"Instead of gravity pointing down, it's now pointing up": Enhancing physics students' connection between mathematics and mechanism

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Digital simulations are especially helpful in physics education, but most simulations provide only a visualization of a phenomenon while obscuring the mathematical relationships that model its behavior. Our team is developing a suite of online simulations called DynamicsLab, which combine visual representations with an ability to input and alter the governing physics equations. Here, we share excerpts from a group of clinical interviews, in which intermediate physics students explored the first iterations of a DynamicsLab simulation of a characteristic problem in Classical Mechanics: the bead on a spinning hoop. The students were given predict-observe-explain prompts to investigate the way they connected the mathematical representation to the physical phenomenon. We highlight three episodes in which students had to revise unsuccessful predictions, and how these instances indicate that engaging with the DynamicsLab simulation encouraged the students to draw upon a more diverse range of knowledge elements to support their physical and mathematical reasoning.

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I. INTRODUCTION

Digital simulations have become a powerful component of the modern physics educator's toolkit [1]. The ability to explore physics in a virtual space can be especially powerful for phenomena that are difficult for students to envision in their heads alone [2]. Simulations can empower students by providing multiple lines of simultaneous representation (e.g. visuals alongside graphs) and allowing for flexible engagement capable of accommodating multiple levels of expertise [3]. Empirical studies using simulations as alternatives and/or supplements to classroom and laboratory instruction have found them to be effective means for evoking conceptual change in students [4–6]. Given these powerful affordances, it is unsurprising that online simulations have become a staple across many levels of physics education.

Most popular physics simulation tools provide a visual example of a phenomenon and allow users to set the value of relevant parameters to observe the effect on the system [7]. We feel that this format, while surely useful in many contexts, has one critical shortcoming: it obscures the actual relationship between the parameters – the equations and formulas modeling the physical mechanisms – behind the scenes and away from students' access. As such, simulations of this nature will likely do very little to impact the way physics students connect the realities of physical systems to the mathematical representations used to model them.

The ability to connect between physical concepts and their associated mathematical form is a vital component of physics problem-solving [8, 9]. For more basic mathematics, students have been shown to have a rich intuition about the common forms used in equation building (e.g. addition and subtraction) [10]. However, as the complexity of mathematical forms rises, students can struggle to overcome the novel challenges [11]. In the context of physics, the mathematical complexity is exacerbated by the immense variety of conceptual complexity that physics equations can contain [12]. In the face of such obstacles, students are in need of instructional scaffolding to assist them in navigating the confusing (and often counter-intuitive) topics found in upper-division physics.

Our research group has been developing a set of online simulations called DynamicsLab [13] to assist in teaching challenging concepts in upper-division physics courses. Importantly, these simulations include the option for students to input their own equations and compare them to the simulation's model. We believe this additional degree of interaction will enable intermediate physics students to construct a more detailed perspective on how various mathematical symbols and operations reflect the realities of physical mechanisms.

This work pertains to a simulation of a bead threaded onto a wire hoop, in which the hoop is spun around a vertical axis (seen in Fig. 1). The bead-on-hoop problem (p.261 in [14]) is commonly used in Classical Mechanics courses to explore and practice Lagrangian mechanics – an advanced physics procedure which constructs equations of motion for constrained phenomena using energy functions and differential equations. Pilot interviews found that students could often perform the calculations of Lagrangians with a high degree of confidence and accuracy, but they often have far less sense for the conceptual meaning of their solutions in the physical context [15]. In this work, we will share the accounts of three intermediate physics students using the DynamicsLab bead-on-hoop simulation to improve their understanding of the problem's equations of motion.

II. THEORETICAL FRAMEWORK

This work utilizes the epistemological perspective *knowledge in pieces* (KiP) [16]. KiP models knowledge as a collection of individual knowledge elements within a web of context-based connections. When a learner embarks on sensemaking in a novel situation, salient details of the problem context will 'activate' a subset of knowledge elements based on their apparent relevance to the current task. Novices often activate elements that, while useful in other cases, may not be applicable to the problem at hand. Likewise, they may overlook elements that would indeed be useful if they have not yet developed (or moreover, reinforced) the elements' connections to the problem. Through instruction and practice, learners refine these contextual connections to better activate the most productive and relevant knowledge elements.

