Students reasoning with multi-variable expressions in the context of potential difference

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This study probes student reasoning with multi-variable expressions in the context of potential difference, $\Delta V_{AB}=-W_{AB}/q_{test}$. Many students reason that if the value of a test charge being moved in an external field between points A and B is increased, the potential difference between the two points is decreased proportionally. It could be argued that these students rely on intuitive ideas informed by their prior practices with similar expressions (*e.g.,* $y=a/x$) in which a and x represent a constant and a variable, respectively. This automated prior knowledge yields an intuitive mental model based on the assumption that W_{AB} is a constant, which many accept without giving it a second thought. It is also possible that the knowledge necessary to check for the validity of this assumption is weak or missing. This study is guided by the dual-process theories of reasoning and probs the likely sources of reasoning errors with multi-variable expressions.

2024 PERC Proceedings edited by Ryan, Pawl, and Zwolak; Peer-reviewed, doi.org/10.1119/perc.2024.pr.Ismael

I. INTRODUCTION

Researchers in physics education have identified many physics student reasoning patterns, both productive and unproductive [1–5]. Some unproductive reasoning patterns persist even after targeted instruction [6–10]. One example is student reasoning with multi-variable expressions. Consider, for instance, Adam, a hypothetical introductory physics student in the calculus-based *Electricity and Magnetism* course, who gave a typical response. Adam is working on a physics assignment involving the operational definition of the electric field: $\vec{E} = \frac{\vec{F}_{on\ test}}{q_{test}}$. In the problem, a point charge is located in an external electric field. Adam argues that if the value of the point charge is increased, the magnitude of the external electric field will decrease because, according to the definition above, E is inversely proportional to q. It seems that Adam did not consider how the force on the test charge by the external electric field, $\vec{F}_{on\ test}$, is affected by the change in q_{test}. Many students who provided this type of reasoning seem to quickly and incorrectly assume that the numerator in the math expression for E remains the same.

Student reasoning patterns similar to the one above exist in other contexts that involve multi-variable expressions (*e.g.,* waves, ideal gas law, capacitance) [6,11,12]. In this study, we focus on the context of an operational definition of electric potential difference. This study is conducted through the lens of the dual-process theories of reasoning (DPToR) [1,13–17]. We consider several reasoning hazards suggested by DPToR that may be present along reasoning paths. We conduct an investigation to pinpoint more precisely possible sources of student errors.

II. MOTIVATION AND THEORETICAL FRAMEWORK

The potential difference between points A and B (ΔV_{AB}) is defined as the negative work done by the electrostatic force to move a unit charge from one point to another, expressed mathematically as $\Delta V_{AB} = -\frac{W_{AB}}{q}$. Our pilot data suggested that less than 20% of students correctly reason that if the charge being moved between the two points is increased, the potential difference ΔV_{AB} will not be affected. Most students argued that ΔV_{AB} would decrease because of the inverse relationship between ΔV_{AB} and q. It appears that this type of reasoning error can be attributed to students' tendency to jump to conclusions and accept as correct the first available appealing intuitive response without further scrutiny. This type of reasoning error is consistent with the dual-process theories of reasoning that suggest that reasoning involves interactions between two qualitatively different processes, process 1 and process 2. Process 1 is fast, automatic, and always active and is typically referred to as intuition or "gut fillings." Process 1 immediately suggests a mental model of a given situation based on recognition. Experts have a more accurate intuition (*i.e.,* recognition) than novices due to a broader range of relevant experiences. The most crucial aspect of process 1 is that it cannot be turned off. Only after process 1 suggests the first mental model does the model become available for scrutiny by slow, deliberate, and rule-based process 2. In the context of multi-variable expressions, the first available mental model may be informed by the students' prior practices with expressions of similar structure (*e.g., y=a/x*) in which symbol a typically represents a constant and x represents a variable. This practiced (and possibly even automated) prior knowledge may yield the first available mental model for the potential difference that treats W_{AB} as a constant. DPToR identifies inaccurate intuition as the first reasoning hazard on the path to a final response [17]. If the prior knowledge is automated or intuitively appealing, process 2 may be circumvented so that the outcome of process 1 becomes a final response.

