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On the Small-Mass Limit for Stationary Solutions of Stochastic Wave Equations with State Dependent Friction

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Abstract

We investigate the convergence, in the small mass limit, of the stationary solutions of a class of stochastic damped wave equations, where the friction coefficient depends on the state and the noisy perturbation is of multiplicative type. We show that the Smoluchowski–Kramers approximation that has been previously shown to be true in any fixed time interval, is still valid in the long time regime. Namely, we prove that the first marginals of any sequence of stationary solutions for the damped wave equation converge to the unique invariant measure of the limiting stochastic quasilinear parabolic equation. The convergence is proved with respect to the Wasserstein distance associated with the H^{-1} norm.

Keywords Stochastic wave equations · Smoluchowski-Kramers approximation · Convergence of invariant measures · Wasserstein convergence

1 Introduction

In this article we deal with the following stochastic wave equation with state-dependent damping, on a bounded smooth domain $\mathcal{O} \subset \mathbb{R}^d$, with $d \geq 1$,

$$\begin{cases} \mu \partial_t^2 u_\mu(t,x) = \Delta u_\mu(t,x) - \gamma(u_\mu(t,x))\partial_t u_\mu(t,x) \\ +f(x, u_\mu(t,x)) + \sigma(u_\mu(t,\cdot))\partial_t w^Q(t,x), \\ \\ u_\mu(0,x) = \mathfrak{u}_0(x), \quad \partial_t u_\mu(0,x) = \mathfrak{v}_0(x), \qquad u_\mu(t,x) = 0, \;\; x \in \partial\mathcal{O}, \end{cases} \tag{1.1}$$

depending on a parameter $0 < \mu << 1$. The friction coefficient γ is a strictly positive, bounded and continuously differentiable function. The diffusion coefficient

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σ is bounded and Lipschitz-continuous and the noise $w^Q(t)$ is a cylindrical Q-Wiener process, white in time and colored in space. The nonlinearity $f : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is Lipschtz-continuous with respect to the second variable and the identically zero function is globally asymptotically stable in the absence of the stochastic perturbation. Here and in what follows, we denote $H := L^2(\mathcal{O})$, $H^{-1} := H^{-1}(\mathcal{O})$, and $H^1 := H_0^1(\mathcal{O})$ (the set of functions in the Sobolev space $H^1(\mathcal{O})$ with zero-trace).

The solution $u_\mu(t,x)$ of Eq. (1.1) can be interpreted as the displacement of the particles of a material continuum in a domain $\mathcal{O}$, subject to a random external force field $\partial_t w^Q(t,x)$ and a damping force which is proportional to the velocity field and depends on the state u_μ. The second order differential operator takes into account of the interaction forces between neighboring particles, in the presence of a non-linear reaction given by f. Here μ represents the constant density of the particles and we are interested in the regime when $\mu \to 0$, known as the Smoluchowski–Kramers approximation limit (Refs. [25, 31]).

In [3, 4] it has been proven that, when γ is constant, for every $T > 0$ and $\eta > 0$

$$\lim_{\mu\to 0} \mathbb{P}\left(\sup_{t\in[0,T]} \|u_\mu(t) - u(t)\|_H > \eta \right) = 0, \tag{1.2}$$

where $u \in L^2(\Omega; C([0,T]; H) \cap L^2(0,T; H^1))$ is the solution of the parabolic problem

$$\begin{cases} \gamma\,\partial_t u(t,x) = \Delta u(t,x) + f(x, u(t,x)) + \sigma(u(t,\cdot))\partial_t w^Q(t,x), \\ u(0,x) = \mathfrak{u}_0(x), \qquad u(t,\cdot)\big|_{\partial\mathcal{O}} = 0. \end{cases} \tag{1.3}$$

When the friction coefficient γ is state-dependent, the situation is more complicated and, because of the interplay between the noise and the non-constant friction, in the small-mass limit an extra drift term is created. In this regard, in [10] it has been proven that for every $\mathfrak{u}_0 \in H^1$, $T > 0$ and $p < \infty$, and for every $\eta > 0$

$$\lim_{\mu\to 0} \mathbb{P}\left(\int_0^T \|u_\mu(t) - u(t)\|_H^p \, dt > \eta \right) = 0, \tag{1.4}$$

where u is the unique solution of the stochastic quasi-linear equation

$$\begin{cases} \gamma(u(t,x))\partial_t u(t,x) = \Delta u(t,x) + f(x, u(t,x)) \\ -\frac{\gamma'(u(t,x))}{2\gamma^2(u(t,x))} \sum_{i=1}^\infty |\sigma(u(t,\cdot))\, Q e_i(x)|^2 \\ \qquad\qquad\qquad + \sigma(u(t,\cdot))\partial_t w^Q(t,x) \\ u(0,x) = \mathfrak{u}_0(x), \qquad u(t,\cdot)\big|_{\partial\mathcal{O}} = 0. \end{cases} \tag{1.5}$$

Notice that the case of a non-constant damping coefficient is not the sole instance in which, within the context of a small mass limit, an additional drift term appears. For example, in the case of a damped stochastic wave equation, constrained to live on the unitary sphere of H, in the limit the Smoluchowski–Kramers approximation yields a stochastic parabolic problem also constrained to live on the unitary sphere of H, where an extra-drift term emerges, and that drift does not encompass the Itô-to-Stratonovich correction (see [2]). For a partial list of references where this type of limit has been addressed in a variety of different contexts, see [1, 5, 16, 17, 22–24, 26, 32], in finite dimension, and [2–4, 9, 10, 27–29], in infinite dimension.

After establishing the validity of the small mass limits within a fixed time interval $[0, T]$, the next step of interest is to compare the long-term dynamics of the second-order system with that of the first-order system (to this purpose, see e.g. [6–8, 11, 12, 24]).

In [3], a comparative analysis of the long-term behavior of equations (1.1) (with a constant γ) and (1.3) was conducted, assuming both systems to be of gradient type. Notably, in the case where the noise is white in both space and time ($Q = I$) and the dimension is $d = 1$, an explicit expression for the Boltzmann distribution of the process $z_\mu(t) := (u_\mu(t), \partial u_\mu/\partial t(t))$ in the phase space $\mathcal{H} := L^2(0, L) \times H^{-1}(0, L)$ was derived. Since there is no equivalent of the Lebesgue measure in the functional space $\mathcal{H}$, an auxiliary Gaussian measure was introduced, and the density of the Boltzmann distribution was then expressed with respect to such auxiliary Gaussian measure, which itself corresponds to the stationary measure of the linear wave equation associated with problem (1.1). In particular, it was shown that the first marginal of the invariant measure linked to the process $z_\mu(t)$ remains independent of $\mu > 0$ and coincides with the invariant measure for the heat equation (1.3).

In the case of non-gradient systems, that is when the noise is colored in space and/or of multiplicative type, there is no explicit expression for the invariant measure ν_μ associated with system (1.1) and there is no reason to expect that the first marginal of ν_μ does not depend on μ or coincides with the invariant measure ν of system (1.3). Nonetheless, in [6] it was proved that, as the mass parameter μ tends to zero, the first marginal of any invariant measure ν_μ associated with the second-order system (1.1) converges in a suitable manner to the invariant measure ν of the first-order system (1.3). Specifically, the following convergence was established

$$\lim_{\mu \to 0} \mathcal{W}_\alpha \left((\Pi_1 \nu_\mu)', \nu \right) = 0, \tag{1.6}$$

where $(\Pi_1 \nu_\mu)'$ denotes the extension of the first marginal of the invariant measure ν_μ to H, and the metric $\mathcal{W}_\alpha$ corresponds to the Wasserstein metric on $\mathcal{P}(H)$ associated with a distance metric α in H, which was determined based on the characteristics of the non-linearity function f under consideration.

In the present paper, we want to see if any of the results proved in [6] in the case of a constant friction γ, can be proven for a state-dependent γ, where the Smoluchowski–Kramers approximation gives the stochastic quasi-linear parabolic problem (1.5), instead of the simpler parabolic semi-linear problem (1.3).

One of the key ingredients used in [6] for the proof of (1.6) is the fact that the transition semigroup P_t^H associated with equation (1.3) admits a unique invariant measure $\nu \in \mathcal{P}(H)$ and the following contraction property holds

$$\mathcal{W}_\alpha\left((P_t^H)^\star\nu_1,(P_t^H)^\star\nu_2\right)\leq c\,e^{-\delta t}\,\mathcal{W}_\alpha(\nu_1,\nu_2),\quad t\geq 0,\quad \nu_1,\nu_2\in\mathcal{P}(H),\tag{1.7}$$

for some $c,\delta>0$. In the case of Eq. (1.3), these types of problems have been studied extensively and a wide variety of results is available. However, in the case of the quasi-linear problem (1.5) the situation is considerably more delicate and several fundamental facts are not known, as for one whether the semigroup associated is Feller in H or not. In particular, even the use of the Krylov–Bogoliubov theorem for the proof of the existence of an invariant measure in H is not possible. Thus, in the present paper we have to follow a different path, that in particular brings us to study Eqs. (1.1) and (1.5) in spaces of lower regularity than $H^1\times H$ and H, respectively.

In [10], it has been proved that Eq. (1.1) is well-posed in $\mathcal{H}_1:=H^1\times H$, for every $\mu>0$, so that the associated Markov transition semigroup $P_t^{\mu,\mathcal{H}_1}$ can be introduced. Our first step is showing that in fact (1.1) is well-posed also in $\mathcal{H}:=H\times H^{-1}$, for every $\mu>0$, and there exists an invariant measure $\nu^{\mu,\mathcal{H}}$ for the corresponding transition semigroup $P_t^{\mu,\mathcal{H}}$. We show that such invariant measure is supported in $\mathcal{H}_1$ and its restriction to $\mathcal{H}_1$ is invariant for $P_t^{\mu,\mathcal{H}_1}$. Moreover, we prove suitable uniform bounds for the moments of $\nu^{\mu,\mathcal{H}}$ and $\nu^{\mu,\mathcal{H}_1}$, which are fundamental for the proof of the limit.

Next, we move our analysis to the limiting equation (1.5). To this purpose, we do not work directly with (1.5), but rather with its equivalent formulation

$$\begin{cases}\partial_t\rho(t,x)=\operatorname{div}\left(\frac{1}{\gamma(g^{-1}(\rho(t,x)))}\nabla\rho(t,x)\right)\\ +f(x,g^{-1}(\rho(t,x)))+\sigma(g^{-1}(\rho(t,\cdot)))\partial_t w^Q(t,x),\\ \rho(0,x)=g(\mathfrak{u}_0(x)),\qquad \rho(t,\cdot)|_{\partial\mathcal{O}}=0,\end{cases}\tag{1.8}$$

where g is the antiderivative of γ vanishing at zero. Since we are assuming that γ is strictly positive, bounded and continuously differentiable, the mappings

$$h\in H\mapsto g\circ h\in H,\qquad h\in H^1\mapsto g\circ h\in H^1,$$

are both homeomorphisms and the coefficients in (1.8) are all well defined and regular. Moreover, as shown in [10], by using a generalized Itô's formula, for every $\mathfrak{r}_0=g(\mathfrak{u}_0)\in H^1$ and $t\geq 0$ we have that

$$\rho^{\mathfrak{r}_0}(t)=g(u^{\mathfrak{u}_0}(t)),\qquad g^{-1}(\rho^{\mathfrak{r}_0}(t))=u^{\mathfrak{u}_0}(t).\tag{1.9}$$

In particular, Eq. (1.8) is well posed in $C([0,T];H)\cap L^2(0,T;H^1)$ if and only if Eq. (1.5) is.

As a consequence of limit (1.4), we have that for every initial condition $\mathfrak{r}_0\in H^1$ Eq. (1.8) has a unique solution $\rho^{\mathfrak{r}_0}\in L^2(\Omega;L^p(0,T;H^1))$, with $p<\infty$. However, since the long time behavior of (1.8) in H^1 and H is not well understood, we need to study its well-posedness in H and H^{-1}, so that we can introduce the corresponding transition semigroups R_t^H and $R_t^{H^{-1}}$. Due to the equivalence of problems (1.5) and

(1.8) in H this allows us to introduce the transition semigroup P_t^H associated with Eq. (1.5).

Next, we prove that there exists some constant $\lambda > 0$ such that for every $\mathfrak{r}_1, \mathfrak{r}_2 \in H^{-1}$ and $t \geq 0$

$$\mathbb{E} \left\| \rho^{\mathfrak{r}_1}(t) - \rho^{\mathfrak{r}_1}(t) \right\|_{H^{-1}}^2 \leq e^{-\lambda t} \left\| \mathfrak{r}_1 - \mathfrak{r}_2 \right\|_{H^{-1}}^2, \quad t \geq 0. \tag{1.10}$$

To this purpose, we would like to mention that in [19, 20], it was proved that under suitable conditions on the initial conditions, the following property holds

$$\mathbb{E} \left\| \rho^{\mathfrak{r}_1}(t) - \rho^{\mathfrak{r}_1}(t) \right\|_{L^1(\mathcal{O})}^2 \leq \left\| \mathfrak{r}_1 - \mathfrak{r}_2 \right\|_{L^1(\mathcal{O})}^2, \qquad t \geq 0. \tag{1.11}$$

Such bound gives in particular the Feller property in $L^1(\mathcal{O})$ but, unfortunately, this is not useful to our analysis, as it is not clear how to handle the proof of our limiting problem in a $L^1(\mathcal{O})$ setting. As far as we know, it is not clear if such a bound like (1.11) is satisfied in H. As we already mentioned above, this is why it becomes important to work in H^{-1}, where we have the validity of even stronger condition (1.10).

As a consequence of (1.10), we have that $R_t^{H^{-1}}$ is Feller. This, together with suitable uniform bounds in H^1, allows to conclude that $R_t^{H^{-1}}$ has an invariant measure $\nu^{H^{-1}}$, supported in H^1. Moreover (1.10) implies that for every $\varphi \in \mathrm{Lip}_b(H^{-1})$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in H^{-1}$

$$\left| R_t^{H^{-1}} \varphi(\mathfrak{r}_1) - R_t^{H^{-1}} \varphi(\mathfrak{r}_2) \right| \leq [\varphi]_{\mathrm{Lip}_{H^{-1}}} e^{-\lambda t/2} \left\| \mathfrak{r}_1 - \mathfrak{r}_2 \right\|_{H^{-1}}, \qquad t \geq 0, \tag{1.12}$$

so that $\nu^{H^{-1}}$ is the unique invariant measure of $R_t^{H^{-1}}$, and ν^H, its restriction to H, turns out to be the unique invariant measure of R_t^H. Finally, due to the *equivalence* between Eqs. (1.5) and (1.8), we show that this implies that P_t^H has a unique invariant measure ν.

By using a general argument developed in [15], and already used in [6] in a similar context, all this allows to obtain our main result. The idea introduced in [15] is quite simple and general. If $\{\nu_n\}_{n \in \mathbb{N}}$ is a sequence of invariant measures for a sequence of Markov processes $\{X_n(t)\}_{n \in \mathbb{N}}$ on some Banach space E, with transition semigroups $\{P_t^n\}_{n \in \mathbb{N}}$, and ν is the invariant measure for a Markov process $X(t)$ on E, with transition semigroup P_t, in order to study the convergence of ν_n to ν with respect to some Wasserstein distance ρ_α, associated to some distance α on E, we first notice that, due to the invariance of ν_n and ν,

$$\rho_\alpha(\nu_n, \nu) \leq \rho_\alpha\left((P_t^n)^\star \nu_n, P_t^\star \nu_n\right) + \rho_\alpha\left((P_t)^\star \nu_n, P_t^\star \nu\right). \tag{1.13}$$

Thus, if there exists some $\delta > 0$ such that for every probability measures ν^1 and ν^2 on E

$$\rho_\alpha\left(P_t^\star \nu^1, P_t^\star \nu^2\right) \leq c\, e^{-\delta t} \rho_\alpha(\nu^1, \nu^2), \qquad t \geq 0,$$

from (1.13) we get

$$\rho_\alpha(\nu_n,\nu) \leq \rho_\alpha\left((P_t^n)^\star\nu_n,\, P_t^\star\nu_n\right) + ce^{-\delta t}\rho_\alpha(\nu_n,\nu)\,.$$

This implies that, if we pick $t_\star > 0$ such that $c\,e^{-\delta t_\star} < 1/2$, we have

$$\rho_\alpha(\nu_n,\nu) \leq 2\,\rho_\alpha\left((P_t^n)^\star\nu_n,\, P_t^\star\nu_n\right), \qquad t \geq t_\star.$$

Now, thanks to the Kantorovich–Rubinstein duality, we have

$$\rho_\alpha\left((P_t^n)^\star\nu_n,\, P_t^\star\nu_n\right) \leq \mathbb{E}\,\alpha(X_n^{\gamma_n}(t), X^{\gamma_n}(t)),$$

where γ_n is a E-valued random variable, distributed as ν_n, and $X_n^{\gamma_n}(t)$ and $X^{\gamma_n}(t)$ are the processes $X_n(t)$ and $X(t)$ with initial condition γ_n. In particular, this implies that the proof of the converge of ν_n to ν with respect to the Wasserstein distance ρ_α reduces to the proof of the following limit

$$\lim_{n\to\infty}\mathbb{E}\,\rho_\alpha\left((P_t^n)^\star\nu_n,\, P_t^\star\nu_n\right) \leq \mathbb{E}\,\alpha(X_n^{\gamma_n}(t), X^{\gamma_n}(t)) = 0,$$

for a fixed time t sufficiently large.

In the present paper, our goal is showing that if we define

$$\alpha(\mathfrak{u}_1,\mathfrak{u}_2) := \|\mathfrak{u}_1-\mathfrak{u}_2\|_{H^{-1}}, \qquad \mathfrak{u}_1,\mathfrak{u}_2 \in H^{-1},$$

then it holds

$$\lim_{\mu\to 0}\mathcal{W}_\alpha\left(\Pi_1\nu_\mu^{\mathcal{H}},\nu\right) = 0. \tag{1.14}$$

Due to (1.12) and the invariance of $\nu_\mu^{\mathcal{H}}$ and $\nu^{H^{-1}}$ we have

$$\begin{aligned}\mathcal{W}_\alpha\left(\left[\left(\Pi_1\nu_\mu^{\mathcal{H}}\right)\circ g^{-1}\right]',\nu^{H^{-1}}\right) &\leq \mathcal{W}_\alpha\left(\left[\Pi_1((P_t^{\mu,\mathcal{H}})^\star\nu_\mu^{\mathcal{H}})\circ g^{-1}\right]',(R_t^{H^{-1}})^*\left[\left(\Pi_1\nu_\mu^{\mathcal{H}}\right)\circ g^{-1}\right]'\right)\\ &\quad + c\,e^{-\lambda t}\,\mathcal{W}_\alpha\left(\left[\left(\Pi_1\nu_\mu^{\mathcal{H}}\right)\circ g^{-1}\right]',\nu^{H^{-1}}\right),\end{aligned}$$

and then, if we pick $\bar t > 0$ such that $ce^{-\lambda\bar t} \leq 1/2$, we obtain

$$\begin{aligned}&\mathcal{W}_\alpha\left(\left[\left(\Pi_1\nu_\mu^{\mathcal{H}}\right)\circ g^{-1}\right]',\nu^{H^{-1}}\right)\\ &\leq 2\,\mathcal{W}_\alpha\left(\left[(\Pi_1((P_{\bar t}^{\mu,\mathcal{H}})^\star\nu_\mu^{\mathcal{H}})\circ g^{-1}\right]',(R_{\bar t}^{H^{-1}})^*\left[\left(\Pi_1\nu_\mu^{\mathcal{H}}\right)\circ g^{-1}\right]'\right),\end{aligned}$$

(here we are using the notation $[\cdot]'$ to denote the extension to H^{-1} of an arbitrary probability measure defined in H). As we have seen above, the Kantorovich–Rubinstein duality gives

$$\mathcal{W}_\alpha\left(\left[(\Pi_1((P_t^{\mu,\mathcal{H}})^\star\nu_\mu^{\mathcal{H}})\circ g^{-1}\right]', (R_t^{H^{-1}})^*\left[\left(\Pi_1\nu_\mu^{\mathcal{H}}\right)\circ g^{-1}\right]'\right)$$
$$\leq \mathbb{E}\,\alpha(g(u_\mu^{\zeta_\mu}(t)), \rho^{g(\xi_\mu)}(t)),$$

for every $\mathcal{F}_0$-measurable $\mathcal{H}_1$-valued random variable $\zeta_\mu := (\xi_\mu, \eta_\mu)$, distributed as the invariant measure $\nu_\mu^{\mathcal{H}}$. Hence, once we prove that for every $t \geq 0$ large enough

$$\lim_{\mu\to 0}\,\mathbb{E}\,\alpha(g(u_\mu^{\zeta_\mu}(t)), \rho^{g(\xi_\mu)}(t)) = \lim_{\mu\to 0}\,\mathbb{E}\,\|g(u_\mu^{\zeta_\mu}(t)) - \rho^{g(\xi_\mu)}(t)\|_{H^{-1}} = 0, \quad (1.15)$$

we obtain that

$$\lim_{\mu\to 0}\mathcal{W}_\alpha\left(\left[\left(\Pi_1\nu_\mu^{\mathcal{H}}\right)\circ g^{-1}\right]', \nu^{H^{-1}}\right) = 0.$$

Our last steps consists in showing that this implies (1.14), which also implies that $\Pi_1\nu_\mu^{\mathcal{H}}$ converges to ν, weakly in H, as $\mu \downarrow 0$.

