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From intersection bodies to dual centroid bodies: A stochastic approach to isoperimetry

Dedicated to the memory of Nicole Tomczak-Jaegermann, 1945–2022.

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Abstract. We establish a family of isoperimetric inequalities for sets that interpolate between intersection bodies and dual L_p-centroid bodies. This provides a bridge between the Busemann intersection inequality and the Lutwak–Zhang inequality. The approach depends on new empirical versions of these inequalities.

Keywords: affine isoperimetric inequalities, polar centroid bodies, radial summation, stochastic approximation, star-shaped sets.

1. Introduction

The focus of this paper is on connections between fundamental inequalities in Brunn–Minkowski theory and dual Brunn–Minkowski theory. The former details the behavior of the volume of Minkowski sums of convex bodies. The standard isoperimetric inequality is emblematic of deep principles within Alexandrov's theory of mixed volumes [87]. A central line of research is on affine-invariant strengthenings of kindred isoperimetric principles, especially around projections of convex sets; as a sample, see Lutwak's survey [62], Schneider's monograph [87], the fundamental papers [64, 68, 69], and [75] for

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a recent breakthrough. In dual Brunn–Minkowski theory, the emphasis is on star-shaped sets and radial addition. Dual mixed volumes, put forth by Lutwak [57], parallel many aspects of mixed volumes. They provide a rich framework for studying intersections of star bodies with subspaces; for example, see [58, 59] for foundational results; the monographs by Koldobsky [51] and Gardner [27] for the resolution of the Busemann–Petty problem and interplay with geometric tomography; the papers [7, 9, 41, 42] for striking new developments. Establishing an important family of isoperimetric inequalities linking the two theories has remained a principle challenge.

A common root for the inequalities we treat is the *Busemann intersection inequality* [15] for the volume of central slices of a compact set $K \subseteq \mathbb{R}^n$:

$$\int_{S^{n-1}} |K \cap u^{\perp}|^n du \leq \frac{\omega_{n-1}^n}{\omega_n^{n-1}} |K|^{n-1}, \tag{1.1}$$

where du denotes integration with respect to the normalized Haar probability measure on the sphere S^{n-1}, $|\cdot|$ is volume and ω_n is the volume of the Euclidean unit ball B_2^n. The result itself (with hindsight) is an invariant inequality for the volume of the *intersection body* $I(K)$ of K, which is defined by its radial function via $\rho(I(K), u) = |K \cap u^{\perp}|$ (see Section 3.1 for notation and definitions). Intersection bodies were introduced by Lutwak [59] in connection with the Busemann–Petty problem and play a crucial role in dual Brunn–Minkowski theory [27, 51]. The proof of (1.1) used an essential ingredient known as the Busemann random simplex inequality, which says that the expected volume of certain random simplices in a convex body is minimal for ellipsoids. Petty used the latter to establish a conjecture of Blaschke on the volume of centroid bodies [83], which is now known as the Busemann–Petty centroid inequality. Geometrically, given an origin-symmetric convex body K in $\mathbb{R}^n$, the centroids of halves of K cut by hyperplanes through the origin form the surface of its centroid body. Centroid bodies are zonoids, i.e., Hausdorff limits of Minkowski sums of segments, and thus naturally belong to Brunn–Minkowski theory. Zonoids play an important role in functional analysis and related fields, e.g., [5, 10, 76, 88].

Lutwak raised the question of connecting the Busemann intersection inequality and the Busemann–Petty centroid inequality in [62]. The latter is one of several fundamental results that lead to strengthenings of the standard isoperimetric inequality; in particular, it is equivalent to an inequality of Petty [84] on *polar projection bodies*, as shown in [62]. Projection bodies are also zonoids and play a central role in Brunn–Minkowski theory [27].

A functional analytic perspective has shaped the development of both intersection bodies and polar projection bodies. Early work in the isometric theory of Banach spaces, going back to Lévy, introduced stable laws in connection with embeddings in L_p for $p \in (0, 2]$. Positive definite distributions, stable laws and associated change of density arguments play a central role [4, 72, 78, 89]; see also the survey [43] and references therein. Koldobsky developed a parallel theory, based on a Fourier-analytic approach, for embedding in L_p, for $p < 0$. This led to fundamental characterizations of intersection bodies and their higher-dimensional analogs [48, 50, 51]. With this view, intersection bodies are unit "balls" of finite-dimensional subspaces of L_{-1}. At the other end, polar projection bodies arise naturally as unit balls of subspaces of L_1 [5]. In between L_{-1} and L_1 there is a continuum of spaces that are no longer Banach spaces. A result of Koldobsky shows that the classes in between decrease as p varies from -1 to 1; in particular, every polar projection body is an intersection body [49,51]. A longstanding open problem of Kwapień [54], in geometric form, asks if every intersection body is isomorphic to a polar projection body; see work of Kalton and Koldobsky [44] for progress on this question.

A rich theory of isoperimetric inequalities has flourished around centroid bodies and polar projection bodies. Two fundamental papers in this development are those of Lutwak–Zhang [70] and Lutwak–Yang–Zhang [64]. For a star-shaped body K and $1 \leq p \leq \infty$, the L_p-centroid body $Z_p(K)$ is defined by its support function (see Section 3.1) for unit vectors u by

$$h^p(Z_p(K), u) = \frac{1}{|K|} \int_K |\langle x, u \rangle|^p \, dx.$$

Equivalently, $h(Z_p(K), \cdot)$ is the norm associated to the polar of $Z_p(K)$, denoted here by $Z_p^\circ(K)$. Lutwak and Zhang proved that for $1 \leq p \leq \infty$,

$$|Z_p^\circ(K)| \leq |Z_p^\circ(K^*)|, \tag{1.2}$$

where K^* is the dilate of the unit ball centered at the origin of the same volume as K. When $p = \infty$, (1.2) is the Blaschke–Santaló inequality, which is equivalent to the affine isoperimetric inequality [62]. When $p = 1$, (1.2) follows from the Busemann–Petty centroid inequality. Lutwak, Yang and Zhang [64] later proved a stronger inequality for $Z_p(K)$ itself. These are central results within the framework of L_p-Brunn–Minkowski theory, which is governed by a different elemental notion of summation, called L_p-addition [26, 61, 63]. This theory provides a basis for wide-ranging inequalities in geometry, analysis and probability, e.g., [37, 38, 40, 65–67]. Campi and Gronchi developed an alternate approach to isoperimetric inequalities for L_p-centroid bodies in [16,17]. In particular, they further developed the notion and applications of shadow systems, as introduced by Rogers and Shephard [86]. These systems generalize Steiner symmetrization and have far-reaching extensions and applications; see, e.g., [18, 19]. There is significant interest in L_p-Brunn–Minkowski theory for the challenging setting of $p < 1$ [8]; see the survey [7], and recent advances in [53, 74], and the references therein.

A common framework for polar projection bodies and intersection bodies has been pursued from several perspectives. Drawing on [70], the notion of the dual L_p-centroid body was extended by Gardner and Giannopoulos in [28] to $p \in (-1, 1)$ via

$$\rho^{-p}(Z_p^\diamond(K), u) = \frac{1}{|K|} \int_K |\langle x, u \rangle|^p dx.$$

The bodies $Z_p^\diamond(K)$ interpolate between intersection bodies and polar L_p-centroid bodies using

$$\rho(I(K), u) = |K \cap u^\perp| = \lim_{p \to -1^+} \frac{p+1}{2} \int_K |\langle x, u \rangle|^p dx;$$

see [28, 32, 34, 51]. For $p < 1$, $Z_p^{\diamond}(K)$ need not be convex, which we emphasize here by the use of the $\cdot^{\diamond}$ notation. Busemann–Petty type volume comparison problems for $Z_p^{\diamond}(K)$, motivated by earlier work of Grinberg and Zhang [32] and Lutwak [60], were treated by Yaskin and Yaskina in [92]. For $p < 0$, these bodies have also been termed L_p-intersection bodies and characterizations of such operators as radial valuations were established by Haberl and Ludwig [36]; see also [35] for $p > -1$. Properties of L_p-intersection bodies were further developed by Haberl in [34]. When K is an origin-symmetric convex body, a result of Berck [3] shows that $Z_p^{\diamond}(K)$ is actually convex for $-1 < p < 1$, which extends Busemann's seminal result for intersection bodies [14].

We develop methods to bridge the gap between the Busemann intersection inequality (1.1) and the Lutwak–Zhang theorem (1.2). Each of these can be proved using Steiner symmetrization, but in very different ways. The former applies to the star bodies $I(K)$ and uses integral geometric identities (of Blaschke–Petkantschin type) that are particular to slices of K. The latter relies on convexity of the polar centroid bodies $Z_p^{\circ}(K)$ for $p \geq 1$. We develop a new approach that applies to star bodies in between these two classes, that sees (1.1) and (1.2) from the same viewpoint. We will show that (1.1) is one of a large family of inequalities for unit balls of finite-dimensional subspaces of L_p. We merge several techniques that have been used for $p = \pm 1$. These include symmetrization, embedding via random linear operators, and a classical change of density technique used in Koldobsky's Fourier-analytic treatment of intersection bodies.

We follow a probabilistic approach in which L_p-centroid bodies are attached to probability densities rather than sets. This view was put forth by the second-named author [79] in the study of high-dimensional measures and their concentration properties; see also [46, 55]. Fundamental inequalities of Lutwak, Yang and Zhang, in [64], were extended to probability measures by the second- and third-named authors in [80, 81]. An empirical approach to dual L_p-centroid bodies, for $p \geq 1$, was developed in further joint work with Cordero-Erausquin and Fradelizi [21], motivated by [18]. To fix the notation, we set

$$\mathcal{P}_n = \left\{ f\colon \mathbb{R}^n \to [0,\infty) : \int_{\mathbb{R}^n} f(x)dx = 1,\ \|f\|_\infty < \infty \right\},$$

where $\|f\|_\infty$ denotes the essential supremum. For $f \in \mathcal{P}_n$, the empirical L_p-centroid body $\mathcal{Z}_{p,N}(f)$ is defined by its support function via

$$h^p(\mathcal{Z}_{p,N}(f), u) = \frac{1}{N}\sum_{i=1}^{N} |\langle X_i, u\rangle|^p, \tag{1.3}$$

where $X_1, \ldots, X_N$ are independent random vectors with density f. In [21], a stronger stochastic version of (1.2) was established for radial measures ν with decreasing densities,

$$\mathbb{E}\nu(\mathcal{Z}_{p,N}^{\circ}(f)) \leq \mathbb{E}\nu(\mathcal{Z}_{p,N}^{\circ}(f^*)), \tag{1.4}$$

where f^* is the symmetric decreasing rearrangement of f (see Section 3). By the law of large numbers, (1.4) implies the Lutwak–Zhang inequalities (1.2) when $N \to \infty$ and $f = \chi_K/|K|$.

The empirical inequality (1.4) follows from a general theorem about random operators acting in normed spaces [21]. The random operator viewpoint is from the asymptotic theory of normed spaces. In seminal work [29], Gluskin used random operators to construct counter-examples to a longstanding question on the maximal Banach–Mazur distance between finite-dimensional spaces. The expository article of Mankiewicz and Tomczak-Jaegermann [71] details its far-reaching extensions in Banach space theory. This viewpoint was also fruitful in developing stochastic versions of a number of isoperimetric inequalities [81, 82]. However, inherent in the method was a restriction to *convex* sets. The main new feature we develop here is its applicability to *star-shaped* sets. We will show how this change provides a bridge between the aforementioned inequalities in Brunn–Minkowski theory and dual Brunn–Minkowski theory.

2. Main results

Our first result establishes a sharp isoperimetric inequality that extends the Lutwak–Zhang inequality (1.2) to the case $p \in (0,1)$. For $f \in \mathcal{P}_n$ and $p \in (0,1)$, define the dual L_p-centroid body $Z_p^{\diamond}(f)$ via its radial function,

$$\rho^{-p}(Z_p^{\diamond}(f), u) = \int_{\mathbb{R}^n} |\langle x, u\rangle|^p f(x)dx.$$

To define the empirical version $\mathcal{Z}_{p,N}^{\diamond}(f)$, we let $N > n$ and consider independent random vectors $X_1, \dots, X_N$ according to f as above, and set

$$\rho^{-p}(\mathcal{Z}_{p,N}^{\diamond}(f), u) = \frac{1}{N}\sum_{i=1}^{N} |\langle X_i, u\rangle|^p. \tag{2.1}$$

As above, we also associate such bodies to the symmetric decreasing rearrangement f^*.

Theorem 2.1. *Let* $f \in \mathcal{P}_n$ *and let* $0 < p < 1$. *Then*

$$|Z_p^{\diamond}(f)| \leq |Z_p^{\diamond}(f^*)|. \tag{2.2}$$

Moreover,

$$\mathbb{E}|\mathcal{Z}_{p,N}^{\diamond}(f)| \leq \mathbb{E}|\mathcal{Z}_{p,N}^{\diamond}(f^*)|. \tag{2.3}$$

Theorem 2.1 relies on first establishing the empirical version (2.3), while (2.2) is derived as a consequence. This is a key difference from the empirical approach in [21,81,82] in which (non-random) inequalities of Lutwak, Yang and Zhang [64, 68, 70] inspired the development of their empirical versions (e.g., (1.2) motivated its stochastic form (1.4)). Recently, Yaskin [91] proved (2.2) and extensions raised in [52] in the case when $f = \chi_K$, where K is an origin-symmetric star-body sufficiently close to the Euclidean ball.

Our original inspiration is a recent volume formula for sections of finite-dimensional L_p-balls by Nayar and Tkocz [77] that builds on ideas involving Gaussian mixtures of random variables from [25]. Kindred probabilistic representations have been indispensable in the study of sections of convex bodies, e.g., [2,47,73]. In our case, such a formula allows for a reduction from star-shaped sets to convex sets that interfaces well with the empirical approach from [21,81,82]; see [1] for recent applications of related formulas in stochastic geometry.

The methods we develop here go beyond centroid bodies to sets generated by families of subspaces of L_p. For $f \in \mathcal{P}_n$, an origin-symmetric convex body C in $\mathbb{R}^m$, $m \geq 1$, and $p \neq 0$, we define $Z^{\diamond}_{p,C}(f) \subseteq \mathbb{R}^n$ by its radial function (see Section 3.1): for $p \neq 0$,

$$\rho^{-p}(Z^{\diamond}_{p,C}(f), u) = \int_{(\mathbb{R}^n)^m} h^p(C, (\langle x_i, u\rangle)_{i=1}^m) \prod_{i=1}^m f(x_i) d\overline{x}, \tag{2.4}$$

where $d\overline{x} = dx_1 \cdots dx_m$, and for $p = 0$,

$$\log \rho(Z^{\diamond}_{0,C}(f), u) = -\int_{(\mathbb{R}^n)^m} \log h(C, (\langle x_i, u\rangle)_{i=1}^m) \prod_{i=1}^m f(x_i) d\overline{x}.$$

As p decreases, the sets $Z^{\diamond}_{p,C}(f)$ increase with respect to inclusion and $\rho(Z^{\diamond}_{p,C}(f), u)$ is finite for almost every $u \in S^{n-1}$ whenever $p > -1$ and $f \in \mathcal{P}_n$ (see Lemma 4.1).

