

RESEARCH ARTICLE

Validity of steady Prandtl layer expansions

Yan Guo¹ | Sameer Iyer²¹Brown University, Providence, Rhode Island, USA²University of California, Davis, Davis, California, USA**Correspondence**

Sameer Iyer, University of California, Davis, Davis, California, USA.

Email: sameer@math.ucdavis.edu**Abstract**

Let the viscosity $\varepsilon \rightarrow 0$ for the 2D steady Navier-Stokes equations in the region $0 \leq x \leq L$ and $0 \leq y < \infty$ with no slip boundary conditions at $y = 0$. For $L \ll 1$, we justify the validity of the steady Prandtl layer expansion for scaled Prandtl layers, including the celebrated Blasius boundary layer. Our uniform estimates in ε are achieved through a fixed-point scheme:

$$[u^0, v^0] \xrightarrow{\text{DNS}^{-1}} v \xrightarrow{\mathcal{L}^{-1}} [u^0, v^0]$$

for solving the Navier-Stokes equations, where $[u^0, v^0]$ are the tangential and normal velocities at $x = 0$, DNS stands for ∂_x of the vorticity equation for the normal velocity v , and $\mathcal{L}$ the compatibility ODE for $[u^0, v^0]$ at $x = 0$.

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1 | INTRODUCTION AND NOTATION

We consider the steady, incompressible Navier-Stokes equations on the two-dimensional domain, $(x, Y) \in \Omega = (0, L) \times (0, \infty)$. Denoting the velocity $\mathbf{U}^{NS} := (U^{NS}, V^{NS})$, the equations read for

$\varepsilon > 0$:

$$\left. \begin{aligned} \mathbf{U}^{NS} \cdot \nabla \mathbf{U}^{NS} + \nabla P^{NS} &= \varepsilon \Delta \mathbf{U}^{NS} \\ \nabla \cdot \mathbf{U}^{NS} &= 0 \end{aligned} \right\} \text{ in } \Omega \quad (1.1)$$

The system above is taken with the no-slip boundary condition on $\{Y = 0\}$:

$$[U^{NS}, V^{NS}]|_{Y=0} = [0, 0]. \quad (1.2)$$

In this article, we fix an outer Euler shear flow of the form $[u_e^0(Y), 0, 0]$, (satisfying generic smoothness and decay assumptions). Such a shear flow solves the Euler equations ((1.1) with $\varepsilon = 0$). A fundamental question is to describe the asymptotic behavior of solutions to (1.1) as the viscosity, ε , vanishes (or equivalently, as the Reynolds number goes to infinity). Generically, there is a mismatch of the tangential velocity at the boundary $\{Y = 0\}$ of the viscous flows, (1.2), and inviscid flows. Thus, one cannot expect $[U^{NS}, V^{NS}] \rightarrow [u_e^0, 0]$ in a sufficiently strong norm (for instance, L^∞).

To rectify this mismatch, it was proposed in 1904 by Ludwig Prandtl that there exists a thin fluid layer of size $\sqrt{\varepsilon}$ near the boundary $Y = 0$ that bridges the velocity of $U^{NS}|_{Y=0} = 0$ with the nonzero Eulerian velocity ([44]). This layer is known as the Prandtl boundary layer, and mathematically corresponds to an asymptotic expansion in ε as shown below in (1.5). We emphasize that Prandtl's original hypothesis was made in the 2D, stationary setting, which is precisely the setting we are addressing in this paper.

We work with the scaled boundary layer variable and the corresponding scaled differential operators:

$$y = \frac{Y}{\sqrt{\varepsilon}}, \quad \nabla_\varepsilon = (\sqrt{\varepsilon} \partial_x, \partial_y), \quad \Delta_\varepsilon := \varepsilon \partial_{xx} + \partial_{yy}. \quad (1.3)$$

Define the scaled Navier-Stokes velocities:

$$U^\varepsilon(x, y) = U^{NS}(x, Y), \quad V^\varepsilon = \frac{V^{NS}(x, Y)}{\sqrt{\varepsilon}}, \quad P^\varepsilon(x, y) = P^{NS}(x, Y). \quad (1.4)$$

Equation (1.1) now becomes:

$$\begin{aligned} U^\varepsilon U_x^\varepsilon + V^\varepsilon U_y^\varepsilon + P_x^\varepsilon &= \Delta_\varepsilon U^\varepsilon \\ U^\varepsilon V_x^\varepsilon + V^\varepsilon V_y^\varepsilon + \frac{P_y^\varepsilon}{\varepsilon} &= \Delta_\varepsilon V^\varepsilon \\ U_x^\varepsilon + V_y^\varepsilon &= 0 \end{aligned} \quad (1.5)$$

We expand the solution in ε as:

$$\begin{aligned} U^\varepsilon &= u_e^0 + u_p^0 + \sum_{i=1}^n \sqrt{\varepsilon}^i (u_e^i + u_p^i) + \varepsilon^{N_0} u^{(\varepsilon)} := u_s + \varepsilon^{N_0} u^{(\varepsilon)}, \\ V^\varepsilon &= v_p^0 + v_e^1 + \sum_{i=1}^{n-1} \sqrt{\varepsilon}^i (v_p^i + v_e^{i+1}) + \sqrt{\varepsilon}^n v_p^n + \varepsilon^{N_0} v^{(\varepsilon)} := v_s + \varepsilon^{N_0} v^{(\varepsilon)}, \\ P^\varepsilon &= p_e^0 + p_p^0 + \sum_{i=1}^n \sqrt{\varepsilon}^i (p_e^i + p_p^i) + \varepsilon^{N_0} P^{(\varepsilon)} := P_s + \varepsilon^{N_0} P^{(\varepsilon)}, \end{aligned} \quad (1.6)$$

where the coefficients are independent of ε . Here $[u_e^i, v_e^i]$ are Euler correctors, and $[u_p^i, v_p^i]$ are Prandtl correctors. These are constructed in the paper [26], culminating in Theorem A.3. For our analysis, we will take $n = 4$ and $N_0 = 1+$. Let us also introduce the following notation:

$$\bar{u}_p^i := u_p^i - u_p^i|_{y=0}, \quad \bar{v}_p^i := v_p^i - v_p^i|_{y=0}, \quad \bar{v}_e^i := v_e^i - v_e^i|_{Y=0}. \quad (1.7)$$

The profile $\bar{u}_p^0, \bar{v}_p^0$ from (1.7) is classically known as the “boundary layer”; one sees from (1.6) that it is the leading order approximation to the Navier-Stokes velocity, U^ε . We will sometimes use the notation $u_{||} := \bar{u}_p^0$, and $v_{||} := \bar{v}_p^0$. The final layer,

$$[u^{(\varepsilon)}, v^{(\varepsilon)}, P^{(\varepsilon)}] = [\mathbf{u}^{(\varepsilon)}, P^{(\varepsilon)}]$$

are called the “remainders” and importantly, they depend on ε . Controlling the remainders uniformly in ε is the fundamental challenge in order to establish the validity of (1.6), and the centerpiece of our article.

Thanks to the elliptic feature of the steady Navier-Stokes equations, *the set-up of our program is to assume the remainders* $[u^{(\varepsilon)}, v^{(\varepsilon)}]$ are bounded in a suitable sense at the boundaries $\{x = 0\}$ and $\{x = L\}$ and to prove that they remain bounded for $x \in (0, L)$. It is important to note that there are no natural boundary conditions for the Navier-Stokes equations in a channel at $\{x = 0\}, \{x = L\}$, and thus part of the mathematical challenge is to impose boundary conditions for $[u^{(\varepsilon)}, v^{(\varepsilon)}]$ which ensure its solvability for $x \in (0, L)$.

We begin by briefly discussing the approximations, $[u_s, v_s]$. The particular equations satisfied by each term in $[u_s, v_s]$ is derived in the appendix. Theorem A.3 summarizes the estimates available for each of the approximate terms, and is proven in [26]. We are prescribed the shear Euler flow, u_e^0 . The profiles $[u_p^i, v_p^i]$ are Prandtl boundary layer correctors. Importantly, these layers are rapidly decaying functions of the boundary layer variable, y . At the leading order, $[u_p^0, v_p^0]$ solve the nonlinear Prandtl equation:

$$\begin{aligned} \bar{u}_p^0 u_{px}^0 + \bar{v}_p^0 u_{py}^0 - u_{pyy}^0 + P_{px}^0 &= 0, \\ u_{px}^0 + v_{py}^0 &= 0, \quad P_{py}^0 = 0, \quad u_p^0|_{x=0} = U_p^0, \quad u_p^0|_{y=0} = -u_e^0|_{Y=0}. \end{aligned} \quad (1.8)$$

Soon after Prandtl’s seminal 1904 paper, Blasius discovered the celebrated self-similar solution to (1.8) (with zero pressure). This solution reads

$$[\bar{u}_p^0, \bar{v}_p^0] = \left[f'(\eta), \frac{1}{2\sqrt{x+x_0}} \{ \eta f'(\eta) - f(\eta) \} \right], \quad \text{where } \eta = \frac{y}{\sqrt{x+x_0}}, \quad (1.9)$$

where f satisfies

$$f f'' + 2f''' = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \quad \frac{f(\eta)}{\eta} \xrightarrow{n \rightarrow \infty} 1. \quad (1.10)$$

Here, $x_0 > 0$ is a free parameter. It is well known that $f''(\eta)$ has a Gaussian tail, and that the following hold:

$$0 \leq f' \leq 1, \quad f''(\eta) \geq 0, \quad f''(0) > 0, \quad f'''(\eta) < 0.$$

Such a Blasius profile has been confirmed by experiments with remarkable accuracy as the main validation of the Prandtl theory (see [47] for instance). These profiles are also canonical from a mathematical standpoint in the following sense: the work, [48], has proven that when x gets large (downstream), solutions to the Prandtl equation, (1.8), converge to an appropriately renormalized Blasius profile. Validating the expansions (1.6) for the Blasius profile is the main objective and motivation in our study.

It is well known that the Prandtl equations (1.8) admit the two parameter scaling invariance:

$$[\bar{u}^{\lambda,\sigma}, \bar{v}^{\lambda,\sigma}] := \left[\frac{\lambda^2}{\sigma} \bar{u}_p^0(\sigma x, \lambda y), \lambda \bar{v}_p^0(\sigma x, \lambda y) \right], \quad (1.11)$$

meaning that if $[\bar{u}_p^0, \bar{v}_p^0]$ solve (1.8), then so do $[\bar{u}^{\lambda,\sigma}, \bar{v}^{\lambda,\sigma}]$ (with appropriately modified initial data).

Typically in boundary layer analyses, the central mathematical analysis concerns the linearized Navier-Stokes operator. Such an operator has coefficients $[u_s, v_s]$, which are the approximate Navier-Stokes solutions defined as in (1.6). The unknown that this operator acts on is the “remainders”, $[u^{(\varepsilon)}, v^{(\varepsilon)}, P^{(\varepsilon)}]$. In vorticity formulation, the operator reads

$$\begin{aligned} -R[q^{(\varepsilon)}] - u_{yyy}^{(\varepsilon)} + 2\varepsilon v_{xyy}^{(\varepsilon)} + \varepsilon^2 v_{xxx}^{(\varepsilon)} + v_s \Delta_\varepsilon u^{(\varepsilon)} - u^{(\varepsilon)} \Delta_\varepsilon v_s \\ = \varepsilon^{N_0} \{u^{(\varepsilon)} \Delta_\varepsilon v^{(\varepsilon)} - v^{(\varepsilon)} \Delta_\varepsilon u^{(\varepsilon)}\} + F_R, \end{aligned} \quad (1.12)$$

Here, F_R is a forcing term defined in (A.22), and where we have defined the Rayleigh operator

$$R[q^{(\varepsilon)}] = \partial_y \{u_s^2 \partial_y q^{(\varepsilon)}\} + \varepsilon \partial_x \{u_s^2 q_x^{(\varepsilon)}\}, \quad q^{(\varepsilon)} := \frac{v^{(\varepsilon)}}{u_s}. \quad (1.13)$$

We define the (unknown) functions

$$u^{(\varepsilon)}|_{x=0} = u^0(y), \quad v^{(\varepsilon)}|_{x=0} = v^0(y). \quad (1.14)$$

The boundary condition we take are the following

$$\begin{aligned} v_x^{(\varepsilon)}|_{x=L} = a_1^\varepsilon(y), \quad v_{xx}^{(\varepsilon)}|_{x=0} = a_2^\varepsilon(y), \quad v_{xxx}^{(\varepsilon)}|_{x=0} = a_3^\varepsilon(y) \\ v_y^0 + u^0 = h(y) \in C^\infty(e^y), \quad h(0) = 0, \\ v^{(\varepsilon)}|_{y=0} = v_y^{(\varepsilon)}|_{y=0} = u^{(\varepsilon)}|_{y=0} = 0, \quad v_y^{(\varepsilon)}|_{y \uparrow \infty} = 0. \end{aligned} \quad (1.15)$$

Here, the $a_i^\varepsilon(y)$ are prescribed boundary data which we assume satisfy

$$\|\partial_y^j a_i^\varepsilon \left\{ \frac{1}{\varepsilon^{\frac{1}{2}}} \langle y \rangle \langle Y \rangle^m \right\}\| \leq o(1) \text{ for } j = 0, \dots, 4, \text{ and } m \text{ large}, \quad (1.16)$$

which is a quantitative statement that the expansion (1.6) is valid at $\{x = 0\}$ and $\{x = L\}$.

We are now able to state our main result, so long as we remain vague regarding the space $\mathcal{X}$ that appears below. A discussion of this norm will be in Subsection 1.2.

Theorem 1.1 (Main Theorem). Assume boundary data for the approximate layers in u_s are prescribed as in Theorem A.3, boundary data for the remainders are prescribed satisfying (1.15) and (1.16). Assume $0 < \sigma \ll 1$ in (1.11). Then let $0 < \varepsilon \ll L \ll 1$. Take $N_0 = 1+$ and $n = 4$ in (1.6). Then the remainders, $[u^{(\varepsilon)}, v^{(\varepsilon)}]$ exists uniquely in the space $\mathcal{X}$ and satisfy

$$\|\mathbf{u}^{(\varepsilon)}\|_{\mathcal{X}} \lesssim 1. \quad (1.17)$$

The Navier-Stokes solutions satisfy

$$\|U^{NS} - u_e^0 - u_p^0\|_{\infty} \lesssim \sqrt{\varepsilon} \text{ and } \|V^{NS} - \sqrt{\varepsilon}v_p^0 - \sqrt{\varepsilon}v_e^1\|_{\infty} \lesssim \varepsilon. \quad (1.18)$$

Theorem A.3 establishes that all terms in the expansion (1.6) exist and are regular: $\|u_s, v_s\|_{W^{k,\infty}} \lesssim 1$ for a large, universal k . Upon establishing the uniform bound (1.17), the result (1.18) follows from the following inequalities: $\|v^{(\varepsilon)}\|_{\infty} \lesssim \varepsilon^{-\frac{1}{2}} \|\mathbf{u}^{(\varepsilon)}\|_{\mathcal{X}}$, and $\|u^{(\varepsilon)}\|_{\infty} \lesssim \|\mathbf{u}^{(\varepsilon)}\|_{\mathcal{X}}$. These are established in Lemmas 2.4, 5.4 together with the definitions in (1.22).