2 Notations and Assumptions

Throughout the present paper $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, with $d \geq 1$, having a smooth boundary. We denote by H the Hilbert space $L^2(\mathcal{O})$ and by $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$ the corresponding norm and inner product.

Given the domain $\mathcal{O}$, we denote by A the realization of the Laplace operator Δ, endowed with Dirichlet boundary conditions. As known there exists a complete orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of H which diagonalizes A. In what follows, we denote by $\{-\alpha_i\}_{i\in\mathbb{N}}$ the corresponding sequence of eigenvalues, and for every $\delta\in\mathbb{R}$, we define H^δ as the completion of $C_0^\infty(\mathcal{O})$ with respect to the norm

$$\|h\|_{H^\delta}^2 := \sum_{i=1}^\infty \alpha_i^\delta\langle h, e_i\rangle_H^2.$$

Notice that with this definition $H^0 = H$ and, if $\delta_1 < \delta_2$, then $H^{\delta_2} \hookrightarrow H^{\delta_1}$ with compact embedding. We also define

$$\mathcal{H}_\delta := H^\delta \times H^{\delta-1}, \qquad \mathcal{H} := H \times H^{-1}.$$

Next, for every two separable Hilbert spaces E and F, we denote by $\mathcal{L}(E,F)$ the space of bounded linear operators from E into F and by $\mathcal{L}_2(E,F)$ the subspace of Hilbert–Schmidt operators. $\mathcal{L}_2(E,F)$ is a Hilbert space, endowed with the inner product

$$\langle B, C\rangle_{\mathcal{L}_2(E,F)} = \mathrm{Tr}_E\,[B^\star C] = \mathrm{Tr}_F[CB^\star],$$

and, as well known, $\mathcal{L}_2(E, F) \subset \mathcal{L}(E, F)$, with

$$\|B\|_{\mathcal{L}(E,F)} \leq \|B\|_{\mathcal{L}_2(E,F)}.$$

Finally, if X is any Polish space, we denote by $B_b(X)$ the space of bounded Borel measurable functions $\varphi : X \to \mathbb{R}$, endowed with the sup-norm

$$\|\varphi\|_\infty := \sup_{h \in X} |\varphi(h)|.$$

Moreover, we denote by $C_b(X)$ the subspace of uniformly continuous and bounded functions.

2.1 Assumptions

We assume that $w^Q(t)$ is a cylindrical Q-Wiener process, for some $Q \in \mathcal{L}(H)$, defined on a complete stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. This means that $w^Q(t)$ can be formally written as

$$w^Q(t) = \sum_{i=1}^{\infty} Qe_i \beta_i(t),$$

where $\{\beta_i\}_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and $\{e_i\}_{i\in\mathbb{N}}$ is the complete orthonormal system introduced above that diagonalizes the Laplace operator, endowed with Dirichlet boundary conditions. When $Q = I$, the process $w^I(t)$ will be denoted by $w(t)$. In particular, we have $w^Q(t) = Qw(t)$.

In what follows we shall denote by H_Q the set $Q(H)$. H_Q is the reproducing kernel of the noise w^Q and is a Hilbert space, endowed with the inner product

$$\langle h, k\rangle_{H_Q} = \langle Q^{-1}h, Q^{-1}k\rangle_H, \quad h, k \in H_Q.$$

Notice that the sequence $\{Qe_i\}_{i\in\mathbb{N}}$ is a complete orthonormal system in H_Q. Moreover, if U is any Hilbert space containing H_Q such that the embedding of H_Q into U is Hilbert–Schmidt, we have that

$$w^Q \in C([0, T]; U).$$

Hypothesis 1 *The mapping* $\sigma : H \to \mathcal{L}_2(H_Q, H)$ *is defined by*

$$[\sigma(h)Qe_i](x) = \sigma_i(x, h(x)), \quad x \in \mathcal{O}, \quad h \in H, \quad i \in \mathbb{N},$$

for some measurable mappings $\sigma_i : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$. We assume that there exists $L_\sigma > 0$ such that

$$\sup_{x\in\mathcal{O}} \sum_{i=1}^{\infty} |\sigma_i(x, y_1) - \sigma_i(x, y_2)|^2 \leq L_\sigma \, |y_1 - y_2|^2, \qquad y_1, y_2 \in \mathbb{R}. \tag{2.1}$$

Moreover, we assume σ is bounded, that is,

$$\sigma_\infty := \sup_{h\in H} \|\sigma(h)\|_{\mathcal{L}_2(H_Q,H)} < \infty. \tag{2.2}$$

Remark 2.1 1. Condition (2.1) implies that $\sigma : H \to \mathcal{L}_2(H_Q, H)$ is Lipschitz continuous. Namely, for any $h_1, h_2 \in H$

$$\|\sigma(h_1) - \sigma(h_2)\|_{\mathcal{L}_2(H_Q,H)} \leq \sqrt{L_\sigma} \, \|h_1 - h_2\|_H. \tag{2.3}$$

2. If the noise is additive, then Hypothesis 1 is satisfied when Tr $Q^2 < +\infty$.

Hypothesis 2 *The mapping γ belongs to $C_b^1(\mathbb{R})$ and there exist γ_0 and γ_1 such that*

$$0 < \gamma_0 \leq \gamma(r) \leq \gamma_1, \qquad r \in \mathbb{R}. \tag{2.4}$$

If we define

$$g(r) := \int_0^r \gamma(\sigma)\, d\sigma, \qquad r \in \mathbb{R}, \tag{2.5}$$

the function $g : \mathbb{R} \to \mathbb{R}$ is differentiable, strictly increasing and invertible so that its inverse $g^{-1} : \mathbb{R} \to \mathbb{R}$ is differentiable, with

$$\sup_{r\in\mathbb{R}} (g^{-1})'(r) \leq \frac{1}{\gamma_0}. \tag{2.6}$$

Hypothesis 3 *The mapping $f : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is measurable and there exists a positive constant L_f such that*

$$L_f < \frac{\alpha_1 \gamma_0}{\gamma_1}, \tag{2.7}$$

where α_1 is the smallest eigenvalue of $-A$, and

$$\sup_{x\in\mathcal{O}} |f(x, r) - f(x, s)| \leq L_f |r - s|, \quad r, s \in \mathbb{R}. \tag{2.8}$$

Moreover,

$$\sup_{x\in\mathcal{O}} |f(x, 0)| < \infty.$$

In what follows, for every $x \in \mathcal{O}$ and $r \in \mathbb{R}$ we denote

$$\mathfrak{f}(x, r) := \int_0^r f(x, s) ds,$$

and for every function $h : \mathcal{O} \to \mathbb{R}$, we denote

$$F(h)(x) := f(x, h(x)), \qquad x \in \mathcal{O}.$$

Remark 2.2 1. Condition (2.8) implies that $F : H \to H$ is Lipschitz continuous. Namely for any $h_1, h_2 \in H$

$$\|F(h_1) - F(h_2)\|_H \leq L_f \, \|h_1 - h_2\|_H \, .$$

Moreover, there exists $c > 0$ such that

$$\|F(h)\|_H \leq L_f \, \|h\|_H + c. \tag{2.9}$$

2. If the friction coefficient γ is constant, then $\gamma_0 = \gamma_1$, and condition (2.7) becomes

$$L_f < \alpha_1.$$

3. It is immediate to check that if for every $h \in H$ we define

$$\Lambda(h) := \int_{\mathcal{O}} f(x, h(x)) \, dx,$$

then $\Lambda : H \to \mathbb{R}$ is differentiable and its differential is given by

$$[D\Lambda(h)](x) = f(x, h(x)), \qquad x \in \mathcal{O}. \tag{2.10}$$

Hypothesis 4 *We assume*

$$L_f + \frac{L_\sigma}{2\gamma_0} < \frac{\alpha_1 \gamma_0}{\gamma_1}. \tag{2.11}$$

Remark 2.3 Condition (2.11) is assumed in order to have the well-posedness of Eq. (1.8) in H^{-1} and to prove limit (1.15). If the diffusion coefficient σ is constant, then $L_\sigma = 0$ and Hypothesis 4 reduces to condition (2.7) in Hypothesis 3. However, in the case σ is non constant, condition (2.7) alone is not enough, as also the Lipschitz constant of g has to be small, compared to the eigenvalue α_1 and the constants γ_0 and γ_1.

3 The Main Result

For every $\mu > 0$, we denote $v_\mu := \partial_t u_\mu$, and rewrite Eq. (1.1) as the following system

$$\begin{cases} du_\mu(t) = v_\mu(t)dt, \\ \mu dv_\mu(t) = \left[Au_\mu(t) - \gamma(u_\mu(t))v_\mu(t) + F(u_\mu(t))\right]dt + \sigma(u_\mu(t))dw^Q(t), \\ u_\mu(0) = \mathfrak{u}_0, \qquad v_\mu(0) = \mathfrak{v}_0, \end{cases} \tag{3.1}$$

where A is the realization in H of the Laplacian Δ, endowed with Dirichlet boundary conditions. In [10] it has been proven that, under Hypotheses 1, 2 and 3 (without condition (2.7)), for every $(\mathfrak{u}_0, \mathfrak{v}_0) \in L^2(\Omega; \mathcal{H}_1)$, and for every $\mu, T > 0$, there exists a unique process $z_\mu \in L^2(\Omega; C([0, T]; \mathcal{H}_1))$ which solves system (3.1), in the sense that

$$\begin{cases} u_\mu(t) = \mathfrak{u}_0 + \displaystyle\int_0^t v_\mu(s)\, ds \\ \mu\, v_\mu(t) = \mu\, \mathfrak{v}_0 + \displaystyle\int_0^t \left[Au_\mu(s) - \gamma(u_\mu(s))v_\mu(s) + F(u_\mu(s))\right] ds \\ + \int_0^t \sigma(u_\mu(s))\, dw(s). \end{cases} \tag{3.2}$$

In particular, we can introduce the transition semigroup $P_t^{\mu,\mathcal{H}_1}$ associated with Eq. (3.1) in $\mathcal{H}_1$, by setting

$$P_t^{\mu,\mathcal{H}_1}\varphi(\mathfrak{z}) = \mathbb{E}\,\varphi\big(z_\mu^{\mathfrak{z}}(t)\big), \qquad t \geq 0,$$

for every $\varphi \in B_b(\mathcal{H}_1)$ and $\mathfrak{z} \in \mathcal{H}_1$.

In what follows, we will need to study system (3.1) also in the space of lower regularity $\mathcal{H}$, and for this reason we introduce the following notion of *generalized solution*.

Definition 3.1 For every $\mu, T > 0$ and every $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathcal{H}$, we say that the process $z_\mu \in L^2(\Omega; C([0, T]; \mathcal{H}))$ is a generalized solution of system (3.1) if for every sequence $\{\mathfrak{u}_{0,n}, \mathfrak{v}_{0,n}\}_{n\in\mathbb{N}} \subset \mathcal{H}_1$ converging to $(\mathfrak{u}_0, \mathfrak{v}_0)$ in $\mathcal{H}$, as $n \to +\infty$, it holds

$$\lim_{n\to+\infty} \mathbb{E}\,\|z_{\mu,n} - z_\mu\|^2_{C([0,T];\mathcal{H})} = 0,$$

where $z_{\mu,n} \in L^2(\Omega; C([0, T]; \mathcal{H}_1))$ is the unique solution of Eq. (3.1) with initial conditions $(\mathfrak{u}_{0,n}, \mathfrak{v}_{0,n})$.

Notice that if $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathcal{H}_1$, then the weak solution coincides with the solution defined above in $\mathcal{H}_1$ in the sense of (3.2).

In Sect. 5 we will prove that, under Hypotheses 1, 2 and 3, for every $\mu, T > 0$ there exists a unique generalized solution $z_\mu \in L^2(\Omega; C([0,T]; \mathcal{H}))$ for system (3.1). This will allow us to introduce the transition semigroup associated with (3.1) in $\mathcal{H}$, which will be denoted by $P_t^{\mu,\mathcal{H}}$. Clearly, if $\varphi \in B_b(\mathcal{H})$, for every $\mu > 0$ and $\mathfrak{z} \in \mathcal{H}_1$ we have

$$P_t^{\mu,\mathcal{H}_1}\varphi(\mathfrak{z}) = P_t^{\mu,\mathcal{H}}\varphi(\mathfrak{z}), \qquad t \geq 0.$$

In Sect. 5 we will also show that under the same Hypotheses, for every $\mu > 0$, the semigroup $P_t^{\mu,\mathcal{H}}$ admits an invariant measure $\nu_\mu^{\mathcal{H}}$ in $\mathcal{H}$, with supp $(\nu_\mu^{\mathcal{H}}) \subset \mathcal{H}_1$. In particular, since supp $(\nu_\mu^{\mathcal{H}}) \subset \mathcal{H}_1$ and $\mathcal{B}(\mathcal{H}_1) \subset \mathcal{B}(\mathcal{H})$, we will have that $\nu_\mu^{\mathcal{H}}$ is also a probability measure on $\mathcal{H}_1$. In what follows, it will be convenient to denote the restriction of $\nu_\mu^{\mathcal{H}}$ to $\mathcal{H}_1$ by $\nu_\mu^{\mathcal{H}_1}$.

Now, we recall that, given a lower semicontinuous metric α on H^{-1}, it is possible to introduce the distance $\mathcal{W}_\alpha : \mathcal{P}(H^{-1}) \times \mathcal{P}(H^{-1}) \to [0, +\infty]$ defined by

$$\mathcal{W}_\alpha(\nu_1, \nu_2) = \sup_{[\varphi]_{\mathrm{Lip}^\alpha_{H^{-1}}} \leq 1} \left| \int_{H^{-1}} \varphi(\mathfrak{r})\nu_1(d\mathfrak{r}) - \int_{H^{-1}} \varphi(\mathfrak{r})\nu_2(d\mathfrak{r}) \right|,$$

where

$$[\varphi]_{\mathrm{Lip}^\alpha_{H^{-1}}} = \sup_{\substack{\mathfrak{r}_1, \mathfrak{r}_2 \in H \\ \mathfrak{r}_1 \neq \mathfrak{r}_2}} \frac{|\varphi(\mathfrak{r}_1) - \varphi(\mathfrak{r}_2)|}{\alpha(\mathfrak{r}_1, \mathfrak{r}_2)}.$$

Notice that, if $\mathcal{C}(\nu_1, \nu_2)$ is the set of all couplings of (ν_1, ν_2), the following Kantorovich–Rubinstein identity holds

$$\mathcal{W}_\alpha(\nu_1, \nu_2) = \inf_{\lambda \in \mathcal{C}(\nu_1, \nu_2)} \int\int \alpha(\mathfrak{r}_1, \mathfrak{r}_2)\, \lambda(d\mathfrak{r}_1, d\mathfrak{r}_2), \tag{3.3}$$

and in fact it is possible to prove that the infimum above is attained at some $\bar{\lambda}$.

At this point, we are ready to state the main result of this paper.

Theorem 3.2 *Assume Hypotheses 1 to 4, and define*

$$\alpha(\mathfrak{u}_1, \mathfrak{u}_2) := \|\mathfrak{u}_1 - \mathfrak{u}_2\|_{H^{-1}}, \qquad \mathfrak{u}_1, \mathfrak{u}_2 \in H^{-1}.$$

Then we have

$$\lim_{\mu \to 0} \mathcal{W}_\alpha\left(\Pi_1 \nu_\mu^{\mathcal{H}}, \nu\right) = 0,$$

where ν is the unique invariant measure for P_t^H, the transition semigroup associated to the limiting equation (1.5). *Moreover,*

$$\lim_{\mu\to 0} \Pi_1 \nu_\mu^{\mathcal{H}} = \nu, \qquad \textit{weakly in } H.$$

4 Plan of the Paper and List of Symbols

In Sect. 5, we will study the well-posedness of system (3.1) in $\mathcal{H}$. Namely, we will prove that for every initial condition $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathcal{H}$ system (3.1) admits a unique generalized solution in $\mathcal{H}$. This will allow us to introduce the transition semigroup $P_t^{\mu\,\mathcal{H}}$, $t \geq 0$, for every $\mu > 0$. Moreover, we will prove that $P_t^{\mu\,\mathcal{H}}$ admits an invariant measure $\nu_\mu^{\mathcal{H}}$ supported in $\mathcal{H}_1$.

In Sect. 6, we will prove suitable a-priori bounds for the solutions of system (3.1). In particular, we will prove some uniform bounds for the momenta of the invariant measures $\nu_\mu^{\mathcal{H}}$.

In Sect. 7, we will consider the limiting problem (1.5) in the space H. To this purpose, we will introduce the auxiliary problem (1.8) and we will first study its well-posedness in H. Due to (1.9), we will get the analogous results for problem (1.5). Moreover, we will study the well-posedness of (5.1) in H^{-1}.

In Sect. 8, we will investigate the ergodic behavior of the limiting Eq. (1.5) in H^{-1}. We will prove that the corresponding semigroup has a contractive property in H^{-1}. In particular, we it admits a unique invariant measure in H^{-1}. Moreover, we will show that such invariant measure is supported in H and its restriction to H is the unique invariant measure for the semigroup associated with Eq. (1.5) in H. Finally, we will show that this implies that the semigroup P_t^H admits a unique invariant measure.

In Sect. 9, we will finally prove the main result of this paper, Theorem 3.2.

Symbols Used Throughout the Paper

Almost all the symbols listed below are introduced for the first time in the Introduction. However, they are introduced again with all the needed details throughout the paper. In what follows, we describe what is their meaning and where their definition is given.

