplicative relationship between the areas of the image and pre-image objects (i.e., the ratio between the area of the image object and the area of the pre-image object) under a linear transformation is consistent regardless of which units the students use or which shapes' areas they measure. Once the whole-class discussion has narrowed to focus on the multiplicative change in area under the matrix transformations, the instructor should ensure that there is a consensus that any matrix transformation distorts space by scaling the area of shapes in the domain by a single factor. That is, for a given matrix transformation and any shape in the domain, the ratio $\frac{area\,of\,image}{area\,of\,pre-image}$ will be constant.

4.2 Task 2: Finding a Formula

The second task in the sequence asks students to calculate the area of a parallelogram created from transforming the unit square by an arbitrary 2×2 matrix. The goal of the task is to have students recreate the formula $det(A) = ad - bc$ for the determinant of a general 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. By reinventing the determinant formula, we hope that students will connect the geometric conceptualization of determinants as a ratio of areas developed in Task 1 to the newly derived formula from Task 2 (Figure 3). It is important for the instructor to constitute Task 2 by supporting a conversation that (1) establishes the value of considering the image of the unit square and (2) connects the entries of the 2×2 matrix to the sides of the parallelogram image of the unit square. A substantial portion of this reasoning tends to be shared among the students in the class based on their work in the Italicizing N unit, specifically that the image of each standard basis vector is the corresponding column vector of the transformation matrix. The unit square pre-image serves as a reasonable starting point because it has a unit area and maps to a geometric figure on which points can be labeled in terms of the transformation matrix entries. At the end of Task 1, the students should be focused on the ratio between the areas of the parallelogram in the image and the square in the pre-image. Because the area of the square in the pre-image is 1, this reduces the problem of creating a formula for the determinant to an exercise of calculating the area of the parallelogram in terms of a, b, c, and d.

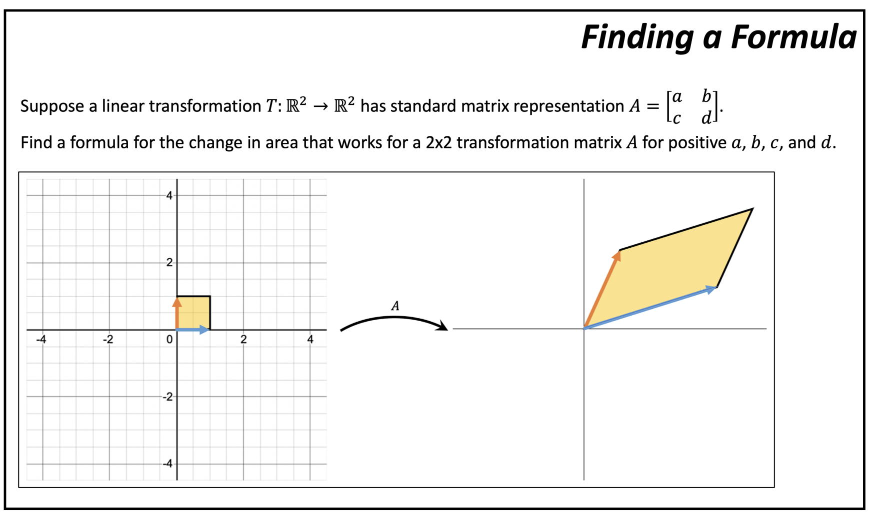
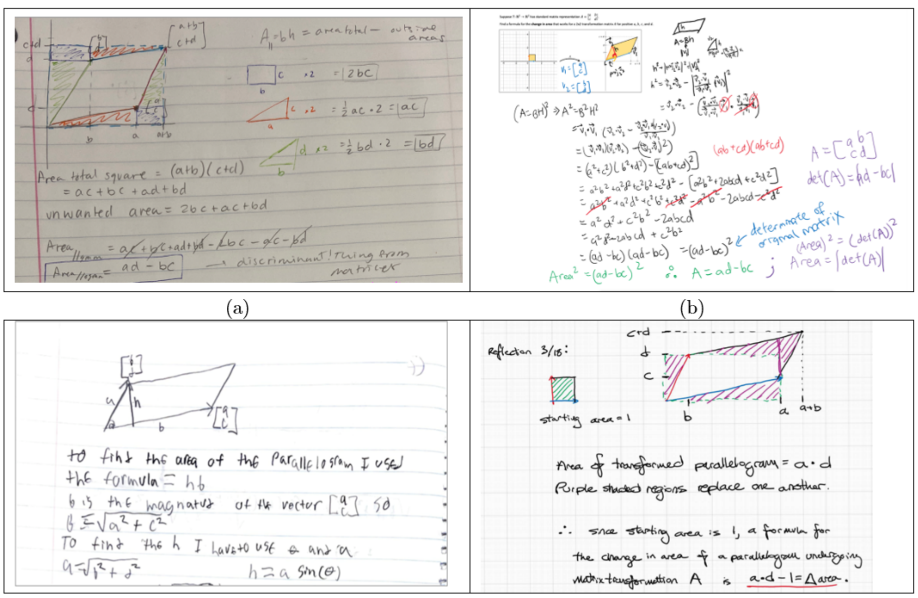


