Distributed Optimal Time-Varying Resource Allocation for Networked High-Order Systems

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***Abstract*—This article investigates the optimal distributed time-varying resource allocation problem for networked high-order systems with time-varying quadratic cost functions. Due to the coexistence of challenges caused by nonidentical Hessian matrices and more complicated agents' dynamics, the extension from existing related results on single-integrator agents is nontrivial. First, a centralized algorithm is proposed to address the optimal time-varying resource allocation problem for high-order-integrator agents. Then, based on the centralized algorithm, two distributed algorithms are designed to achieve the exact optimum tracking. The main difference between the two distributed algorithms is whether a virtual system is required to be constructed for each agent, which results in a tradeoff between economical efficiency and favorable applicability to privacy-sensitive applications during implementations. Then, by using the estimation-tracking method, these two distributed algorithms are applied to solve the resource allocation problem for agents with high-order dynamics. Finally, examples are provided to illustrate the effectiveness of the proposed algorithms.**

***Index Terms*—Continuous-time optimization, distributed time-varying resource allocation, high-order systems.**

I. Introduction

In the optimal resource allocation problem, a certain amount of resource is distributed among a group of agents while minimizing the sum of all the agents' local cost functions. This problem can be found in various fields of research including power systems [1], [2], distributed computer systems [3], sensor networks [4], robot networks [5], and economic systems [6]. Different from optimal consensus problems (see, for example, [7]), where each agent owns a local cost function depending on a common decision variable, the agents own their own local decision variables in the resource allocation problem. Centralized algorithms have been proposed to solve the resource allocation problem, which requires a central unit connected to all the agents. It is generally acknowledged that such a centralized structure may not be suitable or effective in large-scale systems with numerous agents. Recently, a number of distributed algorithms (see, for example, [2], [8], [9], [10], [11], [12], [13], [14]) have been developed to address the optimal resource allocation problem by using local information and communication.

The aforementioned distributed approaches address the problem with time-invariant cost functions and a fixed amount of resource to be distributed. In practical applications, however, the cost functions and/or the amount of resource to be distributed might be time varying, and hence the optimal solutions are trajectories changing over time instead of fixed points. For example, in the economic dispatch problem, a group of power generators aims to meet a power demand and minimize the total generation cost in the meantime. The power demand treated as resources and the generation costs will undoubtedly change over time in a day owing to the unpredictability of residences' activities and the fluctuated prices and availability of energy sources. This results in time-varying cost functions and resources. It is therefore of great importance to investigate the time-varying optimal resource allocation problem.

In literature, there are a few results on the distributed time-varying optimal resource allocation problem. In [15] and [16], the authors proposed distributed algorithms based on prediction-correction methods to solve the constrained time-varying optimization problem. The authors in [17] addressed the distributed resource allocation problem for open multiagent systems, where the replacements of agents lead to variations in both the cost functions and the budget. These results establish discrete-time distributed approaches, and there are usually nonzero tracking errors between the local decision variables and the optimal ones. There is another body of literature that devotes to derive continuous-time distributed algorithms to solve the resource allocation problem, and the established results can be used for robotic systems with continuous-time dynamics to accomplish certain tasks. For example, multiple unmanned aerial vehicle systems are becoming a promising robotic platform for aerial transportation [18], [19]. The works [20] and [21] proposed continuous-time algorithms to solve the resource allocation problem with time-invariant cost functions and time-varying resources. When implementing the results in [20], there exist nonzero tracking errors, and the results in [21] are applicable to the case of quadratic cost functions. In [19], [22], and [23], the optimal time-varying resource allocation problem is solved for the case where both the cost functions and the resource vectors

Manuscript received 4 September 2023; revised 5 September 2023; accepted 20 January 2024. Date of publication 24 January 2024; date of current version 29 August 2024. The work of Y. Ding and W. Ren was supported by National Science Foundation under Grant ECCS-2129949. The work of Z. Meng was supported in part by the National Key R&D Program of China under Grant 2022ZD0119601 and in part by the National Natural Science Foundation of China under Grant 62273195. Recommended by Associate Editor I. Lestas. *(Corresponding author: Wei Ren.)*

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Digital Object Identifier 10.1109/TAC.2024.3358099

are time varying. Specifically, in [22] and [23], it is assumed that the cost functions have identical Hessian matrices, and in [19], the case of nonidentical time-varying diagonal Hessian matrices was addressed.

Notice the fact that a broad class of vehicles can be modeled by high-order dynamics. For instance, unmanned ground and aerial vehicles may be modeled as second-order or higher order systems [24], [25]. Moreover, the results about the time-varying resource allocation problem mentioned above essentially assume single-integrator dynamics for the agents. These results cannot be directly applied to high-order dynamics. To this end, in this article, the optimal time-varying resource allocation problem with time-varying quadratic cost functions is investigated for networked high-order systems. First, high-order integrator dynamics are considered for the agents. A centralized approach is established, where a central virtual system is constructed to track the optimal Lagrange multiplier, and the central state information is used to design control inputs for each agent to track its own optimal decision trajectory. To remove the requirement of a central node, two distributed resource allocation algorithms only using local information and communication are proposed to achieve exact optimal-decision tracking. Specifically, in the first distributed algorithm, each agent has a virtual system to track the optimal Lagrange multiplier, and the local virtual state is used in the controller design. The second distributed algorithm does not require any virtual system to be constructed; instead it needs each agent to use its gradient as a local estimate of the optimal Lagrange multiplier. Consequently, the algorithm needs the agents to exchange the gradients of their local cost functions for controller implementation. Both distributed algorithms have their own advantages. For instance, due to the additional construction of virtual systems, the first one might not be as economically efficient as the second one, and because of the gradient information exchange, the second one might not be as favorable as the first one for privacy-sensitive applications. Then, these two distributed algorithms are applied to solve the time-varying resource allocation problem for networked nonlinear agents with parametric uncertainties. Such extensions are inspired by the estimation-tracking method, where virtual systems are introduced at a higher level to solve the resource allocation problem, and then controllers are designed such that the agents' physical states are capable of tracking their local virtual states.

Comparison With Related Works: The nature of the time-varying resource allocation problem makes this article different from most of the existing results that focus on time-invariant cost functions and/or time-invariant resource vectors. In contrast to nonzero optimum-tracking errors by implementing discrete-time resource allocation algorithms, the proposed algorithms here guarantee exact tracking while taking into account continuous-time high-order dynamics. The closest related results are [19], [22], and [23]. Compared with those three works, this article considers cost functions with nonidentical time-varying Hessian matrices, which is more general and includes them as special cases. Moreover, the agents are with high-order dynamics, which is more complicated and general than the single-integrator systems considered in [19], [22], and [23]. It is worth pointing out that the results obtained in this article are not simple extensions from the existing results established for single-integrator systems. The algorithm designs in [22] and [23] rely heavily on the assumption of identical Hessian matrices, for which the gradients of the cost functions reduce to a decision-variable-dependent term with an identical structure and a decision-variable-free term. When considering nonidentical Hessian matrices, such a reduction cannot be achieved. In [19], the authors propose to apply distributed estimators to track several averaged terms including the average of all the inverses of the Hessian matrices. Although such an idea works in [19] due to the assumption of time-varying diagonal Hessian matrices, it might not be suitable when time-varying nondiagonal Hessian matrices are considered since it is too expensive and inefficient, and hence impractical, to transmit the whole matrices through communication channels. Furthermore, implementing the method in [19] is more computational demanding since it requires multiple virtual systems/estimators to be constructed. In addition, the complexity of high-order (nonlinear) dynamics complicates the algorithm design and convergence analysis when addressing the optimal time-varying resource allocation problem. Therefore, the coexistence of the nonidentical time-varying Hessian matrices and the complex dynamics makes the extension here challenging and worthwhile.

Some preliminary results have been presented in [26], where a special version of the algorithm in Section IV-B is tailored to solve the time-varying resource allocation problem for networked double-integrator agents. In addition to the extensions to the case of high-order (nonlinear) dynamics, the current article introduces another distributed approach (i.e., the algorithm in Section IV-C) to solve the problem. Besides, this article contains more detailed proofs and additional simulation results.

II. Preliminaries

A. Notation

Throughout this article, let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_+$, and $\mathbb{N}_+$ denote the sets of all real numbers, all nonnegative real numbers, all positive real numbers, and all positive integers, respectively. For a set $\mathcal{S}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$, and for a real number $x \in \mathbb{R}$, $|x|$ denotes the absolute value of x. The transpose of matrix A is denoted by $A^\top$. For a given vector $x = [x_1, \ldots, x_p]^\top \in \mathbb{R}^p$, define $\|x\|_1 = \sum_{i=1}^{p} |x_i|$, $\|x\|_2 = \sqrt{|x_1|^2 + \cdots + |x_p|^2}$, and $\|x\|_\infty = \max_{i=1,\ldots,p} |x_i|$. For a symmetric matrix $A \in \mathbb{R}^{p\times p}$, let $\lambda_1(A) \leq \cdots \leq \lambda_p(A)$ denote its eigenvalues. The Kronecker product of matrices A and B is denoted by $A \otimes B$. Let $\mathrm{diag}\{A_1, \ldots, A_p\}$ represent the block diagonal matrix with the ith block in the main diagonal being A_i, where $A_i \in \mathbb{R}^{n_i \times m_i}$. For a vector $x = [x_1, \ldots, x_p]^\top \in \mathbb{R}^p$, define $\mathrm{sgn}(x) = [\mathrm{sgn}(x_1), \ldots, \mathrm{sgn}(x_p)]^\top$, where $\mathrm{sgn}(x_i) = 1$ if $x_i > 0$, $\mathrm{sgn}(x_i) = 0$ if $x_i = 0$, and $\mathrm{sgn}(x_i) = -1$ if $x_i < 0$. Let $\mathbf{0}_{m\times d}$ and $\mathbf{1}_{m\times d}$ denote the $m \times d$-dimensional zero and all-ones matrices, respectively, and for simplicity, let $\mathbf{0}_m = \mathbf{0}_{m\times 1}$ and $\mathbf{1}_m = \mathbf{1}_{m\times 1}$. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix. The gradient of a time-varying function $f(q,t)$ is denoted by $\nabla f(q,t)$, which is the partial derivative of $f(q,t)$ with respect to q, i.e., $\frac{\partial}{\partial q} f(q,t)$. The Hessian matrix of the same function is denoted by $\nabla^2 f(q,t)$, which is the partial derivative of the gradient $\nabla f(q,t)$ with respect to q. For a time-varying signal x, let the kth time derivative of x be denoted by $x^{(k)}$, where k is a nonnegative integer, and in particular, $x^{(0)} = x$ and $x^{(1)} = \dot{x}$. In this article, $\dot{x}$ and $x^{(1)}$ are used interchangeably. For nonnegative integers s and k, let $\binom{s}{k} = \frac{s!}{k!\,(s-k)!}$, where $s! = 1 \times \cdots \times s$ is the factorial function.

B. Nonsmooth Analysis

Some concepts from nonsmooth analysis are introduced in this subsection, which are exploited in the convergence analysis later.

Definition 1 (Filippov Solution) ([27]): Consider the vector differential equation

$$\dot{x} = f(x,t) \tag{1}$$

where $f : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a vector field. A vector function $x(\cdot)$ is called a Filippov solution of (1) on $[t_0, t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and for almost all $t \in [t_0, t_1]$, $\dot{x} \in \mathcal{K}[f](x,t)$, where $\mathcal{K}[f](x,t) := \bigcap_{\Lambda>0} \bigcap_{\mu(\mathscr{N})=0} \overline{\text{co}}\{f[\mathcal{B}(x,\Lambda) - \mathscr{N}, t]\}$ is the Filippov set-valued map of $f(x,t)$, $\overline{\text{co}}$ denotes closed convex hull, $\mathcal{B}(x,\Lambda)$ is the open ball centered at x with radius $\Lambda > 0$, and $\bigcap_{\mu(\mathscr{N})=0}$ denotes the intersection over all sets $\mathscr{N}$ of the Lebesgue measure zero.

Lemma 1 ([28]): Consider the vector differential equation (1), and let $f(x,t)$ be measurable and locally essentially bounded, that is, bounded on a bounded neighborhood of every point excluding sets of measure zero. Then, for any $x_0 \in \mathbb{R}^d$, there exists a Filippov solution of (1) with the initial condition $x(0) = x_0$.

Definition 2 (Clark's Generalized Gradient): Given a locally Lipschitz function $V(x) : \mathbb{R}^d \to \mathbb{R}$, the generalized gradient of the function V at x is given by $\partial V(x) = \overline{\text{co}}\{\lim \nabla V(x) \mid x_i \to x, x_i \notin \Omega_V\}$, where Ω_V is the set of measure zero where the gradient of V is not defined.

Lemma 2 (Chain Rule) ([27]): Let $x(\cdot)$ be a Filippov solution of the vector differential equation (1) and $V(x) : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz continuous function. Then, for almost all t, $\dot{V}[x(t)] \in \dot{\tilde{V}}$, where $\dot{\tilde{V}}$ is the set-valued Lie derivative defined as $\dot{\tilde{V}} := \bigcap_{\xi \in \partial V} \xi^\top \mathcal{K}[f](x,t)$.

C. Graph Theory

For a multiagent system consisting of N agents, the interaction topology can be modeled by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denote the node set and edge set, respectively. An edge denoted by $(i,j) \in \mathcal{E}$, means that agent i and j can obtain information from each other. In an undirected graph, the edges (i,j) and (j,i) are equivalent. It is assumed that $(i,i) \notin \mathcal{E}$. The neighbor set of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j,i) \in \mathcal{E}\}$. By arbitrarily assigning an orientation for every edge in $\mathcal{G}$, let $B = [B_{ij}] \in \mathbb{R}^{N \times |\mathcal{E}|}$ denote the incidence matrix associated with graph $\mathcal{G}$, where $B_{ij} = -1$ if edge e_j leaves node i, $B_{ij} = 1$ if it enters node i, and $B_{ij} = 0$ otherwise.

An undirected path between node i_1 and i_k is a sequence of edges of the form (i_1, i_2), (i_2, i_3), $\ldots$, (i_{k-1}, i_k), where $i_j \in \mathcal{V}, j = 1, \ldots, k$. A connected graph means that there exists an undirected path between any pair of nodes in $\mathcal{V}$.

Assumption 1: The graph $\mathcal{G}$ is connected.

III. Problem Statement

Consider a multiagent system consisting of N high-order-integrator agents, and the agents' dynamics are described as

$$q_i^{(n)} = u_i, \quad i \in \mathcal{V} \tag{2}$$

where $n \in \mathbb{N}_+$, $q_i^{(n)} \in \mathbb{R}^d$ is the nth time derivative of the decision variable q_i, and $u_i \in \mathbb{R}^d$ denotes the control input of agent i. In Section IV-D, heterogeneous high-order nonlinear dynamics are considered. In the distributed resource allocation problem, each agent aims to cooperatively track its optimal trajectory determined by the group objective function and the coupled equality constraint. Let $q = [q_1^\top, \ldots, q_N^\top]^\top$, $f_i(q_i, t) : \mathbb{R}^d \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the local cost function associated with agent $i \in \mathcal{V}$, and $c_i(t) \in \mathbb{R}^d$ be agent i's time-varying resource vector. It is assumed that $f_i(q_i,t)$ and $c_i(t)$ are only available to agent i. The optimal trajectories for the agents, $q^\star(t) = [q_1^{\star\top}(t), \ldots, q_N^{\star\top}(t)]^\top \in \mathbb{R}^{Nd}$, are defined as

$$q^\star(t) = \arg\min_q \left\{ \sum_{i=1}^N f_i(q_i, t) \right\} \tag{3}$$

$$\text{subject to} \sum_{i=1}^N q_i = \sum_{i=1}^N c_i(t). \tag{4}$$

The objective is to design control inputs for the agents such that the agents cooperatively determine and track the optimal trajectories defined in (3) and (4), i.e., $q_i(t) \to q_i^\star(t)\ \forall i \in \mathcal{V}$ subject to local information and communication. We present two examples to clarify the problem setting.