- $H = L^2(\mathcal{O})$, $H^1 = H_0^1(\mathcal{O})$, and $H^{-1} = H^{-1}(\mathcal{O})$, Sect. 2.
- $\mathcal{H} = H \times H^{-1}$, and $\mathcal{H}_1 = H^1 \times H$, Sect. 2.
- For every $\mu > 0$, $z_\mu = (u_\mu, v_\mu)$ denotes the solution of system (3.1), both in $\mathcal{H}$ and $\mathcal{H}_1$, Sect. 3.
- For every $\mu > 0$, $P_t^{\mu,\mathcal{H}_1}$ is the semigroup associated with system (3.1) in $\mathcal{H}_1$, and $P_t^{\mu,\mathcal{H}}$ is the semigroup associated with system (3.1) in $\mathcal{H}$, Sect. 3.
- For every $\mu > 0$, $\zeta_\mu = (u_\mu, \eta_\mu)$ denotes the solution of system (1.8), both in $\mathcal{H}$ and $\mathcal{H}_1$, Sect. 5.
- For every $\mu > 0$, $\nu_\mu^{\mathcal{H}}$ is an invariant measure for the semigroup $P_t^{\mu,\mathcal{H}}$, and $\nu_\mu^{\mathcal{H}_1}$ is its restriction to $\mathcal{H}_1$, which is invariant for $P_t^{\mu,\mathcal{H}_1}$, Sect. 5.3.

- For every $\mu > 0$, $\mathcal{N}_\mu$ is the Kolmogorov operator associated with system (3.1) in $\mathcal{H}_1$ and its associated semigroup $P_t^{\mu,\mathcal{H}_1}$, Sect. 6.
- u denotes the solution of the limiting problem (1.5), both in H and H^{-1}, Sect. 7.
- P_t^H is the transition semigroup associated with the limiting problem (1.5) in H, Sect. 7.2.
- ρ denotes the solution of the limiting problem (1.8), both in H and H^{-1}, Sect. 7.
- R_t^H is the transition semigroup associated with the auxiliary problem (1.8) in H and $R_t^{H^{-1}}$ is the transition semigroup associated with (1.8) in H^{-1}, Sect. 8.
- ν^H is the unique invariant measure of the semigroup R_t^H and $\nu^{H^{-1}}$ is the unique invariant measure of the semigroup $R_t^{H^{-1}}$, Sect. 8.
- $\nu^H \circ g$ is the unique invariant measure of the semigroup P_t^H, Sect. 9.

5 Generalized Solutions for System (3.1) and Invariant Measures

We have seen in Sect. 3 that Eq. (1.1) is equivalent to system (3.1). In fact, we can give another equivalent formulation for system (3.1). Actually, if g is the function introduced in (2.5) and we define

$$\eta := \frac{1}{\sqrt{\mu}}\big(\mu\partial_t u + g(u)\big), \quad \zeta = (u, \eta),$$

then system (3.1) can be rewritten as

$$d\zeta_\mu(t) = \mathcal{A}_\mu(\zeta_\mu(t))dt + \Sigma_\mu(\zeta_\mu(t))dw^Q(t), \quad \zeta_\mu(0) = \Big(\mathfrak{u}_0, \sqrt{\mu}\mathfrak{v}_0 + \frac{g(\mathfrak{u}_0)}{\sqrt{\mu}}\Big), \tag{5.1}$$

where we have denoted

$$\mathcal{A}_\mu(\zeta) := \frac{1}{\sqrt{\mu}}\Big(\eta - \frac{g(u)}{\sqrt{\mu}}, Au + F(u)\Big), \quad \zeta = (u, \eta) \in D(\mathcal{A}_\mu) = \mathcal{H}_1,$$

and

$$\Sigma_\mu(\zeta) := \frac{1}{\sqrt{\mu}}\big(0, \sigma(u)\big), \quad \zeta = (u, \eta) \in \mathcal{H}.$$

This means that, for every $\mu > 0$ and every $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathcal{H}_1$, the adapted $\mathcal{H}_1$-valued process $\zeta_\mu = (u_\mu, \eta_\mu)$ is the unique solution of Eq. (5.1), with $\zeta_\mu(0) = \big(\mathfrak{u}_0, \sqrt{\mu}\mathfrak{v}_0 + g(\mathfrak{u}_0)/\sqrt{\mu}\big)$, if and only if the adapted $\mathcal{H}_1$-valued process

$$z_\mu(t) := (u_\mu(t), v_\mu(t)) = (u_\mu(t), \eta_\mu(t)/\sqrt{\mu} - g(u_\mu(t))/\mu), \qquad t \geq 0,$$

is the unique solution of system (3.1), with $z_\mu(0) = \mathfrak{z}_0 := (\mathfrak{u}_0, \mathfrak{v}_0)$. The reason why we have introduced the equivalent problem (5.1) is that, in the presence of a non-constant friction γ, while it is not clear how to study the well-posedness of system (3.1), the analogous problem for (5.1) can be handled in a more direct way, both in $\mathcal{H}_1$ and in $\mathcal{H}$, thanks to the theory of non-linear quasi-dissipative operators.

As a matter of fact, in [10] it has been proven that, under Hypotheses 1, 2 and 3 (without condition (2.7)), for every $\zeta^0 \in L^2(\Omega; \mathcal{H}_1)$ and for every $\mu, T > 0$, there exists a unique solution $\zeta_\mu \in L^2(\Omega; C([0,T]; \mathcal{H}_1))$ for Eq. (5.1), with $\zeta_\mu(0) = \zeta^0$, and this has allowed to conclude that for every $(\mathfrak{u}_0, \mathfrak{v}_0) \in L^2(\Omega; \mathcal{H}_1)$, and for every $\mu, T > 0$, there exists a unique solution $z_\mu \in L^2(\Omega; C([0,T]; \mathcal{H}_1))$ to Eq. (3.1), with $z_\mu(0) = (\mathfrak{u}_0, \mathfrak{v}_0)$.

5.1 Generalized Solutions for System (5.1)

In order to study the existence and uniqueness of generalized solutions for system (3.1), we study the analogous problem for system (5.1). As for (3.1), we have the following definition of generalized solution for system (5.1).

Definition 5.1 For every $\mu, T > 0$ and every $\zeta^0 \in \mathcal{H}$, we say that $\zeta_\mu \in L^2(\Omega; C([0,T]; \mathcal{H}))$ is a generalized solution of problem (5.1), with initial condition ζ^0, if for every sequence $\{\zeta^0_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_1$ converging to ζ^0 in $\mathcal{H}$, as $n \to +\infty$, it holds

$$\lim_{n\to+\infty} \mathbb{E}\,\|\zeta_{\mu,n} - \zeta_\mu\|^2_{C([0,T];\mathcal{H})},$$

where $\zeta_{\mu,n} \in L^2(\Omega; C([0,T]; \mathcal{H}_1))$ is the unique solution of Eq. (5.1) with initial condition ζ^0_n.

The following result holds.

Lemma 5.2 *Under Hypotheses* 1, 2 and 3*, for every* $\mu, T > 0$ *and every* $\zeta^0 \in \mathcal{H}$*, there exists a unique generalized solution* $\zeta_\mu \in L^2(\Omega; C([0,T]; \mathcal{H}))$ *for Eq.* (5.1)*. Moreover, if* ζ^1_μ, ζ^2_μ *are two generalized solutions of* (5.1)*, with initial conditions* $\zeta^1, \zeta^2 \in \mathcal{H}$*, respectively, then*

$$\mathbb{E} \sup_{t\in[0,T]} \|\zeta^1_\mu(t) - \zeta^2_\mu(t)\|^2_{\mathcal{H}} \le e^{c_\mu T}\,\|\zeta^1 - \zeta^2\|^2_{\mathcal{H}}, \tag{5.2}$$

for some constant c_μ*.*

Proof Without any loss of generality, we assume $\mu = 1$, and for simplicity of notation, we denote $\mathcal{A}_1$ and Σ_1 by $\mathcal{A}$ and Σ, respectively. In [10], it is proved that the operator $\mathcal{A}$ is quasi-m-dissipative in $\mathcal{H}$. Namely, there exists $\eta \ge 0$ such that for every $\zeta, \theta \in D(\mathcal{A})$

$$\left\langle \mathcal{A}(\zeta) - \mathcal{A}(\theta), \zeta - \theta \right\rangle_{\mathcal{H}} \le \eta\,\|\zeta - \theta\|^2_{\mathcal{H}}, \tag{5.3}$$

and there exists $\lambda_0 > 0$ such that

$$\text{Range}(I - \lambda\mathcal{A}) = \mathcal{H}, \quad \lambda \in (0, \lambda_0).$$

Now, let $\zeta^0 \in \mathcal{H}$ and let $\{\zeta_n^0\}_{n\in\mathbb{N}} \subset \mathcal{H}_1$ be any sequence converging to ζ^0 in $\mathcal{H}$. For every $n \in \mathbb{N}$, we denote by ζ_n the unique solution of Eq. (5.1) with initial condition $\zeta_n(0) = \zeta_n^0$. By applying Itô's formula, thanks to (2.1) and (5.3), we get

$$\begin{aligned}
&\frac{1}{2}d\,\|\zeta_n(t)-\zeta_m(t)\|_{\mathcal{H}}^2\\
&\quad=\big\langle \mathcal{A}(\zeta_n(t))-\mathcal{A}(\zeta_m(t)),\zeta_n(t)-\zeta_m(t)\big\rangle_{\mathcal{H}}dt+\frac{1}{2}\,\|\Sigma(\zeta_n(t))-\Sigma(\zeta_m(t))\|_{\mathcal{L}_2(H_Q,\mathcal{H})}^2\,dt\\
&\qquad+\big\langle \zeta_n(t)-\zeta_m(t),\big[\Sigma(\zeta_n(t))-\Sigma(\zeta_m(t))\big]dw^Q(t)\big\rangle_{\mathcal{H}}\\
&\quad\leq c\,\|\zeta_n(t)-\zeta_m(t)\|_{\mathcal{H}}^2\,dt+\big\langle \zeta_n(t)-\zeta_m(t),\big[\Sigma(\zeta_n(t))-\Sigma(\zeta_m(t))\big]dw^Q(t)\big\rangle_{\mathcal{H}}.
\end{aligned}\tag{5.4}$$

Due to (2.3) we have

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \big\langle \zeta_n(r)-\zeta_m(r),\big[\Sigma(\zeta_n(r))-\Sigma(\zeta_m(r))\big]dw^Q(r)\big\rangle_{\mathcal{H}}\right|\\
&\quad\leq c\,\mathbb{E}\left(\int_0^t \|\zeta_n(s)-\zeta_m(s)\|_{\mathcal{H}}^4\,ds\right)^{\frac{1}{2}}\leq\frac{1}{4}\,\mathbb{E}\sup_{s\in[0,t]}\|\zeta_n(s)-\zeta_m(s)\|_{\mathcal{H}}^2\\
&\qquad+c\int_0^t \mathbb{E}\|\zeta_n(s)-\zeta_m(s)\|_{\mathcal{H}}^2\,ds.
\end{aligned}$$

Thus, if we first integrate both sides in (5.4) with respect to time, and then take the supremum and the expectation, we get

$$\mathbb{E}\sup_{s\in[0,t]}\|\zeta_n(s)-\zeta_m(s)\|_{\mathcal{H}}^2\leq\left\|\zeta_n^0-\zeta_m^0\right\|_{\mathcal{H}}^2+c\int_0^t \mathbb{E}\,\|\zeta_n(s)-\zeta_m(s)\|_{\mathcal{H}}^2\,ds,$$

and the Gronwall's inequality gives

$$\mathbb{E}\sup_{s\in[0,t]}\|\zeta_n(s)-\zeta_m(s)\|_{\mathcal{H}}^2\leq e^{c\,t}\left\|\zeta_n^0-\zeta_m^0\right\|_{\mathcal{H}}^2,\qquad t\geq 0,$$

for some constant c. In particular, this implies that the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ is Cauchy in the space $L^2(\Omega; C([0,T];\mathcal{H}))$, so there exists a limit $\zeta \in L^2(\Omega; C([0,T];\mathcal{H}))$. It is easy to see that the limit ζ does not depend on the choice of the sequence $\{\zeta_n^0\} \subset \mathcal{H}_1$, which implies the uniqueness of generalized solutions. Finally, by using a similar argument as above, we obtain (5.2). □

Remark 5.3 When $\zeta^0 \in \mathcal{H}_1$, the unique generalized solution ζ_μ of Eq. (5.1) coincides with its unique classical solution.

5.2 Generalized Solutions for System (3.1)

Due to Hypothesis 2, it is immediate to check that $z_\mu = (u_\mu, v_\mu)$ is a generalized solution to (3.1), with initial condition $\mathfrak{z}_0 = (\mathfrak{u}_0, \mathfrak{v}_0)$, if and only if $\zeta_\mu = \big(u_\mu, \sqrt{\mu}v_\mu + g(u_\mu)/\sqrt{\mu}\big)$ is a generalized solution for system (5.1), with initial condition $\zeta_\mu(0) = (\mathfrak{u}_0, \sqrt{\mu}\mathfrak{v}_0 + g(\mathfrak{u}_0)/\sqrt{\mu})$. In this case, we have

$$u_\mu = \Pi_1 \zeta_\mu, \quad v_\mu = \frac{1}{\mu}\Big(- g(u_\mu) + \sqrt{\mu}\, \Pi_2 \zeta_\mu \Big), \quad \mu > 0. \tag{5.5}$$

Thus, as a consequence of Lemma 5.2 and Remark 5.3, we have the following result.

Proposition 5.4 *Fix* $(\mathfrak{u}_0, \mathfrak{v}_0) \in \mathcal{H}$ *and assume Hypotheses 1, 2 and 3. Then, for every* $\mu, T > 0$ *there exists a unique generalized solution* $z_\mu \in L^2(\Omega; C([0,T]; \mathcal{H}))$ *for system* (3.1).

5.3 Existence of Invariant Measures for System (3.1)

We are proving now that, for every fixed $\mu > 0$, system (3.1) admits an invariant measure $\nu_\mu^{\mathcal{H}}$ in $\mathcal{H}$, which is supported on $\mathcal{H}_1$.

Proposition 5.5 *Assume Hypotheses 1, 2 and 3. Then, for every* $\mu > 0$*, the semigroup* $P_t^{\mu,\mathcal{H}}$ *admits an invariant measure* $\nu_\mu^{\mathcal{H}}$ *in* $\mathcal{H}$*, with supp* $(\nu_\mu^{\mathcal{H}}) \subset \mathcal{H}_1$.

Proof First, if $z_\mu^{\mathfrak{z}_1}$ and $z_\mu^{\mathfrak{z}_2}$ are generalized solutions to system (3.1), with initial conditions $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathcal{H}$, respectively, then due to (5.2) and (5.5), it is easy to see that for every $t \geq 0$

$$\mathbb{E}\left\| z_\mu^{\mathfrak{z}_1}(t) - z_\mu^{\mathfrak{z}_2}(t))\right\|_{\mathcal{H}} \leq c_\mu(t)\, \|\mathfrak{z}_1 - \mathfrak{z}_2\|_{\mathcal{H}}\,,$$

for some $c_\mu(t) > 0$. This means that the transition semigroup $P_t^{\mu,\mathcal{H}}$ is Feller on $\mathcal{H}$.

Now, for every $\mathfrak{z} \in \mathcal{H}$ we introduce the following family of measures on $\mathcal{H}$

$$\Gamma_t^\mu(\mathfrak{z}, \cdot) := \frac{1}{t}\int_0^t (P_t^{\mu,\mathcal{H}})^\star \delta_{\mathfrak{z}}\, dt, \qquad t > 0,$$

and for every $R > 0$ we define the set

$$B_R := \Big\{ \mathfrak{z} \in \mathcal{H}_1 : \|\mathfrak{z}\|_{\mathcal{H}_1} \leq R \Big\}.$$

Then, from (6.1) and (6.2) with $\mu = 1$, we have

$$\Gamma_t^\mu(0, B_R^c) = \frac{1}{t}\int_0^t \mathbb{P}\Big(\left\| z_\mu^0(s)\right\|_{\mathcal{H}_1} > R\Big) ds \leq \frac{c}{R^2}, \quad t > 0, \qquad R > 0,$$

and, due to the compactness of the embedding of $\mathcal{H}_1$ into $\mathcal{H}$, this implies that the family of measures $\{\Gamma_t^\mu(0,\cdot)\}_{t\geq 0}$ is tight in $\mathcal{H}$. By the Prokhorov theorem, there exists some sequence $t_n \uparrow \infty$ such that $\Gamma_{t_n}^\mu(0,\cdot)$ converges weakly, as $n \to +\infty$, to a probability measure $\nu_\mu^{\mathcal{H}}$ that is invariant for $P_t^{\mu,\mathcal{H}}$. Moreover, since

$$\nu_\mu^{\mathcal{H}}(B_R^c) \leq \frac{c}{R^2}, \quad R > 0,$$

it follows that supp $(\nu_\mu^{\mathcal{H}}) \subset \mathcal{H}_1$. □

Remark 5.6 Since supp $(\nu_\mu^{\mathcal{H}}) \subset \mathcal{H}_1$ and $\mathcal{B}(\mathcal{H}_1) \subset \mathcal{B}(\mathcal{H})$, we have that $\nu_\mu^{\mathcal{H}}$ is also a probability measure on $\mathcal{H}_1$. In what follows, it will be convenient to denote the restriction of $\nu_\mu^{\mathcal{H}}$ to $\mathcal{H}_1$ by $\nu_\mu^{\mathcal{H}_1}$.

6 Some Uniform Bounds for System (3.1)

We have seen that for every $\mu > 0$ Eq. (3.1) has an invariant measure. In this section we will prove some uniform bounds for the moments of such family of invariant measures. To this purpose, we need to start with suitable uniform bounds for the solution (u_μ, v_μ) of system (3.1). Some of them have been already proved in [10, Proposition 4.2, Remark 4.3]. In what follows, we show how those bounds depend on time and on random initial conditions in $L^2(\Omega;\mathcal{H}_1)$.