Figure 3. Task 2 of the determinants unit.

Because the task is constituted in such a way as to reduce it to calculating the area of the image parallelogram, the problem necessitates that students use a wide variety of conceptual resources from both linear algebra and earlier mathematics courses, specifically geometry. One of the key approaches students used to find the area is to find the area of a rectangle surrounding the area of the parallelogram and subtract off portions of the rectangle to find the resulting area (Figure 4(a)). For instance, the student whose work is shown in Figure 4(a) computed the areas of four triangles and two smaller rectangles surrounding the parallelogram and subtracted it from the area of the larger rectangle to find the area of the parallelogram. Note the student called this the "discriminant" and not the determinant, indicating that they had at least some sense of familiarity with the notion of determinant.

Another common approach to finding the area of the parallelogram was to try to use the familiar formula $A = bh$, which involves finding the height of the parallelogram. Students may try to do this using either the dot product (Figure 4(b)) or the Pythagorean theorem/trigonometry (Figure 4(c)), both of which can be complex or time-consuming. Some students might try to equate certain areas, allowing them to compute the area of a simpler shape (Figure 4(d)). This student also deduced that the change in area was $ad-1 = \Delta area$. Here they conceptualized the determinant additively as opposed to multiplicatively. Some students found it useful to return to Task 1 to discuss this idea. Upon computing the change in area as a difference, they realized it was inconsistent across all the images.



(a) (b)

(c) (d)

Figure 4. Examples of Student Work on Task 2

Although students may have seen $ad - bc$ in prior settings, they might not have realized that $det(A) = ad-bc$. This realization helps students solidify a sense that this multiplicative change in area is connected to the traditional formula from Advanced Calculus or Physics classes. The formula can also serve as a tool to explore their conjectures from Task 1 and to justify conjectures they will make in future tasks. If the instructor has the time, this discussion is a helpful way to tie the formula back to the students' geometric reasoning in Task 1. For example, they may wish to connect the color change in the first task to transforming with a negative determinant. This can also be explored later using the applets. It is important to establish a shared understanding of the determinant, so we recommend that the instructor conclude this lesson by explicitly introducing the term *determinant* and defining the 2×2 matrix determinant using the four following equivalent statements:

- The determinant of a 2×2 matrix A is the signed area scaling factor [i.e., the multiplicative change in area] of the linear transformation defined by the matrix.
- The determinant of a 2×2 matrix A is the signed area of the parallelogram that is the image of the unit square in $\mathbb{R}^2$.
- The determinant of a 2×2 matrix A is the signed area of the parallelogram created by the column vectors of the matrix .
- The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is calculated by $det(A) = ad - bc$

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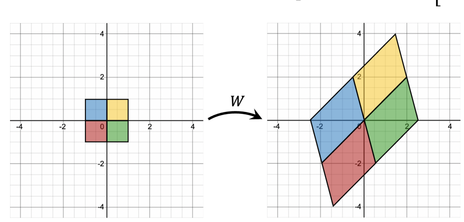
4.3 Task 3: The Windows Task