Related knowledge elements can activate each other through a process called 'cuing' – a knowledge element which bears a strong 'cuing priority' with another will be more likely to activate in conjunction with its companion. Knowledge elements which are often activated together in similar contexts will develop strong cuing priority on their own through standard use. In contrast, when a problem requires drawing upon disparate sectors of knowledge simultaneously (such as physics problems requiring an understanding of both causal mechanisms and abstract representations), it is far less likely for the activation of elements in one area to activate elements in the other domain. Our goal is to use targeted activities which emphasize or draw attention to the connections between these otherwise-distant knowledge elements to enhance their cuing priority and improve the global coherence of learners' knowledge systems.

III. METHODS

Physics students enrolled in the Classical Mechanics course at our university were invited to volunteer for clinical interviews to test out early iterations of the DynamicsLab bead-on-hoop simulation. In total, six students agreed and met with the lead author for roughly 1.5 hours each. After discussing their current understanding of Lagrangian mechanics, the students (if they had not done so in class already) performed the derivation for the equation of motion for the bead-on-hoop problem [14]. The solution comes out to:

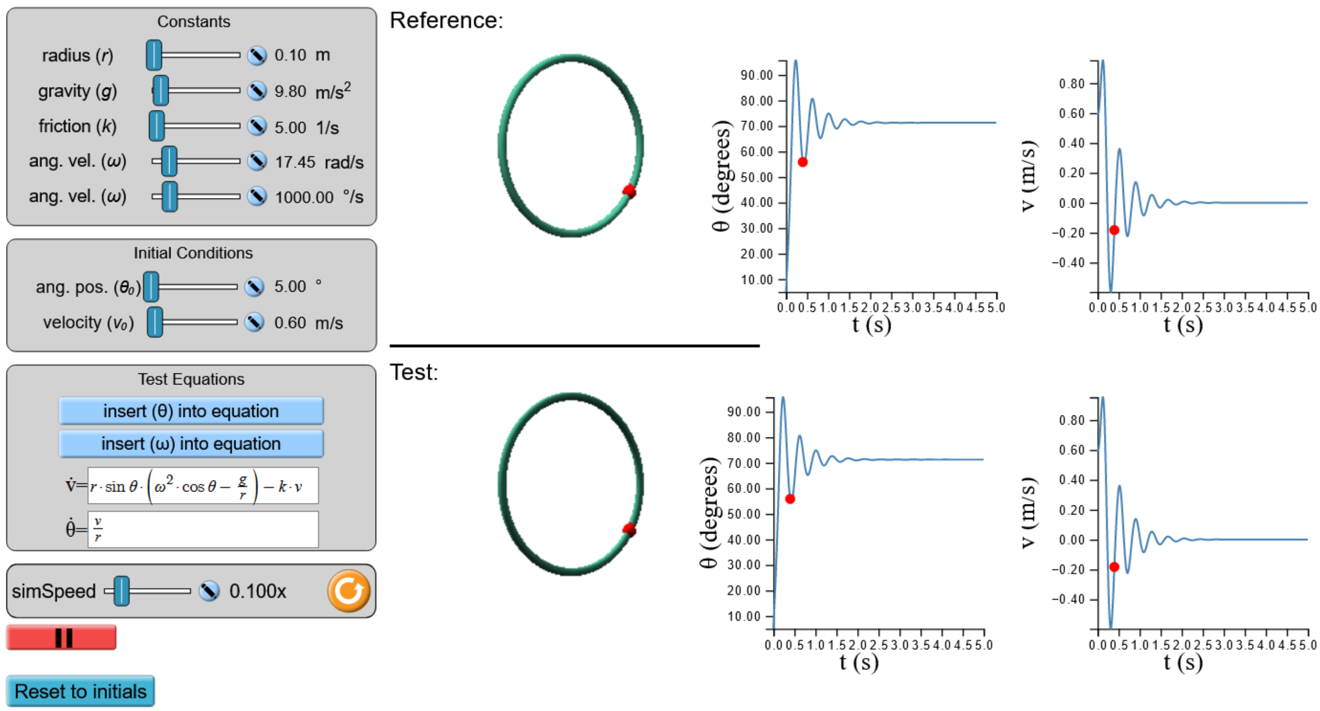

FIG. 1. Screenshot of the DynamicsLab bead-on-hoop simulation [13]. Users can alter the parameters (such as gravity and angular velocity), as well as enter their own equations of motion to govern the behavior of the bottom "test hoop." The graphs on the right plot the velocity and angular position of the reference and test hoop for visual comparison.

$$\ddot{\theta} = sin\theta(\omega^2 cos\theta - g/r), \tag{1}$$

where θ is the angular position of the bead (measured from the bottom of the hoop), ω is the angular velocity of the hoop, g is the gravitational constant, and r is the radius of the hoop. The students then explained how Eq. 1 could be used to find the positions on the hoop at which a stationary bead would remain stationary (called 'critical' or 'equilibrium' points).

Then, the students were introduced to DynamicsLab and allowed to explore freely. When the students were sufficiently comfortable with the interface, they were asked to input equations of motion into the 'test equations' boxes. After converting Eq. 1 from a second-order derivative to two first-order derivatives (using the known relationship $\dot{\theta} = v/r$) and adding a friction term, the final equation is:

$$\dot{v} = rsin\theta(\omega^2 cos\theta - g/r) - kv. \tag{2}$$