Lemma 6.1 *Assume Hypotheses 1, 2 and 3, and fix $(\xi,\eta) \in L^2(\Omega;\mathcal{H}_1)$. For every $\mu, T > 0$, let $(u_\mu, v_\mu) \in L^2(\Omega; C([0,T];\mathcal{H}_1))$ be the unique solution to system (3.1) with initial conditions (ξ,η). Then there exist two constants $\mu_0 \in (0,1)$ and $c > 0$, independent of $T > 0$, such that for every $\mu \in (0,\mu_0)$ and $t \in [0,T]$*

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}\left\|u_\mu(s)\right\|_{H^1}^2 + \mu\,\mathbb{E}\sup_{s\in[0,t]}\left\|v_\mu(s)\right\|_H^2 + \int_0^t \mathbb{E}\left\|v_\mu(s)\right\|_H^2 ds\\ &\leq c\left(\frac{t}{\mu}+1\right) + c\left(\mathbb{E}\left\|\xi\right\|_{H^1}^2 + \mu\,\mathbb{E}\left\|\eta\right\|_H^2\right),\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}\left\|u_\mu(s)\right\|_H^2 + \int_0^t \mathbb{E}\left\|u_\mu(s)\right\|_{H^1}^2 ds\\ &\leq c\left(1 + t + \mathbb{E}\left\|\xi\right\|_H^2 + \mu\,\mathbb{E}\left\|\xi\right\|_{H^1}^2 + \mu^2\,\mathbb{E}\left\|\eta\right\|_H^2\right).\end{aligned} \tag{6.2}$$

Proof Let $(\mathfrak{u}_0, \mathfrak{v}_0) \in L^2(\Omega;\mathcal{H}_1)$ and let $(u_\mu, v_\mu) \in L^2(\Omega; C([0,T];\mathcal{H}_1))$ be the unique solution to system (3.1). By proceeding as in the proof of [10, Lemma 4.2], we have for every $\mu \in (0, 1)$

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} \left\|u_\mu(s)\right\|^2_{H^1} + \mu\, \mathbb{E} \sup_{s\in[0,t]} \left\|v_\mu(s)\right\|^2_H + \int_0^t \mathbb{E} \left\|v_\mu(s)\right\|^2_H ds \\
&\quad \leq c \left(\frac{t}{\mu} + \left(1 + \mathbb{E}\, \|\xi\|^2_{H^1} + \mu \mathbb{E}\, \|\eta\|^2_H \right) + \int_0^t \mathbb{E} \left\|u_\mu(s)\right\|^2_H ds \right).
\end{aligned} \tag{6.3}$$

Moreover, by proceeding as in the proof of [10, Lemma 4.1], we have $\mathbb{P}$-a.s.

$$\begin{aligned}
\frac{\gamma_0}{4} \left\|u_\mu(t)\right\|^2_H &\leq c \left(\|\xi\|^2_H + \mu^2 \|\eta\|^2_H\right) + c\, \mu^2 \left\|v_\mu(t)\right\|^2_H + \mu \int_0^t \left\|v_\mu(s)\right\|^2_H ds \\
&\quad - \int_0^t \left\|u_\mu(s)\right\|^2_{H^1} ds + \int_0^t \left\langle F(u_\mu(s)), u_\mu(s)\right\rangle_H ds + \int_0^t \left\langle u_\mu(s), \sigma(u_\mu(s)) dw^Q(s)\right\rangle_H.
\end{aligned}$$

Due to (2.9), for every $\delta > 0$ we have

$$\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \left\langle F(u_\mu(r)), u_\mu(r)\right\rangle_H dr \right| \leq \frac{1}{\alpha_1}(L_f + \delta) \int_0^t \mathbb{E} \left\|u_\mu(s)\right\|^2_{H^1} ds + c_\delta t.$$

Moreover, due to (2.2) we have

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \left\langle u_\mu(r), \sigma(u_\mu(r)) dw^Q(r)\right\rangle_H \right| \\
&\quad \leq c \left(\mathbb{E} \int_0^t \|u_\mu(s)\|^2_H ds \right)^{\frac{1}{2}} \leq \frac{\delta}{\alpha_1} \int_0^t \mathbb{E} \left\|u_\mu(t)\right\|^2_{H^1} dt + c_\delta.
\end{aligned}$$

According to (2.7), we can fix $\delta > 0$ such that

$$\frac{1}{\alpha_1}\left(L_f + 2\,\delta\right) < 1,$$

and this yields

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} \left\|u_\mu(s)\right\|_H^2 + \int_0^t \mathbb{E}\left\|u_\mu(s)\right\|_{H^1}^2\,ds \\
&\quad \le c\left(1+t+\mathbb{E}\,\|\xi\|_H^2+\mu^2\mathbb{E}\,\|\eta\|_H^2\right)+c\,\mu^2\,\mathbb{E}\sup_{s\in[0,t]}\left\|v_\mu(s)\right\|_H^2 \\
&\qquad +c\,\mu\int_0^t \mathbb{E}\left\|v_\mu(s)\right\|_H^2\,ds.
\end{aligned}
\tag{6.4}
$$

Thus (6.2) holds by combining (6.4) with (6.3). Finally, by combining (6.2) with (6.3), we complete the proof of (6.1). □

Lemma 6.2 *Let $\{(\xi_\mu, \eta_\mu)\}_{\mu\in(0,1)} \subset L^2(\Omega; \mathcal{H}_1)$ be a family of random variables such that*

$$
\sup_{\mu\in(0,1)} \mathbb{E}\Big(\left\|\xi_\mu\right\|_{H^1}^2+\mu\left\|\eta_\mu\right\|_H^2\Big)<\infty. \tag{6.5}
$$

If $(u_\mu, v_\mu) \in L^2(\Omega; C([0,T]; \mathcal{H}_1))$ is the solution to system (3.1) with initial condition (ξ_μ, η_μ), then there exist $\mu_T \in (0, \mu_0)$ and $c_T > 0$ such that for every $\mu \in (0, \mu_T)$

$$
\mathbb{E}\sup_{t\in[0,T]}\Big(\left\|u_\mu(t)\right\|_{H^1}^2+\mu\left\|v_\mu(t)\right\|_H^2\Big)\le\frac{c_T}{\sqrt{\mu}}+\Big(\mathbb{E}\left\|\xi_\mu\right\|_{H^1}^2+\mu\,\mathbb{E}\left\|\eta_\mu\right\|_H^2\Big). \tag{6.6}
$$

Proof If for every $\mu \in (0, \mu_0)$ and $t \in [0, T]$, we define

$$
L_\mu(t):=\left\|u_\mu(t)\right\|_{H^1}^2+\mu\left\|v_\mu(t)\right\|_H^2-\Big(\left\|\xi_\mu\right\|_{H^1}^2+\mu\left\|\eta_\mu\right\|_H^2\Big),
$$

then (6.6) is equivalent to

$$
\sqrt{\mu}\,\mathbb{E}\sup_{t\in[0,T]} L_\mu(t)\le c_T,\qquad \mu\in(0,\mu_T), \tag{6.7}
$$

for some constants $\mu_T \in (0, 1)$ and $c_T > 0$.

Now, if we assume (6.7) is not true, there exists a sequence $(\mu_k)_{k\in\mathbb{N}} \subset (0, \mu_0)$ converging to 0, as $k \to \infty$, such that

$$
\lim_{k\to\infty}\sqrt{\mu_k}\,\mathbb{E}\sup_{t\in[0,T]} L_{\mu_k}(t)=+\infty. \tag{6.8}
$$

For every $k \in \mathbb{N}$, the mapping $t \mapsto L_{\mu_k}(t)$ is continuous $\mathbb{P}$-a.s., so that there exists a random time $t_k \in [0, T]$ such that

$$
L_{\mu_k}(t_k)=\sup_{t\in[0,T]} L_{\mu_k}(t).
$$

As a consequence of Itô's formula, we have

$$\frac{1}{2}d\left(\|u_\mu(t)\|_{H^1}^2+\mu\,\|v_\mu(t)\|_H^2\right)$$

$$=\left(\langle F(u_\mu(t),v_\mu(t)\rangle_H-\langle\gamma(u_\mu(t))v_\mu(t),v_\mu(t)\rangle_H+\frac{1}{2\mu}\|\sigma(u_\mu(t))\|^2_{\mathcal{L}_2(H_Q,H)}\right)dt$$

$$+\langle v_\mu(t),\sigma(u_\mu(t))dw^Q(t)\rangle_H$$

$$\leq\left(c\,\left(\|u_\mu(t)\|_H^2+1\right)-\frac{\gamma_0}{2}\|v_\mu(t)\|_H^2+\frac{\sigma_\infty^2}{2\mu}\right)dt+\langle v_\mu(t),\sigma(u_\mu(t))dw^Q(t)\rangle_H.$$

Hence, if s is any random time such that $\mathbb{P}(s\leq t_k)=1$, we have

$$L_{\mu_k}(t_k)-L_{\mu_k}(s)\leq\frac{\sigma_\infty^2}{\mu_k}(t_k-s)+c\int_s^{t_k}\left(1+\left\|u_{\mu_k}(r)\right\|_H^2\right)dr+2\big(M_k(t_k)-M_k(s)\big),$$

where

$$M_k(t):=\int_0^t\left\langle v_{\mu_k}(r),\sigma(u_{\mu_k}(r))dw^Q(r)\right\rangle_H.$$

If we define

$$U_k:=c\int_0^T\left\|u_{\mu_k}(t)\right\|_H^2\,dt,\qquad M_k:=\sup_{t\in[0,T]}|M_k(t)|,$$

this implies that there exists some constant $\lambda>0$, independent of k, such that

$$L_{\mu_k}(t_k)-L_{\mu_k}(s)\leq\frac{\lambda}{\mu_k}(t_k-s)+U_k+4M_k,\tag{6.9}$$

and since $L_{\mu_k}(0)=0$, if we take $s=0$ we get

$$t_k\geq\frac{\mu_k}{\lambda}\left(L_{\mu_k}(t_k)-U_k-4M_k\right)=:\frac{\mu_k\theta_k}{\lambda}.$$

Now, on the set $E_k:=\{\theta_k>0\}$, we fix an arbitrary $s\in\left[t_k-\mu_k\theta_k/(2\lambda),t_k\right]$. Since $t_k-s\leq\mu_k\theta_k/(2\lambda)$, by using again (6.9) and recalling the definition of θ_k, we have

$$L_{\mu_k}(s)\geq L_{\mu_k}(t_k)-\frac{1}{2}\theta_k-U_k-4M_k=\frac{1}{2}\theta_k>0.\tag{6.10}$$

Hence, if we define

$$I_k := \int_0^T L^+_{\mu_k}(s)\, ds,$$

due to (6.10) we have

$$I_k \geq \int_{t_k - \frac{\mu_k \theta_k}{2\lambda}}^{t_k} L_{\mu_k}(s) ds \geq \frac{\mu_k}{4\lambda}\, \theta_k^2, \qquad \text{on } E_k,$$

so that

$$\mathbb{E}(I_k;\, E_k) \geq \mathbb{E}\bigg(\frac{\mu_k}{4\lambda}\theta_k^2;\, E_k\bigg). \tag{6.11}$$

Now, according to (6.1), (6.2) and (6.5)

$$\mathbb{E}\, U_k \leq c\Big(1 + T + \mathbb{E}\, \left\|\xi_{\mu_k}\right\|_H^2 + \mu_k\, \mathbb{E}\, \left\|\xi_{\mu_k}\right\|_{H^1}^2 + \mu_k^2\, \mathbb{E}\, \left\|\eta_{\mu_k}\right\|_H^2\Big) \leq c_T,$$

and

$$\begin{aligned}\mathbb{E}\, M_k \ &\leq c\Big(\int_0^T \mathbb{E}\left\|v_{\mu_k}(t)\right\|_H^2 dt\Big)^{\frac{1}{2}}\\ &\leq c\left(1 + \frac{T}{\mu_k} + \mathbb{E}\left\|\xi_{\mu_k}\right\|_{H^1}^2 + \mu_k\, \mathbb{E}\left\|\eta_{\mu_k}\right\|_H^2\right)^{\frac{1}{2}} \leq c_T\Big(1 + \frac{1}{\mu_k}\Big)^{\frac{1}{2}},\end{aligned}$$

so that

$$\limsup_{k\to\infty} \sqrt{\mu_k}\, (\mathbb{E}\, U_k + 4\, \mathbb{E}\, M_k) < +\infty.$$

Thanks to (6.8) this gives

$$\lim_{k\to\infty} \sqrt{\mu_k}\, \mathbb{E}(\theta_k) = +\infty,$$

and hence

$$\lim_{k\to\infty} \sqrt{\mu_k}\, \mathbb{E}(\theta_k;\, E_k) = +\infty. \tag{6.12}$$

Now, according to (6.11), we have

$$\mathbb{E}(I_k;\, E_k) \geq \frac{\mu_k}{4\lambda}\mathbb{E}\big(\theta_k^2;\, E_k\big) \geq \frac{\mu_k}{4\lambda}\, (\mathbb{E}(\theta_k;\, E_k))^2\,,$$

and due to (6.12), this implies

$$\lim_{k\to\infty} \mathbb{E}(I_k; E_k) = +\infty.$$

However, as a consequence of (6.1), (6.2) and (6.5), we have

$$\begin{aligned}\sup_{k\in\mathbb{N}} \mathbb{E}\, I_k &\leq \sup_{k\in\mathbb{N}} \int_0^T \mathbb{E}\,|L_{\mu_k}(s)|ds\\ &\leq c_T \sup_{k\in\mathbb{N}} \Big(1 + T + \mathbb{E}\,\|\xi_{\mu_k}\|^2_{H^1} + \mu_k \mathbb{E}\,\|\eta_{\mu_k}\|^2_H\Big) < +\infty,\end{aligned}$$

and this gives a contradiction, since $\mathbb{E}(I_k; E_k) \leq \mathbb{E}(I_k)$ for every $k \in \mathbb{N}$. In particular, this means that claim (6.7) is true, and (6.6) holds. □

Lemma 6.3 *For every $\mu > 0$, if $\nu_\mu^{\mathcal{H}} \in \mathcal{P}(\mathcal{H})$ is any invariant measure for $P_t^{\mu,\mathcal{H}}$ supported in $\mathcal{H}_1$, then $\nu_\mu^{\mathcal{H}_1} \in \mathcal{P}(\mathcal{H}_1)$ is invariant for $P_t^{\mu,\mathcal{H}_1}$. Moreover,*

$$\sup_{\mu\in(0,1)} \int_{\mathcal{H}_1} \Big(\|\mathfrak{u}\|^2_{H^1} + \mu\,\|\mathfrak{v}\|^2_H\Big)\nu_\mu^{\mathcal{H}_1}(d\mathfrak{u}, d\mathfrak{v}) < \infty. \tag{6.13}$$

Proof First, we show the invariance of $\nu_\mu^{\mathcal{H}_1}$ for $P_t^{\mu,\mathcal{H}_1}$. Due to the invariance of $\nu_\mu^{\mathcal{H}}$ in $\mathcal{H}$, for every $\varphi \in C_b(\mathcal{H})$ we have

$$\int_{\mathcal{H}} P_t^{\mu,\mathcal{H}}\varphi(\mathfrak{z})\nu_\mu^{\mathcal{H}}(d\mathfrak{z}) = \int_{\mathcal{H}} \varphi(\mathfrak{z})\nu_\mu^{\mathcal{H}}(d\mathfrak{z}).$$

Thus, since supp $(\nu_\mu^{\mathcal{H}}) \subset \mathcal{H}_1$ and $\mathcal{B}(\mathcal{H}_1) \subset \mathcal{B}(\mathcal{H})$, for every $\varphi \in C_b(\mathcal{H})$ we get

$$\int_{\mathcal{H}_1} P_t^{\mu,\mathcal{H}}\varphi(\mathfrak{z})\nu_\mu^{\mathcal{H}_1}(d\mathfrak{z}) = \int_{\mathcal{H}_1} \varphi(\mathfrak{z})\nu_\mu^{\mathcal{H}_1}(d\mathfrak{z}).$$

If $(\hat{e}_i)_{i\in\mathbb{N}} \subset \mathcal{H}_1$ is an orthonormal basis of $\mathcal{H}$, for every $n \in \mathbb{N}$ we denote by Π_n the projection of $\mathcal{H}$ onto $\mathcal{H}(n) := \text{span}(\hat{e}_1, \ldots, \hat{e}_n)$. We have that $\Pi_n : \mathcal{H} \to \mathcal{H}_1$ is continuous and

$$\|\pi_n h\|_{\mathcal{H}_1} \leq c_n\,\|h\|_{\mathcal{H}}, \quad h \in \mathcal{H}, \qquad \lim_{n\to\infty} \|\Pi_n h - h\|_{\mathcal{H}_1} = 0, \qquad h \in \mathcal{H}_1.$$

Hence, if for any $\varphi \in C_b(\mathcal{H}_1)$ and $n \in \mathbb{N}$, we define $\varphi_n := \varphi \circ \Pi_n$, we have $\varphi_n \in C_b(\mathcal{H})$ and

$$\lim_{n\to\infty} |\varphi_n(h) - \varphi(h)| = 0, \quad h \in \mathcal{H}_1.$$

For every $n \in \mathbb{N}$, we have $\sup_{n\in\mathbb{N}} \|\varphi_n\|_\infty \leq \|\varphi\|_\infty$, and the dominated convergence theorem implies that for any given $\mu > 0$ and $\varphi \in C_b(\mathcal{H}_1)$

$$\lim_{n\to\infty} P_t^{\mu,\mathcal{H}}\varphi_n(\mathfrak{z}) = \lim_{n\to\infty} \mathbb{E}\,\varphi_n\big(z_\mu^{\mathfrak{z}}(t)\big) = \mathbb{E}\,\varphi\big(z_\mu^{\mathfrak{z}}(t)\big) = P_t^{\mu,\mathcal{H}_1}\varphi(\mathfrak{z}), \quad \mathfrak{z} \in \mathcal{H}_1, \quad t \geq 0.$$

In particular, by taking the limit as n goes to infinity in both sides of

$$\int_{\mathcal{H}_1} P_t^{\mu,\mathcal{H}}\varphi_n(\mathfrak{z})\nu_\mu^{\mathcal{H}_1}(d\mathfrak{z}) = \int_{\mathcal{H}_1} \varphi_n(\mathfrak{z})\nu_\mu^{\mathcal{H}_1}(d\mathfrak{z}), \quad \varphi \in C_b(\mathcal{H}_1),$$

we conclude that

$$\int_{\mathcal{H}_1} P_t^{\mu,\mathcal{H}_1}\varphi(\mathfrak{z})\nu_\mu^{\mathcal{H}_1}(d\mathfrak{z}) = \int_{\mathcal{H}_1} \varphi(\mathfrak{z})\nu_\mu^{\mathcal{H}_1}(d\mathfrak{z}), \qquad \varphi \in C_b(\mathcal{H}_1),$$

and this implies the invariance of $\nu_\mu^{\mathcal{H}_1}$.