In Task 3 (see Figure 5), students are presented with a new linear transformation U from $\mathbb{R}^2$ to $\mathbb{R}^2$. They are not told what it or its associated matrix representation W is; rather, students are given information about how it transforms space by showing a pre-image of a four-paned "window" and the associated image of this "window" as mapped by the transformation W. In Question 1, students are prompted to determine the measure of the transformation U's change in the area of objects in $\mathbb{R}^2$, or, in other words, $det(W)$. In Question 2, students are prompted to determine the same information for the matrix transformation that "undoes" the transformation in Question 1. Notably, Question 2 does not explicitly name the "undoing" matrix as W^{-1}. That meaning tends to be available to students because of their prior work, so the connection to the concept of inverse is left for students to make. Finally, in Question 3, students are prompted to determine the values of the W matrix and the "undoing" matrix, compare those determinants with their answers to Questions 1 and 2, and state something they noticed and why it might be sensible.

The Window Task

Suppose you know there is a linear transformation $U: \mathbb{R}^2 \to \mathbb{R}^2$ that graphically distorts the 4-paned "window" as shown in the diagram, but you don't yet know the transformation or its matrix representation $W = [\quad]$.



1. Without knowing the exact transformation or matrix, based on our previous work, what is the measure of this transformation's change in area for objects in $\mathbb{R}^2$? In other words: without knowing W, what is $det(W)$? Why? Show your work.
2. You now want to know about the transformation that "undoes" the effect of matrix transformation W. What is the change in area measure (i.e., the determinant) of that matrix, and why?
3. If you haven't done so already, determine the entries of the matrix W and the "undoing" matrix from #2. Then find the determinants of these matrices and compare them with your measures of the change in area from #1 and #2 above. Describe at least one thing you notice and why you think that might be sensible.

Figure 5. Task 3 of the Determinants unit

The main goal of Task 3 is for students to reinvent the property $det(A^{-1}) = \frac{1}{det(A)}$ for an invertible 2×2 matrix A by coordinating their knowledge of invertible linear transformations with their developing understanding of determinants as a measure of change in area. Through reasoning about the determinant of a matrix transformation and the determinant of the inverse matrix transformation as ratios of figure areas represented in the respective domains and codomains, Task 3 helps students make sense of the determinant of a matrix's inverse as the reciprocal of the original matrix's determinant through comparing ratios of area; i.e., $det(A^{-1}) = \frac{1}{det(A)}$ for an invertible 2×2 matrix A. We recommend not mentioning the word inverse as students may immediately go to previously established formulas and we want the students to focus on the geometry and the definition of the determinant developed in Task 1. We recommend using the word "undoing" to refer to W^{-1} until the inverse is directly mentioned or discussed by the class. Students might also note that the undoing matrix composed with W results in the identity.

Students may demonstrate a variety of rich and creative ways to engage with Task 3. The notion of the determinant is still relatively new, which may be evident as some students work with the concept at first. For instance, a student may make a statement such as "the determinant of the window." This statement may imply that a student thinks of the determinant as being representative of the images drawn on the coordinate grid with said vectors, rather than as being representative of the matrix that changes one set of vectors to another. This line of thinking could be used to further explore the difference between the matrix, the linear transformation it represents, and its inputs and outputs. For Task 3 #1, students are able to engage in calculating the areas of the pre-image and image objects in a variety of ways. For example, in Figure 6(a), student groups created two right triangles, called A and B, above the x-axis (see Figure 6(a)). They created lengths a and c (length of a triangle's base) and b (length of both triangles' height) and use them and the formula $\frac{1}{2}(base)(height)$ to determine the areas of A and B to be 8 and 2, respectively. They then determined the area of the parallelogram to be twice the sum of these values, which they notated as $2(AA + AB) = 20$. Finally, the group calculated $det(W)$ as $\frac{20}{4} = 5$. They do not show how they got 4, but we assume they calculated the area of the pre-image square to be 4. Other groups also notated this as " $\frac{(area\,of\,new)}{(area\,of\,old)} = \frac{20}{4} = 5$."

