

To What Extent do Open-loop and Feedback Nash Equilibria Diverge in General-Sum Linear Quadratic Dynamic Games?

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Abstract—Dynamic games offer a versatile framework for modeling the evolving interactions of strategic agents, whose steady-state behavior can be captured by the *Nash equilibria* of the games. Nash equilibria are often computed in *feedback*, with policies depending on the state at each time, or in *open-loop*, with policies depending only on the initial state. Empirically, open-loop Nash equilibria (OLNE) could be more efficient to compute, while feedback Nash equilibria (FBNE) often encode more complex interactions. However, it remains unclear exactly which dynamic games yield FBNE and OLNE that differ significantly and which do not. To address this problem, we present a principled comparison study of OLNE and FBNE in linear quadratic (LQ) dynamic games. Specifically, we prove that the OLNE strategies of an LQ dynamic game can be synthesized by solving the coupled Riccati equations of an auxiliary LQ game with perturbed costs. The construction of the auxiliary game allows us to establish conditions under which OLNE and FBNE coincide and derive an upper bound on the deviation between FBNE and OLNE of an LQ game.

Index Terms—Game Theory, Linear Systems, Optimal Control.

I. INTRODUCTION

Dynamic game theory [1] provides a powerful mathematical framework for modeling strategic decision-making over time in multi-agent interactions, such as autonomous path planning [2–5]. In dynamic games, steady-state modes of interaction between strategic agents are described by *Nash equilibria* [1, 6], strategy profiles at which each agent’s behavior is unilaterally optimal with respect to their objective. Different Nash equilibria concepts can be defined with respect to different *information structures*, which prescribe the system state information accessible to agents for control design at each time, and thus influence their strategic interactive behavior.

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In [5, 7–12], iterative algorithms were developed for computing Nash equilibria under the *feedback* information structure, wherein each agent observes the current system state at each time and adapts their actions accordingly. Other works solve for Nash equilibria in *open-loop*, in which each agent selects control actions using only the state information available at the start of the planning horizon [3, 4, 7, 11–13].

Prior works [1, 9, 14] suggest that the feedback Nash equilibrium (FBNE) and open-loop Nash equilibrium (OLNE) trajectories of some dynamic games can diverge wildly, even when the OLNE are generated in a receding horizon fashion [5, 14]. Moreover, in dynamic games whose FBNE and OLNE diverge significantly, FBNE often describes nuanced multi-agent interactions more accurately than OLNE [5, 9]. However, some studies indicate that OLNE is more computationally tractable to solve than FBNE, and that FBNE and OLNE are closely aligned for many dynamic games [3]. These empirical phenomena lead naturally to the following question: *For which dynamic games do FBNE diverge sufficiently from OLNE to warrant the computational burden of computing FBNE?*

To answer the above question, we characterize the difference between the FBNE and OLNE solutions, henceforth referred to as the FBNE-to-OLNE gap, for a class of linear quadratic (LQ) dynamic games. First, we identify sufficient conditions for LQ dynamic games under which their FBNE and OLNE coincide. Then, for LQ games whose FBNE and OLNE solutions diverge, we establish an upper bound for their FBNE-to-OLNE gap in terms of the degree to which each agent’s cost depends on other agents’ state values. Our core method involves proving that the OLNE of any LQ game $\mathcal{G}$ satisfying certain structural properties can be synthesized by constructing an *auxiliary* game $\tilde{\mathcal{G}}$ with modified cost matrices and solving the Riccati equations of $\tilde{\mathcal{G}}$. The construction of the auxiliary game allows us to recast the comparison between the OLNE and FBNE of $\mathcal{G}$ as a comparison between the Riccati equation solutions for $\mathcal{G}$ and $\tilde{\mathcal{G}}$, which we then characterize by contrasting the state costs of $\mathcal{G}$ and $\tilde{\mathcal{G}}$.

Below, Section II presents the class of LQ games considered in this work, and describes the FBNE and OLNE solutions of LQ Games. Section III then presents sufficient conditions under which the FBNE and OLNE of a given LQ game coincide. Section IV upper bounds the FBNE-to-OLNE gap of LQ games in terms of their state cost matrices. Section V presents

numerical simulations supporting our theoretical results.

Notation: For each positive integer n , set $[n] := \{1, \dots, n\}$. Given matrices $(A^1, \dots, A^n)$, let $\text{diag}\{A^1, \dots, A^n\}$ be the block diagonal matrix whose i -th block is A^i , $\forall i \in [n]$. Given a matrix M with a block structure, let $[M]_{ij}$, $[M]_{i,:}$, and $[M]_{:,j}$ respectively denote the i -th row j -th column block, the i -th row block (with all columns), and the j -th column block (with all rows) of M . Let I and O denote the identity and zero matrix, respectively.

II. PRELIMINARIES

A. Linear Quadratic Games

We denote by $\mathcal{G} := (A^i, B^i, Q^i, R^i : i \in [N])$ an N -agent, T -stage discrete-time dynamic LQ game with structure described as follows. Each agent $i \in [N]$ is associated with the states $x_t^i \in \mathbb{R}^{n_i}$ and controls $u_t^i \in \mathbb{R}^{m_i} \forall t \in [T]$, which evolve according to the dynamics:

$$\Sigma_i : x_{t+1}^i = A^i x_t^i + B^i u_t^i, \forall t \in [T], \quad (1)$$

where $A^i \in \mathbb{R}^{n_i \times n_i}$ and $B^i \in \mathbb{R}^{n_i \times m_i}$. We define $x_t := (x_t^1, \dots, x_t^N) \in \mathbb{R}^n$ to be the *system state* at each time $t \in [T]$, where $n := \sum_{i \in [N]} n_i$, and $\mathbf{x} := (x_1, \dots, x_T) \in \mathbb{R}^{nT}$ to be the *system state trajectory*. The initial state x_1 is assumed to be fixed. Similarly, we define $u_t := (u_t^1, \dots, u_t^N) \in \mathbb{R}^m$ to be the *system control input* at time $t \in [T]$, where $m := \sum_{i \in [N]} m_i$. We denote each agent i 's control inputs over time by $u^i := (u_1^i, \dots, u_T^i) \in \mathbb{R}^{m_i T}$, and the tuple of all agents' control inputs over time by $u := (u^1, \dots, u^N) \in \mathbb{R}^{mT}$. Each agent $i \in [N]$ is associated with the cost function $C^i : \mathbb{R}^{nT} \times \mathbb{R}^{m_i T} \rightarrow \mathbb{R}$:

$$C^i(\mathbf{x}, u^i) := \sum_{t \in [T]} [u_t^{i\top} R^i u_t^i + x_{t+1}^{i\top} Q^i x_{t+1}^i]. \quad (2)$$

where $R^i \in \mathbb{R}^{m_i \times m_i}$ and $Q^i \in \mathbb{R}^{n \times n}$ denote the control cost matrices and state cost matrices of agent i , respectively.

Remark 1: In this work, we focus our analysis on LQ games with *decoupled* dynamics (1) and *coupled* state costs (2). Thus, the state cost of each agent i may depend on the states of other agents, but their control cost and the dynamics depend only on their own control u_t^i . This assumption reasonably describes many multi-agent interaction applications, such as autonomous driving and multi-robot coordination.

Remark 2: The terms $x_{t+1}^{i\top} Q^i x_{t+1}^i$ in (2) encode the dependence of each agent's cost on the entire state trajectory, thus enforcing that the agents' costs are coupled.

System-level dynamics for the game $\mathcal{G}$ can be constructed by concatenating the system dynamics Σ_i in (1), as follows:

$$\Sigma : x_{t+1} = Ax_t + Bu_t = Ax_t + \sum_{i \in [N]} \hat{B}^i u_t^i, \quad (3)$$

where we have $A := \text{diag}\{A^1, \dots, A^N\} \in \mathbb{R}^{n \times n}$, $B := \text{diag}\{B^1, \dots, B^N\} \in \mathbb{R}^{n \times m}$. We define $\hat{B}^i$ to be the i -th block columns of B , i.e., $\hat{B}^i := [B]_{:,i} \forall i \in [N]$.

