Journal of Functional Analysis 286 (2024) 110418

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Nonlinear random perturbations of PDEs and quasi-linear equations in Hilbert spaces depending on a small parameter

Sandra Cerrai [a,*,1], Giuseppina Guatteri [b,2], Gianmario Tessitore [c,2]

[a] *Department of Mathematics, University of Maryland, 4176 Campus Drive, College Park, MD 20742, United States*
[b] *Dipartimento di Matematica F. Brioschi, Politecnico di Milano, via Bonardi 9, 20133 Milano, Italy*
[c] *Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Via Roberto Cozzi 55, 20125 Milano, Italy*

ARTICLE INFO

Article history:
Received 28 August 2022
Accepted 7 March 2024
Available online 26 March 2024
Communicated by Pietro Caputo

MSC:
60F10
60H15
35K55

Keywords:
PDEs in infinite dimension

ABSTRACT

We study a class of quasi-linear parabolic equations defined on a separable Hilbert space, depending on a small parameter in front of the second order term. Through the nonlinear semigroup associated with such equation, we introduce the corresponding SPDE and we study the asymptotic behavior of its solutions, depending on the small parameter. We show that a large deviations principle holds and we give an explicit description of the action functional.

* Corresponding author.
E-mail addresses: cerrai@umd.edu (S. Cerrai), giuseppina.guatteri@polimi.it (G. Guatteri), gianmario.tessitore@unimib.it (G. Tessitore).

[1] Partially supported by the NSF Grant DMS-1954299 (2020-2024), *Multiscale Analysis of Infinite-Dimensional Stochastic Systems*.
[2] Partially supported by GNAMPA Grant CUP-E53C22001930001, *Riduzione di modelli in sistemi stocastici in dimensione infinita a due scale temporali*.

https://doi.org/10.1016/j.jfa.2024.110418
0022-1236/

Large Deviations
Stochastic PDEs

1. Introduction

Consider the partial differential equation

$$\frac{dX^x}{dt}(t) = AX^x(t) + b(X^x(t)), \qquad X^x(0) = x \in H, \tag{1.1}$$

defined on a separable Hilbert space H, endowed with the scalar product $\langle \cdot, \cdot \rangle_H$ and the corresponding norm $\| \cdot \|_H$. Here $A : D(A) \subset H \to H$ is the generator of a strongly continuous semigroup and $b : D(b) \subseteq H \to H$ is some non-linear mapping. Next, consider the following stochastic perturbation of (1.1)

$$dX^x_\epsilon(t) = [AX^x_\epsilon(t) + b(X^x_\epsilon(t))]\, dt + \sqrt{\epsilon}\, \sigma(X^x_\epsilon(t))\, dW_t, \qquad X^x_\epsilon(0) = x \in H, \tag{1.2}$$

where $\epsilon > 0$ is a small parameter, W_t, $t \geq 0$, is a cylindrical Wiener process and σ is a mapping, defined on H and taking values in some space of bounded linear operators defined on the reproducing kernel of the noise into H. We assume that the differential operator A, the coefficients b and σ and the noise W_t are such that both (1.1) and (1.2) are well-posed.

If the parameter ϵ is small, the trajectories of the perturbed system (1.2) remain close to those of the unperturbed system (1.1) on any bounded time interval. In particular, if there exist a domain $G \subset H$ and a point $x_0 \in G$ such that any trajectory of (1.1) starting in G remains in G and converges to x_0, as time goes to infinity, then with overwhelming probability the trajectories of (1.2) starting from any $x \in G$ enter any neighborhood of x_0, before eventually leaving the domain G because of the effect of the noise. As know this is a consequence of the large deviations of $X_\epsilon(t)$ from $X(t)$ which are described by the *action functional*

$$I^x_T(f) = \frac{1}{2} \inf \left\{ \int_0^T \|\varphi(t)\|^2_H\, dt \,:\, f = X^{x,\varphi} \right\},$$

where we have denoted by $X^{x,\varphi}$ the solution of the controlled version of (1.1)

$$\frac{dX^{x,\varphi}}{dt}(t) = AX^{x,\varphi}(t) + b(X^{x,\varphi}(t)) + \sigma(X^{x,\varphi}(t))\varphi(t), \qquad X^{x,\varphi}(0) = x,$$

and by the *quasi-potential*

$$V(x_0, x) = \inf \{ I_T(f) \,:\, f \in C([0,T]; H),\ f(0) = x_0,\ f(T) = x,\ T > 0 \}.$$

It is known that the stochastic PDE (1.2) is related to the linear Kolmogorov equation on the Hilbert space H

$$\begin{cases} D_t u_\epsilon(t,x) = \dfrac{\epsilon}{2}\mathrm{Tr}\left[\sigma\sigma^\star(x) D_x^2 u_\epsilon(t,x)\right] + \langle Ax + b(x), Du_\epsilon(t,x)\rangle_H, \quad x \in H, \quad t > 0, \\ u_\epsilon(0,x) = g(x), \quad x \in H. \end{cases} \tag{1.3}$$

Actually, under suitable conditions on the operator A, the coefficients b and σ and the initial condition g, equation (1.3) admits a unique classical solution u_ϵ, which can be written in terms of the linear transition semigroup P_t^ϵ associated with (1.2). Namely

$$u_\epsilon(t,x) = P_t^\epsilon g(x) = \mathbb{E} g(X_\epsilon(t,x)), \qquad t \geq 0, \quad x \in H.$$

In particular, the description of the small noise asymptotics of the solutions of equation (1.2) provided by the theory of large deviations allows to give a detailed description of the long-time behavior of the solutions of infinite dimensional PDE (1.3).

In [10], Freidlin and Koralov have considered more general stochastic perturbations of the dynamical system (1.1), when $H = \mathbb{R}^d$, $A = 0$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz-continuous mapping. They have introduced the following quasi-linear parabolic problem

$$\begin{cases} \partial_t u_\epsilon(t,x) = \dfrac{\epsilon}{2}\displaystyle\sum_{i,j=1}^d a_{i,j}(x, u_\epsilon(t,x))\, \partial_{ij} u_\epsilon(t,x) + \sum_{i=1}^d b_i(x)\, \partial_i u_\epsilon(t,x), \quad x \in \mathbb{R}^d, \quad t > 0, \\ u_\epsilon(0,x) = g(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{1.4}$$

where $a_{ij}(x,r) = (\sigma\sigma^\star)_{ij}(x,r)$, and by invoking the classical theory of quasi-linear PDEs, they have shown that, under reasonable assumptions on the coefficients f and σ, equation (1.4) admits a unique classical solution u_ϵ. Next, for every $t > 0$ and $x \in \mathbb{R}^d$, they have introduced the following randomly perturbed system

$$\begin{cases} dX_\epsilon^{t,x}(s) = b(X_\epsilon^{t,x}(s))\, ds + \sqrt{\epsilon}\, \sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\, dB_s, \\ X_\epsilon^{t,x}(0) = x, \end{cases} \tag{1.5}$$

where B_t, $t \geq 0$, is a d-dimensional Brownian motion. As in the linear case, the PDE (1.4) and the SDE (1.5) are related by the following relation

$$u_\epsilon(t,x) = \mathbb{E} g(X_\epsilon^{t,x}(t)) =: T_t^\epsilon g(x), \tag{1.6}$$

but now T_t^ϵ is a non-linear semigroup. This is in fact the reason why equation (1.5) can be seen as a *non-linear* perturbation of the deterministic system.

The study of the large deviation principle and of the quasi-potential for (1.5), has allowed Freidlin and Koralov to study the long-time behavior of the solutions to equation

(1.4), restricted to the domain G (that now is a bounded domain in $\mathbb{R}^d$) and endowed with the boundary condition $u_\epsilon(t,x) = g(x)$, for every $x \in \partial G$. In this case

$$u_\epsilon(t,x) = \mathbb{E} g(X_\epsilon^{t,x}(t \wedge \tau_\epsilon^x)),$$

where τ_ϵ^x is the first exit time of $X_\epsilon^{t,x}$ from the domain G. In particular, the asymptotic description of τ_ϵ^x in terms of the quasi-potential has made possible to study the asymptotic behavior of u_ϵ on exponential time scales $t(\epsilon) \sim \exp(\lambda/\epsilon)$. Freidlin and Koralov's idea is to introduce a family of linear equations obtained from (1.4) by freezing the second variable in $\sigma\sigma^\star$ and putting it equal to a constant c. This allows them to describe the asymptotics of $u_\epsilon(\exp(\lambda/\epsilon), x)$, for different values of $\lambda \in (0,\infty)$, in terms of some function $c(\lambda)$ obtained from $V_G(c)$, the minimum of the quasi-potential in G for the linear problem corresponding to c, and from $g(x^\star(c))$, where $x^\star(c)$ is the point of ∂G where the quasi-potential attains its minimum, for different values of c.

The present paper represents the beginning of a longer term project where we aim to develop an analogous theory for infinite dimensional dynamical systems described by PDEs. As in the finite dimensional case studied in [10], also here, as a first and fundamental step, we need to be able to study the well-posedness of the following quasi-linear equations

$$\begin{cases} D_t u_\epsilon(t,x) = \dfrac{\epsilon}{2} \mathrm{Tr}\left[\sigma\sigma^\star(x, u_\epsilon(t,x)) D_x^2 u_\epsilon(t,x)\right] + \langle Ax + b(x), Du_\epsilon(t,x)\rangle_H, \\ \quad x \in H, \quad t > 0, \\ u_\epsilon(0,x) = g(x), \quad x \in H. \end{cases} \tag{1.7}$$

However, unlike in finite dimension, where a well-established theory of deterministic quasi-linear PDEs is available, it seems that the current literature does not provide any Hilbert space counterpart to such classical theory, and everything has to be done.

In our analysis we will proceed in several steps and here we are considering the case when $\sigma : H \times \mathbb{R} \to \mathcal{L}(H)$ is Lipschitz continuous and there exist a bounded and non-negative symmetric operator Q, a continuous mapping f defined on $H \times \mathbb{R}$ with values in the space of trace-class operators and a constant $\delta > 0$ such that

$$\sigma^\star\sigma(x,r) = Q + \delta\, f(x,r), \qquad x \in H, \quad r \in \mathbb{R}.$$

This allows to rewrite equation (1.7) as

$$\begin{cases} D_t u_\epsilon(t,x) = \mathcal{L}_\epsilon u_\epsilon(t,x) + \dfrac{\epsilon}{2} \mathrm{Tr}\left[\delta\, f(x,u_\epsilon)(t,x) D_x^2 u_\epsilon(t,x)\right] + \langle b(x), Du_\epsilon(t,x)\rangle_H, \\ u_\epsilon(0,x) = g(x), \quad x \in H, \end{cases}$$

where

$$\mathcal{L}_\epsilon\varphi(x) = \frac{\epsilon}{2}\mathrm{Tr}\left[QD_x^2\varphi(x)\right] + \langle Ax, D_x\varphi(x)\rangle_H.$$

In particular, if we denote by R_t^ϵ the Ornstein-Uhlenbeck semigroup associated with the operator $\mathcal{L}_\epsilon$, we can rewrite equation (1.7) in mild form as

$$u_\epsilon(t,x) = R_t^\epsilon g(x) + \int_0^t R_{t-s}^\epsilon \left(\frac{\epsilon}{2}\mathrm{Tr}\left[\delta F(u_\epsilon(s,\cdot))D_x^2 u_\epsilon(s,\cdot)\right] + \langle b(\cdot), Du_\epsilon(s,\cdot)\rangle_H\right)(x)\,ds. \tag{1.8}$$

We can then introduce the stochastic PDE

$$\begin{cases} dX_\epsilon^{t,x}(s) = \left[AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))\right]ds + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), u_\epsilon(t-s, X_\epsilon^{t,x}(s)))\,dW_s, \\ X_\epsilon^{t,x}(0) = x, \end{cases} \tag{1.9}$$

where W_t is a cylindrical Wiener process in H, defined on some stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Due to the regularity of the coefficients and of the function u_ϵ, we can show that there exists $\bar\delta > 0$ such that, for every $\delta \leq \bar\delta$ and for every $t > 0$ and $x \in H$, equation (1.9) admits a unique mild solution in $L^2(\Omega; C([0,t]; H))$. Moreover, we show that, as in the finite dimensional case, the quasi-linear equation (1.7) and the stochastic PDE are related through formula (1.6) and, in particular, a maximum principle holds for equation (1.7).

It is worth noticing that as a consequence of the Markov property, the following relation holds

$$u_\epsilon(t-s, X_\epsilon^{t,x}(s))) = \mathbb{E}(g(X_\epsilon^{t-s,y}(t-s)))\big|_{y=X_\epsilon^{t,x}(s)} = \mathbb{E}(g(X_\epsilon^{t,x}(t))|\mathcal{F}_s),$$

for every $s \in [0,t]$ and $x \in H$, so that equation (1.5) reads as

$$\begin{cases} dX_\epsilon^{t,x}(s) = \left[AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))\right]ds + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), \mathbb{E}(g(X_\epsilon^{t,x}(t))|\mathcal{F}_s))\,dW_s, \\ X_\epsilon^{t,x}(0) = x. \end{cases} \tag{1.10}$$

Setting $Y^{t,x}(s) := \mathbb{E}(g(X_\epsilon^{t,x}(t))|\mathcal{F}_s))$, the equation above can be further rewritten as a coupled forward backward infinite dimensional stochastic system

$$\begin{cases} dX_\epsilon^{t,x}(s) = \left[AX_\epsilon^{t,x}(s) + b(X_\epsilon^{t,x}(s))\right]ds + \sqrt{\epsilon}\,\sigma(X_\epsilon^{t,x}(s), Y^{t,x}(s)))\,dW_s, \quad 0 \leq s \leq t \\ -d_sY^{t,x}(s) = -Z^{t,x}(s)dW_s, \quad 0 \leq s \leq t \\ Y^{t,x}(t) = g(X_\epsilon^{t,x}(t)) \\ X_\epsilon^{t,x}(0) = x. \end{cases} \tag{1.11}$$

Coupled forward-backward systems of stochastic equations of the general form

$$\begin{cases} dX(s) = b(X(s), Y(s), Z(s))ds + \sigma(X(s), Y(s))\, dW_s, \qquad 0 \le s \le t \\ -d_s Y(s) = \psi((X(s), Y(s), Z(s))\, ds - Z(s)dW_s, \qquad 0 \le s \le t \\ Y(t) = g(X(t)) \\ X_\epsilon(0) = x, \end{cases} \tag{1.12}$$

have been extensively studied in the finite dimensional case, see [17] where several results are collected. Since [1], it has been clear that arbitrary forward-backward stochastic systems do not always admit a solution. Different techniques have been developed to prove existence and uniqueness both locally in time and in arbitrarily long time intervals. In particular the classical theory of PDEs, applied to the corresponding nonlinear Kolmogorov equations, offers a wide range of results stating well posedness of system (1.12) (see, for instance [17] [13] or [14]) that include existence and uniqueness of a global solution to the finite dimensional analogue of system (1.11) when σ is not degenerate. In the infinite dimensional case, in which large part of the analytic techniques are not available any more, very few results on existence and uniqueness of a solution to system (1.11) in arbitrary time interval are at hand (for local existence and uniqueness see [12]). It seems that the techniques more likely to be extended in infinite dimensions are the ones introduced in [18] where quantitative conditions on dissipativity of b and bounds on the Lipschitz norm of σ and g are required. Such restrictions go in the same direction as the condition on δ that we have to impose here, see above. We finally notice that, if we show that system (1.11) is well posed, then we can define a candidate solution to the PDE (1.7) by setting

$$\hat{u}_\epsilon(t-s, \xi) = \mathbb{E}(Y^{t,x}(s) | X^{t,x}(s) = \xi)$$

but, unless we have a satisfactory analytic theory for equation (1.11), the proof that $\hat{u}_\epsilon$ is the unique solution of (1.7) (in which formulation?) is still to be done and does not seem obvious at all. Once such relation would be understood, it could also be possible to study the large deviations principle, see below, for more general nonlinear perturbations of (1.1) defined through systems like (1.12) (see [4] for a similar approach in the finite dimensional case where the connection between $\hat{u}_\epsilon$ and equation (1.11) is a straightforward consequence of existence and uniqueness of a regular solution to (1.11) and Ito rule).

As we mentioned at the beginning of this introduction, we are interested in applying our results to the study of the asymptotic behavior of (1.9) and (1.7), as $\epsilon \downarrow 0$. This is a multi-step project and here we are addressing the problem of the validity of a large deviation principle for the trajectories of the solutions of equation (1.9). Thus, in the last section of our paper we prove that the family of laws $\{\mathcal{L}(X_\epsilon^{t,x})\}_{\epsilon \in (0,1)}$ satisfies a large deviation principle in the space $C([0,t]; H)$, which is governed by the action functional

$$I_{t,x}(X) = \frac{1}{2} \inf \left\{ \int_0^t \|\varphi(s)\|_H^2 \, ds \, : \, X(s) = X_\varphi^{t,x}(s), \ s \in [0,t] \right\},$$

where $X_\varphi^{t,x}$ is the unique mild solution of problem

$$X'(s) = AX(s) + b(X(s)) + \sigma(X(s), g(Z^{X(s)}(t-s)))\varphi(s), \qquad X(0) = x,$$

and for every $y \in H$

$$Z^y(s) = e^{sA}y + \int_0^s e^{(s-r)A} b(Z^y(r)) \, dr.$$

2. Notations and preliminaries

Throughout this paper, H is a separable Hilbert space, endowed with the scalar product $\langle \cdot, \cdot \rangle_H$ and the corresponding norm $\| \cdot \|_H$. In what follows we shall introduce some notations and preliminary results (we refer to [2], [6] and [15] for all details).

2.1. Operator spaces

We denote by $\mathcal{L}(H)$ the Banach space of all bounded linear operators $A : H \to H$, endowed with the sup-norm

$$\|A\|_{\mathcal{L}(H)} = \sup_{\|x\|_H \leq 1} \|Ax\|_H.$$

An operator $A \in \mathcal{L}(H)$ is symmetric if it coincides with its adjoint $A^\star$, that is if $\langle Ax, y \rangle_H = \langle x, Ay \rangle_H$, for all $x, y \in H$. Moreover, it is non-negative if $\langle Ax, x \rangle_H \geq 0$, for all $x \in H$. We shall denote by $\mathcal{L}^+(H)$ the subspace of all non-negative and symmetric operators in $\mathcal{L}(H)$.

An operator $A \in \mathcal{L}(H)$ is called an *Hilbert-Schmidt* operator if there exists an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of H such that

$$\sum_{i=1}^\infty \|Ae_i\|_H^2 < \infty.$$

The subspace of Hilbert-Schmidt operators, denoted by $\mathcal{L}_2(H)$, is a Hilbert space, endowed with the scalar product

$$\langle A, B \rangle_{\mathcal{L}_2(H)} = \sum_{i=1}^\infty \langle Ae_i, Be_i \rangle_H.$$

As know n, for every $B \in \mathcal{L}^+(H)$ there exists a unique $C \in \mathcal{L}(H)$, denoted by $\sqrt{B}$ such that $C^2 = B$. Thus, for any $A \in \mathcal{L}(H)$ we can define

$$|A| := \sqrt{A^\star A}.$$

We recall that an operator $A \in \mathcal{L}(H)$ is compact if and only if $|A|$ is compact. Moreover, if A is a symmetric compact operator, then there exists an orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H and a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ converging to zero such that $Ae_i = \alpha_i e_i$, for all $i \in \mathbb{N}$. With these notations, we say that a compact operator $A \in \mathcal{L}(H)$ is *nuclear* or *trace-class* if there exists an orthonormal basis of H consisting of eigenvectors of $|A|$ corresponding to the eigenvalues $\{\alpha_i\}_{i\in\mathbb{N}}$, such that

$$\sum_{i=1}^{\infty} \alpha_i < \infty.$$

In particular, if the operator A is symmetric, it is nuclear if and only if there exists an orthonormal basis of H consisting of eigenvectors of A corresponding to the eigenvalues $\{\alpha_i\}_{i\in\mathbb{N}}$, such that

$$\sum_{i=1}^{\infty} |\alpha_i| < \infty.$$

We denote by $\mathcal{L}_1(H)$ the set of nuclear operators.

It is possible to prove that for every $A \in \mathcal{L}_1(H)$ the series

$$\mathrm{Tr}A := \sum_{i=1}^{\infty} \langle Ae_i, e_i\rangle_H$$

does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. Moreover, a symmetric operator A belongs to $\mathcal{L}_1(H)$ if and only if the series above converges absolutely for every orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. The space $\mathcal{L}_1(H)$ is a Banach space, endowed with the norm

$$\|A\|_{\mathcal{L}_1(H)} = \mathrm{Tr}\,|A|,$$

and

$$|\mathrm{Tr}A| \le \|A\|_{\mathcal{L}_1(H)}. \tag{2.1}$$

It is possible to prove that $\mathcal{L}_1(H) \subset \mathcal{L}_2(H) \subset \mathcal{L}(H)$ with

$$\|A\|_{\mathcal{L}(H)} \le \|A\|_{\mathcal{L}_2(H)} \le \|A\|_{\mathcal{L}_1(H)},$$

and for $j = 1, 2$ it holds

$$\|AB\|_{\mathcal{L}_j(H)} \leq \|A\|_{\mathcal{L}_j(H)}\|B\|_{\mathcal{L}(H)}, \qquad \|AB\|_{\mathcal{L}_j(H)} \leq \|B\|_{\mathcal{L}_j(H)}\|A\|_{\mathcal{L}(H)}.$$

Moreover, if $A, B \in \mathcal{L}_2(H)$, then $AB \in \mathcal{L}_1(H)$, with

$$\|AB\|_{\mathcal{L}_1(H)} \leq \|A\|_{\mathcal{L}_2(H)}\|B\|_{\mathcal{L}_2(H)}.$$

Finally if E and K are arbitrary Banach spaces we denote by $\mathcal{L}^l(K; E)$ the space of l-linear bounded operators $K^l \to E$. When $l = 1$ and $E = K$ we just denote $\mathcal{L}^l(K; E)$ by $\mathcal{L}(K)$. Finally when K is an Hilbert space and $E = \mathbb{R}$ we identify $\mathcal{L}^1(K; \mathbb{R})$ with K and $\mathcal{L}^2(K; \mathbb{R})$ with $\mathcal{L}(K)$.

2.2. Functional spaces

If E is an arbitrary Banach space, endowed with the norm $\|\cdot\|_E$, we denote by $B_b(H; E)$ the space of Borel and bounded functions $\varphi : H \to E$. $B_b(H; E)$ is a Banach space, endowed with the sup-norm

$$\|\varphi\|_0 = \sup_{x \in H} \|\varphi(x)\|_E.$$

Moreover, we denote by $C_b(H; E)$ the closed subspace of uniformly continuous and bounded functions.