With respect to solving for the values of the four components of the matrix W, students tend to leverage their prior knowledge of matrices acting on input vectors to produce output vectors to solve for W. The groupwork in Figure 6(b) shows that students took two input-output pairs (the upper-right and lower-right corner points of the pre-image square and their corresponding points in the resulting parallelogram) and used them in two matrix equations with unknown W components, converted them to two systems of equations in four unknowns, and solved for the entries of W. Most likely the group then took the W they had solved for and wrote in under their sketches that the determinant of that specific W was 5. The group wrote "area = 4" and "area = 20" next to the two respective sketches, and wrote "×5" under the arrow that indicates the transformation from the square to the parallelogram. This seems to correspond with the area of the image being five times the area of the pre-image. In Figure 6(c), a group leveraged their knowledge that the columns of W are the images of the standard basis vectors in order to solve for the component entries of W; this approach not only also connects to prior knowledge of matrices as linear

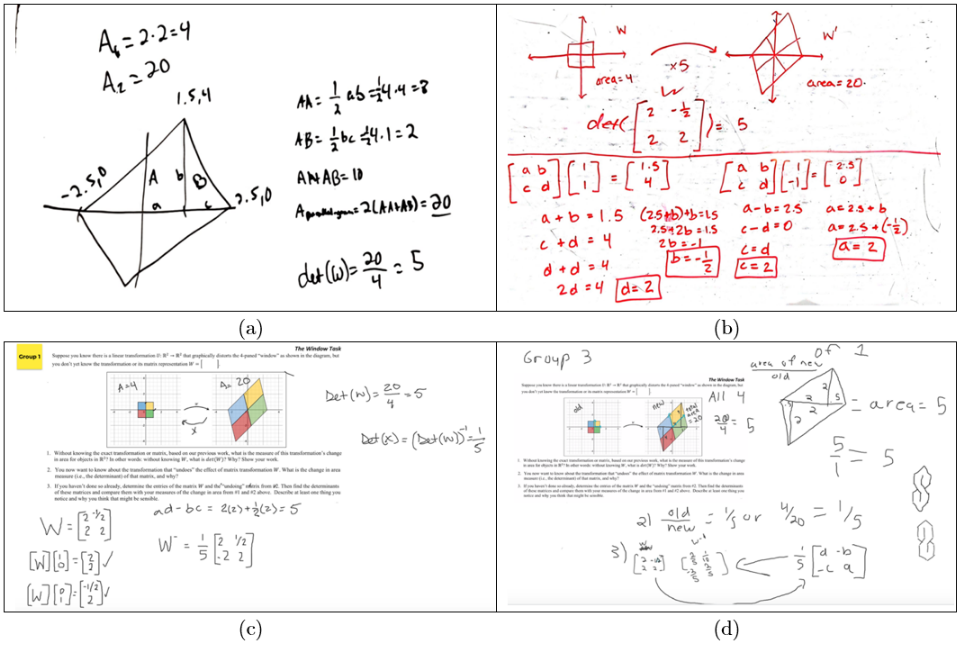


Figure 6. Examples of Student Work on Task 3

transformations, but it also connects to Task 2 (see Figure 3), which focused on the case of the pre-image object being the unit square and the column vectors of the transformation matrix corresponding to the image parallelogram that the unit square transformed into.