B. Feedback Nash Equilibria (FBNE)

The state feedback information structure prescribes that, at each time t , each agent is allowed to access the value of the system state $x_t \in \mathbb{R}^n$ to guide the selection of their current

control input $u_t^i \in \mathbb{R}^{m_i}$. Thus, each agent i designs *feedback strategies* $\gamma_t^i : \mathbb{R}^n \rightarrow \mathbb{R}^{m_i}$ over the time horizon $[T]$ a priori, and subsequently deploys the control input $u_t^i := \gamma_t^i(x_t)$ at each time t after obtaining the value of x_t . We define Γ_t^i to be the strategy space of each agent i at each time t , and Γ to be the joint strategy space of all agents over the time horizon. Given a tuple of state feedback strategies $\gamma \in \Gamma$, for each agent $i \in [N]$, for any $\tau, \bar{\tau} \in [T]$, define $\gamma_{\tau, \bar{\tau}}^i$ to be $(\gamma_\tau^i, \dots, \gamma_{\bar{\tau}}^i)$ if $\tau < \bar{\tau}$, and empty otherwise. Finally, for each agent $i \in [N]$, let the notation “ $-i$ ” denote “all agents except agent i ”, e.g., for any $\tau, \bar{\tau} \in [T]$ such that $\tau < \bar{\tau}$, we have $\gamma_{\tau, \bar{\tau}}^{-i} := (\gamma_\tau^j \in \Gamma_t^j : \tau \leq t \leq \bar{\tau}, j \in [N] \setminus \{i\})$.

Below, we define the Nash equilibrium of a dynamic game under the state feedback information structure using notation introduced in [1]. For each $i \in [N]$, let $J^{i, FB} : \Gamma \rightarrow \mathbb{R}$ denote the cost obtained by unrolling the system dynamics (1) from the fixed initial state x_1 with control inputs prescribed by a given state feedback strategy, i.e., for each feedback strategy $\gamma \in \Gamma$, we have:

$$J^{i, FB}(\gamma) := C^i(\mathbf{x}, \gamma_1^i(x_1), \dots, \gamma_T^i(x_T))$$

where $\mathbf{x}$ satisfies (1) with $u_t^i = \gamma_t^i(x_t) \forall i \in [N], t \in [T]$.

Definition 1 (Feedback Nash Equilibrium (FBNE)): ([1], Chapter 6) The strategy tuple $\gamma^* \in \Gamma$ is called a *feedback Nash equilibrium* (FBNE) if it is unilaterally optimal, i.e., if for all $\gamma \in \Gamma$, at each $i \in [N]$ and $t \in [T]$:

$$\begin{aligned} & J^{i, FB}(\gamma_{1:t-1}^i, \gamma_t^{i*}, \gamma_{t+1:T}^{i*}; \gamma_{1:t-1}^{-i}, \gamma_t^{-i*}, \gamma_{t+1:T}^{-i*}) \\ & \leq J^{i, FB}(\gamma_{1:t-1}^i, \gamma_t^i, \gamma_{t+1:T}^i; \gamma_{1:t-1}^{-i}, \gamma_t^{-i*}, \gamma_{t+1:T}^{-i*}). \end{aligned}$$

Def. 1 imposes *strong time consistency* ([1], Def. 5.14), i.e., a FBNE is unilaterally optimal for any subgame starting at any intermediate stage $t \in [T]$ from any state.

It is well known that the FBNE of LQ games with structure described in Section II-A can be characterized in closed-form [1]. Let $\mathcal{G} = (A^i, B^i, Q^i, R^i : i \in [N])$ denote an LQ game of the form described in Section II-A. Then $Z_t^i \in \mathbb{R}^{n \times n}$, $K_t^i \in \mathbb{R}^{m_i \times n}$, and $F_t \in \mathbb{R}^{n \times n} \forall i \in [N], t \in [T]$ can be *recursively computed backwards in time from $t = T$* using the following coupled Riccati equations:

$$Z_{T+1}^i = Q^i, \quad (4)$$

$$R^i K_t^i + \hat{B}^{i\top} Z_{t+1}^i \sum_{j \in [N]} \hat{B}^j K_t^j = \hat{B}^{i\top} Z_{t+1}^i A, \quad (5)$$

$$F_t = A - \sum_{j \in [N]} \hat{B}^j K_t^j, \quad (6)$$

$$Z_t^i = Q^i + F_t^\top Z_{t+1}^i F_t + K_t^{i\top} R^i K_t^i. \quad (7)$$

In particular, (4) provides boundary conditions for $(Z_t^i : i \in [N])$ at $t = T + 1$. Then, given $(Z_{t+1}^i : i \in [N])$ at some time t , (5) provides a system of linear equations for computing K_t^i . Subsequently, (6) computes F_t using K_t^i , and (7) computes Z_t^i using F_t and K_t^i . We then decrement t by 1 and return to (5) to solve for $(K_{t-1}^i : i \in [N])$.

For convenience, let $P_t \in \mathbb{R}^{m \times m}$ and $K_t \in \mathbb{R}^{m \times n}$, $S_t \in \mathbb{R}^{m \times n}$ be block-wise defined such that $[P_t]_{ii} = R^i + B^{i\top} Z_{t+1}^i B^i$ and $[P_t]_{ij} = B^{i\top} Z_{t+1}^i B^j$, $[K_t]_{i,:} := K_t^i$, and $[S_t]_{i,:} := B^{i\top} Z_{t+1}^i A$, $\forall i, j \in [N], i \neq j$. Then (5) can be rewritten compactly as:

$$P_t K_t = S_t. \quad (8)$$

Below, we present Assumption 1, which is maintained throughout the paper to ensure that the dynamic LQ games defined above yield a unique FBNE ([1], Remark 6.4).

Assumption 1: Q^i is positive semi-definite and R^i is positive definite $\forall i \in [N]$. Moreover, P_t^{-1} exists $\forall t \in [T]$.

Assumption 1 ensures that the coupled Riccati equations have a unique solution, as stated in the following proposition.

Proposition 1 ([1], Corollary 6.1): If a dynamic LQ game $\mathcal{G} = (A^i, B^i, Q^i, R^i : i \in [N])$ satisfies Assumption 1, it admits unique FBNE strategies and trajectory given as follows, $\forall i \in [N], t \in [T]$:

$$\gamma_t^{i*}(x_t) = -K_t^i x_t, \quad (9)$$

$$x_{t+1} = F_t x_t. \quad (10)$$

C. Open-Loop Nash Equilibria (OLNE)

In contrast to the state feedback information structure, the *open-loop information structure* requires each agent i to only use the initial state $x_1 \in \mathbb{R}^n$ to design their control u_t^i at each time t . In this case, each agent i devises *open-loop* strategies $\phi_t^i : \mathbb{R}^n \rightarrow \mathbb{R}^{m_i}$ over the time horizon $[T]$ a priori, and then deploys the control input $u_t^i = \phi_t^i(x_1)$ at each time t . We define Φ to be the open-loop joint strategy space of all agents.

Below, we define the Nash equilibrium corresponding to the open-loop information structure. For each agent $i \in [N]$, let $J^{i,OL} : \Phi \rightarrow \mathbb{R}$ denote the cost obtained by unrolling the system dynamics (1) with control inputs prescribed by a given open-loop strategy, i.e.,

$$J^{i,OL}(\phi) := C^i(\mathbf{x}, \phi_1^i(x_1), \dots, \phi_T^i(x_1))$$

where $\mathbf{x}$ satisfies (1) with $u_t^i = \phi_t^i(x_1) \forall i \in [N], t \in [T]$.