Example 1: The cooperative multirobot systems are becoming a promising robotic platform for transportation given a single robot has limited load capacities. In practice, a group of unmanned aerial vehicles are used for slung or cable-suspended load transportation [18], [19], [29], [30], [31], [32] and, to be energy-efficient, unmanned aerial vehicles are to cooperatively determine their own thrusts required to sustain flying height and forward speed. Take the application of multiquadrotor hose transportation in [19] as an example. There are a team of N quadrotors transporting hoses in a spraying system and it is to determine the optimal thrust to be generated by each quadrotor and minimize the total power consumption of the team. From [19], the power consumption of the ith quadrotor can be represented by $P_i(\mathtt{Th}_i, t) = \mathtt{Th}_i\{v_i \sin(\varsigma_i) + \frac{2\mathtt{Th}_i}{\pi_i \sqrt{[v_i \cos(\varsigma_i)]^2 + [v_i \sin(\varsigma_i) + v_{in}]^2}}\}$, where $\mathtt{Th}_i$ is the generated thrust, v_i is the total free steam speed (including translational velocity and wind velocity), ς_i is the angle of attack for steady flight, π_i is a known function of the density of the surrounding air and the number and diameter of rotors, and v_{in} is the induced velocity. For a quadrotor to sustain the flying height and forward velocity, its thrust is required to counter the gravity and drag forces due to the translation motion and air flow [19], i.e., $\mathtt{Th}_{i,c} = \sqrt{F_{i,w}^2 + F_{i,h}^2}$, where $F_{i,w}$ is the gravity induced by the quadrotor and payload and $F_{i,h}$ is the drag force caused by the airflow in the horizontal direction. See [19], [33], and [34] for details. Then, the thrusts generated by the quadrotors should equal the total required thrusts, i.e., $\sum_{i=1}^N \mathtt{Th}_i = \sum_{i=1}^N \mathtt{Th}_{i,c}$.

Therefore, to optimize the total power consumption of a team of N quadrotors transporting a hose, it is to solve the time-varying resource allocation problem of the form (3) to (4) [19]. Once the optimal thrust trajectories are obtained, the desired flying positions can be determined by using the methods presented in [35] and [36].

Example 2 (Economic dispatch problem of the power systems): Consider N turbine generators that supply power to some regions. The power generated by the N generators should meet the total power demand of these regions $\mathtt{D}$, which are usually time varying, i.e., $\sum_{i=1}^{N} \mathtt{P}_i = \mathtt{D}(t)$. Without loss of generality, there are N regions and the ith generator observes the demand of the ith region $\mathtt{D}_i(t)$. That is, $\sum_{i=1}^{N} \mathtt{P}_i = \sum_{i=1}^{N} \mathtt{D}_i(t)$. In the meantime, it is to minimize the total generation cost. The generation costs are represented by quadratic functions [37]. In addition, due to the heterogeneity of the generation systems and time-varying nature of the costs, the resulting quadratic functions are given by $f_i(\mathtt{P}_i, t) = a_i \mathtt{P}_i^2 + b_i \mathtt{P}_i + m_i$ with time-varying cost coefficients a_i, b_i, and $m_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$. These coefficient functions and demand functions can be empirically fitted to k times continuously differentiable functions and the value of k depends on the orders of the system dynamics. Thus, this problem can be expressed as (3) and (4). Assume that all the generators are run at synchronous speed without relative speed and neglect the mechanical and electromagnetic losses, then the dynamics of the ith generator can be represented as [38] $T_{mi}\dot{\mathtt{P}}_i = -\mathtt{P}_i + K_{mi}\mathtt{X}_{ei}$ and $T_{ei}\dot{\mathtt{X}}_{ei} = -\mathtt{X}_{ei} + v_i$, where T_{mi} and K_{mi} are, respectively, the time constant and gain of the machine's turbine, T_{ei} is the time constant of the machine's speed governor, $\mathtt{X}_{ei}$ is the valve opening of the generation system, and $\mathtt{P}_i$ and v_i are, respectively, the output electrical power and control input of the generation system. Since the dynamics are not exactly high-order integrators, to solve the economic dispatch problem, one can follow the results in Section IV-D.

We make the following assumptions on the cost functions and resource vectors.

Assumption 2: For any $i \in \mathcal{V}$, the cost function $f_i(q_i, t)$ is twice continuously differentiable with respect to q_i for all t. Moreover, it is also uniformly strongly convex with respect to q_i for all $t \geq 0$, i.e., there exists some positive constant $\underline{m}$ such that $\lambda_k(\nabla^2 f_i(q_i, t)) > \underline{m}$, $k = 1, \ldots, d$.

Assumption 3: The nth derivatives of the resource vectors $c_i(t)$, $i \in \mathcal{V}$, exist. There exists a positive constant $\bar{c}$ such that $\sup_{t\in[0,\infty)} \|c_i^{(k)}(t)\|_2 \leq \bar{c} \quad \forall k = 0, 1, \ldots, n \quad \forall i \in \mathcal{V}$.

Assumption 4: For any $i \in \mathcal{V}$, the gradient of the cost function $f_i(q_i, t)$ can be written as $\nabla f_i(q_i, t) = H_i(t) q_i + g_i(t)$, where $H_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^{d\times d}$ and $g_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ are time-varying functions. The nth derivatives of $H_i(t)$ and $g_i(t)$ exist. In addition, there exist positive constants $\bar{H}$ and $\bar{g}$ such that $\sup_{t\in[0,\infty)} \|H_i^{(l)}(t)\|_2 \leq \bar{H}$ and $\sup_{t\in[0,\infty)} \|g_i^{(l)}(t)\|_2 \leq \bar{g}$ hold for any $l = 0, 1, \ldots, n$ and any $i \in \mathcal{V}$.

Remark 1: Note that Assumptions 2–4 are related to and can be satisfied in many real-world applications, especially, robotic systems, e.g., Examples 1 and 2. In addition, the distributed average tracking problem, which has found several applications in region following formation control [39] and coordinated path planning [40], can be transformed to a distributed time-varying optimization problem by formulating the cost functions as $f_i(q_i, t) = \|q_i - g_i(t)\|_2^2$, and hence it can be solved by time-varying optimization algorithms. These assumptions ensure the convergence analysis of the exact optimum tracking. Similar assumptions have been applied in recent works on distributed time-varying optimization [41], [42], [43], especially on time-varying resource allocation [19], [22], [23], and Assumption 4 includes the case considered in all aforementioned results as special cases. Although Assumption 4 seems restrictive, in Section V-C, the proposed algorithms are empirically validated on a broad class of functions (e.g., nonquadratic) such that agents' decision variables track the optimal ones.

IV. Resource Allocation for Networked High-Order Agents

In this section, first, we design a centralized algorithm to solve the optimal time-varying resource allocation problem for high-order integrators, which necessitates the existence of a central server. Inspired by the centralized algorithm, we then develop two distributed algorithms, which can be implemented by using only local information and interaction. Then, we extend the distributed algorithms to solve the time-varying resource allocation problem for networked nonlinear systems with parametric uncertainties by using the estimation-tracking method. Before moving on to the algorithm design, we first provide basic results on the optimality condition of the optimal time-varying resource allocation problem by using the Lagrange function.

Define the Lagrange function associated with the optimization problem in (3) and (4) as

$$\mathcal{L}(q, \mu, t) = \sum_{i=1}^{N} f_i(q_i, t) + \mu^\top(t) \sum_{i=1}^{N} [q_i - c_i(t)] \tag{5}$$

where $\mu(t) \in \mathbb{R}^d$ is the Lagrange multiplier. From Assumption 2, it follows that the Lagrange function (5) is strongly convex in $q(t)$ and concave in $\mu(t)$. Then, by the linear constraint (4) with full-row-rank coefficient matrix, the optimal primal-dual pair $\{q^\star(t), \mu^\star(t)\}$ is unique at all time $t \geq 0$ [44, Sec. 5] and satisfies the following Karush–Kuhn–Tucker (KKT) condition:

$$\frac{\partial}{\partial q}\mathcal{L}(q^\star, \mu^\star, t) = \frac{\partial}{\partial q}\left[\sum_{i=1}^{N} f_i(q_i^\star, t)\right] + \mathbf{1}_N \otimes \mu^\star = \mathbf{0}_{Nd} \tag{6}$$

$$\frac{\partial}{\partial \mu}\mathcal{L}(q^\star, \mu^\star, t) = \sum_{i=1}^{N} [q_i^\star - c_i(t)] = \mathbf{0}_d. \tag{7}$$

Define $z = [q^\top, \mu^\top]^\top$. The Lagrange function (5) can be rewritten as $\mathcal{L}(z, t)$, and the conditions (6), (7) can be rewritten as $\frac{\partial}{\partial z}\mathcal{L}(z^\star, t) = \mathbf{0}_{(N+1)d}$, where $z^\star = [q^{\star\top}, \mu^{\star\top}]^\top$. Then, we have the following lemma adapted from [45].

Lemma 3: Suppose that Assumption 2 holds. If $\frac{\partial}{\partial z}\mathcal{L}(z, t) \to \mathbf{0}_{(N+1)d}$ as $t \to \infty$, i.e., $\nabla f_i(q_i, t) + \mu(t) \to \mathbf{0}_d$ and $\sum_{i=1}^{N}[q_i(t) - c_i(t)] \to \mathbf{0}_d$, then $q_i(t) \to q_i^\star(t) \quad \forall i \in \mathcal{V}$ and $\mu(t) \to \mu^\star(t)$ as $t \to \infty$.

A. Centralized Algorithm

We establish a centralized algorithm for (2) and assume that there exists a central server that can exchange information with all the agents. On the central server's end, we construct a dynamical system that estimates the optimal Lagrange multiplier using the cost function and state information obtained from the agents. In the meanwhile, we design the controllers on the agents' end by using the generated Lagrange multiplier estimate μ such that the agents can determine and track their own optimal trajectories. Specifically, construct a virtual system for the central server as

$$\mu^{(n)} = -\sum_{k=1}^{n-1} \alpha_k \left(\mu^{(k)} - \widetilde{\mu}^{(k-1)}\right) + \widetilde{\mu}^{(n-1)} \tag{8}$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are positive constants to be determined

$$\widetilde{\mu} = -\beta\mu + \left\{\sum_{j=1}^{N} \left[\nabla^2 f_j(q_j, t)\right]^{-1}\right\}^{-1} \sum_{j=1}^{N} \left\{\beta(q_j - c_j) - \dot{c}_j - \left[\nabla^2 f_j(q_j, t)\right]^{-1} \left[\beta\nabla f_j(q_j, t) + \frac{\partial}{\partial t}\nabla f_j(q_j, t)\right]\right\} \tag{9}$$

and β is a positive constant to be determined. Design the control input for agent $i \in \mathcal{V}$ as

$$u_i = -\sum_{k=1}^{n-1} \alpha_k \left[q_i^{(k)} + F_i^{(k-1)}(q_i, \mu, \dot{\mu}, t)\right] - F_i^{(n-1)}(q_i, \mu, \dot{\mu}, t) \tag{10}$$

where the vector-valued function $F_i : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ is defined as

$$F_i(q_i, \mu, \dot{\mu}, t) = \left[\nabla^2 f_i(q_i, t)\right]^{-1} \left[\frac{\partial}{\partial t}\nabla f_i(q_i, t) + \beta\nabla f_i(q_i, t) + \beta\mu + \dot{\mu}\right]. \tag{11}$$

By Assumption 2, it holds that the Hessian matrices, $\nabla^2 f_i(q_i, t)$, $i \in \mathcal{V}$, are positive definite, which implies that all $[\nabla^2 f_i(q_i, t)]^{-1}$, $i \in \mathcal{V}$, exist and are positive definite. Then, the matrix $\sum_{j=1}^{N}[\nabla^2 f_j(q_j, t)]^{-1}$ is also positive definite, and hence invertible. Thus, the definition of $\widetilde{\mu}$ in (9) is justified. By Assumptions 2–4, it holds that $\widetilde{\mu}^{(k)}$, $k = 0, 1, \ldots, n-1$, exist. Then, $F_i^{(k)}(q_i, \mu, \dot{\mu}, t)$, $k = 0, 1, \ldots, n-1$, exist, and hence, the controller (10) is well defined.

The virtual system (8) and the controller (10) are designed to drive $\dot{\mu}$ and $\dot{q}_i$ to $\widetilde{\mu}$ and $F_i(q_i, \mu, \dot{\mu}, t)$, respectively. The terms $F_i(q_i, \mu, \dot{\mu}, t)$, $i \in \mathcal{V}$, and $\widetilde{\mu}$ collectively produce desired stable linear systems (of the form $\dot{x} = -\beta x$) with states $\nabla f_i(q_i, t) + \mu(t)$ and $\sum_{i=1}^{N}[q_i(t) - c_i(t)]$ resulting in the fact that $\nabla f_i(q_i, t) + \mu(t) \to \mathbf{0}_d\ \forall i \in \mathcal{V}$ and $\sum_{i=1}^{N}[q_i(t) - c_i(t)] \to \mathbf{0}_d$ as $t \to \infty$, and hence serve as the changing rates of the desired decision and Lagrange multiplier estimate trajectories that converge to optimal decisions and optimal Lagrange multiplier (by Lemma 3), respectively. As a result, such design collectively ensures $\frac{\partial}{\partial z}\mathcal{L}(z, t) \to \mathbf{0}_{(N+1)d}$, which, by Lemma 3, implies exact optimum tracking. See the following result for detailed explanation.

Proposition 1: Suppose that Assumptions 2–4 hold and let $\beta \in \mathbb{R}_+$ and $\alpha_1, \ldots, \alpha_{n-1}$ be positive constants chosen such that the polynomial $\lambda^{n-1} + \alpha_{n-1}\lambda^{n-2} + \cdots + \alpha_1 = 0$ is Hurwitz. Using (10) with $\mu^{(j)}$, $j = 0, 1, \ldots, n$, generated/given by (8) for the high-order-integrator agents (2) solves the resource allocation problem, i.e., $q_i(t) \to q_i^\star(t)\quad \forall i \in \mathcal{V}$ as $t \to \infty$.

Proof: From (2) and (10), it follows that

$$q_i^{(n)} + F_i^{(n-1)}(q_i, \mu, \dot{\mu}, t) = u_i + F_i^{(n-1)}(q_i, \mu, \dot{\mu}, t) = -\sum_{k=1}^{n-1} \alpha_k \left[q_i^{(k)} + F_i^{(k-1)}(q_i, \mu, \dot{\mu}, t)\right]$$

which is a stable system. This means that $\dot{q}_i + F_i(q_i, \mu, \dot{\mu}, t) \to \mathbf{0}_d\ \forall i \in \mathcal{V}$ as $t \to \infty$.