The central objective of Task 3 is for students to develop a sense of a geometric meaning for the determinant of a 2×2 matrix inverse. The task wording in #2 and #3 does not explicitly say "inverse"; rather, it says "the transformation that 'undoes' the effect of matrix transformation W" and "the undoing matrix." In our study, students have not demonstrated any issues with this and have labeled that as matrix W^{-1} without any prompting or direction from their instructor. We have seen two key ways that students have reasoned about the determinant of W^{-1}: as $\frac{(area\, of\, old)}{(area\, of\, new)}$; and as the multiplicative inverse of $det(W)$, meaning $(det(W))^{-1}$. For example, in Figure 6(d), a group wrote "$\frac{old}{new} = \frac{1}{5}$ or $\frac{4}{20} = \frac{1}{5}$" as their answer for #2. In Figure 6(c), a group drew an arrow going from the image parallelogram to the pre-image square and labeled in "X". The group also wrote $det(X) = (det(W))^{-1} = \frac{1}{5}$. When asked to explain to the class, a student from Figure 6(c)'s group said, "We found that like we, um, the shape becomes five times bigger so to get it back down to its normal size you'd have to divide it by five, so one-fifth." The instructor captured this line of reasoning by writing "multiplicative reciprocal of 5 is $\frac{1}{5}$." Finally, some students do calculate the components of W^{-1}, possibly via the formula for a 2×2 inverse (see Figure 6(d)), and use the newly developed formula for the determinant of a 2×2 to compute $det(W^{-1}) = \frac{2}{5} * \frac{2}{5} - \frac{-2}{5} * \frac{1}{10} = \frac{1}{5}$. Whether it be over time or at the end, we encourage the instructor to have various student groups share their approaches and solutions to Questions 1 and 2, aiming to have a couple of different solution techniques for each question. The instructor is encouraged to also have student groups share how they determined the matrix values for matrices W and W^{-1}.

To move towards Task 3's conclusion, we strongly suggest that instructors gather the various student responses to Question 3's prompt: "Describe at least one thing you notice and why you think that might be sensible." For instance, students may share: The determinants of W and W^{-1} are reciprocals of each other; The determinants of W and W^{-1} are multiplicative inverses of each other; $det(W^{-1}) = \frac{1}{det(W)}$; or $det(W^{-1}) = (det(W))^{-1}$. This student thinking can be leveraged towards the reinvention of the task's key learning goals, which the instructor could then write as their collective overarching conclusions:

- For an invertible matrix A, the determinant of its undoing matrix (or inverse matrix), is the reciprocal of $det(A)$.
- For an invertible matrix A, $det(A^{-1}) = \frac{1}{det(A)}$.

Students could be encouraged to develop a proof for the latter statement using the definition of the determinant developed in Task 1 that is independent of coordinates and thus can be generalized for any 2×2 matrix.

The second learning goal central to this task revolves around $det(A)det(A^{-1}) = det(I)$, which is of course a special case of $det(AB) = det(A)det(B)$, but that may or may not have been derived or discussed yet. If $det(AB) = det(A)det(B)$ was discussed in Task 1 or Task 2, then it could be used together with $AA^{-1} = I$ to justify the validity of learning goal 2. If $det(AB) = det(A)det(B)$ had not yet been discussed, the instructor could guide students through considering the known information $AA^{-1} = A^{-1}A = I$ and $det(A^{-1}) = \frac{1}{det(A)}$. The latter can be rearranged to $det(A^{-1})det(A) = 1$ or $det(A)det(A^{-1}) = 1$. The students could be asked to think about what $det(I)$ would equal and why. Finally, the instructor could state that $(AA)^{-1} = I \implies det(AA^{-1}) = det(I)$, so, all together, $det(A)det(A^{-1}) = det(I)$.

4.4 Task 4: Exploring Determinants Dynamically

In Task 4, students explore the properties of the determinant using two different GeoGebra applets, specifically created for this task, that allow students to manipulate matrices and see the resulting changes to the determinant and changes to the geometric transformation of the unit square or cube. The goal of this task is to support students' reinvention of some of the key properties of the determinant by generalizing across multiple examples created from exploring the 2×2 and 3×3 applets:

2×2 applet: https://www.geogebra.org/m/zqpftxx3

3×3 applet: https://www.geogebra.org/m/anugjjjz

Each applet displays a matrix whose individual entries can be manipulated using sliders and a calculation of the determinant. The 2D applet also displays the unit square with an inscribed circle and its transformation under the represented matrix. The circle is included so that students can see how the matrix transformation distorts a variety of objects and not just the unit square. It was also designed to link to the smiley face example from Task 1 in case students wanted to test different examples. The sliders adjust each matrix entry between -4 and 4 to ensure the distortion can be adequately captured on one screen. The image of the circle changes colors from red to green depending on the determinant, consistent with how the images from Task 1 change colors. The pre-image unit vectors can be manipulated for students to see the impact on the image. This is to help ensure that students do not make generalizations specific to the unit square pre-image.