Definition 2 (Open-Loop Nash Equilibrium (OLNE)): ([1], Chapter 6) We call $(u^{1*}, \dots, u^{N*}) \in \mathbb{R}^{mT}$ an *open-loop Nash Equilibrium (OLNE)* if for any open-loop strategy $\phi \in \Phi$, and agent $\forall i \in [N]$, we have:

$$J^{i,OL}(\phi^{i,*}; \phi^{-i,*}) \leq J^{i,OL}(\phi^i; \phi^{-i,*}).$$

The OLNE of LQ games with the structure described in Section II-A can likewise be characterized in closed-form [1]. Let $\Lambda_t^i \in \mathbb{R}^{n \times n}$, $M_t^i \in \mathbb{R}^{m_i \times n}$, and $L_t^i \in \mathbb{R}^{n \times n} \forall i \in [N], t \in [T]$ be recursively computed backwards in time from $t = T$ using the following Riccati equations:

$$M_{T+1}^i = Q^i, \quad (11)$$

$$\Lambda_t = I + \sum_{i \in [N]} \hat{B}^i (R^i)^{-1} \hat{B}^{i\top} M_{t+1}^i, \quad (12)$$

$$L_t^i = (R^i)^{-1} \hat{B}^{i\top} M_{t+1}^i \Lambda_t^{-1} A, \quad (13)$$

$$M_t^i = Q^i + A^\top M_{t+1}^i \Lambda_t^{-1} A. \quad (14)$$

Proposition 2 ([1], Thm. 6.2): Under Assumption 1, an LQ game $\mathcal{G} := (A^i, B^i, Q^i, R^i : i \in [N])$ admits a unique OLNE if Λ_t^{-1} exists $\forall t \in [T]$ and (11)-(14) admit a unique solution, in which case the OLNE strategies and trajectory are, $\forall i \in [N], t \in [T]$:

$$\phi_t^{i*}(x_1) = -L_t^i x_t = -L_t^i \cdot \Pi_{\tau=1}^{t-1} (\Lambda_\tau^{-1} A) x_1, \quad (15)$$

$$x_{t+1} = \Lambda_t^{-1} A x_t. \quad (16)$$

Throughout the rest of this paper, we assume that Assumption 1 holds, and that Λ_t^{-1} exists for all $t \in [T]$.

III. SUFFICIENT CONDITIONS FOR ALIGNMENT OF FBNE AND OLNE TRAJECTORIES

Empirical studies suggest that FBNE and OLNE strategies are more closely aligned when the state cost matrix of each agent has minimal dependence on the states and inputs of other agents. In Section III-A, we explicitly quantify this observation by associating a given LQ game $\mathcal{G} := (A^i, B^i, Q^i, R^i : i \in [N])$ with an *auxiliary* game $\tilde{\mathcal{G}} := (A^i, B^i, \tilde{Q}^i, R^i : i \in [N])$. The auxiliary game $\tilde{\mathcal{G}}$ is a copy of the original LQ game $\mathcal{G}$ whose state cost matrices $\tilde{Q}^i$ have been modified, to reduce the dependency of each agent's cost on other agents' states. We then use auxiliary games to contrast the FBNE and OLNE solutions for general-sum LQ dynamic games in Section III-B.

A. Auxiliary Game

Formally, for a given LQ dynamic game, we define our auxiliary game as follows.

Definition 3 (Auxiliary LQ Game): Given an LQ Game $\mathcal{G} := (A^i, B^i, Q^i, R^i : i \in [N])$, we define the **auxiliary game** associated with $\mathcal{G}$ to be $\tilde{\mathcal{G}} := (A^i, B^i, \tilde{Q}^i, R^i : i \in [N])$, where $\tilde{Q}^i$ is defined by retaining the elements of Q^i in the i -th row block while setting all other elements to 0, i.e., $[\tilde{Q}^i]_{i,:} = [Q^i]_{i,:}$ and $[\tilde{Q}^i]_{j,:} = 0 \forall j \neq i$.

Remark 3: Auxiliary LQ games in general do not have symmetric Q^i , and thus are not LQ games and do not have equilibria in the conventional sense. Rather, auxiliary games are artificial constructs defined to assist us in establishing algebraic conditions under which the FBNE and OLNE trajectories of an LQ game are aligned. Nonetheless, for ease of exposition, we will refer to matrices satisfying (4)-(8) for an auxiliary game $\tilde{\mathcal{G}} := (A^i, B^i, \tilde{Q}^i, R^i : i \in [N])$, if they exist, as characterizing the “FBNE” of $\tilde{\mathcal{G}}$, and denote them by $(\tilde{Z}_t^i, \tilde{K}_t^i, \tilde{F}_t^i, \tilde{P}_t^i, \tilde{S}_t^i : i \in [N], t \in [T])$. Similarly, we will refer to matrices satisfying (11)-(14) for $\tilde{\mathcal{G}}$, if they exist, as characterizing the “OLNE” of $\tilde{\mathcal{G}}$, and denote them by $(\tilde{M}_t^i, \tilde{\Lambda}_t^i, \tilde{L}_t^i : i \in [N], t \in [T])$.

B. FBNE/OLNE of LQ Games vs. Auxiliary LQ Games

First, we show that the OLNE of an LQ game $\mathcal{G}$ satisfies (11)-(14) for $\tilde{\mathcal{G}}$, i.e., loosely speaking, the OLNE of $\mathcal{G}$ and of $\tilde{\mathcal{G}}$ coincide.

Lemma 1: Under Assumption 1, let $\tilde{\mathcal{G}}$ be the auxiliary game associated with $\mathcal{G}$. Suppose that Λ_t^{-1} exists for all t , then the OLNE of $\mathcal{G}$ satisfies (11)-(14) for $\tilde{\mathcal{G}}$. In other words, $\tilde{L}_t^i = L_t^i, \forall i \in [N], t \in [T]$.

Proof: From (12)-(16), we find it suffices to prove that $\hat{B}^{i\top} \tilde{M}_{t+1}^i = \hat{B}^{i\top} M_{t+1}^i \forall t \in [T], i \in [N]$. Due to the block structure of $\hat{B}^{i\top}$, it suffices to prove that $[\tilde{M}_{t+1}^i]_{i,:} = [M_{t+1}^i]_{i,:}$. We proceed by backward induction from time T . When $t = T$, we have $[\tilde{M}_{T+1}^i]_{i,:} = [\tilde{Q}^i]_{i,:} = [Q^i]_{i,:} = [M_{T+1}^i]_{i,:}$. Suppose $[\tilde{M}_{t+1}^i]_{i,:} = [M_{t+1}^i]_{i,:}$ for some $t \in [T]$. From the block structures of each $\hat{B}^i$ and of A , we have:

$$\hat{B}^{i\top} \tilde{M}_{t+1}^i = B^{i\top} [\tilde{M}_{t+1}^i]_{i,:} = B^{i\top} [M_{t+1}^i]_{i,:} = \hat{B}^{i\top} M_{t+1}^i.$$

From (12), we obtain $\tilde{\Lambda}_t = \Lambda_t$, and $\tilde{\Lambda}_t^{-1}$ exists. Then:

$$[\tilde{M}_t^i]_{i,:} = [\tilde{Q}^i]_{i,:} + [A^\top \tilde{M}_{t+1}^i]_{i,:} \Lambda_t^{-1} A \quad (17)$$

$$= [\tilde{Q}^i]_{i,:} + A^{i\top} [\tilde{M}_{t+1}^i]_{i,:} \Lambda_t^{-1} A \quad (18)$$

$$= [Q^i]_{i,:} + A^{i\top} [M_{t+1}^i]_{i,:} \Lambda_t^{-1} A \quad (19)$$

$$= [Q^i]_{i,:} + [A^\top M_{t+1}^i]_{i,:} \Lambda_t^{-1} A \quad (20)$$

where (18) follows from the block structure of A , and (19) follows from the fact that $[\tilde{Q}^i]_{i,:} = [Q^i]_{i,:}$, and the induction hypothesis that $[\tilde{M}_{t+1}^i]_{i,:} = [M_{t+1}^i]_{i,:}$. This completes the induction step and thus proves that $\tilde{L}_t^i = L_t^i, \forall i \in [N], t \in [T]$, and $\Lambda_t^{-1} A = \tilde{\Lambda}_t^{-1} A \forall t \in [T]$. ■

Remark 4: From (17)-(20), we also find that $\tilde{M}_{t+1}^i$ has nonzero entries only in the i -th row block, $\forall i \in [N], t \in [T]$. Indeed, $[\tilde{M}_{t+1}^i]_{j,:} = [\tilde{Q}^i]_{j,:} = O$. Thus, if $[\tilde{M}_{t+1}^i]_{j,:} = O, \forall j \neq i$, for some $t \in [T]$, then, we have:

$$\begin{aligned} [\tilde{M}_t^i]_{j,:} &= [\tilde{Q}^i]_{j,:} + [A^\top \tilde{M}_{t+1}^i]_{j,:} \Lambda_t^{-1} A \\ &= O + A^{j\top} [\tilde{M}_{t+1}^i]_{j,:} \Lambda_t^{-1} A \\ &= O, \end{aligned}$$

which implies that $[\tilde{M}_t^i]_{j,:} = O \forall j \neq i, t \in [T]$.