It follows from the definition of $\chi_{i,1}$ that

$$\begin{aligned}[\nabla f_i(q_i, t)]^{(1)} + \dot{\mu} &= \nabla^2 f_i(q_i, t)\dot{q}_i + \frac{\partial}{\partial t}\nabla f_i(q_i, t) + \dot{\mu} \\ &= \nabla^2 f_i(q_i, t)\left[\dot{q}_i + F_i(q_i, \mu, \dot{\mu}, t)\right] - \nabla^2 f_i(q_i, t)F_i(q_i, \mu, \dot{\mu}, t) + \frac{\partial}{\partial t}\nabla f_i(q_i, t) + \dot{\mu} \\ &= -\beta[\nabla f_i(q_i, t) + \mu] + \nabla^2 f_i(q_i, t)\left[\dot{q}_i + F_i(q_i, \mu, \dot{\mu}, t)\right]\end{aligned} \tag{12}$$

where the definition of $F_i(q_i, \mu, \dot{\mu}, t)$ in (11) has been used to obtain the last equality. Note that $[\nabla f_i(q_i, t)]^{(1)} + \dot{\mu} = -\beta[\nabla f_i(q_i, t) + \mu]$ is a standard exponentially stable linear time-invariant (LTI) system. Recall that $\dot{q}_i + F_i(q_i, \mu, \dot{\mu}, t) \to \mathbf{0}_d$ as $t \to \infty$. Then, it follows from the property of the input-to-state stability [46, p. 175] and Assumption 4 (i.e., the existence of the upper bound of $\|\nabla^2 f_i(q_i, t)\|_2\quad \forall i \in \mathcal{V}$) that $\nabla f_i(q_i, t) + \mu \to \mathbf{0}_d\quad \forall i \in \mathcal{V}$ as $t \to \infty$. Hence, it holds that $\frac{\partial}{\partial q}\mathcal{L}(q, \mu, t) \to \mathbf{0}_{Nd}$ as $t \to \infty$.

By (8), it holds that

$$\mu^{(n)} - \widetilde{\mu}^{(n-1)} = -\sum_{k=1}^{n-1} \alpha_k \left(\mu^{(k)} - \widetilde{\mu}^{(k-1)}\right) \tag{13}$$

which is a stable system. It then holds that $\mu^{(k)} - \widetilde{\mu}^{(k-1)} \to \mathbf{0}_d\ \forall k = 1\ldots, n-1$ as $t \to \infty$.

By the definition of $\widetilde{\mu}$ in (9), it holds that $\sum_{i=1}^{N}[\nabla^2 f_i(q_i, t)]^{-1}\left[\frac{\partial}{\partial t}\nabla f_i(q_i, t) + \beta\nabla f_i(q_i, t) + \beta\mu + \widetilde{\mu}\right] = \beta\sum_{i=1}^{N}(q_i - c_i) - \sum_{i=1}^{N}\dot{c}_i$. Then, it follows from (11) that

$$-\sum_{i=1}^{N} F_i(q_i, \mu, \dot{\mu}, t) = -\beta\sum_{i=1}^{N}(q_i - c_i) + \sum_{i=1}^{N}\dot{c}_i - \sum_{i=1}^{N}\left[\nabla^2 f_i(q_i, t)\right]^{-1}(\dot{\mu} - \widetilde{\mu}). \tag{14}$$

It then follows that

$$\sum_{i=1}^{N}(\dot{q}_i - \dot{c}_i) = -\beta\sum_{i=1}^{N}(q_i - c_i) + \sum_{i=1}^{N}[\dot{q}_i + F_i(q_i, \mu, \dot{\mu}, t)] - \sum_{i=1}^{N}\left[\nabla^2 f_i(q_i, t)\right]^{-1}(\dot{\mu} - \widetilde{\mu}). \tag{15}$$

Recall that $\dot{q}_i + F_i(q_i, \mu, \dot{\mu}, t) \to \mathbf{0}_d$ and $\dot{\mu} - \widetilde{\mu} \to \mathbf{0}_d\quad \forall i \in \mathcal{V}$ as $t \to \infty$, and note that $\sum_{i=1}^{N}(\dot{q}_i - \dot{c}_i) = -\beta\sum_{i=1}^{N}(q_i - c_i)$ is a standard exponentially stable LTI system. It then follows from the property of the input-to-state stability [46, p. 175] and Assumption 2 (i.e., the existence of the lower bound of $\|\nabla^2 f_i(q_i,t)\|_2 \quad \forall i \in \mathcal{V}$) that $\zeta \to \mathbf{0}_d$ as $t \to \infty$. Hence, it holds that $\frac{\partial}{\partial \mu}\mathcal{L}(q,\mu,t) \to \mathbf{0}_d$ as $t \to \infty$.

From the analysis above, it holds that $\frac{\partial}{\partial z}\mathcal{L}(z,t) \to \mathbf{0}_{(N+1)d}$ as $t \to \infty$. Therefore, the statement in Proposition 1 follows by Lemma 3. ∎

From the proof of Proposition 1, it holds that $\sum_{i=1}^N [q_i(t) - c_i(t)] \to \mathbf{0}_d$ as $t \to \infty$, which implies that feasibility is achieved asymptotically. Note from (13) that $\mu^{(k)}(t) - \widetilde{\mu}^{(k-1)}(t) = \mathbf{0}_d \quad \forall t \geq 0 \quad \forall k = 1,\ldots,n-1$ if $\mu^{(k)}(0) - \widetilde{\mu}^{(k-1)}(0) = \mathbf{0}_d \quad \forall k = 1,\ldots,n-1$. From (14), (10), and (2), it follows that $\sum_{i=1}^N (q_i^{(n)} - c_i^{(n)}) = -\beta\alpha_1 \sum_{i=1}^N (q_i - c_i) - (\alpha_1 + \beta\alpha_2)\sum_{i=1}^N (q_i^{(1)} - c_i^{(1)}) - \cdots - (\alpha_{n-2} + \beta\alpha_{n-1})\sum_{i=1}^N (q_i^{(n-2)} - c_i^{(n-2)}) - (\alpha_{n-1} + \beta) \sum_{i=1}^N (q_i^{(n-1)} - c_i^{(n-1)}) - \sum_{k=0}^{n-1}\sum_{l=0}^{k}\binom{k}{l}\{\sum_{i=1}^N [\nabla^2 f_i(q_i,t)]^{-1}\}^{(l)}(\mu^{(k)} - \widetilde{\mu}^{(k-1)})$. It then follows that $\sum_{i=1}^N [q_i^{(k)}(t) - c_i^{(k)}(t)] = \mathbf{0}_d \; \forall t \geq 0 \quad \forall k = 0,1,\ldots,n-1$ if $\sum_{i=1}^N [q_i^{(k)}(0) - c_i^{(k)}(0)] = \mathbf{0}_d \quad \forall k = 0,1,\ldots,n-1$ and $\mu^{(l)}(0) - \widetilde{\mu}^{(l-1)}(0) = \mathbf{0}_d \quad \forall l = 1,\ldots,n-1$. Therefore, if the initial decision variables satisfy $\sum_{i=1}^N [q_i^{(k)}(0) - c_i^{(k)}(0)] = \mathbf{0}_d \quad \forall k = 0,1,\ldots,n-1$, one can select appropriate values for $\mu^{(k)}$, $k = 0,1,\ldots,n-1$, such that, for any $l = 1,\ldots,n-1$,

$$\begin{aligned}\mu^{(l)}(0) = &-\left(\left\{\sum_{i=1}^N [\nabla^2 f_i(q_i,t)]^{-1}\right\}^{-1} \sum_{i=1}^N \dot{c}_i(t)\right)^{(k-1)}\Bigg|_{t=0} \\ &- \left(\left\{\sum_{i=1}^N [\nabla^2 f_i(q_i,t)]^{-1}\right\}^{-1} \sum_{i=1}^N [\nabla^2 f_i(q_i,t)]^{-1} \right. \\ &\left. \times \left[\frac{\partial}{\partial t}\nabla f_i(q_i,t) + \beta \nabla f_i(q_i,t)\right]\right)^{(k-1)}\Bigg|_{t=0} \\ &- \beta\mu^{(l-1)}(0) \end{aligned} \tag{16}$$

then the decision variables are feasible all the time, i.e., $\sum_{i=1}^N [q_i(t) - c_i(t)] = \mathbf{0}_d \quad \forall t \geq 0$. Specifically, one can first select a value for $\mu(0)$, then use (16) to derive the values of $\mu^{(l)}$, $l = 1,\ldots,n-1$, iteratively.

Remark 2: Note that from the centralized algorithm, the optimal Lagrange multiplier is estimated by μ, and it requires global information, especially all the Hessian matrices $\nabla^2 f_i(q_i,t) \quad \forall i \in \mathcal{V}$, to calculate the terms $G_1^{(k)}$, $G_2^{(k)}$, $k = 0,1,\ldots,n-1$, where $G_1 = \sum_{j=1}^N [\nabla^2 f_j(q_j,t)]^{-1}$ and $G_2 = \sum_{j=1}^N \{H_j^{-1}(q_j,t)[\beta\nabla f_j(q_j,t) + \frac{\partial}{\partial t}\nabla f_j(q_j,t)] + \dot{c}_j - \beta(q_j - c_j)\}$. To derive the distributed counterpart, one can use distributed average tracking algorithms to estimate those terms in a distributed manner, which is done in this way in [19] to address the time-varying resource allocation problem with diagonal Hessian matrices. However, when it comes to nondiagonal Hessian matrices, exchanging matrices (e.g., inverses of the Hessian matrices) among the agents is expensive and not practical. In the following, we propose distributed algorithms for networked high-order-integrator agents (2) to cooperatively solve the time-varying resource allocation problem without exchanging matrices.

B. Estimator-Based Distributed Optimal Time-Varying Resource Allocation Algorithm

Inspired by the centralized algorithm in Section IV-A, we design a local virtual system [see (17)] for each agent $i \in \mathcal{V}$ to estimate the optimal Lagrange multiplier $\mu^\star$ (or the central estimate μ of the optimal Lagrange multiplier) instead of requiring a central virtual system as (8). Then, such local virtual states are used to design the controllers (22) for the agents, as in (10). Construct agent i's local virtual system as

$$\begin{aligned}\mu_i^{(n)} = &-\sum_{k=1}^{n-1}\alpha_k\left(\mu_i^{(k)} - \widetilde{\mu}_i^{(k-1)}\right) + \widetilde{\mu}_i^{(n-1)} \\ &- \gamma\nabla^2 f_i(q_i,t)\sum_{j\in\mathcal{N}_i}\rho_{i,j} + \Gamma_i(n)\end{aligned} \tag{17}$$

where $\alpha_1,\ldots,\alpha_{n-1}$, and γ are positive constants to be determined

$$\begin{aligned}\widetilde{\mu}_i = &-\beta\mu_i + \beta\nabla^2 f_i(q_i,t)(q_i - c_i) - \nabla^2 f_i(q_i,t)\dot{c}_i \\ &- \frac{\partial}{\partial t}\nabla f_i(q_i,t) - \beta\nabla f_i(q_i,t)\end{aligned} \tag{18}$$

$$\rho_{i,j} = \operatorname{sgn}\left[\sum_{k=1}^{n}\alpha_k\left(\mu_i^{(k-1)} - \mu_j^{(k-1)}\right)\right] \tag{19}$$

$$\Gamma_i(n) = -\sum_{s=1}^{n-1}\alpha_{s+1}\Omega_i(s) \tag{20}$$

$$\begin{aligned}\Omega_i(s) = &\nabla^2 f_i(q_i,t)\sum_{k=0}^{s-1}\binom{s}{k}\left\{\left[\nabla^2 f_i(q_i,t)\right]^{-1}\right\}^{(s-k)} \\ &\times\left(\mu_i^{(k+1)} - \widetilde{\mu}_i^{(k)}\right)\end{aligned} \tag{21}$$

and β and α_n are positive constants to be determined. Design the control input for agent $i \in \mathcal{V}$ as

$$\begin{aligned}u_i = &-\sum_{j=1}^{n-1}\alpha_j\left[q_i^{(j)} + F_i^{(j-1)}(q_i,\mu_i,\dot{\mu}_i,t)\right] \\ &- F_i^{(n-1)}(q_i,\mu_i,\dot{\mu}_i,t)\end{aligned} \tag{22}$$

where the function $F_i(\cdot,\cdot,\cdot,\cdot)$ is given in (11). From (19), each agent exchange $\mu_i^{(k)}$, $k = 0,1,\ldots,n-1$, with its neighbors.

Similar to the design of the centralized algorithm in Section IV-A, $\widetilde{\mu}_i$ and $F_i(q_i,\mu_i,\dot{\mu}_i,t)$ sever as the changing rate of agent i's local desired signals that converge to the optimal solutions, and the virtual systems (17) and controllers (22) are designed such that $\dot{\mu}_i$ and $\dot{q}_i$ track $\widetilde{\mu}_i$ and $F_i(q_i,\mu_i,\dot{\mu}_i,t)$, respectively. Compared with (8), the virtual system (17) has two additional terms, $-\gamma\nabla^2 f_i(q_i,t)\sum_{j\in\mathcal{N}_i}\rho_{i,j}$ and $\Gamma_i(n)$. The term $-\gamma\nabla^2 f_i(q_i,t)\sum_{j\in\mathcal{N}_i}\rho_{i,j}$ is used to force all the local Lagrange multiplier estimates μ_i, $i \in \mathcal{V}$, to follow the same trajectory, i.e., $\lim_{t\to\infty}[\mu_i(t) - \mu_j(t)] = \mathbf{0}_d \quad \forall i,j \in \mathcal{V}$ and $\gamma\nabla^2 f_i(q_i,t)$ is the Hessian-dependent gain guaranteeing the convergence under nonidentical Hessian matrices. The term $\Gamma_i(n)$ is introduced to compensate the existence of nonidentical Hessian matrices.

1) Selections of $\alpha_1,\ldots,\alpha_n$: Once the order of the system (2), i.e., the value of n, is determined, one can set $\alpha_n = 1$ and obtain a set of α_k, $k = 1,\ldots,n-1$, by the following steps.

1) Define

$$\mathcal{D}_1 := \left[\begin{array}{c:c} \mathbf{0}_{n-2} & I_{n-2} \\ \hdashline \multicolumn{2}{c}{\mathcal{D}_{1,n-1}} \end{array}\right]$$

$$\mathcal{D}_{1,n-1} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} \end{bmatrix}. \tag{23}$$

Construct a matrix $T \in \mathbb{R}^{(n-1)\times(n-1)}$ by following

$$T_{k+1} = \begin{cases} \left[1 \,\vdots\, \mathbf{0}_{n-2}^\top\right], & k = 0 \\ T_k\left(\mathcal{D}_1 + \beta_k I_{n-1}\right), & k = 1,\ldots,n-2 \end{cases} \tag{24}$$

where T_k is the kth row of T, and $\beta_1,\ldots,\beta_{n-2}$ are positive constants.