Even if process 2 is engaged, error detection in the first available mental model is not guaranteed due to other reasoning hazards. Process 2 suffers from its own analytical biases. For example, reasoners tend to search for evidence to support a solution they already believe is correct (*i.e.,* confirmation bias) [18]. These analytical biases present another hazard on a reasoning path. Even if a possibility of error is detected, if a reasoner does not possess relevant knowledge and skills (often referred to in psychology as *mindware*), the intuitive response will not likely be replaced by a correct alternative solution [19]. Weak or absent mindware presents yet another hazard on a reasoning path.

Adam and other students who engage in similar reasoning with multi-variable expressions may be falling into the trap of hazard 1. It is also possible that their relevant mindware is weak or absent. To develop effective instruction that addresses persistent incorrect reasoning patterns, we need to understand the sources of student errors. This study aimed to identify the degree to which the observed reasoning patterns could be attributed to (1) students' tendency to rely on intuitively appealing responses produced by process 1, (2) weak or absent relevant mindware, or (3) both. We asked the following research questions.

1. Do students possess relevant mathematics knowledge and reasoning, referred to in this study as *math mindware?* Namely, can they articulate what it means for quantity z to depend on x, and if so, can they apply that understanding in math contexts?
2. Can students transfer their math mindware to the context of the potential difference if the operational definition of ΔV_{AB} is presented to them explicitly? Can students use the presented operational definition to correctly analyze a physics situation that elicits intuitively appealing but incorrect responses?
3. Can students utilize the math and physics mindware on a course exam to correctly analyze a physics situation that elicits intuitively appealing but incorrect responses?

III. METHODOLOGY

This study was conducted in the introductory E&M calculus-based physics course at North Dakota State University, a mid-size R1 institution. The course was taught in person, with an option to attend via web-conference. To answer research questions 1 and 2, two assignments were designed (referred to as *math* and *physics* assignments) and administered in a web-based format outside of class. Students completed each assignment in one sitting without a time limit. They were asked to answer each question and explain their reasoning. Credit was awarded based on the completeness of responses rather than their correctness. Both assignments were followed by an instructor-led classroom discussion that addressed common incorrect responses, as shown in Fig. 1. The math and physics assignments were given before and after relevant instruction on potential difference, respectively, with one week in between.

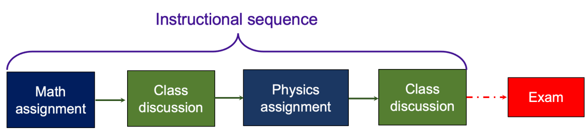

FIG 1. Instructional sequence

The final assessment, designed to answer research question 3, was included in the course exam (referred to here as the *exam*). Exam responses were graded based on the correctness of their answers and reasoning. The exam was also administered in a web-based format outside of class. Students completed the exam in one sitting within 24 hours and were allowed to use notes.

The analyzed data set includes responses from students who completed all three tasks (N=100). The tasks were validated through feedback from the physics faculty, including PER researchers.

A. Description of math assignment intended to answer research question 1

The math assignment contained three questions. In question 1, students were asked to explain in their own words what it means when someone says that *"quantity F depends on x."* Then, students considered quantity $z = \frac{W}{x}$, where W is unknown. They were asked to give an example of W such that (1) z depends on x and (2) z does not depend on x. It was expected that those students who answered the first question correctly would also give correct examples of W for case 1 (*e.g.,* $W=const$, $W=x^2$) and case 2 (*e.g.,* $W=2x$).