Next, we prove that the OLNE solution of any auxiliary LQ Game $\tilde{\mathcal{G}}$ always satisfies the Riccati equations (4)-(7) for $\tilde{\mathcal{G}}$. Loosely speaking, the OLNE and FBNE of $\tilde{\mathcal{G}}$ coincide.

Lemma 2: Under Assumption 1, the OLNE solution of $\tilde{\mathcal{G}} := (A^i, B^i, \tilde{Q}^i, R^i : i \in [N])$ satisfies (4)-(7), with each Q^i, Z_t^i, K_t^i, F_t replaced by $\tilde{Q}^i, \tilde{Z}_t^i, \tilde{K}_t^i, \tilde{F}_t$, respectively. In other words, $\tilde{K}_t^i = \tilde{L}_t^i$ and $\tilde{F}_t = \tilde{\Lambda}_t^{-1} A, \forall i \in [N], t \in [T]$.

Proof: By definition, $\tilde{M}_{T+1}^i = \tilde{Q}^i$, which confirms (4). Next, from (12) and (13), we have $\forall i \in [N], t \in [T]$:

$$\begin{aligned} R^i \tilde{L}_t^i + \hat{B}^{i\top} \tilde{M}_{t+1}^i \sum_{j \in [N]} \hat{B}^j \tilde{L}_t^j \\ = \hat{B}^{i\top} \tilde{M}_{t+1}^i \tilde{\Lambda}_t^{-1} A \\ + \hat{B}^{i\top} \tilde{M}_{t+1}^i \sum_{j \in [N]} \hat{B}^j (R^j)^{-1} \hat{B}^{j\top} \tilde{M}_{t+1}^j \tilde{\Lambda}_t^{-1} A \\ = \hat{B}^{i\top} \tilde{M}_{t+1}^i \left(I + \sum_{j \in [N]} \hat{B}^j (R^j)^{-1} \hat{B}^{j\top} \tilde{M}_{t+1}^j \right) \tilde{\Lambda}_t^{-1} A \\ = \hat{B}^{i\top} \tilde{M}_{t+1}^i A, \end{aligned}$$

so $\tilde{L}_t$ and $\tilde{M}_{t+1}^i$ satisfy (5). Set $\tilde{F}_t := A - \sum_{j \in [N]} \hat{B}^j \tilde{L}_t^j$ so as to satisfy (6). We proceed to verify (7). First, $\forall t \in [T]$:

$$\tilde{\Lambda}_t \tilde{F}_t = \left(I + \sum_{i \in [N]} \hat{B}^i (R^i)^{-1} \hat{B}^{i\top} \tilde{Z}_{t+1}^i \right) \tilde{F}_t \quad (21)$$

$$= \tilde{F}_t + \sum_{i \in [N]} \hat{B}^i (R^i)^{-1} R^i \tilde{K}_t^i \quad (22)$$

$$\begin{aligned} &= \tilde{F}_t + \sum_{i \in [N]} \hat{B}^i \tilde{K}_t^i \\ &= A, \end{aligned} \quad (23)$$

where (22) follows from (24), while (23) follows from (6). Next, by rearranging (5), we obtain that:

$$R^i \tilde{K}_t^i = \hat{B}^{i\top} \tilde{Z}_{t+1}^i \left(A_t - \sum_{j \in [N]} \hat{B}^j \tilde{K}_t^j \right) = \hat{B}^{i\top} \tilde{Z}_{t+1}^i \tilde{F}_t. \quad (24)$$

Moreover:

$$\begin{aligned} \tilde{Q}^i + \tilde{F}_t^\top \tilde{M}_{t+1}^i \tilde{F}_t + \tilde{L}_t^{i\top} R^i \tilde{L}_t^i \\ = \tilde{Q}^i + \tilde{F}_t^\top \tilde{M}_{t+1}^i \tilde{F}_t + \tilde{L}_t^{i\top} \hat{B}^{i\top} \tilde{M}_{t+1}^i \tilde{F}_t \end{aligned} \quad (25)$$

$$= \tilde{Q}^i + \left(\tilde{F}_t + \sum_{j \in [N]} \hat{B}^j \tilde{L}_t^j \right)^\top \tilde{M}_{t+1}^i \tilde{F}_t \quad (26)$$

$$= \tilde{Q}^i + A^\top \tilde{M}_{t+1}^i \tilde{F}_t \quad (27)$$

$$= \tilde{Q}^i + A^\top \tilde{M}_{t+1}^i \tilde{\Lambda}_t^{-1} A \quad (28)$$

$$= \tilde{M}_t^i, \quad (29)$$

so $\tilde{L}_t$ and $\tilde{M}_{t+1}^i$ satisfy (6). Above, (25) follows from (24), (26) follows from Remark 4, (27) follows from (6), (28) follows from (21)-(23), and (29) follows from (14). ■

Recall that Lemmas 1 and 2 respectively imply that, loosely speaking, the OLNE of $\mathcal{G}$ and of $\tilde{\mathcal{G}}$ coincide, and the OLNE of $\tilde{\mathcal{G}}$ satisfies the Riccati equations (4)-(7) for $\tilde{\mathcal{G}}$. Together, Lemmas 1 and 2 establish that the OLNE trajectory of $\mathcal{G}$ matches the FBNE trajectory of $\tilde{\mathcal{G}}$ (Lemma 3). Thus, if the FBNE of $\tilde{\mathcal{G}}$ and of $\mathcal{G}$ coincide, then the OLNE and FBNE trajectories of $\mathcal{G}$ are aligned (Thm. 1).

Lemma 3: If Assumption 1 holds and Λ_t^{-1} exists $\forall t$, then the OLNE of $\mathcal{G}$ satisfies (4)-(7) for $\tilde{\mathcal{G}}$, and $\Lambda_t^{-1} A = \tilde{F}_t, \forall t$.

Theorem 1: Consider an LQ game $\mathcal{G}$ and its auxiliary game $\tilde{\mathcal{G}}$. Under Assumption 1, suppose Λ_t^{-1} exists $\forall t$ and $\tilde{K}_t^i = K_t^i \forall i \in [N], t \in [T]$, then $F_t = \Lambda_t^{-1} A, \forall t$ and the FBNE and OLNE trajectories of $\mathcal{G}$ are identical from any initial state.

IV. BOUNDING THE FBNE-TO-OLNE GAP OF LQ GAMES

Given an LQ game $\mathcal{G} := (A^i, B^i, Q^i, R^i : i \in [N])$, Thm. 1 suggests that the FBNE-to-OLNE gap of $\mathcal{G}$ can be characterized using the difference between the FBNE of $\mathcal{G}$, i.e., $(K_t^i : i \in [N], t \in [T])$, and of its auxiliary game $\tilde{\mathcal{G}} := (A^i, B^i, \tilde{Q}^i, R^i : i \in [N])$, i.e., $(\tilde{K}_t^i : i \in [N], t \in [T])$, as quantified by $\|\tilde{K}_t - K_t\|_2$. Below, Prop. 3 complements Thm. 1 by upper bounding the deviation of the FBNE strategy of a dynamic LQ game when its state cost matrices are perturbed. Together, Thm. 1 and Prop. 3 upper bound the the FBNE-to-OLNE gap of $\mathcal{G}$, as summarized in Thm. 2.