For every integer $n \geq 1$, we denote by $C_b^n(H; E)$ the space of all functions $\varphi \in C_b(H; E)$ which are n-times Fréchet differentiable, with uniformly continuous and bounded Fréchet derivatives $D^l\varphi : H \to \mathcal{L}^l(H; E)$ for all $l \leq n$. We have that $C_b^n(H; E)$ is a Banach space, endowed with the norm

$$\|\varphi\|_n = \|\varphi\|_0 + \sum_{l=1}^{n} \|D^l\varphi\|_0.$$

Next, for every $\vartheta \in (0, 1)$ we denote by $C_b^\vartheta(H; E)$ the space of all functions $\varphi \in C_b(H; E)$ such that

$$[\varphi]_\vartheta := \sup_{\substack{x,y \in H \\ x \neq y}} \frac{\|\varphi(x) - \varphi(y)\|_E}{\|x - y\|_H^\vartheta} < \infty.$$

$C_b^\vartheta(H; E)$ is a Banach space, endowed with the norm

$$\|\varphi\|_\vartheta = \|\varphi\|_0 + [\varphi]_\vartheta.$$

Finally, for every integer $n \in \mathbb{N}$ and $\vartheta \in (0, 1)$, we denote by $C_b^{n+\vartheta}(H; E)$ the space of all functions $\varphi \in C_b^n(H; E)$ such that

$$[D^n\varphi]_\vartheta := \sup_{\substack{x,y\in H\\ x\neq y}} \frac{\|D^n\varphi(x)-D^n\varphi(y)\|_{\mathcal{L}^n(H;E)}}{\|x-y\|_H^\vartheta} < \infty.$$

$C_b^{n+\vartheta}(H;E)$ is a Banach space, endowed with the norm

$$\|\varphi\|_{n+\vartheta} = \|\varphi\|_0 + \sum_{l=1}^n \|D^l\varphi\|_0 + [D^n\varphi]_\vartheta = \|\varphi\|_n + [D^n\varphi]_\vartheta.$$

Notice that in case $E=\mathbb{R}$, we simply write $B_b(H)$ instead of $B_b(H;E)$, and for every $\alpha\geq 0$ we write $C_b^\alpha(H)$ instead of $C_b^\alpha(H;\mathbb{R})$.

Now, we want to see how classical interpolatory estimates for functions defined on $\mathbb{R}^n$ are still valid for functions defined on the infinite dimensional Hilbert space H. To this purpose, we recall that, as shown in [6, Theorem 2.3.5], for every $0\leq\alpha<\beta<\gamma$ there exists a constant $c=c(\alpha,\beta,\gamma)>0$ such that for every $\varphi\in C_b^\gamma(H)$

$$\|\varphi\|_\beta \leq c\,\|\varphi\|_\alpha^{\frac{\gamma-\beta}{\gamma-\alpha}}\|\varphi\|_\gamma^{\frac{\beta-\alpha}{\gamma-\alpha}}. \tag{2.2}$$

However, in what follows we will need the following additional interpolatory estimates.

Lemma 2.1. *Let us fix $\vartheta\in(0,1)$. Then, for every $\varphi\in C_b^1(H)$ we have*

$$[\varphi]_\vartheta \leq c_{1,\vartheta}\,\|\varphi\|_0^{1-\vartheta}\,\|D\varphi\|_0^\vartheta, \tag{2.3}$$

and, for every $\rho\in(0,\vartheta)$

$$[\varphi]_\vartheta \leq [\varphi]_\rho^{\frac{1-\vartheta}{1-\rho}}\,\|D\varphi\|_0^{\frac{\vartheta-\rho}{1-\rho}}. \tag{2.4}$$

Moreover, for every $\varphi\in C_b^{2+\vartheta}(H)$ we have

$$\|D^2\varphi\|_0 \leq c_{2,\vartheta}\,\|D\varphi\|_0^{\frac{\vartheta}{1+\vartheta}}\,[D^2\varphi]_\vartheta^{\frac{1}{1+\vartheta}}, \tag{2.5}$$

and

$$\|D\varphi\|_0 \leq c_{3,\vartheta}\,\|\varphi\|_0^{\frac{1+\vartheta}{2+\vartheta}}\,[D^2\varphi]_\vartheta^{\frac{1}{2+\vartheta}}. \tag{2.6}$$

Proof. Let us fix $\varphi\in C_b^1(H)$ and $x,y\in H$. Then, for every $\vartheta\in(0,1)$ we have

$$|\varphi(x+y)-\varphi(x)| \leq 2\,\|\varphi\|_0^{1-\vartheta}\left|\int_0^1 \langle D\varphi(x+\lambda y),y\rangle_H\,d\lambda\right|^\vartheta \leq 2\,\|\varphi\|_0^{1-\vartheta}\,\|D\varphi\|_0^\vartheta\|y\|_H^\vartheta,$$

so that (2.3) follows. In an analogous way we deduce (2.4), since

$$|\varphi(x+y)-\varphi(x)| \le [\varphi]_\rho^{\frac{1-\vartheta}{1-\rho}} \left| \int_0^1 \langle D\varphi(x+\lambda y), y\rangle_H \, d\lambda \right|^{\frac{\vartheta-\rho}{1-\rho}} \|y\|_H^{\frac{\rho(1-\vartheta)}{1-\rho}}.$$

Now, if we fix $\varphi \in C_b^{2+\vartheta}(H)$, for every $\mu > 0$ and $x, z \in H$, with $\|z\|_H = 1$, we have

$$\begin{aligned} \varphi(x+\mu z) = \varphi(x) + \mu \langle D\varphi(x), z\rangle_H + \frac{\mu^2}{2}\langle D^2\varphi(x)z, z\rangle_H \\ + \mu^2 \int_0^1 (1-r)\langle [D^2\varphi(x+r\mu z) - D^2\varphi(x)]z, z\rangle_H \, dr. \end{aligned} \tag{2.7}$$

By proceeding as in [6, proof of Theorem 2.3.5], we use (2.7) to prove (2.5). Actually, thanks to (2.7) we have

$$\begin{aligned} \frac{\mu^2}{2}\left|\langle D^2\varphi(x)z, z\rangle_H\right| \le |\varphi(x+\mu z) - \varphi(x) - \mu\langle D\varphi(x), z\rangle_H| \\ + \mu^{2+\vartheta}[D^2\varphi]_\vartheta \int_0^1 (1-r)r^\vartheta \, dr, \end{aligned}$$

so that

$$\|D^2\varphi\|_0 \le \frac{2}{\mu}\|D\varphi\|_0 + c_\vartheta \, \mu^\vartheta \, [D^2\varphi]_\vartheta, \qquad \mu > 0.$$

If we take the minimum over $\mu > 0$, we get (2.5).

Finally, by using again (2.7), we have

$$\mu\,|\langle D\varphi(x), z\rangle_H| \le |\varphi(x+\mu z) - \varphi(x)| + \frac{\mu^2}{2}\left|\langle D^2\varphi(x)z, z\rangle_H\right| + \mu^{2+\vartheta}[D^2\varphi]_\vartheta \int_0^1 (1-r)r^\vartheta \, dr,$$

so that, in view of (2.5), we get

$$\begin{aligned} \|D\varphi\|_0 &\le \frac{2}{\mu}\|\varphi\|_0 + \frac{c_{2,\vartheta}\mu}{2}\|D\varphi\|_0^{\frac{\vartheta}{1+\vartheta}}\,[D^2\varphi]_\vartheta^{\frac{1}{1+\vartheta}} + c_\vartheta\mu^{1+\vartheta}[D^2\varphi]_\vartheta \\ &\le \frac{2}{\mu}\|\varphi\|_0 + \frac{1}{2}\|D\varphi\|_0 + c_\vartheta\mu^{1+\vartheta}[D^2\varphi]_\vartheta. \end{aligned}$$

This implies that

$$\|D\varphi\|_0 \le \frac{4}{\mu}\|\varphi\|_0 + c_\vartheta\mu^{1+\vartheta}[D^2\varphi]_\vartheta, \qquad \mu > 0,$$

and if we minimize once again with respect to $\mu > 0$ we obtain (2.6). □

Remark 2.2. As a consequence of (2.3), (2.5) and (2.6), we have that for every $\vartheta \in (0,1)$ there exists some $c_\vartheta > 0$ such that for every $\varphi \in C_b^{2+\vartheta}(H)$

$$[\varphi]_\vartheta \, \|D^2\varphi\|_0 \le c_\vartheta \, \|\varphi\|_0 \, [D^2\varphi]_\vartheta. \tag{2.8}$$

Indeed, from (2.5) and (2.6), we have

$$\|D^2\varphi\|_0 \le c_{2,\vartheta} \left(c_{3,\vartheta} \, \|\varphi\|_0^{\frac{1+\vartheta}{2+\vartheta}} \, [D^2\varphi]_\vartheta^{\frac{1}{2+\vartheta}} \right)^{\frac{\vartheta}{1+\vartheta}} \, [D^2\varphi]_\vartheta^{\frac{1}{1+\vartheta}} \le c_{2,\vartheta} \, c_{3,\vartheta}^{\frac{1}{1+\vartheta}} \, \|\varphi\|_0^{\frac{\vartheta}{2+\vartheta}} \, [D^2\varphi]_\vartheta^{\frac{2}{2+\vartheta}}. \tag{2.9}$$

Moreover, thanks to (2.3) and (2.6) we have

$$[\varphi]_\vartheta \le c_{1,\vartheta} \, \|\varphi\|_0^{1-\vartheta} \left(c_{3,\vartheta} \, \|\varphi\|_0^{\frac{1+\vartheta}{2+\vartheta}} \, [D^2\varphi]_\vartheta^{\frac{1}{2+\vartheta}} \right)^\vartheta = c_{1,\vartheta} \, c_{3,\vartheta}^\vartheta \, \|\varphi\|_0^{\frac{2}{2+\vartheta}} \, [D^2\varphi]_\vartheta^{\frac{\vartheta}{2+\vartheta}}.$$

Therefore, if we combine together this last inequality with (2.9), we obtain (2.8).

2.3. The Ornstein-Uhlenbeck semigroup

By following [6, Chapter 6], we recall here some results about the Ornstein-Uhlenbeck semigroup and the associated Kolmogorov equation.

Let $A : D(A) \subset H \to H$ be the generator of an asymptotically stable C_0-semigroup e^{tA}. We assume that there exist $M, \omega > 0$ such that

$$\|e^{tA}\|_{\mathcal{L}(H)} \le M e^{-\omega t}.$$

Moreover let Q be an operator in $\mathcal{L}^+(H)$. For every $t \ge 0$ we define

$$Q_t := \int_0^t e^{sA} Q e^{sA^\star} \, ds,$$

and we assume that $Q_t \in \mathcal{L}_1(H)$, for every $t \ge 0$. Thus, we can introduce the centered Gaussian measure $\mathcal{N}_{Q_t}$ defined on H with covariance Q_t, and we can define

$$R_t\varphi(x) := \int_H \varphi(e^{tA}x + y) \, \mathcal{N}_{Q_t}(dy), \qquad x \in H, \quad t \ge 0, \tag{2.10}$$

for every φ in $B_b(H)$. R_t is the *Ornstein-Uhlenbeck* semigroup associated with A and Q. In what follows, we assume that

$$e^{tA}(H) \subset Q_t^{1/2}(H), \qquad t > 0, \tag{2.11}$$

and we define

$$\Lambda_t := Q_t^{-1/2} e^{tA}, \qquad t > 0,$$

where $Q_t^{-1/2}$ is the left pseudo-inverse of $Q_t^{1/2}$.

As shown e.g. in [6, Theorem 6.2.2], as a consequence of assumption (2.11) we have that $R_t\varphi \in C_b^\infty(H)$, for every $\varphi \in B_b(H)$ and $t > 0$, and for every $n \in \mathbb{N} \cup \{0\}$ there exists some $c_n > 0$ such that

$$\|D^n R_t\varphi\|_0 \le c_n \|\Lambda_t\|^n_{\mathcal{L}(H)} \|\varphi\|_0$$

Moreover, if we fix $\alpha \in (0,1)$ and assume $\varphi \in C_b^\alpha(H)$ we have

$$[D^n R_t\varphi]_\alpha \le c_n \|\Lambda_t\|^n_{\mathcal{L}(H)} \|e^{tA}\|^\alpha_{\mathcal{L}(H)} [\varphi]_\alpha, \tag{2.12}$$

so that we conclude that for all $\alpha \in [0,1)$ and $\varphi \in C_b^\alpha(H)$

$$\|D^n R_t\varphi\|_\alpha \le c_n \|\Lambda_t\|^n_{\mathcal{L}(H)} \|\varphi\|_{t,\alpha}, \qquad t > 0, \tag{2.13}$$

where, for every $t \ge 0$ and $\alpha \in (0,1)$,

$$\|\varphi\|_{t,\alpha} := \left(\|\varphi\|_0 + e^{-\omega\alpha t} [\varphi]_\alpha\right). \tag{2.14}$$

For every $n \in \mathbb{N} \cup \{0\}$ and $0 \le \alpha \le \beta < 1$ and for every $\varphi \in C_b^\beta(H)$ and $t > 0$ we have, by the interpolation inequality (2.2) applied to the function $D^n R_t\phi$ with constants respectively β, $1+\alpha$, $1+\beta$:

$$\begin{aligned}\|D^n R_t\varphi\|_{1+\alpha} &\le \|D^n R_t\varphi\|^{\beta-\alpha}_\beta \|D^n R_t\varphi\|^{1-(\beta-\alpha)}_{1+\beta} \\ &= \|D^n R_t\varphi\|^{\beta-\alpha}_\beta \left(\|D^n R_t\varphi\|_0 + \|D^{n+1} R_t\varphi\|_\beta\right)^{1-(\beta-\alpha)}.\end{aligned}$$

Hence, thanks to (2.13), we get

$$\begin{aligned}&\|D^n R_t\varphi\|_{1+\alpha} \\ &\le c_n^{\beta-\alpha} \|\Lambda_t\|^{n(\beta-\alpha)}_{\mathcal{L}(H)} \|\varphi\|^{\beta-\alpha}_{t,\beta} \left(c_n \|\Lambda_t\|^n_{\mathcal{L}(H)} \|\varphi\|_0 + c_{n+1} \|\Lambda_t\|^{n+1}_{\mathcal{L}(H)} \|\varphi\|_{t,\beta}\right)^{1-(\beta-\alpha)} \\ &\le c_{\alpha,\beta,n} \|\varphi\|_{t,\beta} \left(\|\Lambda_t\|^n_{\mathcal{L}(H)} + \|\Lambda_t\|^{n+1-(\beta-\alpha)}_{\mathcal{L}(H)}\right).\end{aligned}$$

In particular, recalling that $\|D\psi\|_\alpha \le \|\psi\|_{\alpha+1}$, $\psi \in C^{1+\alpha}$, and $D^n R_t\phi = DD^{n-1} R_t\phi$ this allows to conclude that for every $n \in \mathbb{N}$ and $0 \le \alpha \le \beta < 1$ and for every $\varphi \in C_b^\beta(H)$

$$\|D^n R_t\varphi\|_\alpha \le c_{\alpha,\beta,n} \left(\|\Lambda_t\|^{n-1}_{\mathcal{L}(H)} + \|\Lambda_t\|^{n-(\beta-\alpha)}_{\mathcal{L}(H)}\right) \|\varphi\|_{t,\beta}. \tag{2.15}$$

Next, we recall that in [6, Proposition 6.2.9] it is shown that for every $\varphi \in C_b^1(H)$ and $x \in H$

$$\mathrm{Tr}\,[QD^2R_t\varphi(x)] = \int_H \langle Q_t^{-1/2}y, \Lambda_t Qe^{tA^\star}D\varphi(e^{tA}x+y)\rangle_H \,\mathcal{N}_{Q_t}(dy),$$

so that, if we assume that

$$\Lambda_t Qe^{tA^\star} \in \mathcal{L}_2(H), \tag{2.16}$$

we have

$$\sup_{x\in H}\|QD^2R_t\varphi(x)\|_{\mathcal{L}_1(H)} \le \|\Lambda_t Qe^{tA^\star}\|_{\mathcal{L}_2(H)}\,\|D\varphi\|_0, \qquad t>0.$$

Therefore, since $QD^2R_t\phi = QD^2R_{t/2}(R_{t/2}(\phi))$, (2.15) allows to conclude that for every $\varphi\in C_b^\beta(H)$

$$\sup_{x\in H}\|QD^2R_t\varphi(x)\|_{\mathcal{L}_1(H)} \le c\,\|\Lambda_{t/2}Qe^{tA^\star/2}\|_{\mathcal{L}_2(H)}\Big(1+\|\Lambda_{t/2}\|^{1-\beta}_{\mathcal{L}(H)}\Big)\|\varphi\|_\beta, \qquad t>0. \tag{2.17}$$

Moreover, we recall that in [6, Proposition 6.2.5] it is shown that if the operator $\Lambda_t A$ has a continuous extension $\overline{\Lambda_t A}$ to H, for every $t>0$, then for every $\varphi\in B_b(H)$ and $x\in H$

$$DR_t\varphi(x)\in D(A^\star), \qquad \|A^\star DR_t\varphi\|_0 \le \|\overline{\Lambda_t A}\|_{\mathcal{L}(H)}\,\|\varphi\|_0, \qquad t>0. \tag{2.18}$$

Now, we introduce the parabolic equation in H

$$D_tu(t,x) = \frac{1}{2}\mathrm{Tr}\left[QD_x^2u(t,x)\right] + \langle x, A^\star D_xu(t,x)\rangle_H, \qquad u(0,x)=\varphi(x). \tag{2.19}$$

Definition 1. A function $u:[0,+\infty)\times H\to\mathbb{R}$ is a *classical solution* of problem (2.19) if

1. u is continuous in $[0,+\infty)\times H$ and $u(0,\cdot)=\varphi$.
2. $u(t,\cdot)\in C_b^2(H)$, for all $t>0$, and $QD_x^2u(t,x)\in\mathcal{L}_1(H)$, for all $t>0$ and $x\in H$.
3. $D_xu(t,x)\in D(A^\star)$, for all $t>0$ and $x\in H$.
4. $u(\cdot,x)$ is differentiable in $(0,+\infty)$ for every $x\in H$ and u satisfies equation (2.19).

In [6, Theorem 6.2.4] it is shown that if we assume conditions (2.11) and (2.16) and we assume that the operator $\Lambda_t A$ has a continuous extension to H, then for every $\varphi\in B_b(H)$ the function

$$u(t,x) = R_t\varphi(x)$$

is the unique classical solution of equation (2.19).

3. Assumptions and main results

3.1. Assumptions

In what follows, we shall make the following hypotheses.

Hypothesis 1.

(1) The mapping $\sigma : H \times \mathbb{R} \to \mathcal{L}(H)$ is Lipschitz continuous and there exist an operator $Q \in \mathcal{L}^+(H)$, a continuous mapping $f : H \times \mathbb{R} \to \mathcal{L}_1(H)$ and a constant $\delta > 0$ such that

$$\sigma^\star\sigma(x,r) = Q + \delta\, f(x,r), \qquad x \in H, \quad r \in \mathbb{R}. \tag{3.1}$$

(2) For every fixed $x \in H$, the function $f(x,\cdot) : \mathbb{R} \to \mathcal{L}_1(H)$ is differentiable. Both f and $\partial_r f$ are Lipschitz continuous in both variables, uniformly with respect to the other. Moreover

$$\sup_{x\in H} \|f(x,r)\|_{\mathcal{L}_1(H)} \le c\,(1+|r|)\,, \qquad r \in \mathbb{R}. \tag{3.2}$$

Remark 3.1. (1) Let $H = L^2(\mathcal{O})$, for some smooth and bounded domain $\mathcal{O} \subset \mathbb{R}^d$, with $d \ge 1$. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of H and let $\{\lambda_i\}_{i\in\mathbb{N}}$ be a sequence of non-negative real numbers. We assume that $e_i \in L^\infty(\mathcal{O})$, for every $i \in \mathbb{N}$, and

$$\sum_{i=1}^{\infty} \lambda_i \|e_i\|_{L^\infty(\mathcal{O})} < \infty. \tag{3.3}$$

For every $x, y \in H$ and $r \in \mathbb{R}$, we define

$$[f(x,r)y](\xi) = \sum_{i=1}^{\infty} \mathfrak{f}_i(x(\xi),r)\lambda_i \langle y, e_i\rangle_H e_i(\xi), \qquad \xi \in \mathcal{O},$$

for some continuous functions $\mathfrak{f}_i : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $\mathfrak{f}_i(s,\cdot) : \mathbb{R} \to \mathbb{R}$ is differentiable, for every $s \in \mathbb{R}$ and $i \in \mathbb{N}$. We assume that both $\mathfrak{f}_i$ and $\partial_r \mathfrak{f}_i$ are Lipschitz continuous in both variables, uniformly with respect to the other variable, and uniformly with respect to $i \in \mathbb{N}$. Moreover, we assume that

$$\sup_{i\in\mathbb{N}} \sup_{s\in\mathbb{R}} |\mathfrak{f}_i(s,r)| \le c\,(1+|r|)\,, \qquad r \in \mathbb{R}. \tag{3.4}$$

With this choice of H and f, we have that condition 2 in Hypothesis 1 holds.

Indeed, since $\mathfrak{f}_i(\cdot,r) : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, uniformly with respect to $r \in \mathbb{R}$ and $i \in \mathbb{N}$, for every $x, y \in H$ and $r \in \mathbb{R}$ we have

$$\begin{aligned}
&\|f(x,r)-f(y,r)\|_{\mathcal{L}_1(H)}\\
&\quad\le\sum_{i=1}^{\infty}|\langle[f(x,r)-f(y,r)]\,e_i,e_i\rangle_H|\le\sum_{i=1}^{\infty}\lambda_i\,|\langle[\mathfrak{f}_i(x(\cdot),r)-\mathfrak{f}_i(y(\cdot),r)]\,e_i,e_i\rangle_H|\\
&\quad\le\sum_{i=1}^{\infty}\|\mathfrak{f}_i(x(\cdot),r)-\mathfrak{f}_i(y(\cdot),r)\|_H\|e_i\|_{L^\infty(\mathcal{O})}\lambda_i\le c\,\|x-y\|_H\ \sum_{i=1}^{\infty}\lambda_i\|e_i\|_{L^\infty(\mathcal{O})}.
\end{aligned}$$

In particular, thanks to (3.3), we can conclude that $f(\cdot,r):H\to\mathcal{L}_1(H)$ is Lipschitz continuous, uniformly with respect to $r\in\mathbb{R}$. Moreover, thanks to (3.4), the same argument also yields (3.2). In view of our assumptions, the same is true for $\partial_r f$.

The Lipschitz continuity of $f(x,\cdot)$ and $\partial_r f(x,\cdot):\mathbb{R}\to\mathcal{L}_1(H)$ with respect to r, uniform with respect to $x\in H$, is proved in a similar way. However, in this case (3.3) is not required and we only need the weaker condition

$$\sum_{i=1}^{\infty}\lambda_i<\infty.$$

(2) In case H is an arbitrary Hilbert space, we fix $T\in\mathcal{L}_1(H)$ and $\lambda:H\times\mathbb{R}\to\mathbb{R}$ and we define

$$f(x,r):=\lambda(x,r)T,\qquad (x,r)\in H\times\mathbb{R}.$$

If we assume that λ and $\partial_r\lambda$ are Lipschitz continuous in both variables, uniformly with respect to the other, then Hypothesis 1 2. is satisfied.

Now, we see some consequences of Hypothesis 1.