Next, in order to prove (6.13), we consider the Kolmogorov operator associated to $P_t^{\mu,\mathcal{H}_1}$ in $\mathcal{H}_1$

$$\mathcal{N}_\mu\varphi(\mathfrak{u},\mathfrak{v}) = \frac{1}{2\mu^2}\mathrm{Tr}_H\Big[\big(\sigma(\mathfrak{u})Q\big)(\sigma(\mathfrak{u})Q)^* D_{\mathfrak{v}}^2\varphi(\mathfrak{u},\mathfrak{v})\Big] + \big\langle \mathfrak{v}, D_{\mathfrak{u}}\varphi(\mathfrak{u},\mathfrak{v})\big\rangle_{H^1}$$
$$+\frac{1}{\mu}\big\langle A\mathfrak{u} - \gamma(\mathfrak{u})\mathfrak{v} + F(\mathfrak{u}), D_{\mathfrak{v}}\varphi(\mathfrak{u},\mathfrak{v})\big\rangle_H.$$

If, with the notations of Sect. 2, we define

$$\varphi_\mu(\mathfrak{u},\mathfrak{v}) := \frac{1}{2}\Big(\|\mathfrak{u}\|_{H^1}^2 + \mu\,\|\mathfrak{v}\|_H^2\Big) - \int_{\mathcal{O}} \mathfrak{f}(x,\mathfrak{u}(x))dx = \frac{1}{2}\Big(\|\mathfrak{u}\|_{H^1}^2 + \mu\,\|\mathfrak{v}\|_H^2\Big) - \Lambda(\mathfrak{u}),$$

due to (2.10) and to the fact that $\|\mathfrak{u}\|_{H^1}^2 = \langle(-A)\mathfrak{u},\mathfrak{u}\rangle_H$, we have

$$D_{\mathfrak{u}}\varphi_\mu(\mathfrak{u},\mathfrak{v}) = (-A)\mathfrak{u} - f(\cdot,\mathfrak{u}), \qquad D_{\mathfrak{v}}\varphi_\mu(\mathfrak{u},\mathfrak{v}) = \mu\,\mathfrak{v}, \qquad D_{\mathfrak{v}}^2\varphi_\mu(\mathfrak{u},\mathfrak{v}) = \mu\, I_H.$$

Then, we have

$$\begin{aligned}
&\mathcal{N}_\mu\varphi_\mu(\mathfrak{u},\mathfrak{v})\\
&= \frac{1}{2\mu}\mathrm{Tr}_H\Big[\big(\sigma(\mathfrak{u})Q\big)(\sigma(\mathfrak{u})Q)^*\Big] + \big\langle\mathfrak{v},\mathfrak{u} - (-A)^{-1}F(\mathfrak{u})\big\rangle_{H^1} + \frac{1}{\mu}\big\langle A\mathfrak{u} - \gamma(\mathfrak{u})\mathfrak{v} + F(\mathfrak{u}),\mu\mathfrak{v}\big\rangle_H\\
&= \frac{1}{2\mu}\,\|\sigma(\mathfrak{u})\|_{\mathcal{L}_2(H_Q,H)}^2 + \big\langle\mathfrak{v}, -A\mathfrak{u} - F(\mathfrak{u})\big\rangle_H + \frac{1}{\mu}\big\langle A\mathfrak{u} - \gamma(\mathfrak{u})\mathfrak{v} + F(\mathfrak{u}),\mu\mathfrak{v}\big\rangle_H\\
&= \frac{1}{2\mu}\,\|\sigma(\mathfrak{u})\|_{\mathcal{L}_2(H_Q,H)}^2 - \big\langle\gamma(\mathfrak{u})\mathfrak{v},\mathfrak{v}\big\rangle_H \leq \frac{1}{2\mu}\,\|\sigma(\mathfrak{u})\|_{\mathcal{L}_2(H_Q,H)}^2 - \gamma_0\,\|\mathfrak{v}\|_H^2.
\end{aligned} \tag{6.14}$$

By the invariance of $\nu_\mu^{\mathcal{H}_1}$ in $\mathcal{H}_1$, we have

$$\int_{\mathcal{H}_1} \mathcal{N}_\mu \varphi_\mu(\mathfrak{u},\mathfrak{v})\, \nu_\mu^{\mathcal{H}_1}(d\mathfrak{u}, d\mathfrak{v}) = 0,$$

and thus, due to (6.14) and (2.2),

$$\sup_{\mu\in(0,1)} \mu \int_{\mathcal{H}_1} \|\mathfrak{v}\|_H^2\, \nu_\mu^{\mathcal{H}_1}(d\mathfrak{u}, d\mathfrak{v}) < \infty. \tag{6.15}$$

Next, we consider the function

$$\psi_\mu(\mathfrak{u},\mathfrak{v}) := \frac{1}{2}\Big(\mu\, \|\mathfrak{u}\|_{H^1}^2 + \|g(\mathfrak{u}) + \mu\mathfrak{v}\|_H^2\Big).$$

We have

$$\begin{aligned}
&D_\mathfrak{u}\psi(\mathfrak{u},\mathfrak{v}) = \mu\,(-A)\mathfrak{u} + \gamma(\mathfrak{u})\,(g(\mathfrak{u}) + \mu\,\mathfrak{v})\,,\\
&D_\mathfrak{v}\psi(\mathfrak{u},\mathfrak{v}) = \mu\,(g(\mathfrak{u}) + \mu\,\mathfrak{v})\,, \quad D^2_\mathfrak{v}\psi(\mathfrak{u},\mathfrak{v}) = \mu^2\, I_H,
\end{aligned}$$

so that

$$\begin{aligned}
\mathcal{N}_\mu\psi_\mu(\mathfrak{u},\mathfrak{v}) &= \frac{1}{2}\mathrm{Tr}_H\Big[\big(\sigma(\mathfrak{u})Q\big)\big(\sigma(\mathfrak{u})Q\big)^*\Big] + \big\langle\mathfrak{v}, \mu\mathfrak{u} + (-A)^{-1}\gamma(\mathfrak{u})g(\mathfrak{u}) + \mu(-A)^{-1}\gamma(\mathfrak{u})\mathfrak{v}\big\rangle_{H^1}\\
&+\frac{1}{\mu}\big\langle A\mathfrak{u} - \gamma(\mathfrak{u})\mathfrak{v} + F(\mathfrak{u}), \mu^2\mathfrak{v} + \mu g(\mathfrak{u})\big\rangle_H\\
&= \frac{1}{2}\,\|\sigma(\mathfrak{u})\|^2_{\mathcal{L}_2(H_Q,H)} + \mu\big\langle\mathfrak{v}, -A\mathfrak{u} + \frac{\gamma(\mathfrak{u})}{\mu}g(\mathfrak{u}) + \gamma(\mathfrak{u})\mathfrak{v}\big\rangle_H\\
&+\big\langle A\mathfrak{u} - \gamma(\mathfrak{u})\mathfrak{v} + F(\mathfrak{u}), \mu\mathfrak{v} + g(\mathfrak{u})\big\rangle_H\\
&= \frac{1}{2}\,\|\sigma(\mathfrak{u})\|^2_{\mathcal{L}_2(H_Q,H)} - \big\langle\gamma(\mathfrak{u})\nabla\mathfrak{u}, \nabla\mathfrak{u}\big\rangle_H + \mu\big\langle F(\mathfrak{u}), \mathfrak{v}\big\rangle_H + \big\langle F(\mathfrak{u}), g(\mathfrak{u})\big\rangle_H.
\end{aligned}$$

Note that for every $\delta > 0$ and $\mu \in (0, 1)$

$$\mu\big|\big\langle F(\mathfrak{u}), \mathfrak{v}\big\rangle_H\big| \leq \mu\,\|F(\mathfrak{u})\|_H\,\|\mathfrak{v}\|_H \leq \delta\big(1 + \|\mathfrak{u}\|^2_{H^1}\big) + c_\delta\,\mu^2\,\|\mathfrak{v}\|_H^2\,,$$

and

$$\big|\big\langle F(\mathfrak{u}), g(\mathfrak{u})\big\rangle_H\big| \leq \big(L_f\gamma_1 + \delta\big)\,\|\mathfrak{u}\|_H^2 + c_\delta \leq \frac{1}{\alpha_1}\big(L_f\gamma_1 + \delta\big)\,\|\mathfrak{u}\|^2_{H^1} + c_\delta,$$

so that, due to the invariance of $\nu_\mu^{\mathcal{H}}$, we have

$$\gamma_0 \int_{\mathcal{H}_1} \|\mathfrak{u}\|_{H^1}^2 \, \nu_\mu^{\mathcal{H}}\,(d\mathfrak{u}, d\mathfrak{v})$$

$$\leq \frac{\sigma_\infty^2}{2} + \left(\frac{L_f\,\gamma_1}{\alpha_1} + 2\,\delta\right) \int_{\mathcal{H}_1} \|\mathfrak{u}\|_{H^1}^2 \, \nu_\mu^{\mathcal{H}}\,(d\mathfrak{u}, d\mathfrak{v}) + c_\delta + c_\delta\,\mu^2 \int_{\mathcal{H}_1} \|\mathfrak{v}\|_H^2 \, \nu_\mu^{\mathcal{H}}\,(d\mathfrak{u}, d\mathfrak{v}).$$

Thanks to (2.7), this implies that we can take $\delta > 0$ sufficiently small so that

$$\frac{L_f\,\gamma_1}{\alpha_1} + 2\,\delta < \gamma_0,$$

and then

$$\int_{\mathcal{H}_1} \|\mathfrak{u}\|_{H^1}^2 \, \nu_\mu^{\mathcal{H}_1}(d\mathfrak{u}, d\mathfrak{v}) \leq c \left(1 + \mu^2 \int_{\mathcal{H}_1} \|\mathfrak{v}\|_H^2 \, \nu_\mu^{\mathcal{H}_1}(d\mathfrak{u}, d\mathfrak{v})\right), \quad \mu \in (0, 1).$$

By combining this with (6.15), we complete the proof of (6.13). □

Remark 6.4 1. In Proposition 5.5, we have seen that for every $\mu > 0$ the semigroup $P_t^{\mu,\mathcal{H}}$ admits an invariant measure. Thanks to Lemma 6.3, this implies that for every $\mu > 0$ the transition semigroup $P_t^{\mu,\mathcal{H}_1}$ admits an invariant measure in $\mathcal{H}_1$.

2. As a consequence of (6.13), we have

$$\sup_{\mu \in (0,1)} \int_{\mathcal{H}} \left(\|\mathfrak{u}\|_{H^1}^2 + \mu\,\|\mathfrak{v}\|_H^2\right) \nu_\mu^{\mathcal{H}}(d\mathfrak{u}, d\mathfrak{v}) < \infty. \tag{6.16}$$

7 The Limiting Equation

As we have mentioned in the Introduction, in order to study the limiting problem

$$\begin{cases} \gamma(u(t,x))\partial_t u(t,x) = \Delta u(t,x) + f(x, u(t,x)) - \dfrac{\gamma'(u(t,x))}{2\gamma^2(u(t,x))} \displaystyle\sum_{i=1}^{\infty} |\sigma(u(t,\cdot))\, Qe_i(x)|^2 \\ \qquad\qquad +\sigma(u(t,\cdot))\partial_t w^Q(t,x) \\ u(0,x) = \mathfrak{u}_0(x), \qquad u(t,\cdot)\big|_{\partial\mathcal{O}} = 0, \end{cases} \tag{7.1}$$

we consider first the following quasilinear stochastic parabolic equation

$$\begin{cases} \partial_t \rho(t,x) = \operatorname{div}\big(b(\rho(t,x))\nabla\rho(t,x)\big) + f_g(x, \rho(t,x)) + \sigma_g(\rho(t,\cdot))\partial_t w^Q(t,x), \\ \rho(0,x) = \mathfrak{r}_0(x), \qquad \rho(t,\cdot)|_{\partial\mathcal{O}} = 0, \end{cases} \tag{7.2}$$

where for every $r \in \mathbb{R}$ and $x \in \mathcal{O}$

$$b(r) := \frac{1}{\gamma(g^{-1}(r))}, \qquad f_g(x,r) := f(x, g^{-1}(r)),$$

and for every $h \in H$

$$\sigma_g(h) := \sigma\big(g^{-1} \circ h\big).$$

The rationale behind this approach stems from the inherent advantage of initially establishing the small-mass limit of $g(u_\mu)$ to ρ, alongside their stationary counterparts, before moving back to the original problem involving u_μ and u. As explained in [10], due to a generalized Itô's formula, the solutions u and ρ of Eqs. (7.1) and (7.2), respectively, are related by

$$\rho^{\mathfrak{r}_0}(t) = g(u^{\mathfrak{u}_0}(t)), \qquad t \geq 0, \qquad \mathfrak{r}_0 := g(\mathfrak{u}_0). \tag{7.3}$$

From Hypothesis 2, we know

$$\frac{1}{\gamma_1} \leq b(r) \leq \frac{1}{\gamma_0}, \quad r \in \mathbb{R}.$$

Moreover, if we define

$$F_g(h)(x) := f_g(x, h(x)), \qquad x \in \mathcal{O},$$

due to Hypotheses 1, 2 and 3, and due to (6.14), for every $h_1, h_2 \in H$ we have

$$\big\|F_g(h_1) - F_g(h_2)\big\|_{H^{-1}} \leq \frac{1}{\sqrt{\alpha_1}} \big\|F_g(h_1) - F_g(h_2)\big\|_H \leq \frac{L_f}{\sqrt{\alpha_1}\gamma_0} \|h_1 - h_2\|_H \,,$$

and

$$\begin{aligned}&\big\|\sigma_g(h_1) - \sigma_g(h_2)\big\|_{\mathcal{L}_2(H_Q, H^{-1})}\\ &\quad \leq \frac{1}{\sqrt{\alpha_1}} \big\|\sigma_g(h_1) - \sigma_g(h_2)\big\|_{\mathcal{L}_2(H_Q, H)} \leq \frac{\sqrt{L_\sigma}}{\sqrt{\alpha_1}\gamma_0} \|h_1 - h_2\|_H \,.\end{aligned}$$

Moreover, for every $\delta > 0$

$$\big|\langle F_g(h), h\rangle_H\big| \leq \left(\frac{L_f}{\gamma_0} + \delta\right) \|h\|_H^2 + c_\delta \leq \frac{1}{\alpha_1}\left(\frac{L_f}{\gamma_0} + \delta\right) \|h\|_{H^1}^2 + c_\delta, \quad h \in H^1, \tag{7.4}$$

and

$$\big|\langle F_g(h), h\rangle_{H^{-1}}\big| \leq \frac{1}{\alpha_1} \big\|F_g(h)\big\|_H \|h\|_H \leq \frac{1}{\alpha_1}\left(\frac{L_f}{\gamma_0} + \delta\right) \|h\|_H^2 + c_\delta, \quad h \in H. \tag{7.5}$$

Finally, thanks to (2.2) we have

$$\left\|\sigma_g(h)\right\|_{\mathcal{L}_2(H_Q,H)} \leq \sigma_\infty, \qquad h \in H. \tag{7.6}$$

7.1 Well-Posedness of Eq. (7.2) in *H*

Throughout this subsection we will not need to assume condition (2.7) in Hypothesis 3. Namely, we will just assume that the mapping $f : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ is measurable, with

$$\sup_{x\in\mathcal{O}} |f(x,0)| < \infty, \qquad \sup_{x\in\mathcal{O}} |f(x,r) - f(x,s)| \leq c\,|r-s|, \quad r,s \in \mathbb{R}. \tag{7.7}$$

As a consequence of the limiting result proved in [10], the well-posedness of Eq. (7.2) has been established when the initial condition $\mathfrak{r}_0 \in H^1$. Here we want to prove the existence and uniqueness of the solution of (7.2) when $\mathfrak{r}_0 \in L^2(\Omega; H)$.

Definition 7.1 Let $\mathfrak{r}_0 \in L^2(\Omega; H)$. An adapted process $\rho \in L^2(\Omega; C([0,T]; H) \cap L^2(0,T; H^1))$ is a solution of Eq. (7.2) if for every $\varphi \in C_0^\infty(\mathcal{O})$

$$\begin{aligned}\left\langle \rho(t), \varphi \right\rangle_H &= \left\langle \mathfrak{r}_0, \varphi \right\rangle_H - \int_0^t \left\langle b(\rho(s))\nabla\rho(s), \nabla\varphi \right\rangle_H ds \\ &\quad + \int_0^t \left\langle F_g(\rho(s)), \varphi \right\rangle_H ds + \int_0^t \left\langle \varphi, \sigma_g(\rho(s)) dw^Q(s) \right\rangle_H, \quad \mathbb{P}\text{-a.s.}\end{aligned} \tag{7.8}$$

In order to study Eq. (7.2), we first consider the following approximating problem

$$\begin{cases} \partial_t \rho^\epsilon(t,x) = \operatorname{div}\big(b(\rho^\epsilon(t,x))\nabla\rho^\epsilon(t,x)\big) \\ -\epsilon\Delta^2\rho^\epsilon(t,x) + f(x,\rho(t,x)) + \sigma_g(\rho(s,\cdot))\partial_t w^Q(t,x), \\ \\ \rho^\epsilon(0,x) = \mathfrak{r}_0, \quad \rho^\epsilon(t,\cdot)|_{\partial\mathcal{O}} = 0, \end{cases} \tag{7.9}$$

with $0 < \epsilon << 1$ (for a similar approach see e.g. [14]).

Lemma 7.2 *Assume Hypotheses* 1 and 2 *and condition* (7.7). *Then, for every* $\epsilon, T > 0$ *and every* $\rho_0 \in L^2(\Omega; H)$*, Eq.* (7.9) *admits a unique solution*

$$\rho_\epsilon \in L^2(\Omega; C([0,T]; H) \cap L^2(0,T; H^2)).$$

Moreover, there exists some $c_T > 0$ *such that for every* $\epsilon > 0$

$$\begin{aligned}&\mathbb{E} \sup_{t\in[0,T]} \|\rho^\epsilon(t)\|_H^2 + \frac{2}{\gamma_1}\int_0^t \mathbb{E}\|\nabla\rho^\epsilon(s)\|_H^2 ds \\ &+2\epsilon\int_0^t \mathbb{E}\|\Delta\rho^\epsilon(s)\|_H^2 ds \leq c_T\left(1 + \mathbb{E}\,\|\mathfrak{r}_0\|_H^2\right).\end{aligned} \tag{7.10}$$

Proof The uniqueness and the existence of solutions for Eq. (7.9) can be proven by proceeding as in the proof of [18, Theorem 5.1].

In order to prove the energy estimate (7.10), we apply Itô's formula and we get

$$\frac{1}{2}\left\|\rho^\epsilon(t)\right\|_H^2=\frac{1}{2}\left\|\mathfrak{r}_0\right\|_H^2+\int_0^t\left\langle\operatorname{div}\left(b(\rho^\epsilon(s))\nabla\rho^\epsilon(s)\right),\rho^\epsilon(s)\right\rangle_H ds$$
$$-\epsilon\int_0^t\left\langle\Delta^2\rho^\epsilon(s),\rho^\epsilon(s)\right\rangle_H ds$$
$$+\int_0^t\left\langle F_g(\rho^\epsilon(s)),\rho^\epsilon(s)\right\rangle_H ds+\frac{1}{2}\int_0^t\left\|\sigma_g(\rho^\epsilon(s))\right\|_{\mathcal{L}_2(H_Q,H)}^2 ds$$
$$+\int_0^t\left\langle\rho^\epsilon(s),\sigma_g(\rho^\epsilon(s))dw^Q(s)\right\rangle_H$$
$$\leq\frac{1}{2}\left\|\mathfrak{r}_0\right\|_H^2-\frac{1}{\gamma_1}\int_0^t\left\|\nabla\rho^\epsilon(s)\right\|_H^2 ds-\epsilon\int_0^t\left\|\Delta\rho^\epsilon(s)\right\|_H^2 ds$$
$$+c\int_0^t\left(1+\left\|\rho^\epsilon(s)\right\|_H^2\right)ds+2\int_0^t\left\langle\rho^\epsilon(s),\sigma_g(\rho^\epsilon(s))dw^Q(s)\right\rangle_H.$$

Note that

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s\left\langle\rho^\epsilon(r),\sigma_g(\rho^\epsilon(r))dw^Q(r)\right\rangle_H\right|\leq c\int_0^t\mathbb{E}\left\|\rho^\epsilon(s)\right\|_H^2 ds+c_T,$$

and hence

$$\mathbb{E}\sup_{s\in[0,t]}\left\|\rho^\epsilon(s)\right\|_H^2+\frac{2}{\gamma_1}\mathbb{E}\int_0^t\left\|\nabla\rho^\epsilon(s)\right\|_H^2 ds+2\epsilon\,\mathbb{E}\int_0^t\left\|\Delta\rho^\epsilon(s)\right\|_H^2 ds$$
$$\leq\mathbb{E}\left\|\mathfrak{r}_0\right\|_H^2+c\int_0^t\mathbb{E}\left\|\rho^\epsilon(s)\right\|_H^2 ds+c_T.$$

Therefore, the Gronwall lemma gives (7.10). □

Proposition 7.3 *Assume Hypotheses 1, 2 and condition (7.7), and fix* $\mathfrak{r}_0\in L^2(\Omega;H)$. *Then, for every* $T>0$*, there exists a unique solution*

$$\rho\in L^2(\Omega;C([0,T];H))\cap L^2(0,T;H^1)),$$

of Eq. (7.2). *Moreover, there exists some constant* $c_T > 0$ *such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|\rho(t)\|_H^2 + \mathbb{E} \int_0^T \|\nabla\rho(s)\|_H^2 ds \leq c_T \left(1 + \mathbb{E}\|\mathfrak{r}_0\|_H^2\right). \tag{7.11}$$

Proof By proceeding as in [11, Theorem 6.2], we can show that equation (7.2) admits at most one solution in $L^2(\Omega; C([0,T]; H)) \cap L^2(0,T; H^1))$. Hence, if we show that there exists a probabilistically weak solution

$$(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}\}_t, \hat{\mathbb{P}}, \hat{w}^Q, \hat{\rho}),$$

such that $\hat{\rho} \in L^2(\hat{\Omega}; C([0,T]; H)) \cap L^2(0,T; H^1))$, the existence and uniqueness of a probabilistically strong solution for Eq. (7.2) follows.