Proposition 3: Given $\mathcal{G} := (A^i, B^i, Q^i, R^i : i \in [N])$ and $\tilde{\mathcal{G}} := (A^i, B^i, \tilde{Q}^i, R^i : i \in [N])$, let $\epsilon := \max_{i \in [N]} \|\tilde{Q}^i - Q^i\|_2$, and let $(\tilde{Z}_t^i, \tilde{K}_t^i, \tilde{F}_t, \hat{P}_t, \hat{S}_t : i \in [N], t \in [T])$ be the solutions of (4)-(8) for $\tilde{\mathcal{G}}$. Set $\delta \hat{P}_t := \|\hat{P}_t - P_t\|_2, \delta \hat{S}_t := \|\hat{S}_t - S_t\|_2, \delta \hat{K}_t := \|\hat{K}_t - K_t\|_2, \delta \hat{Z}_t^i := \|\hat{Z}_t^i - Z_t^i\|_2 \forall t \in [T], i \in [N]$. If $\delta \hat{P}_t < 1/\|P_t^{-1}\|_2 \forall t \in [T]$, the following recursive upper bounds for $\delta \hat{P}_t, \delta \hat{S}_t, \delta \hat{K}_t, \delta \hat{Z}_t^i$ hold:

$$\delta \hat{Z}_{t+1}^i = \epsilon, \forall i \in [N], \quad (30)$$

$$\delta \hat{P}_t \leq \sqrt{N} \|B\|_2^2 \max_{i \in [N]} \delta \hat{Z}_{t+1}^i, \quad (31)$$

$$\delta \hat{S}_t \leq \sqrt{N} \|A\|_2 \|B\|_2 \max_{i \in [N]} \delta \hat{Z}_{t+1}^i \quad (32)$$

$$\delta \hat{K}_t \leq \frac{\|P_t^{-1}\|_2}{1 - \|P_t^{-1}\|_2 \delta \hat{P}_t} \cdot (\|K_t\|_2 \delta \hat{P}_t + \delta \hat{S}_t) \quad (33)$$

$$\begin{aligned} \delta \hat{Z}_t^i &\leq (\|K_t^i\|_2^2 + 1) \epsilon + \|B\|_2 (\|Z_{t+1}^i\|_2 + \delta \hat{Z}_{t+1}^i) \\ &\quad \cdot (2\|A - BK_t\|_2 + \|B\|_2 \delta \hat{K}_t) \delta \hat{K}_t \\ &\quad + \|A - BK_t\|_2^2 \cdot \delta \hat{Z}_{t+1}^i \\ &\quad + (\|R^i\|_2 + \epsilon) (2\|K_t^i\|_2 + \delta \hat{K}_t) \delta \hat{K}_t. \end{aligned} \quad (34)$$

Proof: The proof follows by carefully applying known results from the perturbation theory of linear equations ([15], Section 2.2), as well as the triangle and Cauchy-Schwarz inequalities, to the FBNE Riccati equations (4)-(7), as well as (8). Details are omitted for brevity. ■

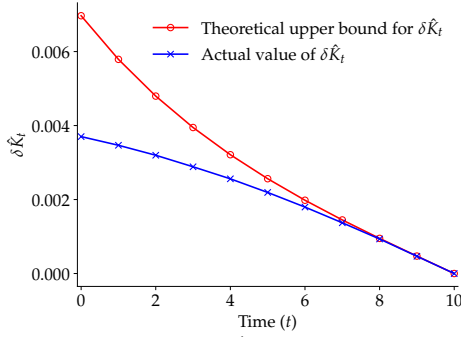


Fig. 1. The actual values of $\delta \tilde{K}_t$ and the theoretical upper bound of $\delta \tilde{K}_t$ as given by (30)-(34) (red, with circle marks), corresponding to the LQ games $\mathcal{G}$ and $\hat{\mathcal{G}}$ described in Example 1. The theoretical upper bound for $\delta \tilde{K}_t$ closely matches the actual value of $\delta \tilde{K}_t$ throughout the first few iterates of the backward iteration process, i.e., when t is close to $T = 10$. Significant divergence between the theoretical upper bound and actual value of $\delta \tilde{K}_t$ only occurs near the start of the time horizon.

The following example shows the theoretical upper bound for $\delta \tilde{K}_t$ in (30)-(34) is nearly tight for some $\mathcal{G}$ and $\hat{\mathcal{G}}$.

Example 1: Consider 2-agent, 10-stage LQ games $\mathcal{G} := (A^i, B^i, Q^i, R^i : i \in \{1, 2\})$ and $\hat{\mathcal{G}} := (A^i, B^i, \hat{Q}^i, R^i : i \in \{1, 2\})$, with $A^i = B^i = 1$, $R^i = 300$, $\forall i \in \{1, 2\}$, $Q^i = I_{2 \times 2}$ (the 2×2 identity matrix), for each $i \in [2]$, and:

$$\hat{Q}^1 = 1.1 I_{2 \times 2}, \quad \hat{Q}^2 = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix}.$$

Fig. 1 plots the theoretical upper bound (from (30)-(34)) and actual values of $\delta \tilde{K}_t$, i.e., the FBNE-to-OLNE gap of $\mathcal{G}$ and $\hat{\mathcal{G}}$. The theoretical upper bound for $\delta \tilde{K}_t$ closely tracks the $\delta \tilde{K}_t$ value during initial steps of the backward recursion from $T = 10$, i.e., when t is close to T .

The following result follows directly from Prop. 3.

Theorem 2: Given an LQ game $\mathcal{G} := (A^i, B^i, Q^i, R^i : i \in [N])$ and its auxiliary game $\hat{\mathcal{G}} := (A^i, B^i, \hat{Q}^i, R^i : i \in [N])$, let $\epsilon := \max_{i \in [N]} \|\hat{Q}^i - Q^i\|_2$. Then the FBNE-to-OLNE gap of $\mathcal{G}$ can be bounded as:

$$\|L_t - K_t\|_2 \leq \frac{\|P_t^{-1}\|_2}{1 - \|P_t^{-1}\|_2 \delta \tilde{P}_t} \cdot (\|K_t\|_2 \delta \tilde{P}_t + \delta \tilde{S}_t),$$

in conjunction with variants of (30)-(32), (34) with $\delta \tilde{K}_t := \|\tilde{K}_t - K_t\|_2 = \|L_t - K_t\|_2$, $\delta \tilde{S}_t := \|\tilde{S}_t - S_t\|_2$, $\delta \tilde{P}_t := \|\tilde{P}_t - P_t\|_2$, $\delta \tilde{Z}_t^i := \|\tilde{Z}_t^i - Z_t^i\|_2$, $\forall t \in [T], i \in [N]$, where $L_t \in \mathbb{R}^{m \times n}$ is defined block-wise by $[L_t]_{i,:} = L_t^i, \forall i \in [N]$.

V. SIMULATION STUDIES

We present a Monte Carlo study to validate the sufficient condition in Thm. 1 for the alignment of the FBNE and OLNE of dynamic LQ games. We then study the FBNE-to-OLNE gaps of two specific games from our Monte Carlo study that violate the sufficient condition to varying degrees of severity. Our observations confirm our conclusions in Thm. 2 that the more severely a dynamic LQ game violates the sufficient condition $\delta \tilde{K}_t = 0$, the more significantly its FBNE and OLNE solutions diverge.