Lemma 3.2. *For any function $\varphi:H\to\mathbb{R}$ we define*

$$F(\varphi)(x)=f(x,\varphi(x)),\qquad x\in H. \tag{3.5}$$

Then, under Hypothesis 1 we have that F maps $C_b^\vartheta(H)$ into $C_b^\vartheta(H;\mathcal{L}_1(H))$ and for every $\varphi\in C_b^\vartheta(H)$

$$\|F(\varphi)\|_\vartheta\le c\,(1+\|\varphi\|_\vartheta)\,. \tag{3.6}$$

Moreover for every $\varphi_1,\ \varphi_2\in C_b^\vartheta(H)$ it holds

$$\|F(\varphi_1)-F(\varphi_2)\|_\vartheta\le c\,(1+\|\varphi_1\|_\vartheta+\|\varphi_2\|_\vartheta)\,\|\varphi_1-\varphi_2\|_\vartheta. \tag{3.7}$$

Proof. Due to (3.2), if $\varphi\in C_b^\vartheta(H)$ we have

$$\|F(\varphi)\|_0\le\sup_{x\in H}\|f(x,\varphi(x))\|_{\mathcal{L}_1(H)}\le c\,(1+\|\varphi\|_0)\,. \tag{3.8}$$

Moreover, for every $x, y \in H$

$$\begin{aligned}
&\|f(x,\varphi(x)) - f(y,\varphi(y))\|_{\mathcal{L}_1(H)} \\
&\leq \|f(x,\varphi(x)) - f(y,\varphi(x))\|^{\vartheta}_{\mathcal{L}_1(H)} \, \|f(x,\varphi(x)) - f(y,\varphi(x))\|^{1-\vartheta}_{\mathcal{L}_1(H)} \\
&+\|f(y,\varphi(x)) - f(y,\varphi(y))\|_{\mathcal{L}_1(H)} \leq c\,\|x-y\|^{\vartheta}_H \, \left(1 + \|\varphi\|^{1-\vartheta}_0\right) + c\,|\varphi(x) - \varphi(y)| \\
&\leq c\,\|x-y\|^{\vartheta}_H \, \left(1 + \|\varphi\|^{1-\vartheta}_0 + [\varphi]_\vartheta\right) \leq c\,\|x-y\|^{\vartheta}_H \, \left(1 + \|\varphi\|_0 + [\varphi]_\vartheta\right),
\end{aligned}$$

so that

$$[F(\varphi)]_\vartheta \leq c\,\left(1 + \|\varphi\|_0 + [\varphi]_\vartheta\right). \tag{3.9}$$

This, together with (3.8) allows to conclude that $F(\varphi) \in C^\vartheta_b(H)$ and (3.6) holds.

Concerning (3.7), for every $\varphi_1, \varphi_2 \in C^\vartheta_b(H)$ we have

$$\|F(\varphi_1) - F(\varphi_2)\|_0 = \sup_{x\in H} \|f(x,\varphi_1(x)) - f(x,\varphi_2(x))\|_{\mathcal{L}_1(H)} \leq c\,\|\varphi_1 - \varphi_2\|_0. \tag{3.10}$$

Moreover, for every $x, y \in H$ we have

$$\begin{aligned}
&(f(x,\varphi_1(x)) - f(x,\varphi_2(x))) - (f(y,\varphi_1(y)) - f(y,\varphi_2(y))) \\
&= \int_0^1 \left[\gamma(s\varphi_1 + (1-s)\varphi_2)(x)(\varphi_1-\varphi_2)(x) - \gamma(s\varphi_1 + (1-s)\varphi_2)(y)(\varphi_1-\varphi_2)(y)\right] ds, \\
&= \int_0^1 \gamma(s\varphi_1 + (1-s)\varphi_2)(x)\left[(\varphi_1-\varphi_2)(x) - (\varphi_1-\varphi_2)(y)\right] ds \\
&\quad + \int_0^1 \left[\gamma(s\varphi_1 + (1-s)\varphi_2)(x) - \gamma(s\varphi_1 + (1-s)\varphi_2)(y)\right](\varphi_1-\varphi_2)(y)\, ds,
\end{aligned}$$

where we have defined

$$\gamma(\varphi)(x) = \partial_r f(x,\varphi(x)), \qquad x \in H.$$

This implies that:

$$\begin{aligned}
&||(f(x,\varphi_1(x)) - f(x,\varphi_2(x))) - (f(y,\varphi_1(y)) - f(y,\varphi_2(y)))||_{\mathcal{L}_1(H)} \\
&\leq |x-y|^\theta \int_0^1 \left[\|\gamma(s\varphi_1 + (1-s)\varphi_2)\|_0 [\varphi_1-\varphi_2]_\theta + [\gamma(s\varphi_1 + (1-s)\varphi_2]_\theta \, \|\varphi_1-\varphi_2\|_0\right] ds
\end{aligned}$$

and consequently that:

$$[f(\cdot,\varphi_1)-f(\cdot,\varphi_2)]_\vartheta \le \int_0^1 \|\gamma(s\varphi_1+(1-s)\varphi_2)\|_\vartheta\, ds\, \|\varphi_1-\varphi_2\|_\vartheta.$$

Since, we are assuming that $\partial_r f$, like f, is Lipschitz continuous with respect to each variable, uniformly with respect to the other, and is clearly uniformly bounded, by using the same arguments we have used to prove (3.8) and (3.9), we have

$$\|\gamma(s\varphi_1+(1-s)\varphi_2)\|_\vartheta \le c\,(1+s\,\|\varphi_1\|_\vartheta+(1-s)\,\|\varphi_2\|_\vartheta)\,,$$

and hence

$$[F(\varphi_1)-F(\varphi_2)]_\vartheta \le c\,(1+\|\varphi_1\|_\vartheta+\|\varphi_2\|_\vartheta)\,\|\varphi_1-\varphi_2\|_\vartheta.$$

This, together with (3.10), implies (3.7). □

Hypothesis 2.

(1) The operator $A : D(A) \subset H \to H$ generates a C_0-semigroup e^{tA} and there exist $M, \omega > 0$ such that

$$\|e^{tA}\|_{\mathcal{L}(H)} \le Me^{-\omega t}. \tag{3.11}$$

(2) If Q is the operator introduced in Hypothesis 1 and if we define

$$Q_t := \int_0^t e^{sA}Qe^{sA^\star}\, ds, \qquad t \ge 0,$$

we have that $Q_t \in \mathcal{L}_1^+(H)$, for every $t \ge 0$.

(3) For every $t > 0$, we have

$$e^{tA}(H) \subset Q_t^{1/2}(H). \tag{3.12}$$

(4) If we define

$$\Lambda_t := Q_t^{-1/2}e^{tA}, \qquad t > 0,$$

(see the discussion after (2.11)) there exists some $\lambda > 0$ such that

$$\|\Lambda_t\|_{\mathcal{L}(H)} \le c\,(t\wedge 1)^{-1/2}e^{-\lambda t}, \qquad t > 0. \tag{3.13}$$

(5) For every $t > 0$ we have that $\Lambda_t Qe^{tA^\star} \in \mathcal{L}_2(H)$ and for every $\vartheta \in (0,1)$ there exist $\beta_\vartheta < 1$ and $\alpha_\vartheta > 0$ such that

$$\kappa_\vartheta(t) := \|\Lambda_t Qe^{tA^\star}\|_{\mathcal{L}_2(H)}\Big(\|\Lambda_t\|_{\mathcal{L}(H)}^{1-\vartheta}+1\Big) \le c\,(t\wedge 1)^{-\beta_\vartheta}e^{-\alpha_\vartheta t}, \qquad t > 0. \tag{3.14}$$

Hypothesis 3. For every $(x,r) \in H \times \mathbb{R}$ and $t > 0$ we have

$$e^{tA}\sigma(x,r) \in \mathcal{L}_2(H).$$

Moreover,

$$\|e^{tA}\sigma(x,r)\|_{\mathcal{L}_2(H)} \leq c\,(t \wedge 1)^{-\frac{1}{4}}\,(1 + \|x\|_H + |r|)\,, \qquad t > 0, \tag{3.15}$$

and for every $(x,r),(y,s) \in H \times \mathbb{R}$

$$\|e^{tA}\sigma(x,r) - e^{tA}\sigma(y,s)\|_{\mathcal{L}_2(H)} \leq c\,(t \wedge 1)^{-\frac{1}{4}}\,(\|x-y\|_H + |r-s|)\,, \qquad t > 0. \tag{3.16}$$

Remark 3.3. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis in H and assume that $Ae_i = -\alpha_i e_i$ and $Qe_i = \gamma_i\, e_i$, for every $i \in \mathbb{N}$, with $\alpha_i,\ \gamma_i > 0$, and $\alpha_i \uparrow +\infty$, as $i \to \infty$. By proceeding as in [6, Example 6.2.11], we have that

$$Q_t e_i = \frac{\gamma_i}{2\alpha_i}\left(1 - e^{-2\alpha_i t}\right) e_i, \qquad i \in \mathbb{N},$$

so that $Q_t \in \mathcal{L}_1(H)$ if and only if

$$\sum_{i=1}^{\infty} \frac{\gamma_i}{\alpha_i} < \infty. \tag{3.17}$$

Moreover,

$$\Lambda_t e_i = \left(\frac{2\,\alpha_i t\, e^{-\alpha_i t}}{\gamma_i\,(1 - e^{-2\alpha_i t})}\right)^{1/2} t^{-1/2} e^{-\frac{\alpha_i}{2}t} e_i, \qquad i \in \mathbb{N}.$$

In particular, if $\gamma_i \geq \gamma_0 > 0$, we have

$$\|\Lambda_t\|_{\mathcal{L}(H)} \leq c\, t^{-1/2} e^{-\frac{\alpha_1}{2}t}, \qquad t > 0,$$

so that (3.13) holds. Furthermore,

$$\|\Lambda_t Q e^{tA^\star}\|^2_{\mathcal{L}_2(H)} = 2\sum_{i=1}^{\infty} \frac{\alpha_i\,\gamma_i\, e^{-2\alpha_i t}}{e^{2\alpha_i t} - 1} \leq c\, t^{-1} e^{-2\alpha_1 t}. \tag{3.18}$$

Moreover if $\alpha_i \sim i^p$ for some $p > 0$ then

$$\|e^{tA}\|^2_{\mathcal{L}_2(H)} = \sum_{i=1}^{\infty} e^{-2i^p t} \sim t^{-1/p}. \tag{3.19}$$

Thus (3.16) hold whenever $p \geq 2$ and σ is Lipschitz. Moreover, (3.15) follows since

$$\sup_{t\geq 0}\|e^{tA}\sigma(0,0)\|^2_{\mathcal{L}_2(H)}<\infty.$$

In the special case when A is the realization of the Laplace operator in an interval, endowed with Dirichlet boundary conditions, we have that $\alpha_i\sim i^2$ and (3.17) is satisfied, for every choice of $Q\in\mathcal{L}(H)$. If we assume that $Q=I$, we have that (3.13) holds. Moreover, thanks to (3.18) we have

$$\begin{aligned}\|\Lambda_t Q e^{tA^\star}\|_{\mathcal{L}_2(H)}\left(\|\Lambda_t\|^{1-\vartheta}_{\mathcal{L}(H)}+1\right)&\leq c\,t^{-\frac{1}{2}}\,e^{-\alpha_1 t}\,\left(t^{-\frac{1-\vartheta}{2}}e^{-\frac{\alpha_1(1-\vartheta)t}{2}}+1\right)\\&\leq c\,(t\wedge 1)^{-(1-\frac{\vartheta}{2})}e^{-\frac{\alpha_1}{2}t},\end{aligned}$$

and Condition (5) in Hypothesis 2 holds for every $\vartheta\in(0,1)$. Also notice that in this case Hypothesis 3 is satisfied, due to (3.19) with $p=2$.

Hypothesis 4. The mapping $b:H\to H$ is Lipschitz continuous and bounded.

3.2. Main results

As we have done in Section 2 for the linear Kolmogorov equation (2.19), we introduce here the notion of *classical solution* for the quasi-linear problem

$$\begin{cases}D_t u_\epsilon(t,x)=\dfrac{\epsilon}{2}\mathrm{Tr}\left[\sigma^\star\sigma(x,u_\epsilon(t,x))D^2_x u_\epsilon(t,x)\right]+\langle Ax+b(x),Du_\epsilon(t,x)\rangle_H,\quad x\in D(A),\\ u_\epsilon(0,x)=g(x),\quad x\in H.\end{cases}\tag{3.20}$$

We recall, see (3.1), that the above equation depends on parameter δ and can be rewritten as:

$$\begin{cases}D_t u_\epsilon(t,x)=\dfrac{\epsilon}{2}\mathrm{Tr}\left[(Q+\delta f(x,u_\epsilon(t,x)))D^2_x u_\epsilon(t,x)\right]\\ \qquad\qquad\qquad+\langle Ax+b(x),Du_\epsilon(t,x)\rangle_H,\quad x\in D(A),\\ u_\epsilon(0,x)=g(x),\quad x\in H.\end{cases}$$

Definition 2. A function $u_\epsilon:[0,+\infty)\times H\to\mathbb{R}$ is a *classical solution* of problem (3.20) if the following conditions are satisfied.

1. It is continuous in $[0,+\infty)\times H$ and $u_\epsilon(0,\cdot)=g$.
2. $u_\epsilon(t,\cdot)\in C^2_b(H)$, for all $t>0$, and $QD^2_x u_\epsilon(t,x)\in\mathcal{L}_1(H)$, for all $(t,x)\in(0,+\infty)\times H$.
3. $u_\epsilon(\cdot,x)$ is differentiable in $(0,+\infty)$, for every $x\in D(A)$.
4. It satisfies equation (3.20), for every $(t,x)\in(0,+\infty)\times D(A)$.

In what follows, for every $\epsilon \in (0,1)$, $0 < \vartheta < \eta < 1$, $\varrho \in (0,1/2)$ and $T > 0$, we denote by $C_{\epsilon,\varrho,\eta}((0,T];C_b^{2+\vartheta}(H))$ the space of all functions $u \in C([0,T];C_b^{\eta}(H)) \cap C((0,T];C_b^{2+\vartheta}(H))$ such that

$$\|u\|_{\epsilon,\varrho,\eta,\vartheta,T} := \sup_{t\in(0,T]} \left(\|u(t,\cdot)\|_\eta + \epsilon^\varrho (t\wedge 1)^\varrho \|D_x u(t,\cdot)\|_\vartheta + \epsilon^{\varrho+\frac{1}{2}} (t\wedge 1)^{\varrho+\frac{1}{2}} \|D_x^2 u(t,\cdot)\|_\vartheta \right) < \infty.$$

Theorem 3.4. *Assume Hypotheses 1 to 4, and fix an arbitrary $g \in C_b^\eta(H)$, for some $\eta \in (1/2,1)$. Moreover fix $\vartheta \in (0, \eta - 1/2)$ and we define*

$$\varrho = \frac{1-(\eta-\vartheta)}{2}. \tag{3.21}$$

Then there exists $\bar\delta > 0$ such that for every $\delta \le \bar\delta$, $\epsilon \in (0,1)$ and $T > 0$ there exists a unique classical solution $u_\epsilon \in C_{\epsilon,\varrho,\eta}((0,T];C_b^{2+\vartheta}(H))$ for equation (3.20).

Finally

$$\|u_\epsilon\|_{\epsilon,\varrho,\eta,\vartheta,T} \le c_{\epsilon,\delta} \|g\|_\eta, \qquad \epsilon \in (0,1), \tag{3.22}$$

for some constant $c_{\epsilon,\delta} > 0$ independent of $T > 0$.

Next, for every $\epsilon > 0$, we fix arbitrary $t > 0$ and $x \in H$ and we introduce the following stochastic PDE

$$\begin{cases} dX(s) = [AX(s) + b(X(s)) + \sigma(X(s), u_\epsilon(t-s, X(s)))\,\varphi(s)]\,ds \\ \qquad\qquad + \sqrt{\epsilon}\,\sigma(X(s), u_\epsilon(t-s,X(s)))\,dW_s, \\ X(0) = x. \end{cases} \tag{3.23}$$

Here W_t, $t \ge 0$, is a cylindrical Wiener process on H, defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$, such that for every $h,k \in H$ and $t,s \ge 0$

$$\mathbb{E}\,\langle W_t, h\rangle_H \langle W_s, k\rangle_H = (t\wedge s)\,\langle h,k\rangle_H,$$

and φ is a predictable process in $L^2(\Omega; L^2(0,t;H))$.

Definition 3. An adapted process $X^{t,x}_{\varphi,\epsilon} \in L^2(\Omega; C([0,t];H))$ is a *mild solution* for equation (3.23) if for every $s \in [0,t]$

$$X^{t,x}_{\varphi,\epsilon}(s) = e^{sA}x + \int_0^s e^{(s-r)A} b(X^{t,x}_{\varphi,\epsilon}(r))\,dr + \int_0^s e^{(s-r)A}\sigma(X^{t,x}_{\varphi,\epsilon}(r), u_\epsilon(t-r, X^{t,x}_{\varphi,\epsilon}(r)))\,\varphi(r)\,dr + \sqrt{\epsilon}\int_0^s e^{(s-r)A}\sigma(X^{t,x}_{\varphi,\epsilon}(r), u_\epsilon(t-r, X^{t,x}_{\varphi,\epsilon}(r)))\,dW_r. \tag{3.24}$$

Theorem 3.5. *Suppose that Hypotheses 1 to 4 hold, and fix any $g \in C^\eta_b(H)$, with $\eta \in (1/2, 1)$, $\epsilon \in (0,1)$ and $\delta \in [0, \bar\delta)$, where $\bar\delta$ is the constant introduced in Theorem 3.4. Moreover, fix an arbitrary predictable process in $L^2(\Omega; L^2(0,T;H))$ such that*

$$\int_0^t \|\varphi(s)\|^2_H\,ds \le \mathfrak{M}, \qquad \mathbb{P}-a.s., \tag{3.25}$$

for some $\mathfrak{M} > 0$. Then equation (3.23) admits a unique mild solution $X^{t,x}_{\varphi,\epsilon} \in L^2(\Omega; C([0,t];H))$, for every $x \in H$ and $t > 0$.

In what follows, the solution of the uncontrolled version of equation (3.23), corresponding to $\varphi = 0$, will be denoted by $X^{t,x}_\epsilon$.

Once proved Theorem 3.5, we are interested in studying the limiting behavior of $X^{t,x}_\epsilon$ as $\epsilon \downarrow 0$. More precisely, we want to prove that for every fixed $t > 0$ and $x \in H$ the family $\{\mathcal{L}(X^{t,x}_\epsilon)\}_{\epsilon\in(0,1)}$ satisfies a large deviation principle in the space $C([0,t];H)$ (with speed ϵ) with respect to a suitable action functional $I_{t,x}$ that we will describe explicitly. For all definitions and details we refer e.g. to [8] and [11].

In order to state our result, we have to introduce some notations. First, we introduce the unperturbed problem

$$Z'(s) = AZ(s) + b(Z(s)), \qquad Z(0) = y \in H. \tag{3.26}$$

Since we are assuming that $b : H \to H$ is Lipschitz continuous, for every $T > 0$ and $y \in H$ there exists a unique $Z^y \in C([0,T];H)$ such that

$$Z^y(s) = e^{sA}y + \int_0^s e^{(s-r)A} b(Z^y(r))\,dr.$$

Next, for every $x \in H$, $t > 0$ and $\varphi \in L^2(0,t;H)$ we introduce the controlled problem

$$X'(s) = AX(s) + b(X(s)) + \sigma(X(s), g(Z^{X(s)}(t-s)))\varphi(s), \qquad X(0) = x. \tag{3.27}$$

In Section 8 we will see that under the same assumptions of Theorem 3.5, equation (3.27) admits a unique mild solution $X_\varphi^{t,x} \in C([0,t];H)$. This will allow to state the last main result of this paper.

Theorem 3.6. *In addition to the conditions assumed in Theorem 3.5, suppose that $g : H \to \mathbb{R}$ is Lipschitz-continuous. Moreover, suppose that the semigroup e^{tA} is analytic. Then, for every fixed $t > 0$ and $x \in H$ the family $\{\mathcal{L}(X_\epsilon^{t,x})\}_{\epsilon\in(0,1)}$ satisfies a large deviation principle in the space $C([0,t];H)$, with speed ϵ, with respect to the action functional*

$$I_{t,x}(X) = \frac{1}{2}\inf\left\{\int_0^t \|\varphi(s)\|_H^2\,ds \,:\, X(s) = X_\varphi^{t,x}(s),\ s\in[0,t]\right\}, \tag{3.28}$$

where $X_\varphi^{t,x}$ is the unique mild solution of problem (3.27).

4. The well-posedness of the stochastic PDE (3.23)

In this section we will, for the moment, assume that, for some $T > 0$, $\eta \in (1/2,1)$ and $\vartheta \in (0,\eta-1/2)$, $\varrho < 1/4$, and $\epsilon \in (0,1)$ there exists a solution $u_\epsilon \in C_{\epsilon,\varrho,\eta}((0,T];C_b^{2+\vartheta}(H))$ for equation (3.20). We will show how this allows to prove Theorem 3.5 for every $t \in (0,T]$.