We also define empirical versions involving multiple bodies C and densities f. Specifically, let $C_1, \ldots, C_N$ be origin-symmetric convex bodies with $m_i = \dim(C_i) \geq 1$ for $i \in [N] = \{1, \ldots, N\}$, where $N > n$. Let (X_{ij}), $i \in [N]$, $j \in [m_i]$, be independent random vectors with X_{ij} distributed according to $f_{ij} \in \mathcal{P}_n$. Write $\mathcal{C} = (C_1, \ldots, C_N)$ and $\mathcal{F} = ((f_{ij})_j)_i$. For $p \neq 0$, we define a star-shaped set $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}) \subseteq \mathbb{R}^n$ by

$$\rho^{-p}(\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}), u) = \frac{1}{N} \sum_{i=1}^N h^p(C_i, (\langle X_{ij}, u\rangle)_{j=1}^{m_i}); \tag{2.5}$$

for $p = 0$, we define $\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F}) \subseteq \mathbb{R}^n$ by its radial function

$$\rho^{-N}(\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F}), u) = \prod_{i=1}^N h(C_i, (\langle X_{ij}, u\rangle)_{j=1}^{m_i}). \tag{2.6}$$

With these definitions, we note the following:

(i) For $p \geq 1$, the convexity of the p-norm ensures that (2.4) and (2.5) define radial functions of origin-symmetric convex sets. Thus we may naturally define

$$Z_{p,C}(f) = (Z^{\diamond}_{p,C}(f))^{\circ} \quad \text{and} \quad \mathcal{Z}_{p,\mathcal{C}}(\mathcal{F}) = (\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}))^{\circ}.$$

In this way, $\cdot^{\diamond}$ coincides with usual polarity whenever $Z^{\diamond}_{p,C}(f)$ and $\mathcal{Z}^{\diamond}_{p,C}(f)$ are compact, convex and have non-empty interior.

(ii) For $p > 0$ and $C = [-1, 1]$, we have $Z^{\diamond}_p(f) = Z^{\diamond}_{p,[-1,1]}(f)$; similarly, if $\mathcal{F} = (f)_{i=1}^N$ and $\mathcal{C} = ([-1, 1])_{i=1}^N$, then $\mathcal{Z}^{\diamond}_{p,N}(f) = \mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$. The latter identities should be taken as definitions in the case $p = 0$.

For $p \geq 0$, we have the following generalization of Theorem 2.1, going from $\mathcal{F} = (f_{ij})$ to the family of rearranged densities $\mathcal{F}^\# = (f_{ij}^*)$.

Theorem 2.2. *Let $f \in \mathcal{P}_n$ and let $p \geq 0$. If C is an origin-symmetric convex body of dimension $m \geq 1$, then*

$$|Z_{p,C}^{\diamond}(f)| \leq |Z_{p,C}^{\diamond}(f^*)|. \tag{2.7}$$

Moreover, if $\mathcal{F} = (f_{ij}) \subseteq \mathcal{P}_n$ and $\mathcal{C} = (C_1, \ldots, C_N)$, where each C_i is an origin-symmetric convex body of dimension $m_i \geq 1$, then

$$\mathbb{E}|\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F})| \leq \mathbb{E}|\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F}^\#)|.$$

The theorem is new for all values of p. For $p \geq 1$, the proof uses tools that have already been developed in [21]. The main novelty here is in techniques to deal with the star-shaped sets $\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F})$ in the range $p \in [0, 1)$. In particular, we provide a separate treatment for $p = 0$ including a new volume formula for $\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F})$. For $p < 0$, the expected volume of the empirical bodies $\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F})$ need not be finite when $\dim(C_i) < n$ (see Remark 5.8); here the use of higher-dimensional convex bodies $C_1, \ldots, C_N$ is essential. For certain values of p, namely when $p \in [-1, 0)$ and n/p is an integer, we establish the following theorem.

Theorem 2.3. *Let $f \in \mathcal{P}_n$ and let $p \in [-1, 0)$. Let C be an origin-symmetric convex body with $\dim(C) \geq 1$. If $p > -1$ and $n/|p| \in \mathbb{N}$, then*

$$|Z_{p,C}^{\diamond}(f)| \leq |Z_{p,C}^{\diamond}(f^*)|.$$

Furthermore, let $\mathcal{F} = (f_{ij}) \subseteq \mathcal{P}_n$ and $\mathcal{C} = (C_1, \ldots, C_N)$, where each C_i is an origin-symmetric convex body of dimension $m_i \geq n + 1$. If $p \geq -1$ and $n/|p| \in \mathbb{N}$, then

$$\mathbb{E}|\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F})| \leq \mathbb{E}|\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F}^\#)|. \tag{2.8}$$

Empirical versions of isoperimetric inequalities from [21, 81, 82] have involved operations in Brunn–Minkowski theory; e.g., for $p \geq 1$, the sets $\mathcal{Z}_{p,N}(f)$ in (1.3) are L_p-sums of random line segments (see Section 3.1). Theorems 2.1–2.3 are the first to treat empirical forms of inequalities for star-shaped sets in dual Brunn–Minkowski theory. In particular, we develop randomized analogs of approximation results of Goodey and Weil [30], and Kalton, Koldobsky, Yaskin and Yaskina [45], in which intersection bodies and their L_p-analogs are limits of radial sums of ellipsoids. The use of higher-dimensional bodies C_i in Theorem 2.3 is needed for this purpose and such bodies are crucial for establishing the corresponding isoperimetric inequalities. In particular, we define a variant of the L_p-intersection body as follows: for $f \in \mathcal{P}_n$, $\alpha > 0$ and $p \in [-1, 0)$, we set

$$\rho^{|p|}(I_{|p|}^{\alpha}(f), u) = \int_{\mathbb{R}^n} (|\langle x, u\rangle|^2 + \alpha^2 \|u\|_2^2)^{-|p|/2} f(x) dx.$$

For the empirical version, we consider $N > n$ independent random vectors $X_1, \ldots, X_N$ from $f \in \mathcal{P}_n$ and define $\mathcal{I}_{|p|,N}^{\alpha}(f)$ via

$$\rho^{|p|}(\mathcal{I}_{|p|,N}^{\alpha}(f), u) = \frac{1}{N} \sum_{i=1}^{N} (|\langle X_i, u\rangle|^2 + \alpha^2 \|u\|_2^2)^{-|p|/2}.$$

The star-shaped bodies $\mathcal{I}^{\alpha}_{|p|,N}(f)$ are L_p-radial sums of ellipsoids (see Section 3.1). In fact, the bodies $\mathcal{I}^{\alpha}_{|p|,N}(f)$ are special (limiting) cases of $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$ for a suitable choice of $\mathcal{C}$ and $\mathcal{F}$, involving ellipsoids and uniform measures on balls.

Corollary 2.4. *Let $f \in \mathcal{P}_n$, $\alpha > 0$, $p \in [-1, 0)$ and $n/|p| \in \mathbb{N}$. Then*

$$|I^{\alpha}_{|p|}(f)| \leq |I^{\alpha}_{|p|}(f^*)|.$$

Moreover,

$$\mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(f)| \leq \mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(f^*)|. \tag{2.9}$$

When $p = -1$, (2.9) is a stochastic form of the Busemann intersection inequality (1.1), as it implies the latter when $N \to \infty$ and $\alpha \to 0$. Indeed, if $f \in \mathcal{P}_n$, we write $I(f)$ for the intersection body of f, defined by its radial function via

$$\rho(I(f), u) = \int_{u^{\perp}} f(x)dx,$$

and (2.9) implies the following functional version of (1.1).

Corollary 2.5. *Let f be a continuous and compactly supported function in $\mathcal{P}_n$. Then*

$$|I(f)| \leq |I(f^*)|.$$

Thus the Busemann intersection inequality (1.1) is one limiting case of a family of extremal inequalities about L_p-radial sums in Theorem 2.3. For (non-random) functional versions of the Busemann intersection inequality, see [22], and [39] for recent developments.

Lastly, we can further reduce inequalities to uniform measures on balls in each of the above theorems whenever the convex bodies C and C_i are unconditional, i.e., invariant under reflections in the coordinate hyperplanes.

Theorem 2.6. *Let $f \in \mathcal{P}_n$. Suppose that $p \in [0, 1]$, or $p \in [-1, 0)$ and $n/|p| \in \mathbb{N}$. Let C be an unconditional convex body in $\mathbb{R}^m$, $m \geq 1$. Set $g = \|f\|_{\infty}\chi_{rB_2^n}$, where $r > 0$ satisfies $\int g = 1$. Then for $p > -1$,*

$$|Z^{\diamond}_{p,C}(f)| \leq |Z^{\diamond}_{p,C}(g)|,$$

while for $p \geq -1$ and $\alpha > 0$,

$$\mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(f)| \leq \mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(g)|.$$

Furthermore, assume that $\mathcal{F} = (f_{ij}) \subseteq \mathcal{P}_n$ and $\mathcal{G} = (g_{ij})$, where $g_{ij} = \|f_{ij}\|_{\infty}\chi_{r_{ij}B_2^n}$ with $r_{ij} > 0$ satisfying $\int g_{ij} = 1$. Let $\mathcal{C} = (C_1, \ldots, C_N)$, where each C_i is an unconditional convex body of dimension m_i. If $p \geq 0$ and $m_i \geq 1$, or $p \in [-1, 0)$ and $m_i \geq n + 1$, then

$$\mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})| \leq \mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{G})|.$$

As with Theorem 2.3, the condition that $n/|p| \in \mathbb{N}$ when $p < 0$ is a by-product of our approach; we make no claim that it is a necessary condition and hope our work attracts interest in resolving the non-integer values.

The paper is organized as follows: Section 3 introduces notation and basic tools; Section 4 is devoted to the non-random bodies $Z^{\diamond}_{p,C}(f)$ and variants of L_p-intersection bodies; Section 5 develops the randomized versions of these objects. New volume formulas and representations for radial functions are developed in Section 6. The theorems are proved in Section 7.

3. Preliminaries

3.1. Notation and definitions

For a compact set $K \subseteq \mathbb{R}^n$, we denote its convex hull by $\operatorname{conv}(K)$. The set of all compact, convex sets in $\mathbb{R}^n$ will be denoted by $\mathcal{K}^n$. For $K \in \mathcal{K}^n$, its support function is defined by $h(K,u) = \sup_{x\in K}\langle x,u\rangle$, $u \in \mathbb{R}^n$. The Hausdorff metric on $\mathcal{K}^n$ is defined by

$$\delta^H(K,L) = \sup_{\theta\in S^{n-1}} |h(K,\theta) - h(L,\theta)|,$$

where S^{n-1} is the unit sphere. We call $K \in \mathcal{K}^n$ a convex body if it has interior points. We say that $K \in \mathcal{K}^n$ is origin-symmetric if $-x \in K$ whenever $x \in K$. The set of all origin-symmetric convex bodies in $\mathbb{R}^n$ will be denoted by $\mathcal{K}^n_s$. Each $K \in \mathcal{K}^n_s$ gives rise to a norm on $\mathbb{R}^n$ given by

$$\|u\|_K = \inf\{\lambda > 0 : u \in \lambda K\}.$$

The polar body of $K \in \mathcal{K}^n_s$ is defined by $K^\circ = \{u \in \mathbb{R}^n : h_K(u) \le 1\}$.

For measurable sets $A \subseteq \mathbb{R}^n$, we use $|A|$ for the Lebesgue measure of A. By ω_n, we mean the volume of the Euclidean ball in $\mathbb{R}^n$ with radius 1, i.e.,

$$\omega_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

We will call a set K in $\mathbb{R}^n$ star-shaped if $0 \in K$ and $\alpha x \in K$ whenever $x \in K$ and $\alpha \in [0,1]$. The radial function of a star-shaped set K is defined as $\rho(K,u) = \sup\{r \ge 0 : ru \in K\}$ for $u \in S^{n-1}$. Throughout, we use the same symbol for the (-1)-homogeneous extension of $\rho(K,\cdot)$ defined for $u \in \mathbb{R}^n \backslash \{0\}$ by

$$\rho(K,u) = \|u\|_2^{-1}\rho\Big(K, \frac{u}{\|u\|_2}\Big).$$

Here we allow K to be unbounded and $\rho(K,u)$ may take the value $+\infty$. As our focus is on volumetric inequalities, we are particularly interested in radial functions of star-shaped sets K with $\rho(K,\cdot) \in L_n(S^{n-1},\sigma)$; in this case, we write

$$\|\rho(K,\cdot)\|_n = \Big(\int_{S^{n-1}} \rho^n(K,u)du\Big)^{1/n} = \omega_n^{-1/n}|K|^{1/n}, \tag{3.1}$$

which follows by expressing the volume in spherical coordinates (e.g., [87, p. 57]). Note that we use du for $d\sigma(u)$, where σ is the normalized Haar probability measure on S^{n-1}.

We will call K a star-body if it is a compact, star-shaped set with the origin in its interior and its radial function is continuous. When $K \in \mathcal{K}_s^n$, we have for $u \in \mathbb{R}^n \backslash \{0\}$,

$$\rho(K, u) = \|u\|_K^{-1} \quad \text{and} \quad \rho(K^\circ, u) = h^{-1}(K, u).$$

We recall a core notion of addition of convex bodies from L_p-Brunn–Minkowski theory, e.g., [26, 61, 63]. For $K, L \in \mathcal{K}^n$ containing the origin and $p \geq 1$, we will write $K +_p L$ for their L_p-sum, i.e.,

$$h^p(K +_p L, u) = h^p(K, u) + h^p(L, u) \quad (u \in \mathbb{R}^n). \tag{3.2}$$

In dual Brunn–Minkowski theory (e.g., [70, 87]), for star-bodies K, L, and $p \neq 0$, their L_p-radial sum $K \mathbin{\tilde{+}_p} L$ is defined by

$$\rho^p(K \mathbin{\tilde{+}_p} L, u) = \rho^p(K, u) + \rho^p(L, u) \quad (u \in S^{n-1}).$$

For a measurable set A in $\mathbb{R}^n$ with finite volume, we define its rearrangement A^* to be the (open) Euclidean ball centered at the origin satisfying $|A^*| = |A|$. We will use the following bracket notation for indicator functions:

$$[u \in A] = \chi_A(u).$$

For a non-negative integrable function f on $\mathbb{R}^n$, its layer-cake representation is given by

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x)dt = \int_0^\infty [x \in \{f > t\}]dt.$$

The symmetric decreasing rearrangement of a non-negative integrable function f on $\mathbb{R}^n$ is defined using rearrangement of its level sets $\{x \in \mathbb{R}^n : f(x) > t\} = \{f > t\}$, $t > 0$, via

$$f^*(x) = \int_0^\infty \chi_{\{f>t\}^*}(x)dt = \int_0^\infty [x \in \{f > t\}^*]dt.$$

For a general reference on rearrangements, we refer the reader to [56]. We will use the fact that f and f^* are equimeasurable; in particular, f^* preserves all L_p-norms of f. Note also that if $f \leq g$, then $f^* \leq g^*$. Moreover, rearrangements satisfy the following contractive property (see [56, Theorem 3.5]): for $1 \leq p \leq \infty$ and for $f, g \in L_p$,

$$\|f^* - g^*\|_p \leq \|f - g\|_p. \tag{3.3}$$

For $f \in \mathcal{P}_n$, the marginal density of f on a subspace E of dimension k is defined as

$$\pi_E(f)(x) = \int_{E^\perp + x} f(y)dy, \tag{3.4}$$

where $E^\perp$ denotes the orthogonal complement of E. Note that when $f \in \mathcal{P}_n$ and has compact support, then $\pi_E(f)$ is also bounded and has compact support.