In our Monte Carlo study, we generated 10000 instances of a 2-agent dynamic LQ game with $T = 10$ by independently sampling state cost matrices Q^i while holding all other game parameters fixed at the following values:

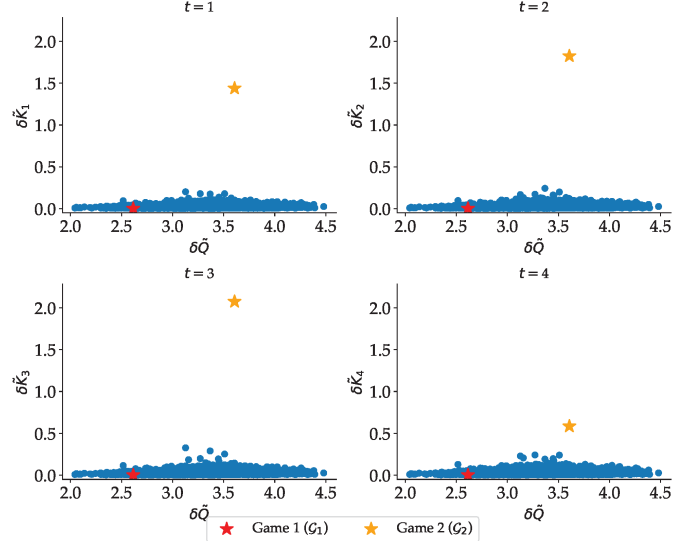


Fig. 2. Scatter plot of $\delta \tilde{K}_t := \|\tilde{K}_t - K_t\|_2$ vs. $\delta \tilde{Q} := \|\tilde{Q} - Q\|_2$, at each $t \in [4]$, for each of the 10000 sampled LQ dynamic games. Each sampled game admits a **unique** FBNE and a **unique** OLNE. These samples violate to different degrees the sufficient condition $\delta \tilde{K}_t = 0$ for the FBNE and OLNE trajectories to be aligned (Thm. 1). Specifically, the game $\mathcal{G}_1$ barely violates the condition, while the game $\mathcal{G}_2$ violates the condition much more severely. Moreover, many sampled games correspond to large values of $\delta \tilde{Q}$ but tiny values of $\delta \tilde{K}_t$, which shows the difficulty of obtaining nontrivial lower bounds for $\delta \tilde{K}_t$ in terms of $\delta \tilde{Q}$.

$$A^1 = A^2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, B^1 = B^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, R^1 = 3, R^2 = 2.$$

Fig. 2 presents $\delta \tilde{K}_t$ vs. $\delta \tilde{Q}$ scatter plots for all sampled games, for $t \in [4]$. Many of these sampled games correspond to large values of $\delta \tilde{Q}$ and very small values of $\delta \tilde{K}_t$, showcasing the difficulty of obtaining a uniform lower bound.

To study the sufficient condition in Thm. 1 in more detail, we sample two games that violate the sufficient condition of the game to different degrees of severity. Without loss of generality, we reindex the game instance marked by the red star in Fig. 2 as $\mathcal{G}_1$, which has state cost matrices given by:

$$Q^1 = \begin{bmatrix} 1.86 & 0.85 & 0.57 & 0.44 \\ 0.85 & 1.99 & 0.35 & 0.25 \\ 0.57 & 0.35 & 1.53 & 0.70 \\ 0.44 & 0.25 & 0.70 & 1.07 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} 1.29 & 0.92 & 0.98 & 0.57 \\ 0.92 & 1.16 & 0.78 & 0.53 \\ 0.98 & 0.78 & 1.63 & 0.53 \\ 0.57 & 0.53 & 0.53 & 1.84 \end{bmatrix}.$$

We observe from Fig. 2 that $\mathcal{G}_1$ only barely violates the sufficient condition. Next, we denote by $\mathcal{G}_2$ the game marked by the orange star in Fig. 2, with the state cost matrices:

$$Q^1 = \begin{bmatrix} 1.48 & 1.41 & 1.09 & 0.94 \\ 1.41 & 1.66 & 0.89 & 1.46 \\ 1.09 & 0.89 & 1.64 & 0.87 \\ 0.94 & 1.46 & 0.87 & 1.95 \end{bmatrix},$$

$$Q^2 = \begin{bmatrix} 1.83 & 0.44 & 1.43 & 0.81 \\ 0.44 & 1.97 & 0.63 & 0.87 \\ 1.43 & 0.63 & 1.29 & 0.85 \\ 0.81 & 0.87 & 0.85 & 0.73 \end{bmatrix}.$$

We observe from Fig. 2 that $\mathcal{G}_2$ violates the sufficient condition much more severely compared to $\mathcal{G}_1$.

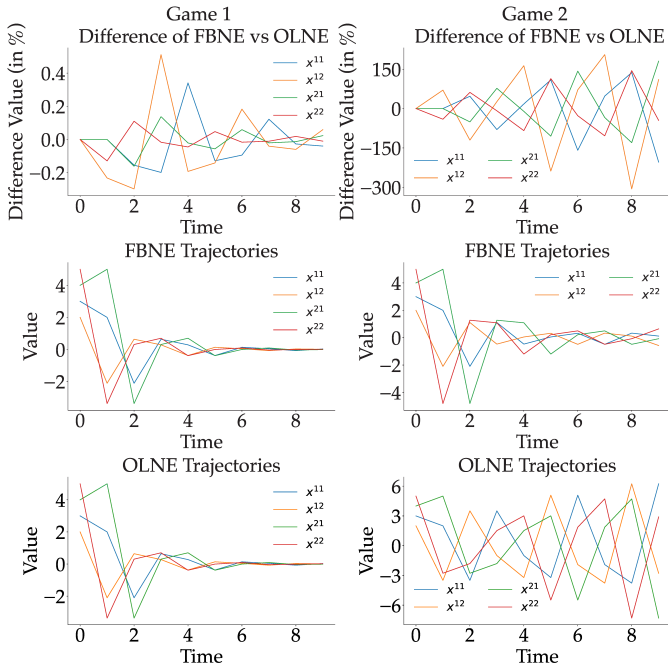


Fig. 3. Plots, for $\mathcal{G}_1$ and $\mathcal{G}_2$, of the FBNE trajectories, OLNE trajectories, and the differences between the FBNE and OLNE trajectories. We observe that the FBNE and OLNE trajectories of $\mathcal{G}_1$, which only slightly violates the sufficient condition in Thm. 1 (i.e., $\delta \tilde{K}_t = 0$), are within 0.25% of each other. In contrast, $\mathcal{G}_2$ violates the sufficient condition more severely, and its FBNE and OLNE trajectories differ up to 300%.

Fig. 3 plots the FBNE and OLNE trajectories of $\mathcal{G}_1$ and of $\mathcal{G}_2$, as well as the percent difference between the FBNE and OLNE of $\mathcal{G}_1$ and of $\mathcal{G}_2$, normalized with respect to the respective initial state values. We observe that the FBNE and OLNE trajectories for $\mathcal{G}_1$ are within 0.5% of each other since $\mathcal{G}_1$ only slightly violates the sufficient condition. In contrast, $\mathcal{G}_2$, which violates the sufficient condition by a wider margin, has FBNE and OLNE trajectories that differ up to 300%. Our results illustrate that the degree to which a dynamic LQ game violates the sufficient condition heavily influences how much its FBNE and OLNE trajectories diverge¹.

VI. CONCLUSION

We presented a principled approach for quantifying the difference between the FBNE and OLNE of a dynamic LQ game. Given a dynamic LQ game $\mathcal{G}$, we constructed an auxiliary game $\tilde{\mathcal{G}}$, a copy of $\mathcal{G}$ with modified state costs, and proved that the OLNE of $\mathcal{G}$ can be synthesized by solving for the FBNE of $\tilde{\mathcal{G}}$. We then quantified the FBNE-to-OLNE gap of $\mathcal{G}$ by bounding the difference between the FBNE strategies of $\mathcal{G}$ and $\tilde{\mathcal{G}}$, i.e., $\|K_t - \tilde{K}_t\|_2$, using the difference between the state cost matrices of $\mathcal{G}$ and $\tilde{\mathcal{G}}$. We presented numerical results that validate our theoretical claims and illustrate the difficulty of obtaining matching lower bounds. Our results provide insight into the FBNE-to-OLNE gap for dynamic games, and facilitate the selection of appropriate equilibrium solutions, with promising applications in areas such as autonomous driving and human-robot interaction.

¹For plots of the FBNE-to-OLNE gap over a wider selection of games, please see the Appendix in the ArXiv version of this work [16].