We fix $t \in (0,T]$, a predictable process $\varphi \in L^2(\Omega;L^2(0,t;H))$ satisfying (3.25), a regular enough function ψ defined on $[0,t]\times H$ and we consider the stochastic equation in $[0,t]$

$$\begin{cases} dX(s) = [AX(s) + b(X(s)) + \sigma(X(s),\psi(t-s,X(s)))\varphi(s)]\,ds \\ \qquad\qquad + \sqrt{\epsilon}\,\sigma(X(s),\psi(t-s,X(s)))\,dW_s, \\ X(0) = x. \end{cases} \tag{4.1}$$

For every $s \in [0,t]$ and $x \in H$, we define

$$\Sigma_t(s,x) := \sigma(x,\psi(t-s,x)). \tag{4.2}$$

Definition 4. We say that a process $X \in L^2(\Omega;C([0,t];H))$ is a mild solution of equation (4.1) if for all $s \in [0,t]$ it holds:

$$X(s) := e^{sA}x + \int_0^s e^{(s-r)A}b(X(r))\,ds + \int_0^s e^{(s-r)A}\Sigma_t(r,X(r))\,\varphi(r)\,dr$$
$$+\sqrt{\epsilon}\int_0^s e^{(s-r)A}\Sigma_t(r,X(r))\,dW_r.$$

Theorem 4.1. *Suppose that Hypotheses 1 to 4 hold. Fix $\eta \in (1/2)$, $\epsilon \in (0,1)$, $\vartheta \in (0, \eta - 1/2)$ and define ϱ by (3.21). Moreover, fix an arbitrary predictable process $\varphi \in L^2(\Omega; C([0,t]; H))$ verifying (3.25) and function $\psi \in C_{\epsilon,\varrho,\eta}((0,T]; C_b^{2+\vartheta}(H))$. Then equation (3.23) admits a unique mild solution $X \in L^2(\Omega; C([0,t]; H))$, for every $x \in H$.*

Proof. We start by noticing that a process $X \in L^2(\Omega; C([0,t]; H))$ is a mild solution of equation (4.1) if it is a fixed point of the mapping $\Lambda_{t,\epsilon}$ defined by

$$\Lambda_{t,\epsilon}(X)(s) := e^{sA}x + \int_0^s e^{(s-r)A} b(X(r))\, ds + \int_0^s e^{(s-r)A} \Sigma_t(r, X(r))\, \varphi(r)\, dr + \sqrt{\epsilon} \int_0^s e^{(s-r)A} \Sigma_t(r, X(r))\, dW_r.$$

According to Hypothesis 3, for every $\tau > 0$, $s \in [0,t]$ and $x, y \in H$ we have

$$\|e^{\tau A} (\Sigma_t(s,x) - \Sigma_t(s,y))\|_{\mathcal{L}_2(H)} \le c\, (\tau \wedge 1)^{-\frac{1}{4}} \left(\|x - y\|_H + |\psi(t-s,x) - \psi(t-s,y)|\right).$$

Since $\psi \in C_{\epsilon,\varrho,\eta}((0,T]; C_b^{2+\vartheta}(H))$, we have

$$\begin{aligned} |\psi(t-s,x) - \psi(t-s,y)| &\le \|D_x \psi(t-s,\cdot)\|_0 \, \|x - y\|_H \\ &\le \epsilon^{-\varrho}\, ((t-s) \wedge 1)^{-\varrho} \|\psi\|_{\epsilon,\varrho,\eta,\vartheta,T} \|x - y\|_H, \end{aligned}$$

so that

$$\begin{aligned} &\|e^{\tau A} (\Sigma_t(s,x) - \Sigma_t(s,y))\|_{\mathcal{L}_2(H)} \\ &\quad \le c\, (\tau \wedge 1)^{-\frac{1}{4}} \left(1 + \epsilon^{-\varrho} ((t-s) \wedge 1)^{-\varrho} \|\psi\|_{\epsilon,\varrho,\eta,\vartheta,T}\right) \|x - y\|_H. \end{aligned} \tag{4.3}$$

Now, for every $\beta \ge 0$ we denote by $\mathcal{K}_{\beta,t}(H)$ the Banach space of all H-valued predictable processes X such that

$$\|X\|^2_{\mathcal{K}_{\beta,t}(H)} := \sup_{s \in [0,t]} e^{-\beta s}\, \mathbb{E}\, \|X(s)\|_H^2 < \infty.$$

In what follows, we need to show that $\Lambda_{t,\epsilon}$ maps the space $\mathcal{K}_{\beta,t}(H)$ into itself and, for some $\beta > 0$, is a contraction. In fact, we are only going to prove the contraction property, as the proof that $\Lambda_{t,\epsilon}$ maps $\mathcal{K}_{\beta,t}(H)$ into itself follows from analogous arguments.

If $X_1, X_2 \in \mathcal{K}_{\beta,t}(H)$, in view of (4.3) and (3.25) we have

$$\mathbb{E} \left\| \int_0^s e^{(s-r)A} \left[\Sigma_t(r, X_1(r)) - \Sigma_t(r, X_2(r))\right] \varphi(r)\, dr \right\|_H^2$$

$$\leq \mathbb{E}\int_0^s \left\| e^{(s-r)A}\left[\Sigma_t(r,X_1(r))-\Sigma_t(r,X_2(r))\right]\right\|^2_{\mathcal{L}_2(H)}\,dr \int_0^s \|\varphi(r)\|_H^2\,dr$$

$$\leq c\,\mathfrak{M}\,\mathbb{E}\int_0^s ((s-r)\wedge 1)^{-\frac{1}{2}}\left(1+\epsilon^{-2\varrho}((t-r)\wedge 1)^{-2\varrho}\|\psi\|^2_{\epsilon,\varrho,\eta,\vartheta,T}\right)\|X_1(r)-X_2(r)\|_H^2\,dr$$

$$\leq c\,\mathfrak{M}\,\|X_1-X_2\|^2_{\mathcal{K}_{\beta,t}(H)}\int_0^s ((s-r)\wedge 1)^{-\frac{1}{2}}\left(1+e^{-2\varrho}((t-r)\wedge 1)^{-2\varrho}\|\psi\|^2_{\epsilon,\varrho,\eta,\vartheta,T}\right)e^{\beta r}\,dr.$$

Since we are assuming that $\varrho<1/4$, for every $s\in[0,t]$

$$\int_0^s ((s-r)\wedge 1)^{-\frac{1}{2}}\left(1+e^{-2\varrho}((t-r)\wedge 1)^{-2\varrho}\|\psi\|^2_{\epsilon,\varrho,\eta,\vartheta,T}\right)e^{-\beta(s-r)}\,dr\leq c_{\epsilon,\beta,t}(s),$$

for some continuous increasing function $c_{\epsilon,\beta,t}:[0,t]\to[0,+\infty)$ such that

$$\lim_{\beta\to\infty}\sup_{s\in[0,t]} c_{\epsilon,\beta,t}(s)=0.$$

Therefore, we pick $\beta_1=\beta_1(\epsilon,t)>0$ such that

$$c\,\mathfrak{M}\sup_{s\in[0,t]} c_{\epsilon,\beta_1,t}(s)\leq\frac{1}{6},$$

we have

$$\sup_{s\in[0,t]} e^{-\beta_1 s}\,\mathbb{E}\left\|\int_0^s e^{(s-r)A}\left[\Sigma_t(r,X_1(r))-\Sigma_t(r,X_2(r))\right]\varphi(r)\,dr\right\|_H^2\leq\frac{1}{6}\|X_1-X_2\|^2_{\mathcal{K}_{\beta_1,t}(H)}.$$

Moreover by (4.3) with $\tau=s-r$ and $s=r$, we have:

$$\mathbb{E}\left\|\int_0^s e^{(s-r)A}\left[\Sigma_t(r,X_1(r))-\Sigma_t(r,X_2(r))\right]dW_r\right\|_H^2$$

$$\leq c\int_0^s ((s-r)\wedge 1)^{-\frac{1}{2}}\left(1+\epsilon^{-2\varrho}((t-r)\wedge 1)^{-2\varrho}\|\psi\|^2_{\epsilon,\varrho,\eta,\vartheta,T}\right)\mathbb{E}\|X_1(r)-X_2(r)\|_H^2\,dr$$

$$\leq c\,\|X_1-X_2\|^2_{\mathcal{K}_{\beta,t}(H)}\int_0^s ((s-r)\wedge 1)^{-\frac{1}{2}}\left(1+\epsilon^{-2\varrho}((t-r)\wedge 1)^{-2\varrho}\|\psi\|^2_{\epsilon,\varrho,\eta,\vartheta,T}\right)e^{\beta r}\,dr.$$

Then, by proceeding as above

$$\sup_{s\in[0,t]} e^{-\beta_1 s}\,\mathbb{E}\left\|\int_0^s e^{(s-r)A}\left[\Sigma_t(r,X_1(r))-\Sigma_t(r,X_2(r))\right]\,dW_r\right\|_H^2$$
$$\leq \frac{1}{6}\,\|X_1-X_2\|^2_{\mathcal{K}_{\beta_1,t}(H)}.$$

Finally, due to the Lipschitz-continuity of b, we have that there exists $\beta_2>0$ such that

$$\sup_{s\in[0,t]} e^{-\beta_2 s}\mathbb{E}\left\|\int_0^s e^{(s-r)A}\left[b(X_1(r))-b(X_2(r))\right]\,dr\right\|_H^2 \leq \frac{1}{6}\,\|X_1-X_2\|^2_{\mathcal{K}_{\beta_2,t}(H)}.$$

Therefore, if we take $\bar\beta:=\beta_1\vee\beta_2$, we have that $\Lambda_{t,\epsilon}$ is a contraction in $\mathcal{K}_{\bar\beta,t}(H)$ and its fixed point is the unique mild solution of equation (4.1).

Finally, by using a stochastic factorization argument, it is possible to prove that $X^{t,x}_\varphi$ belongs to $L^2(\Omega;C([0,t];H))$ (for all details about stochastic factorization see [7, Subsection 5.3.1]). □

Remark 4.2. If we still assume that, $T>0$, $\eta\in(1/2,1)$ and $\vartheta\in(0,\eta-1/2)$, $\varrho<1/4$, and $\epsilon\in(0,1)$ but now we also suppose that $u_\epsilon\in C_{\epsilon,\varrho,\eta}((0,T];C_b^{2+\vartheta}(H))$ is a solution for equation (3.20) and we set $\psi=u_\epsilon$ in Theorem 4.1 then, for all $t\in(0,T]$, the process X provided Theorem 4.1 is the process $X^{t,x}_{\varphi,\epsilon}$ required in Theorem 3.5.

5. Local existence of mild solutions for the quasi-linear problem

In this section we will prove that the quasi-linear problem (3.20) admits a local mild solution, for every $\epsilon\in(0,1)$.

In view of (3.1) and (3.5), problem (3.20) can be rewritten as

$$\begin{cases} D_t u_\epsilon(t,x)=\mathcal{L}_\epsilon u_\epsilon(t,x)+\dfrac{\epsilon}{2}\mathrm{Tr}\left[\delta F(u_\epsilon)(t,x)D_x^2u_\epsilon(t,x)\right]+\langle b(x),Du_\epsilon(t,x)\rangle_H,\\ u_\epsilon(0,x)=g(x),\quad x\in H,\end{cases}\tag{5.1}$$

where $\mathcal{L}_\epsilon$ is the linear Kolmogorov operator

$$\mathcal{L}_\epsilon\varphi(x)=\frac{\epsilon}{2}\mathrm{Tr}\left[QD_x^2\varphi(x)\right]+\langle Ax,D_x\varphi(x)\rangle_H.$$

As we have recalled in Subsection 2.3, see, in particular (2.18) and (2.19) for the definition of operator $\mathcal{L}_\epsilon$, for every $\varphi\in B_b(H)$ the unique classical solution of the linear problem

$$D_t v_\epsilon(t,x)=\mathcal{L}_\epsilon v_\epsilon(t,x),\qquad v_\epsilon(0,x)=\varphi(x),$$

is given by the Ornstein-Uhlenbeck semigroup

$$v_\epsilon(t,x) = R_t^\epsilon \varphi(x) = \int_H \varphi(e^{tA}x + y)\mathcal{N}_{\epsilon Q_t}(dy).$$

Before proceeding with the study of equation (5.1), we show how, in view of Hypothesis 2, the properties of the Ornstein-Uhlenbeck semigroup described in Subsection 2.3 apply to the semigroup R_t^ϵ.

Thanks to (3.11) and (3.13), inequality (2.13) gives for every $n \in \mathbb{N}\cup\{0\}$ and $\theta \in (0,1)$

$$\|D^n R_t^\epsilon \varphi\|_\theta \le c_{n,\theta}\, \epsilon^{-\frac{n}{2}} (t\wedge 1)^{-\frac{n}{2}} e^{-\lambda n t} \|\varphi\|_{t,\theta}, \qquad t>0, \quad \epsilon \in (0,1), \tag{5.2}$$

where

$$\|\varphi\|_{t,\theta} := \left(\|\varphi\|_0 + e^{-\omega\theta t}\,[\varphi]_\theta\right).$$

Moreover, thanks again to (3.11) and (3.13), inequality (2.15) gives for every $n \in \mathbb{N}$ and $0 \le \theta \le \rho \le 1$

$$\|D^n R_t^\epsilon \varphi\|_\theta \le c_{n,\theta,\rho}\, \epsilon^{-\frac{n-(\rho-\theta)}{2}} (t\wedge 1)^{-\frac{n-(\rho-\theta)}{2}} e^{-\lambda n t} \|\varphi\|_{t,\rho}, \qquad t>0, \quad \epsilon \in (0,1). \tag{5.3}$$

Finally,

$$\|R_t^\epsilon \varphi\|_0 \le \|\varphi\|_0, \qquad [R_t^\epsilon \varphi]_\rho \le e^{-\omega\rho t}[\varphi]_\rho, \qquad \epsilon>0. \tag{5.4}$$

By (2.4), with $\theta=\alpha$ and $\rho=\beta$, we have

$$[R_t^\epsilon\varphi]_\alpha \le \|DR_t^\epsilon\varphi\|_0^{\frac{\alpha-\beta}{1-\beta}} [R_t^\epsilon\varphi]_\beta^{\frac{1-\alpha}{1-\beta}},$$

so that, thanks to (5.3), with $\theta=0$, $\rho=\beta$ and $n=1$, and (5.4), with $\rho=\beta$, we get

$$[R_t^\epsilon\varphi]_\alpha \le c_{\alpha,\beta}\, \epsilon^{-\frac{\alpha-\beta}{2}} (t\wedge 1)^{-\frac{\alpha-\beta}{2}} e^{-\omega_{\alpha,\beta} t} \|\varphi\|_{t,\beta}, \qquad t>0, \quad \epsilon\in(0,1), \tag{5.5}$$

for some $\omega_{\alpha,\beta}>0$. In particular, due to (5.4), this implies

$$\|R_t^\epsilon\varphi\|_\alpha \le (c_{\alpha,\beta}+1)\, \epsilon^{-\frac{\alpha-\beta}{2}} (t\wedge 1)^{-\frac{\alpha-\beta}{2}} \|\varphi\|_{t,\beta}, \qquad t>0, \quad \epsilon\in(0,1). \tag{5.6}$$

Now, we introduce the notion of mild solution for equation (3.20).

Definition 5. A function $u_\epsilon \in C([0,+\infty);H)$ such that $u_\epsilon(t,\cdot)\in C_b^2(H)$, for every $t>0$, is a mild solution for problem (3.20) if for every $(t,x)\in[0,+\infty)\times H$

$$u_\epsilon(t,x) = R_t^\epsilon g(x) + \int_0^t R_{t-s}^\epsilon \left(\frac{\epsilon}{2} Tr\left[\delta F(u_\epsilon(s,\cdot)) D_x^2 u_\epsilon(s,\cdot)\right] + \langle b(\cdot), Du_\epsilon(s,\cdot)\rangle_H\right)(x)\, ds.$$

For every $R>0$, $\eta\in(1/2,1)$, $\theta\in(0,\eta-1/2)$, $\rho\in(0,1/4)$, $T>0$ we define

$$\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}:=\left\{u\in C_{\epsilon,\varrho,\eta}((0,T];C_b^{2+\vartheta}(H))\ :\ \|u\|_{\epsilon,\varrho,\eta,\vartheta,T}\le R\right\},$$

and for every $v\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$ and $\delta>0$ we define

$$\Gamma_{\epsilon,\delta}(v)(t,x):=\int_0^t R^\epsilon_{t-s}\gamma_{\epsilon,\delta}(v,s)(x)\,ds,\qquad t\in[0,T],\quad x\in H,$$

where

$$\gamma_{\epsilon,\delta}(v,s)(x):=\frac{\epsilon}{2}\mathrm{Tr}\left[\delta F(v(s,\cdot))(x)D_x^2v(s,x)\right]+\langle b(x),Dv(s,x)\rangle_H.$$

In particular, u_ϵ is a mild solution for problem (3.20) if and only if

$$u_\epsilon(t,x)=R^\epsilon_tg(x)+\Gamma_{\epsilon,\delta}(u_\epsilon)(t,x).$$

First, we investigate the dependence of $\gamma_{\epsilon,\delta}$ on $v\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$.

Lemma 5.1. *For every $v\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$ and $\delta>0$*

$$\|\gamma_{\epsilon,\delta}(v,s)\|_\vartheta\le c\,\epsilon^{\frac12-\varrho}\,\delta R\,(1+R)\,(s\wedge1)^{-(\varrho+\frac12)}+c\,\epsilon^{-\varrho}R(s\wedge1)^{-\varrho},\qquad s\in(0,T].\tag{5.7}$$

Moreover, for every $v_1,v_2\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$ and $\delta>0$

$$\begin{aligned}&\|\gamma_{\epsilon,\delta}(v_1,s)-\gamma_{\epsilon,\delta}(v_2,s)\|_\vartheta\le c\,\epsilon^{\frac12-\varrho}\delta\,R(1+R)\,(s\wedge1)^{-(\varrho+\frac12)}\|v_1(s,\cdot)-v_2(s,\cdot)\|_\vartheta\\&+c\,\epsilon\,\delta(1+R)\,\|D_x^2v_1(s,\cdot)-D_x^2v_2(s,\cdot)\|_\vartheta+c\,\|D_xv_1(s,\cdot)-D_xv_2(s,\cdot)\|_\vartheta.\end{aligned}\tag{5.8}$$

Proof. In view of (3.6) and Hypothesis 4, we have

$$\begin{aligned}\|\gamma_{\epsilon,\delta}(v,s)\|_\vartheta&\le c\,\epsilon\,\delta\,\|F(v(s,\cdot))\|_\vartheta\,\|D_x^2v(s,\cdot)\|_\vartheta+\|b\|_\vartheta\,\|D_xv(s,\cdot)\|_\vartheta\\&\le c\,\epsilon\,\delta\,(1+\|v(s,\cdot)\|_\vartheta)\,\|D_x^2v(s,\cdot)\|_\vartheta+c\,\|D_xv(s,\cdot)\|_\vartheta,\end{aligned}$$

and since we are assuming that $v\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$, this implies (5.7).

Next, if $v_1,v_2\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$ and $\delta>0$ we have

$$\begin{aligned}&\|\gamma_{\epsilon,\delta}(v_1,s)-\gamma_{\epsilon,\delta}(v_2,s)\|_\vartheta\le c\,\epsilon\,\delta\,\|F(v_1(s,\cdot))-F(v_2(s,\cdot))\|_\vartheta\,\|D_x^2v_1(s,\cdot)\|_\vartheta\\&+c\,\epsilon\,\delta\,\|F(v_2(s,\cdot))\|_\vartheta\,\|D_x^2v_1(s,\cdot)-D_x^2v_2(s,\cdot)\|_\vartheta+\|b\|_\vartheta\,\|D_xv_1(s,\cdot)-D_xv_2(s,\cdot)\|_\vartheta.\end{aligned}$$

Thus, according to (3.6) and (3.7),

$$\begin{aligned}&\|\gamma_{\epsilon,\delta}(v_1,s)-\gamma_{\epsilon,\delta}(v_2,s)\|_\vartheta\\&\leq c\,\epsilon\,\delta\,\|v_1(s,\cdot)-v_2(s,\cdot)\|_\vartheta\,(1+\|v_1(s,\cdot)\|_\vartheta+\|v_2(s,\cdot)\|_\vartheta)\,\|D_x^2v_1(s,\cdot)\|_\vartheta\\&+c\,\epsilon\,\delta\,(1+\|v_2(s,\cdot)\|_\vartheta)\,\|D_x^2v_1(s,\cdot)-D_x^2v_2(s,\cdot)\|_\vartheta+c\,\|D_xv_1(s,\cdot)-D_xv_2(s,\cdot)\|_\vartheta.\end{aligned}$$

Recalling that $v_1,v_2\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$, this implies (5.8). □

Remark 5.2. If for every fixed $\epsilon,\delta>0$ we define

$$\alpha_{\epsilon,\delta}(R,s):=\epsilon^{\frac12-\varrho}\delta R(1+R)(s\wedge1)^{-(\varrho+\frac12)}+\epsilon^{-\varrho}R(s\wedge1)^{-\varrho},\qquad s>0,\quad R>0,$$

and

$$\mathfrak{a}_{\epsilon,\delta}(R,s):=\epsilon^{\frac12-\varrho}\delta(1+R)^2(s\wedge1)^{-(\varrho+\frac12)}+\epsilon^{-\varrho}(s\wedge1)^{-\varrho},\qquad s>0,\quad R>0,$$

due (5.7) and (5.8) we have that for every $v,v_1,v_2\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$ and $s\in(0,T]$

$$\|\gamma_{\epsilon,\delta}(v,s)\|_\vartheta\leq c\,\alpha_{\epsilon,\delta}(R,s),\tag{5.9}$$

and

$$\|\gamma_{\epsilon,\delta}(v_1,s)-\gamma_{\epsilon,\delta}(v_2,s)\|_\vartheta\leq c\,\mathfrak{a}_{\epsilon,\delta}(R,s)\,\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T}.\tag{5.10}$$

Notice that for all $\beta<1$ and $\mu>0$ and for all $t\geq0$

$$\begin{aligned}&\int_0^t((t-s)\wedge1)^{-\beta}e^{-\mu(t-s)}\alpha_{\epsilon,\delta}(R,s)\,ds\\&\leq\epsilon^{\frac12-\varrho}\delta R(1+R)\int_0^t((t-s)\wedge1)^{-\beta}e^{-\mu(t-s)}(s\wedge1)^{-(\varrho+\frac12)}\,ds\\&+\epsilon^{-\varrho}R\int_0^t((t-s)\wedge1)^{-\beta}e^{-\mu(t-s)}(s\wedge1)^{-\varrho}\,ds,\end{aligned}$$

and this implies that there exists some constant $c>0$ only dependent on β and μ such that

$$\int_0^t((t-s)\wedge1)^{-\beta}e^{-\mu(t-s)}\alpha_{\epsilon,\delta}(R,s)\,ds\leq c\,(t\wedge1)^{\frac12-(\varrho+\beta)}\epsilon^{\frac12-\varrho}\lambda_{\epsilon,\delta}(R,t),\tag{5.11}$$

where

$$\lambda_{\epsilon,\delta}(R,t):=\delta R(1+R)\,+\epsilon^{-\frac12}R\,(t\wedge1)^{\frac12}.$$

To get the above we have observed that, when $t \geq 1$ the right hand side of (5.11) reduces to

$$c\,\epsilon^{\frac{1}{2}-\varrho}\left[\delta R(1+R)\ +\epsilon^{-\frac{1}{2}}R\right]$$

and (5.11) follows since

$$\alpha_{\epsilon,\delta}(R,s) \leq \epsilon^{\frac{1}{2}-\varrho}(s\wedge 1)^{-(\frac{1}{2}+\varrho)}\left(\delta R(1+R)+\epsilon^{-\frac{1}{2}}R\right),$$

and

$$\sup_{t\geq 1}\int_0^t((t-s)\wedge 1)^{-\beta}e^{-\mu(t-s)}(s\wedge 1)^{-(\frac{1}{2}+\varrho)}\,ds<\infty.$$

In the other case, that is when $t<1$, we have $(t-s)\wedge 1=(t-s)$, $s\in[0,t]$ and

$$\int_0^t(t-s)^{-\beta}e^{-\mu(t-s)}\alpha_{\epsilon,\delta}(R,s)\,ds \leq \epsilon^{\frac{1}{2}-\varrho}\delta R(1+R)\int_0^t(t-s)^{-\beta}s^{-(\varrho+\frac{1}{2})}\,ds + \epsilon^{-\varrho}\delta R\int_0^t(t-s)^{-\beta}s^{-\varrho}\,ds$$

and

$$\int_0^t(t-s)^{-\beta}s^{-(\varrho+\frac{1}{2})}\,ds \leq ct^{-\beta-\varrho+\frac{1}{2}},\qquad \int_0^t(t-s)^{-\beta}s^{-\varrho}\,ds\leq ct^{-\beta-\varrho+1}$$

by a standard change of variable.

In an analogous way

$$\int_0^t((t-s)\wedge 1)^{-\beta}e^{-\mu(t-s)}\mathfrak{a}_{\epsilon,\delta}(R,s)\,ds \leq c\,(t\wedge 1)^{\frac{1}{2}-(\varrho+\beta)}\epsilon^{\frac{1}{2}-\varrho}\mathfrak{l}_{\epsilon,\delta}(R,t), \tag{5.12}$$

where

$$\mathfrak{l}_{\epsilon,\delta}(R,t):=\delta(1+R)^2\ +\epsilon^{-\frac{1}{2}}(t\wedge 1)^{\frac{1}{2}}.$$

Next we prove the following estimates for $\Gamma_{\epsilon,\delta}$ on $\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$.