2) Solve

$$T_{n-1}\left(\mathcal{D}_1 + \beta_{n-1} I_{n-1}\right) = \mathbf{0}_{n-1}^\top \tag{25}$$

for $\alpha_1,\ldots,\alpha_{n-1}$, where β_{n-1} is a positive constant.

The resulting values of $\alpha_1,\ldots,\alpha_n$ are some functions of $\beta_1,\ldots,\beta_{n-1}$. One can verify that $T_{k,k} = 1$ for $k = 1,\ldots,n-1$, where $T_{k,j}$ is the jth entry of T_k.

Lemma 4: Let $x = [x_1^\top,\ldots,x_{n-1}^\top]^\top$ with $x_i \in \mathbb{R}^d$. The LTI system $\dot{x} = (\mathcal{D}_1 \otimes I_d)x$ is stable, where $\mathcal{D}_1$ is given in (23) with $\beta_1,\ldots,\beta_{n-1} \in \mathbb{R}_+$.

Proof: Note from (24) and (25) that $T_k\mathcal{D}_1 = T_{k+1} - \beta_k T_k$ for $k = 1,2,\ldots,n-2$, and that $T_{n-1}\mathcal{D}_1 = -\beta_{n-1}T_{n-1}$. Define $y_k = (T_k \otimes I_d)x$ for $k = 1,2,\ldots,n-1$. It then follows that

$$\dot{y}_{n-1} = -\beta_{n-1} y_{n-1} \tag{26}$$

$$\dot{y}_k = -\beta_k y_k + y_{k+1}, \quad k = n-2, \quad n-1,\ldots,1. \tag{27}$$

From (26), it holds that $y_{n-1} \to \mathbf{0}_d$ as $t \to \infty$. Then, it follows from (27) that $y_k \to \mathbf{0}_d$ for $k = n-2, n-1,\ldots,1$ as $t \to \infty$. Let $y = [y_1^\top,\ldots,y_{n-1}^\top]^\top$. It holds that $y = (T \otimes I_d)x$, and hence $x = (T^{-1} \otimes I_d)y$, which implies that $x \to \mathbf{0}_{(n-1)d}$ as $t \to \infty$. ∎

For the sake of notational and analytical simplicity, we set $\beta_1,\ldots\beta_{n-1}$ to β and let

$$\alpha_k = \binom{n-1}{k-1}\beta^{n-k}, \quad k = 1,\ldots,n. \tag{28}$$

In addition, the (k,j)th entry of T is given by

$$T_{k,j} = \begin{cases} \binom{k-1}{j-1}\beta^{k-j}, & \text{if } k \le j \\ 0, & \text{if } k > j \end{cases} \tag{29}$$

and hence, the kth row of T is given by

$$T_k = \left[\binom{k-1}{0}\beta^{k-1} \quad \cdots \quad \binom{k-1}{k-2}\beta \quad 1 \,\vdots\, \mathbf{0}_{n-1-k}^\top\right]. \tag{30}$$

Define $\Pi_1 := \mathcal{D}_1 + \beta I_{n-1}$. One can verify that $T_{k+1} = T_k\Pi_1$, $k = 1,\ldots,n-2$ and $T_{n-1}\Pi_1 = \mathbf{0}_{n-1}^\top$ hold. By using a similar analysis to the proof of Lemma 4, the following result holds straightforwardly.

Corollary 1: Given any positive integer m, let $x = [x_1^\top,\ldots,x_m^\top]^\top$ with $x_i \in \mathbb{R}^d$. The LTI system $\dot{x} = (\mathcal{A} \otimes I_d)x$ is stable, where

$$\mathcal{A} = \left[\begin{array}{c:c} \mathbf{0}_{m-1} & I_{m-1} \\ \hdashline \multicolumn{2}{c}{\mathcal{A}_m} \end{array}\right]$$

$$\mathcal{A}_m = \begin{bmatrix} -\binom{m}{0}\beta^m & -\binom{m}{1}\beta^{m-1} & \cdots & -\binom{m}{m-1}\beta \end{bmatrix}.$$

Note that such a setting/procedure is just an example to determine the values of $\alpha_1,\ldots,\alpha_n$, and there are other ways to find appropriate values for $\alpha_1,\ldots,\alpha_n$ such that the following convergence analysis holds by minor revisions.

2) Convergence Analysis: Before moving on to the convergence analysis, a preliminary lemma is presented.

Lemma 5: Assume that $\nabla^2 f_i(q_i,t)$ is invertible and its nth order derivative exists. It holds that for $s = 1,\ldots,n-1$

$$\Omega_i(s) = -\sum_{j=0}^{s-1}\binom{s}{j}\left[\nabla^2 f_i(q_i,t)\right]^{(s-j)} e_{i,1}^{(j)} \tag{31}$$

where $e_{i,1} = [\nabla^2 f_i(q_i,t)]^{-1}(\dot{\mu}_i - \widetilde{\mu}_i)$.

Proof: For notational simplicity, let $H_i = \nabla^2 f_i(q_i,t)$. Since H_i is invertible, H_i^{-1} exists. Then, for any $j = 1,\ldots,s$, it holds that $\mathbf{0}_{d\times d} = I_d^{(j)} = (H_iH_i^{-1})^{(j)} = \sum_{k=0}^{j}\binom{j}{k}H_i^{(j-k)}(H_i^{-1})^{(k)}$. Then, it holds that

$$H_i\left(H_i^{-1}\right)^{(j)} = -\sum_{k=0}^{j-1}\binom{j}{k}H_i^{(j-k)}\left(H_i^{-1}\right)^{(k)}. \tag{32}$$

For nonnegative integer k, define

$$E_{j,k} = \begin{cases} \binom{j}{k}\left(H_i^{-1}\right)^{(j-k)}, & \text{if } k \le j \\ \mathbf{0}_{d\times d}, & \text{if } k > j. \end{cases}$$

By the definition of $e_{i,1}$, it holds that $e_{i,1}^{(j)} = \sum_{k=0}^{j} E_{j,k}(\mu_i^{(k+1)} - \widetilde{\mu}_i^{(k)})$. Note that $\binom{s}{j}\binom{j}{k} = \frac{s!}{(j-k)!\,(s-j)!\,k!} = \binom{s}{k}\binom{s-k}{l}$ holds for $l = 0,\ldots,s-k-1$. Then, for any $s = 1,\ldots,n-1$, and any $k = 0,1,\ldots,s-1$, it holds that

$$\begin{aligned} \sum_{j=0}^{s-1}\binom{s}{j}H_i^{(s-j)}E_{j,k} &= \sum_{j=k}^{s-1}\binom{s}{j}H_i^{(s-j)}\binom{j}{k}\left(H_i^{-1}\right)^{(j-k)} \\ &= \sum_{l=0}^{s-k-l}\binom{s}{k}\binom{s-k}{l}H_i^{(s-k-l)}\left(H_i^{-1}\right)^{(l)} \\ &= -\binom{s}{k}H_i\left(H_i^{-1}\right)^{(s-k)} \end{aligned} \tag{33}$$

where the definition of $E_{j,k}$ and (32) have been used to obtain the second and third equalities, respectively. Then, it holds that

$$\begin{aligned} &-\sum_{j=0}^{s-1}\binom{s}{j}H_i^{(s-j)}e_{i,1}^{(j)} \\ &= -\sum_{j=0}^{s-1}\binom{s}{j}H_i^{(s-j)}\sum_{k=0}^{j}E_{j,k}\left(\mu_i^{(k+1)} - \widetilde{\mu}_i^{(k)}\right) \\ &= -\sum_{k=0}^{s-1}\sum_{j=0}^{s-1}\binom{s}{j}H_i^{(s-j)}E_{j,k}\left(\mu_i^{(k+1)} - \widetilde{\mu}_i^{(k)}\right) \end{aligned}$$

$$= \sum_{k=0}^{s-1} \binom{s}{k} H_i \left(H_i^{-1}\right)^{(s-k)} \left(\mu_i^{(k+1)} - \widetilde{\mu}_i^{(k)}\right)$$

where the fact that $E_{i,k} = \mathbf{0}_{d\times d}$ for any $k > j$ has been used to obtain the second equality, and (33) has been used to obtain the last equality. This implies that (31) holds. ∎

For $k = 1, \ldots, n$, and $s = 1, \ldots, n-1$, define

$$\zeta_{i,k}(t) := [q_i(t) - c_i(t)]^{(k-1)} \tag{34}$$

$$e_{i,s}(t) := \left\{ \left[\nabla^2 f_i(q_i,t)\right]^{-1} (\dot{\mu}_i - \widetilde{\mu}_i) \right\}^{(s-1)}. \tag{35}$$

It follows from Assumptions 2 and 4, (17), and (22) that (34) and (35) are well defined. The following theorem shows the exact convergence of the proposed algorithm in this subsection to the optimal trajectories.

Theorem 1: Suppose that Assumptions 1–4 hold and let positive constants β and γ be selected to satisfy that

$$\gamma \bar{H} \geq 2|\mathcal{E}| d\bar{\omega}\sqrt{Nd} \tag{36}$$

where

$$\begin{aligned}\bar{\omega} &= \bar{H}\bar{e}\sum_{k=1}^{n-1}\sum_{s=k}^{n-1}\binom{n-1}{s}\frac{(1+\beta)^{n-k-1}}{\beta^{s-k}} + \bar{g}(1+\beta)^n \\ &\quad + \bar{H}\bar{\zeta}\sum_{k=1}^{n}\sum_{s=k}^{n}\binom{n-1}{s-k}\frac{(1+\beta)^{n-k}}{\beta^{n-s}} + \bar{H}\bar{c}(2+\beta)^n \\ &\quad + \bar{H}\gamma N\sqrt{d}\left[\left(1+\frac{1}{\beta}\right)^n - 1\right]\end{aligned} \tag{37}$$

$$\bar{\zeta} = \max_{i\in\mathcal{V}}\left\{\max_{k=1,\ldots,n}\left\{\left\|\zeta_{i,k}(t)\Big|_{t=0}\right\|_2\right\}\right\} \tag{38}$$

$$\bar{e} = \max_{i\in\mathcal{V}}\left\{\max_{k=1,\ldots,n-1}\left\{\left\|e_{i,k}(t)\Big|_{t=0}\right\|_2\right\}\right\} \tag{39}$$

and $\zeta_{i,k}$ and $e_{i,k}$ are given in (34) and (35), respectively. Using (22) with $\mu_i^{(k)}$, $k = 0, 1, \ldots, n$, generated/given by (17) for the integrator agents (2) solves the distributed resource allocation problem, i.e., $q_i(t) \to q_i^\star(t)\ \ \forall i \in \mathcal{V}$ as $t \to \infty$.

Proof: The proof is divided into four steps.

In *Step 1*, it is proved that $\nabla f_i(q_i,t) + \mu_i \to \mathbf{0}_d\ \ i \in \mathcal{V}$ as $t \to \infty$. For $k = 1, \ldots, n-1$, define $\chi_{i,k} = q_i^{(k)} + F_i^{(k-1)}(q_i, \mu_i, \dot{\mu}_i, t)$, and let $\chi_i = [\chi_{i,1}^\top, \ldots, \chi_{i,n-1}^\top]^\top$. It follows from (22) that

$$\dot{\chi}_i = (\mathcal{D}_1 \otimes I_d)\chi_i \tag{40}$$

where $\mathcal{D}_1$ is given in (23) with

$$\mathcal{D}_{1,n-1} = \left[-\binom{n-1}{0}\beta^{n-1} \quad -\binom{n-1}{1}\beta^{n-2} \quad \cdots \quad -\binom{n-1}{n-2}\beta\right].$$

It follows from Lemma 4 that the system (40) is stable. Hence, it holds that $\chi_i \to \mathbf{0}_{(n-1)d}\ \ \forall i \in \mathcal{V}$ as $t \to \infty$.

Define $\psi_i := \nabla f_i(q_i,t) + \mu_i$. By (12), it holds that $\dot{\psi}_i = -\alpha\psi_i + H_i(q_i,t)\chi_{i,1}$. Recall the fact that $\chi_i \to \mathbf{0}_{(n-1)d}\ \ \forall i \in \mathcal{V}$ as $t \to \infty$. Then, it holds that $\psi_i \to \mathbf{0}_d\ \ \forall i \in \mathcal{V}$ as $t \to \infty$, which completes Step 1.

In *Step 2*, it is proved that $\sum_{i=1}^N [q_i(t) - c_i(t)] \to \mathbf{0}_d$ as $t \to \infty$. From (11), it holds that

$$\begin{aligned}&F_i(q_i, \mu_i, \dot{\mu}_i, t) \\ &= \left[\nabla^2 f_i(q_i,t)\right]^{-1}(\dot{\mu}_i - \widetilde{\mu}_i) + \left[\nabla^2 f_i(q_i,t)\right]^{-1}\left\{\frac{\partial}{\partial t}\nabla f_i(q_i,t)\right. \\ &\quad \left. + \beta\nabla f_i(q_i,t) + \beta\mu_i + \widetilde{\mu}_i\right\} \\ &= e_{i,1} + \beta(q_i - c_i) - \dot{c}_i\end{aligned} \tag{41}$$

where $e_{i,1}$ is defined in (35), and the definition of $\widetilde{\mu}_i$ in (18) has been used to obtain the second equality. It follows from the definition of $\zeta_{i,k}$ in (34), (2), (22), and (41) that

$$\begin{aligned}\dot{\zeta}_{i,n} &= -\binom{n-1}{0}\beta^n\zeta_{i,1} - \sum_{k=1}^{n-1}\left[\binom{n-1}{k-1} + \binom{n-1}{k}\right]\beta^{n-k} \\ &\qquad \times \zeta_{i,k+1} - \sum_{k=1}^{n-1}\binom{n-1}{k-1}\beta^{n-k}e_{i,1}^{(k-1)} - e_{i,1}^{(n-1)} \\ &= -\sum_{k=1}^{n}\binom{n}{k-1}\beta^{n+1-k}\zeta_{i,k} - \sum_{k=1}^{n}\binom{n-1}{k-1}\beta^{n-k}e_{i,1}^{(k-1)}\end{aligned}$$

where (28) and the fact that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$ have been used. By the definition of $e_{i,k}$ in (35), (17), (20), and (21), it holds that

$$\begin{aligned}\dot{e}_{i,n-1} &= \sum_{k=0}^{n-2}\binom{n-1}{k}\left\{\left[\nabla^2 f_i(q_i,t)\right]^{-1}\right\}^{(n-1-k)}\left(\mu_i^{(k+1)}\right. \\ &\qquad \left. - \widetilde{\mu}_i^{(k)}\right) + \left[\nabla^2 f_i(q_i,t)\right]^{-1}\left(\mu_i^{(n)} - \widetilde{\mu}_i^{(n-1)}\right) \\ &= -\sum_{k=1}^{n-1}\binom{n-1}{k-1}\beta^{n-k}e_{i,k} - \gamma\sum_{j\in\mathcal{N}_i}\rho_{i,j}\end{aligned} \tag{42}$$

where $\rho_{i,j}$ is given in (19). Hence, it holds that

$$\dot{\zeta}_{i,n} = -\sum_{k=1}^{n}\binom{n}{k-1}\beta^{n+1-k}\zeta_{i,k} + \gamma\sum_{j\in\mathcal{N}_i}\rho_{i,j}. \tag{43}$$

Define $x_k := \sum_{i=1}^N \zeta_{i,k}$ for $k = 1, \ldots, n$, and $x = [x_1^\top, \ldots, x_n^\top]^\top$. It then holds that

$$\dot{x} = (\mathcal{D}_2 \otimes I_d)\, x \tag{44}$$

where

$$\mathcal{D}_2 = \left[\begin{array}{c:c}\mathbf{0}_{n-1} & I_{n-1} \\ \hdashline \multicolumn{2}{c}{\mathcal{D}_{2,n}}\end{array}\right]$$

$$\mathcal{D}_{2,n} = \left[-\binom{n}{0}\beta^n \quad -\binom{n}{1}\beta^{n-1} \quad \cdots \quad -\binom{n}{n-1}\beta\right] \tag{45}$$

and the fact that $\sum_{i=1}^N\sum_{j\in\mathcal{N}_i}\rho_{i,j} = \mathbf{0}_d$ according to Assumption 1 has been used. It follows from Corollary 1 that (44) is stable. Then, it holds that $x \to \mathbf{0}_{nd}$ as $t \to \infty$. That is, $\sum_{i=1}^N \zeta_{i,1} \to \mathbf{0}_d$ as $t \to \infty$, which completes Step 2.