Following the math assignment, the instructor reviewed the solutions and discussed how this type of math reasoning could be applied to the physics concepts. The instructor reminded the students that although q_{test} appears in the expression for the operational definition of the electric field, the value of the field does not depend on the test charge used to measure it since F_{test} is a linear function of q_{test} as well, as shown in Eq. 1 for the electric field due to a point charge Q:

$$\vec{E} = \frac{\vec{F}_{test}}{q_{test}} = k\frac{Qq_{test}}{r^2}\frac{1}{q_{test}}\hat{r} = k\frac{Q}{r^2}\hat{r}. \quad (1)$$

The instructor also reminded the students that in physics, a symbol may represent a variable, function, or constant. Therefore, they need to consider each quantity carefully and not assume it is a constant, as illustrated by Eq. 1.

B. Description of physics assignment intended to answer research question 2

The first task of the physics assignment began with a reminder of how the potential difference between points A and B, ΔV_{AB}, is defined operationally. Fig. 2 shows a region in space with a uniform electric field. A positive charge Q is moved a distance l from point A to point B. Two statements were given: (1) ΔV_{AB} is defined as the negative of the work done by the electrostatic force to move a unit charge from A to B, and (2) the potential difference between the two points, however, does not depend on the work done or the charge being moved. The students were asked to comment on whether the expression was consistent with each statement. It was expected that the students would recognize that this expression is a direct translation of statement 1 into a symbolic form. By applying reasoning similar to that practiced in the math assignment and discussed in class, the students would also recognize that for the expression to be consistent with statement 2, W_{AB} must be a linear function of Q such that Q cancels. This question was designed to probe the transferability of math mindware to a physics context.

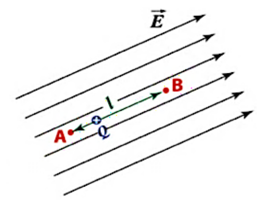

FIG.2. Situation considered on physics assignment

In the following question, students were asked to apply the operational definition of ΔV_{AB} to a modified scenario. The goal was to probe how successfully the students would apply the operational definition to a situation that elicits intuitively appealing but incorrect responses. They were asked, "If the value of the charge moved from A to B is increased by a factor of 4, $Q_{new}=4Q$, does the potential difference between points A and B *increase, decrease,* or *remain the same?*" We refer to this question as a *target* question since it elicits common incorrect responses targeted in this investigation.

Following this assignment, the instructor reviewed the solutions and discussed (again) how the math reasoning practiced in the math assignment applies to physics concepts.

C. Description of the exam tasks intended to answer research question 3

The final assessment included in the course exam consisted of 3 questions. Students considered a region of uniform electric field, as shown in Fig. 3. They were told that a positive charge Q=1μC is moved from point A to point B, and the work done by the electrostatic force to move the charge is 8 μV·C. In question 1, students were asked to determine the potential difference between points A and B. This question probed whether the students could apply the operational definition of ΔV_{AB} in a straightforward situation (*i.e.,* possess basic mindware about the potential difference).

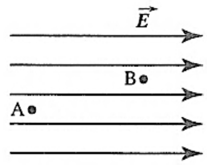

FIG. 3. Situation considered on exam

The second question asked students to determine the magnitude of the electric field in the region if the horizontal distance between points A and B is 0.2 m. The students were expected to use a common relationship:

$$\Delta V_{AB} = -\frac{W_{AB}}{Q} = -\frac{\vec{F}\cdot\vec{r}}{Q} = -\frac{\vec{E}\cdot\vec{r}Q}{Q} = -E_x\Delta x. \qquad (2)$$

This relationship was derived and practiced in class to emphasize that ΔV_{AB} does not depend on either W_{AB} or Q. Instead, ΔV_{AB} could change if $\vec{E}$ is altered (*i.e.,* the source of the field is changed).