Lemma 5.3. *For every $v\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$ and $\epsilon,\delta\in(0,1)$ it holds*

$$\|\Gamma_{\epsilon,\delta}(v)\|_{\epsilon,\varrho,\eta,\vartheta,T}\leq c\,\lambda_{\epsilon,\delta}(R,T)\left[\epsilon^{\frac{1-(\eta-\vartheta)}{2}-\varrho}\,(T\wedge 1)^{\frac{1-(\eta-\vartheta)}{2}-\varrho}\,(T\vee 1)+1\right]. \tag{5.13}$$

Proof. *Step 1.* We have

$$\|\Gamma_{\epsilon,\delta}(v)(t)\|_\eta \le c\,\epsilon^{\frac{1-(\eta-\vartheta)}{2}-\varrho}\lambda_{\epsilon,\delta}(R,t)\,(t\wedge 1)^{\frac{1-(\eta-\vartheta)}{2}-\varrho}(t\vee 1), \qquad t\in[0,T]. \tag{5.14}$$

Proof of Step 1. In view of (5.6), applied for $\beta=\vartheta$ and $\alpha=\eta$, we have that, for every $t\in[0,T]$:

$$\begin{aligned}\|\Gamma_{\epsilon,\delta}(v)(t)\|_\eta &\le \int_0^t \|R^\epsilon_{t-s}\gamma_{\epsilon,\delta}(v,s)\|_\eta\,ds \le c\,\epsilon^{-\frac{\eta-\vartheta}{2}}\int_0^t((t-s)\wedge 1)^{-\frac{\eta-\vartheta}{2}}\|\gamma_{\epsilon,\delta}(v,s)\|_\vartheta\,ds\\ &\le c\,\epsilon^{-\frac{\eta-\vartheta}{2}}\int_0^t((t-s)\wedge 1)^{-\frac{\eta-\vartheta}{2}}\alpha_{\epsilon,\delta}(R,s)\,ds.\end{aligned}$$

Then, by adapting (5.11) to the case $\mu=0$, we get

$$\|\Gamma_{\epsilon,\delta}(v)(t)\|_\eta \le c\,\epsilon^{-\frac{\eta-\vartheta}{2}}\epsilon^{\frac12-\varrho}\lambda_{\epsilon,\delta}(R,t)(t\wedge 1)^{\frac{1-(\eta-\vartheta)}{2}-\varrho}\,(t\vee 1),$$

and (5.14) follows.

Step 2. We have

$$(t\wedge 1)^\varrho\,\|D\Gamma_{\epsilon,\delta}(v)(t)\|_\vartheta \le c\,\epsilon^{-\varrho}\left[\delta\,R(1+R)+\epsilon^{-\frac12}R(t\wedge 1)^{\frac12}\right]. \tag{5.15}$$

Proof of Step 2. According to (5.2), we have that

$$\int_0^t \|DR^\epsilon_{t-s}\gamma_{\epsilon,\delta}(v,s)\|_\vartheta\,ds \le c\epsilon^{-\frac12}\int_0^t((t-s)\wedge 1)^{-\frac12}e^{-\lambda(t-s)}\|\gamma_{\epsilon,\delta}(v,s)\|_\vartheta\,ds.$$

Then, thanks to (5.9) and (5.11) we conclude

$$\int_0^t \|DR^\epsilon_{t-s}\gamma_{\epsilon,\delta}(v,s)\|_\vartheta\,ds \le c\,\epsilon^{-\frac12}(t\wedge 1)^{-\varrho}\epsilon^{\frac12-\varrho}\lambda_{\epsilon,\delta}(R,t),$$

and (5.15) follows.

Step 3. We have

$$(t\wedge 1)^{\varrho+\frac12}\,\|D^2\Gamma_{\epsilon,\delta}(v)(t)\|_\vartheta \le c\,\epsilon^{-(\frac12+\varrho)}\lambda_{\epsilon,\delta}(R,t). \tag{5.16}$$

Proof of Step 3. By proceeding as in the proof of Step 2, we have, using (5.3) with $n=2$, $\alpha=0$ and $\beta=\theta$:

$$\|D^2\Gamma_{\epsilon,\delta}(v)(t)\|_0 \le c\epsilon^{-1+\frac{\vartheta}{2}} \int_0^t ((t-s)\wedge 1)^{-1+\frac{\vartheta}{2}} e^{-\lambda(t-s)} \alpha_{\epsilon,\delta}(R,s)\, ds$$
$$\le c\,\epsilon^{-1+\frac{\vartheta}{2}} \epsilon^{\frac{1}{2}-\varrho} (t\wedge 1)^{-(\varrho+\frac{1}{2})+\frac{\vartheta}{2}} \lambda_{\epsilon,\delta}(R,t),$$

and this implies

$$(t\wedge 1)^{\varrho+\frac{1}{2}}\, \|D^2\Gamma_{\epsilon,\delta}(v)(t)\|_0 \le c\,\epsilon^{-(\frac{1}{2}+\varrho)+\frac{\vartheta}{2}} \lambda_{\epsilon,\delta}(R,t)(t\wedge 1)^{\frac{\vartheta}{2}}. \tag{5.17}$$

Now, for every $x, h \in H$ and $t \in [0,T]$, we have

$$(t\wedge 1)^{\varrho+\frac{1}{2}} \|D^2\Gamma_{\epsilon,\delta}(v)(t,x+h) - D^2\Gamma_{\epsilon,\delta}(v)(t,x)\|_{\mathcal{L}(H)} \le c\,\epsilon^{-(\frac{1}{2}+\varrho)+\frac{\vartheta}{2}} \lambda_{\epsilon,\delta}(R,t)\,(t\wedge 1)^{\frac{\vartheta}{2}}.$$

Hence, if we assume that $\|h\|_H^2 > \epsilon\,(t\wedge 1)/2$ we get

$$(t\wedge 1)^{\varrho+\frac{1}{2}}\, \|D^2\Gamma_{\epsilon,\delta}(v)(t,x+h) - D^2\Gamma_{\epsilon,\delta}(v)(t,x)\|_{\mathcal{L}(H)} \le c\,\epsilon^{-(\frac{1}{2}+\varrho)} \lambda_{\epsilon,\delta}(R,t)\,\|h\|_H^{\vartheta}. \tag{5.18}$$

When $\|h\|_H^2 \le \epsilon\,(t\wedge 1)/2$ we have

$$D^2\Gamma_{\epsilon,\delta}(v)(t,x) = \int_0^{\epsilon^{-1}\|h\|_H^2} D^2 R_s^\epsilon \gamma_{\epsilon,\delta}(v,t-s)(x)\, ds + \int_{\epsilon^{-1}\|h\|_H^2}^t D^2 R_s^\epsilon \gamma_{\epsilon,\delta}(v,t-s)(x)\, ds$$
$$=: a_{\epsilon,\delta}(h,t,x) + b_{\epsilon,\delta}(h,t,x).$$

Due to (5.3), from (5.9) we have to evaluate the Hölderianity

$$\begin{aligned}
&\|a_{\epsilon,\delta}(h,t,x+h) - a_{\epsilon,\delta}(h,t,x)\|_{\mathcal{L}(H)} \\
&\le c\,\epsilon^{-1+\frac{\vartheta}{2}} \int_0^{\epsilon^{-1}\|h\|_H^2} (s\wedge 1)^{-1+\frac{\vartheta}{2}} e^{-2\lambda s} \|\gamma_{\epsilon,\delta}(v,t-s)\|_\vartheta\, ds \\
&\le c\,\epsilon^{-1+\frac{\vartheta}{2}} \int_0^{\epsilon^{-1}\|h\|_H^2} (s\wedge 1)^{-1+\frac{\vartheta}{2}} e^{-\lambda s} \alpha_{\epsilon,\delta}(R,t-s)\, ds \\
&\le c\,\epsilon^{-(\frac{1}{2}+\varrho)} \lambda_{\epsilon,\delta}(R,t)\,(t\wedge 1)^{-(\varrho+\frac{1}{2})}\,\|h\|_H^{\vartheta}.
\end{aligned} \tag{5.19}$$

To get the last inequality we have observed that,

$$\int_0^{\epsilon^{-1}||h||_H^2} (s\wedge 1)^{-1+\frac{\theta}{2}} e^{-\lambda s} \alpha_{\epsilon,\delta}(R,t-s)\, ds$$

$$\leq \epsilon^{\frac{1}{2}-\varrho}\delta R(1+R)\int_0^{\epsilon^{-1}||h||_H^2}(s\wedge 1)^{-1+\frac{\theta}{2}}((t-s)\wedge 1)^{-(\varrho+\frac{1}{2})}e^{-\lambda s}\,ds$$

$$+\epsilon^{-\varrho}R\int_0^{\epsilon^{-1}||h||_H^2}(s\wedge 1)^{-1+\frac{\theta}{2}}((t-s)\wedge 1)^{-\varrho}e^{-\lambda s}\,ds$$

and recalling that $\epsilon^{-1}\|h\|_H^2 \leq (t\wedge 1)/2 \leq t/2$ and hence $(t-s)\wedge 1 \geq t/2\wedge 1$ we deduce that

$$\int_0^{\epsilon^{-1}\|h\|_H^2}(s\wedge 1)^{-1+\frac{\theta}{2}}e^{-\lambda s}\alpha_{\epsilon,\delta}(R,t-s)\,ds \leq c\,\epsilon^{\frac{1}{2}-\varrho-\frac{\vartheta}{2}}\lambda_{\epsilon,\delta}(R,t)\,(t\wedge 1)^{-(\varrho+\frac{1}{2})}\,\|h\|_H^{\vartheta}.$$

As for $b_{\epsilon,\delta}(h,t,\cdot)$, we have

$$b_{\epsilon,\delta}(h,t,x+h)-b_{\epsilon,\delta}(h,t,x) = \int_{\epsilon^{-1}\|h\|_H^2}^t \left[D^2R_s^\epsilon\gamma_{\epsilon,\delta}(v,t-s)(x+h)-D^2R_s^\epsilon\gamma_{\epsilon,\delta}(v,t-s)(x)\right]\,ds.$$

Hence, due again to (5.3), we have

$$\begin{aligned}
&\|b_{\epsilon,\delta}(h,t,x+h)-b_{\epsilon,\delta}(h,t,x)\|_{\mathcal{L}(H)}\\
&\leq \int_{\epsilon^{-1}\|h\|_H^2}^t \|D^2R_s^\epsilon\gamma_{\epsilon,\delta}(v,t-s)(x+h)-D^2R_s^\epsilon\gamma_{\epsilon,\delta}(v,t-s)(x)\|_{\mathcal{L}(H)}\,ds\\
&\leq c\int_{\epsilon^{-1}\|h\|_H^2}^t \|R_s^\epsilon\gamma_{\epsilon,\delta}(v,t-s)\|_3\,ds\,\|h\|_H\\
&\leq c\,\epsilon^{-\frac{3-\vartheta}{2}}\epsilon^{\frac{1}{2}-\varrho}\int_{\epsilon^{-1}\|h\|_H^2}^t (s\wedge 1)^{-\frac{3-\vartheta}{2}}e^{-3\lambda s}((t-s)\wedge 1)^{-(\varrho+\frac{1}{2})}\lambda_{\epsilon,\delta}(R,t-s)\,ds\,\|h\|_H.
\end{aligned}$$

Since we are assuming $\|h\|_H^2 \leq \epsilon\,(t\wedge 1)/2$, we have

$$\begin{aligned}
&\int_{\epsilon^{-1}\|h\|_H^2}^t (s\wedge 1)^{-\frac{3-\vartheta}{2}}e^{-3\lambda s}((t-s)\wedge 1)^{-(\varrho+\frac{1}{2})})\,ds\\
&= \int_{\epsilon^{-1}\|h\|_H^2}^{t/2} (s\wedge 1)^{-\frac{3-\vartheta}{2}}e^{-3\lambda s}((t-s)\wedge 1)^{-(\varrho+\frac{1}{2})}\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t/2}^{t}(s\wedge 1)^{-\frac{3-\vartheta}{2}}e^{-3\lambda s}((t-s)\wedge 1)^{-(\varrho+\frac12)}\,ds\\
&\le c\,(t\wedge 1)^{-(\varrho+\frac12)}\epsilon^{\frac{1-\theta}{2}}\,\|h\|_H^{-1+\vartheta}+c\,(t\wedge 1)^{-(1+\varrho)+\frac{\theta}{2}}\\
&= c\,(t\wedge 1)^{-(\varrho+\frac12)}\left(\epsilon^{\frac{1-\vartheta}{2}}\|h\|_H^{-1+\vartheta}+(t\wedge 1)^{-\frac{1-\vartheta}{2}}\right)\\
&\le c\,\epsilon^{\frac{1-\vartheta}{2}}(t\wedge 1)^{-(\varrho+\frac12)}\,\|h\|_H^{-1+\vartheta}.
\end{aligned}
$$

Moreover, in the same way we have

$$
\int_{\epsilon^{-1}\|h\|_H^2}^{t}(s\wedge 1)^{-\frac{3-\vartheta}{2}}e^{-3\lambda s}((t-s)\wedge 1)^{-\varrho}\,ds\le c\,\epsilon^{\frac{1-\vartheta}{2}}(t\wedge 1)^{-\varrho}\,\|h\|_H^{-1+\vartheta},
$$

so that

$$
(t\wedge 1)^{\varrho+\frac12}\,\|b_{\epsilon,\delta}(h,t,x+h)-b_{\epsilon,\delta}(h,t,x)\|_{\mathcal{L}(H)}\le c\,\epsilon^{-(\frac12+\varrho)}\lambda_{\epsilon,\delta}(R,t)\|h\|_H^{\vartheta}.
$$

This, together with (5.19) and (5.18), implies that for every $h\in H$

$$
(t\wedge 1)^{\varrho+\frac12}\|D^2\Gamma_{\epsilon,\delta}(v)(t,x+h)-D^2\Gamma_{\epsilon,\delta}(v)(t,x)\|_{\mathcal{L}(H)}\le c\,\epsilon^{-(\frac12+\varrho)}\lambda_{\epsilon,\delta}(R,t)\,\|h\|_H^{\vartheta}.
$$

Thus, thanks to (5.17), we obtain (5.16).

Conclusion. Estimate (5.13) is a consequence of (5.14), (5.15) and (5.16). □

Remark 5.4. From the proof of the previous lemma, we easily see that for every $t\in(0,T]$ and $\epsilon\in(0,1)$

$$
\begin{aligned}
&\epsilon^{\frac{\eta-\vartheta}{2}}[\Gamma_{\epsilon,\delta}(v)(t)]_\eta+\epsilon^{\varrho}(t\wedge 1)^{\varrho}\|D\Gamma_{\epsilon,\delta}(v)(t)\|_\vartheta+\epsilon^{\frac12+\varrho}(t\wedge 1)^{\varrho+\frac12}\|D^2\Gamma_{\epsilon,\delta}(v)(t)\|_\vartheta\\
&\quad\le c\,\lambda_{\epsilon,\delta}(R,t)\left(\epsilon^{\frac12-\varrho}\,(t\wedge 1)^{\frac{1-(\eta-\vartheta)}{2}-\varrho}+1\right),
\end{aligned}
\tag{5.20}
$$

for some constant $c>0$ independent of $T>0$. Indeed, in view of (5.5), with $\alpha=\eta$ and $\beta=\theta$ and (5.9), we have

$$
\begin{aligned}
[\Gamma_{\epsilon,\delta}(v)(t)]_\eta&\le c\,\epsilon^{-\frac{\eta-\vartheta}{2}}\int_0^t e^{-\omega_{\vartheta,\eta}(t-s)}((t-s)\wedge 1)^{-\frac{\eta-\vartheta}{2}}\|\gamma_{\epsilon,\delta}(v,s)\|_\vartheta\,ds\\
&\le c\,\epsilon^{-\frac{\eta-\vartheta}{2}}\int_0^t e^{-\omega_{\vartheta,\eta}(t-s)}((t-s)\wedge 1)^{-\frac{\eta-\vartheta}{2}}\alpha_{\epsilon,\delta}(R,s)\,ds\\
&\le c\,\lambda_{\epsilon,\delta}(R,t)\left(\epsilon^{\frac{1-(\eta-\vartheta)}{2}-\varrho}(t\wedge 1)^{\frac{1-(\eta-\vartheta)}{2}-\varrho}\right).
\end{aligned}
$$

This, together with (5.15) and (5.16), implies (5.20).

Now we are ready to prove the existence of a local mild solution.

Theorem 5.5. *Fix $\eta \in (1/2, 1)$ and $\vartheta \in (0, \eta - 1/2)$ and define*

$$\varrho := \frac{1 - (\eta - \vartheta)}{2}. \tag{5.21}$$

Then, there exist δ_1 such that for every $\epsilon \in (0,1)$ there exists $T_1(\epsilon) > 0$ so that problem (3.20) has a mild solution u_ϵ in $C_{\epsilon,\varrho,\eta}((0, T_1(\epsilon)]; C_b^{2+\vartheta}(H))$, for every $\delta \le \delta_1$.

Proof. A function u_ϵ is a mild solution of equation (3.20) if and only if it is a fixed point for the mapping $\Gamma^g_{\epsilon,\delta}$ defined by

$$\Gamma^g_{\epsilon,\delta}(v)(t) = R^\epsilon_t g + \Gamma_{\epsilon,\delta}(v)(t), \qquad t \in [0, T].$$

Thus, we will prove the existence of a local mild solution for equation (3.20) by showing that there exist some $T_1, R > 0$ and $\delta_1 > 0$ such that $\Gamma^g_{\epsilon,\delta}$ maps $\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T_1}$ into itself as a contraction, for every $\delta \le \delta_1$.

Thanks to (5.6) we have

$$\|R^\epsilon_t g\|_\eta \le c\, \|g\|_\eta. \tag{5.22}$$

Moreover, thanks to (5.3)

$$\begin{aligned} \|DR^\epsilon_t g(v)(t)\|_\vartheta &\le c\, \epsilon^{-\frac{1-(\eta-\vartheta)}{2}} (t \wedge 1)^{-\frac{1-(\eta-\vartheta)}{2}} e^{-\lambda t}\, \|g\|_\eta, \\ \|D^2 R^\epsilon_t g(v)(t)\|_\vartheta &\le c\, \epsilon^{-(1-\frac{\eta-\vartheta}{2})} (t \wedge 1)^{-(1-\frac{\eta-\vartheta}{2})} e^{-2\lambda t}\, \|g\|_\eta. \end{aligned} \tag{5.23}$$

Therefore, if we define ϱ as in (5.21), we have that $\varrho \in (0, 1/4)$ and $1 - \frac{\eta-\theta}{2} - \varrho = 0$, so from (5.22) and (5.23), it follows

$$\|R^\epsilon_t g\|_\eta + \epsilon^\varrho (t \wedge 1)^\varrho \|DR^\epsilon_t g(t)\|_\vartheta + \epsilon^{\frac{1}{2}+\varrho} (t \wedge 1)^{\varrho+\frac{1}{2}} \|D^2 R^\epsilon_t g(t)\|_\vartheta \le c\, \|g\|_\eta. \tag{5.24}$$

With ϱ defined as in (5.21), together with (5.13) this implies that for every $v \in \mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$

$$\|\Gamma^g_{\epsilon,\delta}(v)\|_{\epsilon,\varrho,\eta,\vartheta,T} \le c\, \|g\|_\eta + c \left[\delta\, R(1+R) + \epsilon^{-\frac{1}{2}} R\, (T \wedge 1)^{\frac{1}{2}}\right] (T \vee 1).$$

In particular, if we first take $R := 3c\, \|g\|_\eta$ and $\delta' > 0$ small enough such that

$$c\, \delta'\, R(1+R) \le \frac{R}{3},$$

and then fix $T' \le 1$ small enough so that

$$c\,\epsilon^{-\frac{1}{2}}R(T'\wedge 1)^{\frac{1}{2}}\leq\frac{R}{3},$$

we conclude that for every $\delta\leq\delta'$ and $T\leq T'$

$$\|\Gamma^g_{\epsilon,\delta}(v)\|_{\epsilon,\varrho,\eta,\vartheta,T}\leq R,$$

so that $\Gamma^g_{\epsilon,\delta}$ maps $\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$ into itself.