In *Step 3*, it is proved that $\mu_i - \mu_j \to \mathbf{0}_d\ \ \forall i,j \in \mathcal{V}$ as $t \to \infty$. It follows from the definition of $\widetilde{\mu}_i$ in (18) that the system

(17) can be rewritten as

$$\mu_i^{(n)} = -\sum_{k=1}^{n}\binom{n}{k-1}\beta^{n+1-k}\mu_i^{(k-1)} - \gamma\nabla^2 f_i(q_i,t)\sum_{j\in\mathcal{N}_i}\rho_{i,j} + \omega_i(q_i,t) \tag{46}$$

where $\omega_i(q_i,t) = \mathcal{P}_i(n) + \Gamma_i(n)$, $\mathcal{P}_i(n) = \sum_{k=1}^{n}\binom{n-1}{k-1}\beta^{n-k} P_i^{(k-1)}$, $P_i = \beta\nabla^2 f_i(q_i,t)(q_i - c_i) - \nabla^2 f_i(q_i,t)\dot{c}_i - \frac{\partial}{\partial t}\nabla f_i(q_i,t) - \beta\nabla f_i(q_i,t)$, and $\Gamma_i(n)$ is given in (20).

Define $e_i = \left[e_{i,1}^\top,\ldots,e_{i,n-1}^\top\right]^\top$. It follows from the definitions of $e_{i,k}$, $k = 1,\ldots,n-1$, and (42) that

$$\dot{e}_i = (\mathcal{D}_1\otimes I_d)e_i - \underbrace{\begin{bmatrix}\mathbf{0}_{(n-2)d\times d}\\ \hdashline I_d\end{bmatrix}}_{:=\Delta_1}\gamma\sum_{j\in\mathcal{N}_i}\rho_{i,j}. \tag{47}$$

Define $y_{i,k} := (T_{n-1}\otimes I_d)e_i$, $k = 1,\ldots,n-1$, where T_k are given in (30). Note from (29) that $T_{n-1,n-1} = 1$. Then, it holds that $(T_{n-1}\otimes I_d)\Delta_1 = T_{n-1,n-1}\otimes I_d = I_d$. It follows from (42), (47), and (55) that

$$\begin{aligned}\dot{y}_{i,n-1} &= (T_{n-1}\otimes I_d)\dot{e}_i\\ &= (T_{n-1}\mathcal{D}_1\otimes I_d)e_i - \gamma\sum_{j\in\mathcal{N}_i}\rho_{i,j}\\ &= -\beta y_{i,n-1} - \gamma\sum_{j\in\mathcal{N}_i}\rho_{i,j}.\end{aligned} \tag{48}$$

Similarly, it holds that

$$\dot{y}_{i,k} = -\beta y_{i,k} + y_{i,k+1},\quad k = 1,\ldots,n-2. \tag{49}$$

It follows from (48) that

$$y_{i,n-1}(t) = e^{-\beta t}y_{i,n-1}(0) - \gamma\int_0^t e^{-\beta(t-\tau)}\sum_{j\in\mathcal{N}_i}\rho_{i,j}(\tau)\mathrm{d}\tau.$$

Note that for any $i\in\mathcal{V}$, it holds that $\|\sum_{j\in\mathcal{N}_i}\rho_{i,j}\|_2 \le \sum_{j\in\mathcal{N}_i}\|\rho_{i,j}\|_2 \le \sqrt{d}\sum_{j\in\mathcal{N}_i}\|\rho_{i,j}\|_\infty \le N\sqrt{d}$. It follows that

$$\begin{aligned}&\|y_{i,n-1}(t)\|_2\\ &\le e^{-\beta t}\left[\|y_{i,n-1}(0)\|_2 + \gamma\left\|\int_0^t e^{\beta\tau}\sum_{j\in\mathcal{N}_i}\rho_{i,j}(\tau)\mathrm{d}\tau\right\|_2\right]\\ &\le e^{-\beta t}\left[\|y_{i,n-1}(0)\|_2 + \left(\gamma N\sqrt{d}\right)\frac{e^{\beta\tau}}{\beta}\bigg|_0^t\right]\\ &\le \|y_{i,n-1}(0)\|_2 + \frac{\gamma N\sqrt{d}}{\beta}\end{aligned} \tag{50}$$

where the fact that $e^{-\alpha t}\le 1\quad\forall t\in\mathbb{R}_{\ge 0}$ has been used to obtain the last inequality. Similarly, it follows from (49) that

$$\|y_{i,k}(t)\|_2 \le \sum_{j=0}^{n-1-k}\frac{1}{\beta^{n-1-k-j}}\|y_{i,n-1-j}(0)\|_2 + \frac{\gamma N\sqrt{d}}{\beta^{n-k}}$$
$$k = n-1,\ldots,1. \tag{51}$$

From Lemma 5, it follows that

$$\begin{aligned}\Gamma_i(n) &= \sum_{s=1}^{n-1}\binom{n-1}{s}\beta^{n-1-s}\sum_{j=0}^{s-1}\binom{s}{j}\left[\nabla^2 f_i(q_i,t)\right]^{(s-j)}e_{i,1}^{(j)}\\ &= \sum_{k=1}^{n-1}\left[\nabla^2 f_i(q_i,t)\right]^{(k)}\mathcal{S}_k\end{aligned}$$

where $\mathcal{S}_k = \sum_{s=k}^{n-1}\binom{n-1}{s}\binom{s}{s-k}\beta^{n-1-s}e_{i,1}^{(s-k)}$. For $k = 1,\ldots,n-1$, it holds that

$$\begin{aligned}\mathcal{S}_k &= \sum_{s=k}^{n-2}\binom{s}{s-k}\binom{n-1}{s}\beta^{n-1-s}e_{i,1}^{(s-k)} + \binom{n-1}{n-1-k}\\ &\quad\times\left[y_{i,n-k} - \sum_{s=k}^{n-2}\binom{n-k-1}{s-k}\beta^{n-1-s}e_{i,1}^{(s-k)}\right]\\ &= \binom{n-1}{n-1-k}y_{i,n-k}\end{aligned}$$

where the definition of $y_{i,n-k}$ has been used to obtain the first equality, and the second equality follows by noting that $\binom{s}{s-k}\binom{n-1}{s} = \frac{(n-1)!}{(n-1-s)!\,(s-k)!\,k!} = \binom{n-1}{n-1-k}\binom{n-k-1}{s-k}$ for $s = k,\ldots,n-2$. Hence, it holds that

$$\Gamma_i(n) = \sum_{k=1}^{n-1}\left[\nabla^2 f_i(q_i,t)\right]^{(k)}\binom{n-1}{n-1-k}y_{i,n-k}.$$

Then, it follows from (51) that

$$\|\Gamma_i(n)\|_2 \le \bar{H}\sum_{k=1}^{n-1}\|y_{i,n-k}(0)\|_2\sum_{s=k}^{n-1}\binom{n-1}{n-1-s}\frac{1}{\beta^{s-k}} + \bar{H}\gamma N\sqrt{d}\sum_{k=1}^{n-1}\binom{n-1}{k}\frac{1}{\beta^k}. \tag{52}$$

Define $\zeta_i = [\zeta_{i,1}^\top,\ldots,\zeta_{i,n}^\top]^\top$. It follows from (43) that

$$\dot{\zeta}_i = (\mathcal{D}_2\otimes I_d)\zeta_i + \underbrace{\begin{bmatrix}\mathbf{0}_{(n-1)d\times d}\\ \hdashline I_d\end{bmatrix}}_{:=\Delta_2}\gamma\sum_{j\in\mathcal{N}_i}\rho_{i,j} \tag{53}$$

where $\mathcal{D}_2$ is given in (45). Define a matrix $\mathcal{T}\in\mathbb{R}^{n\times n}$, whose jth row is given as

$$\mathcal{T}_j = \begin{cases}\left[T_j \;\vdots\; \mathbf{0}_{n-j}^\top\right], & \text{if } j\ne n,\\ \left[\binom{n-1}{0}\beta^{n-1}\quad\binom{n-1}{1}\beta^{n-2}\quad\cdots\quad\binom{n-1}{n-2}\beta\quad 1\right] & \\ \text{if } j = n.\end{cases} \tag{54}$$

Define a matrix $\Pi_2 := \mathcal{D}_2 + \beta I_n$. It can be verified that

$$\begin{cases}\mathcal{T}_k\Pi_2 = \mathcal{T}_{k+1}, & k = 1,\ldots,n-1\\ \mathcal{T}_n\Pi_2 = \mathbf{0}_n^\top.\end{cases} \tag{55}$$

Define $z_{i,n} = (\mathcal{T}_n\otimes I_d)\zeta_i$, and it follows from (53) and (43) that

$$\dot{z}_{i,n} = -\beta z_{i,n} + \gamma\sum_{j\in\mathcal{N}_i}\rho_{i,j}.$$

By using a similar analysis to that for system (48), it holds that $\|z_{i,n}(t)\|_2 \le \|z_{i,n}(0)\|_2 + \frac{\gamma N\sqrt{d}}{\alpha}$. Define $z_{i,k} := (\mathcal{T}_k\otimes I_d)\zeta_i$

for $k = n-1, \ldots, 1$. It holds that

$$\dot{z}_{i,k} = -\beta z_{i,k} + z_{i,k+1}, \quad k = 1, \ldots, n-1. \tag{56}$$

It then holds that $\|z_{i,k}(t)\|_2 \le \sum_{j=0}^{n-k} \frac{1}{\beta^{n-k-j}} \|z_{i,n-j}(0)\|_2 + \frac{\gamma N\sqrt{d}}{\beta^{n-k+1}}$, $k = 1, \ldots, n$.

From Assumption 4, it follows that $P_i = -\beta H_i c_i - H_i \dot{c}_i - \dot{H}_i q_i - \dot{g}_i - \beta g_i$. Then,

$$\mathcal{P}_i(n) = \omega_{i,1} + \omega_{i,2} \tag{57}$$

where $\omega_{i,1} = -\sum_{k=1}^{n} \binom{n-1}{k-1} \beta^{n-k} \sum_{s=0}^{k-1} \binom{k-1}{s} H_i^{(k-s)} q_i^{(s)}$ and $\omega_{i,2} = -\sum_{k=1}^{n} \binom{n-1}{k-1} \beta^{n-k} [\sum_{s=0}^{k-1} \binom{k-1}{s} H_i^{(k-1-s)} (\beta c_i^{(s)} + c_i^{(s+1)}) + g_i^{(k)} + \beta g_i^{(k-1)}]$. Note that

$$\begin{aligned}
\omega_{i,1} &= -\sum_{k=1}^{n} H_i^{(k)} \sum_{s=k}^{n} \binom{n-1}{s-1} \beta^{n-s} \binom{s-1}{s-k} q_i^{(s-k)} \\
&= -\sum_{k=1}^{n} H_i^{(k)} \Bigg\{ \sum_{s=k}^{n-1} \binom{n-1}{s-1} \beta^{n-s} \binom{s-1}{s-k} q_i^{(s-k)} \\
&\quad + \binom{n-1}{n-k} \Bigg[z_{i,n-k+1} - \sum_{s=k}^{n-1} \binom{n-k}{s-k} \beta^{n-s} \\
&\quad \times \left(q_i^{(s-k)} - c_i^{(s-k)} \right) + c_i^{(n-k)} \Bigg] \Bigg\} \\
&= -\sum_{k=1}^{n} H_i^{(k)} \binom{n-1}{n-k} z_{i,n-k+1} + \omega_{i,3}
\end{aligned} \tag{58}$$

where $\omega_{i,3} = -\sum_{k=1}^{n} H_i^{(k)} \binom{n-1}{n-k} \sum_{s=k}^{n} \binom{n-k}{s-k} \beta^{n-s} c_i^{(s-k)}$, and the definition of $\mathcal{T}_{n-k+1}$ and the fact that $\binom{n-1}{s-1}\binom{s-1}{s-k} = \frac{(n-1)!}{(k-1)!\,(n-s)!\,(s-k)!} = \binom{n-1}{n-k}\binom{n-k}{s-k}$ have been used to obtain the second and third equalities, respectively. It follows from Assumption 4 that

$$\begin{aligned}
\|\omega_{i,2}\|_2 &\le \sum_{k=1}^{n} \binom{n-1}{k-1} \beta^{n-k} \Bigg[(\beta+1)\bar{H}\bar{c} \sum_{s=0}^{k-1} \binom{k-1}{s} \\
&\quad + (\beta+1)\bar{g} \Bigg] \\
&= \bar{H}\bar{c}(\beta+1)(\beta+2)^{n-1} + \bar{g}(\beta+1)^n.
\end{aligned} \tag{59}$$

Similarly, it holds that

$$\|\omega_{i,3}\|_2 \le \bar{H}\bar{c}(\beta+2)^{n-1}. \tag{60}$$

It also follows that

$$\begin{aligned}
&\left\| \sum_{k=1}^{n} H_i^{(k)} \binom{n-1}{n-k} z_{i,n-k+1} \right\|_2 \\
&\le \bar{H} \sum_{k=1}^{n} \|z_{i,n-k+1}(0)\|_2 \sum_{s=k}^{n} \binom{n-1}{s-k} \frac{1}{\beta^{n-s}} \\
&\quad + \bar{H}\gamma N\sqrt{d} \sum_{k=1}^{n} \binom{n-1}{n-k} \frac{1}{\beta^k}.
\end{aligned} \tag{61}$$

From the definition of $y_{i,n-k}$, it holds that $\|y_{i,n-k}(0)\|_2 = \|\sum_{j=1}^{n-1} T_{n-k,j} e_{i,j}(0)\|_2 \le \sum_{j=1}^{n-1} T_{n-k,j} \|e_{i,j}(0)\|_2 \le \bar{e}\|T_{n-k}\|_1 = \bar{e}(1+\beta)^{n-k-1}$, where $\bar{e}$ is given in (39). Similarly,

$$\|z_{i,n-k+1}(0)\|_2 \le \bar{\zeta}(1+\beta)^{n-k}$$

where $\bar{\zeta}$ is given in (38). In addition, it holds that

$$\begin{aligned}
&\bar{H}\gamma N\sqrt{d} \sum_{k=1}^{n-1} \binom{n-1}{k} \frac{1}{\beta^k} + \bar{H}\gamma N\sqrt{d} \sum_{k=1}^{n} \binom{n-1}{n-k} \frac{1}{\beta^k} \\
&= \bar{H}\gamma N\sqrt{d} \left[\left(1 + \frac{1}{\beta}\right)^n - 1 \right].
\end{aligned}$$

Thus, it follows from (52), (57)–(61) that

$$\|\omega_i(q_i,t)\|_\infty \le \|\omega_i(q_i,t)\|_2 \le \bar{\omega} \tag{62}$$

where $\bar{\omega}$ is given in (37).