Our main result thus ensures a local in space ($L \ll 1$) validity for the Prandtl expansion, (1.6). This marks an important first step to study the optimal bound for $\sup L$. Such a study would address the phenomenon of “boundary layer detachment” (which would correspond to $\sup L < \infty$) versus global in x validity (in the sense of [32–34]).

Regarding our scaling, (1.11), it is important to note that λ can be arbitrary. This covers rich structures in the Prandtl equation. In particular, when $\lambda^2 = \sigma$, the scaling of $\lambda \ll 1$ is equivalent to $x_0 \gg 1$ in (1.9). Letting η_λ denote the rescaled self-similar variable, one has by definition

$$\eta_\lambda := \frac{\lambda y}{\sqrt{\lambda^2 x + x_0}} = \frac{y}{\sqrt{x + \lambda^{-2} x_0}}.$$

For this reason, we interpret our main theorem as being *asymptotic*, that is for large values of x_0 : in the particular case of $\lambda = \sigma^2$, setting σ small is equivalent to taking x_0 large. Moreover, in light of [52], general solutions to the Prandtl equation converge to the Blasius profile as $x_0 \rightarrow \infty$. We thus expect that the validity of (1.6) holds for generic Prandtl data without rescaling, for $x_0 \gg 1$. Furthermore, we remark the L may not necessarily need to be small in this case.

1.1 | Notation

Before we state the main ideas of the proof, we will discuss our notation. Since we use the L^2 norm extensively in the analysis, we use $\|\cdot\|$ to denote the L^2 norm. It will be clear from context whether we mean $L^2(\mathbb{R}_+)$ or $L^2(\Omega)$. When there is a potential confusion (e.g., when changing coordinates), we will take care to specify with respect to which variable the L^2 norm is being taken (for instance, L_y^2 means with respect to dy , whereas L_Y^2 will mean with respect to dY). Similarly, when there is potential confusion, we will distinguish L^2 norms along a one-dimensional surface (say $\{x = 0\}$) by $\|\cdot\|_{x=0}$. Analogously, we will often use inner products $(\cdot, \cdot)$ to denote the L^2 inner product. When unspecified, it will be clear from context if we mean $L^2(\mathbb{R}_+)$ or $L^2(\Omega)$. When there is potential confusion, we will distinguish inner products on a one-dimensional surface (say $\{x = 0\}$) by writing $(\cdot, \cdot)_{x=0}$. Given a weight function w , we use the notation $\|\cdot\|_{L^2(w)} := \|\cdot w\|$, and $L^2(w)$ to refer to the corresponding weighted L^2 space.

We will often use scaled differential operators

$$\nabla_\varepsilon := (\partial_x, \sqrt{\varepsilon} \partial_y), \quad \Delta_\varepsilon := \partial_{yy} + \varepsilon \partial_{xx}.$$

Define also the integration operator, $I_x[g] := \int_0^x g(x') dx'$. For functions $w : \mathbb{R}_+ \rightarrow \mathbb{R}$, we distinguish between w' which means differentiation with respect to its argument versus w_y which refers to differentiation with respect to y .

Regarding unknowns, the central object of study in our paper are the remainders, $[u^{(\varepsilon)}, v^{(\varepsilon)}]$. By a standard homogenization argument, we may move the inhomogeneous boundary terms a_i^ε to the forcing and consider the homogeneous problem. Specifically, we homogenize $v^{(\varepsilon)}$ to v using the following:

$$\begin{aligned} \tilde{v} &:= v^0 + x \left\{ a_1^\varepsilon - L a_2^\varepsilon - \frac{L^2}{2} a_3^\varepsilon \right\} + x^2 \frac{a_2^\varepsilon}{2} + x^3 \frac{a_3^\varepsilon}{6} =: v^0 + a^\varepsilon(x, y), \\ v &:= v^\varepsilon - \tilde{v} = v^\varepsilon - v^0 - a^\varepsilon, \quad u := u^\varepsilon + \int_0^x \tilde{v}_y = u^\varepsilon + x v_y^0 + I_x[a_y^\varepsilon]. \end{aligned} \quad (1.19)$$

We call the new unknowns $[u, v]$ ($= \mathbf{u}$), and these are actually the objects we will analyze throughout the paper.

When we write $a \lesssim b$, we mean there exists a number $C < \infty$ such that $a \leq Cb$, where C is independent of small L, ε but could depend on $[u_s, v_s]$. We write $o_L(1)$ to refer to a constant that is bounded by some unspecified, perhaps small, power of L : that is, $a = o_L(1)$ if $|a| \leq CL^\delta$ for some $\delta > 0$.

We will, at various times, require localizations. All such localizations will be defined in terms of the following fixed C^∞ cutoff function:

$$\chi(y) := \begin{cases} 1 & \text{on } y \in [0, 1) \\ 0 & \text{on } y \in (2, \infty) \end{cases} \quad \chi'(y) \leq 0 \text{ for all } y > 0. \quad (1.20)$$

We will use $\|\cdot\|_{loc}$ to mean localized L^2 norms. More specifically we take for concreteness $\|\cdot\|_{loc} := \|\cdot\| \chi(\frac{y}{10})$. We adopt the notation that $\langle a \rangle = 1 + a$. Define the weight

$$w_0 := \langle y \rangle \langle Y \rangle^m, \text{ for } m \text{ sufficiently large, universal number.} \quad (1.21)$$

We will define now the key norms that appear throughout our analysis:

Definition 1.2. Given a weight function $w = w(y)$, define:

$$\begin{aligned} \|v\|_{X_w} &:= \varepsilon^{-\frac{3}{16}} |||v|||_w + |||q|||_w, \\ \|v\|_{Y_w} &:= |||v|||_w + \sqrt{\varepsilon} |||q|||_w, \\ [u^0, v^0]_B &:= \|u^0\| + \|u_y^0\| + \|u_{yy}^0 w\| + \|u_{yyy}^0 w\| + \|q_y^0\| + \left\| \frac{q^0}{y} \right\| \\ &\quad + \|\sqrt{u_s} q_{yy}^0 w_0\| + \|q_y^0\|_{y=0} + \|v_{yyy}^0 w_0\| + \|v_{yyyy}^0 w_0\|, \end{aligned}$$

$$\begin{aligned}
\|\mathbf{u}\|_{\mathcal{X}} &:= \|v, u^0, v^0\|_{\mathcal{X}} := [u^0, v^0]_B + \varepsilon^{\frac{1}{4}} \|v\|_{X_1} + \varepsilon^{\frac{1}{4}} \|v\|_{Y_{w_0}}, \\
\|q\|_w &:= \|\nabla_{\varepsilon} q_x \cdot u_s w\| + \|\sqrt{u_s} \{q_{yyy}, q_{xyy}, \sqrt{\varepsilon} q_{xxy}, \varepsilon q_{xxx}\} w\| + |q|_{\partial, 2, w} \\
\|v\|_w &:= \left\| v_{yyy}, \sqrt{\varepsilon} \sqrt{u_s} v_{xyy}, \varepsilon v_{xxy}, \varepsilon^{\frac{3}{2}} v_{xxxy}, \varepsilon^2 v_{xxxx} \right\| w + |v|_{\partial, 3, w} \\
|q|_{\partial, 2, w} &:= \|u_s q_{xy} w\|_{x=0} + \|q_{xy} w\|_{y=0} + \|\sqrt{\varepsilon} u_s q_{xx} w\|_{x=L} + \|q_{yy} w\|_{y=0} \\
|v|_{\partial, 3, w} &:= \|\varepsilon^{\frac{3}{2}} \sqrt{u_s} v_{xxx} w\|_{x=0} + \|\sqrt{\varepsilon} u_s v_{xyy} w\|_{x=0} + \|\varepsilon u_s v_{xxy} w\|_{x=L}.
\end{aligned} \tag{1.22}$$

Note above that we identify the vector $\mathbf{u}$ with the triple (v, u^0, v^0) . We will use the above set of norms with either the choice $w = 1$ or $w = w_0$ (see (1.21)). We also define now the space $\mathcal{X}$:

Definition 1.3. The space $\mathcal{X}$ is defined via

$$\begin{aligned}
\mathcal{X} &:= \{(v, u^0, v^0) \in L^2(\Omega) \times L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+) : \|v, u^0, v^0\|_{\mathcal{X}} < \infty, \\
v|_{y=0} &= v_y|_{y=0} = v_x|_{x=L} = v_{xx}|_{x=0} = v_{xxx}|_{x=L} = v|_{y=\infty} = 0, \\
v^0(0) &= v_y^0(0) = \partial_y^k v(\infty) = 0 \text{ for } k \geq 1, \quad u^0 + v_y^0 = h(y), \quad u^0(0) = 0.\}
\end{aligned} \tag{1.23}$$

1.2 | Overview of proof

Let us first recap the ideas introduced in [27], which treated the case when the boundary $\{y = 0\}$ was moving with velocity $u_b > 0$. First, let us extract:

$$\text{Leading order operators in (1.12)} = -R[q] - u_{yyy}. \tag{1.24}$$

Due to the nonzero velocity at the $\{y = 0\}$ boundary, the quantity $\bar{u}|_{y=0} > 0$. A central idea introduced by [26] is the coercivity of $R[q]$ over $\|\nabla_{\varepsilon} q\|$. This coercivity relied on the fact that $q = \frac{v}{u_s} = 1 \notin \text{Ker}(R)$, thanks to the non-zero boundary velocity of $\bar{u}|_{y=0}$. Extensive efforts without success have been made to extracting coercivity from $R[q]$ in the present, motionless boundary, case. However, it appears that this procedure interacts poorly with the operator $\partial_{yyy} u$, producing singularities too severe to handle. In fact, the natural multiplier for the Rayleigh operator is q itself, which produces $(R[q], q) = \|u_s q_y\|^2$. However, due to the degeneracy of u_s at $y = 0$ (which is notably absent when $u_s|_{y=0} > 0$ as in [24]) this is too weak of a contribution to control the interaction term (u_{yy}, q_y) .

Our main idea is based on the observation that the x derivative of (1.24) produces, at leading order:

$$-\partial_x R[q] + v_{yyy}. \tag{1.25}$$

Unlike (1.24), these two operators enjoy better interaction properties. To see this on a preliminary level, consider the interaction between v_{yyy} and the multiplier $-q_{xx}$ (ignoring boundary

contributions at $x = 0, x = L$:

$$\begin{aligned} (v_{yyyy}, -q_{xx}) &\sim -(v_{yyyx}, q_{xy}) \\ &\sim -(u_s q_{yyyx} + 3u_{sy} q_{yyx} + 3u_{syy} q_{yx} + u_{syys} q_x, q_{xy}) \\ &\sim \|\sqrt{u_s} q_{yyyx}\|^2 + \frac{3}{2} \|\sqrt{u_{sy}} q_{xy}\|_{y=0}^2, \end{aligned} \quad (1.26)$$

which is a crucial favorable boundary contribution at $\{y = 0\}$ as $u_{sy}|_{y=0} \sim \bar{u}_{py}^0 > 0$.

To this end, we split the equation (1.12) into two pieces that are linked together. First, we study the boundary trace, $[u^0, v^0] = [u, v]|_{x=0}$. By evaluating the vorticity equation (1.12) at $\{x = 0\}$ and using the relation (1.15), we obtain the following system for v^0 :

$$\begin{aligned} \mathcal{L}v^0 &= F_{(v)} + F_R^a + \mathcal{Q} + \mathcal{H}, \\ \mathcal{L}v^0 &:= v_{yyyy}^0 - \partial_y \{u_s^2 q_y^0\} - \{v_s v_{yyy}^0 - v_y^0 v_{syy}^0\} + \varepsilon u_{sxx} v^0 + \varepsilon v_{sxx} v_y^0, \\ F_{(v)} &:= -2\varepsilon u_s u_{sx} q_x|_{x=0} - 2\varepsilon v_{xyy}|_{x=0} - \varepsilon^2 v_{xxx}|_{x=0} - \varepsilon v_s u_{xx}|_{x=0}, \\ v^0(0) &= 0, \quad v_y^0(0) = 0, \quad \partial_y^k v^0(\infty) = 0 \text{ for } k \geq 1, \\ v_y^0 + u^0 &= h(y). \end{aligned} \quad (1.27)$$

The $F_R^a + \mathcal{Q} + \mathcal{H}$ terms above contain contributions of $h(y)$, quadratic nonlinearities in v^0 , and pure forcing terms. We refrain from discussing these terms further in the introduction; the full equations are specified in (A.23). The important point is that the forcing term $F_{(v)}$ in (1.27) depends on (derivatives of v) $|_{x=0}$.

Second, we take ∂_x of (1.12) (call this “DNS” for Derivative Navier-Stokes) to obtain:

$$\begin{aligned} \text{DNS}(v) &:= -\partial_x R[q] + \Delta_\varepsilon^2 v + J(v) = -B_{v^0} + \varepsilon^{N_0} \mathcal{N} + F_{(q)} \\ v|_{x=0} &= v_x|_{x=L} = v_{xx}|_{x=0} = v_{xxx} = 0, \\ v|_{y=0} &= v|_{y=0} = 0. \end{aligned} \quad (1.28)$$

Here, the $\varepsilon^{N_0} \mathcal{N} + F_{(q)}$ terms are quadratic and forcing terms which shall remain unspecified for the moment. Note the change in notation as we have dropped the superscript ε , and homogenized the boundary conditions on the sides $\{x = 0\}, \{x = L\}$. Above, B_{v^0} is the result of homogenizing the boundary condition $v|_{x=0} = v^0$ as well as using $u = u^0 - I_x[v_y]$. The operators J, B_{v^0} are defined:

$$\begin{aligned} J(v) &:= -v_{sx} I_x[v_{yyy}] - v_s v_{yyy} - \varepsilon v_{sx} v_{xy} \\ &\quad - \varepsilon v_s v_{xxy} + v_y \Delta_\varepsilon v_s + I_x[v_y] \Delta_\varepsilon v_{sx}, \end{aligned} \quad (1.29)$$

$$\begin{aligned} B_{v^0} &:= v_{yyyy}^0 - 2\partial_y \{u_s u_{sx} q_y^0\} + [v_{yyy}^0 \partial_x \{(x+1)v_s\} \\ &\quad - v_y^0 \partial_x \{(x+1)v_{syy}\}] - \varepsilon v_y^0 \partial_x \{(x+1)v_{sxx}\}. \end{aligned} \quad (1.30)$$

Thus, the approach we take is to analyze (1.27) in order to control the boundary trace $[u^0, v^0]$ in terms of v , and subsequently analyze (1.28) to control v in terms of the boundary trace, $[u^0, v^0]$.