Now, if we fix $v_1,v_2\in\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,T}$, we have, in view of Definition 5 and estimates (5.3) and (5.6)

$$\begin{aligned}
&\|\Gamma^g_{\epsilon,\delta}(v_1)-\Gamma^g_{\epsilon,\delta}(v_2)\|_{\epsilon,\varrho,\eta,\vartheta,T}\\
&\leq c\,\epsilon^{-\frac{\eta-\vartheta}{2}}\sup_{t\in(0,T]}\int_0^t((t-s)\wedge 1)^{-\frac{\eta-\vartheta}{2}}\|\gamma_{\epsilon,\delta}(v_1,s)-\gamma_{\epsilon,\delta}(v_2,s)\|_\vartheta\,ds\\
&+\epsilon^\varrho\sup_{t\in(0,T]}(t\wedge 1)^\varrho\int_0^t\epsilon^{-\frac{1}{2}}((t-s)\wedge 1)^{-\frac{1}{2}}e^{-\lambda(t-s)}\|\gamma_{\epsilon,\delta}(v_1,s)-\gamma_{\epsilon,\delta}(v_2,s)\|_\vartheta\,ds\\
&+\epsilon^{\frac{1}{2}+\varrho}\sup_{t\in(0,T]}(t\wedge 1)^{\varrho+\frac{1}{2}}\\
&\times\int_0^t\epsilon^{-(1-\frac{\vartheta}{2})}((t-s)\wedge 1)^{-(1-\frac{\vartheta}{2})}e^{-2\lambda(t-s)}\|\gamma_{\epsilon,\delta}(v_1,s)-\gamma_{\epsilon,\delta}(v_2,s)\|_\vartheta\,ds\\
&+\epsilon^{\frac{1}{2}+\varrho}\sup_{t\in(0,T]}(t\wedge 1)^{\varrho+\frac{1}{2}}\,[D^2\Gamma^g_{\epsilon,\delta}(v_1)(t)-D^2\Gamma^g_{\epsilon,\delta}(v_2)(t)]_\vartheta=:\sum_{i=1}^4 I_{\delta,i}(\epsilon).
\end{aligned}$$

Then, according to (5.10) and (5.12), we have, since $\eta-\vartheta<1$.

$$I_{\delta,1}(\epsilon)\leq c\,\mathfrak{l}_{\epsilon,\delta}(R,T)\,\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T}.\tag{5.25}$$

In the same way,

$$\begin{aligned}
I_{\delta,2}(\epsilon)&\leq c\,\epsilon^\varrho\sup_{t\in(0,T]}(t\wedge 1)^\varrho\int_0^t\epsilon^{-\frac{1}{2}}((t-s)\wedge 1)^{-\frac{1}{2}}e^{-\lambda(t-s)}\mathfrak{a}_{\epsilon,\delta}(R,s)\,ds\,\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T}\\
&\leq c\,\mathfrak{l}_{\epsilon,\delta}(R,T)\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T},
\end{aligned}\tag{5.26}$$

and

$$
\begin{aligned}
I_{\delta,3}(\epsilon) &\le c\,\epsilon^{\frac{1}{2}+\varrho} \sup_{t\in(0,T]} (t\wedge 1)^{\varrho+\frac{1}{2}} \\
&\quad \times \int_0^t \epsilon^{-(1-\frac{\vartheta}{2})}((t-s)\wedge 1)^{-(1-\frac{\vartheta}{2})} e^{-2\lambda(t-s)} \mathfrak{a}_{\epsilon,\delta}(R,s)\,ds\,\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T} \\
&\le c\,\epsilon^{\frac{\vartheta}{2}}\,\mathfrak{l}_{\epsilon,\delta}(R,T)\,\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T}.
\end{aligned}
\tag{5.27}
$$

As for $I_{\delta,4}(\epsilon)$, due to (5.27), if we fix any $x,h\in H$ and assume $\|h\|_H^2 > \epsilon\,t/2$ we have

$$
\begin{aligned}
&\epsilon^{\varrho+\frac{1}{2}}(t\wedge 1)^{\varrho+\frac{1}{2}}\,\|D^2\Gamma^g_{\epsilon,\delta}(v_1)(t,x+h) - D^2\Gamma^g_{\epsilon,\delta}(v_2)(t,x)\|_{\mathcal{L}(H)} \\
&\le c\,\epsilon^{\frac{\vartheta}{2}}(t\wedge 1)^{\frac{\vartheta}{2}}\mathfrak{l}_{\epsilon,\delta}(R,t)\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T} \le c\,\mathfrak{l}_{\epsilon,\delta}(R,t)\,\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T}\|h\|_H^{\vartheta}.
\end{aligned}
\tag{5.28}
$$

On the other hand, if we assume that $\|h\|_H^2 \le \epsilon\,t/2$ we write

$$
D^2\Gamma^g_{\epsilon,\delta}(v_1)(t,x) - D^2\Gamma^g_{\epsilon,\delta}(v_2)(t,x) = a_{\epsilon,\delta}(h,t,x) + b_{\epsilon,\delta}(h,t,x),
$$

where

$$
a_{\epsilon,\delta}(h,t,x) := \int_0^{\epsilon^{-1}\|h\|_H^2} D^2R^\epsilon_s\left(\gamma_{\epsilon,\delta}(v_1,t-s) - \gamma_{\epsilon,\delta}(v_2,t-s)\right)\,ds,
$$

and

$$
b_{\epsilon,\delta}(h,t,x) =: \int_{\epsilon^{-1}\|h\|_H^2}^t D^2R^\epsilon_s\left(\gamma_{\epsilon,\delta}(v_1,t-s) - \gamma_{\epsilon,\delta}(v_2,t-s)\right)\,ds.
$$

Then, thanks to (5.10) and (5.12), we can proceed as in Step 3 of the proof of Lemma 5.3 and we obtain that (5.28) holds also when $\epsilon^{-1}\|h\|_H^2 \le t/2$. In particular, we obtain that

$$
\epsilon^{\varrho+\frac{1}{2}}(t\wedge 1)^{\varrho+\frac{1}{2}}\,[D^2\Gamma^g_{\epsilon,\delta}(v_1)(t) - D^2\Gamma^g_{\epsilon,\delta}(v_2)(t)]_\vartheta \le c\,\mathfrak{l}_{\epsilon,\delta}(R,t)\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T},
$$

so that

$$
I_{\delta,4}(\epsilon) \le c\,\mathfrak{l}_{\epsilon,\delta}(R,t)\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T}. \tag{5.29}
$$

Therefore, if we combine (5.25), (5.26), (5.27) and (5.29), we obtain that

$$
\|\Gamma^g_{\epsilon,\delta}(v_1) - \Gamma^g_{\epsilon,\delta}(v_2)\|_{\epsilon,\varrho,\eta,\vartheta,T} \le c\,\mathfrak{l}_{\epsilon,\delta}(R,t)\|v_1-v_2\|_{\epsilon,\varrho,\eta,\vartheta,T}.
$$

This means that if we first choose $\delta_1 \le \delta'$ such that

$$
c\,\delta_1\,(1+R)^2 < \frac{1}{2},
$$

and then $T_1 \leq T'$ such that

$$c\,\epsilon^{-\frac{1}{2}}\,(T_1 \wedge 1)^{\frac{1}{2}} < \frac{1}{2},$$

we can conclude that $\Gamma_{\delta,g}$ maps $\mathcal{Y}^R_{\varrho,\eta,\vartheta,T_1}$ into itself as a contraction, for every $\delta \leq \delta_1$. □

6. Further properties of mild solutions of the quasi-linear problem

We will show that any mild solution u_ϵ of equation (3.20) that belongs to $C_{\epsilon,\varrho,\eta}((0,T];C_b^{2+\vartheta}(H))$ is in fact a classical solution in the sense of Definition 2. Moreover, by using its probabilistic interpretation in terms of equation (3.23), we will prove that a maximum principle holds for equation (3.20). This will imply that the local mild solution we have found in Section 5 is the unique global classical solution of Theorem 3.4.

We start by proving that $QD_x^2u_\epsilon(t,x)$ is a trace-class operator.

Lemma 6.1. *For every $t \in (0,T]$ and $x \in H$, we have that $QD_x^2u_\epsilon(t,x) \in \mathcal{L}_1(H)$.*

Proof. If u_ϵ is a mild solution, with the notations we have introduced in Section 5 we have

$$u_\epsilon(t,x) = R^\epsilon_t g(x) + \Gamma_{\epsilon,\delta}(u_\epsilon)(t,x).$$

According to (2.17) we have that $QD_x^2R^\epsilon_t g(x) \in \mathcal{L}_1(H)$, for every $\epsilon \in (0,1)$, $t > 0$ and $x \in H$, and thanks to (2.17) and (3.14)

$$\|QD_x^2R^\epsilon_t g\|_0 \leq \kappa_{\epsilon,\eta}(t/2)\,\|g\|_\eta.$$

Here $\kappa_{\epsilon,\eta}$ is the constant defined in (3.14) with Q replaced by $\sqrt{\epsilon}Q$. As far as $\Gamma_{\epsilon,\delta}(u_\epsilon)$ is concerned, if $R = \|u_\epsilon\|_{\epsilon,\varrho,\eta,\vartheta,T}$, thanks to (2.17), (3.14) and (5.9), we have

$$\begin{aligned}
&\|QD^2\Gamma_{\epsilon,\delta}(u_\epsilon)(t,x)\|_{\mathcal{L}_1(H)}\\
&\leq \int_0^t \|QD_x^2R^\epsilon_{t-s}\gamma_{\epsilon,\delta}(u_\epsilon,s)(x)\|_{\mathcal{L}_1(H)}\,ds \leq c\int_0^t \kappa_{\epsilon,\vartheta}((t-s)/2)\|\gamma_{\epsilon,\delta}(u_\epsilon,s)\|_\vartheta\,ds\\
&\leq c\,e^{-\beta_\vartheta}\int_0^t ((t-s)/2\wedge 1)^{-\beta_\vartheta}e^{-\alpha_\vartheta\frac{(t-s)}{2}}\epsilon^{-\varrho+\frac{1}{2}}(s\wedge 1)^{-(\varrho+\frac{1}{2})}\lambda_{\epsilon,\delta}(R,s)\,ds \leq c_\epsilon(R,T).
\end{aligned}$$

This allows to conclude that $QD_x^2u_\epsilon(t,x) \in \mathcal{L}_1(H)$ for every $\epsilon \in (0,1)$, $t \in (0,T]$ and $x \in H$. □

Next, we show that u_ϵ is differentiable with respect to $t \in (0,T]$ and $x \in D(A)$ and is a classical solution of equation (3.20). In Subsection 2.3, we have seen that for every $\varphi \in B_b(H)$ and $x \in D(A)$ the mapping

$$t \in (0,+\infty) \mapsto R^\epsilon_t \varphi(x) \in \mathbb{R},$$

is differentiable and

$$D_t R^\epsilon_t \varphi(x) = \mathcal{L}_\epsilon R^\epsilon_t \varphi(x).$$

See (2.18) and (2.19). Hence, since

$$\begin{aligned} u_\epsilon(t,x) &= R^\epsilon_t g(x) + \int_0^t R^\epsilon_{t-s}\left(\frac{\epsilon}{2}\mathrm{Tr}\left[\delta\, F(u_\epsilon(s,\cdot))D^2_x u_\epsilon(s,\cdot)\right] + \langle b(\cdot), Du_\epsilon(s,\cdot)\rangle_H\right)(x)\, ds \\ &= R^\epsilon_t g(x) + \Gamma_{\epsilon,\delta}(u_\epsilon)(t,x), \end{aligned}$$

thanks to Lemma 6.1, for every $x \in D(A)$ we can differentiate both sides with respect to $t > 0$, and we get

$$\begin{aligned} D_t u_\epsilon(t,x) &= \mathcal{L}_\epsilon R^\epsilon_t g(x) + \frac{\epsilon}{2}\mathrm{Tr}\left[\delta\, F(u_\epsilon(t,x))D^2_x u_\epsilon(t,x)\right] + \langle b(x), Du_\epsilon(t,x)\rangle_H \\ &\quad + \mathcal{L}_\epsilon \Gamma_{\epsilon,\delta}(u_\epsilon)(t,x) \\ &= \mathcal{L}_\epsilon u_\epsilon(t,x) + \frac{\epsilon}{2}\mathrm{Tr}\left[\delta\, F(u_\epsilon(t,x))D^2_x u_\epsilon(t,x)\right] + \langle b(x), Du_\epsilon(t,x)\rangle_H. \end{aligned}$$

Thus, we have proven the following result.

Theorem 6.2. *Under Hypotheses 1 to 4, if u_ϵ is a mild solution of equation (3.20) that belongs to $C_{\epsilon,\varrho,\eta}((0,T];C^{2+\vartheta}_b(H))$, then it is a classical solution.*

Next, we show how any solution of equation (3.20) is related to the stochastic PDE (3.23).

Theorem 6.3. *Assume Hypotheses 1 to 4. Then if $u_\epsilon \in C_{\epsilon,\varrho,\eta}((0,T];C^{2+\vartheta}_b(H))$ is a solution of equation (3.20) and $X^{t,x}_\epsilon \in L^2(\Omega;C([0,t];H))$ is a solution of equation (3.23), we have*

$$u_\epsilon(t,x) = \mathbb{E}g(X^{t,x}_\epsilon(t)). \tag{6.1}$$

Proof. The natural way to prove (6.1) is by applying the Itô formula to the function $(s,x) \in [0,t] \times H \mapsto u_\epsilon(t-s,x)$ and to the process $X^{t,x}_\epsilon(s)$. However, we cannot do this directly first because u_ϵ satisfies equation (3.20) in classical sense only for $x \in D(A)$ and second because $X^{t,x}_\epsilon$ is only a mild solution of equation (3.23), and not a strong solution,

as required when Itô's formula is used. To overcome these difficulties, we introduce a suitable approximation of u_ϵ and $X_\epsilon^{t,x}$, by adapting an argument introduced in [7, Proof of Theorem 9.25].

For every $m \in \mathbb{N}$ we define $J_m = m(m-A)^{-1}$ and

$$u_{\epsilon,m}(t,x) = u_\epsilon(t, J_m x), \qquad (t,x) \in [0,T] \times H.$$

Since $J_m x \to x$, as $m \to \infty$, and $u_\epsilon \in C_{\epsilon,\varrho,\eta}((0,T]; C_b^{2+\vartheta}(H))$, we have that

$$\lim_{m\to\infty} \sup_{t\in[0,T]} |u_{\epsilon,m}(t,x) - u_\epsilon(t,x)| = \lim_{m\to\infty} \sup_{t\in[0,T]} |u_{\epsilon,m}(t,J_m x) - u_\epsilon(t,x)| = 0, \qquad x \in H. \tag{6.2}$$

For all the details about the Yoshida approximants J_m we refer to [19]. Moreover,

$$D_x u_{\epsilon,m}(t,x) = J_m^\star D_x u_\epsilon(t, J_m x), \qquad D_x^2 u_{\epsilon,m}(t,x) = J_m^\star D_x^2 u_\epsilon(t, J_m x) J_m. \tag{6.3}$$

Next, for every $m \in \mathbb{N}$ we introduce the stochastic PDE

$$\begin{cases} dX_{\epsilon,m}^{t,x}(s) = \left[AX_{\epsilon,m}^{t,x}(s) + J_m b(X_{\epsilon,m}^{t,x}(s))\right] ds + \sqrt{\epsilon} J_m \Sigma_t(s, X_{\epsilon,m}^{t,x}(s))\, dW_s^m, \\ X^{t,x}(0) = J_m x, \end{cases} \tag{6.4}$$

where Σ_t is the operator introduced in (4.2) and W_t^m is the projection of the cylindrical Wiener process W_t onto $H_m := \mathrm{span}\{e_1, \dots, e_m\}$. By proceeding as in Section 4, we can prove that equation (6.4) admits a unique mild solution $X_{\epsilon,m}^{t,x} \in L^2(\Omega; C([0,t]; H))$. Since J_m maps H into $D(A)$ and W_t^m is a finite dimensional noise, it is immediate to check that $X_{\epsilon,m}^{t,x}$ is a strong solution. Namely, writing $X_{\epsilon,m}^{t,x}$ in its mild form,

$$X_{\epsilon,m}^{t,x}(s) = e^{sA} J_m x + \int_0^s e^{(s-r)A} J_m b(X_{\epsilon,m}^{t,x}(r))\, dr + \int_0^s e^{(s-r)A} \sqrt{\epsilon}\, J_m \Sigma_t(r, X_{\epsilon,m}^{t,x}(r))\, dW_r^m$$

we see that all the terms lives in $D(A)$.

At the end of this section we will prove that

$$\lim_{m\to\infty} \sup_{s\in[0,t]} \mathbb{E}\|J_m X_{\epsilon,m}^{t,x}(s) - X_\epsilon^{t,x}(s)\|_H = \lim_{m\to\infty} \sup_{s\in[0,t]} \mathbb{E}\|X_{\epsilon,m}^{t,x}(s) - X_\epsilon^{t,x}(s)\|_H = 0. \tag{6.5}$$

Now we apply Itô's formula to $u_{\epsilon,m}$ and $X_{\epsilon,m}^{t,x}$ and thanks to (6.3) we get

$$\begin{aligned} & d_s u_{\epsilon,m}(t-s, X_{\epsilon,m}^{t,x}(s)) \\ & = -D_t u_\epsilon(t-s, J_m X_{\epsilon,m}^{t,x}(s))\, ds \\ & + \frac{\epsilon}{2} \mathrm{Tr}\left[J_m^\star D_x^2 u_\epsilon(t-s, J_m X_{\epsilon,m}^{t,x}(s)) J_m (J_m \Sigma_t(s, X_{\epsilon,m}^{t,x}(s)))(J_m \Sigma_t(s, X_{\epsilon,m}^{t,x}(s)))^\star\right] ds \\ & + \langle A J_m X_{\epsilon,m}^{t,x}(s) + J_m^2 b(X_{\epsilon,m}^{t,x}(s)), D_x u_\epsilon(t-s, J_m X_{\epsilon,m}^{t,x}(s))\rangle_H\, ds \end{aligned}$$

$$+\sqrt{\epsilon}\langle J_m^2\Sigma_t(s,X^{t,x}_{\epsilon,m}(s))\,dW^m_s, D_x u_\epsilon(t-s,J_mX^{t,x}_{\epsilon,m}(s))\rangle_H.$$

Therefore, recalling that $u_\epsilon(t,x)$ satisfies equation (5.1), for every $x\in D(A)$, since $J_mX^{t,x}_{\epsilon,m}(s)\in D(A)$ we have

$$\begin{aligned} &d_s u_{\epsilon,m}(t-s,X^{t,x}_{\epsilon,m}(s))\\ &=\sqrt{\epsilon}\langle J_m^2\Sigma_t(s,X^{t,x}_{\epsilon,m}(s))\,dW^m_s, D_x u_\epsilon(t-s,J_mX^{t,x}_{\epsilon,m}(s))\rangle_H+[I^\epsilon_{m,1}(s)+I^\epsilon_{m,2}(s)]\,ds, \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} I^\epsilon_{m,1}(s)&:=\frac{\epsilon}{2}\mathrm{Tr}\left[J_m^\star D_x^2u_\epsilon(t-s,J_mX^{t,x}_{\epsilon,m}(s))J_m(J_m\Sigma_t(s,X^{t,x}_{\epsilon,m}(r)))(J_m\Sigma_t(s,X^{t,x}_{\epsilon,m}(s)))^\star\right]\\ &-\frac{\epsilon}{2}\mathrm{Tr}\left[D_x^2u_\epsilon(t-s,J_mX^{t,x}_{\epsilon,m}(s))(\Sigma_t(s,J_mX^{t,x}_{\epsilon,m}(s)))(\Sigma_t(s,J_mX^{t,x}_{\epsilon,m}(s)))^\star\right], \end{aligned}$$

and

$$I^\epsilon_{m,2}(s):=\langle J_m^2b(X^{t,x}_{\epsilon,m}(s))-b(J_mX^{t,x}_{\epsilon,m}(s)),D_xu_\epsilon(t-s,J_mX^{t,x}_{\epsilon,m}(s))\rangle_H.$$

If we take the expectation of both sides in (6.6) and integrate with respect to $s\in[0,t]$ we get

$$\mathbb{E}g(J_mX^{t,x}_{\epsilon,m}(t))=u_{\epsilon,m}(t,J_mx)+\int_0^t\mathbb{E}\left(I^\epsilon_{m,1}(s)+I^\epsilon_{m,2}(s)\right)\,ds. \tag{6.7}$$

In view of (6.2) and (6.5), recalling that g is bounded, we have that

$$\lim_{m\to\infty}\mathbb{E}g(J_mX^{t,x}_{\epsilon,m}(t))=\mathbb{E}g(X^{t,x}_\epsilon(t)),\qquad\qquad \lim_{m\to\infty}u_{\epsilon,m}(t,J_mx)=u_\epsilon(t,x).$$

Moreover, since $u_\epsilon\in C_{\epsilon,\varrho,\eta}((0,T];C_b^{2+\vartheta}(H))$, by using again (6.2) and (6.5), since we have pointwise and dominated convergence as $m\to+\infty$ of all terms appearing in $I^\epsilon_{m,1}$ and $I^\epsilon_{m,2}$, we get

$$\lim_{m\to\infty}\int_0^t\mathbb{E}\left(|I^\epsilon_{m,1}(s)|+|I^\epsilon_{m,2}(s)|\right)\,ds=0.$$

Therefore, if we take the limit of both sides in (6.7), as $m\to\infty$, we obtain (6.1). □

Remark 6.4. Thanks to the representation formula (6.1) of u_ϵ, we have that

$$\sup_{t\in[0,T]}\|u_\epsilon(t,\cdot)\|_0\le\|g\|_0. \tag{6.8}$$

Now, we conclude this section with the proof of (6.5).

Lemma 6.5. *If $X^{t,x}_{\epsilon,m}$ is the solution of problem (6.4), we have*

$$\lim_{m\to\infty}\sup_{s\in[0,t]}\mathbb{E}\|J_mX^{t,x}_{\epsilon,m}(s)-X^{t,x}_{\epsilon}(s)\|^2_H=\lim_{m\to\infty}\sup_{s\in[0,t]}\mathbb{E}\|X^{t,x}_{\epsilon,m}(s)-X^{t,x}_{\epsilon}(s)\|^2_H=0. \quad (6.9)$$

Proof. If we denote $\rho_{\epsilon,m}(s):=X^{t,x}_{\epsilon,m}(s)-X^{t,x}_{\epsilon}(s)$ and $\hat{W}^m_r:=W_r-W^m_r$, we have

$$\begin{aligned}\rho_{\epsilon,m}(s)=e^{sA}(J_mx-x)&+\int_0^s e^{(s-r)A}\left(J_mb(X^{t,x}_{\epsilon,m}(r))-b(X^{t,x}_{\epsilon}(r))\right)dr\\&+\sqrt{\epsilon}\int_0^s e^{(s-r)A}\left(J_m\Sigma_t(r,X^{t,x}_{\epsilon,m}(r))-\Sigma_t(r,X^{t,x}_{\epsilon}(r))\right)dW^m_r\\&+\sqrt{\epsilon}\int_0^s e^{(s-r)A}\Sigma_t(r,X^{t,x}_{\epsilon}(r))d\hat{W}^m_r.\end{aligned}$$

Therefore, since $\|J_m\|_{\mathcal{L}(H)}\le 1$, we have

$$\begin{aligned}\mathbb{E}\|\rho_{\epsilon,m}(s)\|^2_H\le c\|J_mx-x\|^2_H&+c_t\int_0^s\mathbb{E}\|\rho_{\epsilon,m}(r)\|^2_H\,dr\\&+c_t\int_0^s\mathbb{E}\|J_mb(X^{t,x}_{\epsilon}(r))-b(X^{t,x}_{\epsilon}(r))\|^2_H\,dr\\&+\epsilon\,c\int_0^s\mathbb{E}\|e^{(s-r)A}J_m\left(\Sigma_t(r,X^{t,x}_{\epsilon,m}(r))-\Sigma_t(r,X^{t,x}_{\epsilon}(r))\right)\|^2_{\mathcal{L}_2(H)}\,dr\\&+\epsilon\,c\int_0^s\mathbb{E}\|J_me^{(s-r)A}\Sigma_t(r,X^{t,x}_{\epsilon}(r))-e^{(s-r)A}\Sigma_t(r,X^{t,x}_{\epsilon}(r))\|^2_{\mathcal{L}_2(H)}\,dr\\&+\epsilon\,c\int_0^s\mathbb{E}\|e^{(s-r)A}\Sigma_t(r,X^{t,x}_{\epsilon}(r))S_m\|^2_{\mathcal{L}_2(H)}\,dr=:\sum_{i=1}^6 I^{\epsilon}_{m,i}(s),\end{aligned}$$

where $S_m:=I-P_m$ is the projection of H onto $\mathrm{span}\{e_{m+1},e_{m+2},\dots\}$.