3.2. Probabilistic tools

We will make repeated use of the following fact about uniformly integrable collections of random variables (e.g., [90, p. 189]).

Proposition 3.1. *Let $\eta, \eta_1, \eta_2, \ldots$ be non-negative random variables on a probability space $(\Omega, \mathcal{M}, \mathbb{P})$ such that $\eta_k \to \eta$ as $k \to \infty$ almost surely. If $\{\eta_k\}$ is uniformly integrable, then*

$$\lim_{k\to\infty} \mathbb{E}\eta_k = \mathbb{E}\eta < \infty.$$

Remark 3.2. A sufficient condition for uniform integrability of a family of random variables $\{\eta_k\}$ is boundedness in $L_{1+\delta}(\Omega, \mathcal{M}, \mathbb{P})$, for some $\delta > 0$ [90, p. 190].

We will also use Kolmogorov's strong law of large numbers [90, p. 391].

Proposition 3.3. *Let $\eta_1, \eta_2, \ldots$ be independent identically distributed random variables on a probability space $(\Omega, \mathcal{M}, \mathbb{P})$ such that $\mathbb{E}|\eta_1| < \infty$. Then, almost surely, as $N \to \infty$,*

$$\frac{1}{N}\sum_{k=1}^{N} \eta_k \to \mathbb{E}\eta_1.$$

We will frequently use *a.s.* as an abbreviation for almost sure convergence; similarly, we use *i.i.d.* for a sequence of independent identically distributed random variables.

3.3. Volume in terms of Gaussian integrals

We will use the following elementary lemma which relates the volume of star-shaped sets to certain Gaussian integrals.

Lemma 3.4. *Let K be a star-shaped set with $0 \in \operatorname{int}(K)$ and $\rho(K,\cdot) \in L_n(S^{n-1}, \sigma)$. If ξ is a standard Gaussian vector in $\mathbb{R}^n$, and $s \in (0, n)$, then*

$$\mathbb{E}_\xi \rho^s(K,\xi) = b_{n,s} \int_{S^{n-1}} \rho^s(K,u)du, \tag{3.5}$$

where

$$b_{n,s} = \mathbb{E}_\xi \|\xi\|_2^{-s} = \frac{n\Gamma((n-s)/2)}{2^{s/2+1}\Gamma(n/2+1)}. \tag{3.6}$$

Furthermore, if $\rho(K,\cdot)$ is additionally the pointwise limit of an increasing sequence of radial functions $\{\rho(K_\ell,\cdot)\}$ of star-shaped sets $\{K_\ell\}$, then

$$|K| = \omega_n \lim_{\ell\to\infty} \frac{\mathbb{E}_\xi \rho^{n-1/\ell}(K_\ell,\xi)}{b_{n,n-1/\ell}}.$$

Proof. Using polar coordinates, we have for $0 < s < n$,

$$\begin{aligned}\mathbb{E}_\xi \rho^s(K,\xi) &= \frac{n\omega_n}{(2\pi)^{n/2}} \int_0^\infty r^{n-s-1} e^{-r^2/2} dr \int_{S^{n-1}} \rho^s(K,u)du \\ &= \frac{n\Gamma((n-s)/2)}{2^{s/2+1}\Gamma(n/2+1)} \int_{S^{n-1}} \rho^s(K,u)du.\end{aligned}$$

The conditions $0 \in \mathrm{int}(K)$ and $\rho(K,\cdot) \in L_n(S^{n-1},\sigma)$ ensure that $\rho(K,u)$ is positive and finite for all u outside of a null set on S^{n-1}. For such u, since $\rho(K_\ell,u) \to \rho(K,u)$, we have

$$\rho^{n-1/\ell}(K_\ell,u) = \rho^n(K_\ell,u)\exp\Big(-\frac{\log\rho(K_\ell,u)}{\ell}\Big) \to \rho^n(K,u).$$

Next, since $\{\rho(K_\ell,u)\}$ is increasing,

$$\rho^{n-1/\ell}(K_\ell,u) \le \max(1,\rho^n(K_\ell,u)) \le \max(1,\rho^n(K,u)) \le 1+\rho^n(K,u). \tag{3.7}$$

By dominated convergence and (3.5), we get

$$\begin{aligned}\omega_n^{-1}|K| &= \int_{S^{n-1}} \rho^n(K,u)du = \lim_{\ell\to\infty}\int_{S^{n-1}} \rho^{n-1/\ell}(K_\ell,u)du \\ &= \lim_{\ell\to\infty} \frac{\mathbb{E}_\xi \rho^{n-1/\ell}(K_\ell,\xi)}{b_{n,n-1/\ell}}.\end{aligned}$$

■

4. Dual $L_{p,C}$-centroid bodies

Let $f \in \mathcal{P}_n$, $p > -1$ and let C be an origin-symmetric convex body in $\mathbb{R}^m$, $m \ge 1$. For ease of reference, we recall that for $p \ne 0$,

$$\rho^{-p}(Z^{\diamondsuit}_{p,C}(f),u) = \int_{(\mathbb{R}^n)^m} h^p(C,(\langle x_i,u\rangle)_{i=1}^m)\prod_{i=1}^m f(x_i)d\overline{x}$$

and for $p = 0$,

$$\log\rho(Z^{\diamondsuit}_{0,C}(f),u) = -\int_{(\mathbb{R}^n)^m} \log h(C,(\langle x_i,u\rangle)_{i=1}^m)\prod_{i=1}^m f(x_i)d\overline{x}.$$

As noted in the introduction, the latter bodies are not convex in general. We will use the term *dual $L_{p,C}$-centroid body* as these bodies fit within dual Brunn–Minkowski theory. This agrees with the convex case when $p \ge 1$, however, the term here is meant in a broader sense than duality for convex bodies. When $p \ge 1$, $Z^{\diamondsuit}_{p,C}(f) = Z^{\circ}_{p,C}(f)$.

We start by noting a few elementary properties of the bodies $Z^{\diamondsuit}_{p,C}(f)$. It will be useful to compare $Z^{\diamondsuit}_{p,C}(f)$ and $Z^{\diamondsuit}_{p,D}(f)$ for $C \in \mathcal{K}_s^m$, $D \in \mathcal{K}_s^{m_1}$, where $m \le m_1$; in such cases we use the standard embedding of $\mathbb{R}^m$ into $\mathbb{R}^{m_1}$.

Lemma 4.1. *Let $f \in \mathcal{P}_n$, $p, p_1, p_2 > -1$ and $C \in \mathcal{K}_s^m$, $m \ge 1$.*

(a) *If $p_1 \le p_2$, then*

$$Z^{\diamondsuit}_{p_2,C}(f) \subseteq Z^{\diamondsuit}_{p_1,C}(f).$$

(b) *If $D \in \mathcal{K}_s^{m_1}$ for some $m_1 \ge m$, and $C \subseteq D$, then*

$$Z^{\diamondsuit}_{p,C}(f) \supseteq Z^{\diamondsuit}_{p,D}(f).$$

(c) $\rho(Z^{\diamond}_{p,C}(f),\cdot) \in L_{|p|}(S^{n-1},\sigma)$.

(d) *For $k \in \mathbb{N}$ such that $\int_{kB_2^n} f(x)dx > 0$, let $\varphi^{(k)} = f \cdot \chi_{kB_2^n}$ and $\phi^{(k)} = \varphi^{(k)}/\int \varphi^{(k)}$. Then for a.e. $u \in S^{n-1}$,*

$$\rho(Z^{\diamond}_{p,C}(f),u) = \lim_{k\to\infty} \rho(Z^{\diamond}_{p,C}(\phi^{(k)}),u).$$

Proof. Part (a) is a consequence of Hölder's inequality. For (b), the condition $C \subseteq D$ is equivalent to $h(C,\cdot) \leq h(D,\cdot)$, hence $\rho(Z^{\diamond}_{p,D}(f),u) \leq \rho(Z^{\diamond}_{p,C}(f),u)$ for each $u \in S^{n-1}$.

By using (a), it is sufficient to treat (c) for $p \in (-1,0)$. Since $C \in \mathcal{K}^m_s$, we can assume there exists $r_0 > 0$ such that $r_0[-e_1,e_1] \subseteq C$, hence

$$\rho(Z^{\diamond}_{p,C}(f),u) \leq r_0^{-1}\rho(Z^{\diamond}_{p,[-1,1]}(f),u)$$

for $u \in S^{n-1}$. For $p \in (-1,0)$, we have for each $u \in S^{n-1}$,

$$\|x\|_2^{-|p|} = \beta_{n,p}\int_{S^{n-1}} |\langle x,u\rangle|^{-|p|}du,$$

where $\beta_{n,p} = b_{n,|p|}/b_{1,|p|}$ (cf. (3.6)). Since $x \mapsto \|x\|_2^{-|p|}$ is locally integrable and $f \in \mathcal{P}_n$, we have

$$\int_{\mathbb{R}^n} \|x\|_2^{-|p|} f(x)dx \leq \|f\|_\infty \int_{B_2^n} \|x\|_2^{-|p|}dx + \int_{\mathbb{R}^n\setminus B_2^n} f(x)dx < \infty.$$

Thus part (c) follows from

$$\begin{aligned}\int_{S^{n-1}} \rho^{|p|}(Z^{\diamond}_{p,[-1,1]}(f),u)du &= \int_{S^{n-1}}\int_{\mathbb{R}^n} |\langle x,u\rangle|^{-|p|} f(x)dxdu\\ &= \beta_{n,p}^{-1}\int_{\mathbb{R}^n} \|x\|_2^{-|p|} f(x)dx.\end{aligned}$$

To prove (d), we note that part (c) implies $\rho(Z^{\diamond}_{p,C}(f),u) < \infty$ for a.e. $u \in S^{n-1}$. Since $\varphi^{(k)} \to f$, and $f \in \mathcal{P}_n$, we have $\int \varphi^{(k)} \to \int f = 1$. For $p \neq 0$, we have by monotone convergence,

$$\begin{aligned}&\int_{(\mathbb{R}^n)^m} h^p(C,(\langle x_i,u\rangle)_{i=1}^m)\prod_{i=1}^m \varphi^{(k)}(x_i)d\overline{x}\\ &\quad\to \int_{(\mathbb{R}^n)^m} h^p(C,(\langle x_i,u\rangle)_{i=1}^m)\prod_{i=1}^m f(x_i)d\overline{x}; \qquad (4.1)\end{aligned}$$

the latter holds even when the right-hand side of (4.1) is infinite (when $p > 0$, this entails $\rho(Z^{\diamond}_{p,C}(f),u) = 0$). Since $\phi^{(k)} = \varphi^{(k)}/\int\varphi^{(k)} \to f$, and $(\int \varphi^{(k)})^m \to 1$ as $k \to \infty$, convergence in (4.1) remains valid when $\varphi^{(k)}$ is replaced by $\phi^{(k)}$. Thus part (d) holds for $p \neq 0$. To treat $p = 0$, we set

$$P_1(u) = \{(x_i)_{i=1}^m \in (\mathbb{R}^n)^m : h(C,(\langle x_i,u\rangle)_{i=1}^m) > 1\}$$

and $P_2(u) = (\mathbb{R}^n)^m \setminus P_1(u)$. For u outside of a null set on S^{n-1}, we can apply the same argument to each factor in

$$\rho(Z_{0,C}^{\diamond}(\phi^{(k)}), u) = \prod_{i=1}^{2} \exp\Big(- \int_{P_i(u)} \log h(C, (\langle x_i, u\rangle)_{i=1}^m) \prod_{i=1}^{m} \phi^{(k)}(x_i) d\overline{x} \Big),$$

to conclude that (d) holds for $p = 0$ as well. ∎

4.1. L_p^α-intersection bodies

For $f \in \mathcal{P}_n$, we write $I(f)$ for its intersection body, defined by its radial function via

$$\rho(I(f), u) = \int_{u^\perp} f(x) dx,$$

where the integration is with respect to Lebesgue measure on $u^\perp$; for background on intersection bodies, see [27, 51, 59]. Motivated by approximation results for intersection bodies involving radial sums of ellipsoids [30, 32, 45], we define a variant of $I(f)$: for $\alpha > 0$ and $p \in [-1, 0)$, the L_p^α-intersection body of f is given by

$$\rho^{|p|}(I_{|p|}^\alpha(f), u) = \int_{\mathbb{R}^n} (|\langle x, u\rangle|^2 + \alpha^2 \|u\|_2^2)^{-|p|/2} f(x) dx.$$

As mentioned, when f is the indicator of a star-body and $\alpha = 0$, the latter bodies were studied in [34, 36, 92]. When $p = -1$ and $\alpha > 0$, we write $I^\alpha(f) = I_1^\alpha(f)$.

Proposition 4.2. *Let f be a continuous compactly supported function in $\mathcal{P}_n$. For $\alpha > 0$, let $s_\alpha = \sinh^{-1}(1/\alpha)$. Then*

$$|I(f)| = \lim_{\alpha \to 0} (2s_\alpha)^{-n} |I^\alpha(f)|.$$

We will prove this using an approximate identity, i.e., a family of non-negative functions $(k_\alpha)_{\alpha \in (0,1)}$ on $\mathbb{R}$ satisfying the following conditions, for each $\alpha \in (0, 1)$:

(i) $\int_{\mathbb{R}} k_\alpha(t) dt = 1$;

(ii) for any $\delta > 0$, $\lim_{\alpha \to 0} \int_{|t| > \delta} k_\alpha(t) dt = 0$.

In this case, if g is continuous and supported on a compact set K, then

$$\|(k_\alpha * g) - g\|_{L_\infty(K)} \to 0;$$

see, e.g., [31, p. 27].

Proof of Proposition 4.2. For $\alpha > 0$, let

$$k_\alpha(t) = (2s_\alpha)^{-1}(t^2 + \alpha^2)^{-1/2} \chi_{[-1,1]}(t).$$

Standard computations show that $(k_\alpha)_\alpha$ is an approximate identity. Fix $u \in S^{n-1}$ and recall the notation for the marginal of f on $[u] = \mathrm{span}\{u\}$ (cf. (3.4)), and set

$$f_u(t) = \pi_{[u]}(f)(t).$$

Then f_u is compactly supported and

$$\begin{aligned}(2s_\alpha)^{-1}\rho(I^\alpha(f),u) &= (2s_\alpha)^{-1}\int_{\mathbb{R}}(t^2+\alpha^2)^{-1/2}f_u(t)dt\\ &= \int_{|t|\le 1} k_\alpha(t)f_u(t)dt + (2s_\alpha)^{-1}\int_{|t|>1}(t^2+\alpha^2)^{-1/2}f_u(t)dt\\ &= (k_\alpha * f_u)(0) + (2s_\alpha)^{-1}\int_{|t|>1}(t^2+\alpha^2)^{-1/2}f_u(t)dt. \qquad (4.2)\end{aligned}$$

We have $k_\alpha * f_u(0)\to f_u(0)=\rho(I(f),u)$. Since $\int_{\mathbb{R}} f_u(t)dt = 1$ and $s_\alpha\to\infty$ as $\alpha\to 0$, we have

$$\lim_{\alpha\to 0}(2s_\alpha)^{-1}\int_{|t|>1}(t^2+\alpha^2)^{-1/2}f_u(t)dt = 0.$$

It follows that

$$(2s_\alpha)^{-n}\rho^n(I^\alpha(f),u)\to\rho^n(I(f),u).$$

Using formula (4.2), we have that the latter convergence is dominated on (S^{n-1},σ) by $(\sup_u\|f_u\|_\infty+(2s_1)^{-1})^n$, hence

$$|I(f)| = \omega_n\int_{S^{n-1}}\lim_{\alpha\to 0}\rho^n((2s_\alpha)^{-1}I^\alpha(f),u)du = \lim_{\alpha\to 0}(2s_\alpha)^{-n}|I^\alpha(f)|. \qquad \blacksquare$$

5. Empirical dual $L_{p,C}$-centroid bodies

An empirical approach to L_p-centroid bodies was initiated in [81] and developed further in [21,82]. It relies on random linear operators acting on various sets in finite-dimensional normed spaces. In this section, we recall the main theorem from [21]. We lay the groundwork to re-interpret the random star-shaped bodies $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$ of our main theorems in terms of random sections of ℓ_p-balls. We also develop new notions of randomly generated intersection bodies.