We may schematize this procedure via:

$$[u^0, v^0] \xrightarrow{\text{DNS}^{-1}} v \xrightarrow{\mathcal{L}^{-1}} [u^0, v^0]. \quad (1.31)$$

We then recover a solution to the original Navier-Stokes equation (NS) via a fixed point of (1.31). This structure of analysis gives rise to a linked set of inequalities which we summarize here:

$$\left. \begin{aligned} [u^0, v^0]_B^2 &\lesssim \varepsilon \|v\|_{Y_{w_0}}^2 + \varepsilon^{\frac{1}{2} + \frac{3-}{16}} \|v\|_{X_1}^2 + \text{Data} \\ \|v\|_{X_1}^2 &\lesssim \varepsilon^{-\frac{1}{2}} [u^0, v^0]_B^2 + \text{Data} \\ \|v\|_{Y_{w_0}}^2 &\lesssim \|v\|_{X_1}^2 + [u^0, v^0]_B^2 + \text{Data}. \end{aligned} \right\} \quad (1.32)$$

Above w_0 is the specific weight given in (1.21). Since the above inequalities imply $[u^0, v^0]_B \lesssim \varepsilon^{\frac{3}{16}} \|\mathbf{u}\|_X + \text{Data}$, it is clear that the above scheme of estimates closes to yield control over $\|\mathbf{u}\|_X$.

As shown in Section 5 that $B_{(v^0)}$ (Lemma 5.2), $F_{(v)}$ (Lemma 5.3) and the nonlinearity (Lemma 5.4) can be controlled with a small constant. We therefore turn our attention to the following two linear problems.

Section 2: Study of linear problem $\mathcal{L}v^0 = F$

Let us turn now to the system, (1.27). The main estimate we prove is:

$$[u^0, v^0]_B \lesssim \|F_{(v)} w_0\| + \text{Data}. \quad (1.33)$$

Upon recalling the terms in $F_{(v)}$ shown in (1.27) and analyzing the resulting expressions, such an estimate produces the first bound shown in (1.32).

By evaluating the vorticity equation, (1.12) at $\{x = 0\}$, one obtains a compatibility equation that must be satisfied by the tuple, $[u^0, v^0]$. However, it is important to realize that we have the freedom to prescribe the relationship between these two boundary data. We do so by selecting $u^0 + v_y^0 = h(y)$ as shown in (1.15). This boundary condition is natural from the setup of our program, since both u^0 and v_y^0 should be individually bounded in Sobolev norms. The selection of this boundary condition results in a fourth order equation $v_{yyyy}^0 - \partial_y \{u_s^2 q_y^0\}$, which enjoys similar favorable properties to DNS and similar corresponding quotient estimates as in (1.26).

Estimate (1.33) is obtained in two steps. The first step is to apply the multiplier $q^0 = \frac{v}{u_s}$, and the second is to apply the multiplier $v_{yyyy} w_0^2$. The multiplier q^0 leads to a delicate interaction between the ∂_y^4 operator and the Rayleigh term $-\partial_y \{u_s^2 \partial_y q^0\}$. The key estimate we prove in this direction is the positivity

$$(\partial_y^4 v^0 - \partial_y \{u_s^2 q_y^0\}, q^0) \gtrsim \|\sqrt{u_s} q_{yy}^0\|^2 + \|u_s q_y^0\|^2 + \|q_y^0\|_{y=0}^2.$$

It is for this lower bound that we require the assumption that $\sigma \ll 1$ in (1.11). Once this is established, the remaining terms may be treated perturbatively. Overall, the upshot of the selection of boundary condition (1.15) is to capitalize on similar favorable structures to the DNS analysis.

Section 3 and 4: Study of linear problem $\text{DNS}(v) = F$

We now turn our attention to (1.28). The goal is to establish control over the norms $\|\cdot\|_{Y_{w_0}}, \|\cdot\|_{Y_1}, \|\cdot\|_{X_1}$. Consulting (1.22), the basic building blocks of these norms are the fourth and third order norms, $|||v|||_w, |||q|||_w$. Hence, our discussion will be centered on the control of $|||v|||_w, |||q|||_w$. Let us also emphasize that we require $L \ll 1$ to establish these controls and ultimately solve the DNS equation.

Based on the crucial quotient estimate (1.26), we perform a cascade of five estimates which culminate in the following:

$$\begin{aligned} |||v|||_1^2 &\lesssim \varepsilon^{\frac{3}{8}} |||q|||_1^2 + \text{Data}, \\ |||q|||_1^2 &\lesssim |||v|||_1^2 + \text{Data}. \end{aligned} \quad (1.34)$$

Let us discuss the important features of the above scheme. The top (fourth order) bound in (1.34) consists of two estimates, first using the multiplier $\varepsilon^2 v_{xxxx}$ and second using the multiplier $\varepsilon u_s v_{xxyy}$. These estimates are possible due to carefully designed boundary conditions at $x = 0$ and $x = L$ for v (see (1.28)). Our central observation at this level is that the $\varepsilon u_s v_{xxyy}$ estimate is essentially *standalone at the top order*, up to $|||q|||$, thanks to the crucial weight u_s .

The bottom (third order) bound in (1.34) consists of three delicate estimates, using the multipliers successively q_x, q_{xx}, q_{yy} . First, we emphasize that the multipliers at this stage are derivatives of the quotient, q . This is because the main coercivity is extracted from the Rayleigh operator, $R[q]$. The key feature we capitalize on in this scheme is that the estimates using multiplier q_x, q_{yy} are *standalone up to $o_L(1)$ contributions*. It is important to note that since $q = \frac{v}{u_s}$, despite the presence of $\sqrt{u_s}$ weight in $|||q|||_w$ (see (1.22)), this is still significantly stronger at $\{y = 0\}$ than a classical scaled H_{loc}^4 norm to measure v itself. In turn, to facilitate estimates near $\{y = 0\}$, we establish careful embedding estimates in (3.2).

The weighted analog of the scheme (1.34) is, for any given $w(y)$ (satisfying reasonable hypotheses):

$$\begin{aligned} |||v|||_w^2 &\lesssim \varepsilon^{\frac{3}{8}} |||q|||_1^2 + \varepsilon |||q|||_w^2 + \sqrt{\varepsilon} |||q|||_w |||q|||_{w_y} + \text{Data}, \\ |||q|||_w^2 &\lesssim o_L(1) |||v|||_w^2 + o_L(1) ||q_{xx}||_{w_y}^2 + \text{Data}. \end{aligned} \quad (1.35)$$

Apart from the key elements discussed above in the unweighted case, the new features here is a *gain of ε* when going from $|||v|||_w$ to $|||q|||_w$. This crucial gain of ε is what ultimately enables us to relate the weighted estimate for $\|v\|_{Y_w}$ back to the $\|v\|_{X_1}$ unweighted norm.

As a final remark, we note that the appendix of this paper contains the summary of the construction of the profiles, $[u_s, v_s]$, in Theorem A.3. This theorem is proven in [26].

1.3 | Other works

Let us now place this result in the context of the existing literature. To organize the discussion, we will focus on the setting of stationary flows in dimension 2. This setting in particular occupies a fundamental role in the theory, as it was the setting in which Prandtl first formulated and introduced the idea of boundary layers for Navier-Stokes flows in his seminal 1904 paper [48].

In this context, one fundamental problem is to establish the validity of the expansions (1.6). This was first achieved under the assumption of a moving boundary in [27] for $x \in [0, L]$, for L sufficiently small. The method of [27] is to establish a positivity estimate to control $\|\nabla_\varepsilon v\|_{L^2}$, which crucially used the assumed motion of the boundary. Several generalizations were obtained in [30–34]. First, [30] considered flows over a rotating disk, in which geometric effects were seen [31–33], considered flows globally in the tangential variable, and [34] considered outer Euler flows that are non-shear. All of these works are under the crucial assumption of a moving boundary.

The classical setting of a motionless boundary with the no-slip condition is treated by the present work, as well as the exciting result of [17]. It is our understanding that our present work is mutually exclusive with the work of [17]. Our work here, and main concern, treats the classical self-similar Blasius solution which appears to not be covered by [17]. On the other hand, our result does not cover a pure shear boundary layer of the form $(U_0(y), 0)$ since such shears are not a solution to the homogeneous Prandtl equation.

A related question is that of wellposedness of the Prandtl equation (the equation for $\bar{u}_p^0$, as defined in (1.7)). This investigation was by Oleinik in [46, 47]. In the 2D, stationary setting, it is shown that under local monotonicity assumptions, solutions exist in $[0, L]$. In the case where the pressure gradient is favorable, it is shown that L can be taken arbitrarily large. The recent work of [8] addresses the related issue of blowup of the Prandtl equation under the assumption of an unfavorable pressure gradient. The regularity results obtained in the present paper can be viewed as an extension of Oleinik's local-in- x result: assuming strong decay at $y \rightarrow \infty$, we can obtain enhanced regularity of Oleinik's solutions.

For unsteady flows, expansions of the form (1.6) have been verified in the analyticity framework: [45, 46], in the Gevrey setting: [14], for the initial vorticity distribution assumed away from the boundary: [39]. The reader should also see [2, 11, 34, 54], [40, 51, 52] for related results. There have also been several works ([17–22, 25]) establishing generic invalidity of expansions of the type (1.6) in Sobolev spaces in the unsteady setting.

In the unsteady setting, there is again the related matter of wellposedness of the Prandtl equation. This was also initiated by Oleinik, who under the monotonicity assumption, $\partial_y U(t=0) > 0$, obtained global-in-time regular solutions on $[0, L] \times \mathbb{R}_+$ for L small, and local-in-time solutions on $[0, L] \times \mathbb{R}_+$ for arbitrarily large L by finite L . Global-in-time weak solutions were obtained by [53] for arbitrary L under the monotonicity assumption and a favorable pressure gradient of the Euler flow: $\partial_x P^E(t, x) \leq 0$ for $t \geq 0$.

The works mentioned above use the Crocco transform, which is available in the monotonic setting. Still assuming monotonicity, local wellposedness was proven in [1] and [41] without using the Crocco transform, and in [35] for multiple monotonicity regions. [41] introduced a good unknown which enjoys a crucial cancelation, whereas [1] performed energy estimates on a transformed quantity together with a Nash-Moser iteration.

When the assumption of monotonicity is removed, the wellposedness results are largely in the analytic or Gevrey setting. The reader should consult [8, 13, 28, 37, 39, 47, 48] for some results in this direction. In the Sobolev setting without monotonicity, the equations are linearly and nonlinearly ill-posed (see [12] and [16]). A finite-time blowup result was obtained in [10] when the outer Euler flow is taken to be zero, in [38] for a particular, periodic outer Euler flow, and in [27] for both the inviscid and viscous Prandtl equations.

The related question of L^2 (in space) convergence of Navier-Stokes flows to an Euler flow has been studied by several authors. We refer the reader to [4–7, 36], and [53] for some works in this direction.

The above discussion is not comprehensive; we refer the reader to the review articles, [9, 50] and references therein for a more thorough review of the wellposedness theory.

2 | $\mathcal{L}^{-1}$ AND BOUNDARY ESTIMATES FOR $[u^0, v^0]$

2.1 | Setup and basic inequalities

In this section we analyze $[u^0, v^0]$. Recall (A.23) and the definition of w_0 , (1.21). We thus consider

$$\begin{aligned} \mathcal{L}v^0 &= F \in L^2(w_0), \\ \mathcal{L}v^0 &:= v_{yyyy}^0 - \{u_s v_{yy}^0 - u_{syy} v^0\} - \{v_s v_{yyy}^0 - v_y^0 v_{syy}\} \\ &\quad + \varepsilon u_{sxx} v^0 + \varepsilon v_{sxx} v_y^0, \\ v^0(0) &= v_y^0(0) = 0, \quad \partial_y^k v^0(\infty) = 0 \text{ for } k \geq 1. \end{aligned} \quad (2.1)$$

Above, we take F as an abstract forcing term. We also write $\mathcal{L}$ as shown in (1.27). Define the unknown $q^0 = \frac{v^0}{u_s}$, which satisfies the boundary condition $q^0(0) = 0$. As all of the analysis in this section will be on $\{x = 0\}$, we will use $(\cdot, \cdot)$ to refer to the $L^2(x = 0)$ inner product for this section.

We now introduce norms in which we control $[u^0, v^0]$ (recall the definition (1.21)):

$$[[q^0]] := \|\sqrt{u_s} q_{yy}^0\| + \|u_s q_y^0\| + \|\sqrt{u_s} q_y^0\|_{y=0}, \quad (2.2)$$

$$[[[v^0]]] := \|u_s v_{yy}^0 w_0\| + \|\sqrt{u_s} v_{yyy}^0 w_0\| + \|v_{yyyy}^0 w_0\|. \quad (2.3)$$

We also now define the $[\cdot]_B$ norms in which we control the solution:

$$\begin{aligned} [u^0, v^0]_B &:= \|u^0\| + \|u_y^0\| + \|u_{yy}^0 w_0\| + \|u_{yyy}^0 w_0\| + \|q_y^0\| + \left\| \frac{q^0}{y} \right\| \\ &\quad + \|\sqrt{u_s} q_{yy}^0 w_0\| + \|q_y^0\|_{y=0} + \|v_{yyy}^0 w_0\| + \|v_{yyyy}^0 w_0\|. \end{aligned} \quad (2.4)$$

We also define the space B via

$$B = \left\{ [u^0, v^0] \in L^2 \times L^2 \left(\frac{1}{\langle y \rangle} \right) : u^0 + v_y^0 = h(y), \quad [u^0, v^0]_B < \infty \right\} \quad (2.5)$$

The main result of Section 2 is

Proposition 2.1. *There exists a unique solution v^0 (and thus u^0 according to (1.15)) to (2.1) such that $[u^0, v^0] \in B$, and the following estimate holds*

$$[u^0, v^0]_B^2 \lesssim |(F, q^0)| + \|F w_0\|^2. \quad (2.6)$$

Note that the quantity $|(F, q^0)|$ is finite for $[u^0, v^0] \in B$ and $F \in L^2(w_0)$ by Hardy's inequality.

The first task is to generate inequalities relating the norms (2.2), (2.3), and (2.4) to various other quantities that will arise in the analysis.

Lemma 2.2. For any $0 < \sigma \ll 1$ in (1.11),

$$\|q_y^0\| \lesssim \sigma^{2/3} \lambda^{-2} [[q^0]], \quad (2.7)$$

$$|q^0| \leq \sigma^{2/3} \lambda^{-2} \langle y \rangle^{1/2} [[q^0]]. \quad (2.8)$$

Proof. Fix a $\delta \ll 1$ to be selected later. We split at scale $\lambda y = \delta$ via

$$\begin{aligned} \|q_y^0\| &\leq \|q_y^0 \chi\left(\frac{\lambda y}{\delta}\right)\| + \|q_y^0 \left\{1 - \chi\left(\frac{\lambda y}{\delta}\right)\right\}\| \\ &= \|q_y^0 \chi\left(\frac{\lambda y}{\delta}\right)\| + \left\|\frac{1}{u_s} q_y^0 \left\{1 - \chi\left(\frac{\lambda y}{\delta}\right)\right\}\right\| \\ &\lesssim \|q_y^0 \chi\left(\frac{\lambda y}{\delta}\right)\| + \frac{\sigma}{\lambda^2 \delta} \|u_s q_y^0\|. \end{aligned}$$

Above, we have used that $u_s \gtrsim \frac{\lambda^2 \delta}{\sigma}$ when $\lambda y \geq \delta$ by (1.11).