When Eq. 2 is executed in the DynamicsLab simulation (as seen in Fig. 1), the motion of the bead on the test hoop exactly matches the motion of the bead on the reference hoop.

To conclude, the students were given a set of instructional prompts, which used the *predict-observe-explain* format and either asked 1) how would the behavior of the bead be affected by a given change in the equations, or 2) what kind of change in the equations would generate a target behavior? Audio, video, and screen recordings of the interviews were captured and analyzed using an iterative observe-schematize-systematize procedure [17] in accordance with Knowledge Analysis [18, 19], a KiP-aligned methodology for studying the structure and dynamics of knowledge.

IV. FINDINGS

Overall, the students displayed a strong intuitive sense for the relationship between the equations and the behavior of the bead. In response to the prompts, the majority of the students' predictions were nearly or entirely accurate. This suggests that many intermediate physics students have already begun to develop a robust capacity for connecting abstract and concrete conceptualizations.

For this work, we chose to focus on the minority of instances in which students had to revise unsuccessful predictions. These occurrences shed light on the ways in which the simulation and activity design encouraged the students to draw upon a wider assortment of sensemaking resources. Here, we present examples taken from the interviews of three students: Armando, Alena, and Youssef.

A. Flipping Gravity

The first prompt was to predict how the bead would behave if the negative sign in front of the g/r term in Eq. 2 were flipped to a positive sign. Armando initially approached this task by relying solely on mathematical reasoning. He understood from working through the derivations that the locations of the bead's equilibrium points were found by setting Eq. 1 equal to zero (as there would be no angular acceleration at these points). Armando recalled that when the parenthetical term was zero, the resulting θ described the equilibrium point shown in Fig. 1, where the bead comes to rest just below 90 degrees. This mathematical reasoning led Armando to conclude that the flipping of the sign of g/r would disrupt the stability that arose from that portion of the equation of motion:

> "You have this relationship between the omega and the gravity that are contrary to each other. As the gravity increases, it kind of steals away from this omega. But if we add to it, then I'm guessing the critical point... I think it'll disappear entirely... because this term will never be zero."

But when Armando implemented the equation change in DynamicsLab, the equilibrium point was not eliminated. Instead, it shifted from just below 90 degrees to just above 90 degrees on the test hoop.