5.1. *Tools from the empirical approach*

It will be useful to fix some notation for matrices acting as linear operators. For an $n\times N$ matrix $\boldsymbol{X}=[x_1\dots x_N]$, we write $\boldsymbol{X}^\intercal$ for the transpose of $\boldsymbol{X}$ and we view $\boldsymbol{X}:\mathbb{R}^N\to\mathbb{R}^n$ and $\boldsymbol{X}^\intercal:\mathbb{R}^n\to\mathbb{R}^N$ as linear operators. In particular, for an origin-symmetric convex body $C\subseteq\mathbb{R}^N$,

$$\boldsymbol{X}C=\{\boldsymbol{X}c : c\in C\}=\left\{\sum_{i=1}^N c_i x_i : c=(c_i)\in C\right\}.$$

Principal examples include

$$C=B_1^N=\mathrm{conv}\{\pm e_1,\dots,\pm e_N\}\quad\text{and}\quad C=B_\infty^N=[-1,1]^N$$

in which case

$$\boldsymbol{X}B_1^N=\mathrm{conv}\{\pm x_1,\dots,\pm x_N\}\quad\text{and}\quad \boldsymbol{X}B_\infty^N=\sum_{i=1}^N[-x_i,x_i].$$

Volumetric inequalities for convex hulls of random points and random zonotopes [11, 33] motivated work in [81] to interpolate between these two extremes and led to an empirical study of L_p-centroid bodies; see the survey [82] and the references therein.

All of the theorems in Section 2 will be derived from the following result about polars of convex bodies from [21]. It concerns radial measures with decreasing densities ("decreasing" is meant in a non-strict sense).

Theorem 5.1. *Let $\boldsymbol{X}$ and $\boldsymbol{X}^{\#}$ be $n \times N$ random matrices with independent columns drawn from $\mathcal{F} = (f_i)_{i=1}^N \subseteq \mathcal{P}_n$ and $\mathcal{F}^{\#} = (f_i^*)_{i=1}^N$, respectively. Let ν be a radial measure with a decreasing density, i.e., $d\nu(x) = h(\|x\|_2)dx$ with $h\colon [0,\infty) \to [0,\infty)$ decreasing. Then for any origin-symmetric convex body C in $\mathbb{R}^N$,*

$$\mathbb{E}\nu((\boldsymbol{X}C)^\circ) \leq \mathbb{E}\nu((\boldsymbol{X}^{\#}C)^\circ). \tag{5.1}$$

Assume additionally that $\boldsymbol{Z}$ is an $n \times N$ random matrix with independent columns drawn from $g_i = \|f_i\|_\infty \chi_{r_i B_2^n}$, where $r_i > 0$ satisfies $\int g_i = 1$. Then for any unconditional convex body C in $\mathbb{R}^N$,

$$\mathbb{E}\nu((\boldsymbol{X}C)^\circ) \leq \mathbb{E}\nu((\boldsymbol{Z}C)^\circ). \tag{5.2}$$

The latter theorem relies on rearrangement inequalities of Rogers [85], Brascamp–Lieb–Luttinger [13] and Christ [20]. It also relies on the Borell–Brascamp–Lieb inequalities [6, 12]. It was motivated by the work of Campi and Gronchi on symmetrization of polar convex bodies [18].

The following lemma is a useful re-interpretation of the bodies $(\boldsymbol{X}C)^\circ$, stated in terms of the transpose $\boldsymbol{X}^\top$ and the pre-image of C°, i.e., $\boldsymbol{X}^{-\top}[C^\circ] = \{x \in \mathbb{R}^n : \boldsymbol{X}^\top x \in C^\circ\}$ (square brackets are used here to avoid multiple nested parentheses in subsequent expressions).

Lemma 5.2. *Let $\boldsymbol{X}$ be an $n \times N$ matrix of full rank, viewed as a linear operator $\boldsymbol{X}\colon \mathbb{R}^N \to \mathbb{R}^n$. Then for $C \in \mathcal{K}_s^N$,*

$$(\boldsymbol{X}C)^\circ = \boldsymbol{X}^{-\top}[C^\circ].$$

Proof. Observe that

$$\begin{aligned}(\boldsymbol{X}C)^\circ &= \{x \in \mathbb{R}^n : \langle x, \boldsymbol{X}c\rangle \leq 1 \text{ for all } c \in C\}\\ &= \{x \in \mathbb{R}^n : \langle \boldsymbol{X}^\top x, c\rangle \leq 1 \text{ for all } c \in C\}\\ &= \{x \in \mathbb{R}^n : \boldsymbol{X}^\top x \in C^\circ\} = \boldsymbol{X}^{-\top}[C^\circ].\end{aligned}$$

■

Remark 5.3. The lemma will be applied for $\boldsymbol{X}$ of various dimensions. When $N = n$, the full rank assumption entails that $\boldsymbol{X}$ and $\boldsymbol{X}^\top$ are both invertible. When $N < n$, $(\boldsymbol{X}C)^\circ$ denotes polarity in $\mathbb{R}^n$ and $(\boldsymbol{X}C)^\circ$ may be unbounded. When $N \geq n$, $\boldsymbol{X}^\top$ is injective and $\boldsymbol{X}^{-\top}$ is also the inverse of $\boldsymbol{X}^\top$ on $\mathrm{Im}(\boldsymbol{X}^\top) = \ker(\boldsymbol{X})^\perp$, in which case

$$\boldsymbol{X}^{-\top}[C^\circ] = \boldsymbol{X}^{-\top}[C^\circ \cap \mathrm{Im}(\boldsymbol{X}^\top)],$$

hence

$$|(\boldsymbol{X}C)^\circ| = \det(\boldsymbol{X}\boldsymbol{X}^\top)^{-1/2}|C^\circ \cap \mathrm{Im}(\boldsymbol{X}^\top)|. \tag{5.3}$$

5.2. *Random slices of finite-dimensional ℓ_p-balls*

For $p \geq 1$, the centroid body $Z_p(f)$ can be viewed in terms of limits of images of finite-dimensional ℓ_q-balls, where $1/p + 1/q = 1$. To fix the notation, for $p \neq 0$, we denote by B_p^N the ℓ_p-ball in $\mathbb{R}^N$, i.e.,

$$B_p^N = \left\{ x \in \mathbb{R}^N : \left(\sum_{i=1}^N |\langle x, e_i \rangle|^p \right)^{1/p} \leq 1 \right\},$$

where $\{e_1, \ldots, e_N\}$ is the standard unit vector basis for $\mathbb{R}^N$. For $p = 0$, we set

$$B_0^N = \left\{ x \in \mathbb{R}^N : \left(\prod_{i=1}^N |\langle x, e_i \rangle| \right)^{1/N} \leq 1 \right\}.$$

Note that B_p^N is a convex body when $p \in [1, \infty)$ and a star-body when $p > 0$. When $p \leq 0$, B_p^N is unbounded but remains star-shaped.

Let $\boldsymbol{X}$ be an $n \times N$ random matrix with independent column vectors $X_1, \ldots, X_N$ drawn from $f \in \mathcal{P}_n$. For $1 \leq p < \infty$, the empirical L_p-centroid body $\mathcal{Z}_{p,N}(f)$ defined above in (1.3) has the equivalent description

$$\mathcal{Z}_{p,N}(f) = N^{-1/p} \boldsymbol{X} B_q^N,$$

where $1/p + 1/q = 1$. Indeed,

$$h(\boldsymbol{X} B_q^N, u) = h(B_q^N, \boldsymbol{X}^\mathsf{T} u) = \left(\sum_{i=1}^N |\langle X_i, u \rangle|^p \right)^{1/p}.$$

Using Lemma 5.2 and $1/p + 1/q = 1$, we have

$$\mathcal{Z}_{p,N}^\circ(f) = N^{1/p} \boldsymbol{X}^{-\mathsf{T}}[B_p^N], \tag{5.4}$$

where, as above, $\boldsymbol{X}^{-\mathsf{T}}[A]$ denotes the pre-image of A under $\boldsymbol{X}^\mathsf{T}$. We will mimic identity (5.4) to realize the bodies $\mathcal{Z}_{p,N}^\diamond(f)$ defined in (2.1) as sections of B_p^N for $p \in (0, 1)$.

Lemma 5.4. *Let $\boldsymbol{X}$ be an $n \times N$ random matrix with independent columns distributed according to $f \in \mathcal{P}_n$. Then for $p \in (0, 1)$,*

$$\mathcal{Z}_{p,N}^\diamond(f) = N^{1/p} \boldsymbol{X}^{-\mathsf{T}}[B_p^N].$$

Proof. Fix $u \in S^{n-1}$. We have by (2.1),

$$\rho(\mathcal{Z}_{p,N}^\diamond(f), u) = \left(\frac{1}{N} \sum_{i=1}^N |\langle X_i, u \rangle|^p \right)^{-1/p} = N^{1/p} \rho(B_p^N, \boldsymbol{X}^\mathsf{T} u).$$

On the other hand,

$$\begin{aligned} \rho(B_p^N, \boldsymbol{X}^T u) &= \sup\{r \geq 0 : r\boldsymbol{X}^\mathsf{T} u \in B_p^N\} = \sup\{r \geq 0 : ru \in \boldsymbol{X}^{-\mathsf{T}}[B_p^N]\} \\ &= \rho(\boldsymbol{X}^{-\mathsf{T}}[B_p^N], u). \end{aligned}$$

The lemma now follows from $N^{1/p} \rho(\boldsymbol{X}^{-\mathsf{T}}[B_p^N], u) = \rho(N^{1/p} \boldsymbol{X}^{-\mathsf{T}}[B_p^N], u)$. ∎

We can similarly view the bodies $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$ (cf. (2.5)) using random linear operators. For $\mathcal{C} = (C_1, \ldots, C_N)$ with $C_i \in \mathcal{K}_s^{m_i}$, we place them in orthogonal subspaces $\mathbb{R}^{m_i} = \mathrm{span}\{e_{ij}\}_{j=1}^{m_i}$, $i = 1, \ldots, N$. Then for $p \neq 0$, we define

$$\mathcal{B}_p^N(\mathcal{C}) = \left\{(x_1, \ldots, x_N) \in \bigoplus_{i=1}^N \mathbb{R}^{m_i} : \left(\sum_{i=1}^N h^p(C_i, x_i)\right)^{1/p} \leq 1\right\};$$

when $p = \infty$, we replace the sum by $\max_i h(C_i, x_i)$. For $p = 0$, we set

$$\mathcal{B}_0^N(\mathcal{C}) = \left\{(x_1, \ldots, x_N) \in \bigoplus_{i=1}^N \mathbb{R}^{m_i} : \left(\prod_{i=1}^N h(C_i, x_i)\right)^{1/N} \leq 1\right\}.$$

When the C_i's are all equal to $[-1, 1]$, we have

$$B_p^N = \mathcal{B}_p^N(([-e_1, e_1], \ldots, [-e_N, e_N])).$$

As for B_p^N, the set $\mathcal{B}_p^N(\mathcal{C})$ is a convex body, star-body or unbounded star-shaped set, according to whether $p \geq 1$, $p \in (0, 1)$ or $p \leq 0$, respectively. Note that we have defined $\mathcal{B}_p^N(\mathcal{C})$ using support functions $h(C_i, \cdot)$ rather than norms associated to the C_i's, as some computations are more convenient with this convention. By standard duality arguments, for $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$, we have for $\mathcal{C} = (C_1, \ldots, C_N)$,

$$(\mathcal{B}_p^N(\mathcal{C}))^\circ = \mathcal{B}_q^N(\mathcal{C}^\circ), \tag{5.5}$$

where we have set $\mathcal{C}^\circ = (C_1^\circ, \ldots, C_N^\circ)$ (see [23, p. 97]). We will use the particular case of $p = 1$ and $q = \infty$, combined with Lemma 5.2 in the following form.

Lemma 5.5. *Let $\mathcal{C} = (C_1, \ldots, C_N)$, where $C_i \in \mathcal{K}_s^{m_i}$, $m_i \geq 1$, and $\mathcal{C}^\circ = (C_1^\circ, \ldots, C_N^\circ)$. Set $M = m_1 + \cdots + m_N$. Let $\boldsymbol{X} = [\boldsymbol{X}_1 \ldots \boldsymbol{X}_N]$ be an $n \times M$ matrix with $n \times m_i$ blocks $\boldsymbol{X}_i$ of full rank. Then*

$$\bigcap_{i=1}^N (\boldsymbol{X}_i C_i)^\circ = (\boldsymbol{X}\mathcal{B}_1^N(\mathcal{C}^\circ))^\circ.$$

Proof. By Lemma 5.2,

$$\bigcap_{i=1}^N (\boldsymbol{X}_i C_i)^\circ = \bigcap_{i=1}^N \boldsymbol{X}_i^{-\mathsf{T}}[C_i^\circ] = \bigcap_{i=1}^N \{u \in \mathbb{R}^n : \boldsymbol{X}_i^{\mathsf{T}} u \in C_i^\circ\},$$

while

$$(\boldsymbol{X}\mathcal{B}_1^N(\mathcal{C}^\circ))^\circ = \boldsymbol{X}^{-\mathsf{T}}[\mathcal{B}_\infty^N(\mathcal{C})] = \left\{u \in \mathbb{R}^n : \max_{i \leq N} h(C_i, \boldsymbol{X}_i^{\mathsf{T}} u) \leq 1\right\}. \qquad \blacksquare$$

Using the above notation, the empirical bodies $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$ defined in (2.5) and (2.6) can be realized as sections of $\mathcal{B}_p^N(\mathcal{C})$ as follows.