It thus remains to examine the localized contribution, for which we integrate by parts:

$$\begin{aligned} \|q_y^0 \chi\left(\frac{\lambda y}{\delta}\right)\|^2 &= \left(\partial_y \{y\} q_y^0, q_y^0 \chi\left(\frac{\lambda y}{\delta}\right)^2\right) \\ &= - \left(2y q_y^0, q_{yy}^0 \chi\left(\frac{\lambda}{\delta} y\right)^2\right) - \left(2y q_y^0, q_y^0 \frac{\lambda}{\delta} \chi'\left(\frac{\lambda}{\delta} y\right) \chi\left(\frac{\lambda}{\delta} y\right)\right) \\ &\lesssim \frac{\sqrt{\sigma \delta}}{\lambda^2} \|q_y^0 \chi\left(\frac{\lambda y}{\delta}\right)\| \|\sqrt{u_s} q_{yy}^0\| + \frac{\sigma^2}{\lambda^4 \delta^2} \|u_s q_y^0\|^2 \\ &\leq \frac{1}{2} \|q_y^0 \chi\left(\frac{\lambda y}{\delta}\right)\|^2 + \mathcal{O}(1) \left\{ \frac{\sigma \delta}{\lambda^4} \|\sqrt{u_s} q_{yy}^0\|^2 + \frac{\sigma^2}{\lambda^4 \delta^2} \|u_s q_y^0\|^2 \right\}. \end{aligned} \quad (2.9)$$

Above, for the first term from (2.9), we used that in the support of the cut-off $\chi(\frac{\lambda}{\delta} y)$, one has $y \leq \frac{\delta}{\lambda}$, so recalling (1.11) we obtain

$$y \chi\left(\frac{\lambda}{\delta} y\right) \leq \sqrt{y} \sqrt{\frac{\delta}{\lambda}} \chi\left(\frac{\lambda}{\delta} y\right) \leq \frac{\sqrt{\sigma}}{\lambda^{\frac{3}{2}}} \sqrt{u_s} \frac{\sqrt{\delta}}{\sqrt{\lambda}}.$$

For the second term from (2.9), we have used

$$y \frac{\lambda}{\delta} \chi'\left(\frac{\lambda}{\delta} y\right) \sim 1 \text{ and } u_s^2 \chi'\left(\frac{\lambda}{\delta} y\right) \gtrsim \frac{\lambda^4 \delta^2}{\sigma^2} \chi'\left(\frac{\lambda}{\delta} y\right).$$

In summary, we have

$$\|q_y^0\| \lesssim o(1) \|q_y^0\| + \frac{\sqrt{\sigma \delta}}{\lambda^2} \|\sqrt{u_s} q_{yy}^0\| + \frac{\sigma}{\lambda^2 \delta} \|u_s q_y^0\|.$$

We optimize above using $\delta = \sigma^{1/3}$ which gives:

$$\|q_y^0\| \lesssim \sigma^{2/3} \lambda^{-2} [[q^0]]. \quad (2.10)$$

To conclude the proof, the q^0 bound, (2.8), follows via integration

$$q^0 = \int_0^y q_y^0 \leq \sqrt{y} \|q_y^0\| \leq \sqrt{y} \sigma^{2/3} \lambda^{-2} [[q^0]].$$

□

Lemma 2.3. *The following estimates hold*

$$\|v_{yyy}^0 w_0\| \lesssim \sigma^{1/3} \lambda^{-1} [[[v^0]]], \quad (2.11)$$

$$\|v_y^0\| \lesssim (\sigma^{-1/3} \lambda^{1/2} + 1) [[q^0]] \quad (2.12)$$

$$\|v_{yy}^0\| \lesssim \left(\lambda^{\frac{3}{2}} \sigma^{-\frac{1}{3}} + \lambda \sigma^{-\frac{1}{3}} + 1 \right) [[q^0]]. \quad (2.13)$$

Proof. Proof of (2.11): We again let $\delta \ll 1$ to be selected below, and localize to regions $\lambda y \geq \delta$ and $\lambda y \leq \delta$ by introducing the cutoff $\chi(\frac{\lambda}{\delta} y)$:

$$\begin{aligned} \|v_{yyy}^0 w_0\| &\leq \|v_{yyy}^0 w_0 \chi\left(\frac{\lambda}{\delta} y\right)\| + \|v_{yyy}^0 w_0 \left\{1 - \chi\left(\frac{\lambda}{\delta} y\right)\right\}\| \\ &= \|v_{yyy}^0 w_0 \chi\left(\frac{\lambda}{\delta} y\right)\| + \left\| \frac{1}{\sqrt{u_s}} \sqrt{u_s} v_{yyy}^0 w_0 \left\{1 - \chi\left(\frac{\lambda}{\delta} y\right)\right\} \right\| \\ &\leq \|v_{yyy}^0 w_0 \chi\left(\frac{\lambda}{\delta} y\right)\| + \frac{\sqrt{\sigma}}{\lambda \sqrt{\delta}} \left\| \sqrt{u_s} v_{yyy}^0 w_0 \left\{1 - \chi\left(\frac{\lambda}{\delta} y\right)\right\} \right\| \\ &\lesssim \left\| v_{yyy}^0 w_0 \chi\left(\frac{\lambda}{\delta} y\right) \right\| + \frac{\sqrt{\sigma}}{\lambda \sqrt{\delta}} \left\| \sqrt{u_s} v_{yyy}^0 w_0 \left\{1 - \chi\left(\frac{\lambda}{\delta} y\right)\right\} \right\|. \end{aligned}$$

Above, we have used that $u_s \gtrsim \frac{\lambda^2}{\sigma} \lambda y \gtrsim \frac{\lambda^2}{\sigma} \delta$ on the region where $\lambda y \geq \delta$.

For the first integral above, we integrate by parts

$$\begin{aligned} &\left(\partial_y \{I_y[w_0^2]\} |v_{yyy}^0|^2, \chi\left(\frac{\lambda}{\delta} y\right)^2 \right) \\ &= - \left(2I_y[w_0^2] v_{yyy}^0, v_{yyy}^0 \chi\left(\frac{\lambda}{\delta} y\right)^2 \right) - \left(I_y[w_0^2] |v_{yyy}^0|^2, \frac{\lambda}{\delta} \chi'\left(\frac{\lambda}{\delta} y\right) \chi\left(\frac{\lambda}{\delta} y\right) \right) \\ &\lesssim \frac{\delta}{\lambda} \|\sqrt{u_s} v_{yyy}^0 \chi\left(\frac{\lambda}{\delta} y\right) w_0\| \|v_{yyy}^0 w_0\| + \frac{\sigma}{\lambda^2 \delta} \|\sqrt{u_s} v_{yyy}^0 w_0\|^2. \end{aligned} \quad (2.14)$$

In the first term of (2.14), we have used that $y \leq \frac{\delta}{\lambda}$ on the support of the cut-off function. For the second term, we have used that $|y \frac{\lambda}{\delta}| \lesssim 1$ on the support of $\chi'(\frac{\lambda}{\delta}y)$. Moreover, we have used by (1.11) that $u_s \gtrsim \frac{\lambda^2 \delta}{\sigma}$ when $\lambda y \geq \delta$. We thus take $\delta = \sqrt{\lambda}$. We now optimize the constant $\frac{\delta}{\lambda} + \frac{\sqrt{\sigma}}{\lambda \sqrt{\delta}}$ with a choice of $\delta = \sigma^{1/3}$.

Proof of (2.12): We have, upon recalling (2.8) and (1.11),

$$\begin{aligned} \|v_y^0\| &= \|\{u_s q^0\}_y\| \leq \|u_{sy} q^0\| + \|u_s q_y^0\| \\ &\leq \|u_{sy} \sqrt{y}\|_{\infty} \sigma^{2/3} \lambda^{-2} [[q^0]] + [[q^0]] \\ &\lesssim \left(\frac{\lambda^3}{\sigma} \lambda^{-1/2} \sigma^{2/3} \lambda^{-2} + 1 \right) [[q^0]] \\ &\lesssim (\sigma^{-1/3} \lambda^{1/2} + 1) [[q^0]]. \end{aligned}$$

Proof of (2.13): We have, upon recalling (2.8) and (1.11),

$$\begin{aligned} \|v_{yy}^0\| &\leq \|u_{syy} q^0\| + 2 \|u_{sy} q_y^0\| + \|u_s q_{yy}^0\| \\ &\lesssim \|u_{syy} \sqrt{y}\|_{\infty} \sigma^{2/3} \lambda^{-2} [[q^0]] + \|u_{sy}\|_{\infty} \sigma^{2/3} \lambda^{-2} [[q^0]] + [[q^0]] \\ &\lesssim \left(\frac{\lambda^4}{\sigma} \lambda^{-1/2} \sigma^{2/3} \lambda^{-2} + \frac{\lambda^3}{\sigma} \sigma^{2/3} \lambda^{-2} + 1 \right) [[q^0]] \\ &= \left(\lambda^{\frac{3}{2}} \sigma^{-\frac{1}{3}} + \lambda \sigma^{-\frac{1}{3}} + 1 \right) [[q^0]]. \end{aligned}$$

□

We will also need the following embedding results for later use:

Lemma 2.4. *The following inequality is valid*

$$\varepsilon^{\frac{1}{4}} \|v^0\|_{\infty} \leq C_{\lambda, \sigma} [u^0, v^0]_B. \quad (2.15)$$

Proof. We compute by Sobolev interpolation, Hardy's inequality (as $v^0(0) = 0$), and using $\sqrt{\varepsilon}y = Y$,

$$\begin{aligned} \|v^0\|_{\infty} &\leq \left\| \frac{v^0}{y} \right\|^{\frac{1}{2}} \left\| y v_y^0 \right\|^{\frac{1}{2}} \lesssim \|v_y^0\|^{\frac{1}{2}} \|y v_y^0\|^{\frac{1}{2}} \lesssim \|v_y^0\|^{\frac{1}{2}} \|y^2 v_{yy}^0\|^{\frac{1}{2}} \\ &\lesssim \varepsilon^{-\frac{1}{4}} \|v_y^0\|^{\frac{1}{2}} \|\langle y \rangle \langle Y \rangle v_{yy}^0\|^{\frac{1}{2}} \lesssim \varepsilon^{-\frac{1}{4}} [u^0, v^0]_B. \end{aligned}$$

□

For later use, we shall record the following:

Corollary 2.5. *For a constant $C = C_{\lambda, \sigma}$ depending on the parameters (λ, σ) ,*

$$[u^0, v^0]_B \leq C_{\lambda, \sigma} ([[q^0]] + [[[v^0]]]) + C(h). \quad (2.16)$$

2.2 | Estimates for $[[q^0]]$ and $[[[v^0]]]$

Define the following:

$$a_0 := \frac{3}{2}u_{sy}v_s - \frac{3}{2}u_s v_{sy}, \quad a_1 := \frac{1}{2}u_{sy}v_{syy} - \frac{1}{2}u_{syyy}v_s. \quad (2.17)$$

Recall the estimates available on u_s, v_s according to Theorem A.3.

Lemma 2.6. *The following estimates are valid*

$$\|u_{syyy}\langle y \rangle\|_1 \lesssim \lambda^4 \sigma^{-1}, \quad (2.18)$$

$$\|a_0\|_\infty + \|a_1\langle y \rangle\|_1 \lesssim \lambda^4 \sigma^{-1}. \quad (2.19)$$

Proof. We decompose the profiles

$$\begin{aligned} u_s &= \bar{u}_p^0 + \bar{u}_e^0 + \sqrt{\varepsilon}u_e + \sqrt{\varepsilon}u_p, \\ v_s &= \bar{v}_p^0 + \bar{v}_e^1 + \sqrt{\varepsilon}v_p + \sqrt{\varepsilon}v_e. \end{aligned} \quad (2.20)$$

The quantities u_e, u_p, v_e, v_p have been defined according to (1.6). The chief properties are that u_e, v_e and u_p decay rapidly in their arguments, whereas v_p is bounded.

Using the decompositions (2.20), we have

$$\begin{aligned} \|u_{syyy}\langle y \rangle\|_1 &\leq \|\bar{u}_{pyyyy}^0\langle y \rangle\|_1 + \|\varepsilon^2 \bar{u}_{eYYY}^0\langle y \rangle\|_1 \\ &\quad + \|\varepsilon^{5/2} u_{eYYY}\langle y \rangle\|_1 + \|\sqrt{\varepsilon} u_{pyyyy}\langle y \rangle\|_1 \\ &\lesssim \lambda^4 \sigma^{-1} + \sqrt{\varepsilon}. \end{aligned}$$

Above, we have used the scaling

$$\|\bar{u}_{pyyyy}^0\langle y \rangle\|_1 = \|\partial_y^4 \left\{ \frac{\lambda^2}{\sigma} u_p(\sigma x, \lambda y) \right\} \langle y \rangle\|_1 = \frac{\lambda^6}{\sigma} \lambda^{-2} = \frac{\lambda^4}{\sigma}.$$

Recall the definition of a_1 in (2.17). Recall further the expansions given in (2.20).

$$\begin{aligned} \|v_s u_{syyy}\langle y \rangle\|_1 &= \|[\bar{v}_p^0 + \bar{v}_e^1 + \sqrt{\varepsilon}v_p + \sqrt{\varepsilon}v_e] \\ &\quad \times [\bar{u}_{pyyy}^0 + \varepsilon^{3/2} \bar{u}_{eYYY}^0 + \varepsilon^2 u_{eYYY} + \sqrt{\varepsilon} u_{pyyy}]\langle y \rangle\|_1 \\ &\leq \|\bar{v}_p^0 \bar{u}_{pyyy}^0\langle y \rangle\|_1 + \|\varepsilon^{3/2} \bar{v}_e^1 \bar{u}_{eYYY}^0\langle y \rangle\|_1 + \sqrt{\varepsilon} \\ &\lesssim \lambda^4 \sigma^{-1} + \sqrt{\varepsilon}. \end{aligned}$$

Note above that

$$\begin{aligned} \|\bar{v}_e^1 \bar{u}_{pyyy}^0\langle y \rangle\|_1 &\leq \|\bar{v}_e^1 \bar{u}_{pyyy}^0\langle y \rangle \chi(Y)\|_1 + \|\bar{v}_e^1 \bar{u}_{pyyy}^0\langle y \rangle \{1 - \chi(Y)\}\|_1 \\ &\lesssim \sqrt{\varepsilon} \|\bar{u}_{pyyy}^0\langle y \rangle^2\|_1 + \varepsilon^\infty, \end{aligned}$$

since $\bar{v}_e^1 \lesssim \sqrt{\varepsilon}$ for $Y \lesssim 1$ while $\bar{u}_{pyyy}^0 \lesssim \varepsilon^\infty$ for $Y \gtrsim 1$.

Next,

$$\begin{aligned}
 \|u_{sy}v_{syy}\langle y \rangle\|_1 &= \|[\bar{u}_{py}^0 + \sqrt{\varepsilon}\bar{u}_{eY}^0 + \varepsilon u_{eY} + \sqrt{\varepsilon}u_{py}] \\
 &\quad \times [\bar{v}_{pyy}^0 + \varepsilon\bar{v}_{eYY}^1 + \sqrt{\varepsilon}v_{pyyy} + \varepsilon^{3/2}v_{eYY}]\langle y \rangle\|_1 \\
 &\lesssim \|\bar{u}_{py}^0\bar{v}_{pyy}^0\langle y \rangle\|_1 + \|\varepsilon^{3/2}\bar{u}_{eY}^0\bar{v}_{eYY}^1\langle y \rangle\|_1 + \sqrt{\varepsilon} \\
 &\lesssim \lambda^4\sigma^{-1} + \sqrt{\varepsilon}.
 \end{aligned}$$

The above computations account for all of the terms in a_1 .