Step 1 There exists a filtered probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}\}_t, \hat{\mathbb{P}})$, a cylindrical Wiener process $\hat{w}^Q$ associated with $\{\hat{\mathcal{F}}\}_t$ and a process $\hat{\rho} \in L^2(\Omega; L^\infty(0,T; H)) \cap L^2(0,T; H^1))$ such that

$$\langle\hat{\rho}(t), \varphi\rangle_H = \langle\mathfrak{r}_0, \varphi\rangle_H - \int_0^t \langle b(\hat{\rho}(s))\nabla\hat{\rho}(s), \nabla\varphi\rangle_H ds$$
$$+ \int_0^t \langle F_g(\hat{\rho}(s)), \varphi\rangle_H ds + \int_0^t \langle \varphi, \sigma_g(\hat{\rho}(s)) d\hat{w}^Q(s)\rangle_H, \quad \hat{\mathbb{P}}\text{-a.s.},$$

for every $\varphi \in C_0^\infty(\mathcal{O})$.

Proof of Step 1. According to Proposition 7.2, we know that for every $\epsilon > 0$ there exists a unique solution ρ_ϵ to Eq. (7.9), and

$$\sup_{\epsilon\in(0,1)} \left(\mathbb{E} \sup_{t\in[0,T]} \|\rho^\epsilon(t)\|_H^2 + \mathbb{E}\int_0^T \|\rho^\epsilon(t)\|_{H^1}^2 dt\right) < \infty. \tag{7.12}$$

For every $h \in (0,T)$ and $t \in [0, T-h]$ we have

$$\rho^\epsilon(t+h) - \rho^\epsilon(t) = \int_t^{t+h} \operatorname{div}\left(b(\rho^\epsilon(s))\nabla\rho^\epsilon(s)\right) ds - \epsilon \int_t^{t+h} \Delta^2\rho^\epsilon(s)\, ds$$
$$+ \int_t^{t+h} F_g(\rho^\epsilon(s))\, ds + \int_t^{t+h} \sigma_g(\rho^\epsilon(s)) dw^Q(s) =: \sum_{k=1}^4 I_k^\epsilon(t,h).$$

We have

$$\sup_{t\in[0,T-h]} \|I_1^\epsilon(t,h)\|_{H^{-1}} \leq c \sup_{t\in[0,T]} \int_t^{t+h} \|\rho^\epsilon(s)\|_{H^1}\, ds \leq c \left(\int_0^T \|\rho^\epsilon(s)\|_{H^1}^2\, ds\right)^{1/2} h^{1/2}. \tag{7.13}$$

For $I_2^\epsilon(t,h)$, if $\epsilon \in (0,1)$ we have

$$\begin{aligned}&\sup_{t\in[0,T-h]} \|I_2^\epsilon(t,h)\|_{H^{-3}}\\ &\leq \sup_{t\in[0,T-h]} \int_t^{t+h} \|\rho^\epsilon(s)\|_{H^1}\, ds \leq c \left(\int_0^T \|\rho^\epsilon(s)\|_{H^1}^2\, ds\right)^{1/2} h^{1/2},\end{aligned} \tag{7.14}$$

and for $I_3^\epsilon(t,h)$ we have

$$\begin{aligned}&\sup_{t\in[0,T-h]} \|I_3^\epsilon(t,h)\|_{H}\\ &\leq c \sup_{t\in[0,T-h]} \int_t^{t+h} \left(1+\|\rho^\epsilon(s)\|_H\right) ds \leq c_T \left(1+\sup_{t\in[0,T]} \|\rho^\epsilon(s)\|_H\right) h.\end{aligned} \tag{7.15}$$

Finally, for $I_4^\epsilon(t,h)$, by using a factorization argument as in [13, Theorems 5.11, 5.15], due to the boundedness of σ_g in $\mathcal{L}_2(H_Q, H)$ we obtain that for some $\theta \in (0,1)$

$$\sup_{\epsilon\in(0,1)} \mathbb{E}\, \|I_4^\epsilon(t,h)\|_{C^\theta([0,T];H)} < \infty. \tag{7.16}$$

Therefore, by putting together (7.13), (7.14), (7.15) and (7.16), thanks to (7.12) we conclude

$$\sup_{t\in[0,T-h]} \|\rho^\epsilon(t+h)-\rho^\epsilon(t)\|_{H^{-3}} \leq c\, h^{\frac{1}{2}\wedge\theta}, \qquad h \in [0,T),$$

and together with the bound

$$\sup_{\epsilon\in(0,1)} \mathbb{E} \sup_{t\in[0,T]} \|\rho^\epsilon(t)\|_H < \infty,$$

due to [30, Theorem 7] this implies that $\{\rho^\epsilon\}_{\epsilon\in(0,1)}$ is tight in $L^\infty(0,T;H^{-\alpha})$, for every $\alpha > 0$. Moreover, since for every $\beta \in (-3,1)$ we have

$$\|u\|_{H^\beta} \leq \|u\|_{H^1}^{\frac{3+\beta}{4}} \|u\|_{H^{-3}}^{\frac{1-\beta}{4}},$$

and the bound

$$\sup_{\epsilon \in (0,1)} \int_0^T \mathbb{E}\, \|\rho^\epsilon(s)\|_{H^1}^2\, ds < \infty,$$

holds, thanks again to [30, Theorem 7], we have that the family $\{\rho^\epsilon\}_{\epsilon \in (0,1)}$ is tight also in the space $L^{8/(3+\beta)}(0, T; H^\beta)$, for every $\beta \in (-3, 1)$.

In what follows, for every $\alpha > 0$ and $\beta \in (-3, 1)$ we denote

$$\mathcal{X}_{\alpha,\beta}(T) := \left[L^\infty(0, T; H^{-\alpha}) \cap L^{8/(3+\beta)}(0, T; H^\beta)\right] \times C([0, T]; U),$$

where U is any Hilbert space such that the embedding $H_Q \hookrightarrow U$ is Hilbert–Schmidt. Due to the tightness of $\{\rho^\epsilon, w^Q\}_{\epsilon \in (0,1)}$ in $\mathcal{X}_{\alpha,\beta}(T)$, there exists a sequence $\epsilon_n \downarrow 0$ such that $\mathcal{L}(\rho^{\epsilon_n}, w^Q)$ is weakly convergent in $\mathcal{X}_{\alpha,\beta}(T)$. Due to Skorohod's Theorem this implies that there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, a sequence of $\mathcal{X}_{\alpha,\beta}(T)$-valued random variables $\mathcal{Y}_n = (\hat{\rho}_n, \hat{w}_n^Q)$ and a $\mathcal{X}_{\alpha,\beta}(T)$-valued random variable $\mathcal{Y} = (\hat{\rho}, \hat{w}^Q)$, all defined on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, such that

$$\mathcal{L}(\mathcal{Y}_n) = \mathcal{L}(\rho^{\epsilon_n}, w^Q), \tag{7.17}$$

and

$$\lim_{n\to\infty} \left(\|\hat{\rho}_n - \hat{\rho}\|_{L^\infty(0,T;H^{-\alpha})} + \|\hat{\rho}_n - \hat{\rho}\|_{L^{8/(3+\beta)}(0,T;H^\beta)} + \|\hat{w}_n^Q - \hat{w}^Q\|_{C([0,T];U)}\right) = 0,\ \hat{\mathbb{P}} - \text{a.s.} \tag{7.18}$$

Now, we have

$$\int_0^t \left\langle \text{div}\left(b(\hat{\rho}_n(s))\nabla\hat{\rho}_n(s)\right), \varphi\right\rangle_H ds = -\int_0^t \left\langle b(\hat{\rho}_n(s))\nabla\hat{\rho}_n(s), \nabla\varphi\right\rangle_H ds$$

$$= -\int_0^t \left\langle \nabla(B(\hat{\rho}_n(s)), \nabla\varphi\right\rangle_H ds = \int_0^t \left\langle B(\hat{\rho}_n(s)), \Delta\varphi\right\rangle_H ds,$$

and thanks to (7.17) and (7.18), this gives for every $\varphi \in C_0^\infty(\mathcal{O})$,

$$\left\langle \hat{\rho}_n(t), \varphi\right\rangle_H = \left\langle \mathfrak{r}_0, \varphi\right\rangle_H + \int_0^t \left\langle B(\hat{\rho}_n(s)), \Delta\varphi\right\rangle_H ds - \epsilon_n \int_0^t \left\langle \hat{\rho}_n(s), \Delta^2\varphi\right\rangle_H ds$$

$$+ \int_0^t \left\langle F_g(\hat{\rho}_n(s)), \varphi\right\rangle_H ds + \int_0^t \left\langle \varphi, \sigma_g(\hat{\rho}_n(s))d\hat{w}_n^Q(s)\right\rangle_H.$$

Thus, by using the general argument introduced in [14, proof of Theorem 4.1], thanks to (7.18) we can take the limit as $n \to \infty$ of both sides in the equality above, and we obtain that $\hat{\rho}$ satisfies (7.8), with w^Q replaced by $\hat{w}^Q$. Moreover, $\hat{\rho} \in L^2(\hat{\Omega}; L^\infty(0,T;H)) \cap L^2(0,T;H^1))$ and satisfies (7.11), with $\mathbb{E}$ replaced by $\hat{\mathbb{E}}$.

Step 2 We have that there exists a unique solution $\rho \in L^2(\Omega; C([0,T];H) \cap L^2(0,T;H^1))$ that satisfies (7.11).

Proof of Step 2 Due to what we have seen above, there exists a unique solution

$$\rho \in L^2(\Omega; L^\infty(0,T;H) \cap L^2(0,T;H^1)),$$

that satisfies (7.11). It only remains to prove that $\rho \in C([0,T];H)$, $\mathbb{P}$-a.s. By proceeding as in [14, Sect. 4.3] we consider the problem

$$\begin{cases} \partial_t \xi(t,x) = \Delta \xi(t,x) + \sigma_g(\rho(t,\cdot))\partial_t w^Q(t,x), \\ \xi(0,x) = \mathfrak{r}_0(x), \qquad \xi(t,\cdot)|_{\partial\mathcal{O}} = 0, \end{cases}$$

whose unique solution ξ belongs to $L^2(\Omega; C([0,T];H) \cap L^2(0,T;H^1))$. Then, if we denote $\eta(t) := \rho(t) - \xi(t)$, we have that $\eta \in L^\infty(0,T;H) \cap L^2(0,T;H^1)$, $\mathbb{P}$-a.s., and solves

$$\begin{cases} \partial_t \eta(t,x) = \operatorname{div}\big(b(\rho(t,x))\nabla\eta(t,x)\big) + \operatorname{div}\big[(b(\rho(t,x)) - I)\nabla\xi(t,x)\big] + f_g(x,\rho(t,x)), \\ \eta(0,x) = 0, \qquad \eta(t,\cdot)|_{\partial\mathcal{O}} = 0. \end{cases} \tag{7.19}$$

Now, if we denote by $U(t,s)$ the evolution family associated with the time-dependent differential operator

$$\mathcal{L}_t\varphi(x) = \operatorname{div}\left[b(\rho(t,x))\nabla\varphi(x)\right], \qquad x \in \mathcal{O},$$

we have that

$$\eta(t,x) = \int_0^t U(t,s)\left[\operatorname{div}\left[b(\rho(t,\cdot)) - I\right]\nabla\xi(s,\cdot) + f_g(\cdot,\rho(s,\cdot))\right](x)\,ds,$$

and since $\xi \in L^2(0,T;H^1)$ and $\rho \in L^\infty(0,T;H)$, $\mathbb{P}$-a.s., we get that $\eta \in C([0,T];H)$, $\mathbb{P}$-a.s. In particular, $\rho = \eta + \xi$ belongs to $C([0,T];H)$, $\mathbb{P}$-a.s. □

7.2 Well-Posedness of Eq. (7.1) in *H*

From the well-posedness of the quasilinear stochastic parabolic Eq. (7.2), we get the well-poseness of Eq. (7.1) in H. By proceeding as in the proof of [10, Theorem 7.1], we can show that $u \in L^2(\Omega; C([0,T];H) \cap L^2([0,T];H^1))$ is a solution to Eq. (7.1) with initial condition $\mathfrak{u}_0 \in L^2(\Omega; H)$ if and only if $\rho := g(u)$ is a weak solution to equation (7.2) with initial value $\mathfrak{r}_0 = g(\mathfrak{u}_0) \in L^2(\Omega; H)$. Moreover, as a consequence of the Lipschitz continuity of g and g^{-1} on $\mathbb{R}$, we have the following result.

Proposition 7.4 *Assume Hypotheses 1, 2 and condition (7.7). For every $T > 0$ and every $\mathfrak{u}_0 \in L^2(\Omega; H)$, there exists a unique weak solution $u \in L^2(\Omega; C([0, T]; H) \cap L^2(0, T; H^1))$, to Eq. (7.1) such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|u(t)\|_H^2 + \mathbb{E}\int_0^T \|u(t)\|_{H^1}^2\, dt \leq c_T \left(1 + \mathbb{E}\|\mathfrak{u}_0\|_H^2\right).$$

In what follows, we shall denote

$$P_t^H \varphi(\mathfrak{u}) := \mathbb{E}\,\varphi(u^{\mathfrak{u}}(t)), \qquad \mathfrak{u} \in H, \qquad t \geq 0,$$

for every $\varphi \in B_b(H)$.

7.3 Some Bounds for ρ and u

Once established the existence of a unique weak solution in $L^2(\Omega; C([0, T]; H) \cap L^2([0, T]; H^1))$, both for (7.1) and (7.2), we prove some bounds for their solutions h and ρ.

Lemma 7.5 *Under Hypotheses 1, 2 and 3, there exist some $\lambda > 0$ and $c > 0$, such that for every $t \geq 0$*

$$\mathbb{E}\,\|\rho(t)\|_H^2 \leq c\left(1 + e^{-\lambda t}\mathbb{E}\,\|\mathfrak{r}_0\|_H^2\right), \quad \mathbb{E}\int_0^t \|\rho(s)\|_{H^1}^2\, ds \leq c\left(t + \mathbb{E}\,\|\mathfrak{r}_0\|_H^2\right). \tag{7.20}$$

Proof We apply Itô's formula to the process $\rho(t)$ and the function $K(\mathfrak{r}) = \|\mathfrak{r}\|_H^2$ and we get

$$\frac{1}{2} d\,\|\rho(t)\|_H^2 \leq -\gamma_1^{-1}\,\|\rho(t)\|_{H^1}^2\, dt + \left\langle F_g(\rho(t)), \rho(t)\right\rangle_H dt + \frac{1}{2}\left\|\sigma_g(\rho(t))\right\|_{\mathcal{L}_2(H_Q,H)}^2 dt$$
$$+ \langle \rho(t), \sigma_g(\rho(t)) dw^Q(t)\rangle_H.$$

Then thanks to (7.4) and (2.7), together with 7.6, we can find some constant $\lambda > 0$ such that

$$\frac{d}{dt}\mathbb{E}\,\|\rho(t)\|_H^2 + \lambda\mathbb{E}\,\|\rho(t)\|_{H^1}^2 \leq c,$$

and this allows we complete the proof. □

Due to estimates (7.20) and the Lipschitz continuity of g and g^{-1} on $\mathbb{R}$, estimates analogous to (7.20) holds for the solution u.

Proposition 7.6 *Assume Hypotheses 1, 2 and 3. For every $T > 0$ and every $\mathfrak{u}_0 \in L^2(\Omega; H)$, there exists a unique $u \in L^2(\Omega; C([0,T]; H) \cap L^2(0,T; H^1))$ which solves Eq. (7.1) in the following sense*

$$\langle u(t), \psi\rangle_H = \langle \mathfrak{u}_0, \psi\rangle_H - \int_0^t \left\langle \frac{\nabla u(s)}{\gamma(u(s))}, \nabla\psi \right\rangle_H ds - \int_0^t \left\langle \nabla\left(\frac{1}{\gamma(u(s))}\right)\cdot \nabla u(s), \psi \right\rangle_H ds$$

$$+ \int_0^t \left\langle \frac{f(u(s))}{\gamma(u(s))}, \psi \right\rangle_H ds - \int_0^t \left\langle \frac{\gamma'(u(s))}{2\gamma(u(s))^3} \sum_{i=1}^{\infty} (\sigma(u(s))Qe_i)^2, \psi \right\rangle_H ds$$

$$+ \int_0^t \left\langle \frac{\sigma(u(s))}{\gamma(u(s))} dw^Q(s), \psi \right\rangle_H,$$

for any $\varphi \in C_0^\infty(\mathcal{O})$. Moreover, for every $t \geq 0$

$$\mathbb{E}\,\|u(t)\|_H^2 \leq c\left(1 + e^{-\lambda t}\mathbb{E}\,\|\mathfrak{u}_0\|_H^2\right), \qquad \mathbb{E}\int_0^t \|u(s)\|_{H^1}^2\, ds \leq c\left(t + \mathbb{E}\,\|\mathfrak{u}_0\|_H^2\right).$$

7.4 Well-Posedness of Eq. (7.2) in H^{-1}

Here, we will use the results we have just mentioned about the well-posedness of Eq. (7.2) in H, to study its well-posedness in H^{-1}.