Lemma 5.6. *For $i \in [N]$, let $C_i \in \mathcal{K}_s^{m_i}$, $m_i \geq 1$ and let $M = m_1 + \cdots + m_N$. Let $\boldsymbol{X} = [\boldsymbol{X}_1 \cdots \boldsymbol{X}_N]$ be an $n \times M$ random matrix with independent $n \times m_i$ blocks $\boldsymbol{X}_i = [X_{i1} \ldots X_{im_i}]$ having independent columns X_{ij} distributed according to $f_{ij} \in \mathcal{P}_n$. Then for $p \neq 0$,*

$$\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}) = N^{1/p}\boldsymbol{X}^{-\top}[\mathcal{B}_p^N(\mathcal{C})], \tag{5.6}$$

and for $p = 0$,

$$\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F}) = \boldsymbol{X}^{-\top}[\mathcal{B}_0^N(\mathcal{C})]. \tag{5.7}$$

Proof. For $u \in S^{n-1}$,

$$\boldsymbol{X}^{\top}u = (\boldsymbol{X}_1^{\top}u, \ldots, \boldsymbol{X}_N^{\top}u) = ((\langle X_{1j}, u\rangle)_{j=1}^{m_1}, \ldots, (\langle X_{Nj}, u\rangle)_{j=1}^{m_N}).$$

For any set $\mathcal{S}$ in $\mathbb{R}^M$, we have

$$\boldsymbol{X}^{-\top}[\mathcal{S}] = \{u \in \mathbb{R}^n : \boldsymbol{X}^{\top}u \in \mathcal{S}\}.$$

For $p \neq 0$, we have

$$\begin{aligned}\rho(N^{1/p}\boldsymbol{X}^{-\top}[\mathcal{B}_p^N(\mathcal{C})], u) &= \rho(N^{1/p}\mathcal{B}_p^N(\mathcal{C}), \boldsymbol{X}^{\top}u)\\ &= \left(\frac{1}{N}\sum_{i=1}^{N} h^p(C_i, \boldsymbol{X}_i^{\top}u)\right)^{-1/p}\\ &= \rho(\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}), u).\end{aligned}$$

For $p = 0$, we have

$$\rho(\boldsymbol{X}^{-\top}[\mathcal{B}_0^N(\mathcal{C})], u) = \prod_{i=1}^{N} h(C_i, \boldsymbol{X}_i^{\top}u)^{-1/N} = \rho(\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F}), u). \qquad \blacksquare$$

Remark 5.7. For $p \geq 1$, we have by Lemma 5.2 and (5.5)

$$\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}) = \mathcal{Z}^{\circ}_{p,\mathcal{C}}(\mathcal{F}) = N^{1/p}\boldsymbol{X}^{-\top}[\mathcal{B}_p^N(\mathcal{C})] = N^{1/p}(\boldsymbol{X}\mathcal{B}_q^N(\mathcal{C}^{\circ}))^{\circ}.$$

Remark 5.8. For $p \leq 0$, the bodies $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$ are pre-images of slices of unbounded sets and hence need not be bounded. This is reflected in our notation as their radial functions take the value $+\infty$. When $m_j = \dim(C_j) < n$, the matrix $\boldsymbol{X}_j^{\top}$ has a non-trivial kernel and, for $p \neq 0$,

$$\rho(\boldsymbol{X}^{-\top}[\mathcal{B}_p^N(\mathcal{C})], u) = \left(\sum_{i=1}^{N} h^{-|p|}(C_i, \boldsymbol{X}_i^{\top}u)\right)^{1/|p|} \geq h^{-1}(C_j, \boldsymbol{X}_j^{\top}u),$$

which is infinite for $u \in \ker(\boldsymbol{X}_j^{\top})$ and arbitrarily large in any neighborhood of such u. When each C_i has dimension $m_i \geq n$, absolute continuity ensures that the $n \times m_i$ matrix $\boldsymbol{X}_i$ has rank n a.s. This implies that $h(C, \boldsymbol{X}_i^{\top}\cdot) > 0$ a.s., hence each summand in the radial function $\rho(\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}), \cdot)$ is necessarily finite a.s.

Let $\mathcal{C} = (C_1, \ldots, C_N)$ and $\mathcal{D} = (D_1, \ldots, D_N)$ be N-tuples of origin-symmetric convex sets with $\dim(C_i) \leq \dim(D_i)$. We will write

$$\mathcal{C} \subseteq \mathcal{D} \;\Leftrightarrow\; C_i \subseteq D_i \quad \text{for all } i = 1, \ldots, N.$$

Lemma 5.9. *Let $\mathcal{F} = (f_{ij}) \subseteq \mathcal{P}_n$ and $-1 \leq p, p_1, p_2 \leq \infty$. Let $\mathcal{C} = (C_1, \ldots, C_N)$ with $C_i \in \mathcal{K}_s^{m_i}$, $m_i \geq 1$, for $i \in [N]$.*

(a) *If $p_1 \leq p_2$, then*

$$\mathcal{Z}^{\diamond}_{p_2,\mathcal{C}}(\mathcal{F}) \subseteq \mathcal{Z}^{\diamond}_{p_1,\mathcal{C}}(\mathcal{F}). \tag{5.8}$$

(b) *If $\mathcal{D} = (D_1, \ldots, D_N)$ with $D_i \in \mathcal{K}_s^{m_i'}$, $m_i' \geq m_i$, for $i \in [N]$, and $\mathcal{C} \subseteq \mathcal{D}$, then*

$$\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}) \supseteq \mathcal{Z}^{\diamond}_{p,\mathcal{D}}(\mathcal{F}). \tag{5.9}$$

Proof. Part (a) is a consequence of Hölder's inequality, which gives monotonicity of the normalized means in the definition of $\rho(\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}), u)$ (cf. (2.5) and (2.6)).

For part (b), $C_i \subseteq D_i$ is equivalent to $h(C_i, \cdot) \leq h(D_i, \cdot)$ for each i, which implies (5.9). ■

5.3. *Convergence of volumes*

The next proposition details integrability and sufficient conditions to obtain the volume of $Z^{\diamond}_{p,C}(f)$ as a limit of the expected volumes of the random bodies $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$.

Proposition 5.10. *For $i \in \mathbb{N}$, let $C_i \in \mathcal{K}_s^{m_i}$, $m_i \geq 1$, and $(f_{ij}) \subseteq \mathcal{P}_n$, $j \in [m_i]$. For $N \in \mathbb{N}$, let $\mathcal{C}_N = (C_1, \ldots, C_N)$ and $\mathcal{F}_N = ((f_{ij})_{j=1}^{m_i})_{i=1}^{N}$. Assume that*

(a) *there is an $r_0 > 0$ such that $r_0 B_2^{m_i} \subseteq C_i$ for each i;*

(b) *f_{ij} are supported on a common compact set and $\sup_{i,j} \|f_{ij}\|_\infty < \infty$.*

If $p \in [0, 1]$, or $p \in [-1, 0)$ and $m_i \geq n + 1$ for each i, then for any $\varepsilon \in (0, 1)$,

$$\sup_{N \geq n+1} \sup_{u \in S^{n-1}} \mathbb{E}\rho^{n+\varepsilon}(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N), u) < \infty, \tag{5.10}$$

and hence

$$\sup_{N \geq n+1} \mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N)| < \infty. \tag{5.11}$$

Furthermore, if $C_1, C_2, \ldots$ are copies of a given convex body C of dimension m and f_{ij} are identical and satisfy (5.10), then

$$|Z^{\diamond}_{p,C}(f)| = \lim_{N \to \infty} \mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N)|. \tag{5.12}$$

Proof. Without loss of generality, we may assume that $r_0 = 1$. By assumption (b), we can fix a Gaussian density ϕ_α and a constant $A > 0$ such that for each i, j,

$$\frac{1}{A} f_{ij}(x) \leq \phi_\alpha(x) = \frac{1}{(2\pi\alpha^2)^{n/2}} e^{-\|x\|_2^2/2\alpha^2} \quad (x \in \mathbb{R}^n). \tag{5.13}$$

Fix $\varepsilon > 0$ and $u \in S^{n-1}$. Assume first that $p \in [0,1]$. By Lemma 5.9, to prove (5.10) we need only treat the case $p = 0$, $m_i = 1$ for $i = 1,\ldots,N$, and $\mathcal{C}_N = ([-e_i, e_i])_{i=1}^N$. In the notation of Lemma 5.6, this means that $\mathcal{F}_N = (f_{i1})_{i=1}^N$ and $\boldsymbol{X}_i = [X_{i1}]$ are $n \times 1$ matrices. By Fubini's theorem,

$$\mathbb{E}\rho^{n+\varepsilon}\big(Z^{\diamond}_{0,\mathcal{C}_N}(\mathcal{F}_N), u\big) = \prod_{i=1}^{N} \mathbb{E}|\langle X_{i1}, u\rangle|^{-(n+\varepsilon)/N}.$$

Set $\tau = (n+\varepsilon)/N$. Let $g_1,\ldots,g_N$ be i.i.d. standard Gaussian vectors in $\mathbb{R}^n$. Fix $i \in \{1,\ldots,N\}$. Then $\langle g_i, u\rangle$ is a standard Gaussian random variable. Assume first that $N \geq 2(n+\varepsilon)$ so that $\tau \leq 1/2$. By Hölder's inequality,

$$\mathbb{E}_{X_{i1}}|\langle X_{i1}, u\rangle|^{-\tau} \leq \big(\mathbb{E}|\langle X_{i1}, u\rangle|^{-1/2}\big)^{2\tau}.$$

Using (5.13) and the notation for $b_{n,s}$ from (3.6), we have

$$A^{-1}\alpha^{1/2}\mathbb{E}_{X_{i1}}|\langle X_{i1}, u\rangle|^{-1/2} \leq \mathbb{E}_{g_i}|\langle g_i, u\rangle|^{-1/2} = b_{1,1/2} < \infty,$$

hence for $N \geq 2(n+\varepsilon)$,

$$\mathbb{E}\rho^{n+\varepsilon}(Z^{\diamond}_{0,\mathcal{C}_N}(\mathcal{F}_N), u) \leq (A\alpha^{-1/2}b_{1,1/2})^{2\tau N} = (A\alpha^{-1/2}b_{1,1/2})^{2(n+\varepsilon)}. \qquad (5.14)$$

Assume now that $n+1 \leq N < 2(n+\varepsilon)$. Then τ belongs to the interval $J = (1/2, (n+\varepsilon)/(n+1)]$ and $b_{1,\tau} \leq b = \sup_{\kappa\in J} b_{1,\kappa} < \infty$. Using (5.13), we have

$$A^{-1}\alpha^{\tau}\mathbb{E}_{X_{i1}}|\langle X_{i1}, u\rangle|^{-\tau} \leq \mathbb{E}_{g_i}|\langle g_i, u\rangle|^{-\tau} = b_{1,\tau} \leq b.$$

Writing $\underline{\alpha} = \min(\alpha, 1)$ and using $\tau < 1$, we have for $n+1 \leq N < 2(n+\varepsilon)$,

$$\mathbb{E}\rho^{n+\varepsilon}\big(Z^{\diamond}_{0,\mathcal{C}_N}(\mathcal{F}_N), u\big) \leq (A\alpha^{-\tau}b)^N \leq \max(1, (A\underline{\alpha}^{-1}b)^{2(n+\varepsilon)}). \qquad (5.15)$$

Bounds (5.14) and (5.15) are independent of u and N, so we obtain (5.10) for $p \in [0,1]$.

Assume now that $p \in [-1, 0)$. By Lemma 5.9, we can assume that $p = -1$, $m_i = n+1$, for $i = 1,\ldots,N$ and $\mathcal{C}_N = (B_2^{n+1})_{i=1}^N$. By Jensen's inequality,

$$\rho^{n+\varepsilon}(Z^{\diamond}_{-1,\mathcal{C}_N}(\mathcal{F}_N), u) \leq \frac{1}{N}\sum_{i=1}^{N}\|\boldsymbol{X}_i^{\mathsf{T}}u\|_2^{-(n+\varepsilon)}.$$

For $i \in [N]$, let $\boldsymbol{G}_i$ be i.i.d. $n \times (n+1)$ random matrices with i.i.d. standard Gaussian entries. By (5.13), for $i \in [N]$, we have

$$A^{-(n+1)}\alpha^{n+\varepsilon}\mathbb{E}_{\boldsymbol{X}_i}\|\boldsymbol{X}_i^{\mathsf{T}}u\|_2^{-(n+\varepsilon)} \leq \mathbb{E}_{\boldsymbol{G}_i}\|\boldsymbol{G}_i^{\mathsf{T}}u\|_2^{-(n+\varepsilon)} = b_{n+1,n+\varepsilon}.$$

Thus (5.10) now follows from

$$\mathbb{E}_{\boldsymbol{X}}\rho^{n+\varepsilon}(Z^{\diamond}_{-1,\mathcal{C}_N}(\mathcal{F}_N), u) \leq A^{n+1}\alpha^{-(n+\varepsilon)}b_{n+1,n+\varepsilon};$$

here we have used that $m_i = \dim(C_i) = n+1$, which ensures finiteness of $b_{n+1,n+\varepsilon}$.

To justify (5.11), for general $\mathcal{C}_N$ and $\mathcal{F}_N$, set $\delta = \varepsilon/n$ so that $n(1+\delta) = n+\varepsilon$. By Hölder's inequality,

$$\Big(\int_{S^{n-1}} \mathbb{E}\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N),u)du\Big)^{1+\delta} \leq \int_{S^{n-1}} (\mathbb{E}\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N),u))^{1+\delta}du$$
$$\leq \int_{S^{n-1}} \mathbb{E}\rho^{n+\varepsilon}(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N),u)du.$$

Therefore, (5.11) follows from

$$(\mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N)|)^{1+\delta} \leq \omega_n^{1+\delta} \sup_{u\in S^{n-1}} \mathbb{E}\rho^{n+\varepsilon}(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N),u). \tag{5.16}$$

Towards proving (5.12), we fix $u \in S^{n-1}$, identical bodies $C_i = C$ of dimension m and $f_{ij} = f$. For $p \neq 0$, the family of i.i.d. random variables $\{h^p(C, X_i^{\mathsf{T}}u)\}_{i\in\mathbb{N}}$ has finite first moment, i.e.,

$$\mathbb{E}h^p(C, X_i^{\mathsf{T}}u) = \int_{(\mathbb{R}^n)^m} h^p(C, (\langle x_i,u\rangle)_{i=1}^m)\prod_{i=1}^m f(x_i)d\overline{x} < \infty. \tag{5.17}$$

Indeed, for $p > 0$, this is a direct consequence of f being bounded and compactly supported. For $p < 0$, the function $\mathbb{E}h^p(C, X_i^{\mathsf{T}}\cdot) = \rho^{-p}(Z^{\diamond}_{p,C}(f),\cdot)$ is integrable by part (c) of Lemma 4.1; in particular, (5.17) holds for all u outside of a null set on S^{n-1} (henceforth disregarded). Thus by Proposition 3.3, for our fixed $u \in S^{n-1}$,

$$\frac{1}{N}\sum_{i=1}^N h^p(C, X_i^{\mathsf{T}}u) \to \mathbb{E}h^p(C, X_i^{\mathsf{T}}u) = \rho^{-p}(Z^{\diamond}_{p,C}(f),u) \quad \text{(a.s.)};$$

similarly, for $p = 0$, as f has compact support, the i.i.d. collection $\{\log h(C, X_i^{\mathsf{T}}u)\}_{i\in\mathbb{N}}$ satisfies

$$\mathbb{E}|\log h(C, X_i^{\mathsf{T}}u)| = \int_{(\mathbb{R}^n)^m} |\log h(C, (\langle x_i,u\rangle)_{i=1}^m)|\prod_{i=1}^m f(x_i)d\overline{x} < \infty,$$

hence

$$\frac{1}{N}\sum_{i=1}^N \log h(C, X_i^{\mathsf{T}}u) \to \mathbb{E}\log h(C, X_i^{\mathsf{T}}u) \quad \text{(a.s.)}.$$

In all cases, we have

$$\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N),u)\to\rho^n(Z^{\diamond}_{p,C}(f),u) \quad \text{(a.s.)}.$$

Using (5.10), the collection $\{\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N),u) : N \geq n+1\}$ (for our fixed u) is bounded in $L_{1+\delta}$, where, as above, $\delta = \varepsilon/n$. By Proposition 3.1 and Remark 3.2, as $N \to \infty$,

$$\mathbb{E}\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N),u) \to \mathbb{E}\rho^n(Z^{\diamond}_{p,C}(f),u) = \rho^n(Z^{\diamond}_{p,C}(f),u). \tag{5.18}$$

Lastly, the collection $\{\mathbb{E}\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N),\cdot) : N \geq n+1\}$ is uniformly integrable on (S^{n-1},σ) (by the inequality preceding (5.16)). Using (5.18), Proposition 3.1 and Fubini's theorem, we get

$$|Z^{\diamond}_{p,C}(f)| = \omega_n \int_{S^{n-1}} \rho^n(Z^{\diamond}_{p,C}(f),u)du = \omega_n \lim_{N\to\infty} \int_{S^{n-1}} \mathbb{E}\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N),u)du$$
$$= \lim_{N\to\infty} \mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}_N}(\mathcal{F}_N)|,$$

which establishes (5.12) and completes the proof of the proposition. ■

5.4. *Empirical L_p-intersection bodies*

In this section, we show how particular choices of $\mathcal{C}$ and $\mathcal{F}$ in the bodies $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$ lead naturally to empirical versions of L_p-intersection bodies. As mentioned, unit balls of normed spaces that embed in L_p, $p \in [-1,1]$ can be obtained as limits of p-radial sums of ellipsoids [30, 45]. Here we treat complementary volumetric *random* approximations. Since our main interest is when $p=-1$, we develop this only for $p \in [-1,0)$; similar considerations lead to analogous results for $p>0$.