We move now to the pointwise bound of a_0 , from whose definition we obtain

$$\begin{aligned}
 |a_0| &\lesssim |u_{sy}v_s| + |u_s v_{sy}| \\
 &\lesssim |[\bar{u}_{py}^0 + \sqrt{\varepsilon}\bar{u}_{eY}^0 + \varepsilon u_{eY} + \sqrt{\varepsilon}u_{py}]| \times |[v_{\parallel} + \bar{v}_e^1 + \sqrt{\varepsilon}v_p + \sqrt{\varepsilon}v_e]| \\
 &\lesssim |\bar{u}_{py}^0| |\bar{v}_p^0 + \bar{v}_e^1| + \sqrt{\varepsilon} \\
 &\lesssim \lambda^4\sigma^{-1} + |\bar{u}_{py}^0\bar{v}_e^1|\chi(Y) + |\bar{u}_{py}^0\bar{v}_e^1|\{1 - \chi(Y)\} \\
 &\lesssim \lambda^4\sigma^{-1} + \sqrt{\varepsilon} + \varepsilon^\infty.
 \end{aligned}$$

□

We will use these estimates to prove the following lemma.

Lemma 2.7. *Let v^0 be a solution to (2.1). Let $\sigma \ll 1$ in (1.11). Then the following estimate holds*

$$[[q^0]]^2 \lesssim |(F, q^0)|. \quad (2.21)$$

Proof. We use the expression in (1.27). First,

$$\begin{aligned}
 (v_{yyyy}^0 - \{u_s^2 q_y^0\}_y, q^0) &= (u_s q_{yy}^0, q_{yy}^0) + (u_s^2 q_y^0, q_y^0) + (u_{sy} q_y^0, q_y^0)_{y=0} \\
 &\quad - 2(u_{syy} q_y^0, q_y^0) + \frac{1}{2}(u_{yyyy} q^0, q^0) \\
 &\gtrsim [[q^0]]^2.
 \end{aligned} \quad (2.22)$$

Above, we have used (1.11), (2.7), and (2.8) paired with (2.18) and (2.19) to estimate the last two terms by

$$|(2.22.4)| + |(2.22.5)| \lesssim \frac{\lambda^4}{\sigma} (\sigma^{\frac{2}{3}} \lambda^{-2} [[q^0]]^2) = \sigma^{\frac{1}{3}} [[q^0]]^2 = o(1) [[q^0]]^2,$$

upon invoking the assumption that $\sigma \ll 1$.

To prove the identity (2.22) we record

$$\begin{aligned}
 (v_{yyyy}^0, q^0) &= -(v_{yy}^0, q_y^0) \\
 &= (v_{yy}^0, q_{yy}^0) + (v_{yy}^0, q_y^0)_{y=0}
 \end{aligned}$$

$$\begin{aligned}
 &= (\partial_{yy} \{u_s q^0\}, q_{yy}^0) + (2u_{sy} q_y^0, q_y^0)_{y=0} \\
 &= (u_s q_{yy}^0 + 2u_{sy} q_y^0 + u_{syy} q^0, q_{yy}^0) + (2u_{sy} q_y^0, q_y^0)_{y=0} \\
 &= (u_s q_{yy}^0, q_{yy}^0) - (u_{syy} q_y^0, q_y^0) - (u_{sy} q_y^0, q_y^0)_{y=0} \\
 &\quad - (u_{syy} q_y^0, q_y^0) - (u_{syyy} q^0, q_y^0) + (2u_{sy} q_y^0, q_y^0)_{y=0} \\
 &= (u_s q_{yy}^0, q_{yy}^0) + (u_{sy} q_y^0, q_y^0)_{y=0} - (2u_{syy} q_y^0, q_y^0) + \frac{1}{2} (u_{syyy} q^0, q^0).
 \end{aligned}$$

For the next term from (1.27), we record the integration by parts identity and estimate due to (2.7), (2.8), and (2.19)

$$\begin{aligned}
 | -(\{v_s v_{yyy}^0 - v_y^0 v_{syy}\}, q^0) | &= |(a_0 q_y^0, q_y^0) + (a_1 q^0, q^0)| \\
 &\lesssim \lambda^4 \sigma^{-1} \left(\lambda^{-2} \sigma^{\frac{2}{3}} [[q^0]] \right)^2 \\
 &= \sigma^{\frac{1}{3}} [[q^0]]^2 = o(1) [[q^0]]^2,
 \end{aligned} \tag{2.23}$$

upon invoking the assumption that $\sigma \ll 1$.

To prove the equality in (2.23), we record the following integrations by parts:

$$\begin{aligned}
 -(v_s v_{yyy}^0, q^0) &= (v_{sy} v_{yy}^0, q^0) + (v_s v_{yy}^0, q_y^0) \\
 &= (v_{sy} [u_s q_{yy}^0 + 2u_{sy} q_y^0 + u_{syy} q^0], q^0) \\
 &\quad + (v_s [u_s q_{yy}^0 + 2u_{sy} q_y^0 + u_{syy} q^0], q_y^0) \\
 &= -((v_{sy} u_s)_y q_y^0, q^0) - (u_s v_{sy} q_y^0, q_y^0) - ((u_{sy} v_{sy})_y q^0, q^0) \\
 &\quad + (v_{sy} u_{syy} q^0, q^0) - \frac{1}{2} ((u_s v_s)_y q_y^0, q_y^0) + 2(u_{sy} v_s q_y^0, q_y^0) \\
 &\quad - \frac{1}{2} ((u_{syy} v_s)_y q^0, q^0) \\
 &= \frac{1}{2} ((v_{sy} u_s)_{yy} q^0, q^0) - (u_s v_{sy} q_y^0, q_y^0) - ((u_{sy} v_{sy})_y q^0, q^0) \\
 &\quad + (v_{sy} u_{syy} q^0, q^0) - \frac{1}{2} ((u_s v_s)_y q_y^0, q_y^0) + 2(u_{sy} v_s q_y^0, q_y^0) \\
 &\quad - \frac{1}{2} ((u_{syy} v_s)_y q^0, q^0) \\
 &= \frac{1}{2} (\{u_s v_{syyy} - v_s u_{syyy}\} q^0, q^0) + \frac{3}{2} (\{u_{sy} v_s - v_{sy} u_s\} q_y^0, q_y^0)
 \end{aligned} \tag{2.24}$$

We record the second integration by parts:

$$\begin{aligned}
 (v_{syy} v_y^0, q^0) &= (v_{syy} u_{sy} q^0, q^0) + (v_{syy} u_s q_y^0, q^0) \\
 &= (v_{syy} u_{sy} q^0, q^0) - \frac{1}{2} (\partial_y \{u_s v_{syy}\} q^0, q^0).
 \end{aligned} \tag{2.25}$$

Combining (2.24) and (2.25) with the definition of a_0, a_1 given in (2.17) proves the equality in (2.23).

We now treat the final two terms in (1.27). First, we insert (2.8) to obtain

$$\begin{aligned} |(\varepsilon u_{sxx} u_s q^0, q^0)| &\leq (\varepsilon u_{pxx}^i u_s q^0, q^0) + (\varepsilon^{3/2} u_{exx}^i u_s q^0, q^0) \\ &\lesssim (\| \varepsilon u_{pxx}^i \langle y \rangle \|_1 + \varepsilon^{3/2} \| u_{exx}^i \langle y \rangle \|_1) [[q^0]]^2 \\ &\lesssim \sqrt{\varepsilon} [[q^0]]^2. \end{aligned}$$

A similar estimate is available for the final term upon integrating by parts:

$$\begin{aligned} (\varepsilon v_{sxx} u_y^0, q^0) &= (\varepsilon v_{sxx} u_{sy} q^0, q^0) + (\varepsilon v_{sxx} u_s q_y^0, q^0) \\ &= (\varepsilon v_{sxx} u_{sy} q^0, q^0) - \left(\frac{\varepsilon}{2} \partial_y \{v_{sxx} u_s\} q^0, q^0 \right). \end{aligned}$$

The right-hand side clearly contributes $|(F, q^0)|$. This completes the proof. $\square$

Lemma 2.8. *Let v^0 be a solution to (2.1). Then the following estimate holds*

$$[[[v^0]]]^2 \lesssim [[q^0]]^2 + \|F w_0\|^2. \quad (2.26)$$

Proof. We take the inner product of $v_{yyyy}^0 w_0^2$ with (2.1). Clearly, the v_{yyy}^0 term in $\mathcal{L}$ produces coercivity over $\|v_{yyyy}^0 w_0\|^2$.

According to (2.1), the next term from $\mathcal{L}$ is

$$\begin{aligned} -(u_s v_{yy}^0, v_{yyyy}^0 w_0^2) &= (u_s w_0^2 v_{yyy}^0, v_{yyy}^0) + (\{u_s w_0^2\}_y v_{yy}^0, v_{yyy}^0) \\ &= (u_s w_0^2 v_{yyy}^0, v_{yyy}^0) + (u_{sy} w_0^2 v_{yy}^0, v_{yyy}^0) + (u_s \{w_0^2\}_y v_{yy}^0, v_{yyy}^0) \\ &= (u_s w_0^2 v_{yyy}^0, v_{yyy}^0) + (u_{sy} w_0^2 v_{yy}^0, v_{yyy}^0) - \frac{1}{2} (u_{sy} \{w_0^2\}_y v_{yy}^0, v_{yyy}^0) \\ &\quad - \frac{1}{2} (u_s \{w_0^2\}_{yy} v_{yy}^0, v_{yyy}^0) \\ &\gtrsim \sqrt{u_s} v_{yyy}^0 w_0^2 - \|u_{sy} w_0\|_\infty \|v_{yy}^0\| \|v_{yyy}^0 w_0\| \\ &\quad - \|u_{sy} \{w_0^2\}_y\|_\infty \|v_{yy}^0\|^2 - \|u_s\|_\infty \|\{w_0^2\}_{yy} v_{yy}^0\|^2 \\ &\gtrsim \sqrt{u_s} v_{yyy}^0 w_0^2 - \|u_{sy} w_0\|_\infty \|v_{yy}^0\| \|v_{yyy}^0 w_0\| \\ &\quad - \|u_{sy} \{w_0^2\}_y\|_\infty \|v_{yy}^0\|^2 - o(1) \|v_{yy}^0 w_0\|^2 - [[q^0]]^2. \end{aligned} \quad (2.27)$$

The next term from $\mathcal{L}$ in (2.1) is $u_{syy} v^0$, which we combine with $\varepsilon u_{sxx} v^0$ (the sixth term in $\mathcal{L}$ in (2.1)) to produce $v^0 \Delta_\varepsilon u_s$. We treat this via:

$$\begin{aligned} (v^0 \Delta_\varepsilon u_s, v_{yyyy}^0 w_0^2) &= (\Delta_\varepsilon u_s u_s q^0, v_{yyyy}^0 w_0^2) \\ &= (u_{syy} u_s q^0, v_{yyyy}^0 w_0^2) + (\varepsilon u_{sxx} u_s q^0, v_{yyyy}^0 w_0^2) \\ &\leq \|\Delta_\varepsilon u_s w_0 \sqrt{\langle y \rangle}\|_\infty [[q^0]] \|v_{yyyy}^0 w_0\| \\ &\lesssim [[q^0]] \|v_{yyyy}^0 w_0\|. \end{aligned}$$

Next, we integrate by parts, using that $v_{sy}|_{y=0} = 0$, to obtain

$$\begin{aligned} -(v_s v_{yyy}^0, v_{yyyy}^0 w_0^2) &= \frac{1}{2}(\partial_y \{v_s w_0^2\} v_{yyy}^0, v_{yyy}^0) \\ &= -\frac{1}{2}(\partial_{yy} \{v_s w_0^2\} v_{yyy}^0, v_{yy}^0) - \frac{1}{2}(\partial_y \{v_s w_0^2\} v_{yyyy}^0, v_{yy}^0) \\ &\lesssim \|\partial_{yy} \{v_s w_0^2\}\|_\infty \|v_{yyy}^0\| \|v_{yy}^0\| + \|v_{yyyy}^0 w_0\| \|v_{yy}^0\| \|v_{sy} w_0 + v_s w_{0y}\|_\infty \\ &\lesssim o(1)[[v^0]]^2 + [[q^0]]^2. \end{aligned}$$

Next, we arrive at

$$\begin{aligned} |(v_{syy} v_y^0, v_{yyyy}^0 w_0^2)| &\leq (\|v_{syy} u_s q_y^0 w_0\| + \|v_{syy} u_{sy} q^0 w_0\|) \|v_{yyyy}^0 w_0\| \\ &\leq (\|v_{syy} w_0\|_\infty \|u_s q_y^0\| + \|q^0 \langle y \rangle^{-\frac{1}{2}}\|_\infty \|v_{syy} u_{sy} w_0\|) \|v_{yyyy}^0 w_0\| \\ &\lesssim [[q^0]] [[v^0]]. \end{aligned}$$

Next, we arrive at

$$\begin{aligned} |(\varepsilon v_{sxx} v_y^0, v_{yyy}^0 w_0^2)| &= |(\varepsilon v_{sxx} \{u_s q_y^0 + u_{sy} q^0\}, v_{yyy}^0 w_0^2)| \\ &\leq (\sqrt{\varepsilon} \|\sqrt{\varepsilon} w_0 v_{sxx}\|_\infty \|u_s q_y^0\| + \varepsilon^{\frac{1}{2}} \|v_{sxx} u_{sy} w_0 y\|_\infty [[q^0]]) \|v_{yyy}^0 w_0\| \end{aligned}$$

The remaining step is to absorb the $o(1)\|v_{yy}^0 w_0\|$ appearing above in (2.27). Thanks to the $o(1)$ factor, it suffices to rearrange (2.1) to obtain

$$\begin{aligned} \|u_s v_{yy}^0 w_0\| &\leq \|v_{yyyy}^0 w_0\| + \|F w_0\| + \|u_{syy} w_0 \langle y \rangle\| \|\frac{v^0}{\langle y \rangle}\| + \|v_s v_{yyy}^0 w_0\| \\ &\quad + \|v_{syy} w_0\|_\infty \|v_y^0\| + \|\varepsilon u_{sxx} w_0 \langle y \rangle\|_\infty \|\frac{v^0}{\langle y \rangle}\| + \|\varepsilon v_{sxx} w_0\|_\infty \|v_y^0\| \\ &\lesssim \|v_{yyyy}^0 w_0\| + \|v_{yyy}^0 w_0\| + \|F w_0\| + [[q^0]] \\ &\lesssim [[v^0]] + \|F w_0\| + [[q^0]]. \end{aligned}$$

To conclude the proof, the right-hand side clearly contributes $|(F, v_{yyyy}^0 w_0^2)| \lesssim \|F w_0\| \|v_{yyy}^0 w_0\|$. $\square$

2.3 | Existence and uniqueness

We now establish existence and uniqueness for the system (2.1). First, consider the operator:

$$\begin{aligned} \mathcal{L}_0 v^0 &= F, \quad v^0(0) = v_y^0(0) = \partial_y^k v^0(\infty) = 0 \text{ for } k \geq 1, \\ \mathcal{L}_0 v^0 &:= v_{yyyy}^0 - u_{\parallel}^\infty v_{yy}^0. \end{aligned} \tag{2.28}$$

Lemma 2.9. Assume $F \in C_0^\infty$. There exists a unique solution v^0 to the problem (2.28). Moreover, v^0 is given by the expression $v^0 = C_1 + C_2 e^{-\sqrt{u_\parallel^\infty} y} + u_p[F]$, where $u_p[F]$ is the particular solution defined below.