Definition 7.7 For every fixed $\mathfrak{r}_0 \in H^{-1}$ and $T > 0$, an adapted process $\rho \in L^2(\Omega; L^2(0,T; H))$ is a solution of Eq. (7.2) with initial condition ρ_0 if for every $\varphi \in C_0^\infty(\mathcal{O})$

$$\left\langle \rho(t), \varphi\right\rangle_H = \left\langle \mathfrak{r}_0, \varphi\right\rangle_H + \int_0^t \left\langle B(\rho(s)), \Delta\varphi\right\rangle_H ds + \int_0^t \left\langle F_g(\rho(s)), \varphi\right\rangle_H ds$$

$$+ \int_0^t \left\langle \varphi, \sigma_g(\rho(s)) dw^Q(s)\right\rangle_H,$$

$\mathbb{P}$-a.s., where

$$B(r) := \int_0^r b(s)ds, \quad r \in \mathbb{R}.$$

Proposition 7.8 *Assume Hypotheses* 1, 2, 3 and 4. *Then, for every* $\mathfrak{r}_0 \in H^{-1}$ *and every* $T > 0$*, there exists a unique solution*

$$\rho^{\mathfrak{r}_0} \in L^2(\Omega; C([0,T]; H^{-1}) \cap L^2(0,T; H)),$$

to Eq. (7.2). *Moreover, there exist* $c, \lambda > 0$ *independent of* $T > 0$ *such that for every* $t \in [0,T]$

$$\mathbb{E}\left\|\rho^{\mathfrak{r}_0}(t)\right\|_{H^{-1}}^2 \leq c\left(1 + e^{-\lambda t}\,\mathbb{E}\,\|\mathfrak{r}_0\|_{H^{-1}}^2\right),$$
$$\mathbb{E}\int_0^t \left\|\rho^{\mathfrak{r}_0}(s)\right\|_H^2 ds \leq c\left(t + \mathbb{E}\,\|\mathfrak{r}_0\|_{H^{-1}}^2\right). \tag{7.21}$$

Proof We fix an arbitrary sequence $\{\mathfrak{r}_\epsilon\}_{\epsilon>0} \subset H$ converging to $\mathfrak{r}_0$ strongly in H^{-1}, as $\epsilon \to 0$. Thanks to Proposition 7.3, for each $\epsilon > 0$ there exists a solution $\rho_\epsilon \in L^2(\Omega; C([0,T]; H) \cap L^2([0,T]; H^1))$ for problem (7.2) with initial condition $\mathfrak{r}_\epsilon$. If for every $\epsilon, \delta > 0$ we define

$$\vartheta_{\epsilon,\delta}(t) := \rho_\epsilon(t) - \rho_\delta(t), \qquad t \in [0,T],$$

we have

$$\begin{aligned}
&\frac{1}{2}\, d\left\|\vartheta_{\epsilon,\delta}(t)\right\|_{H^{-1}}^2 \\
&\quad = -\left\langle B(\rho_\epsilon(t)) - B(\rho_\delta(t)), \vartheta_{\epsilon,\delta}(t)\right\rangle_H dt + \left\langle F_g(\rho_\epsilon(t)) - F_g(\rho_\delta(t)), \vartheta_{\epsilon,\delta}(t)\right\rangle_{H^{-1}} dt \\
&\qquad + \frac{1}{2}\left\|\sigma_g(\rho_\epsilon(t)) - \sigma_g(\rho_\delta(t))\right\|_{\mathcal{L}_2(H_Q, H^{-1})}^2 dt + \left\langle \vartheta_{\epsilon,\delta}(t), \left[\sigma_g(\rho_\epsilon(t)) - \sigma_g(\rho_\delta(t))\right] dw^Q(t)\right\rangle_{H^{-1}}.
\end{aligned}$$

Since

$$B(r) = \int_0^r b(s)\,ds = \int_0^r \frac{1}{\gamma(g^{-1}(s))}\,ds,$$

we have

$$(B(r_1) - B(r_2))(r_1 - r_2) \geq \frac{1}{\gamma_1}\,|r_1 - r_2|^2, \qquad r_1, r_2 \in \mathbb{R},$$

so that

$$\begin{aligned}
\frac{1}{2}\, d\left\|\vartheta_{\epsilon,\delta}(t)\right\|_{H^{-1}}^2 \leq & -c_0\left\|\vartheta_{\epsilon,\delta}(t)\right\|_H^2 dt \\
& + \left\langle \vartheta_{\epsilon,\delta}(t), \left[\sigma_g(\rho_\epsilon(t)) - \sigma_g(\rho_\delta(t))\right] dw^Q(t)\right\rangle_{H^{-1}},
\end{aligned} \tag{7.22}$$

where

$$c_0 := \left(\frac{1}{\gamma_1} - \frac{L_\sigma}{2\alpha_1\gamma_0^2} - \frac{L_f}{\alpha_1\gamma_0}\right) > 0,$$

last inequality following from (2.11).

Hence, if we first integrate both sides in (7.22) with respect to time and then take the expectation, we get

$$\sup_{s\in[0,T]} \mathbb{E}\,\left\|\vartheta_{\epsilon,\delta}(s)\right\|_{H^{-1}}^2 + 2\,c_0 \int_0^T \mathbb{E}\,\left\|\vartheta_{\epsilon,\delta}(s)\right\|_H^2\,ds \leq \mathbb{E}\,\|\mathfrak{r}_\epsilon - \mathfrak{r}_\delta\|_{H^{-1}}^2,$$

and this implies that the sequence (ρ_ϵ) is Cauchy in $C([0,T];L^2(\Omega;H^{-1})) \cap L^2(\Omega;L^2(0,T;H)))$. In particular, it converges to some ρ in $C([0,T];L^2(\Omega;H^{-1}))\cap L^2(\Omega;L^2(0,T;H)))$, as $\epsilon \to 0$. For every $\epsilon > 0$ and $\varphi \in C_0^\infty(\mathcal{O})$ we have

$$\begin{aligned}\left\langle\rho_\epsilon(t),\varphi\right\rangle_H = \left\langle\mathfrak{r}_\epsilon,\varphi\right\rangle_H + \int_0^t \left(\left\langle B(\rho_\epsilon(s)),\Delta\varphi\right\rangle_H + \left\langle F_g(\rho_\epsilon(s)),\varphi\right\rangle_H\right)ds\\ + \int_0^t \left\langle\varphi,\sigma_g(\rho_\epsilon(s))dw^Q(s)\right\rangle_H.\end{aligned}$$

Then, due to the Lipschitz continuity of B, F_g and σ_g, we can take the limit in both sides of the identity above, as $\epsilon \to 0$, and we get that ρ is a solution for (7.2).

To prove the uniqueness, assume that ρ_1, ρ_2 are two solutions to (7.2). By proceeding as above, we have

$$\sup_{t\in[0,T]} \mathbb{E}\,\|\rho_1(t) - \rho_2(t)\|_{H^{-1}}^2 + c_0 \int_0^T \mathbb{E}\,\|\rho_1(t) - \rho_2(t)\|_H^2\,dt \leq 0,$$

which gives $\rho_1 = \rho_2$.

Next, we prove that $\rho \in L^2(\Omega;C([0,T];H^{-1}) \cap L^2(0,T;H)))$. We apply Itô's formula to $\|\rho\|_{H^{-1}}^2$ and we get

$$\begin{aligned}\frac{1}{2}d\,\|\rho(t)\|^2_{H^{-1}} &= \frac{1}{2}\,\big\|\sigma_g(\rho(t))\big\|^2_{\mathcal{L}_2(H_Q,H^{-1})}\,dt - \langle B(\rho(t)),\rho(t)\rangle_H dt\\ &\quad+\big\langle F_g(\rho(t)),\rho(t)\big\rangle_{H^{-1}}dt + \langle \rho(t),\sigma_g(\rho(t))dw^Q(t)\rangle_{H^{-1}}\\ &\leq c-\gamma_1^{-1}\,\|\rho(t)\|^2_H\,dt + \big\langle F_g(\rho(t)),\rho(t)\big\rangle_{H^{-1}}dt + \langle \rho(t),\sigma_g(\rho(t))dw^Q(t)\rangle_{H^{-1}}.\end{aligned}$$

Due to (2.7) there exists $\bar{\delta} > 0$ such that

$$c_1 := \frac{1}{\gamma_1} - \frac{L_f+\bar{\delta}}{\alpha_1\gamma_0} > 0,$$

so that, thanks to (7.5) we have

$$\frac{1}{2}d\,\|\rho(t)\|^2_{H^{-1}} \leq c - c_1\,\|\rho(t)\|^2_H\,dt + \langle \rho(t),\sigma_g(\rho(t))dw^Q(t)\rangle_{H^{-1}}. \tag{7.23}$$

Now, since we have

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \langle \rho(t),\sigma_g(\rho(t))dw^Q(t)\rangle_{H^{-1}}\right| \leq \frac{1}{4}\mathbb{E}\sup_{s\in[0,t]}\|\rho(s)\|^2_{H^{-1}}+c,$$

if we integrate both sides in (7.23) and then take the supremum with respect to time and the expectation, we get

$$\mathbb{E}\sup_{t\in[0,T]}\|\rho(t)\|^2_{H^{-1}} + \int_0^T \mathbb{E}\,\|\rho(s)\|^2_H\,ds \leq c_T\left(1+\|\mathfrak{r}_0\|^2_{H^{-1}}\right),$$

which, in particular implies that $\rho\in L^2(\Omega;L^\infty(0,T;H^{-1})\cap L^2(0,T;H)))$. Moreover, since ρ solves equation (7.2), it belongs to $C([0,T];H^{-1})$, $\mathbb{P}$-a.s.

Finally, in order to prove (7.21), we take the expectation of both sides of (7.23) and we get

$$\frac{d}{dt}\mathbb{E}\,\|\rho(t)\|^2_{H^{-1}} + 2\,c_1\mathbb{E}\,\|\rho(s)\|^2_H\,ds \leq c,$$

and this implies that there exist some $c,\lambda>0$ such that (7.21) holds. □

8 Ergodic Behavior of the Limiting Equation

We first study the existence of a unique invariant measure for R_t^H and $R_t^{H^{-1}}$, and then we show how this implies the existence of a unique invariant measure for P_t^H.

8.1 Ergodicity of Eq. (7.2)

In what follows, we denote by $R_t^{H^{-1}}$ the transition semigroup associated to Eq. (7.2) on H^{-1}

$$R_t^{H^{-1}}\varphi(\mathfrak{r}) := \mathbb{E}\varphi(\rho^{\mathfrak{r}}(t)), \qquad \mathfrak{r} \in H^{-1}, \qquad t \geq 0,$$

for every $\varphi \in B_b(H^{-1})$. Similarly, we denote by R_t^H the transition semigroup associated to equation (7.2) on H,

$$R_t^H\varphi(\mathfrak{r}) := \mathbb{E}\varphi(\rho^{\mathfrak{r}}(t)), \quad \mathfrak{r} \in H, \quad t \geq 0,$$

for every $\varphi \in B_b(H)$. Clearly, if $\mathfrak{r} \in H$ and $\varphi \in B_b(H^{-1})$, then

$$R_t^H\varphi(\mathfrak{r}) = R_t^{H^{-1}}\varphi(\mathfrak{r}), \qquad t \geq 0.$$

For every $A \in \mathcal{B}(H^{-1})$ we have that $A \cap H \in \mathcal{B}(H)$. Thus, if $\nu \in \mathcal{P}(H)$, we can define its extension $\nu' \in \mathcal{P}(H^{-1})$ by setting

$$\nu'(A) = \nu(A \cap H), \qquad A \in \mathcal{B}(H^{-1}).$$

With this definition, $\text{supp}\,(\nu') \subset H$. Indeed, if we denote by $B_H(\mathfrak{r}, R)$ the closed ball in H centered at $\mathfrak{r} \in H$ with radius $R > 0$, then $B_H^c(\mathfrak{r}, R) \in \mathcal{B}(H)$, so that

$$\lim_{R\to+\infty} \nu'(B_H^c(0, R)) = \lim_{R\to+\infty} \nu(B_H^c(0, R)) = 0,$$

which implies that $\text{supp}\,(\nu') \subset H$.

Proposition 8.1 *Assume Hypotheses* 1, 2, 3 and 4, *and define* $\alpha(\mathfrak{r}, \mathfrak{s}) := |\mathfrak{r} - \mathfrak{s}|_{H^{-1}}$. *Then, there exist some positive constant* λ_0, t_0 *and* c *such that*

$$\mathcal{W}_\alpha\left((R_t^{H^{-1}})^*\nu_1, (R_t^{H^{-1}})^*\nu_2\right) \leq c\, e^{-\lambda_0 t}\, \mathcal{W}_\alpha(\nu_1, \nu_2), \qquad t > t_0. \tag{8.1}$$

Moreover, $R_t^{H^{-1}}$ *has a unique invariant measure* $\nu^{H^{-1}}$ *such that* $\text{supp}\,(\nu^{H^{-1}}) \subset H^1$ *and*

$$\mathcal{W}_\alpha\left((R_t^{H^{-1}})^*\delta_{\mathfrak{r}},\, \nu^{H^{-1}}\right) \leq c\left(1 + \|\mathfrak{r}\|_{H^{-1}}\right) e^{-\lambda_0 t}, \qquad t \geq 0, \quad \mathfrak{r} \in H^{-1}. \tag{8.2}$$

Proof Let $\rho^{\mathfrak{r}_1}, \rho^{\mathfrak{r}_2}$ be two solutions of (7.2), with initial conditions $\mathfrak{r}_1, \mathfrak{r}_2 \in H^{-1}$, respectively. By proceeding as in the proof of Proposition 7.8, we have

$$\mathbb{E}\left\|\rho^{\mathfrak{r}_1}(t) - \rho^{\mathfrak{r}_1}(t)\right\|_{H^{-1}}^2 \leq e^{-\lambda t}\, \|\mathfrak{r}_1 - \mathfrak{r}_2\|_{H^{-1}}^2, \qquad t \geq 0,$$

for some constant $\lambda > 0$. In particular, the semigroup $R_t^{H^{-1}}$ is Feller in H^{-1} and for every $\varphi \in \mathrm{Lip}_b(H^{-1})$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in H^{-1}$

$$\left| R_t^{H^{-1}} \varphi(\mathfrak{r}_1) - R_t^{H^{-1}} \varphi(\mathfrak{r}_2) \right| \leq [\varphi]_{\mathrm{Lip}_{H^{-1},\alpha}} e^{-\lambda t/2} \left\| \mathfrak{r}_1 - \mathfrak{r}_2 \right\|_{H^{-1}}, \qquad t \geq 0. \tag{8.3}$$

As shown e.g. in [21, Theorem 2.5], (8.3) implies (8.1). Moreover, it implies that $R_t^{H^{-1}}$ has at most one invariant measure.

If for every $R > 0$ and $t > 0$ we denote

$$B_R := \left\{ \mathfrak{r} \in H^{-1} : \|\mathfrak{r}\|_{H^1} \leq R \right\}, \qquad \Gamma_t := \frac{1}{t} \int_0^t (R_t^{H^{-1}})^* \, \delta_0 \, dt.$$

Then, thanks to (7.20), for every $R > 0$ and $t > 0$ we have

$$R_t(B_R^c) = \frac{1}{t} \int_0^t \mathbb{P} \left(\left\| \rho^0(s) \right\|_{H^1} > R \right) ds \leq \frac{c}{R^2}. \tag{8.4}$$

Since B_R is compactly embedded in H^{-1}, this implies that the family of measures $\{\Gamma_t\}_{t>0}$, is tight in H^{-1}. Then, by Prokhorov's Theorem, there exists $t_n \uparrow \infty$ such that Γ_{t_n} converges weakly to some probability measure in $\mathcal{P}(H^{-1})$ which is invariant for $R_t^{H^{-1}}$ and, due to what we have seen above, such measure is the unique invariant measure $\nu^{H^{-1}}$ of $R_t^{H^{-1}}$. Moreover, (8.4) gives

$$\nu^{H^{-1}}(B_R^c) \leq \liminf_{n \to \infty} \Gamma_{t_n}(B_R^c) \leq \frac{c}{R^2}, \qquad R > 0,$$

so that $\mathrm{supp}\,(\nu^{H^{-1}}) \subset H^1$.

Finally, in order to prove (8.2), we first notice that due to the invariance of $\nu^{H^{-1}}$ and (7.21)

$$\begin{aligned}
\int_{H^{-1}} \|\mathfrak{r}\|_{H^{-1}}^2 \, \nu^{H^{-1}}(d\mathfrak{r}) &\leq \liminf_{R \to \infty} \int_{H^{-1}} \left(\|\mathfrak{r}\|_{H^{-1}}^2 \wedge R \right) \, \nu^{H^{-1}}(d\mathfrak{r}) \\
&= \liminf_{R \to \infty} \int_{H^{-1}} \left(\mathbb{E} \, \|\rho^{\mathfrak{r}}(t)\|_{H^{-1}}^2 \wedge R \right) \, \nu^{H^{-1}}(d\mathfrak{r}) \\
&\leq c \left(1 + e^{-\lambda t} \int_{H^{-1}} \|\mathfrak{r}\|_{H^{-1}}^2 \nu^{H^{-1}}(d\mathfrak{r}) \right).
\end{aligned}$$

Thus, if we take $\bar{t} > 0$ such that $ce^{-\lambda \bar{t}} = 1/2$, we get

$$\int_{H^{-1}} \|\mathfrak{r}\|^2_{H^{-1}} \, \nu^{H^{-1}}(d\mathfrak{r}) \leq c. \tag{8.5}$$

Then, in view of (8.3), for every $\varphi \in \mathrm{Lip}_b(H^{-1})$ we have

$$\begin{aligned}\mathcal{W}_\alpha\left((R_t^{H^{-1}})^*\delta_{\mathfrak{r}},\ \nu^{H^{-1}}\right) &\leq \left| R_t^{H^{-1}}\varphi(\mathfrak{r}) - \int_{H^{-1}} \varphi(\mathfrak{s})\, \nu^{H^{-1}}(d\mathfrak{s}) \right| \\ &\leq \int_{H^{-1}} \left| R_t^{H^{-1}}\varphi(\mathfrak{r}) - R_t^{H^{-1}}\varphi(\mathfrak{s}) \right| \nu^{H^{-1}}(d\mathfrak{s}) \\ &\leq [\varphi]_{\mathrm{Lip}_{H^{-1},\alpha}}\, e^{-\lambda t/2} \int_{H^{-1}} \|\mathfrak{r} - \mathfrak{s}\|_{H^{-1}} \nu^{H^{-1}}(d\mathfrak{s}),\end{aligned}$$

and (8.5) allows to obtain (8.2), with $\lambda_0 = \lambda/2$. □

Remark 8.2 Based on the fact that $\mathcal{B}(H) \subset \mathcal{B}(H^{-1})$ and the fact that supp $(\nu^{H^{-1}}) \subset H^1$, we have that $\nu^{H^{-1}} \in \mathcal{P}(H^{-1})$ is also a probability measure on H. In what follows, it will be convenient to distinguish the restriction of $\nu^{H^{-1}}$ to H from $\nu^{H^{-1}}$ itself and for this reason we will denote it by ν^H.

Proposition 8.3 *The probability measure ν^H is the unique invariant measure for the transition semigroup R_t^H. Moreover,* supp $(\nu^H) \subset H^1$ *and*

$$\int_{H} \|\mathfrak{r}\|^2_{H^1} \, \nu^{H}(d\mathfrak{r}) < \infty. \tag{8.6}$$

Proof By proceeding as in the proof of Lemma 6.3, it is possible to show that ν^H is invariant for R_t^H, and from Proposition 8.1 we get that supp$(\nu^H) \subset H^1$.

To prove its uniqueness, we notice that if $\nu \in \mathcal{P}(H)$ is any invariant measure for R_t^H, then its extension $\nu' \in \mathcal{P}(H^{-1})$, with the support in H, is invariant for $R_t^{H^{-1}}$. From Proposition 8.1, we have $\nu' = \nu^{H^{-1}}$, and hence

$$\nu(A) = \nu'(A) = \nu^{H^{-1}}(A) = \nu^H(A), \qquad A \in \mathcal{B}(H),$$

which implies that $\nu = \nu^H$.