By proceeding as in Section 4, we have that

$$I^{\epsilon}_{m,4}(s)\le c(\epsilon)\int_0^s((s-r)\wedge 1)^{-\frac{1}{2}}\left(1+((t-r)\wedge 1)^{-2\varrho}\right)\mathbb{E}\|\rho_{\epsilon,m}(r)\|^2_H\,dr,$$

so that

$$\mathbb{E}\|\rho_{\epsilon,m}(s)\|_H^2 \le c(\epsilon)\int_0^s ((s-r)\wedge 1)^{-(2\varrho+\frac{1}{2})}\mathbb{E}\|\rho_{\epsilon,m}(r)\|_H^2\,dr + \Lambda_{\epsilon,m}(s),$$

where

$$\Lambda_{\epsilon,m}(s) := I^\epsilon_{m,1}(s) + I^\epsilon_{m,3}(s) + I^\epsilon_{m,5}(s) + I^\epsilon_{m,6}(s).$$

Since $2\varrho + 1/2 < 1$, thanks to a generalized Gronwall's inequality (see [20, Theorem 1]), this implies that

$$\mathbb{E}\|\rho_{\epsilon,m}(s)\|_H^2 \le c_{\epsilon,t}\,\Lambda_{\epsilon,m}(s) \le c_{\epsilon,t}\,\Lambda_{\epsilon,m}(t), \quad s\in[0,t],$$

and (6.9) follows if we can prove that

$$\lim_{m\to\infty}\Lambda_{\epsilon,m}(t) = 0. \tag{6.10}$$

It is immediate to check that

$$\lim_{m\to\infty} I^\epsilon_{m,1} + I^\epsilon_{m,3}(t) = 0. \tag{6.11}$$

Moreover, according to Hypothesis 3 and to the fact that u is bounded in $[0,t]\times H$, for every $m\in\mathbb{N}$ we have

$$\begin{aligned}&\|J_m e^{(t-r)A}\Sigma_t(r, X^{t,x}_\epsilon(r)) - e^{(t-r)A}\Sigma_t(r, X^{t,x}_\epsilon(r))\|_{\mathcal{L}_2(H)}\\ &\le 2\,\|e^{(t-r)A}\sigma(X^{t,x}_\epsilon(r), u_\epsilon(t-r, X^{t,x}_\epsilon(r))\|_{\mathcal{L}_2(H)} \le c\,(t-r)^{-\frac{1}{4}}\left(1+\|X^{t,x}_\epsilon(r)\|_H\right).\end{aligned}$$

Then, since

$$\lim_{m\to\infty}\|J_m e^{(t-r)A}\Sigma_t(r, X^{t,x}_\epsilon(r)) - e^{(t-r)A}\Sigma_t(r, X^{t,x}_\epsilon(r))\|_{\mathcal{L}_2(H)} = 0,$$

and since the mapping

$$s\in[0,t]\mapsto (t-s)^{-\frac{1}{4}}\left(1+\|X^{t,x}_\epsilon(s)\|_H\right)\in\mathbb{R},$$

belongs to $L^2(\Omega; L^2([0,t]))$, by the dominated convergence theorem we have that

$$\lim_{m\to\infty} I^\epsilon_{m,5}(t) = 0. \tag{6.12}$$

In the same way, by the dominated convergence theorem we have also that

$$\lim_{m\to\infty} I^\epsilon_{m,6}(t) = 0. \tag{6.13}$$

Therefore, combining together (6.11), (6.12) and (6.13), we obtain (6.10) and we get

$$\lim_{m\to\infty}\sup_{s\in[0,t]}\mathbb{E}\|X^{t,x}_{\epsilon,m}(s)-X^{t,x}_{\epsilon}(s)\|^2_H=0. \tag{6.14}$$

Moreover, since $\|J_m\|_{\mathcal{L}(H)}\le 1$, we have

$$\|J_mX^{t,x}_{\epsilon,m}(s)-X^{t,x}_{\epsilon}(s)\|_H\le\|X^{t,x}_{\epsilon,m}(s)-X^{t,x}_{\epsilon}(s)\|_H+\|J_mX^{t,x}_{\epsilon}(s)-X^{t,x}_{\epsilon}(s)\|_H,$$

and due to (6.14), we have

$$\limsup_{m\to\infty}\sup_{s\in[0,t]}\mathbb{E}\,\|J_mX^{t,x}_{\epsilon,m}(s)-X^{t,x}_{\epsilon}(s)\|^2_H\le 2\limsup_{m\to\infty}\sup_{s\in[0,t]}\mathbb{E}\,\|J_mX^{t,x}_{\epsilon}(s)-X^{t,x}_{\epsilon}(s)\|^2_H.$$

Now, since

$$f_m(s):=\mathbb{E}\,\|J_mX^{t,x}_{\epsilon}(s)-X^{t,x}_{\epsilon}(s)\|^2_H,\quad s\in[0,t],\quad m\in\mathbb{N},$$

defines an equicontinuous sequence of functions, pointwise converging to zero, we have that they converge to zero uniformly for $s\in[0,t]$ and we can conclude that (6.9) holds. □

7. Existence and uniqueness of global classical solutions for the quasi-linear problem

In Theorem 5.5 we have proved that for every $\eta\in(1/2,1)$ and $\vartheta\in(0,\eta-1/2)$, there exist $\delta_1,T_1>0$ such that problem (3.20) has a mild solution u_ϵ in $C_{\epsilon,\varrho,\eta}((0,T_1];C^{2+\vartheta}_b(H))$. In Section 6 we have shown that such mild solution is in fact a classical solution. Our purpose here is first proving that u_ϵ is defined on the interval $[0,T]$, for every $T>0$, and then proving that it is the unique solution.

We start with the following a-priori bound.

Lemma 7.1. *There exists $\delta_2\in(0,\delta_1]$, that depends only on $\|g\|_\eta$, such that if u_ϵ is a mild solution of (3.20) for some $\delta\le\delta_2$, belonging to $C_{\epsilon,\varrho,\eta}((0,T];C^{2+\vartheta}_b(H))$, with ϱ given in (5.21), then*

$$\|u_\epsilon\|_{\epsilon,\varrho,\eta,\vartheta,T}\le c_{\epsilon,\delta}\,\|g\|_\eta,\qquad \epsilon\in(0,1), \tag{7.1}$$

for some constant $c_{\epsilon,\delta}$ independent of $T>0$.

Proof. In what follows, for any function $v:[0,T]\times H\to\mathbb{R}$ we define

$$N_\epsilon(v(t)):=[v(t,\cdot)]_\eta+\epsilon^\varrho(t\wedge 1)^\varrho\|Dv(t,\cdot)\|_\vartheta+\epsilon^{\varrho+\frac12}(t\wedge1)^{\varrho+\frac12}\|D^2v(t,\cdot)\|_\vartheta.$$

With the notations we have introduced in Section 5, thanks to (6.8) we have

$$\|u_\epsilon(t,\cdot)\|_0 + N_\epsilon(u_\epsilon(t)) \le \|g\|_0 + N_\epsilon(R^\epsilon_t g) + N_\epsilon(\Gamma^\epsilon_{\delta,1}(u_\epsilon)(t)) + N_\epsilon(\Gamma^\epsilon_2(u_\epsilon)(t)), \tag{7.2}$$

where

$$\Gamma^\epsilon_{\delta,1}(u)(t,x) := \int_0^t R^\epsilon_{t-s}\,\gamma^\epsilon_{\delta,1}(u,s)(x)\,ds := \frac{\epsilon}{2}\int_0^t R^\epsilon_{t-s}\mathrm{Tr}\left[\delta F(u)(s,\cdot)D^2u(s,\cdot)\right](x)\,ds$$

and

$$\Gamma^\epsilon_2(u)(t,x) := \int_0^t R^\epsilon_{t-s}\gamma_2(u,s)(x)\,ds := \int_0^t R^\epsilon_{t-s}\langle b, Du(s,\cdot)\rangle_H(x)\,ds.$$

In (5.24) we have already shown that

$$\sup_{t\in[0,T]} N_\epsilon(R^\epsilon_t g) \le c\,\|g\|_\eta. \tag{7.3}$$

Thus, in order to prove (7.1) we need to estimate $N_\epsilon(\Gamma^\epsilon_{\delta,1}(u_\epsilon)(t))$ and $N_\epsilon(\Gamma^\epsilon_2(u_\epsilon)(t))$.

Thanks to (3.8) and (6.8), we have

$$\|F(u_\epsilon(s,\cdot))\|_0 \le c\,(1+\|u_\epsilon(s,\cdot)\|_0) \le c\,(1+\|g\|_0)\,,$$

and then

$$\|\gamma^\epsilon_{\delta,1}(u_\epsilon,s)\|_0 \le c\,\epsilon\,\delta\|F(u_\epsilon(s,\cdot))\|_0\|D^2u_\epsilon(s,\cdot)\|_0 \le c\,\epsilon\,\delta\,(1+\|g\|_0)\,\|D^2u_\epsilon(s,\cdot)\|_0. \tag{7.4}$$

Moreover, due to (3.9) and (6.8) we have

$$[F(u_\epsilon(s,\cdot))]_\vartheta \le c\,(1+\|u_\epsilon(s,\cdot)\|_0 + [u_\epsilon(s,\cdot)]_\vartheta) \le c\,(1+\|g\|_0 + [u_\epsilon(s,\cdot)]_\vartheta)\,,$$

so that

$$\begin{aligned}[\gamma^\epsilon_{\delta,1}(u_\epsilon,s)]_\vartheta &\le c\,\epsilon\,\delta\,[F(u_\epsilon(s,\cdot))]_\vartheta\|D^2u_\epsilon(s,\cdot)\|_0 + c\,\epsilon\,\delta\|F(u_\epsilon(s,\cdot))\|_0[D^2u_\epsilon(s,\cdot)]_\vartheta\\ &\le c\,\epsilon\,\delta\,(1+\|g\|_0)\,\|D^2u_\epsilon(s,\cdot)\|_\vartheta + c\,\epsilon\,\delta\,[u_\epsilon(s,\cdot)]_\vartheta\|D^2u_\epsilon(s,\cdot)\|_0.\end{aligned}$$

According to (2.8) and (6.8), this implies

$$\begin{aligned}[\gamma^\epsilon_{\delta,1}(u_\epsilon,s)]_\vartheta &\le c\,\epsilon\,\delta\,(1+\|g\|_0)\,\|D^2u_\epsilon(s,\cdot)\|_\vartheta + c\,\epsilon\,\delta\,\|u_\epsilon(s,\cdot)\|_0[D^2u_\epsilon(s,\cdot)]_\vartheta\\ &\le c\,\epsilon\,\delta\,(1+\|g\|_0)\,\|D^2u_\epsilon(s,\cdot)\|_\vartheta.\end{aligned} \tag{7.5}$$

Therefore, if we combine together (7.4) and (7.5) we conclude that

$$\|\gamma^\epsilon_{\delta,1}(u_\epsilon,s)\|_\vartheta \le c\,\epsilon\,\delta\,(1+\|g\|_0)\,\|D^2u_\epsilon(s,\cdot)\|_\vartheta \le c\,\epsilon\,\delta\,(1+\|g\|_0)\,\epsilon^{-(\varrho+\frac12)}(s\wedge 1)^{-(\varrho+\frac12)}\,N_\epsilon(u_\epsilon(s,\cdot)).$$

By proceeding as in the proof of Lemma 5.3 (see also Remark 5.4), this allows to conclude

$$N_\epsilon(\Gamma^\epsilon_{\delta,1}(u_\epsilon)(t)) \le c\,\delta\,(1+\|g\|_0)\sup_{s\in[0,t]} N_\epsilon(u_\epsilon(s,\cdot)), \qquad t\in[0,T]. \tag{7.6}$$

Now, let us estimate $N_\epsilon(\Gamma^\epsilon_2(u_\epsilon)(t))$. We have, by direct computations taking into account Hypothesis 4,

$$\|\gamma_2(u_\epsilon,t)\|_1 \le c\,\left(\|Du_\epsilon(t,\cdot)\|_0+\|D^2u_\epsilon(t,\cdot)\|_0\right).$$

Thus, according to (2.6) and (2.9) and thanks to Young inequality for every $\ell>0$ there exists $\kappa_\ell>0$ such that

$$\begin{aligned}\|\gamma_2(u_\epsilon,t)\|_1 &\le c(\|u_\epsilon(t,\cdot)\|_0^{\frac{1+\vartheta}{2+\vartheta}}[D^2u_\epsilon(t,\cdot)]_\vartheta^{\frac{1}{2+\vartheta}}+\|u_\epsilon(t,\cdot)\|_0^{\frac{\vartheta}{2+\vartheta}}[D^2u_\epsilon(t,\cdot)]_\vartheta^{\frac{2}{2+\vartheta}})\\ &\le \ell\,[D^2u_\epsilon(t,\cdot)]_\vartheta+\kappa_\ell\|u_\epsilon(t,\cdot)\|_0.\end{aligned} \tag{7.7}$$

In view of (5.3), (5.5), (6.8) and (7.7), there exists some $\lambda_\vartheta>0$ and $\hat\kappa_\ell$ such that for every $t\in[0,T]$

$$\begin{aligned}N_\epsilon(\Gamma^\epsilon_2(u_\epsilon)(t)) &\le \ell\,c_{\eta,1}\int_0^t((t-s)\wedge1)^{-\varsigma}e^{-\lambda_\vartheta(t-s)}[D^2u_\epsilon(s,\cdot)]_\vartheta\,ds\\ &+\ell\,c_{1,\theta,1}\,\epsilon^\varrho\,(t\wedge1)^\varrho\int_0^t e^{-\lambda_\vartheta(t-s)}\epsilon^{-\frac{\vartheta}{2}}((t-s)\wedge1)^{-\frac{\vartheta}{2}}[D^2u_\epsilon(s,\cdot)]_\vartheta\,ds\\ &+\ell\,c_{2,\theta,1}\,\epsilon^{\varrho+\frac12}(t\wedge1)^{\varrho+\frac12}\int_0^t e^{-\lambda_\vartheta(t-s)}\epsilon^{-\frac{1+\vartheta}{2}}((t-s)\wedge1)^{-\frac{1+\vartheta}{2}}[D^2u_\epsilon(s,\cdot)]_\vartheta\,ds\\ &+\hat\kappa_\ell\int_0^t e^{-\lambda_\vartheta(t-s)}\epsilon^{-\frac{1+\vartheta}{2}}((t-s)\wedge1)^{-\frac{1+\vartheta}{2}}\,ds\|g\|_0\\ &\le c\,\ell\epsilon^{-(\varrho+1/2)}\sup_{s\in[0,t]}N_\epsilon(u_\epsilon(s,\cdot))+c\,\hat\kappa_\ell\epsilon^{-\frac{1+\vartheta}{2}}\,\|g\|_0\end{aligned} \tag{7.8}$$

Notice that in the last inequality we put $\epsilon^{-(\varrho+1/2)}$ because $\varrho+1/2>(1+\vartheta)/2$. Moreover, in the first inequality of (7.8) we have applied (5.5) with $\alpha=\eta$ and $\beta=\eta-\varsigma$ with ς arbitrarily small together with (7.7) in the first term (recall that $\|\cdot\|_1$ is stronger than $\|\cdot\|_\varsigma$), (5.3) with $n=1$ and $\rho=1$ in the second term and again (5.3) with $n=2$ and $\rho=1$ in the third term. Hence, if we plug (7.3), (7.6) and (7.8) into (7.2), we obtain

$$\|u_\epsilon(t,\cdot)\|_0 + N_\epsilon(u_\epsilon(t,\cdot))$$
$$\leq c\,\|g\|_\eta + c\left[\delta\,(1+\|g\|_0) + \ell\epsilon^{-(\varrho+1/2)}\right]\sup_{s\in[0,t]} N_\epsilon(u_\epsilon(s,\cdot)) + c\,\hat\kappa_\ell\epsilon^{-\frac{1+\vartheta}{2}}\,\|g\|_0,$$

where c only depends on the costants of Hypotheses 1, 2 and 3.

In particular if take $\ell = \delta\epsilon^{\varrho+1/2}$ and $\delta_2 \leq \delta_1$ such that

$$c\,\delta_2\,(2+\|g\|_0) < 1/2,$$

we obtain (7.1) for every $\delta \leq \delta_2$. □

7.1. Conclusion of the proof of Theorem 3.4

Thanks to (7.1), by standard arguments we have that for every $\epsilon \in (0,1)$ the local solution we found in Theorem 5.5 is in fact a global solution. Moreover, this global solution is unique. The arguments used to get a global solution from a local one and the arguments used to get uniqueness are quite similar and both rely on the a-priori bound (7.1). Even though they are well known in the literature, here, for the reader's convenience, we give the proof of uniqueness.

Indeed, if $u_1, u_2 \in C_{\epsilon,\varrho,\eta}((0,T];C_b^{2+\vartheta}(H))$ are two solutions of equation (5.1), for some fixed $\delta \leq \delta_2$, we assume that

$$t_0 := \sup\{t\in(0,T]\ :\ u_1(s) = u_2(s),\ s\in[0,t]\} < T.$$

With the same notations we have used in Section 5, we introduce the problem

$$u(t) = \Gamma^\varphi_{\epsilon,\delta}(u)(t) = R^\epsilon_t\varphi + \Gamma_{\epsilon,\delta}(u)(t), \qquad t \geq t_0, \tag{7.9}$$

where $\varphi := u_1(t_0) = u_2(t_0)$. Due to (7.1), we have that

$$\|\varphi\|_\eta \leq c_{\epsilon,\delta}\,\|g\|_\eta,$$

for some constant $c_{\epsilon,\delta} > 0$ independent of $T > 0$.

As shown in Section 5, there exist $\bar R$, $\bar\tau > 0$ and $\bar\delta \leq \delta_2$ such that the mapping $\Gamma^\varphi_{\epsilon,\delta}$ maps $\mathcal{Y}^{\epsilon,\bar R}_{\varrho,\eta,\vartheta,t_0,\bar\tau}$ into itself as a contraction, for every $\delta \leq \bar\delta$, where

$$\mathcal{Y}^{\epsilon,R}_{\varrho,\eta,\vartheta,t_0,\bar\tau} := \left\{\, u \in C_{\epsilon,\varrho,\eta}((t_0,t_0+\bar\tau];C_b^{2+\vartheta}(H))\ :\ \|u\|_{\epsilon,\varrho,\eta,\vartheta,t_0,\bar\tau} \leq \bar R\right\},$$

and $C_{\epsilon,\varrho,\eta}((t_0,t_0+\bar\tau];C_b^{2+\vartheta}(H))$ is the space of all functions u belonging to $C([t_0,t_0+\bar\tau];C_b^\eta(H)) \cap C((t_0,t_0+\bar\tau];C_b^{2+\vartheta}(H))$ such that the norm

$$\begin{aligned}&\|u\|_{\epsilon,\varrho,\eta,\vartheta,t_0,\bar\tau}\\&:=\sup_{t\in(t_0,t_0+\bar\tau]}\Big(\|u(t,\cdot)\|_\eta+\epsilon^\varrho((t-t_0)\wedge 1)^\varrho\|D_xu(t,\cdot)\|_\vartheta\\&\quad+\epsilon^{\varrho+\frac12}((t-t_0)\wedge 1)^{\varrho+\frac12}\|D_x^2u(t,\cdot)\|_\vartheta\Big)\end{aligned}$$

is finite.

In particular $\Gamma^\varphi_{\epsilon,\delta}$ has a unique fixed point in $\mathcal{Y}^{\epsilon,\bar R}_{\varrho,\eta,\vartheta,t_0,\bar\tau}$ or, equivalently, equation (7.9) has a unique solution on the interval $[t_0,t_0+\bar\tau]$. This implies that

$$u_1(s)=u_2(s),\qquad s\in[0,t_0+\bar\tau],$$

violating the definition of $[0,t_0]$ as the maximal interval where u_1 and u_2 coincide.

8. The large deviation principle

In this last section we give a proof of Theorem 3.6. We follow the well-known method based on weak convergence, as developed in [3]. To this purpose, we need to introduce some notations.

For every $t>0$, we denote by $L^2_w(0,T;H)$ the space $L^2(0,T;H)$ endowed with the weak topology, and by $\mathcal{P}_t$ the set of predictable processes in $L^2(\Omega\times[0,t];H)$, and for every $M>0$ we introduce the sets

$$\mathcal{S}_{t,M}:=\left\{\varphi\in L^2_w(0,t;H)\ :\ \|\varphi\|_{L^2(0,t;H)}\le M\right\},$$

and

$$\Lambda_{t,M}:=\left\{\varphi\in\mathcal{P}_t\ :\ \varphi\in\mathcal{S}_{t,M},\ \mathbb{P}-\text{a.s.}\right\}.$$

In Theorem 3.5 we have shown that for every $M,t>0$ and $\varphi\in\Lambda_{t,M}$ and for every $x\in H$ and $\epsilon\in(0,1)$ there exists a unique mild solution $X^{t,x}_{\varphi,\epsilon}\in L^2(\Omega;C([0,t];H))$ for equation (3.23).

Next, we consider the problem

$$\frac{dX}{ds}(s)=AX(s)+b(X(s))+\sigma(X(s),g(Z^{X(s)}(t-s)))\varphi(s),\qquad X(0)=x,\tag{8.1}$$

where, as we did in Section 3, for every $y\in H$ we denote by Z^y the solution of equation (3.26). We recall that $X\in C([0,t];H)$ is a mild solution for equation (8.1) if

$$X(s)=e^{sA}x+\int_0^s e^{(s-r)A}b(X(r))\,dr+\int_0^s e^{(s-r)A}\Sigma_t(X(r),r)\,\varphi(r)\,dr,\qquad s\in[0,t],$$

where for every $y\in H$ and $s\in[0,t]$ we have defined

$$\Sigma_t(y,s) := \sigma(y, g(Z^y(t-s))).$$

In what follows, we show that the following result holds.

Proposition 8.1. *Assume that $g : H \to \mathbb{R}$ is Lipschitz-continuous. Then, under the same assumptions of Theorem 3.5, for every $t > 0$ and $\varphi \in L^2(0,t;H)$ and for every $x \in H$, there exists a unique mild solution $X^{t,x}_\varphi \in C([0,t];H)$ for equation (8.1).*

Once proved Theorem 3.5 (see Section 4) and Proposition 8.1, we introduce the following two conditions.

C1. Let $\{\varphi_\epsilon\}_{\epsilon>0}$ be an arbitrary family of processes in $\Lambda_{t,M}$ such that

$$\lim_{\epsilon\to 0} \varphi_\epsilon = \varphi, \qquad \text{in distribution in} \quad L^2_w(0,t;H),$$

where $L^2_w(0,t;H)$ is the space $L^2(0,t;H)$ endowed with the weak topology and $\varphi \in \Lambda_{t,M}$. Then we have

$$\lim_{\epsilon\to 0} X^{t,x}_{\varphi_\epsilon,\epsilon} = X^{t,x}_\varphi, \qquad \text{in distribution} \quad C([0,t],H).$$

C2. For every $t, R > 0$, the level sets $\Phi_{t,R} = \{I_{t,x} \le R\}$ are compact in the space $C([0,t];H)$, where we recall that $I_{t,x}$ is the functional defined in (3.28).

As shown in [3], Conditions C1. and C2. imply that the family $\{X^{t,x}_\epsilon\}_{\epsilon\in(0,1)}$ satisfies a Laplace principle with action functional $I_{t,x}$ in the space $C([0,t];H)$. As known, if $I_{t,x}$ has compact level sets, then the large deviation principle with action functional $I_{t,x}$ is equivalent to the Laplace principle with actional functional $I_{t,x}$. Hence, due to the compactness of the level sets $\Phi_{t,R}$ stated in C2, the proof of C1. and C2. is equivalent to the proof of Theorem 3.6.