As in Section 2, for $f \in \mathcal{P}_n$, $p \in [-1,0)$ and $\alpha > 0$, the empirical L_p^{α}-intersection body $\mathcal{I}^{\alpha}_{|p|,N}(f)$ is defined for i.i.d. random vectors $X_1,\ldots,X_N$ with density f by

$$\rho^{|p|}(\mathcal{I}^{\alpha}_{|p|,N}(f),u) = \frac{1}{N}\sum_{i=1}^{N}(|\langle X_i,u\rangle|^2 + \alpha^2\|u\|_2^2)^{-|p|/2}.$$

Forming the ellipsoids $\mathcal{E}^{\alpha}(X_i) = ([-X_i,X_i] +_2 \alpha B_2^n)^{\circ}$ (cf. (3.2)), we have

$$\rho^{|p|}(\mathcal{I}^{\alpha}_{|p|,N}(f),u) = \frac{1}{N}\sum_{i=1}^{N}\rho^{|p|}(\mathcal{E}^{\alpha}(X_i),u).$$

To link $\mathcal{I}^{\alpha}_{|p|,N}(f)$ to $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$, we replace each $\mathcal{E}^{\alpha}(X_i)$ by the *approximate ellipsoid*

$$([-X_i,X_i] +_2 \alpha[B_2^n]_{i,m})^{\circ},$$

where $[B_2^n]_{i,m} = \mathrm{conv}\{\pm Z_{i1},\ldots,\pm Z_{im}\}$, and the Z_{ij} are i.i.d. random vectors with density $\omega_n^{-1/n}\chi_{B_2^n}$. The bodies $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$ naturally accommodate these approximate ellipsoids, as we specify in the next proposition.

Proposition 5.11. *Let f be a compactly supported function in $\mathcal{P}_n$. Let $p \in [-1,0)$ and $\alpha > 0$. Then for $N \geq n+1$,*

$$\mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(f)| = \lim_{m\to\infty}\mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}^{\alpha}_m}(\mathcal{F}_m)|,$$

where $\mathcal{C}^{\alpha}_m = (C^{\alpha}_m)_{i=1}^N$ and $\mathcal{F}_m = ((f_{ij})_{j=1}^{m+1})_{i=1}^N$ are given by

$$C^{\alpha}_m = [-e_1,e_1] +_2 \alpha\,\mathrm{conv}\{\pm e_j\}_{j=2}^{m+1}, \quad f_{ij} = \begin{cases} f & \text{if } i \in [N],\ j=1, \\ \omega_n^{-1}\chi_{B_2^n} & \text{if } i \in [N],\ j>1. \end{cases}$$

Proof. Let $(X_{i1})_{i=1}^{\infty}$ and $(Z_{ij})_{i,j=1}^{\infty}$ be independent collections of i.i.d. random vectors such that X_{i1} has density f and Z_{ij} has density $\omega_n^{-1/n}\chi_{B_2^n}$. For $i = 1, \dots, N$ and $m \geq n$, we let $\boldsymbol{X}_{i,m} = [X_{i1} Z_{i1} \cdots Z_{im}]$. Then

$$\boldsymbol{X}_{i,m} C_m^{\alpha} = [-X_{i1}, X_{i1}] +_2 \alpha [B_2^n]_{i,m}$$

and, as $m \to \infty$, the latter converges a.s. in the Hausdorff metric to $[-X_{i1}, X_{i1}] +_2 \alpha B_2^n$ (see, e.g., [24, Corollary 1]). For $u \in S^{n-1}$, we have as $m \to \infty$

$$\frac{1}{N}\sum_{i=1}^{N} h^{-|p|}(\boldsymbol{X}_i C_m^{\alpha}, u) \to \frac{1}{N}\sum_{i=1}^{N} h^{-|p|}([-X_{i1}, X_{i1}] +_2 \alpha B_2^n, u) \quad \text{(a.s.)},$$

and hence

$$\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_m^{\alpha}}(\mathcal{F}_m), u) \to \rho^n(\mathcal{I}^{\alpha}_{|p|,N}(f), u) \quad \text{(a.s.)}.$$

For $m \geq n$, we have $C_m^{\alpha} \supseteq C_n^{\alpha}$, so (5.9) implies that the latter convergence is dominated by $\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_n^{\alpha}}(\mathcal{F}_n), u)$, which is independent of m. The inradius of C_n^{α} is $\min(1, \alpha/\sqrt{n})$. Using Proposition 5.10 with fixed $N \geq n+1$,

$$\int_{S^{n-1}} \mathbb{E}\rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_n^{\alpha}}(\mathcal{F}_n), u)du < \infty.$$

By dominated convergence, we get

$$\mathbb{E}\int_{S^{n-1}} \rho^n(\mathcal{I}^{\alpha}_{|p|,N}(f), u)du = \lim_{m\to\infty} \mathbb{E}\int_{S^{n-1}} \rho^n(\mathcal{Z}^{\diamond}_{p,\mathcal{C}_m^{\alpha}}(\mathcal{F}_m), u)du. \qquad \blacksquare$$

Proposition 5.12. *Let $f \in \mathcal{P}_n$, $p \in [-1, 0)$, and $\alpha > 0$. Then*

$$|I^{\alpha}_{|p|}(f)| = \lim_{N\to\infty} \mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(f)|.$$

Proof. Fix $u \in S^{n-1}$. Since $f \in \mathcal{P}_n$, the random variables $(|\langle X_i, u\rangle|^2 + \alpha^2\|u\|_2^2)^{-|p|/2}$ have finite first moment. By the law of large numbers, as $N \to \infty$, we have

$$\frac{1}{N}\sum_{i=1}^{N}(|\langle X_i, u\rangle|^2 + \alpha^2\|u\|_2^2)^{-|p|/2} \to \int_{\mathbb{R}^n}(|\langle x, u\rangle|^2 + \alpha^2\|u\|_2^2)^{-|p|/2} f(x)dx \quad \text{(a.s.)},$$

hence

$$\rho^n(\mathcal{I}^{\alpha}_{|p|,N}(f), u) \to \rho^n(I^{\alpha}_{|p|}(f), u) \quad \text{(a.s.)}.$$

Since $\rho(I^{\alpha}_{|p|,N}(f), u) \leq 1/\alpha$ for each u, we can use dominated convergence to get

$$\omega_n \mathbb{E}\int_{S^{n-1}} \rho^n(\mathcal{I}^{\alpha}_{|p|,N}(f), u)du \to \omega_n \mathbb{E}\int_{S^{n-1}} \rho^n(I^{\alpha}_{|p|}(f), u)du = |I^{\alpha}_{|p|}(f)|. \qquad \blacksquare$$

6. Volume formulas

As mentioned, our work is inspired by a formula for the volume of sections of B_p^N, $p \in (0, 2)$, due to Nayar and Tkocz [77]. We will recall the basic ingredients and then derive a formula for the volume of the random sets $\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F})$. For $p \le 0$, we will present an alternative path and complementary volume formulas.

6.1. Volume via Gaussian mixtures for $p > 0$

Recall that for $0 < \alpha < 1$, a positive random variable w is called *normalized positive α-stable* if

$$\mathbb{E}e^{-tw} = e^{-t^\alpha} \quad (t > 0).$$

We will denote the density of such a random variable by g_α; for background on stable random variables, see [93]. The following Nayar–Tkocz volume formula was proved in [77], where it is stated explicitly for $p = 1$ and explained how the same method applies to $p \in (0, 2)$.

Proposition 6.1. *Let $0 < p < 2$ and let X be an $n \times N$ matrix with columns $x_1, \dots, x_N$ spanning $\mathbb{R}^n$. Let $W = (w_1, \dots, w_N)$ be a random vector with i.i.d. entries w_i having common density proportional to $s \mapsto s^{-1/2} g_{p/2}(s)$. Then*

$$\frac{|B_p^N \cap \mathrm{Im}(X^\top)|}{\det(XX^\top)^{1/2}} = a_{N,n,p}\pi^{n/2}\mathbb{E}_W \sqrt{w_1 \cdots w_N}\left(\det\left(\sum_{i=1}^N w_i x_i x_i^\top\right)\right)^{-1/2}, \tag{6.1}$$

where

$$a_{N,n,p} = \pi^{-N/2}\left(2\Gamma\left(1 + \frac{1}{p}\right)\right)^N \Gamma\left(1 + \frac{n}{p}\right)^{-1}.$$

The proof of the formula relies on two ingredients. The first is that the volume of a star-body K in $\mathbb{R}^n$ with radial function $\rho(K, \cdot)$ is given by

$$|K| = c_{n,p} \int_{\mathbb{R}^n} \exp(-\rho^{-p}(K, x))dx, \tag{6.2}$$

where $c_{n,p} = \Gamma(1 + n/p)^{-1}$. The second ingredient is the following fact from [25, Lemma 23]: if ξ is a standard Gaussian random variable, independent of a positive random variable w with density proportional to $t \mapsto t^{-1/2} g_{p/2}(t)$, then $\xi/\sqrt{2w}$ has density $[2\Gamma(1 + 1/p)]^{-1} e^{-|t|^p}$ and

$$e^{-|x|^p} = d_p \mathbb{E}_w \sqrt{w} e^{-wx^2} \quad (x \in \mathbb{R}), \tag{6.3}$$

where $d_p = 2\Gamma(1 + 1/p)/\sqrt{\pi}$ (as can be seen by integrating (6.3) on $\mathbb{R}$).

We will adapt the Nayar–Tkocz argument to derive a volume formula for $\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F})$ for $p \in (0, 2)$, using the pre-image interpretation in (5.6).

Proposition 6.2. *Let $\mathcal{C} = (C_1, \ldots, C_N)$, $\mathcal{F}$ and X be as in Lemma 5.6. Let $0 < p < 2$ and let $W = (w_1, \ldots, w_N)$ be a random vector with i.i.d. entries w_i having a common density proportional to $s \mapsto s^{-1/2} g_{p/2}(s)$. Set $\mathcal{C}_W^\circ = ((\sqrt{w_1}C_1)^\circ, \ldots, (\sqrt{w_N}C_N)^\circ)$. Then*

$$|Z_{p,\mathcal{C}}^\diamond(\mathcal{F})| = a_{N,n,p} c_{n,2}^{-1} N^{n/p} \mathbb{E}_W \sqrt{w_1 \cdots w_N} \left| \left(X \mathcal{B}_2^N(\mathcal{C}_W^\circ) \right)^\circ \right|.$$

Proof. By Lemma 5.6,

$$Z_{p,\mathcal{C}}^\diamond(\mathcal{F}) = N^{1/p} X^{-\top}[\mathcal{B}_p^N(\mathcal{C})]. \tag{6.4}$$

Applying (5.5) with $\mathcal{C}_W = (\sqrt{w_1}C_1, \ldots, \sqrt{w_N}C_N)$, we have $(\mathcal{B}_2^N(\mathcal{C}_W^\circ))^\circ = \mathcal{B}_2^N(\mathcal{C}_W)$. Thus by Lemma 5.2,

$$(X\mathcal{B}_2^N(\mathcal{C}_W^\circ))^\circ = X^{-\top}[\mathcal{B}_2^N(\mathcal{C}_W)]. \tag{6.5}$$

As in the proof of Lemma 5.6,

$$\rho^{-p}(X^{-\top}[\mathcal{B}_p^N(\mathcal{C})], u) = \rho^{-p}(\mathcal{B}_p^N(\mathcal{C}), X^\top u) = \sum_{i=1}^N h^p(C_i, X_i^\top u),$$

while

$$\rho^{-2}(X^{-\top}[\mathcal{B}_2^N(\mathcal{C}_W)], u) = \rho^{-2}(\mathcal{B}_2^N(\mathcal{C}_W), X^\top u) = \sum_{i=1}^N h^2(\sqrt{w_i}C_i, X_i^\top u).$$

Using the volume representation (6.2) and change of density (6.3) with the latter radial functions, we have

$$\begin{aligned}
c_{n,p}^{-1}|X^{-\top}[\mathcal{B}_p^N(\mathcal{C})]| &= \int_{\mathbb{R}^n} \prod_{i=1}^N \exp(-h^p(C_i, X_i^\top u)) du \\
&= d_p^N \int_{\mathbb{R}^n} \mathbb{E}_W \prod_{i=1}^N \sqrt{w_i} \exp(-w_i h^2(C_i, X_i^\top u)) du \\
&= d_p^N \int_{\mathbb{R}^n} \mathbb{E}_W \sqrt{w_1 \cdots w_N} \exp\left(-\sum_{i=1}^N h^2(\sqrt{w_i}\, C_i, X_i^\top u) \right) du \\
&= c_{n,2}^{-1} d_p^N \mathbb{E}_W \sqrt{w_1 \cdots w_N} |X^{-\top}[\mathcal{B}_2^N(\mathcal{C}_W)]|.
\end{aligned}$$

Thus the proposition follows from (6.4), (6.5) and the identity $a_{N,n,p} = c_{n,p} d_p^N$. ■

Remark 6.3. To see that the latter (pointwise) proof also implies (6.1), we take $C_i = [-e_i, e_i]$ and write $X_W = [\sqrt{w_1}X_1, \ldots, \sqrt{w_N}X_N]$ so that $X_W B_2^N = X\mathcal{B}_2^N(\mathcal{C}_W^\circ)$. By (5.3),

$$|(X_W B_2^N)^\circ| = \omega_n \left(\det\left(\sum_{i=1}^N w_i X_i X_i^\top \right) \right)^{-1/2}.$$

When $p = 1$ in (6.1), w_i is the reciprocal of an exponential random variable [77] and we have maintained this convention here, though the exact normalization is immaterial in what follows.