For the 3D applet, students can manipulate the input matrix and see the resultant 3D image of the unit cube under the transformation and a calculation of the resultant determinant. Like the 2D applet, the unit vectors are labeled and the image changes color when the determinant changes signs. When transitioning between the 2D and 3D applet, we recommend the instructor either elicit student thoughts about how the geometric interpretation of determinant might translate to 3D (i.e., the determinant as a ratio of areas in 2D generalizes to 3D volumes) or mention in class that volume is the 3D analog of area to establish a shared understanding of the determinant of a 3x3 matrix as $\frac{(volume\, of\, new)}{(volume\, of\, old)}$. One of the primary advantages of exploring these applets is that they allow students to explore how the various elements affect pre-images under a linear transformation. For example, one student noted:

> When our group was using the applet, we noticed that each slider (a, b, c, d) affected a different aspect of the image. The a affected the width and the d affected the height. While b and c affected the tilt or stretch of the image. The b affected the horizontal or side-to-side stretch of the image and c affected the up-vertical stretch. The respective images are below in order a-d (Figure 7a)

We recommend introducing the 2×2 applet to students and asking them what they observe about relationships between the determinant, the matrix, and the transformed object. Students may generate conjectures related to zero determinants, invariant determinants, switching rows or columns, negative determinants, or manipulating particular entries. If time is limited, the instructor materials include a set of prompts that can be useful. For example, we used the following set of prompts during class after the students' exploration:

> On your own laptop, navigate to the applet for 2×2 matrices. In your small groups, use the applet to explore at least three different matrices that have determinant 0.
>
> 1. What is the geometric impact of the various matrix transformations?
> 2. What are some similarities and differences in these specific matrices?
> 3. Develop some conjectures: under what conditions is the determinant zero?

After conjecturing, students are asked to provide evidence for their conjecture. The applets provide ample reference points for students to utilize to justify their conjectures including vivid imagery, a calculation of the determinant, and the representative matrix. This prompt can be repeated by replacing "determinant 0" with "a negative determinant" to bring out generalizations about changing orientation, or with "determinant 2" to bring out generalizations about under which conditions matrices might have the same determinant.

One group of students found two different matrices that both had determinant 4 [Figure 7b]. They concluded that "determinants can't be used to fully describe a transformation because different matrices can have the same determinant." Another group of students observed that when the columns of a 2×2 matrix are linearly dependent, then the result is a line and the determinant is zero [Figure 7c]. These examples helped students visualize the transformation, understand that the determinant is not 1-1, and connect the idea of linear dependence to a determinant of zero.

After students experience the 2D applet, they might theorize about how linear dependence and a determinant of zero worked in 3D. We strongly recommend gathering these

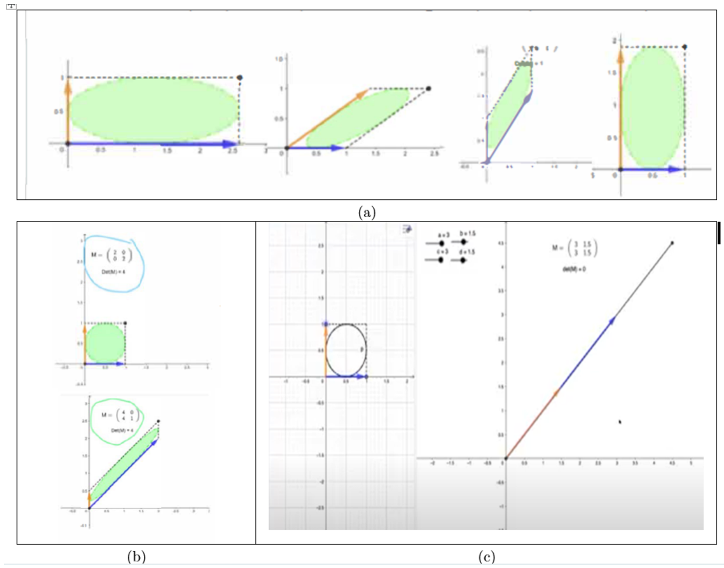


Figure 7. Examples from students exploring the 2×2 applet.