The last question was analogous to the target question introduced in the physics assignment. Students were asked, "If the value of the charge moved from A to B is increased by a factor of 4, Q_{new}=4Q, does the potential difference between points A and B *increase, decrease,* or *remain the same?*" Students were expected to successfully utilize the knowledge emphasized in class in a similar context. In addition, it was hoped that Question 2, which uses Eq. 2, would assist in identifying errors and overriding incorrect responses given by students who initially chose the canonical, intuitively appealing, but wrong answer based on the inverse relationship between Q and ΔV_{AB}.

IV. RESULTS

Table I contains results from the math assignment intended to answer research question 1. Almost all students (~89%) demonstrated that they have at least a minimal understanding of what it means for quantity z to depend on x. Most students gave responses similar to the following: "*F depends on x means if x changes, F must change as well.*" Most were also successful at giving examples of W such that $z = \frac{W}{x}$ depends on x (*e.g., W=const*, $W=x^2$). However, less than a third gave a correct example of W such that $z = \frac{W}{x}$ does not depend on x (*e.g., W=ax,* where *a=const)*.

TABLE I. Results of the math assignment intended to answer research question 1

Tasks	Correct
Question 1	89%
Question 2	68%
Question 3	28%

TABLE II. Results of the physics assignment intended to answer research question 2

Tasks	Correct	Incorrect reasoning with multi-variable expression
Question 1	18%	-
Question 2	33%	34%

The drop in the success rate on the last question may not be surprising, given that there are numerous possibilities for W in question 2 and only one type of response for W in question 3. However, the knowledge necessary to answer question 3 is also necessary for the correct analysis of many multi-variable expressions in physics. It appears that most students in our calculus-based physics courses need extra help developing this type of math mindware.

Table II contains results from student responses to the physics assignment intended to answer research question 2. Only ~18% of students could explain why the potential difference between points A and B does not depend on work to move a charge or the charge being moved. This relatively low success rate is consistent with that on the last question of the math assignment (~28%). A few students could apply their math mindware to respond correctly to the physics task. The instructor-led classroom discussion, which followed the math assignment and aimed to illustrate the relevance of this type of math reasoning to the concept of an electric field, does not appear to be effective at improving performance.

On the target question, ~33% correctly answered that ΔV_{AB} remains the same after Q being moved between the two points is increased by a factor of 4. We speculate that the higher success rate on question 2 could be attributed to the operational definition of ΔV_{AB} provided at the beginning of the assignment. This definition explicitly mentioned that ΔV_{AB} does not depend on W_{AB} or Q. It may be somewhat concerning that only a small fraction of the students used this information. However, this result is consistent with reasoning patterns predicted by DPToR; namely, in the presence of an intuitively appealing idea, relevant knowledge is applied selectively (*i.e.,* confirmation bias). The most common type of incorrect reasoning (~34%) to question 2 was based on the inverse relationship between ΔV_{AB} and Q.

Table III presents the results from the assessment included in the course exam. Almost all students answered the first question correctly, which required the application of the operational definition of the potential difference in a straightforward manner, illustrating that they possess a minimal level of relevant mindware. However, only half of the students could determine the electric field in the region by applying $\Delta V_{AB} = -E_x\Delta x$. This suggests that the type of

TABLE III. Results of the exam intended to answer research question 3

Tasks	Correct	Incorrect reasoning with multi-variable expression
Question 1	91%	-
Question 2	49%	-
Question 3	14%	78%

relevant mindware that could aid in error detection and override for those students who may initially apply canonical incorrect reasoning on question 3 is either missing or weak, which reduces the likelihood of successful reasoning even further. Indeed, only ~14% of the students answered question 3 correctly with correct reasoning (a simple statement that ΔV_{AB} does not depend on the charge being moved between points A and B was considered correct with a correct explanation). The overwhelming majority of the students (~78%) defaulted to the canonical incorrect reasoning based on the inverse relationship between ΔV_{AB} and Q without further considering W_{AB}.

V. DISCUSSIONS

The results of this investigation suggest that students entering our introductory physics calculus-based courses possess some level of relevant math mindware; however, it must be strengthened and developed further to help them analyze relationships between physics quantities described by multi-variable expressions.