6.2. *Volume via Gaussian measure for* $p = 0$

The set $\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F})$ can be treated as a limiting case of $\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})$ when $p \to 0$ but it will be handy to derive a different volume formula using the pre-image representation (5.7) directly. This approach will also be helpful for $p < 0$. The formula involves standard Gaussian measure γ_n and negative moments of the Gaussian random vectors $b_{n,s}$ defined in (3.6).

Proposition 6.4. *Let* $\mathcal{C} = (C_1, \dots, C_N)$*,* $\mathcal{F}$ *and* $\boldsymbol{X}$ *be as in Lemma 5.6. For* $t = (t_1, \dots, t_N)$ *in* $\mathbb{R}^N_+$ *and* $s > 0$*, set* $\mathcal{C}^{\circ}_{s,t} = ((t_1^{N/s} C_1)^{\circ}, \dots, (t_N^{N/s} C_N)^{\circ})$*. Then*

$$\mathbb{E}|\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F})| = \omega_n \lim_{s \to n^-} b_{n,s}^{-1} \int_{\mathbb{R}^N_+} \mathbb{E}_{\boldsymbol{X}} \gamma_n((\boldsymbol{X}\mathcal{B}^N_1(\mathcal{C}^{\circ}_{s,t}))^{\circ}) dt. \tag{6.6}$$

Proof. We will first show that for $u \in \mathbb{R}^n \backslash \{0\}$,

$$\rho^s(\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F}), u) = \int_{\mathbb{R}^N_+} [u \in (\boldsymbol{X}\mathcal{B}^N_1(\mathcal{C}^{\circ}_{s,t}))^{\circ}] dt. \tag{6.7}$$

Note that

$$\rho^s(\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F}), u) = \prod_{i=1}^N h^{-s/N}(C_i, \boldsymbol{X}_i^{\top} u) = \int_{\mathbb{R}^N_+} \prod_{i=1}^N [u \in \{h^{-s/N}(C_i, \boldsymbol{X}_i^{\top} \cdot) > t_i\}] dt.$$

For each $i = 1, \dots, N$, we have (up to a null set)

$$\{h^{-s/N}(C_i, \boldsymbol{X}_i^{\top} \cdot) > t_i\} = \{t_i^{N/s} h(\boldsymbol{X}_i C_i, \cdot) < 1\} = (t_i^{N/s} \boldsymbol{X}_i C_i)^{\circ}.$$

By Lemma 5.5,

$$\bigcap_{i=1}^N (t_i^{N/s} \boldsymbol{X}_i C_i)^{\circ} = (\boldsymbol{X}\mathcal{B}^N_1(\mathcal{C}^{\circ}_{s,t}))^{\circ}.$$

Therefore,

$$\begin{aligned} \rho^s(\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F}), u) &= \int_{\mathbb{R}^N_+} \left[u \in \bigcap_{i=1}^N \{h^{-s/N}(C_i, \boldsymbol{X}_i^{\top} y) > t_i\} \right] dt \\ &= \int_{\mathbb{R}^N_+} [u \in (\boldsymbol{X}\mathcal{B}^N_1(\mathcal{C}^{\circ}_{s,t}))^{\circ}] dt. \end{aligned}$$

Let ξ be a standard Gaussian vector in $\mathbb{R}^n$ and $s \in (0, n)$. By (6.7), we have

$$\mathbb{E}_{\xi} \rho^s(\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F}), \xi) = \int_{\mathbb{R}^N_+} \gamma_n((\boldsymbol{X}\mathcal{B}^N_1(\mathcal{C}^{\circ}_{s,t}))^{\circ}) dt. \tag{6.8}$$

Assume first that

$$\mathbb{E}|\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F})| = \omega_n \mathbb{E} \int_{S^{n-1}} \rho^n(\mathcal{Z}^{\diamond}_{0,\mathcal{C}}(\mathcal{F}), u) du < \infty.$$

Then $\rho(\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F}),\cdot) \in L_n(S^{n-1},\sigma)$ a.s. Arguing as in the proof of Lemma 3.4, as $s \to n^-$,

$$\int_{S^{n-1}} \rho^s(\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F}),u)du \to \int_{S^{n-1}} \rho^n(\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F}),u)du \quad \text{(a.s.)}, \tag{6.9}$$

and the convergence is dominated by $1 + \omega_n^{-1}|\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F})|$ (cf. (3.7)). Thus, by (3.5),

$$\begin{aligned}\mathbb{E}|\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F})| &= \omega_n \mathbb{E}_{\boldsymbol{X}} \lim_{s\to n^-} \int_{S^{n-1}} \rho^s(\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F}),u)du \\ &= \omega_n \lim_{s\to n^-} \mathbb{E}_{\boldsymbol{X}} \int_{S^{n-1}} \rho^s(\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F}),u)du \\ &= \omega_n \lim_{s\to n^-} b_{n,s}^{-1}\mathbb{E}_{\boldsymbol{X}}\mathbb{E}_{\xi}\rho^s(\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F}),\xi).\end{aligned}$$

Applying (6.8) gives the proposition when $\mathbb{E}|\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F})|$ is finite. If $\mathbb{E}|\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F})|$ is infinite, we can replace S^{n-1} in (6.9) by its subset $\{\rho(\mathcal{Z}_{0,\mathcal{C}}^{\diamond}(\mathcal{F}),\cdot) \geq 1\}$, in which case the convergence is monotone and both sides of (6.6) are divergent. ■

6.3. *Volume via Gaussian measure for $p < 0$*

We will start with a volume formula for the non-random bodies $Z_{p,C}^{\diamond}(f)$.

Proposition 6.5. *Let $f \in \mathcal{P}_n$ and $C \in \mathcal{K}_s^m$, where $m \geq 1$. Let $p \in (-1,0)$ and set $n(p) = n/|p| \in \mathbb{N}$. Let $\boldsymbol{X}$ be an $n \times n(p)m$ random matrix with independent columns distributed according to f. For $\ell \in \mathbb{N}$, let $p_\ell = p(1 - 1/(\ell n)) \in (-1,0)$. For $t_1,\dots,t_{n(p)} > 0$ and $\ell \in \mathbb{N}$, let $\mathcal{C}_{t,p_\ell}^{\circ} = ((t_1^{1/|p_\ell|}C)^{\circ},\dots,(t_{n(p)}^{1/|p_\ell|}C)^{\circ})$. Then*

$$|Z_{p,C}^{\diamond}(f)| = \omega_n \lim_{\ell\to\infty} b_{n,n-1/\ell}^{-1} \int_{\mathbb{R}_+^{n(p)}} \mathbb{E}_{\boldsymbol{X}}\gamma_n((\boldsymbol{X}\mathcal{B}_1^{n(p)}(\mathcal{C}_{t,p_\ell}^{\circ}))^{\circ})dt.$$

Proof. Fix $k \in \mathbb{N}$. Let $\boldsymbol{X}_1,\dots,\boldsymbol{X}_k$ be independent $n \times m$ random matrices with independent columns drawn from f. We will first show that for $u \in \mathbb{R}^n\backslash\{0\}$,

$$\rho^{k|p|}(Z_{p,C}^{\diamond}(f),u) = \int_{\mathbb{R}_+^k} \mathbb{E}_{\boldsymbol{X}}[u \in (\boldsymbol{X}\mathcal{B}_1^k(\mathcal{C}_{t,p}^{\circ}))^{\circ}]dt. \tag{6.10}$$

Note that

$$\rho^{k|p|}(Z_{p,C}^{\diamond}(f),u) = (\mathbb{E}_{\boldsymbol{X}_1}h^{-|p|}(C,\boldsymbol{X}_1^{\mathsf{T}}u))^k = \mathbb{E}_{\boldsymbol{X}_1}\cdots\mathbb{E}_{\boldsymbol{X}_k}\prod_{i=1}^k h^{-|p|}(C,\boldsymbol{X}_i^{\mathsf{T}}u)$$

and

$$\prod_{i=1}^k h^{-|p|}(C,\boldsymbol{X}_i^{\mathsf{T}}u) = \int_{\mathbb{R}_+^k}\prod_{i=1}^k[u \in \{h^{-|p|}(C,\boldsymbol{X}_i^{\mathsf{T}}\cdot) > t_i\}]dt.$$

For each $i = 1,\dots,k$, we have (up to a null set)

$$\{h^{-|p|}(C,\boldsymbol{X}_i^{\mathsf{T}}\cdot) > t_i\} = \{h(\boldsymbol{X}_iC,\cdot) < t_i^{-1/|p|}\} = (t_i^{1/|p|}\boldsymbol{X}_iC)^{\circ}.$$

By Lemma 5.5,

$$\bigcap_{i=1}^{k}(t_i^{1/|p|}\boldsymbol{X}_i C)^\circ = (\boldsymbol{X}\mathcal{B}_1^k(\mathcal{C}_{t,p}^\circ))^\circ.$$

Therefore,

$$\mathbb{E}_{\boldsymbol{X}_1}\cdots\mathbb{E}_{\boldsymbol{X}_k}\prod_{i=1}^{k} h^{-|p|}(C,\boldsymbol{X}_i^\top u) = \int_{\mathbb{R}_+^k}\mathbb{E}_{\boldsymbol{X}}[u\in(\boldsymbol{X}\mathcal{B}_1^k(\mathcal{C}_{t,p}^\circ))^\circ]dt,$$

which implies (6.10). If ξ is a standard Gaussian vector in $\mathbb{R}^n$, then

$$\mathbb{E}_\xi\rho^{k|p|}(Z_{p,C}^\diamond(f),\xi) = \int_{\mathbb{R}_+^k}\mathbb{E}_{\boldsymbol{X}}\gamma_n((\boldsymbol{X}\mathcal{B}_1^k(\mathcal{C}_{t,p}^\circ))^\circ)dt. \tag{6.11}$$

Note that $n(p) = n/|p| = (n-1/\ell)/|p_\ell|$. It remains to apply Lemma 3.4 with $K = Z_{p,C}^\diamond(f)$ and the increasing sequence $K_\ell = Z_{p_\ell,C}^\diamond(f)$ (cf. Lemma 4.1 (a)). With an eye on (6.11) with p_ℓ and $n(p)$ in place of p and k, respectively, we conclude by

$$|Z_{p,C}^\diamond(f)| = \omega_n\lim_{\ell\to\infty} b_{n,n-1/\ell}^{-1}\mathbb{E}_\xi\rho^{n(p)|p_\ell|}(Z_{p_\ell,C}^\diamond(f),\xi). \qquad \blacksquare$$

6.4. Radial function representation for $p<0$

The volume formulas for $\mathcal{Z}_{0,\mathcal{C}}^\diamond(\mathcal{F})$ and $Z_{p,C}^\diamond(f)$ each rely on a representation of the radial function as a mixture of indicator functions of origin-symmetric convex bodies. In this subsection, we develop an analogous representation for the radial function of the empirical bodies $\mathcal{Z}_{p,\mathcal{C}}^\diamond(\mathcal{F})$ for $p<0$ and $n/|p|\in\mathbb{N}$. A similar volume formula for $\mathcal{Z}_{p,\mathcal{C}}^\diamond(\mathcal{F})$ holds but the notation becomes lengthy, so we will derive only the radial function for later use.

To fix the notation, for $k\in\mathbb{N}$, we set $[k]_0 = \{0,1,\dots,k\}$. For $\underline{\mathbf{k}} = (k_1,\dots,k_N)\in[k]_0^N$, we define $S(\underline{\mathbf{k}}) = k_1+\cdots+k_N$ and $m(\underline{\mathbf{k}}) = \{i\in[N] : k_i\neq 0\}$; we write $|m(\underline{\mathbf{k}})|$ for the cardinality of $m(\underline{\mathbf{k}})$.

Proposition 6.6. *Let $\mathcal{C} = (C_1,\dots,C_N)$, $\mathcal{F}$ and $\boldsymbol{X}$ be as in Lemma 5.6. Let $p\in(-1,0)$ and $k\in\mathbb{N}$. Then for $u\in\mathbb{R}^n\setminus\{0\}$,*

$$\rho^{k|p|}(\mathcal{Z}_{p,\mathcal{C}}^\diamond(\mathcal{F}),u) = N^{-k}\sum_{\substack{\underline{\mathbf{k}}\in[k]_0^N\\ S(\underline{\mathbf{k}})=k}}\binom{k}{\underline{\mathbf{k}}}\int_{\mathbb{R}_+^{|m(\underline{\mathbf{k}})|}}[u\in(\boldsymbol{X}_{\underline{\mathbf{k}}}\mathcal{B}_1^{|m(\underline{\mathbf{k}})|}(\mathcal{C}_{\underline{\mathbf{k}},t,p}^\circ))^\circ]dt,$$

where $\binom{k}{\underline{\mathbf{k}}} = k!/(k_1!\cdots k_N!)$, $\boldsymbol{X}_{\underline{\mathbf{k}}} = [\boldsymbol{X}_{k_i}]_{i\in m(\underline{\mathbf{k}})}$ and $\mathcal{C}_{\underline{\mathbf{k}},t,p}^\circ = ((t_i^{1/(k_i|p|)}C_i)^\circ)_{i\in m(\underline{\mathbf{k}})}$.

Proof. By Lemma 5.6, we have $\mathcal{Z}_{p,\mathcal{C}}^\diamond(\mathcal{F}) = N^{1/p}\mathbf{X}^{-\top}[\mathcal{B}_p^N(\mathcal{C})]$. Using the fact that $k\in\mathbb{N}$, we have for any $u\in\mathbb{R}^n$,

$$\rho^{k|p|}(\boldsymbol{X}^{-\top}[\mathcal{B}_p^N(\mathcal{C})],u) = \sum_{\substack{\underline{\mathbf{k}}\in[k]_0^N\\ S(\underline{\mathbf{k}})=k}}\binom{k}{\underline{\mathbf{k}}}\prod_{i\in m(\underline{\mathbf{k}})} h^{-k_i|p|}(C_i,\boldsymbol{X}_i^\top u).$$

Fix $\underline{\mathbf{k}} = (k_1, \dots, k_N)$ with $S(\underline{\mathbf{k}}) = k$. Then

$$\prod_{i \in m(\underline{\mathbf{k}})} h^{-k_i|p|}(C_i, X_i^\top u) = \int_{\mathbb{R}_+^{|m(\underline{\mathbf{k}})|}} \prod_{i \in m(\underline{\mathbf{k}})} [u \in \{h^{-k_i|p|}(C_i, X_i^\top \cdot) > t_i\}] dt.$$

For each $i \in m(\underline{\mathbf{k}})$, we have (up to a null set)

$$\{h^{-k_i|p|}(C_i, X_i^\top \cdot) > t_i\} = \{h(X_i C_i, \cdot) < t_i^{-1/(k_i|p|)}\} = (t_i^{1/(k_i|p|)} X_i C_i)^\circ.$$

By Lemma 5.5,

$$\bigcap_{i \in m(\underline{\mathbf{k}})} (t_i^{1/(k_i|p|)} X_i C_i)^\circ = (X_{\underline{\mathbf{k}}} \mathcal{B}_1^{|m(\underline{\mathbf{k}})|}(\mathcal{C}^\circ_{\underline{\mathbf{k}},t,p}))^\circ.$$

Thus the proposition follows from

$$\prod_{i \in m(\underline{\mathbf{k}})} h^{-k_i|p|}(C_i, X_i^\top u) = \int_{\mathbb{R}_+^{|m(\underline{\mathbf{k}})|}} [u \in (X_{\underline{\mathbf{k}}} \mathcal{B}_1^{|m(\underline{\mathbf{k}})|}(\mathcal{C}^\circ_{\underline{\mathbf{k}},t,p}))^\circ] dt. \qquad \blacksquare$$

Each of the proofs of Propositions 6.5 and 6.6 uses a tensorization argument that relies on the condition $n/|p| \in \mathbb{N}$. It would be of great interest to find an approach that extends to non-integer values.