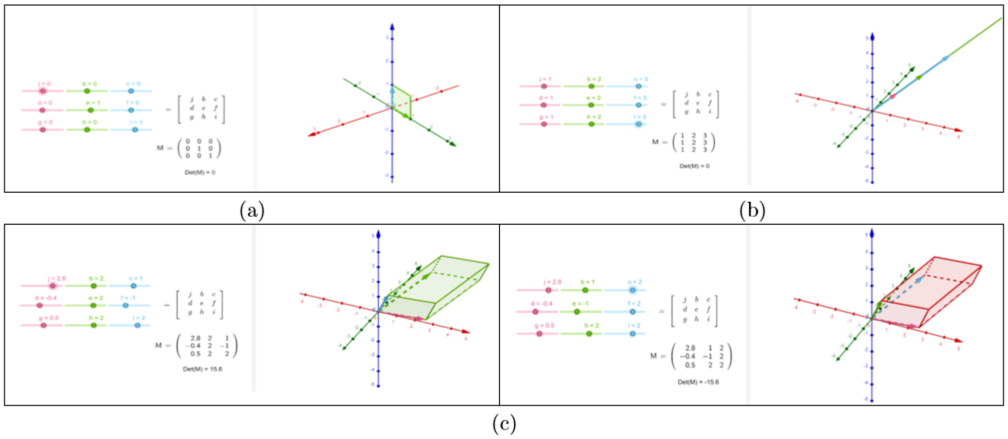
conjectures from the 2D applet and giving students some time to play and analyze their conjectures before introducing the tasks. This allows students to build connections between the 2D and 3D depictions that can be leveraged into generalizations about the determinant in n dimensions. After students experiment with their conjectures, they can be guided to specific questions about the determinant being zero, connections to linear independence, row and column operations, and scaling a column by k. Most students are curious when using the applet and highlight several important connections between the entries in the matrix, the calculation of the determinant, and then image of the unit cube. For example, one student connected matrices with determinant zero to the shape having zero volume (Figure 8a):

> If a column has all zeros, the determinant is 0, and if the determinant is zero, the shape has no volume in the third dimension. Thinking about the determinant as $\frac{new\,volume}{old\,volume}$, we can algebraically see that the numerator must be zero for the quotient to be zero. Also, if one of the column vectors is the zero vector, the shape does not span all three dimensions so it cannot possibly contain a volume.

The student generalized the formula from the first section of the unit to $\frac{new\,volume}{old\,volume}$ and leveraged this to compute the determinant. In addition, the student connected ideas of spanning (or not spanning) to the concept of the determinant.

In a second example, a student looked at linear dependence and the determinant highlighting, connection to the shape, its volume, and the alignment of the vectors:

> One observation I made was that if the set is linearly dependent or has two or more vectors that are scalar multiples, the determinant will be zero. I found out after out [sic] discussion that this is because the vectors will be all on the same line and therefore the shape will have no volume. Here is a link to see the linear dependent set with $\det(A) = 0$ (Figure 8b).



(a) (b) (c)

Figure 8. Student work on linear dependence and the determinant.

This student connected the idea of linear dependence of vectors as recognized in the image by the vectors lying on the same line and the columns of the matrix being multiples of each other to the idea of the determinant being zero. In class, images like 8(a) and 8(b) can be used to demonstrate that the determinant can be zero in multiple ways, but all of them involve collapsing one or more dimensions.

In a third example, a student looked at the connection between the rows and columns of the matrix and the determinant:

> ...I saw was that flipping one row with another or a column with another column will make the determinant negative. I am not too sure why this is but it may be because the shape is flipping on an axis. Here are two links to show that switching 2 columns makes the determinant negative (Figure 8c).