Let $\widehat{\mu}_i = \left[\mu_i^\top, \mu_i^{(1)\top}, \ldots, \mu_i^{(n-1)\top}\right]^\top$. Define $\delta_i = (\mathcal{T}_n \otimes I_d)\widehat{\mu}_i$ and $\delta = [\delta_1^\top, \ldots, \delta_N^\top]^\top$. Then, it holds that

$$\dot{\delta} = -\beta\delta - \gamma \boldsymbol{H}(t)(B \otimes I_d)\operatorname{sgn}\left[\left(B^\top \otimes I_d\right)\delta\right] + \omega(q,t) \tag{63}$$

where $\boldsymbol{H}(t) = \operatorname{diag}\{H_1(t), \ldots, H_N(t)\}$ and $\omega(q,t) = [\omega_1^\top(q_1,t), \ldots, \omega_N^\top(q_N,t)]^\top$. Since the signum function is measurable and locally essentially bounded and $\omega(q,t)$ is bounded, by Lemma 1, the Filippov solutions of (63) exist and are absolutely continuous, that is, δ is continuous. Hence, $\mathcal{K}[-\beta\delta] = \{-\beta\delta\}$ and $\mathcal{K}[\omega(q,t)] \subseteq [-\bar{\omega}, \bar{\omega}]^{Nd}$, where $\bar{\omega}$ is given in (37). It holds that $\mathcal{K}[\dot{\delta}] \subseteq \mathcal{F}_\delta$, where

$$\mathcal{F}_\delta = \{-\beta\delta\} + \mathcal{K}[\omega(q,t)] - \gamma\boldsymbol{H}(B \otimes I_d)\mathcal{K}\left[\operatorname{sgn}\left[(B^\top \otimes I_d)\delta\right]\right].$$

Note that, for any $r = [r_1, \ldots, r_d] \in \mathbb{R}^d$, it holds that $\mathcal{K}[\operatorname{sgn}(r)] = \mathcal{K}[\operatorname{sgn}(r_1)] \times \cdots \times \mathcal{K}[\operatorname{sgn}(r_d)]$ and $\mathcal{K}[\operatorname{sgn}(r_i)] = \{1\}$ if $r_i > 0$, $\mathcal{K}[\operatorname{sgn}(r_i)] = \{-1\}$ if $r_i < 0$, and $\mathcal{K}[\operatorname{sgn}(r_i)] = [-1, 1]$ if $r_i = 0$. Consider the Lyapunov function candidate $V[\delta(t)] = \|(B^\top \otimes I_d)\delta\|_1$. Note that V is locally Lipschitz continuous but nonsmooth at some points. Then, by Lemma 2, it holds that $\dot{V}[\delta(t)] \in \dot{\tilde{V}}$. The generalized gradient of V is given by

$$\partial V = (B \otimes I_d)\mathcal{K}\left[\operatorname{sgn}\left[(B^\top \otimes I_d)\delta\right]\right].$$

From Lemma 2, the set-valued Lie derivative of V is given by

$$\dot{\tilde{V}} \subseteq \bigcap_{\xi \in \mathcal{K}[\operatorname{sgn}[(B^\top \otimes I_d)\delta]]} \xi^\top (B^\top \otimes I_d)\mathcal{F}_\delta. \tag{64}$$

By (64), it holds that if $\dot{\tilde{V}} \neq \emptyset$ and assume that $\tilde{a} \in \dot{\tilde{V}}$, then there exist $\tilde{\eta} \in \mathcal{K}[\operatorname{sgn}[(B^\top \otimes I_d)\delta]]$ and $\tilde{\omega} \in \mathcal{K}[\omega(q,t)]$ such that $\tilde{a} = \xi^\top(B^\top \otimes I_d)[-\beta\delta - \gamma\boldsymbol{H}(B \otimes I_d)\tilde{\eta} + \tilde{\omega}]$ holds for any $\xi \in \mathcal{K}[\operatorname{sgn}[(B^\top \otimes I_d)\delta]]$. Define $\rho = \operatorname{sgn}[(B^\top \otimes I_d)\delta]$, and for such $\tilde{\eta}$ and $\tilde{\omega}$, one can choose $\xi = \tilde{\xi} \in \mathcal{K}[\operatorname{sgn}[(B^\top \otimes I_d)\delta]]$ such that $\tilde{\xi}_i = \rho_i$ if $\rho_i \neq 0$ and $\tilde{\xi}_i = \tilde{\eta}_i$ if $\rho_i = 0$, where $\tilde{\xi}_i$, ρ_i, and $\tilde{\eta}_i$ denote the ith element of the vectors $\tilde{\xi}$, ρ, and $\tilde{\eta}$, respectively. Note that $\rho_i = 0$ if and only if $X_i = 0$, where X_i is the ith element of the vector $X = (B^\top \otimes I_d)\delta$. Then, it holds that $-\beta\tilde{\xi}^\top(B^\top \otimes I_d)\delta = -\beta\|(B^\top \otimes I_d)\delta\|_1$. It also holds that $-\gamma\xi^\top(B^\top \otimes I_d)\boldsymbol{H}(B \otimes I_d)\tilde{\eta} \le -\gamma\lambda_{\min}(\boldsymbol{H})\|(B \otimes I_d)\xi\|_2^2 \le -\gamma\bar{H}\|(B \otimes I_d)\xi\|_2^2$ and $\xi^\top(B^\top \otimes I_d)\tilde{\omega} \le \|(B \otimes$

$I_d)\xi\|_2\|\tilde{\omega}\|_2 \le \bar{\omega}\sqrt{Nd}\|(B\otimes I_d)\xi\|_1 \le 2|\mathcal{E}|d\bar{\omega}\sqrt{Nd}\|\xi\|_\infty \le 2|\mathcal{E}|d\bar{\omega}\sqrt{Nd}$. If there exists an edge $(i,j)\in\mathcal{E}$ such that $\delta_i \ne \delta_j$, then $\|(B\otimes I_d)\xi\|_2 \ge 1$. Then, it holds that

$$\tilde{a} \le -\beta\left\|(B^\top \otimes I_d)\delta\right\|_1 - \gamma\bar{H} + 2|\mathcal{E}|d\bar{\omega}\sqrt{Nd}.$$

Hence, if β and γ are selected to satisfy (36), then it holds that $\tilde{a} \le -\beta\|(B^\top\otimes I_d)\delta\|_1$. Therefore, for any $\tilde{a}\in\dot{\tilde{V}}$, if there exists an edge $(i,j)\in\mathcal{E}$ such that $\delta_i\ne\delta_j$, $\tilde{a}\le -\beta\|(B^\top\otimes I_d)\delta\|_1$. Hence, it follows from that $(B^\top\otimes I_d)\delta\to\mathbf{0}_{|\mathcal{E}|d}$ as $t\to\infty$. That is, $\delta_i-\delta_j\to\mathbf{0}_d \quad \forall (i,j)\in\mathcal{E}$ as $t\to\infty$. It then follows from Assumption 1 that $\delta_i-\delta_j\to\mathbf{0}_d \quad \forall i,j\in\mathcal{V}$ as $t\to\infty$. Define $\theta_{i,j}=\mu_i-\mu_j$. From the definition of δ_i, it follows that

$$\theta_{i,j}^{(n-1)} = -\mathcal{T}_{n,1}\theta_{i,j} - \mathcal{T}_{n,2}\dot{\theta}_{i,j} - \cdots - \mathcal{T}_{n,n-2}\theta_{i,j}^{(n-3)} - \mathcal{T}_{n,n-1}\theta_{i,j}^{(n-2)} + \delta_i - \delta_j.$$

Then, it holds that $\theta_{i,j}\to\mathbf{0}_d \quad \forall i,j\in\mathcal{V}$ as $t\to\infty$, which implies that Step 3 is complete.

In *Step 4*, the statement of Theorem 1 is finally proved. From Step 1, it holds that $\nabla f_i(q_i,t)+\mu_i\to\mathbf{0}_d \quad \forall i\in\mathcal{V}$ as $t\to\infty$. From Step 3, it can be derived that there exists a signal $\mu(t)$ such that $\mu_i(t)-\mu(t)\to\mathbf{0}_d \quad \forall i\in\mathcal{V}$ as $t\to\infty$. Note that $\nabla f_i(q_i,t)+\mu_i = \nabla f_i(q_i,t)+\mu+\mu_i-\mu\to\mathbf{0}_d \quad \forall i\in\mathcal{V}$. Then, it holds that $\nabla f_i(q_i,t)+\mu\to\mathbf{0}_d \quad \forall i\in\mathcal{V}$ as $t\to\infty$, which implies that $\frac{\partial}{\partial q_i}\mathcal{L}(q,\mu,t)\to\mathbf{0}_d \quad \forall i,j\in\mathcal{V}$ as $t\to\infty$. Combining with Step 2, it holds that $\frac{\partial}{\partial z}\mathcal{L}(z,t)\to\mathbf{0}_{(N+1)d}$ as $t\to\infty$. Then, by Lemma 3, the statement in Theorem 1 is proved. ■

In the proof of Theorem 1, if the systems, e.g., (40) and (44), have continuous right-hand sides, there is no need to use the concept of the Filippov solution to avoid symbol redundancy, since the Filippov set-valued map is a singleton and the Filippov solution becomes the classical solution [47].

C. Virtual-System-Free Distributed Optimal Time-Varying Resource Allocation Algorithm

In order to remove the virtual systems used to estimate the Lagrange multiplier in this subsection, inspired by the centralized algorithm and (6), we treat the agents' local negative gradients as the estimates and propose a virtual-system-free distributed algorithm.

Design control input for agent $i\in\mathcal{V}$ as

$$u_i = -\sum_{k=0}^{n-1}\sigma_{k+1}\left(q_i^{(k)}-c_i^{(k)}\right)+c_i^{(n)}+\gamma\widehat{u}_i \tag{65a}$$

$$\mu_i = -\nabla f_i(q_i,t) \tag{65b}$$

$$\widehat{u}_i = \sum_{j\in\mathcal{N}_i}\operatorname{sgn}\left[\sum_{k=1}^{n}\alpha_k\left(\mu_i^{(k-1)}-\mu_j^{(k-1)}\right)\right] \tag{65c}$$

where $\sigma_1,\ldots,\sigma_n$, $\alpha_1,\ldots,\alpha_n$, and γ are positive constants to be determined. From (65c), each agent exchange $\mu_i^{(k)}$, $k=0,1,\ldots,n-1$, with its neighbors.

In (65a), the term $-\sum_{k=0}^{n-1}\sigma_{k+1}(q_i^{(k)}-c_i^{(k)})+c_i^{(n)}$ is designed to establish $\sum_{i=1}^{N}[q_i(t)-c_i(t)]\to\mathbf{0}_d$, that is, to drive $\frac{\partial}{\partial\mu}\mathcal{L}(q,\mu,t)$ to zero. Note from (5) that $\frac{\partial}{\partial q_i}\mathcal{L}(q,\mu,t)=\nabla f_i(q_i,t)+\mu(t)$. By (6), it then holds that $\mu^\star=-\nabla f_i(q_i^\star,t) \quad \forall i\in\mathcal{V}$, where $\mu^\star$ is the optimal Lagrange multiplier. This implies that $-\nabla f_i(q_i,t)$ [or μ_i from (65b)] can act as an estimate of the optimal Lagrange multiplier. Note that, by Lemma 3, the existence of $\mu(t)$ such that $\nabla f_i(q_i,t)+\mu(t)\to\mathbf{0}_d \quad \forall i\in\mathcal{V}$ as $t\to\infty$, implies that $\mu(t)\to\mu^\star(t)$ as $t\to\infty$. The term $\gamma\widehat{u}_i$ in (65a) is designed to establish $\nabla f_i(q_i,t)+\mu_j\to\mathbf{0}_d$ for any $i,j\in\mathcal{V}$, that is, to drive all the agents' Lagrange multiplier estimates $-\nabla f_i(q_i,t)$ (or μ_i) to $\mu^\star(t)$.

1) Selections of $\sigma_1,\ldots,\sigma_n$, $\alpha_1,\ldots,\alpha_n$: First the values of $\alpha_1,\ldots,\alpha_n$ can be determined by following the procedure presented in Section IV-B1. For the values of $\sigma_1,\ldots,\sigma_n$, set

$$\sigma_1=\beta_n\alpha_1,\ \text{and}\ \alpha_k=\alpha_{k-1}+\beta_n\alpha_k,\quad k=2,\ldots,n.$$

where β_n is a positive constant to be determined. For the sake of notational and analytical simplicity, we set $\alpha_1,\ldots,\alpha_n$ as in (28) and

$$\sigma_k=\binom{n}{k-1}\beta^{n+1-k},\quad k=1,\ldots,n \tag{66}$$

where β is a positive constant to be determined.

2) Convergence Analysis: *Theorem 2:* Suppose that Assumptions 1–4 hold. Select the positive constants β and γ to satisfy (36), where

$$\bar{\omega}=\bar{H}\bar{\zeta}\sum_{k=0}^{n-1}\sum_{s=k}^{n-1}\binom{n}{s+1}\frac{(1+\beta)^{n-k-1}}{\beta^{s-k}}+\bar{g}(1+\beta)^n + \bar{H}\gamma N\sqrt{d}\left[\left(1+\frac{1}{\beta}\right)^n-1\right]+\bar{H}\bar{c}(2+\beta)^n \tag{67}$$

and $\bar{\zeta}$ is defined in (38). Using (65) for integrator agent (2) solves the distributed resource allocation problem, i.e., $q_i(t)\to q_i^\star(t) \quad \forall i\in\mathcal{V}$ as $t\to\infty$.

Proof: From Assumption 4, (2), (65a), and (65b), it follows that

$$\begin{aligned}\mu_i^{(n)} &= -\sum_{k=0}^{n-1}\binom{n}{k}H_i^{(n-k)}q_i^{(k)}-H_iq_i^{(n)}-g_i^{(n)}\\ &= -\sum_{k=0}^{n-1}\binom{n}{k}H_i^{(n-k)}q_i^{(k)}-H_ic_i^{(n)}-\gamma H_i\widehat{u}_i-g_i^{(n)}\\ &\quad+\sum_{k=0}^{n-1}\binom{n}{k}\beta^{n-k}H_i\left(q_i^{(k)}-c_i^{(k)}\right).\end{aligned}$$

By Assumption 4 and (65b), it holds that $q_i=-H_i^{-1}(t)[\mu_i+g_i(t)]$, and for $k=1,\ldots,n-1$

$$q_i^{(k)}=-H_i^{-1}\left[\mu_i^{(k)}+\sum_{s=0}^{k-1}\binom{k}{s}H_i^{(k-s)}q_i^{(s)}+g_i^{(k)}\right].$$

Then, it holds that

$$\mu_i^{(n)}=-\sum_{k=0}^{n-1}\binom{n}{k}\beta^{n-k}\mu_i^{(k)}-\gamma H_i\widehat{u}_i+\omega_i(q_i,t) \tag{68}$$

where $\omega_i(q_i,t)=-\omega_{i,1}-\omega_{i,2}$, $\omega_{i,1}=\sum_{k=0}^{n}\binom{n}{k}\beta^{n-k}[H_ic_i^{(k)}+g_i^{(k)}]$, and $\omega_{i,2}=\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k}\sum_{s=0}^{k-1}\binom{k}{s}H_i^{(k-s)}q_i^{(s)}$.