Proof. The characteristic equation is $r^4 - u_\parallel^\infty r^2 = 0$. The roots thus correspond to the basis solutions $\{v_1^0, v_2^0, v_3^0, v_4^0\} := \{1, y, e^{ry}, e^{-ry}\}$ where $r = \sqrt{u_\parallel^\infty}$.

$$\begin{aligned} \mathbf{W}(y) &= \begin{bmatrix} 1 & y & e^{ry} & e^{-ry} \\ 0 & 1 & re^{ry} & -re^{-ry} \\ 0 & 0 & r^2 e^{ry} & r^2 e^{-ry} \\ 0 & 0 & r^3 e^{ry} & -r^3 e^{-ry} \end{bmatrix} & \mathbf{W}_1(y) &= \begin{bmatrix} 0 & y & e^{ry} & e^{-ry} \\ 0 & 1 & re^{ry} & -re^{-ry} \\ 0 & 0 & r^2 e^{ry} & r^2 e^{-ry} \\ F & 0 & r^3 e^{ry} & -r^3 e^{-ry} \end{bmatrix} \\ \mathbf{W}_2(y) &= \begin{bmatrix} 1 & 0 & e^{ry} & e^{-ry} \\ 0 & 0 & re^{ry} & -re^{-ry} \\ 0 & 0 & r^2 e^{ry} & r^2 e^{-ry} \\ 0 & F & r^3 e^{ry} & -r^3 e^{-ry} \end{bmatrix} & \mathbf{W}_3(y) &= \begin{bmatrix} 1 & y & 0 & e^{-ry} \\ 0 & 1 & 0 & -re^{-ry} \\ 0 & 0 & 0 & r^2 e^{-ry} \\ 0 & 0 & F & -r^3 e^{-ry} \end{bmatrix} \\ \mathbf{W}_4(y) &= \begin{bmatrix} 1 & y & e^{ry} & 0 \\ 0 & 1 & re^{ry} & 0 \\ 0 & 0 & r^2 e^{ry} & 0 \\ 0 & 0 & r^3 e^{ry} & F \end{bmatrix} \end{aligned}$$

Let $W(y) = |\mathbf{W}|$ and $W_i(y) = |\mathbf{W}_i|$. Define

$$c_i[F](y) = - \int_y^\infty \frac{W_i(z)}{W(z)} dz$$

As F has compact support, it is clear that c_i and its derivatives decay rapidly at $y = \infty$. The full solution is thus given by $v^0 = C_1 + C_2 e^{-ry} + u_p[F]$, where $u_p[F]$ is the particular solution $u_p[F] := \sum c_i[F] v_i^0$. We achieve the boundary conditions by solving $C_1 + C_2 + u_p[F](0) = 0$ and $C_1 - rC_2 + \partial_y u_p[F](0) = 0$. $\square$

We now quantify the space in which v^0 lives. To do so, define

$$\begin{aligned} \|v^0\|_T &:= \|v_{yyyy}^0\| + \|v_{yyy}^0\| + \|v_{yy}^0\| + \|v_y^0\| + \left\| \frac{v^0}{y} \right\|, \\ \|v^0\|_{T_s} &:= \|v_{yyyy}^0 e^{sy}\| + \|v_{yyy}^0 e^{sy}\| + \|v_{yy}^0 e^{sy}\| + \|v_y^0\|, \\ \|v^0\|_{\hat{T}} &:= \|v_{yyy}^0\| + \|v_{yy}^0\| + \|v_y^0\| + \left\| \frac{v^0}{y} \right\| \end{aligned}$$

Lemma 2.10. Let $F \in C_0^\infty$. Then $v^0 \in T, T_s$ and the following estimate is valid

$$\|v^0\|_T \lesssim \|F w_0\|, \text{ and } \|v^0\|_{T_s} \lesssim \|F e^{sy}\|,$$

for $0 < s < r$.

Proof. We square and integrate the equation $\|\mathcal{L}_0 v^0\|^2 = \|F\|^2$. It is immediate to see that

$$\|\mathcal{L}_0 v^0\|^2 = \|v_{yyyy}^0\|^2 + 2u_{yy}^\infty \|v_{yyy}^0\|^2 + |u_{yy}^\infty|^2 \|v_{yy}^0\|^2 + 2u_{yy}^\infty v_{yy}^0(0)v_{yy}^0(0).$$

Next, one takes inner product with v^0 to obtain control over $\|v_y^0\|^2 + \|v_{yy}^0\|^2$, whereas on the right hand side one uses Hardy inequality via $|(F, v^0)| \lesssim \|F\| \|v_y^0\|$. We may repeat the first step with weights e^{sy} , and all integrations by parts are justified since $s < r$. $\square$

We now remove the compact support assumption on F .

Lemma 2.11. *Let $F \in L^2(w_0)$. Then there exists a solution $v^0 \in T$ satisfying $\|v^0\|_T \lesssim \|Fw_0\|$. Let $F \in L^2(e^{sy})$. Then $v^0 \in T_s$ satisfying the estimate $\|v^0\|_{T_s} \lesssim \|Fe^{sy}\|$.*

Proof. This follows from a straightforward density argument. $\square$

The final step is to add on the perturbations from $\mathcal{L}$ to $\mathcal{L}_0$. To do so, write $\mathcal{L} = \mathcal{L}_0 + K$, where

$$Kv^0 = (u_s - u_s^\infty)v_{yy}^0 - u_{syy}v^0 - v_{syy}v_{yy}^0 - v_{syy}v_y^0 - \varepsilon u_{sxx}v^0 + \varepsilon v_{sxx}v_y^0$$

Lemma 2.12. *Let $F \in L^2(w_0)$. Assume the operator $\mathcal{L}$ satisfies the a-priori bound $\|\mathcal{L}v^0w_0\| \gtrsim \|v^0\|_T$. Then there exists a unique solution $v^0 \in T$ which satisfies the bound $\|v^0\|_T \lesssim \|Fw_0\|$.*

Proof. We note first that $\mathcal{L}_0^{-1}K$ is a compact operator on $\tilde{T}$. Indeed, letting $v^0 \in \tilde{T}$, we see that $Kv^0 \in L^2e^{sy}$ for some $0 < s$. Thus, we may apply $\mathcal{L}_0^{-1}$ which brings $\mathcal{L}_0^{-1}Kv^0$ into T_s , which is compactly embedded in $\tilde{T}$. We thus apply the Fredholm alternative so that we must rule out nontrivial solutions to the homogeneous problem $\mathcal{L}_0v^0 = -Kv^0$. Since $v^0 \in \tilde{T}$, we bootstrap to conclude $v^0 \in T_s$. We may subsequently apply the assumed a-priori bound on $\mathcal{L}$ to conclude that $v^0 = 0$ is the only solution. $\square$

Proof of Proposition 2.1. Estimate (2.6) is obtained by combining (2.16) with (2.21) and (2.26). Together with Lemma 2.12 (whose hypotheses are verified by estimate (2.6)), this concludes the proof of the proposition. $\square$

3 | FORMULATION OF DNS

3.1 | Solvability of DNS

The main object of study in this section, motivated by (A.27), will be the following system:

$$\begin{aligned} -\partial_x R[q] + \Delta_\varepsilon^2 v + J(v) &= F \\ v_{xxx}|_{x=L} &= v_x|_{x=L} = 0 \text{ and } v|_{x=0} = v_{xx}|_{x=0} = 0, \\ v|_{y=0} &= v_y|_{y=0} = v|_{y \uparrow \infty} = 0 \end{aligned} \tag{3.1}$$

The above F serves as an abstract forcing for this section. Recall the definition of $R[q]$ given in (1.13), and of $J(v)$ given in (1.29).

Recall that $q = \frac{v}{u_s}$ from (1.13). Define:

$$\tilde{u} := u - u^0 = \int_0^x -v_y(x', y) dx' := I_x[-v_y]. \quad (3.2)$$

We will record now identities regarding the boundary conditions for q :

$$q_x|_{x=L} = -\frac{u_{sx}}{u_s} q|_{x=L}, \quad q_{xx}|_{x=0} = -2\frac{u_{sx}}{u_s} q_x|_{x=0}. \quad (3.3)$$

Define our ambient function space via:

$$H_0^4 := \{v \in H^4 : (3.1) \text{ is satisfied.}\}$$

We want to establish existence for v as a solution to the system (3.1). We will define now several function spaces which will enable us to state the existence theorem.

Definition 3.1 (Function Spaces). Fix any weight, $w(y) \in C^\infty(\mathbb{R}_+)$.

$$\|v\|_{L^2(w)} := \|v \cdot w\|, \quad \|v\|_{\dot{H}_\varepsilon^k(w)} := \|\nabla_\varepsilon^k v\|_{L^2(w)}, \quad \|v\|_{H_\varepsilon^k(w)} := \sup_{0 \leq j \leq k} \|v\|_{\dot{H}_\varepsilon^j(w)},$$

$$\|v\|_{\dot{H}_{\varepsilon,d}^4(w)} = \|\{v_{yyyy}, \varepsilon v_{xxyy}, \varepsilon^{\frac{3}{2}} v_{xxxy}, \varepsilon^2 v_{xxxx}\} \cdot w\| + \|\sqrt{u_s} \sqrt{\varepsilon} v_{xyyy} \cdot w\|,$$

$$\|v\|_{H_{\varepsilon,d}^4(w)} = \|v\|_{\dot{H}_{\varepsilon,d}^4(w)} + \|v\|_{H_\varepsilon^3(w)}.$$

We adopt the convention that $\|v\|_{H_\varepsilon^0(w)} := \|\sqrt{\varepsilon} v \cdot w\|$, and that when w is left unspecified, $w = 1$. The relevant class of test functions is $C_V^\infty := \{\phi \in C^\infty : \phi(0) = 0 \text{ and } \partial_x \phi = 0 \text{ in a neighborhood of } x = 0, \text{ and are compactly supported in } y\}$. The following spaces are defined: $H_\varepsilon^2(w) := \overline{C_V^\infty}^{\|\cdot\|_{H_\varepsilon^2(w)}}$, and $X_w := \{v \in H_\varepsilon^4 : \|v\|_{X_w} < \infty\}$, where $\|\cdot\|_{X_w}$ has been defined in (1.22).

We now define notation for several operators:

$$\begin{aligned} J^0(v) &:= \partial_x(-[u_s - u_s(\infty)]v_{yy} + u_{syy}v) + \varepsilon \partial_x(-[u_s - u_s(\infty)]v_{xx} \\ &\quad + u_{sxx}v) + \partial_x(v_s(I_x[-v_{yyy}] - \varepsilon v_{xy}) - I_x[-v_y]\Delta_\varepsilon v_s) \end{aligned} \quad (3.4)$$

$$D_N(v) := \Delta_\varepsilon^2 v - u_s(\infty) \chi\left(\frac{y}{N}\right) \Delta_\varepsilon v_x, \quad D(v) := D_\infty(v). \quad (3.5)$$

We now prove the following result, where $\|v\|_{X_1}, \|v\|_{Y_{w_0}}$ are defined in (1.22):

Proposition 3.2. Assume $v \in H^4$ satisfies the a-priori estimate:

$$\|v\|_{X_1} \lesssim C_\varepsilon \|F\| \text{ and } \|v\|_{Y_{w_0}} \lesssim C_\varepsilon \|Fw_0\| \quad (3.6)$$

for solutions $v \in X_1 \cap Y_{w_0}$ to (3.7).

Then there exists a unique solution $v \in X_1 \cap Y_{w_0}$ to the problem:

$$\begin{aligned} \Delta_\varepsilon^2 v - \partial_x R[q] + J(v) &= F, \\ v_x|_{x=L} &= v_{xxx}|_{x=L} = 0, \quad v|_{x=0} = v_{xx}|_{x=0} = 0, \\ v|_{y=0} &= v_y|_{y=0} = 0, \quad v|_{y \rightarrow \infty} = 0. \end{aligned} \quad (3.7)$$

The first step is to invert the highest-order operator, Δ_ε^2 . In so doing, the first point is the existence of a finite-energy solution:

Lemma 3.3. *Given $F \in L^2$, there exists a unique H_ε^4 solution to $\Delta_\varepsilon^2 v = F$ with boundary conditions from (3.7). Moreover, for any w satisfying $|\partial_y^k w| \lesssim |w|$, this v satisfies the following estimates:*

$$\left\| \left\{ \varepsilon v_{xyyy}, \varepsilon^{\frac{3}{2}} v_{xxxy}, \varepsilon^2 v_{xxxx} \right\} w \right\|^2 - \varepsilon \| |q| \|_w^2 \lesssim |(F, \varepsilon^2 v_{xxxx} w)|, \quad (3.8)$$

$$\begin{aligned} & \left\| \left\{ \sqrt{\varepsilon} v_{xyyy}, \varepsilon v_{xxxy}, \varepsilon^{\frac{3}{2}} v_{xxxx} \right\} \sqrt{u_s} w \right\|^2 - \varepsilon \| |q| \|_1^2 \\ & - \| |q| \|_{\sqrt{\varepsilon} w}^2 \lesssim |(F, \varepsilon u_s v_{xyyy} w)|. \end{aligned} \quad (3.9)$$

Proof. Fix $f^m \in C_C^\infty$ such that $\|F - f_m\| \xrightarrow{L^2} 0$. Let $\tilde{f}^m$ denote the even extension over $x = L$, which satisfies $\tilde{f}^m(0) = \tilde{f}^m(2L) = 0$. We may now expand $\tilde{f}^m$ periodically in a Fourier sine series: $\tilde{f}^m = \sum_n \sin(n \frac{\pi}{2L} x)$. Since $\tilde{f}$ is even across $x = L$, only the n -odd coefficients remain. We now solve the equation $\Delta_\varepsilon^2 \tilde{v}^m = \tilde{f}^m$ on $\mathbb{H}$. Thus:

$$\tilde{f}^m = \sum_{n \text{ odd}} f_n^m(y) \sin\left(n \frac{\pi}{2L} x\right), \quad \tilde{v}^m = \sum_{n \text{ odd}} v_n^m(y) \sin\left(n \frac{\pi}{2L} x\right).$$

We thus obtain the following ODEs:

$$(v_n^m)'''' - 2\varepsilon \left(\frac{\pi}{2L}\right)^2 n^2 (v_n^m)'' + \varepsilon^2 n^4 \left(\frac{\pi}{2L}\right)^4 (v_n^m) = f_n^m \text{ for } n \neq 0 \text{ and } n \text{ odd}. \quad (3.10)$$

Note that $f_{n=0}^m = 0$ since $\tilde{f}^m$ is odd. For each fixed n , we solve the above ODE using Lax Milgram. Precisely, define the bilinear form:

$$B_n[v, \phi] := (v'', \phi'') - 2\varepsilon \left(\frac{\pi}{2L}\right)^2 (nv', n\phi') + \varepsilon^2 \left(\frac{\pi}{2L}\right)^4 (n^2 v, n^2 \phi) : H_y^2 \times H_y^2 \rightarrow \mathbb{R}.$$

First, for $n \neq 0$, B_n is coercive over H_y^2 since $B_n[v, v] = |v''|^2 + 2\varepsilon \left(\frac{\pi}{2L}\right)^2 n^2 |v'|^2 + \varepsilon^2 \left(\frac{\pi}{2L}\right)^4 n^2 |v|^2$. Similarly, B_n is bounded on $H_y^2 \times H_y^2$. Summing in n yields the estimate $\|\tilde{v}^m\|_{H_\varepsilon^2} \lesssim \|\tilde{f}^m\|$.