Here the student highlighted the geometric aspect of flipping along an axis to illustrate why the determinant becomes negative. Some students mentioned they noticed how the image of the standard basis vectors changed between the images. For example, in Figure 8c the green and blue vectors switched locations. Some students might connect this imagery to orientation from other math classes, the left/right-hand rules from physics, or chirality from chemistry. We believe the applet helps makes some of the determinant properties real for the students by allowing them to make conjectures about the properties and then test those conjectures. While exploring their conjectures students may connect changes in the matrix, changes in the geometry of the image of the unit square/cube, and changes in the determinant. Students may make generalizations about the determinant that hold for higher dimensions such as the determinant being zero if the columns are linearly dependent and linearly dependent columns collapsing dimensions.

5 Conclusion

Our primary design research goal for the IOLA Determinants unit was to create instructional materials that help take the determinant from something that can be procedural, unintuitive, or even difficult ([2],[3],[15]) and make it experientially real and conceptually rich for students. Throughout the task sequence, students engage in genuine mathematical activity that motivates the need for determinants and builds connections with previous course material. In Task 1, students develop the notion of examining the signed multiplicative change in area, $\frac{area\,of\,image}{area\,of\,pre-image}$, as a quantifiable property of a 2×2 matrix. In Task 2 students connect this notion with the signed area of the parallelogram that is both the image of the unit square in $\mathbb{R}^2$ and is created by the column vectors of the matrix, with the standard formula $ad - bc$. Importantly, the term *determinant* surfaces as a label for students' work in the task setting. This is often an "aha" moment for students who may connect their calculations to formulas previously seen in books or to finding matrix inverses. After having reinvented a formula, students can then validate their previous conjectures by applying their formula to the images in Task 1. In Task 3, students engage with composing linear transformations and "undoing" linear transformations and their associated matrix determinant value. Students gain an intuition for the formula $det(A^{-1}) = \frac{1}{det(A)}$ for an invertible matrix A. The task sequence concludes with an exploration of determinants in 2D and 3D using GeoGebra applets. Through the GeoGebra applets, students make conjectures about how changing a matrix affects the determinant. These may include conjectures about linear independence/dependence, switching rows or columns, scaling rows or columns, making the determinant zero, and making the determinant negative. Importantly, this generalizing activity builds bridges between the determinant and other course topics developing key links in the Invertible Matrix Theorem and providing key intuitions for proofs and explorations in higher dimensions if desired. Finally, the sequence includes refined the GeoGebra applets that have the ability to generate random matrices, automatically switch rows and columns, and engage with the image of the standard basis vectors by dragging:

2×2 applet: https://www.geogebra.org/m/v8hw88fs

3×3 applet: https://www.geogebra.org/m/fppgz9q2

The IOLA task sequence is consistent with the LACSG. 2.0 recommendations [22] and complements the work of Donevska-Todorova [5] on relating determinant and analytic geometry through DGS. The IOLA sequence in its current form focuses on a guided reinvention of the 2×2 determinant formula and properties of determinants through exploring DGS applets for both 2×2 and 3×3 matrices. Although the IOLA sequence does not suggest a guided reinvention of the 3×3 determinant formula or explore the notion of the determinant in higher dimensions, the geometric intuition in lower dimensions that it helps students develop can be referenced as they work in higher dimensions. Our future work involves further refinement of the applets by analyzing student data, exploring student generalizing activity during applet use, and considering how applets might best leveraged in a linear algebra classroom setting.

In conclusion, we highlight three important outcomes from implementing the IOLA determinants task sequence. First, students were able to visualize concepts such as span, linear (in)dependence, and linear transformations in new ways through their engagement with the tasks and their interactions with the applets, affording a deeper conceptual understanding of many aspects of the Invertible Matrix Theorem. Second, we found that the applet could be useful later in topics such as eigen-theory, especially the 2×2 applet which affords adjusting the pre-image vectors. Finally, we have evidence that students were able to internalize the definition of the determinant as a scaling factor through reflections after the tasks:

> I am confident in my understanding of what a determinate [*sic*] is, and how it relates to the change in area. The determinate is simply the ratio of the area of the new mutation compared to the original matrix (new/old). I am also confident in the concept of the inverse determinate. This is simply taking the inverse of the area so that (a being the determinate) $a * \frac{1}{a} = 1$

This illustrates the potential of our sequence to make determinants intuitive, approachable, and real for students.

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