From Assumption 4 and the definitions of $\sigma_j, j = 1,\dots,n+1$, it holds that

$$\|\omega_{i,1}\|_2 \le \left(\bar{H}\bar{c} + \bar{g}\right)\sum_{k=0}^{n}\binom{n}{k}\beta^{n-k} = \left(\bar{H}\bar{c} + \bar{g}\right)(1+\beta)^n. \tag{69}$$

Define $\zeta_{i,k} := q_i^{(k-1)} - c_i^{(k-1)}$ for $k = 1,\dots,n$ and $\zeta_i = [\zeta_{i,1}^\top,\dots,\zeta_{i,n}^\top]^\top$. Define $z_{i,k} := (\mathcal{T}_k \otimes I_d)\zeta_i$ for $k = 1,\dots,n$, where $\mathcal{T}_k$ is given in (54). Note that $\binom{n}{s}\binom{s}{s-k} = \frac{n!}{(n-s)!\,(s-k)!\,k!} = \binom{n}{k}\binom{n-k}{s-k}$ holds for any $k = 1,\dots,n$ and $s = k,\dots,n$. Then, it holds that $\omega_{i,2} = \sum_{k=1}^{n} H_i^{(k)} \sum_{s=k}^{n}\binom{n}{s}\beta^{n-s}\binom{s}{s-k}q_i^{(s-k)} = \sum_{k=1}^{n}\binom{n}{k}H_i^{(k)} \sum_{s=k}^{n}\binom{n-k}{s-k}\beta^{n-s}q_i^{(s-k)} = \sum_{k=1}^{n}\binom{n}{k}H_i^{(k)}\sum_{s=0}^{n-k}\binom{n-k}{s}\beta^{n-k-s} q_i^{(s)}$. Note from the definition of $\mathcal{T}_k$ in (54) that $z_{i,n-k+1} = \mathcal{T}_{n-k+1}\zeta_i = \sum_{s=0}^{n-k}\binom{n-k}{s}\beta^{n-k-s}(q_i^{(s)} - c_i^{(s)})$. Hence, it holds that

$$\omega_{i,2} = \sum_{k=1}^{n}\binom{n}{k}H_i^{(k)} z_{i,n-k+1} + \omega_{i,3}$$

where $\omega_{i,3} = \sum_{k=1}^{n}\binom{n}{k}H_i^{(k)}\sum_{s=0}^{n-k}\binom{n-k}{s}\beta^{n-k-s}c_i^{(s)}$. From Assumption 4, it holds that

$$\|\omega_{i,3}\|_2 \le \bar{c}\bar{H}\sum_{k=1}^{n}\binom{n}{k}(1+\beta)^{n-k}. \tag{70}$$

From (65a) and the definitions of $\mathcal{T}$ and $z_{i,k}$, it holds that

$$\begin{cases} \dot{z}_{i,n} = -\beta z_{i,n} + \gamma\widehat{u}_i \\ \dot{z}_{i,n-1} = -\beta z_{i,n-1} + z_{i,n} \\ \quad\vdots \\ \dot{z}_{i,1} = -\beta z_{i,1} + z_{i,2}. \end{cases} \tag{71}$$

By following a similar analysis as in (50), it holds that $\|z_{i,k}(t)\|_2 \le \sum_{j=0}^{n-k}\frac{1}{\beta^{n-k-j}}\|z_{i,n-j}(0)\|_2 + \frac{\gamma N\sqrt{d}}{\beta^{n-k+1}}$. Then, it holds that

$$\begin{aligned} &\left\|\sum_{k=1}^{n}\binom{n}{k}H_i^{(k)}z_{i,n-k+1}\right\|_2 \\ &\le \bar{H}\sum_{k=0}^{n-1}\|z_{i,n-k}(0)\|_2\sum_{s=k}^{n-1}\binom{n}{s+1}\frac{1}{\beta^{s-k}} \\ &\quad + \frac{\gamma N\sqrt{d}}{\beta^n}\sum_{k=1}^{n}\binom{n}{k}\beta^{n-k} \\ &\le \bar{H}\bar{\zeta}\sum_{k=0}^{n-1}\sum_{s=k}^{n-1}\binom{n}{s+1}\frac{(1+\beta)^{n-k-1}}{\beta^{s-k}} \\ &\quad + \bar{H}\gamma N\sqrt{d}\left[\left(1+\frac{1}{\beta}\right) - 1\right] \end{aligned} \tag{72}$$

where $\bar{\zeta}$ is given in (38). Note that

$$\begin{aligned} &\bar{c}\bar{H}\sum_{k=1}^{n}\binom{n}{k}(1+\beta)^{n-k} + \bar{H}\bar{c}(1+\beta)^n \\ &= \bar{H}\bar{c}(2+\beta)^n. \end{aligned} \tag{73}$$

Then, it follows from (69), (72), (70), and (73) that $\|\omega_i(q_i,t)\|_\infty \le \|\omega_i(q_i,t)\|_2 \le \bar{\omega}$, where $\bar{\omega}$ is given in (67).

Let $\widehat{\mu}_i = [\mu_i^\top, {\mu_i^{(1)}}^\top, \dots, {\mu_i^{(n-1)}}^\top]^\top$. Define $\delta_i = (\mathcal{T}_n \otimes I_d)\widehat{\mu}_i$ and $\delta = [\delta_1^\top,\dots,\delta_N^\top]^\top$. Then, (63) holds. By following the same analysis as in Step 3 of Theorem 1, it holds that $\mu_i(t) - \mu_j(t) \to \mathbf{0}_d \quad \forall i,j \in \mathcal{V}$ as $t \to \infty$, which implies that there exists a signal $\mu(t)$ such that $\mu_i(t) - \mu \to \mathbf{0}_d \quad \forall i \in \mathcal{V}$ as $t \to \infty$.

Define $x = \sum_{i=1}^{N}\zeta_{i,1}$. It follows from (65a) that $x^{(n)} = -\sum_{k=1}\binom{n}{k}\beta^{n-k}x^{(k)} + \gamma\sum_{i=1}^{N}\widehat{u}_i = -\sum_{k=1}\binom{n}{k}\beta^{n-k}x^{(k)}$, where the second equality holds by following Assumption 1. By Corollary 1, this system is stable. Hence, $x \to \mathbf{0}_d$ as $t \to \infty$. That is, $\sum_{i=1}^{N}[q_i - c_i(t)] \to \mathbf{0}_d$ as $t \to \infty$. From (65b), it holds that $\nabla f_i(q_i,t) + \mu_i = \nabla f_i(q_i,t) + \mu + \mu_i - \mu = \mathbf{0}_d \quad \forall i,j \in \mathcal{V}$. Hence, it holds that $\nabla f_i(q_i,t) + \mu = -(\mu_i - \mu) \to \mathbf{0}_d \quad \forall i \in \mathcal{V}$. Then, it holds that $\frac{\partial}{\partial q_i}\mathcal{L}(q,\mu,t) \to \mathbf{0}_d \quad \forall i \in \mathcal{V}$ as $t \to \infty$. Therefore, $\frac{\partial}{\partial z}\mathcal{L}(z,t) \to \mathbf{0}_{(N+1)d}$ as $t \to \infty$. Then, by Lemma 3, the statement in Theorem 2 holds. ■

Remark 3: Both distributed algorithms in Sections IV-B and IV-C solve the time-varying resource allocation problem in a distributed manner for networked integrator agents. However, each algorithm has its own merit. For the algorithm in Section IV-B, each agent has an extra virtual system to estimate the central Lagrange multiplier, which might not be computationally efficient compared with the other distributed algorithm. For the algorithm in Section IV-C, the requirement of constructing virtual systems for the agents is removed, but each agent needs to exchange the exact gradient information with its neighbors, and sometimes, the agents would prefer not to give out such information because it might be sensitive and private.

Remark 4: While implementing either the algorithm in Section IV-B or the one in Section IV-C, it is worth pointing out that one can always find positive constants β and γ satisfying (36). For instance, consider the algorithm in Section IV-B, and one can choose β and γ such that

$$\beta > \frac{1}{\left(\frac{1}{2|\mathcal{E}|d^2N\sqrt{N}} + 1\right)^{1/n} - 1} \tag{74}$$

$$\gamma \ge \frac{2|\mathcal{E}|d\sqrt{N}d\bar{\bar{\omega}}}{1 - 2|\mathcal{E}|d^2N\sqrt{N}\left[\left(1+\frac{1}{\beta}\right)^n - 1\right]} \tag{75}$$

where $\bar{\bar{\omega}} = \bar{H}\bar{e}\sum_{k=1}^{n-1}\sum_{s=k}^{n-1}\binom{n-1}{s}\frac{(1+\beta)^{n-k-1}}{\beta^{s-k}} + \bar{H}\bar{\zeta}\sum_{k=1}^{n}\sum_{s=k}^{n}\binom{n-1}{s-k}\frac{(1+\beta)^{n-k}}{\beta^{n-s}} + \bar{g}(1+\beta)^n + \bar{H}\bar{c}(2+\beta)^n$. From the proofs of Theorems 1 and 2, the value of the β has an essential effect on the convergence rate of the proposed distributed algorithms and the value of γ is closely related to the ultimate optimum-tracking errors. Specifically, a larger β results in a faster convergence of the decision variables to the optimal solutions. However, from (75), one has that a larger γ is required to guarantee the exact optimum tracking if a large value of β is chosen. The performance comparison of the proposed distributed algorithms for different values of the design parameters β and γ is illustrated in Fig. 3 later.

Remark 5: Note that the values of $\bar{\zeta}$ and $\bar{e}$ in (38) and (39), respectively, depend on the initial conditions. These values can be estimated by using maximum consensus algorithms [48]. It is worth pointing out that the design parameters β and γ are constants that can be determined off-line. One can be conservative and choose a large enough value for β and then choose a relatively larger value for γ. Then, one can embed these chosen values into the agents.

Remark 6: The proposed distributed algorithms in Sections IV-B and IV-C can be implemented in a sampled-data setting, where the systems have continuous-time dynamics while the inputs are based on zero-order hold and the interactions with the neighbors are made at discrete sampling times. Then, the choices of β and γ depend on the value of the sampling period T and the ultimate optimum-tracking errors are proportional to T.

D. Application to Nonlinear Dynamics With Parametric Uncertainties

In this section, by the estimation-tracking method, we apply the results obtained in Sections IV-B or IV-C to solve the optimal time-varying resource allocation problem for networked nonlinear agents with parametric uncertainties in a distributed manner. Different from the high-order integrators (2), the dynamics of the ith agent are given as

$$x_{i,1}^{(m)} = \theta_i v_i + \phi_i(x_i,t)\vartheta_i + d_i(t) \tag{76}$$

where $m \in \mathbb{N}_+$, $x_{i,1} \in \mathbb{R}^d$, and $v_i \in \mathbb{R}^d$ are, respectively, the decision variable and control input of agent i, $x_i = \left[x_{i,1}^\top, {x_{i,1}^{(1)}}^\top, \dots, {x_{i,1}^{(m-1)}}^\top\right]^\top$, $\theta_i \in \mathbb{R}$ is the unknown control direction, $\phi_i(x_i,t) \in \mathbb{R}^{d\times p_i}$ is a known bounded Lipshitz continuous function, $\vartheta_i \in \mathbb{R}^{p_i}$ is the constant but unknown parameter vector of agent i, and $d_i(t) \in \mathbb{R}^d$ is the disturbance vector satisfying $\|d_i\|_2 \le d_{\max}$. It is assumed that θ_i can be either positive or negative for all the agents and $0 < \theta_{\min} \le |\theta_i| \le \theta_{\max}$.

The main idea of the estimation-tracking method is described as follows. First, each agent constructs a high-order integrator virtual system of the form (2) with the order $n \ge m$ and u_i given as in (22) or (65), and consequently, the virtual systems solve the optimal time-varying resource allocation problem and the virtual states estimate the optimal trajectories. Then, tracking controllers are designed for the agent such that the physical states track their local virtual states, and hence the optimal trajectories.

Therefore, the distributed time-varying resource allocation algorithm designed for networked nonlinear systems (76) is summarized as follows:

$$\begin{cases} \text{virtual system (2) with (22) or (65),} \\ s_i = \widetilde{x}_{i,m} + \sum_{k=1}\lambda_{i,k}\widetilde{x}_{i,k} \\ v_i = \mathscr{N}(k_i)\left[\alpha_{i,1}s_i + \phi_i\widehat{\vartheta}_i + \sum_{l=1}^{m-1}\lambda_{i,l}\widetilde{x}_{i,l+1}\right. \\ \left. -q_i^{(m)} + \widehat{d}_i\mathrm{sgn}(s_i)\right] \\ \dot{k}_i = \alpha_{i,1}\|s_i\|_2^2 + s_i^\top\left(\phi_i\widehat{\vartheta}_i + \sum_{l=1}^{m-1}\lambda_{i,l}\widetilde{x}_{i,l+1}\right. \\ \left. -q_i^{(m)}\right) + \widehat{d}_i\|s_i\|_1 \\ \dot{\widehat{\vartheta}}_i = \alpha_{i,2}\phi_i^\top s_i \\ \dot{\widehat{d}}_i = \alpha_{i,3}\|s_i\|_1 \end{cases} \tag{77}$$

where $\mathscr{N}(\cdot)$ is an even Nussbaum-type function [49], $\widetilde{x}_{i,k} = x_{i,1}^{(k-1)} - q_i^{(k-1)}$ for $k = 1,\dots,m$, $\lambda_{i,1},\dots,\lambda_{i,m-1}$ are positive constants selected such that the polinomial $\varsigma^{m-1} + \lambda_{i,m-1}\varsigma^{m-2} + \cdots + \lambda_{i,2}\varsigma + \lambda_{i,1} = 0$ is Hurwitz, and $\alpha_{i,1}, \alpha_{i,2}$, and $\alpha_{i,3}$ are positive constants.