Armando's judgement that the parenthetical could never equal zero when the sign of g/r was positive had overlooked a subtle mathematical nuance: the $cos\theta$ in the first term introduces a negative sign in some quadrants of the hoop. When the sign on g/r was reversed to positive, the parenthetical would now equal zero at a position where the value of the $\omega^2 cos\theta$ term was the same but its sign was flipped to negative. This would occur when the horizontal component of the bead's position (the magnitude of $cos\theta$) was unchanged, but it was in the next quadrant – thus reflecting the equilibrium point over the horizontal axis.

The persistence of the test bead's stability surprised Armando, who now had to construct an explanation for the unforeseen result. Interestingly, Armando did not attempt to revise his mathematical intuitions surrounding the equation to explain the behavior. Instead, he drew upon his conceptual understanding of the phenomenon to address the discrepancy:

> "It's flipping the sign of g, right? Nothing else is really changing. But by making that to an addition, you can almost now think that instead of gravity pointing down, it's now pointing up."

Indeed, Armando's envisioning of the flipped sign as a reversal of the direction of gravity was sound, and it sufficiently explained the behavior of the test bead: as the centrifugal force urged the bead away from the axis, it was gravity that prevented the bead from reaching the limit of 90 degrees. With gravity reversed, the same effect would occur, just with the bead being held in the upper quadrant instead of the lower.

B. Flipping Friction

Similar to the first, another prompt had the students consider the effect of changing the sign in front of the friction term (kv) from negative to positive. Alena's first intuition correctly identified that "that force would keep adding to it." However, something about the circular nature of the phenomena and the trigonometric functions involved caused her to temper her predictions:

> "The cosine and sine would help regulate it... as θ got smaller, v might get smaller, which would decrease the amount it's adding to it."

From her statements, it is unclear if Alena believed that the bead would slow down and lose velocity at some point, or that the bead's positive acceleration would simply be lessened while still remaining positive. In either case, it is more important to note that Alena was connecting the trigonometric effects that appeared elsewhere in the equation of motion to the friction term (which had no trigonometric function).

Alena's prediction that the positive friction force would be reigned in was refuted by DynamicsLab; when Alena reversed the sign on the friction term, the test bead quickly took off, accelerating non-stop until it sped around the hoop multiple times each second. As Alena came to terms with the result, she did not mention the mitigating effects of trigonometric terms again, indicating that she had possibly abandoned that connection. Instead, she used a new term to described her updated conceptualization:

> "It gets faster and faster, but it doesn't slow down. It looks like it just keeps going... that makes sense, because as your v is increasing, because we added it, it means the change in v is increasing. It's like a feedback loop, it just makes the v get bigger."

Alena connected the behavior of this additive-friction scenario to a relational ontology seen in many other real-world contexts: the "feedback loop." This new connection suggests that Alena (like Armando in the previous example) was expanding the pool of knowledge resources upon which she drew to inform her interpretations as her mathematical reasoning was challenged.

C. Removing Velocity

In a more open-ended prompt, the students were asked to choose one of the terms or variables in Eq. 2 and remove it. From the options, Youssef decided that he would remove v from both test equations. Like Armando and Alena, Youssef's first prediction followed primarily from mathematical reasoning: as he altered the equations, he stated that deleting a variable from an expression was equivalent to "essentially, holding it fixed at one." Youssef concluded that his actions would cause the velocity of the bead to be held constant indefinitely. In turn, he believed that "this whole equation, $\dot{v}$, is going to be zero."

In general, Youssef's assumption that deleted variables are mathematically equivalent to a substituted constant value of 'one' is correct. Yet, it is an overextension to extrapolate that v would remain at one perpetually. Eq.2 is a differential equation – it states that the rate of change in v is dependent on v itself. When the v is removed, the rate of change of v will no longer depend on v [20], but the formula will still yield some value based on the other parameters (in fact, the initial rate of change would be numerically equivalent to the value found using an unaltered Eq.2 and $v = 1$). This value of $\dot{v}$ would still represent the rate of change of the velocity of the bead regardless of the presence or absence of v in Eq.2. As long as $\dot{v}$ was non-zero, the simulation would calculate new values for v using this function as time progressed.