8.1. Proof of Proposition 8.1

With the notation introduced above, a function in $C([0,t];H)$ is a mild solution for equation (8.1) if it is a fixed point of the mapping Λ_t defined for every $X \in C([0,t];H)$ by

$$\Lambda_t(X)(s) := e^{sA}x + \int_0^s e^{(s-r)A} b(X(r))\,dr + \int_0^s e^{(s-r)A}\Sigma_t(X(r),r)\,\varphi(r)\,dr, \qquad s \in [0,t].$$

It is immediate to check that there exists a continuous increasing function $\kappa(s)$ such that for every $y_1, y_2 \in H$

$$\|Z^{y_1}(s) - Z^{y_2}(s)\|_H \leq \kappa(s)\,\|y_1 - y_2\|_H, \qquad s \geq 0. \tag{8.2}$$

Hence, since we are assuming that $g : H \to \mathbb{R}$ is Lipschitz-continuous, according to Hypothesis 1 for every $y_1, y_2, h \in H$ we have

$$\|[\Sigma_t(y_1, r) - \Sigma_t(y_2, r)]h\|_H \leq c\,(1 + \kappa(t-r))\,\|y_1 - y_2\|_H \|h\|_H, \qquad r \in [0, t].$$

In particular, for every $X_1, X_2 \in C([0,t]; H)$ and $s \in [0,t]$ we have

$$\begin{aligned}\|\Lambda_t(X_1)(s) - \Lambda_t(X_2)(s)\|_H &\leq c \int_0^s (1 + (1 + \kappa(t-r))\,\|\varphi(r)\|_H)\,\|X_1(r) - X_2(r)\|_H\,dr \\ &\leq c_t\left(\|\varphi\|_{L^2(0,t;H)} + 1\right)\|X_1 - X_2\|_{C([0,t];H)}.\end{aligned}$$

This implies that $\Lambda_t : C([0,t]; H) \to C([0,t]; H)$ is Lipschitz continuous and by standard arguments we conclude that Λ_t has a unique fixed point.

8.2. Preliminary results

Lemma 8.2. *Under the same assumptions of Theorem 3.6, for every $p \geq 1$ we have*

$$\sup_{\epsilon \in (0,1)} \mathbb{E} \sup_{s \in [0,t]} \|X^{t,x}_{\varphi_\epsilon,\epsilon}(s)\|^p_H \leq c(t, M, p)\,(1 + \|x\|^p_H)\,. \tag{8.3}$$

Proof. We have

$$\begin{aligned}X^{t,x}_{\varphi_\epsilon,\epsilon}(s) := e^{sA}x &+ \int_0^s e^{(s-r)A} b(X^{t,x}_{\varphi_\epsilon,\epsilon}(r))\,ds + \int_0^s e^{(s-r)A}\Sigma_{t,\epsilon}(r, X^{t,x}_{\varphi_\epsilon,\epsilon}(r))\,\varphi_\epsilon(r)\,dr \\ &+ \sqrt{\epsilon}\int_0^s e^{(s-r)A}\Sigma_{t,\epsilon}(r, X^{t,x}_{\varphi_\epsilon,\epsilon}(r))\,dW_r,\end{aligned}$$

where $\Sigma_{t,\epsilon}$ is the operator defined in (4.2). Hence, for every $s \in [0,t]$ and $p \geq 1$ we have

$$\begin{aligned}\|X^{t,x}_{\varphi_\epsilon,\epsilon}(s)\|^p_H \leq c_p\,\|x\|^p_H &+ c_p \int_0^s \|X^{t,x}_{\varphi_\epsilon,\epsilon}(r)\|^p_H\,dr \\ &+ c_{p,M}\left(\int_0^s \|e^{(s-r)A}\Sigma_{t,\epsilon}(r, X^{t,x}_{\varphi_\epsilon,\epsilon}(r))\|^2_{\mathcal{L}(H)}\,dr\right)^{\frac{p}{2}} \\ &+ c_p\left\|\int_0^s e^{(s-r)A}\Sigma_{t,\epsilon}(r, X^{t,x}_{\varphi_\epsilon,\epsilon}(r))\,dW_r\right\|^p_H + c_{p,t}.\end{aligned} \tag{8.4}$$

According to Hypothesis 3, for every $\tau > 0$, $s \in [0,t]$ and $x \in H$ we have

$$\|e^{\tau A}\Sigma_{t,\epsilon}(s,x)\|^2_{\mathcal{L}_2(H)} \leq c\,(\tau \wedge 1)^{-\frac{1}{2}} \left(\|x\|^2_H + |u_\epsilon(t-s,x)|^2 + 1\right).$$

Moreover, according to (6.1), we have

$$\sup_{(s,x)\in[0,t]\times H} |u_\epsilon(s,x)| \leq \|g\|_0, \qquad \epsilon \in (0,1),$$

so that

$$\sup_{\epsilon\in(0,1)} \|e^{\tau A}\Sigma_{t,\epsilon}(s,x)\|^2_{\mathcal{L}_2(H)} \leq c\,(\tau \wedge 1)^{-\frac{1}{2}} \left(\|x\|^2_H + 1\right). \tag{8.5}$$

In particular, if $p > 4$

$$\begin{aligned} &\mathbb{E} \sup_{r\in[0,s]} \left(\int_0^r \|e^{(r-\rho)A}\Sigma_{t,\epsilon}(\rho, X^{t,x}_{\varphi_\epsilon,\epsilon}(\rho))\|^2_{\mathcal{L}(H)}\, d\rho\right)^{\frac{p}{2}} \\ &\quad \leq c_{p,t} \int_0^s \mathbb{E}\left(\|X^{t,x}_{\varphi_\epsilon,\epsilon}(r)\|^p_H + 1\right) dr. \end{aligned} \tag{8.6}$$

Now, if we fix $p > 4$, we can find $\alpha < 1/4$ such that $(\alpha-1)p/(p-1) > -1$. By using the stochastic factorization, we have

$$\int_0^s e^{(s-r)A}\Sigma_{t,\epsilon}(r, X^{t,x}_{\varphi_\epsilon,\epsilon}(r))\, dW_r = c_\alpha \int_0^s e^{(s-r)A}(s-r)^{\alpha-1} Y_{\alpha,\epsilon}(r)\, dr,$$

where

$$Y_{\alpha,\epsilon}(r) := \int_0^r e^{(r-\rho)A}(r-\rho)^{-\alpha}\Sigma_{t,\epsilon}(\rho, X^{t,x}_{\varphi_\epsilon,\epsilon}(\rho))\, dW_\rho.$$

Then, we obtain

$$\left\| \int_0^s e^{(s-r)A}\Sigma_{t,\epsilon}(r, X^{t,x}_{\varphi_\epsilon,\epsilon}(r))\, dW_r \right\|^p_H \leq c_{\alpha,p} \left(\int_0^s (s-r)^{\frac{(\alpha-1)p}{p-1}} dr\right)^{p-1} \int_0^s \|Y_{\alpha,\epsilon}(r)\|^p_H\, dr,$$

so that, thanks to (8.5) and to the fact that $\alpha < 1/4$

$$\mathbb{E}\sup_{r\in[0,s]}\left\|\int_0^r e^{(r-\rho)A}\Sigma_{t,\epsilon}(\rho,X^{t,x}_{\varphi_\epsilon,\epsilon}(\rho))\,dW_\rho\right\|_H^p \le c_{\alpha,p,t}\int_0^s \mathbb{E}\|Y_{\alpha,\epsilon}(r)\|_H^p\,dr$$

$$\le c_{\alpha,p,t}\int_0^s \mathbb{E}\left(\int_0^r (r-\rho)^{-(\frac12+2\alpha)}\left(\|X^{t,x}_{\varphi_\epsilon,\epsilon}(\rho)\|_H^2+1\right)d\rho\right)^{\frac p2} dr \tag{8.7}$$

$$\le c_{\alpha,p,t}\left(\int_0^s \mathbb{E}\sup_{\rho\in[0,r]}\|X^{t,x}_{\varphi_\epsilon,\epsilon}(\rho)\|_H^p\,dr+1\right).$$

Therefore, thanks to (8.4), (8.6) and (8.7),

$$\mathbb{E}\sup_{r\in[0,s]}\|X^{t,x}_{\varphi_\epsilon,\epsilon}(r)\|_H^p \le c_{t,M,p}\left(\|x\|_H^p+1\right)+c_{p,t}\int_0^s \mathbb{E}\sup_{\rho\in[0,r]}\|X^{t,x}_{\varphi_\epsilon,\epsilon}(\rho)\|_H^p\,dr,$$

and Gronwall's Lemma allows to conclude in case $p>4$. The case $p\in[1,4]$ follows from the Hölder inequality. □

Lemma 8.3. *Under the same assumptions of Theorem 3.6, we have*

$$|u_\epsilon(s,x)-g(Z^x(s))| \le c_t\sqrt{\epsilon}\,(1+\|x\|_H), \qquad s\in[0,t]. \tag{8.8}$$

Proof. Thanks to (6.1), we have

$$u_\epsilon(s,x)-g(Z^x(s)) = \mathbb{E}\left(g(X^{s,x}_\epsilon(s))-g(Z^x(s))\right),$$

so that, since we are assuming that g is Lipschitz-continuous

$$|u_\epsilon(s,x)-g(Z^x(s))| \le c\,\mathbb{E}\,\|X^{s,x}_\epsilon(s)-Z^x(s)\|_H.$$

Now, if we define $\rho^x_\epsilon(s) := X^{s,x}_\epsilon(s)-Z^x(s)$, we have

$$\rho^x_\epsilon(s) = \int_0^s e^{(s-r)A}\left(b(X^{s,x}_\epsilon(r))-b(Z^x(r))\right)dr+\sqrt{\epsilon}\int_0^s e^{(s-r)A}\Sigma_{s,\epsilon}(r,X^{s,x}_\epsilon(r))\,dW_r,$$

where $\Sigma_{s,\epsilon}$ is the operator introduced in (4.2). Due to (8.3) and (8.7), we have

$$\mathbb{E}\|\rho_\epsilon(s)\|_H \le c\int_0^s \mathbb{E}\|\rho_\epsilon(r)\|_H\,dr+c_t\sqrt{\epsilon}\,(1+\|x\|_H),$$

and Gronwall's lemma allows to conclude. □

8.3. Proof of the validity of Condition C1

Now, we are ready to prove condition C1. Let $\{\varphi_\epsilon\}_{\epsilon>0}$ be an arbitrary family of processes in $\Lambda_{t,M}$ converging in distribution, with respect to the weak topology of $L^2(0,t;H)$, to some $\varphi \in \Lambda_{t,M}$. As a consequence of Skorohod theorem (see e.g. [7, Theorem 2.4]), we can assume that the sequence $\{\varphi_\epsilon\}_{\epsilon>0}$ converges $\mathbb{P}$-a.s. to φ, with respect to the weak topology of $L^2(0,t;H)$. We will prove that this implies that

$$\lim_{\epsilon\to 0} \mathbb{E} \sup_{s\in[0,t]} \|X^{t,x}_{\varphi_\epsilon,\epsilon}(s) - X^{t,x}_{\varphi}(s)\|^2_H = 0, \tag{8.9}$$

and in particular $X^{t,x}_{\varphi_\epsilon,\epsilon}$ converges in distribution to $X^{t,x}_{\varphi}$ in $C([0,t];H)$. If we define

$$\rho_\epsilon(s) := X^{t,x}_{\varphi_\epsilon,\epsilon}(s) - X^{t,x}_{\varphi}(s), \qquad s\in[0,t],$$

we have

$$\begin{aligned}
\rho_\epsilon(s) = &\int_0^s e^{(s-r)A}\left[b(X^{t,x}_{\varphi_\epsilon,\epsilon}(r)) - b(X^{t,x}_{\varphi}(r))\right] dr \\
&+ \int_0^s e^{(s-r)A}\Big[\sigma(X^{t,x}_{\varphi_\epsilon,\epsilon}(r), u_\epsilon(t-r, X^{t,x}_{\varphi_\epsilon,\epsilon}(r)))\,\varphi_\epsilon(r) \\
&- \sigma(X^{t,x}_{\varphi}(r), g(Z^{X^{t,x}_{\varphi}(r)}(t-r)))\,\varphi(r)\Big] dr \\
&+ \sqrt{\epsilon}\int_0^s e^{(s-r)A}\Sigma_{t,\epsilon}(r, X^{t,x}_{\varphi,\epsilon}(r))\, dW_r =: \sum_{k=1}^3 I_{k,\epsilon}(s).
\end{aligned} \tag{8.10}$$

For $I_{1,\epsilon}(s)$, due to the Lipschitz continuity of b, we have

$$\|I_{1,\epsilon}(s)\|^2_H \le c_t \int_0^s \|\rho_\epsilon(s)\|^2_H\, ds. \tag{8.11}$$

Concerning $I_{2,\epsilon}(s)$, it can be written as

$$\begin{aligned}
&\int_0^s e^{(s-r)A}\left[\sigma(X^{t,x}_{\varphi_\epsilon,\epsilon}(r), u_\epsilon(t-r, X^{t,x}_{\varphi_\epsilon,\epsilon}(r))) - \sigma(X^{t,x}_{\varphi}(r), g(Z^{X^{t,x}_{\varphi_\epsilon,\epsilon}(r)}(t-r)))\right]\,\varphi_\epsilon(r)\, dr \\
&+ \int_0^s e^{(s-r)A}\left[\sigma(X^{t,x}_{\varphi}(r), g(Z^{X^{t,x}_{\varphi_\epsilon,\epsilon}(r)}(t-r))) - \sigma(X^{t,x}_{\varphi}(r), g(Z^{X^{t,x}_{\varphi}(r)}(t-r)))\right]\,\varphi_\epsilon(r)\, dr \\
&+ \int_0^s e^{(s-r)A}\sigma(X^{t,x}_{\varphi}(r), g(Z^{X^{t,x}_{\varphi}(r)}(t-r)))\,(\varphi_\epsilon(r) - \varphi(r))\, dr =: \sum_{k=1}^3 J_{k,\epsilon}(s).
\end{aligned}$$

According to (8.8), we have

$$\|J_{1,\epsilon}(s)\|_H \le c \int_0^s \left(\|\rho_\epsilon(r)\|_H + c_t\,\sqrt{\epsilon}\,\left(1 + \|X^{t,x}_{\varphi_\epsilon,\epsilon}(r)\|_H\right)\right)\|\varphi_\epsilon(r)\|_H\,dr,$$

so that

$$\|J_{1,\epsilon}(s)\|^2_H \le c_{t,M}\int_0^s \|\rho_\epsilon(r)\|^2_H\,dr + \epsilon\,c_{t,M}\left(1 + \sup_{r\in[0,t]}\|X^{t,x}_{\varphi_\epsilon,\epsilon}\|^2_H\right). \tag{8.12}$$

Moreover, thanks to (8.2), we have

$$\|J_{2,\epsilon}(s)\|^2_H \le c_{t,M}\int_0^s \|Z^{X^{t,x}_{\varphi_\epsilon,\epsilon}(r)}(t-r) - Z^{X^{t,x}_{\varphi}(r)}(t-r)\|^2_H\,dr \le c_{t,M}\int_0^s \|\rho_\epsilon(r)\|^2_H\,dr. \tag{8.13}$$

Finally, for $I_{3,\epsilon}(s)$, thanks to (8.3) and (8.7) we have

$$\mathbb{E}\sup_{s\in[0,t]}\|I_{3,\epsilon}(s)\|^2_H \le c_t\epsilon\left(1 + \|x\|^2_H\right). \tag{8.14}$$

Therefore, if we plug (8.11), (8.12), (8.13) and (8.14) into (8.10), in view of (8.3) we obtain

$$\mathbb{E}\sup_{r\in[0,s]}\|\rho_\epsilon(r)\|^2_H \le c_{t,M}\int_0^s \mathbb{E}\sup_{r\in[0,\rho]}\|\rho_\epsilon(r)\|^2_H\,d\rho + c_{t,M}\,\epsilon\left(1 + \|x\|^2_H\right) + \mathbb{E}\sup_{s\in[0,t]}\|J_{3,\epsilon}(s)\|^2_H,$$

and the Gronwall lemma gives

$$\mathbb{E}\sup_{r\in[0,t]}\|\rho_\epsilon(r)\|^2_H \le c_{t,M}\,\epsilon\left(1 + \|x\|^2_H\right) + c_{t,M}\mathbb{E}\sup_{s\in[0,t]}\|J_{3,\epsilon}(s)\|^2_H\,dr. \tag{8.15}$$

Thus, if we prove that

$$\lim_{\epsilon\to 0}\mathbb{E}\sup_{s\in[0,t]}\|J_{3,\epsilon}(s)\|^2_H = 0, \tag{8.16}$$

by taking the limit as ϵ goes to zero in both sides of (8.15) we obtain (8.9).

Thanks to the stochastic factorization formula, for every $\beta\in(0,1)$ we have

$$J_{3,\epsilon}(s) = c_\beta\int_0^s (s-r)^{\beta-1}e^{(s-r)A}Y_{\beta,\epsilon}(r)\,dr,$$

where

$$Y_{\beta,\epsilon}(r) := \int_0^r (r-\rho)^{-\beta} e^{(r-\rho)A} \sigma(X_\varphi^{t,x}(\rho), g(Z^{X_\varphi^{t,x}(\rho)}(t-\rho)))\,(\varphi_\epsilon(\rho)-\varphi(\rho))\,d\rho.$$

Due to the Young inequality, we get

$$\begin{aligned}\|Y_{\beta,\epsilon}\|^p_{L^p(0,t;H)} &\le c_{t,M} \int_0^t \left(\int_0^r (r-\rho)^{-\beta} \|\varphi_\epsilon(\rho)-\varphi(\rho)\|_H \, d\rho \right)^p dr \\ &\le c_{t,M,p}\, \|\varphi_\epsilon - \varphi\|^p_{L^2(0,T;H)} \left(\int_0^t r^{-\frac{2\beta p}{p+2}}\, dr \right)^{\frac{p+2}{2}}.\end{aligned}$$

Hence, if $\beta < 1/2$, we have

$$\|Y_{\beta,\epsilon}\|_{L^p(0,t;H)} \le c_{t,M,p}\, \|\varphi_\epsilon - \varphi\|_{L^2(0,T;H)}.$$

Now, due to the analyticity of e^{tA}, we have that e^{tA} maps H into $D((-A)^\gamma)$, for every $\gamma > 0$, with

$$\|e^{tA}x\|_{D((-A)^\gamma)} = \|(-A)^\gamma e^{tA}x\|_H \le c_\gamma (t \wedge 1)^{-\gamma} \|x\|_H.$$

Thus, if $\beta > \alpha + 1/p$ the mapping

$$Y \in L^p(0,t;H) \mapsto G(Y) := \int_0^\cdot (\cdot - r)^{\beta-1} e^{(\cdot - r)A} Y(r)\, dr \in C^{\beta-\alpha-1/p}([0,t]; D((-A)^\alpha))$$

is well defined and

$$\|G(Y)\|_{C^{\beta-\alpha-1/p}([0,t];D((-A)^\alpha))} \le c\, \|Y\|_{L^p(0,t;H)}. \tag{8.17}$$

(see the computations in the proof of [5, Proposition A 1.1]). Therefore, since $J_{3,\epsilon} = c_\beta\, G(Y_{\beta,\epsilon})$ and (8.17) holds, we conclude that if $\alpha + 1/p < \beta < 1/2$, then

$$\|J_{3,\epsilon}\|_{C^{\beta-\alpha-1/p}([0,t];D((-A)^\alpha))} \le \tilde{c}_{t,M,p}\, \|\varphi_\epsilon - \varphi\|_{L^2(0,T;H)}, \qquad \mathbb{P}-\text{a.s.}$$

Due to condition 3.12 and point (4) in Hypothesis 2, we have that

$$e^{tA} = Q_t^{1/2} \Lambda_t, \quad t > 0,$$

for some $\Lambda_t \in \mathcal{L}(H)$, and since $Q_t^{1/2}$ is compact, it follows that e^{tA} is compact. As shown in [9, Theorem 4.29] this implies that A has compact resolvent. Since $(-A)^\alpha$ is defined in terms of an integral in $\mathcal{L}(H)$ of the resolvent (see [16, Definition 4.3]), this implies that

$(-A)^\alpha$ is a compact operator. In particular, this means that $C^{\beta-\alpha-1/p}([0,t];D((-A)^\alpha))$ is compact in $C([0,T];H)$ and $D(-A)^\alpha)$ is compactly embedded in H.

Therefore, since $\varphi_\epsilon \to \varphi$, as $\epsilon \to 0$, in $L^2_w(0,t;H)$, we conclude that

$$\lim_{\epsilon\to 0} \|J_{3,\epsilon}\|_{C([0,t];H)} = 0, \qquad \mathbb{P}-\text{a.s.}$$

Moreover, since

$$\sup_{\epsilon\in(0,1)} \|J_{3,\epsilon}\|_{C([0,t];H)} \le c_{M,t}, \qquad \mathbb{P}-\text{a.s.}$$

by the dominated convergence theorem we obtain (8.16).

8.4. Proof of the validity of Condition C2

In the proof of Condition C1. we have seen that if φ_ϵ converges $\mathbb{P}$-a.s. to φ, with respect to the weak topology in $L^2(0,t;H)$, then (8.9) holds. In particular, this holds in the deterministic case, so that the mapping

$$\varphi \in L^2_w(0,t;H) \mapsto X^{t,x}_\varphi \in C([0,t];H),$$

is continuous, and for every $c>0$

$$\bigcap_{\epsilon\in(0,1)} \left\{X^{t,x}_\varphi\,,\ \varphi\in\mathcal{S}_{t,c+\epsilon}\right\} = \left\{X^{t,x}_\varphi\,,\ \varphi\in\mathcal{S}_{t,c}\right\}. \tag{8.18}$$

Moreover, for every $t>0$ and $x\in H$ the set $\mathcal{S}_{t,c}$ is compact in $L^2_w(0,t;H)$, so that

$$\left\{X^{t,x}_\varphi\,,\ \varphi\in\mathcal{S}_{t,c}\right\} \subset C([0,t];H) \ \text{ is compact.}$$

Now, recalling the definition of $I_{t,x}$, for every $R>0$ we have

$$\Phi_{t,R} = \{I_{t,x}\le R\} = \{X^{t,x}_\varphi\,,\ \varphi\in\mathcal{S}_{t,\sqrt{2R}}\}. \tag{8.19}$$

Indeed, if X belongs to $\{X^{t,x}_\varphi \ :\ \varphi\in\mathcal{S}_{t,\sqrt{2R}}\}$, then there exists $\bar\varphi\in\mathcal{S}_{t,\sqrt{2R}}$ such that $X=X^{t,x}_{\bar\varphi}$, so that $I_{t,x}(X)\le R$. On the other hand, if $X\in\{I_{t,x}\le R\}$, then for any $\epsilon>0$ there exists $\varphi_\epsilon\in\mathcal{S}_{t,\sqrt{2R}+\epsilon}$ such that $X=X^{t,x}_{\varphi_\epsilon}$, and together with (8.18) this implies

$$X\in\bigcap_{\epsilon\in(0,1)}\left\{X^{t,x}_\varphi\,,\ \varphi\in\mathcal{S}_{t,\sqrt{2R}+\epsilon}\right\} = \left\{X^{t,x}_\varphi\,,\ \varphi\in\mathcal{S}_{t,\sqrt{2R}}\right\}.$$

Therefore, from (8.19) and the compactness of $\{X^{t,x}_\varphi\,,\ \varphi\in\mathcal{S}_{t,\sqrt{2R}}\}$, we conclude that Condition C2. holds.

Data availability

No data was used for the research described in the article.

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