We now estimate $\|v_{xyyy}^m\|$. Integration by parts in y and appealing to the trace theorem in $\mathbb{R}_+$ produces:

$$\begin{aligned} n^2 \|(v_n^m)_{yyy}\|^2 &= (n^2 v_{yy}^n, v_{yyy}^n) + \left(n^{\frac{1}{2}} v_{yyy}^n(0), n^{\frac{3}{2}} v_{yy}^n(0) \right) \\ &\leq \|n^2 v_{yy}^n\| \|v_{yyy}^n\| + |n^{\frac{1}{2}} v_{yyy}^n(0)| |n^{\frac{3}{2}} v_{yy}^n(0)| \\ &\leq \|n^2 v_{yy}^n\| \|v_{yyy}^n\| + \|n v_{yyy}^n\|^{\frac{1}{2}} \|v_{yyy}^n\|^{\frac{1}{2}} \|n^2 v_{yy}^n\|^{\frac{1}{2}} \|n v_{yy}^n\|^{\frac{1}{2}}. \end{aligned}$$

Taking summation over n gives and applying Young's inequality for products with exponents $\frac{1}{4} + \frac{1}{(4/3)} = 1$:

$$\begin{aligned} \|v_{xyy}^m\|^2 &\lesssim \|v_{xxy}^m\| \|v_{yyy}^m\| + \|v_{xyy}^m\|^{\frac{1}{2}} \|v_{yyy}^m\|^{\frac{1}{2}} \|v_{xxy}^m\|^{\frac{1}{2}} \|v_{xyy}^m\|^{\frac{1}{2}} \\ &\lesssim \|v_{xxy}^m\| \|v_{yyy}^m\| + \kappa \left(\|v_{xyy}^m\|^{\frac{1}{2}} \right)^4 + N_\kappa \left(\|v_{yyy}^m\|^{\frac{1}{2}} \|v_{xxy}^m\|^{\frac{1}{2}} \|v_{xyy}^m\|^{\frac{1}{2}} \right)^{\frac{4}{3}}. \end{aligned}$$

Multiplying by v_{xxx}^m produces the bound: $\|\varepsilon v_{xxy}^m, \varepsilon^{\frac{3}{2}} v_{xxy}^m, \varepsilon^2 v_{xxx}^m\|^2 \lesssim \|\bar{f}^m\|^2$. We use the equation to estimate $\|v_{yyy}^m\|$. This then concludes the full H_ε^4 bound.

That $\bar{v}^m$ is in $C^\infty(\mathbb{H})$ follows by multiplying (3.10) by factors of n^j , summing in n , and using that f^m is smooth to ensure summability of the right-hand side $\sum_n n^{2j} \|f_n^m\|^2 < \infty$.

That $v^m(0) = v_{xx}^m(0) = 0$ is guaranteed by the fact that v^m is a Fourier sine series and $v_x^m(L) = v_{xxx}^m(L) = 0$ is guaranteed by the fact that only odd n coefficients are nonzero.

We turn now to the estimate (3.8). Integrating by parts produces:

$$\begin{aligned} (\Delta_\varepsilon^2 v^m, \varepsilon^2 v_{xxx}^m w^2) &= \|\{\varepsilon v_{xxy}^m, 2\varepsilon^{\frac{3}{2}} v_{xxy}^m, \varepsilon^2 v_{xxx}^m\} w\|^2 \\ &\quad - 4\|\varepsilon v_{xxy}^m \sqrt{(|w_y|^2 + ww_{yy})}\|^2 + \|\varepsilon v_{xx}^m \sqrt{\langle \partial_{yy}\{ww_{yy}\} + \partial_{yy}\{|w_y|^2\}}\|^2 \\ &\quad - 2\|\varepsilon^{\frac{3}{2}} v_{xxx}^m \sqrt{(|w_y|^2 + ww_{yy})}\|^2. \end{aligned}$$

On the right-hand side, we have $(f^m, \varepsilon^2 v_{xxx}^m w^2)$. As $f^m \rightarrow f$ in L^2 , and $v_{xxx}^m \rightharpoonup v_{xxx}$ weakly in L^2 , we may pass to the limit in the inner product. From here, (3.8) follows immediately.

We turn now to (3.9): We integrate by parts the Δ_ε^2 terms:

$$\begin{aligned} (\Delta_\varepsilon^2 v^m, \varepsilon v_{xxy}^m u_s w^2) &= -(\varepsilon v_{xxy}^m w^2, \partial_x \{u_s v_{yyy}^m\}) + 2\|\varepsilon v_{xxy}^m \sqrt{u_s} w\|^2 \\ &\quad - (\varepsilon^3 v_{xxx}^m w^2, \partial_x \{u_s v_{xxy}^m\}) \\ &= -(\varepsilon u_s v_{xxy}^m, v_{xyyy}^m w^2) - (\varepsilon u_{sx} v_{xyy}^m, v_{yyyy}^m w^2) \\ &\quad + 2\|\varepsilon v_{xxy}^m \sqrt{u_s} w\|^2 - (\varepsilon^3 u_{sx} v_{xxx}^m, v_{xxyy}^m w^2) \\ &\quad - (\varepsilon^3 u_s v_{xxx}^m, v_{xxxy}^m w^2) \\ &= \|\sqrt{\varepsilon} v_{xyyy}^m w \sqrt{u_s}\|^2 + (\varepsilon v_{xyyy}^m, v_{xyy}^m \partial_y \{w^2 u_s\}) \\ &\quad + (\varepsilon v_{yyy}^m, v_{xyyy}^m w^2 u_{sx}) + (\varepsilon v_{yyy}^m, v_{xyy}^m \partial_y \{w^2 u_{sx}\}) \\ &\quad + 2\|\varepsilon v_{xxy}^m \sqrt{u_s} w\|^2 + (\varepsilon^3 u_{sx} v_{xxy}^m, v_{xxxy}^m w^2) \\ &\quad + (2\varepsilon^3 u_{sx} v_{xxy}^m, v_{xxx}^m ww_y) + (\varepsilon^3 u_{sxy} v_{xxy}^m, v_{xxx}^m w^2) \\ &\quad + \|\varepsilon^{\frac{3}{2}} \sqrt{u_s} v_{xxxy}^m w\|^2 + (\varepsilon^3 u_{sy} v_{xxx}^m, v_{xxxy}^m w^2) \\ &\quad + (2\varepsilon^3 u_s v_{xxx}^m, v_{xxxy}^m ww_y) \end{aligned}$$

$$\begin{aligned} &\geq |\sqrt{u_s} \{ \sqrt{\varepsilon} v_{xyy}^m, 2\varepsilon v_{xxy}^m, \varepsilon^{\frac{3}{2}} v_{xxx}^m \} w|^2 - \varepsilon |||q^m|||_1^2 \\ &\quad - |||q^m|||_{\sqrt{\varepsilon} w}^2 \end{aligned}$$

We have used the bound $|w_y| \lesssim |w|$ and Young's inequality for products to perform the above estimate. We again pass to the limit as $m \rightarrow \infty$ in the same manner as for (3.8). $\square$

Lemma 3.4. *Let $F \in L^2(\langle y \rangle^m)$ for some $1 \leq m < \infty$. Then $\|v\|_{H_{\varepsilon,d}^4(\langle y \rangle^m)} \leq C_\varepsilon \|F\|_{L^2(\langle y \rangle^m)}$. In particular, in the case when $F \in L^2 \cap L^2(w_0)$, $v = (\Delta_\varepsilon^2)^{-1} F \in X_1 \cap Y_{w_0}$.*

Proof. This follows from standard polynomial-type weighted estimates, and we omit the proof. $\square$

We will now study the perturbation in two steps.

Lemma 3.5. *The map $D^{-1} : L^2 \rightarrow H^4$ is well-defined.*

Proof. Consider the map $D_N(v) = F \in L^2$. By calling $v_0 = \Delta_\varepsilon^2 v$, we may rewrite the equation as $v_0 + \chi_N(y) u_s(\infty) \Delta_\varepsilon \Delta_\varepsilon^{-2} v_0 = F$. We will study this as an equality in L^2 , and it is clear that $\chi_N(y) u_s(\infty) \Delta_\varepsilon \Delta_\varepsilon^{-2}$ is a compact operator on L^2 due to the cutoff function. Therefore, by the Fredholm alternative, to establish solvability of D_N , we must prove uniqueness of the homogeneous solution. This follows by performing an energy estimate:

$$(\Delta_\varepsilon^2 v_x, v_{xx}) - u_s(\infty) (\chi_N(y) \Delta_\varepsilon v_x, v_{xx}) = (F, v_{xx})$$

The Bilaplacian term produces the quantities $- \|v_{xyy}, 2\sqrt{\varepsilon} v_{xxy}, \varepsilon v_{xxx}\|^2$.

Next, assuming $N = \varepsilon^{-\infty}$, we have:

$$\begin{aligned} -u_s(\infty) (\chi_N \Delta_\varepsilon v_x, v_{xx}) &= -\frac{u_s(\infty)}{2} [|v_{xy}(0) \sqrt{\chi_N}|^2 + |\sqrt{\varepsilon} v_{xx}(L) \sqrt{\chi_N}|^2 \\ &\quad + \frac{1}{N} (v_{xy}, v_{xx} \chi_N')]. \end{aligned}$$

Note that $u_s(\infty) > 0$. Thus, the operator is coercive over the quantities

$$\|v_{xyy}, 2\sqrt{\varepsilon} v_{xxy}, \varepsilon v_{xxx}\|^2 + [|v_{xy}(0) \sqrt{\chi_N}|^2 + |\sqrt{\varepsilon} v_{xx}(L) \sqrt{\chi_N}|^2].$$

By Poincaré inequalities this implies that $v = 0$ if $F = 0$. Passing to the limit as $N \uparrow \infty$, we find that D is invertible from $H^4 \rightarrow L^2$. $\square$

Proof of Proposition 3.2. We will now consider the full equation (3.7), which may be written as $D(v) + J^0(v) = F \in L^2$. Again, standard arguments show that $J^0 \circ D^{-1}$ is a compact operator on L^2 or $L^2(e^Y)$. By the Fredholm alternative, it suffices to show uniqueness for the homogeneous solution to (3.7). For this, we apply the assumed *a-priori* estimate, (3.6) to conclude. $\square$

3.2 | Basic estimates

First, we urge the reader to recall the definitions in (1.22). For the weight, w , we will take

$$w = \text{either } 1 \text{ or } w_0, \quad (3.11)$$

where w_0 is defined in (1.21). For both of these choices, the following elementary inequalities hold:

$$|w_y| \lesssim \sqrt{\varepsilon}|w| + 1, \quad |w_y| \lesssim |w|. \quad (3.12)$$

Lemma 3.6 (Hardy-type inequalities). *Let f satisfy $f|_{y=0} = 0$ and $f|_{y \rightarrow \infty} = 0$. Then:*

$$\left\| \frac{f}{y} w \right\| \lesssim \|f_y w\| + \|\sqrt{\varepsilon} f w\|. \quad (3.13)$$

Proof. The case of $w = 1$ follows from the standard Hardy inequality. We thus consider $w = w_0$ (recall (1.21)). We integrate by parts in y in the following manner:

$$\begin{aligned} \left\| \frac{f}{y} w_0 \right\|^2 &= (\partial_y \left\{ y - \frac{1}{y} \right\} f, f \langle Y \rangle^{2m}) \\ &= 2m \left(f \left\{ y - \frac{1}{y} \right\}, f \langle Y \rangle^{2m-1} \sqrt{\varepsilon} \right) - 2 \left(f \left\{ y - \frac{1}{y} \right\}, f_y \langle Y \rangle^{2m} \right) \\ &\lesssim \|f \langle Y \rangle^m\| \left[\|\sqrt{\varepsilon} f_y \langle Y \rangle^{m-1}\| + \|f_y \langle Y \rangle^m\| \right]. \end{aligned}$$

□

Lemma 3.7. *Let $v \in H_{0^+}^4$, let $q = \frac{v}{u_s}$, and let w satisfy $|w_y| \lesssim |w|$. Assume $L \ll 1$.*

1. *The following Poincaré type inequalities hold:*

$$\|\partial_y^j \partial_x^{j_2} v w\| \lesssim L \|\partial_y^j \partial_x^{j_2+1} v w\| \text{ for } j_2 = 0, 1, 2, 3, \quad (3.14)$$

$$\|u_s^k \partial_y^j q w\| \lesssim L \|u_s^k \partial_y^j q_x w\| \text{ for } k = 0, 1, \quad (3.15)$$

$$\|\sqrt{\varepsilon} q_x w\| \lesssim L \|\sqrt{\varepsilon} q_{xx} w\|. \quad (3.16)$$

2. *The following quantities are controlled by the triple norm:*

$$\begin{aligned} &\|\{q_{yy}, q_{xy}, \sqrt{\varepsilon} q_{xx}\} \cdot w\| + \|\nabla_\varepsilon q \cdot w\| + \left\| \frac{q_x}{y} w \right\| \\ &\quad + \|\{v_{yyy}, v_{xyy}, \sqrt{\varepsilon} v_{xxy}, \varepsilon v_{xxx}\} \cdot w\| \\ &\quad + \|\{v_{yy}, v_{xy}, \sqrt{\varepsilon} v_{xx}\} \cdot w\| + \|\nabla_\varepsilon v \cdot w\| \\ &\lesssim \|q\|_w. \end{aligned} \quad (3.17)$$

3. The following quantities can be controlled with a pre-factor of $o_L(1)$:

$$\begin{aligned} \|v_{yy} \cdot w\| + \|\sqrt{\varepsilon}\{q_x, v_x\} \cdot w\| &\leq L\|q\|_w, \\ \|q_{yy}w\| &\lesssim \sqrt{L}\|q\|_w. \end{aligned} \quad (3.18)$$

4. Fix any $\delta > 0$. The following interpolation estimate holds:

$$\|\nabla_\varepsilon q_x \cdot w\| \leq \delta\|q\|_w + N_\delta\|u_s \nabla_\varepsilon q_x \cdot w\|. \quad (3.19)$$