REFERENCES

- [1] Tamer Başar and Geert Jan Olsder. *Dynamic Noncooperative Game Theory*. SIAM, 1998.
- [2] Eugene Vinitsky, Nathan Lichtlé, Kanaad Parvate, and Alexandre Bayen. “Optimizing Mixed Autonomy Traffic Flow with Decentralized Autonomous Vehicles and Multi-agent Reinforcement Learning”. In: *ACM Transactions on Cyber-Physical Systems* 7.2 (2023), pp. 1–22.
- [3] Maulik Bhatt, Yixuan Jia, and Negar Mehr. “Efficient Constrained Multi-agent Trajectory Optimization using Dynamic Potential Games”. In: *2023 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*. IEEE, 2023, pp. 7303–7310.
- [4] Simon Le Cleac’h, Mac Schwager, and Zachary Manchester. “ALGAMES: A Fast Solver for Constrained Dynamic Games”. In: *arXiv preprint arXiv:1910.09713* (2019).
- [5] David Fridovich-Keil, Ellis Ratner, Lasse Peters, Anca D Dragan, and Claire J Tomlin. “Efficient Iterative Linear-Quadratic Approximations for Nonlinear Multi-Player General-Sum Differential Games”. In: *2020 IEEE International Conference on Robotics and Automation (ICRA)* (2020), pp. 1475–1481.
- [6] John F Nash Jr. “Equilibrium Points in N-Person Games”. In: *Proceedings of the National Academy of Sciences* 36.1 (1950), pp. 48–49.
- [7] Alan Wilbor Starr and Yu-Chi Ho. “Nonzero-sum Differential Games”. In: *Journal of Optimization Theory and Applications* 3 (1969), pp. 184–206.
- [8] Ryan Lowe, Yi I Wu, Aviv Tamar, Jean Harb, OpenAI Pieter Abbeel, and Igor Mordatch. “Multi-agent Actor-critic for Mixed Cooperative-competitive Environments”. In: *Advances in Neural Information Processing Systems* 30 (2017).
- [9] Forrest Laine, David Fridovich-Keil, Chih-Yuan Chiu, and Claire Tomlin. “The Computation of Approximate Generalized Feedback Nash Equilibria”. In: *arXiv preprint arXiv:2101.02900* (2021).
- [10] Ben Hambly, Renyuan Xu, and Huining Yang. “Policy Gradient Methods Find the Nash Equilibrium in N-Player General-sum Linear-Quadratic Games”. In: *Journal of Machine Learning Research* 24.139 (2023), pp. 1–56.
- [11] Andrea Monti, Benita Nortmann, Thulasi Mylvaganam, and Mario Sassano. “Feedback and Open-Loop Nash Equilibria for LQ Infinite-Horizon Discrete-Time Dynamic Games”. In: *SIAM Journal on Control and Optimization* 62.3 (2024), pp. 1417–1436.
- [12] Emilio Benenati and Sergio Grammatico. “Linear-Quadratic Dynamic Games as Receding-Horizon Variational Inequalities”. In: *arXiv preprint arXiv:2408.15703* (2024).
- [13] Pudur Viswanadha Reddy and Georges Zaccour. “Open-loop Nash Equilibria in a Class of Linear-Quadratic Difference Games with Constraints”. In: *IEEE Transactions on Automatic Control* 60.9 (2015), pp. 2559–2564.
- [14] Jingqi Li, C-Y. Chiu, Lasse Peters, Somayeh Sojoudi, Claire Tomlin, and David Fridovich-Keil. “Cost Inference for Feedback Dynamic Games from Noisy Partial State Observations and Incomplete Trajectories”. In: *Proceedings of the 2023 International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*. London, United Kingdom: International Foundation for Autonomous Agents and Multiagent Systems, 2023, pp. 1062–1070.
- [15] James W. Demmel. *Applied Numerical Linear Algebra*. Society for Industrial and Applied Mathematics, 1997.
- [16] Chih-Yuan Chiu, Jingqi Li, Maulik Bhatt, and Negar Mehr. “To What Extent do Open-loop and Feedback Nash Equilibria Diverge in General-Sum Linear Quadratic Dynamic Games?” In: *arXiv preprint arXiv:2409.11257* (2024).

APPENDIX I ADDITIONAL SIMULATION RESULTS

A. Heterogeneity among Agent Dynamics

In order to study whether the magnitude of the FBNE-to-OLNE gap of a dynamic LQ game is affected by heterogeneity in agents' dynamics, as encoded by the matrices $\{A^i, B^i : i \in [N]\}$, we randomly sample 10,000 distinct values of A^1, A^2, B^1, B^2, Q^1 and Q^2 while fixing $R^1 = 3$ and $R^2 = 2$.

Fig. 4 presents $\delta\tilde{K}_t$ vs. $\delta\tilde{Q}$ scatter plots for all sampled games, at each $t \in [T]$ with $T = 4$. The corresponding values

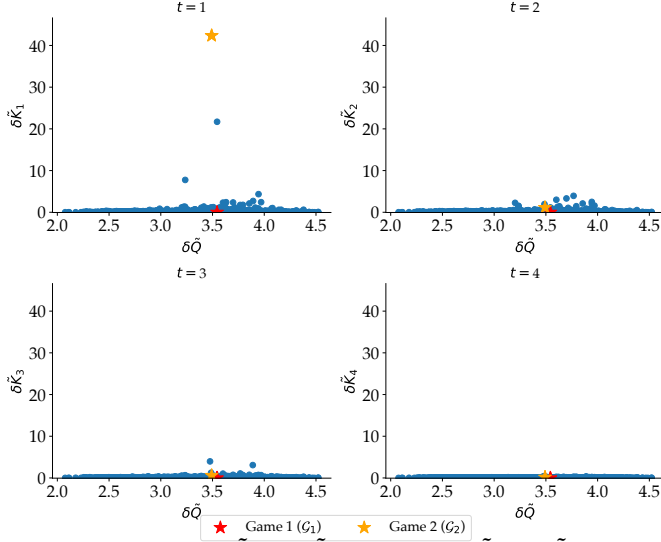


Fig. 4. Scatter plot of $\delta\tilde{K} := \|\tilde{K}_t - K_t\|_2$ vs. $\delta\tilde{Q} := \|\tilde{Q} - Q\|_2$, at each $t \in [4]$, for each of the 10000 sampled LQ dynamic games. The dynamics matrices A^1, A^2, B^1, B^2 and state cost matrices Q^1, Q^2 of each sampled game is independently sampled, while the control cost matrices R^1 and R^2 are fixed across sampled games.

of A^1, A^2, B^1, B^2, Q^1 and Q^2 for $\mathcal{G}_3$ and $\mathcal{G}_4$ are as follows.

- Game $\mathcal{G}_3$:

$$A^1 = \begin{bmatrix} 0.39 & 0.87 \\ 0.15 & 0.37 \end{bmatrix}, A^2 = \begin{bmatrix} 0.07 & 0.38 \\ 0.04 & 0.30 \end{bmatrix},$$

$$B^1 = \begin{bmatrix} 0.78 \\ 0.97 \end{bmatrix}, B^2 = \begin{bmatrix} 0.14 \\ 0.001 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1.88 & 0.67 & 0.49 & 0.75 \\ 0.67 & 1.88 & 1.03 & 1.81 \\ 0.49 & 1.03 & 1.58 & 0.70 \\ 0.75 & 1.81 & 0.70 & 1.98 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.72 & 0.54 & 0.58 & 0.68 \\ 0.54 & 1.20 & 0.79 & 0.39 \\ 0.58 & 0.79 & 1.53 & 0.26 \\ 0.68 & 0.39 & 0.26 & 1.71 \end{bmatrix}$$

- Game $\mathcal{G}_4$

$$A_1 = \begin{bmatrix} 0.16 & 0.96 \\ 0.96 & 0.55 \end{bmatrix}, A_2 = \begin{bmatrix} 0.65 & 0.94 \\ 0.29 & 0.81 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.89 \\ 0.49 \end{bmatrix}, B_2 = \begin{bmatrix} 0.62 \\ 0.56 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 1.04 & 0.32 & 0.80 & 0.66 \\ 0.32 & 1.03 & 0.56 & 1.22 \\ 0.80 & 0.56 & 1.97 & 1.06 \\ 0.66 & 1.22 & 1.06 & 1.79 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 0.94 & 1.03 & 1.07 & 0.90 \\ 1.03 & 1.40 & 1.34 & 1.01 \\ 1.07 & 1.34 & 1.35 & 0.88 \\ 0.90 & 1.01 & 0.88 & 1.61 \end{bmatrix}.$$

In order to further investigate how the heterogeneity across agents' dynamics affects the FBNE-to-OLNE gap of the dynamic game, we perform a Monte Carlo study of 2000 different games with different values of the following quantities:

$$\max_{i \in \{1,2\}} \|A_i - \text{avg}(A)\|, \quad \max_{i \in \{1,2\}} \|B_i - \text{avg}(B)\|,$$

where $\text{avg}(A) := \frac{1}{2}(A_1 + A_2)$ and $\text{avg}(B) := \frac{1}{2}(B_1 + B_2)$.