Youssef's prediction was short-lived as the test bead continued to accelerate and change speeds – albeit in an erratic manner – even after the removal of v from the test equations. The non-constant motion pushed Youssef to reconsider his understanding of the relationship between v and $\dot{v}$. Notably, one resource which Youssef drew upon to accept the bead's unexpected behavior was attending to the other parts of Eq.2 that had remained unchanged. Youssef realized that "we still have omega in play here... the omega squared is still making it rise – the centrifugal force." Youssef's articulation of the centrifugal force, albeit brief, indicates that he was trying to situate his updated explanation in conceptual terms – something that was not present in his previous attempt.

V. DISCUSSION

In each of the examples presented, the students crafted an initial prediction based primarily (or even entirely) on mathematical reasoning. Our KiP perspective urges us to note that the mathematical principles which the students drew upon to justify their predictions were all productive: Armando looking for possible zeros in the function, Alena considering the bounded nature of sines/cosines, and Youssef equating variable elimination to substituting a 'one' are all useful strategies that have wide-ranging relevance across many mathematical domains. Moreover, the students did not activate these resources by chance; each idea had a clear relevance to the salient features of the bead-on-hoop equations of motion, and their contributions were not moot. Productive student ideas such as these should be validated and encouraged to reinforce their activation in potentially relevant contexts.

KiP can also offer an idea as to why these students went to their mathematical knowledge so readily in their explanation-building: the instructional prompts discussed here each started with a proposed change to the structure or composition of the equations. Drawing the students' attention to this inciting alteration at the onset of their sensemaking would cue any knowledge or experience the students had regarding the focal representational construct (e.g. cuing resources about positives and negatives when being asked to flip a sign).

As we described each students' mathematical justification for their initial prediction in the previous section, we pointed out the flaws in the students' reasoning or the subtleties of the application that the students had overlooked. We did not do this to expose the students' deficiencies or make any arguments about their lack of capability in these matters (a starkly anti-KiP endeavor [21, 22]). Instead, we wanted to show that the mathematical arguments used by the students were not unsalvageable – with some careful attention, they could be refined and applied successfully to the bead-on-hoop problem. With that in mind, we find it intriguing that these students did not return to their proposed mathematical ideas when their predictions were refuted by the simulation. Instead, they appeared to move away from their initial mathematical ideas in favor of searching for an explanation elsewhere. In the cases we shared, each student found some amount of clarity in conceptual connections that weren't present in their first articulation: Armando and Youssef pointed to conceptual pieces that were specific to the bead-on-hoop problem (gravity and the centrifugal force, respectively), while Alena mentioned a more generalizable conceptual idea (the feedback loop).

Seeing the students reach for a wider variety of knowledge resources when revising their predictions is a promising sign for the feasibility of simulations like DynamicsLab: drawing the students' attention to the equation in the *predict* step seemed to cue mathematical ideas, whereas attending to the visual behavior in the *observe* step seemed to cue conceptual resources about the physical mechanisms and relationships. That said, it was not our goal to have students relinquish or abandon their previous mathematical thinking, but instead to widen the scope of relevant ideas to include both mathematical and conceptual resources simultaneously. In an ideal instructional environment, students working with DynamicsLab would be encouraged to explore the applicability of mathematical and conceptual ideas in tandem to enhance the connections between these two domains of knowledge.

In short, these preliminary results serve to validate DynamicsLab's motivation to include mathematical manipulation in digital simulations. We believe the diversity of mathematical and conceptual resources seen in the data would not have been demanded by a simulation which only displayed a visualization of a phenomenon without its accompanying mathematical representation. By allowing students to interact directly with mathematical representations alongside coupled visual models, we believe they can construct deeper and more reliable connections between the symbols on the page and the physics of the real world.

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