7. Main proofs

Proof of Theorem 2.2. Suppose that X and $X^\#$ are $n \times M$ random matrices with independent columns drawn from $\mathcal{F} = (f_{ij}) \subseteq \mathcal{P}_n$ and $\mathcal{F}^\# = (f_{ij}^*)$ respectively, where $M = m_1 + \cdots + m_N$. Suppose that each f_{ij} is supported on a Euclidean ball RB_2^n. Denote the expectation in X and $X^\#$ by $\mathbb{E}_X$ and $\mathbb{E}_{X^\#}$, respectively.

For $p \geq 1$, we have by Remark 5.7 and Theorem 5.1,

$$\begin{aligned}\mathbb{E}|Z^\diamond_{p,\mathcal{C}}(\mathcal{F})| &= \mathbb{E}_X |N^{1/p}(X\mathcal{B}_q^N(\mathcal{C}^\circ))^\circ| \leq \mathbb{E}_{X^\#}|N^{1/p}(X^\#\mathcal{B}_q^N(\mathcal{C}^\circ))^\circ| \\ &= \mathbb{E}|Z^\diamond_{p,\mathcal{C}}(\mathcal{F}^\#)|.\end{aligned}$$

For $p \in (0,1)$, using Proposition 6.2, Fubini's theorem and Theorem 5.1,

$$\begin{aligned}\mathbb{E}|Z^\diamond_{p,\mathcal{C}}(\mathcal{F})| &= a_{N,n,p} c_{n,2}^{-1} N^{n/p} \mathbb{E}_W \mathbb{E}_X \sqrt{w_1 \cdots w_N} |(X\mathcal{B}_2^N(\mathcal{C}^\circ_W))^\circ| \\ &\leq a_{N,n,p} c_{n,2}^{-1} N^{n/p} \mathbb{E}_W \mathbb{E}_{X^\#} \sqrt{w_1 \cdots w_N} |(X^\#\mathcal{B}_2^N(\mathcal{C}^\circ_W))^\circ| \\ &= \mathbb{E}|Z^\diamond_{p,\mathcal{C}}(\mathcal{F}^\#)|.\end{aligned}$$

For $p = 0$, we apply Proposition 6.4 and Theorem 5.1 to get

$$\begin{aligned}\mathbb{E}|Z^\diamond_{0,\mathcal{C}}(\mathcal{F})| &= \omega_n \lim_{s\to n^-} b_{n,s}^{-1} \int_{\mathbb{R}_+^N} \mathbb{E}_X \gamma_n((X\mathcal{B}_1^N(\mathcal{C}^\circ_{s,t}))^\circ) dt \\ &\leq \omega_n \lim_{s\to n^-} b_{n,s}^{-1} \int_{\mathbb{R}_+^N} \mathbb{E}_{X^\#} \gamma_n((X^\#\mathcal{B}_1^N(\mathcal{C}^\circ_{s,t}))^\circ) dt \\ &= \mathbb{E}|Z^\diamond_{0,\mathcal{C}}(\mathcal{F}^\#)|.\end{aligned}$$

When the C_i's are identical, we have by (5.12) of Proposition 5.10,

$$|Z_{p,C}^{\diamond}(f)| = \lim_{N\to\infty} \mathbb{E}|\mathcal{Z}_{p,\mathcal{C}_N}^{\diamond}(\mathcal{F}_N)|,$$

which proves (2.7) for f compactly supported. For a general $f \in \mathcal{P}_n$, we define $\varphi^{(k)} = f \cdot \chi_{kB_2^n}$ and $\phi^{(k)} = \varphi^{(k)}/\int \varphi^{(k)}$. By Lemma 4.1 (d) with Fatou's lemma combined with the volume formula (3.1), we have by the compactly supported case,

$$|Z_{p,C}^{\diamond}(f)| \le \liminf_{k\to\infty} |Z_{p,C}^{\diamond}(\phi^{(k)})| \le \liminf_{k\to\infty} |Z_{p,C}^{\diamond}((\phi^{(k)})^*)| = |Z_{p,C}^{\diamond}(f^*)|;$$

here we have used that each set $Z_{p,C}^{\diamond}((\phi^{(k)})^*)$ is a Euclidean ball and (3.3) ensures that $\|(\varphi^{(k)})^* - f^*\|_1 \to 0$, which together with monotonicity in k, gives the final equality.

Lastly, we turn to the case when $\mathcal{F} = (f_{ij})$ consists of functions that are not supported on a common compact set. In the notation of Lemma 4.1 (d), we set $\varphi_{ij}^{(k)} = f_{ij} \cdot \chi_{kB_2^n}$ and $\phi_{ij}^{(k)} = \varphi_{ij}^{(k)}/\int \varphi_{ij}^{(k)}$, and set $\mathcal{F}_k = (\phi_{ij}^{(k)})$. Then

$$\begin{aligned}
&\mathbb{E}|\mathcal{Z}_{p,C}^{\diamond}(\mathcal{F}_k)| \\
&\quad = \omega_n \int_{((\mathbb{R}^n)^m)^N} \int_{S^{n-1}} \left(\frac{1}{N}\sum_{i=1}^{N} h^p(C, (\langle x_{ij}, u\rangle)_{j=1}^{m_i})\right)^{-n/p} \prod_{i,j} \phi_{ij}^{(k)}(x_{ij})\,du\,d\overline{x}.
\end{aligned}$$

Using $\int \varphi_{ij}^{(k)} \to \int f_{ij} = 1$ and monotone convergence for $\varphi_{ij}^{(k)}$,

$$\mathbb{E}|\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F})| = \lim_{k\to\infty} \mathbb{E}|\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F}_k)| \le \lim_{k\to\infty} \mathbb{E}|\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F}_k^{\#})| = \mathbb{E}|\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F}^{\#})|. \qquad \blacksquare$$

Proof of Theorem 2.1. Taking $C = C_i = [-1, 1]$ and $\mathcal{F} = (f)$ gives $\mathcal{Z}_{p,N}^{\diamond}(f) = \mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F})$ and $Z_{p,C}^{\diamond}(f) = Z_p^{\diamond}(f)$, hence Theorem 2.1 follows from Theorem 2.2. $\blacksquare$

Proof of Theorem 2.3. Let $\boldsymbol{X}$ and $\boldsymbol{X}^{\#}$ be $n \times n(p)m$ random matrices with i.i.d. columns drawn from f and f^*, respectively. By Proposition 6.5 and Theorem 5.1,

$$\begin{aligned}
|Z_{p,C}^{\diamond}(f)| &= \omega_n \lim_{\ell\to\infty} b_{n,n-1/\ell}^{-1} \int_{\mathbb{R}_+^{n(p)}} \mathbb{E}_{\boldsymbol{X}}\gamma_n((\boldsymbol{X}\mathcal{B}_1^{n(p)}(\mathcal{C}_{t,p_\ell}^{\circ}))^{\circ})dt \\
&\le \omega_n \lim_{\ell\to\infty} b_{n,n-1/\ell}^{-1} \int_{\mathbb{R}_+^{n(p)}} \mathbb{E}_{\boldsymbol{X}^{\#}}\gamma_n((\boldsymbol{X}^{\#}\mathcal{B}_1^{n(p)}(\mathcal{C}_{t,p_\ell}^{\circ}))^{\circ})dt \\
&= |Z_{p,C}^{\diamond}(f^*)|.
\end{aligned}$$

Next, we prove (2.8). Fix origin-symmetric convex bodies $C_1, \ldots, C_N$ with $\dim(C_i) = m_i \ge n+1$. Set $M = m_1 + \cdots + m_N$. Suppose that $\boldsymbol{X}$ and $\boldsymbol{X}^{\#}$ are $n \times M$ random matrices with independent columns drawn from $\mathcal{F} = (f_{ij})$ and $\mathcal{F}^{\#} = (f_{ij}^*)$, respectively. Fix $k \in \mathbb{N}$ and $p \in [-1, 0)$ with $k|p| < n$.

Assume first that (f_{ij}) are supported on a Euclidean ball RB_2^n. By Proposition 5.10,

$$\mathbb{E}|\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F})| = \omega_n \mathbb{E}_{\boldsymbol{X}} \int_{S^{n-1}} \rho^n(\mathcal{Z}_{p,\mathcal{C}}^{\diamond}(\mathcal{F}), u)du < \infty. \tag{7.1}$$

Applying Proposition 6.6 for a standard Gaussian random vector ξ in $\mathbb{R}^n$, we have

$$\mathbb{E}_\xi \rho^{k|p|}(\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}),\xi) = \sum_{\substack{\underline{\mathbf{k}}\in[k]_0^N\\ S(\underline{\mathbf{k}})=k}} \binom{k}{\underline{\mathbf{k}}} \int_{\mathbb{R}_+^{|m(\underline{\mathbf{k}})|}} \gamma_n((\boldsymbol{X}_{\underline{\mathbf{k}}}\mathcal{B}_1^{|m(\underline{\mathbf{k}})|}(\mathcal{C}^\circ_{\underline{\mathbf{k}},t}))^\circ)dt.$$

Fix $\underline{\mathbf{k}} = (k_1,\dots,k_N) \in [k]_0^N$ with $S(\underline{\mathbf{k}}) = k$ and $t_i \in (0,\infty)$ for $i \in m(\underline{\mathbf{k}})$. By Theorem 5.1,

$$\mathbb{E}_{\boldsymbol{X}_{\underline{\mathbf{k}}}}\gamma_n((\boldsymbol{X}_{\underline{\mathbf{k}}}\mathcal{B}_1^{|m(\underline{\mathbf{k}})|}(\mathcal{C}^\circ_{\underline{\mathbf{k}},t}))^\circ) \leq \mathbb{E}_{\boldsymbol{X}^\#_{\underline{\mathbf{k}}}}\gamma_n((\boldsymbol{X}^\#_{\underline{\mathbf{k}}}\mathcal{B}_1^{|m(\underline{\mathbf{k}})|}(\mathcal{C}^\circ_{\underline{\mathbf{k}},t}))^\circ).$$

Consequently,

$$\mathbb{E}_{\boldsymbol{X}}\mathbb{E}_\xi \rho^{k|p|}(\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}),\xi) \leq \mathbb{E}_{\boldsymbol{X}^\#}\mathbb{E}_\xi \rho^{k|p|}(\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}^\#),\xi). \tag{7.2}$$

As in the proof of Proposition 6.5, when $n/|p| \in \mathbb{N}$, we choose $p_\ell \in \mathbb{Q}\cap(p,0)$ such that $n(p) = n/|p| = (n-1/\ell)/|p_\ell|$ for $\ell \in \mathbb{N}$. For $u \in S^{n-1}$, we have

$$\rho(\mathcal{Z}^{\diamond}_{p_\ell,\mathcal{C}}(\mathcal{F}),u) \to \rho(\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}),u) \quad \text{(a.s.)}.$$

As in the proof of Lemma 3.4, using (3.7) with $K_\ell = \mathcal{Z}^{\diamond}_{p_\ell,\mathcal{C}}(\mathcal{F})$, we have

$$\int_{S^{n-1}} \rho^{n(p)|p_\ell|}(\mathcal{Z}^{\diamond}_{p_\ell,\mathcal{C}}(\mathcal{F}),u)du \to \int_{S^{n-1}} \rho^{n}(\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}),u)du \quad \text{(a.s.)}$$

and the convergence is dominated by $1+\omega_n^{-1}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})|$, which is integrable by (7.1). Thus

$$\begin{aligned}\mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})| &= \omega_n \mathbb{E}_{\boldsymbol{X}} \lim_{\ell\to\infty}\int_{S^{n-1}} \rho^{n(p)|p_\ell|}(\mathcal{Z}^{\diamond}_{p_\ell,\mathcal{C}}(\mathcal{F}),u)du\\ &= \omega_n \lim_{\ell\to\infty}\mathbb{E}_{\boldsymbol{X}}\int_{S^{n-1}} \rho^{n-1/\ell}(\mathcal{Z}^{\diamond}_{p_\ell,\mathcal{C}}(\mathcal{F}),u)du\\ &= \omega_n \lim_{\ell\to\infty} b^{-1}_{n,n-1/\ell}\mathbb{E}_{\boldsymbol{X}}\mathbb{E}_\xi\rho^{n-1/\ell}(\mathcal{Z}^{\diamond}_{p_\ell,\mathcal{C}}(\mathcal{F}),\xi),\end{aligned}$$

where $b_{n,n-1/\ell}$ is the constant in (3.6). The same identities apply for $\boldsymbol{X}^\#$ and $\mathcal{F}^\#$. Thus, applying (7.2) with $k = n(p)$ and $p = p_\ell$, we get

$$\mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F})| \leq \mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}}(\mathcal{F}^\#)|.$$

Lastly, we can remove the assumption that the functions are compactly supported by arguing as in the proof of Theorem 2.2. ■

Proof of Corollary 2.4. By Proposition 5.11 and Theorem 2.3,

$$\mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(f)| = \lim_{m\to\infty}\mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}^\alpha_m}(\mathcal{F}_m)| \leq \lim_{m\to\infty}\mathbb{E}|\mathcal{Z}^{\diamond}_{p,\mathcal{C}^\alpha_m}(\mathcal{F}^\#_m)| = \mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(f^*)|. \tag{7.3}$$

Using (7.3) with Proposition 5.12, we get

$$|I^{\alpha}_{|p|}(f)| = \lim_{N\to\infty}\mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(f)| \leq \lim_{N\to\infty}\mathbb{E}|\mathcal{I}^{\alpha}_{|p|,N}(f^*)| = |I^{\alpha}_{|p|}(f^*)|. \tag{7.4}$$

This completes the proof. ■

Proof of Corollary 2.5. We apply Proposition 4.2 and (7.4) for $p = -1$ to obtain

$$|I(f)| = \lim_{\alpha \to 0+} |(2s_\alpha)^{-1} I_\alpha(f)| \leq \lim_{\alpha \to 0+} |(2s_\alpha)^{-1} I_\alpha(f^*)| = |I(f^*)|. \qquad \blacksquare$$

Proof of Theorem 2.6. We have reduced Theorems 2.1–2.3 and Corollaries 2.4 and 2.5 to a suitable application of (5.1) in Theorem 5.1. If the convex bodies $C_1, \dots, C_N$ are unconditional, we can instead apply (5.2) in Theorem 5.1. ■

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