Figures 4 and 5 illustrate the trickling effect of weak or missing math mindware on reasoning. Fig. 4 shows the performance on the math assignment of those students who explained in their own words what it means for quantity F to depend on x (~89% of the total). We argue that these students possess at least a minimal level of relevant math mindware. The data suggests that even in the context of math (without additional layers of complexity added by the physics contexts), only a third of these students (~31%) could give an example of W such that $z = \frac{W}{x}$ does not depend on x.

The success rate with this type of math reasoning decreases even further once the physics context is added. Fig. 5 illustrates the performance on the physics assignment and the exam of those students who correctly provided examples of W such that $z = \frac{W}{x}$ does not depend on x (~31% in Fig. 4). Despite the explicit inclusion of the operational definition of potential difference in the assignment, only about one-third of these students could apply their relevant math mindware to the physics context. The success rate dropped even further on the exam, with only ~11% of the students providing correct answers with correct reasoning.

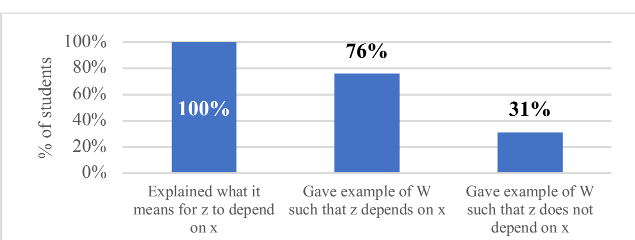

FIG 4. Performance on math assignment of those students who answered Q1 correctly

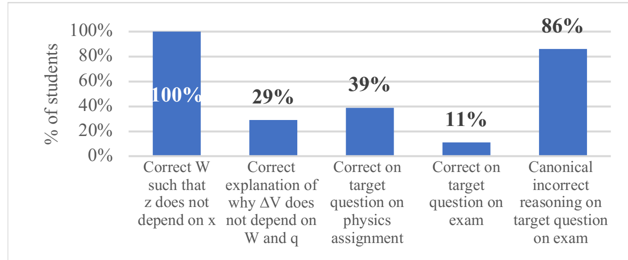

FIG 5. Performance on physics assignment and exam of those students who answered Q3 of math assignment correctly

Most of the remaining students defaulted to canonical incorrect reasoning, illustrating its intuitive appeal that is difficult to override even by those with relevant mindware.

VI. CONCLUSIONS

Dual-process theories of reasoning suggest that relevant mindware is necessary but insufficient to reason productively. Students must also develop skills to use the relevant mindware to navigate reasoning hazards presented along the way to a correct conclusion (*e.g.,* detecting and overriding mistakes).

Our findings suggest that the students' math mindware relevant to topics involving multi-variable expressions is weak or missing. This means that error detection is unlikely for students with weak mindware and very unlikely if the mindware is missing.

Our results are consistent with prior findings that even if relevant mindware is available, its transfer to different concepts and contexts is challenging, even in the same domain [20]. The transfer between domains (e.g., math to physics) adds further complexities to the challenge. Research indicates that solving physics problems using math requires significant expertise, which could be comparable to being bilingual in the languages of math and physics [21]. Therefore, it should be expected that students may start their reasoning in physics with inappropriate (but intuitively appealing) assumptions that various symbols in physics equations could be treated as constants without considering these thoughts further.

Future research should focus on identifying instructional strategies to strengthen relevant math mindware and facilitate its transfer to the context of physics. In addition, instructional efforts guided by DPToR should be dedicated to helping students develop reasoning skills necessary for checking for the validity of assumptions (*e.g., W=const*), which are necessary to detect and override mistakes.

ACKNOWLEDGMENTS

This material is based upon work supported by the National Science Foundation under the grants Nos. DUE-1821390, 1821123, 1821400, 1821511, 1821561.

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