The results of the Monte Carlo study, plotted in Figures 5 and 6, reveal that on average, dynamic LQ games with greater heterogeneity between agents' dynamics matrices empirically exhibit larger FBNE-to-OLNE gaps. This result appears to support the hypothesis that heterogeneity among agents' dynamics may widen the gap between the agents' equilibrium behaviors exhibited under open-loop and feedback equilibrium patterns.

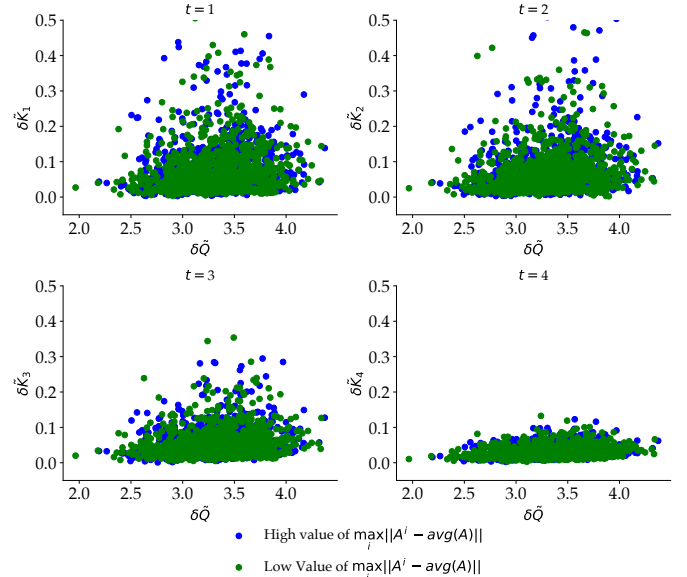


Fig. 5. Scatter plot of $\delta\tilde{K} := \|\tilde{K}_t - K_t\|_2$ vs. $\delta\tilde{Q} := \|\tilde{Q} - Q\|_2$, at each $t \in [4]$, for each of the 2000 sampled LQ dynamic games. Each sample has randomly sampled values of A^1, A^2, Q^1 and Q^2 . 1000 of these samples have high values of $\max_i \|A_i - \text{avg}(A)\|$ while the rest have low values of $\max_i \|A_i - \text{avg}(A)\|$, where $\text{avg}(A) := \frac{1}{2}(A_1 + A_2)$.

B. Dense Sampling Around Game $\mathcal{G}_4$ from Figure 2

As Figure 2 illustrates, Game $\mathcal{G}_4$ corresponds to the case when even for small values of δ , we observe high values of $\delta\tilde{K}$ which results in a large FBNE-to-OLNE gap, as quantified by $\delta\tilde{K}$. A natural question is to ask whether Game $\mathcal{G}_4$ is merely an aberration, or whether we can in general expect large FBNE-to-OLNE gaps under small values of $\delta\tilde{Q}$. To

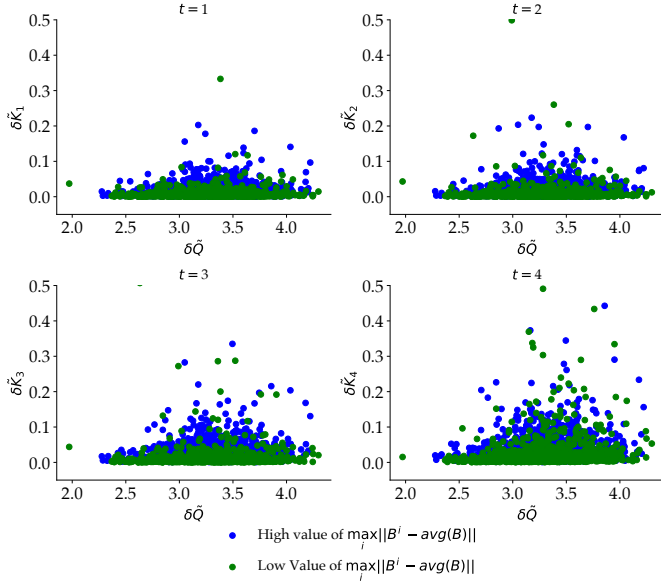


Fig. 6. Scatter plot of $\delta \tilde{K} := \|\tilde{K}_t - K_t\|_2$ vs. $\delta \tilde{Q} := \|\tilde{Q} - Q\|_2$, at each $t \in [4]$, for each of the 2000 sampled LQ dynamic games. Each sample has randomly sampled values of B^1, B^2, Q^1 and Q^2 . 1000 of these samples have high values of $\max_i \|B_i - \text{avg}(B)\|$ while the rest have low values of $\max_i \|B_i - \text{avg}(B)\|$, where $\text{avg}(B) := \frac{1}{2}(B_1 + B_2)$.

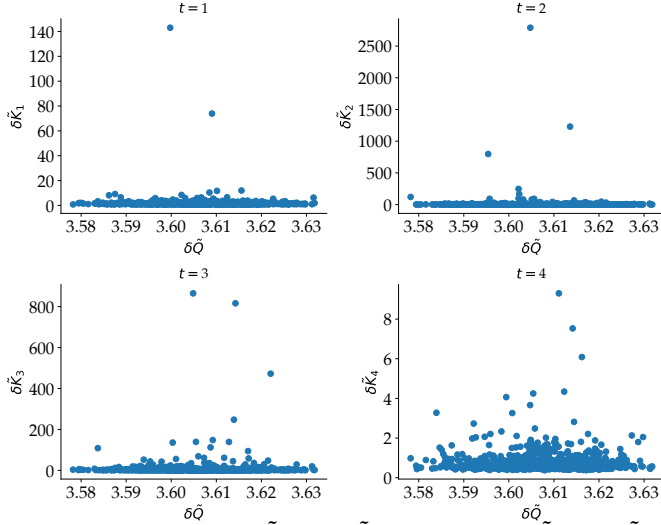


Fig. 7. Scatter plot of $\delta \tilde{K} := \|\tilde{K}_t - K_t\|_2$ vs. $\delta \tilde{Q} := \|\tilde{Q} - Q\|_2$, at each $t \in [4]$, for each of 1000 densely sampled LQ dynamic games around Game $\mathcal{G}_4$ from Figure 2. We note that we observe many additional games that, like $\mathcal{G}_4$, have both small values of $\delta \tilde{K}$ and large values of $\delta \tilde{Q}$.

answer this question, we perform a Monte Carlo study by densely sampling state cost matrices in the vicinity of the Q^1 and Q^2 matrices of Game $\mathcal{G}_4$, and plot our findings in Figure 7. We observe that Game $\mathcal{G}_4$ is not simply an exception, and in fact there exist multiple games for which $\delta \tilde{Q}$ is small but $\delta \tilde{K}$ is large. Furthermore, we discover that there exist many games whose $\delta \tilde{K}$ values are significantly higher than those of Game $\mathcal{G}_4$.