Finally, in order to prove (8.6), we consider the Komolgov operator associated to R_t^H

$$N\varphi(\mathfrak{r}) = \frac{1}{2}\mathrm{Tr}_H\Big[\big(\sigma_g(\mathfrak{r})Q\big)\big(\sigma_g(\mathfrak{r})Q\big)^* D^2\varphi(\mathfrak{r})\Big] + \big\langle \mathrm{div}\big(b(\mathfrak{r})\nabla\mathfrak{r}\big) + F_g(\mathfrak{r}), D\varphi(\mathfrak{r})\big\rangle_H.$$

We consider the function $\varphi(\mathfrak{r}) := \|\mathfrak{r}\|_H^2 /2$, then

$$N\varphi(\mathfrak{r}) = \frac{1}{2}\left\|\sigma_g(\mathfrak{r})\right\|^2_{\mathcal{L}_2(H_Q,H)} - \langle b(\mathfrak{r})\nabla\mathfrak{r}, \nabla\mathfrak{r}\rangle_H + \langle F_g(\mathfrak{r}), \mathfrak{r}\rangle_H \leq \tfrac{1}{2}\sigma_\infty^2 - \gamma_1^{-1}\|\nabla\mathfrak{r}\|_H^2 + \langle F_g(\mathfrak{r}), \mathfrak{r}\rangle_H,$$

so thanks to (7.4) and (2.7), by the invariance of ν^H on H we have

$$\int_H \|\mathfrak{r}\|^2_{H^1}\,\nu^H(d\mathfrak{r}) < \infty.$$

□

Remark 8.4 As a direct consequence of (8.6), we have

$$\int_{H^{-1}} \|\mathfrak{r}\|^2_{H^1}\,\nu^{H^{-1}}(d\mathfrak{r}) < \infty. \tag{8.7}$$

8.2 Ergodicity for Eq. (7.1)

Now, we recall that we denoted by P_t^H the transition semigroup associated to the limiting problem (7.1)

$$P_t^H\varphi(\mathfrak{u}) := \mathbb{E}\,\varphi(u^{\mathfrak{u}}(t)), \qquad \mathfrak{u}\in H, \qquad t\geq 0,$$

for every $\varphi \in B_b(H)$. For every $\mathfrak{r}, \mathfrak{u} \in H$ and $t \geq 0$ we have

$$g^{-1}(\rho^{\mathfrak{r}}(t)) = u^{g^{-1}(\mathfrak{r})}(t), \qquad \rho^{g(\mathfrak{u})}(t) = u^{\mathfrak{u}}(t).$$

Hence, if we define the operator $T_g : C_b(H) \to C_b(H)$ by

$$[T_g\varphi](\mathfrak{u}) = \varphi(g(\mathfrak{u})), \quad \mathfrak{u}\in H,$$

we have $T_g^{-1} := T_{g^{-1}}$,

$$\int_H [T_g\varphi](\mathfrak{u})\,(\nu\circ g)(d\mathfrak{u}) = \int_H \varphi(\mathfrak{r})\,\nu(d\mathfrak{r}), \tag{8.8}$$

and for every $\varphi \in C_b(H)$

$$R_t^H\varphi(\mathfrak{r}) = \mathbb{E}\,\varphi(\rho^{\mathfrak{r}}(t)) = \mathbb{E}\,[T_g\varphi](u^{g^{-1}(\mathfrak{r})}(t)) = P_t^H[T_g\varphi](g^{-1}(\mathfrak{r})), \qquad t\geq 0 \tag{8.9}$$

Lemma 8.5 *$\nu \in \mathcal{P}(H)$ is invariant for P_t^H if and only $\nu\circ g^{-1}$ is invariant for R_t^H. In particular, $\nu^H \circ g$ is the unique invariant measure for P_t^H.*

Proof Assume $\nu \in \mathcal{P}(H)$ is invariant for P_t^H. Then, thanks to (8.9) and (8.8), for every $\varphi \in C_b(H)$ and $t \geq 0$ we have

$$\int_H R_t^H \varphi(\mathfrak{r})\,(\nu \circ g^{-1})(d\mathfrak{r}) = \int_H R_t^H \varphi(g(\mathfrak{u}))\,\nu(d\mathfrak{u}) = \int_H P_t^H[T_g\varphi](\mathfrak{u})\,\nu(d\mathfrak{u})$$

$$= \int_H [T_g\varphi](\mathfrak{u})\,\nu(d\mathfrak{u}) = \int_H \varphi(\mathfrak{r})\,(\nu \circ g^{-1})(d\mathfrak{r}).$$

This implies that $\nu \circ g^{-1}$ is invariant for R_t^H. In the same way, if $\lambda \in \mathcal{P}(H)$ is invariant for R_t^H, then $\lambda \circ g$ is invariant for P_t^H. Hence, we can conclude due to (9.1).

Our statement can be rephrased by saying that there exists a unique invariant measure for R_t^H if and only is there exists a unique invariant measure for P_t^H. Therefore, since we have shown in Corollary 8.3 that ν^H is the unique invariant measure for R_t^H, we obtain that $\nu^H \circ g$ is the unique invariant measure for P_t^H. □

9 Proof of Theorem 3.2

Due to Hypothesis 2, with an abuse of notation in this section we will look at g and g^{-1} as mappings on H

$$[g(h)](x) := g(h(x)), \quad [g^{-1}(h)](x) := g^{-1}(h(x)), \quad x \in \mathcal{O}, \qquad h \in H.$$

For every probability measure $\nu \in \mathcal{P}(H)$, we define probability measures $\nu \circ g$ and $\nu \circ g^{-1} \in \mathcal{P}(H)$ by

$$\big(\nu \circ g\big)(A) := \nu(g(A)), \quad \big(\nu \circ g^{-1}\big)(A) := \nu(g^{-1}(A)), \quad A \in \mathcal{B}(H).$$

Clearly, we have

$$(\nu \circ g) \circ g^{-1} = (\nu \circ g^{-1}) \circ g = \nu. \tag{9.1}$$

We notice that Theorem 3.2 is proved once we can show that if $(\nu_\mu^{\mathcal{H}})_{\mu>0} \subset \mathcal{P}(\mathcal{H})$ is a family of invariant measures for the transition semigroups $P_t^{\mu,\mathcal{H}}$, such that $\text{supp}(\nu_\mu^{\mathcal{H}}) \subset \mathcal{H}_1$, then

$$\lim_{\mu\to 0} \mathcal{W}_\alpha\left(\left[\left(\Pi_1 \nu_\mu^{\mathcal{H}}\right) \circ g^{-1}\right]', \nu^{H^{-1}}\right) = 0, \tag{9.2}$$

where $\nu^{H^{-1}}$ is the unique invariant measure for $R_t^{H^{-1}}$ in H^{-1}.

Actually, in view of (6.16), the family of probability measures $(\Pi_1 \nu_\mu^{\mathcal{H}})_{\mu\in(0,1)}$ is tight in H^δ, for every $\delta < 1$. If ν is any weak limit of $\Pi_1 \nu_\mu^{\mathcal{H}}$ in H, as $\mu \to 0$, we have $(\Pi_1 \nu_\mu^{\mathcal{H}}) \circ g^{-1}$ converges weakly to $\nu \circ g^{-1}$ on H. Due to the continuity of the embedding of H^{-1} into H,

$$\Big[(\Pi_1 \nu_\mu^{\mathcal{H}}) \circ g^{-1}\Big]' \rightharpoonup \big(\nu \circ g^{-1}\big)', \qquad \text{as } \mu \to 0,$$

as measures on H^{-1}. On the other hand, according to (9.2) we have that $\big[(\Pi_1 \nu_\mu^{\mathcal{H}}) \circ g^{-1}\big]'$ converges weakly to $\nu^{H^{-1}}$ in H^{-1}, so that $(\nu \circ g^{-1})' = \nu^{H^{-1}}$ in H^{-1}. This implies that $\nu \circ g^{-1} = \nu^H \in \mathcal{P}(H)$, and thus $\nu = \nu^H \circ g \in \mathcal{P}(H)$. Since this holds for every weak limit ν of $\Pi_1 \nu_\mu^{\mathcal{H}}$, we conclude that $\Pi_1 \nu_\mu^{\mathcal{H}}$ converges weakly to $\nu^H \circ g$ in H, as $\mu \to 0$, and, due to Lemma 8.5, $\nu^H \circ g$ is the unique invariant measure for P_t^H.

9.1 Proof of (9.2)

Due to the invariance of $\nu_\mu^{\mathcal{H}}$ and $\nu^{H^{-1}}$, we have

$$\begin{aligned}\mathcal{W}_\alpha\left(\left[\left(\Pi_1 \nu_\mu^{\mathcal{H}}\right) \circ g^{-1}\right]', \nu^{H^{-1}}\right) &\leq \mathcal{W}_\alpha\left(\left[\Pi_1((P_t^{\mu,\mathcal{H}})^\star \nu_\mu^{\mathcal{H}}) \circ g^{-1}\right]', (R_t^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathcal{H}}\right) \circ g^{-1}\right]'\right)\\ &\quad + \mathcal{W}_\alpha\left((R_t^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathcal{H}}\right) \circ g^{-1}\right]', (R_t^{H^{-1}})^* \nu^{H^{-1}}\right).\end{aligned}$$

According to (8.1), we have

$$\begin{aligned}&\mathcal{W}_\alpha\left((R_t^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathcal{H}}\right) \circ g^{-1}\right]', (R_t^{H^{-1}})^* \nu^{H^{-1}}\right)\\ &\leq c\, e^{-\lambda_0 t} \mathcal{W}_\alpha\left(\left[\left(\Pi_1 \nu_\mu^{\mathcal{H}}\right) \circ g^{-1}\right]', \nu^{H^{-1}}\right),\end{aligned}$$

and then, if we pick $\bar{t} > 0$ such that $ce^{-\lambda_0 \bar{t}} \leq 1/2$, we obtain

$$\begin{aligned}&\mathcal{W}_\alpha\left(\left[\left(\Pi_1 \nu_\mu^{\mathcal{H}}\right) \circ g^{-1}\right]', \nu^{H^{-1}}\right)\\ &\leq 2\, \mathcal{W}_\alpha\left(\left[(\Pi_1((P_{\bar{t}}^{\mu,\mathcal{H}})^\star \nu_\mu^{\mathcal{H}}) \circ g^{-1}\right]', (R_{\bar{t}}^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathcal{H}}\right) \circ g^{-1}\right]'\right).\end{aligned}$$

Now, if we fix a $\mathcal{F}_0$-measurable $\mathcal{H}_1$-valued random variable $\vartheta_\mu := (\xi_\mu, \eta_\mu)$, distributed as the invariant measure $\nu_\mu^{\mathcal{H}}$, the Kantorovich–Rubinstein identity (3.3) gives for every $t \geq 0$

$$\begin{aligned}&\mathcal{W}_\alpha\left(\left[(\Pi_1((P_t^{\mu,\mathcal{H}})^\star \nu_\mu^{\mathcal{H}}) \circ g^{-1}\right]', (R_t^{H^{-1}})^*\left[\left(\Pi_1 \nu_\mu^{\mathcal{H}}\right) \circ g^{-1}\right]'\right)\\ &\leq \mathbb{E}\, \alpha(g(u_\mu^{\vartheta_\mu}(t)), \rho^{g(\xi_\mu)}(t)).\end{aligned}$$

Thus, (9.2) follows once we prove that for every $t \geq 0$ large enough

$$\lim_{\mu \to 0} \mathbb{E}\, \alpha(g(u_\mu^{\vartheta_\mu}(t)), \rho^{g(\xi_\mu)}(t)) = \lim_{\mu \to 0} \mathbb{E}\, \|g(u_\mu^{\vartheta_\mu}(t)) - \rho^{g(\xi_\mu)}(t)\|_{H^{-1}} = 0. \quad (9.3)$$

According to (6.16) we have that $\vartheta_\mu \in L^2(\Omega; \mathcal{H}_1)$, for every $\mu \in (0, 1)$. Hence, if we denote $\rho_\mu(t) := g(u_\mu^{\vartheta_\mu}(t))$, by proceeding as in [10, Sect. 5], we can rewrite equation (3.1) in the following way

$$\rho_\mu(t) + \mu\, v_\mu^{\vartheta_\mu}(t) = g(\xi_\mu) + \mu\, \eta_\mu + \int_0^t \Delta[B(\rho_\mu(s))]ds$$
$$+ \int_0^t F_g(\rho_\mu(s))ds + \int_0^t \sigma_g(\rho_\mu(s))dw^Q(s),$$

where the identity holds in H^{-1} sense. Since $\rho^{g(\xi_\mu)}$ solves Eq. (7.2) with initial condition $g(\xi_\mu) \in L^2(\Omega; H)$ in H^{-1} sense, we have

$$\rho_\mu(t) - \rho^{g(\xi_\mu)}(t) + \mu\, v_\mu^{\vartheta_\mu}(t) = \mu\, \eta_\mu + \int_0^t \Delta\left[B(\rho_\mu(s)) - B(\rho^{g(\xi_\mu)}(s))\right] ds$$

$$+ \int_0^t \left(F_g(\rho_\mu(s)) - F_g(\rho^{g(\xi_\mu)}(s))\right) ds + \int_0^t \left(\sigma_g(\rho_\mu(s)) - \sigma_g(\rho^{g(\xi_\mu)}(s))\right) dw^Q(s).$$

If we define $\vartheta_\mu(t) := \rho_\mu(t) - \rho^{g(\xi_\mu)}(t)$, as a consequence of Itô's formula, we have

$\vartheta_\mu(t) := \rho_\mu(t) - \rho^{g(\xi_\mu)}(t)$, as a consequence of Itô's formula, we have

$$\frac{1}{2}\, \mathbb{E}\, \|\vartheta_\mu(t) + \mu\, v_\mu^{\vartheta_\mu}(t)\|_{H^{-1}}^2$$
$$= \frac{1}{2}\, \mu^2\, \mathbb{E}\, \|\eta_\mu\|_{H^{-1}}^2 - \mathbb{E} \int_0^t \left\langle B\big(\rho_\mu(s)\big) - B\big(\rho^{g(\xi_\mu)}(s)\big), \vartheta_\mu(s) + \mu\, v_\mu^{\vartheta_\mu}(s)\right\rangle_H ds$$
$$+ \mathbb{E} \int_0^t \left\langle F_g\big(\rho_\mu(s)\big) - F_g\big(\rho^{g(\xi_\mu)}(s)\big), \vartheta_\mu(s) + \mu\, v_\mu^{\vartheta_\mu}(s)\right\rangle_{H^{-1}} ds$$
$$+ \frac{1}{2}\, \mathbb{E} \int_0^t \|\sigma_g(\rho_\mu(s)) - \sigma_g(\rho^{g(\xi_\mu)}(s))\|_{\mathcal{L}_2(H_Q, H^{-1})}^2 ds$$

so that

$$\mathbb{E}\,\|\vartheta_\mu(t)+\mu\, v_\mu^{\vartheta_\mu}(t)\|_{H^{-1}}^2 \leq \mu^2\mathbb{E}\,\|\eta_\mu\|_{H^{-1}}^2 - 2\mu\,\mathbb{E}\int_0^t \big\langle B\big(\rho_\mu(s)\big)-B\big(\rho^{\,g(\xi_\mu)}(s)\big),\, v_\mu^{\vartheta_\mu}(s)\big\rangle_H ds$$

$$+2\mu\,\mathbb{E}\int_0^t \big\langle F_g\big(\rho_\mu(s)\big)-F_g\big(\rho^{\,g(\xi_\mu)}(s)\big),\, v_\mu^{\vartheta_\mu}(s)\big\rangle_{H^{-1}} ds - c_0\,\mathbb{E}\int_0^t \|\vartheta_\mu(s)\|_H^2 ds,$$

where

$$c_0 := 2\Big(\frac{1}{\gamma_1}-\frac{L_\sigma}{2\alpha_1\gamma_0^2}-\frac{L_f}{\alpha_1\gamma_0}\Big) > 0.$$

Since B has linear growth, thanks to (6.1) and (6.2) for every $\mu \in (0,\mu_0)$ we have

$$\mu\cdot\mathbb{E}\left|\int_0^t \big\langle B(\rho_\mu(s))-B\big(\rho^{\,g(\xi_\mu)}(s)),\, v_\mu^{\vartheta_\mu}(s)\big\rangle_H ds\right|$$

$$\leq c\left(\int_0^t \Big(1+\mathbb{E}\,\big\|\rho_\mu(t)\big\|_H^2+\mathbb{E}\,\|\rho^{\,g(\xi_\mu)}(t)\|_H^2\Big)\,dt\right)^{\frac12}\left(\int_0^t \mu^2\mathbb{E}\,\|v_\mu^{\vartheta_\mu}(t)\|_H^2 dt\right)^{\frac12}$$

$$\leq c\left(1+t+\mathbb{E}\,\big\|\xi_\mu\big\|_{H^1}^2+\mu^2\mathbb{E}\|\eta_\mu\|_H^2\right)^{\frac12}\left(\mu\, t+\mu^2+\mu^2\mathbb{E}\|\xi_\mu\|_{H^1}^2+\mu^3\mathbb{E}\,\big\|\eta_\mu\big\|_H^2\right)^{\frac12}$$

$$\leq c_t\left(1+\int_{\mathcal{H}}\left(\|\mathfrak{u}\|_{H^1}^2+\mu^2\,\|\mathfrak{v}\|_H^2\right)\nu_\mu^{\mathcal{H}}(d\mathfrak{u},d\mathfrak{v})\right)^{\frac12}$$
$$\left(\mu+\mu^2\int_{\mathcal{H}}\left(\|\mathfrak{u}\|_{H^1}^2+\mu\,\|\mathfrak{v}\|_H^2\right)\nu_\mu^{\mathcal{H}}(d\mathfrak{u},d\mathfrak{v})\right)^{\frac12}$$

$$\leq c_t\sqrt{\mu}\left(1+\int_{\mathcal{H}}\left(\|\mathfrak{u}\|_{H^1}^2+\mu\,\|\mathfrak{v}\|_H^2\right)\nu_\mu^{\mathcal{H}}(d\mathfrak{u},d\mathfrak{v})\right).$$

Similarly, thanks to the linear growth of F_g, we have for every $\mu \in (0,\mu_0)$

$$\mu\cdot\mathbb{E}\left|\int_0^t \big\langle F_g(\rho_\mu(s))-F_g\big(\rho^{\,g(\xi_\mu)}(s)),\, v_\mu^{\vartheta_\mu}(s)\big\rangle_{H^{-1}} ds\right|$$

$$\leq c_t\sqrt{\mu}\left(1+\int_{\mathcal{H}}\left(\|\mathfrak{u}\|_{H^1}^2+\mu\,\|\mathfrak{v}\|_H^2\right)\nu_\mu^{\mathcal{H}}(d\mathfrak{u},d\mathfrak{v})\right).$$

Moreover, due to (6.16) we know the family of random variable ϑ_μ satisfies (6.5). Then, from (6.6) we obtain that for every $\mu \in (0, \mu_t)$

$$\mu^2 \mathbb{E}\,\|v_\mu^{\vartheta_\mu}(t)\|_{H^{-1}}^2 \leq c_t\sqrt{\mu} + c\,\mu\left(\mathbb{E}\,\left\|\xi_\mu\right\|_{H^1}^2 + \mu\,\mathbb{E}\,\left\|\eta_\mu\right\|_H^2\right)$$
$$= c_t\sqrt{\mu} + c\,\mu \int_{\mathcal{H}} \left(\|\mathfrak{u}\|_{H^1}^2 + \mu\,\|\mathfrak{v}\|_H^2\right)\nu_\mu^{\mathcal{H}}(d\mathfrak{u}, d\mathfrak{v}).$$

Therefore, from (2.11) and once again (6.16), we conclude that for every $\mu \in (0, \mu_t)$

$$\frac{1}{2}\,\mathbb{E}\,\|\rho_\mu(t) - \rho^{\,g(\xi_\mu)}(t)\|_{H^{-1}}^2$$
$$\leq \left(\mathbb{E}\,\|\rho_\mu(t) - \rho^{\,g(\xi_\mu)}(t) + \mu v_\mu^{\vartheta_\mu}(t)\|_{H^{-1}}^2 + \mu^2\mathbb{E}\,\|v_\mu^{\vartheta_\mu}(t)\|_{H^{-1}}^2\right)$$
$$\leq c_t\sqrt{\mu}\left(1 + \int_{\mathcal{H}} \left(\|\mathfrak{u}\|_{H^1}^2 + \mu\,\|\mathfrak{v}\|_H^2\right)\nu_\mu^{\mathcal{H}}(d\mathfrak{u}, d\mathfrak{v})\right),$$

and (9.3) follows.

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Declarations

Conflict of interest There are no conflict of interest, nor conflict of interest related to this manuscript.

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