From Theorems 1 or 2, under Assumptions 1–4, the virtual state q_i estimates the optimal decision trajectory $q_i^\star$, i.e., $q_i(t) \to q^\star(t)$ as $t \to \infty$. From the definition of s_i, it holds that $\dot{s}_i = [\theta_i\mathscr{N}(k_i)+1][\alpha_{i,1}s_i + \phi_i\widehat{\vartheta}_i + \sum_{l=1}^{m-1}\lambda_{i,l}\widetilde{x}_{i,l+1} - q_i^{(m)} + \widehat{d}_i\mathrm{sgn}(s_i)] - \alpha_{i,1}s_i - \phi_i\widetilde{\vartheta}_i - [\widehat{d}_i\mathrm{sgn}(s_i) - d_i]$, where $\widetilde{\vartheta}_i = \widehat{\vartheta}_i - \vartheta_i$. Let $\widetilde{d}_i = \widehat{d}_i - d_{\max}$. Consider the Lyapunov function $W_i = \frac{1}{2}s_i^\top s_i + \frac{1}{\alpha_{i,2}}\widetilde{\vartheta}_i^\top\widetilde{\vartheta}_i + \frac{1}{\alpha_{i,3}}\widetilde{d}_i^2$. The derivative of W_i is given as $\dot{W}_i = [\theta_i\mathscr{N}(k_i)+1]\dot{k}_i - \alpha_{i,1}\|s_i\|_2^2 - \widehat{d}_i\|s_i\|_1 + s_i^\top d_i \le [\theta_i\mathscr{N}(k_i)+1]\dot{k}_i - \alpha_{i,1}\|s_i\|_2^2$.[1] By [49, Lemma 1], it can be shown that $k_i(t)$, $W_i(t)$, and $\int_0^t\{\theta_i\mathscr{N}[k_i(\tau)]+1\}\dot{k}_i(\tau)\mathrm{d}\tau$ are bounded on $[0,t_f)$. Then, no finite-time escape phenomenon may happen and $t_f = \infty$. Then, it holds that $\lim_{t\to\infty}\int_0^t\alpha_{i,1}s_i^\top(\tau)s_i(\tau)\mathrm{d}\tau < \infty$, which implies that s_i is integratable. By Barbalat's lemma [46, p. 175], it holds that $s_i \to \mathbf{0}_d$, and hence $x_{i,1}(t) \to q_i(t)\ \ \forall i \in \mathcal{V}$ as $t \to \infty$. Therefore, it holds that $x_{i,1}(t) \to q_i^\star(t)\ \ \forall i \in \mathcal{V}$ as $t \to \infty$.

Remark 7: From the analysis above, it can be verified that the dynamics in (76) can be extended to the case of heterogeneous orders. That is, if all the agents (76) have different orders (e.g., the order of agent i is m_i), then the agents can cooperatively determine the highest order $m = \max_{i\in\mathcal{V}}\{m_i\}$ and then the orders of the virtual systems (i.e., n). Moreover, since the time-varying resource allocation problem for nonlinear systems is transferred to a tracking problem by utilizing the estimation-tracking method and the algorithms in Sections IV-B or IV-C, then the dynamics given in (76) can be made more general. For instance, let agent i's dynamics be described in strict feedback form

$$\begin{cases} \dot{x}_{i,k} = \theta_{i,k}\phi_{i,k}(\bar{x}_{i,k},t)x_{i,k+1} + \varphi_{i,k}(\bar{x}_{i,k},t)\vartheta_{i,k} + d_{i,k}(t) \\ \qquad k = 1,\dots,m-1, \\ \dot{x}_{i,m} = \theta_{i,m}\phi_{i,m}(x_i,t)u_i + \varphi_{i,m}(x_i,t)\vartheta_{i,m} + d_{i,m}(t) \end{cases}$$

where $\bar{x}_{i,k} = [x_{i,1}^\top,\dots,x_{i,k}^\top]^\top$, $\theta_{i,k} \in \mathbb{R}$ is unknown constant, $\vartheta_{i,k} \in \mathbb{R}^{p_{i,k}}$ is a vector of unknown parameters, and $\phi_{i,k}(\bar{x}_{i,k},t)$ and $\varphi_{i,k}(\bar{x}_{i,k},t)$ are known functions with compatible dimensions. Each agent has a virtual system (2) with $n \ge m$. Define $\widetilde{x}_{i,k} = x_{i,k} - q_i^{(k-1)}$, $k = 1,\dots,m$. It then holds that

$$\begin{cases} \dot{\widetilde{x}}_{i,k} = \theta_{i,k}\phi_{i,k}\widetilde{x}_{i,k+1} + \widetilde{\varphi}_{i,k}\widetilde{\vartheta}_{i,k} + d_{i,k} - q_i^{(k)} \\ \qquad k = 1,\dots,m-1, \\ \dot{\widetilde{x}}_{i,m} = \theta_{i,m}\phi_{i,m}u_i + \varphi_{i,m}\vartheta_{i,m} + d_{i,m}(t) - q_i^{(m)} \end{cases}$$

where $\widetilde{\varphi}_{i,k} = [\varphi_{i,k}, \phi_{i,k}q_i^{(k)}]$ and $\widetilde{\vartheta}_{i,k} = [\vartheta_{i,k};\theta_{i,k}]$. By following the design procedures introduced in [50], one can derive the controllers for all the agents.

V. Illustrative Examples

In this section, we provide examples to illustrate the results in this article and the communication graph among the agents is described by a ring topology.

[1]In the analysis, since the Lyapunov function W_i is continuously differentiable, then the generalized gradient of W_i is singleton. Note also that $x^\top\mathcal{K}[\mathrm{sgn}(x)] = \|x\|_1$. Then, the analysis still holds and nonsmooth analysis is not used to avoid symbol redundancy.

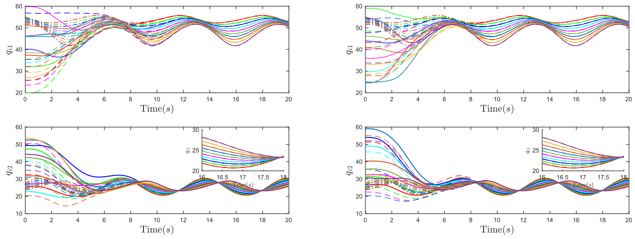

Fig. 1. Decision trajectories of double-integrator agents (2) generated by using the algorithms in Section IV-B (left) and Section IV-C (right).

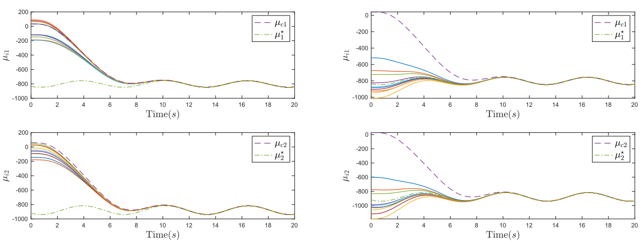

Fig. 2. Trajectories of μ_i generated by using the algorithms in Section IV-B (left) and Section IV-C (right).

A. Fourth-Order Integrators

Consider a group of $N = 10$ fourth-order integrators described by (2) with $n = 4$ and $d = 2$, and assume that each agent $i \in \mathcal{V}$ has a cost function $f_i(q_i, t) = \frac{1}{2} q_i^\top H_i(t) q_i + g_i(t)^\top q_i + h_i(t)$, where $H_i(t) = \begin{bmatrix} 10+0.1i & 10+0.1i \\ 10+0.1i & 13+0.1i\cos(t)+0.1i \end{bmatrix}$, $g_i(t) = [i\cos(t), i\sin(t)]^\top$, and $h_i(t)$ is a time-varying function. The agents aim to cooperatively solve the resource allocation problem defined in (3)–(4), where $c_i(t) = [0.5i\cos(t) + i + 45, 0.5i\sin(t) + i + 20]^\top$. In the simulations of this subsection, the initial values of agents' decision variables and their derivatives are chosen randomly. In Figs. 1 and 2, the solid and dashed lines are the trajectories generated by using the distributed algorithms (in Sections IV-B and IV-C) and the centralized algorithm (in Section IV-A), respectively. The dash-dotted lines are the optimal solutions of $q_i^\star$ and $\mu^\star$, $i \in \mathcal{V}$.

First, we validate the distributed algorithm in Section IV-B. The values of $\mu_i(0)$ and $\dot{\mu}_i(0)$ are generated uniformly randomly. We select $\beta = 1$ and $\gamma = 20$. The trajectories of q_i and μ_i are presented in Fig. 1 (left) and Fig. 2 (left), respectively. It can be seen that q_i and μ_i are able to track the optimal $q_i^\star$ and $\mu_i^\star$, respectively.

Then, we validate the distributed algorithm in Section IV-C. We select $\beta = 1$ and $\gamma = 17$. The trajectories of q_i and μ_i are presented in Fig. 1 (right) and Fig. 2 (right), respectively. It can be seen that q_i and μ_i are able to track the optimal $q_i^\star$ and $\mu_i^\star$, respectively.

We show in Fig. 3 how the total optimum tracking error $\sum_{i=1}^N \|q_i - q_i^\star\|_2$ changes when different values of γ and β are selected while implementing the distributed algorithms in Section IV-B and IV-C. From Fig. 3, it can be seen that the value of β determines the convergence speed of the proposed distributed algorithms and that the value of γ affects the ultimate

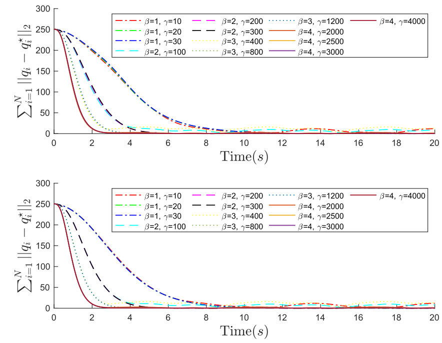

Fig. 3. Total optimum tracking error $\sum_{i=1}^N \|q_i - q_i^\star\|_2$ with time for the fixed initial conditions and various values of β and γ by using the distributed algorithms in Sections IV-B (top) and IV-C (bottom).

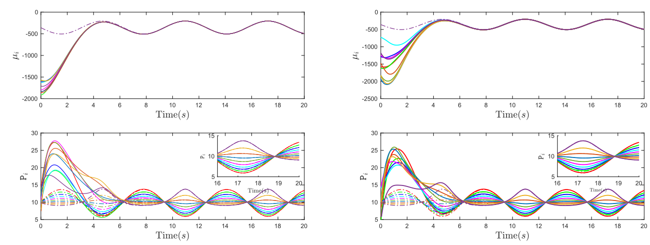

Fig. 4. Trajectories of μ_i and $\mathtt{P}_i$ generated by using the algorithms (77) with (22) (left) and (65) (right).

optimum tracking errors, which coincides with the statement in Remark 5.

B. Economic Dispatch Problem of the Power Systems

Consider Example 2 with ten generators. The generator i's dynamics can be rewritten as [38]

$$\mathtt{P}_i^{(2)} = -\frac{T_{mi} + T_{ei}}{T_{mi}T_{ei}} \mathtt{P}_i^{(1)} - \frac{1}{T_{mi}T_{ei}} \mathtt{P}_i + \frac{K_{mi}}{T_{mi}T_{ei}} v_i \tag{78}$$

which have the form of (76) with $\theta_i = \frac{K_{mi}}{T_{mi}T_{ei}}$, $\vartheta_i = [-\frac{1}{T_{mi}T_{ei}}, -\frac{T_{mi}+T_{ei}}{T_{mi}T_{ei}}]^\top$, and $\phi_i = [\mathtt{P}_i, \mathtt{P}_i^{(1)}]$. In the simulation, let $T_{mi} = 0.34 + 0.02(i-1)$, $T_{ei} = 0.12 + 0.02 * (i-1)$, and $K_{mi} = 1.13 + 0.01(i-1)$, which are unknown and not used in the algorithm design. We also add $d_i(t) = i\cos(t)$ on the right-hand side of (78). By [38], the time-varying cost coefficient functions of the ith generator are assumed to be $a_i(t) = 16.78 + i\sin(t)$, $b_i(t) = 18.3391 + i\cos(t)$, and $m_i = 1.24 + i\sin(t)$, and the resource is given as $\mathtt{D}_i = 10 + 0.2i\sin(t)$. The controllers can be designed by following Section IV-D. In the simulation, we set $n = m = 2$. Construct a virtual system $q_i^{(2)} = u_i$ for generator i with u_i given in (22) or (65) and design control v_i, as in (77) such that $\mathtt{P}_i$ tracks q_i. Select $\alpha_{i,1} = \alpha_{i,2} = \alpha_{i,3} = \lambda_{i,1} = \beta = 1$, $\gamma = 10$. The simulation results are presented in Figs. 4, where the dash-dotted lines are the optimal solutions of $q_i^\star$ and $\mu^\star$, $i \in \mathcal{V}$. It can be seen that $\mathtt{P}_i(t) \to \mathtt{P}_i^\star(t)$ as $t \to \infty$ for all $i \in \mathcal{V}$.

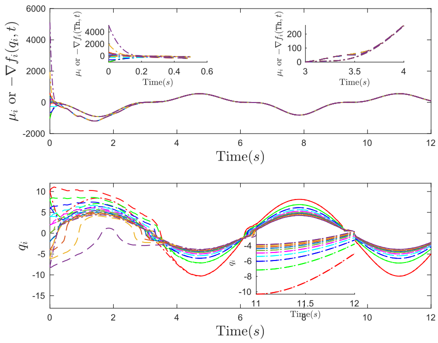

Fig. 5. Trajectories of μ_i or $-\nabla f_i(q_i, t)$ [defined in (65b)] and q_i generated by using the distributed algorithm in Section IV-B (dashed lines) and Section IV-C (dot-dashed lines).

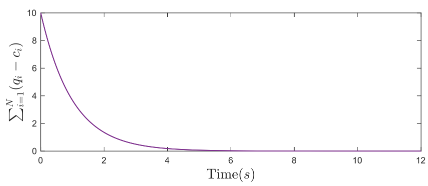

Fig. 6. Trajectories of $\sum_{i=1}^{N}(q_i - c_i)$, where q_i is generated by using the distributed algorithm in Section IV-C.

C. Simulation Results With Nonquadratic Cost Functions

Although the Assumption 4 indicates that the cost functions are assumed to be quadratic, the algorithms proposed in this article can be used for nonquadratic cost functions. In the following, we provide an example with nonquadratic cost functions. Consider ten single-integrator agents with $d = 1$ and assume that the gradient of the ith cost function is given as $\nabla f_i(q_i, t) = [0.5\sin(t) + i]q_i^3 + q_i + i\cos(t)$. In Fig. 5, it shows that the two distributed algorithms ultimately produce the same μ_i and $-\nabla f_i(q_i, t)$, $i \in \mathcal{V}$. It can also be seen that the decision trajectories generated by both distributed algorithms for each agent converge to the same one. From Fig. 6, it shows that $\sum_{i=1}^{N}[q_i(t) - c_i(t)] \to 0$ as $t \to \infty$ by using the distributed algorithm in Section IV-C. Recall from (65b) and Fig. 5 that $\mu_i + \nabla f_i(q_i, t) = 0$ and $\mu_i \to \mu_j$ $\forall i, j \in \mathcal{V}$, and it can be implied that the agents' decision trajectories in Fig. 5 converge to optimal ones.

VI. Conclusion

The distributed time-varying resource allocation problem has been investigated for networked high-order agents. Two distributed algorithms have been proposed for high-order integrator agents to track the optimal decision trajectories with zero errors. There is a tradeoff between economical efficiency and favorable applicability to privacy-sensitive applications while implementing these two algorithms. By using the estimation-tracking method, these two distributed algorithms have been used to solve the resource allocation problem for networked high-order nonlinear agents. Finally, simulation results have been presented to validate the effectiveness of the proposed algorithms.

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