We will often (for the sake of concreteness) apply the above interpolation with the following choices of δ :

$$\|\nabla_\varepsilon q_x w\| \lesssim L^{\frac{\alpha}{2}}\|q\|_w + L^{-\alpha}\|u_s \nabla_\varepsilon q_x w\|. \quad (3.20)$$

5. The following boundary estimates are valid:

$$\|q_x w\|_{x=L} + \|\sqrt{\varepsilon}q_x \cdot w\|_{x=0} + \|v_{yy} \cdot w\|_{x=L} \quad (3.21)$$

$$\begin{aligned} &+ \|\sqrt{\varepsilon}q_{xx}w\|_{x=0} + \|\varepsilon\sqrt{u_s}q_{xx}w\|_{x=L} + \|q_{xy}w\|_{x=L} \lesssim \sqrt{L}\|q\|_w \\ &\|\varepsilon^{\frac{1}{4}}q_{xx} \cdot w\|_{x=0} + \|\varepsilon^{\frac{1}{4}}\frac{q_x}{\langle y \rangle}w\|_{x=0} \lesssim \left(1 + \frac{\varepsilon^{\frac{1}{4}}}{L}\right)\|q\|_w \end{aligned} \quad (3.22)$$

$$\|\sqrt{u_s}q_{xyy} \cdot w\|_{x=L} \lesssim \sqrt{L}\|q\|_w. \quad (3.23)$$

Proof. Step 1:

Proof of (3.14)–(3.16): Fix a function $\tilde{u}_s$ that is a function of y only, and such that $C_0\tilde{u}_s \leq u_s \leq C_1\tilde{u}_s$ for all $(x, y) \in \Omega$. We may take $\tilde{u}_s \gtrsim y$ for $y \leq 1$ as $u_s \gtrsim y$ for $y \leq 1$. For any function g satisfying $g|_{x=0} = 0$ or $g|_{x=L} = 0$, a Poincaré inequality gives:

$$\|u_s g w\| \lesssim \|\tilde{u}_s g w\| \lesssim L\|\tilde{u}_s g_x w\| \lesssim L\|u_s g_x w\|.$$

We will apply the above with $g = \partial_y^j q, \partial_y^{j_1} \partial_y^{j_2} v$ for $j_2 = 0, 1, 2, 3$. We turn now to the following Poincaré-type inequality in the x -direction:

$$\begin{aligned} \|\sqrt{\varepsilon}q_x \cdot w\| &= \|\sqrt{\varepsilon}\left(q_x(L) + \int_L^x q_{xx}\right) \cdot w\| \\ &= \|\sqrt{\varepsilon}\frac{u_{sx}}{u_s}q|_L(y) \cdot w\| + \|\sqrt{\varepsilon}\int_L^x q_{xx} dx' \cdot w\| \\ &= \|\sqrt{\varepsilon}\frac{u_{sx}}{u_s}\int_0^L q_x dx' \cdot w\| + \|\sqrt{\varepsilon}\int_L^x q_{xx} dx' \cdot w\| \\ &\lesssim o_L(1)\|\sqrt{\varepsilon}q_x \cdot w\| + L\|\sqrt{\varepsilon}q_{xx} \cdot w\|. \end{aligned} \quad (3.24)$$

By absorbing the $\|\sqrt{\varepsilon}q_x \cdot w\|$ to the left-hand side, we obtain the desired estimate.

Step 2:

Proof of (3.17): We will work systematically through (3.17). Let us start with the $\nabla_\varepsilon^2 q$ terms. For this, let $\xi > 0$ a free parameter, and we will compute the localized quantity:

$$\begin{aligned}
 \|q_{yy} w \chi\left(\frac{y}{\xi}\right)\|^2 &= \left(\partial_y \{y\}, q_{yy}^2 w^2 \chi\left(\frac{y}{\xi}\right)^2\right) \\
 &= -\left(2y q_{yy}, q_{yyy} w^2 \chi\left(\frac{y}{\xi}\right)^2\right) - \left(y q_{yy}^2, 2w w_y \chi\left(\frac{y}{\xi}\right)^2\right) \\
 &\quad - \left(y q_{yy}^2, w^2 \frac{1}{\xi} \chi'\left(\frac{y}{\xi}\right) \chi\left(\frac{y}{\xi}\right)\right) \\
 &\leq L \|\sqrt{u_s} q_{xyy} w \chi\left(\frac{y}{\xi}\right)\| \|\sqrt{u_s} q_{yyy} w\| \\
 &\quad + L^2 \|\sqrt{u_s} q_{xyy}\|^2 \sup_{y \leq \xi} |w w_y| + \frac{L^2}{\xi} \|\sqrt{u_s} q_{xyy} w\|^2 \\
 &\leq \left(L + \frac{L^2}{\xi}\right) \|q\|_w^2.
 \end{aligned}$$

We have used (3.15). Inserting this below gives:

$$\begin{aligned}
 \|q_{yy} \cdot w\| &\leq \|q_{yy} \cdot w \left[1 - \chi\left(\frac{y}{\xi}\right)\right]\| + \|q_{yy} \cdot w \chi\left(\frac{y}{\xi}\right)\| \\
 &\leq \frac{L}{\sqrt{\xi}} \|\sqrt{u_s} q_{xyy} w\| + \left(\frac{L}{\sqrt{\xi}} + \sqrt{L}\right) \|q\|_w^2 \\
 &\leq \sqrt{L} \|q\|_w^2 \text{ for } \xi = L.
 \end{aligned} \tag{3.25}$$

A similar bound can be performed for the remaining components of $\nabla_\varepsilon^2 q$. However, we must forego the pre-factor of $o_L(1)$ for these terms. Let g be generic for now. For the far-field component, estimate $\|g \cdot w [1 - \chi(\frac{y}{\xi})]\| \leq \frac{1}{\xi} \|u_s g w\|$. For the localized component:

$$\begin{aligned}
 \|g \cdot w \chi\left(\frac{y}{\xi}\right)\|^2 &= -\left(y, \partial_y \left\{g^2 w^2 \chi\left(\frac{y}{\xi}\right)^2\right\}\right) \\
 &= -\left(2y g, g_y w^2 \chi\left(\frac{y}{\xi}\right)^2\right) - \left(2y g^2, w w_y \chi\left(\frac{y}{\xi}\right)^2\right) \\
 &\quad - \left(y g^2, w^2 \chi'\left(\frac{y}{\xi}\right) \chi\left(\frac{y}{\xi}\right) \xi^{-1}\right) \\
 &\lesssim \sqrt{\xi} \mathcal{O}(\sqrt{\text{LHS}}) \|u_s g_y w\| + \sup_{y \leq \xi} |w w_y| \sqrt{\xi} \|g\|^2 \\
 &\quad + \xi^{-1} \|u_s g w\|^2.
 \end{aligned}$$

Accumulating these estimates gives:

$$\|gw\| \leq \xi \|u_s g_y w\|^2 + \xi^{-1} \|u_s g w\|^2 + \sup_{y \leq \xi} |w w_y| \sqrt{\xi} \|g\|^2. \quad (3.26)$$

We will apply the above computation to $g = q_{xy}$ and $g = \sqrt{\varepsilon} q_{xx}$ and take $\xi = 1$. Next, applying (3.13) with $f = q_x$ gives:

$$\left\| \frac{q_x}{y} w \right\| \lesssim \|q_{xy} w\| + \|\sqrt{\varepsilon} q_x w\|. \quad (3.27)$$

Upon using (3.16), this concludes all of the q terms from (3.17).

We now move to v terms from (3.17), for which we expand:

$$\begin{aligned} v_x &= u_s q_x + u_{sx} q, \quad v_y = u_s q_y + u_{sy} q, \\ v_{xy} &= u_{sxy} q + u_{sx} q_y + u_{sy} q_x + u_s q_{xy}, \\ v_{yy} &= u_{syy} q + 2u_{sy} q_y + u_s q_{yy}, \\ v_{xx} &= u_{sxx} q + 2u_{sx} q_x + u_s q_{xx}, \\ v_{yyy} &= u_s q_{yyy} + u_{syyy} q + 3u_{syy} q_y + 3u_{sy} q_{yy} \\ v_{xyy} &= u_s q_{xyy} + u_{sxyy} q + u_{syy} q_x + u_{sx} q_{yy} + 2u_{sxy} q_y + 2u_{sy} q_{xy} \\ v_{xxy} &= u_s q_{xxy} + u_{sxxy} q + u_{sxx} q_y + u_{sy} q_{xx} + 2u_{sxy} q_x + 2u_{sx} q_{xy} \\ v_{xxx} &= u_{sxxx} q + u_s q_{xxx} + 3u_{sxx} q_x + 3u_{sx} q_{xx}. \end{aligned}$$

We turn to the third order terms for v , starting with v_{yyy} . We have already established the required estimates for $u_s q_{yyy}$, q_y , q_{yy} , and so we must estimate using Hardy's inequality:

$$\begin{aligned} \|u_{syyy} q\| &\leq \|u_{pyyy}^i q\| + \|\varepsilon^2 u_{eYYY}^i q\| \\ &\leq \|u_{pyyy}^i y\|_\infty \|q\| \langle y \rangle^{-1} + \varepsilon^{\frac{3}{2}} \|u_{eYYY}^i\|_\infty \|\sqrt{\varepsilon} q_x\| \\ &\lesssim \|q_y\| + \varepsilon^{\frac{3}{2}} \|\sqrt{\varepsilon} q_x\|. \end{aligned}$$

The same argument is performed for the remaining quantities from $\nabla^3 v$. The quantities in $\nabla^2 v$ and ∇v follow immediately upon using (3.1) and Poincaré's inequality. This concludes the proof of (3.17).

Step 3:

Proof of (3.18): The q_{yy} estimate follows from taking $\xi = 1$ in (3.25). For v_{yy} , we use (3.14) and (3.17) which shows that $\|v_{xyy} w\| \lesssim \|q\| \|w\|$. Both q_x and v_x follow from (3.14) to (3.16).

Step 4:

Proof of (3.19), (3.20): This follows immediately from (3.26) upon selecting $g = q_{xy}$ or $g = \sqrt{\varepsilon} q_{xx}$ and with $\delta = \sqrt{\xi}$, $\xi = L^\alpha$.

Step 5:

Proof of (3.21) The estimate for $q_x|_{x=L}$ is obtained by appealing to the boundary condition, (3.1), (3.3):

$$\begin{aligned} \|q_x w\|_{x=L} &= \left\| \frac{u_{sx}}{u_s} q w \right\|_{x=L} \leq \sqrt{L} \left\| \partial_x \left\{ \frac{u_{sx}}{u_s} \right\} q + \frac{u_{sx}}{u_s} q_x \right\| w \\ &\lesssim \sqrt{L} \left\| \left(\partial_x \left\{ \frac{u_{sx}}{u_s} \right\} + \frac{u_{sx}}{u_s} \right) \langle y \rangle \right\|_\infty \left\| \frac{q_x}{\langle y \rangle} w \right\|. \end{aligned}$$

For $q_x|_{x=0}$, we use Fundamental Theorem of Calculus:

$$\|q_x w\|_{x=0} = \|q_x(L, \cdot)w + \int_L^0 q_{xx} w\| \leq \|q_x w\|_{x=L} + \sqrt{L} \|q_{xx} w\|.$$

Next, $|v_{yy} w(L, \cdot)| \leq \sqrt{L} \|v_{xyy} w\|$ by using $v|_{x=0} = 0$. We now move to the q_{xx} terms from (3.21) for which we recall (3.3). From here, we obtain

$$|\sqrt{\varepsilon} q_{xx} w(0, \cdot)| = 2|\sqrt{\varepsilon} \frac{u_{sx}}{u_s} q_x(0, \cdot) w(0, \cdot)|.$$

The result then follows from the q_x estimate. At $x = L$, we use Fundamental theorem of calculus to conclude:

$$\|\varepsilon \sqrt{u_s} q_{xx}\|_{x=L} \leq \|\varepsilon \sqrt{u_s} q_{xx} w\|_{x=0} + \sqrt{L} \|\varepsilon \sqrt{u_s} q_{xx} w\|.$$

We now compute using (3.3):

$$\begin{aligned} \|q_{xy} \cdot w\|_{x=L} &= \|\partial_y \left\{ \frac{u_{sx}}{u_s} q \right\} \cdot w\|_{x=L} \\ &\leq \|\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} q \cdot w\|_{x=L} + \left\| \frac{u_{sx}}{u_s} q_y \cdot w \right\|_{x=L}. \end{aligned}$$

The latter term is estimated using $q|_{x=0} = 0$ so by Fundamental Theorem of Calculus is majorized by $\sqrt{L} \|q_{xy} w\|$. The former term requires a decomposition, upon which we use that $q|_{x=0} = 0$ and Hardy's inequality for the localized and Prandtl component, and the extra $\sqrt{\varepsilon}$ for the Euler component coupled with the Poincaré inequality in (3.16) for the q_x term:

$$\begin{aligned} &\|\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} q w\|_{x=L} + \|\partial_y \left\{ \frac{u_{px}}{u_s} \right\} q w[1 - \chi]\|_{x=L} + \|\partial_y \left\{ \frac{\sqrt{\varepsilon} u_{ex}}{u_s} \right\} q w[1 - \chi]\|_{x=L} \\ &\lesssim \sqrt{L} \left\| \frac{q_x}{y} \right\| + \sqrt{L} \|\partial_y \left\{ \frac{u_{px}}{u_s} \right\} [1 - \chi] y \langle Y \rangle^m\|_{\infty} \left\| \frac{q_x}{y} \right\| + \|\partial_y \{u_{ex}\} \sqrt{\varepsilon} y \langle Y \rangle^m\|_{\infty} \sqrt{L} \|\sqrt{\varepsilon} q_x\|. \end{aligned}$$

Above, we have used $\sqrt{\varepsilon} u_e = \sum_{i=1}^n \sqrt{\varepsilon} u_e^i(x, Y)$ from which $\|\partial_y \{u_{ex}\} \sqrt{\varepsilon} w_0\|_{\infty} < \infty$. This concludes the treatment of (3.21).

Step 6:

Proof of (3.22)

Using (3.3):

$$\|\varepsilon^{\frac{1}{4}} q_{xx} w\|_{x=0} = \|2\varepsilon^{\frac{1}{4}} \frac{u_{sx}}{u_s} q_x w\|_{x=0} \lesssim \|u_{sx} \langle y \rangle\|_{\infty} \|\varepsilon^{\frac{1}{4}} \frac{q_x}{\langle y \rangle} w\|_{x=0}.$$

We use the cutoff function $\chi(\frac{10x}{L})$, which satisfies $|\partial_x \chi(\frac{10x}{L})| \lesssim \frac{1}{L}$, and use the standard Trace inequality to estimate:

$$\|\varepsilon^{\frac{1}{4}} \frac{q_x}{\langle y \rangle} w\|_{x=0} = \|\varepsilon^{\frac{1}{4}} \frac{q_x}{\langle y \rangle} \chi\left(\frac{10x}{L}\right) w\|_{x=0} \lesssim \left\| \frac{q_x}{y} w \right\|^{\frac{1}{2}} \|\sqrt{\varepsilon} q_{xx} w\|^{\frac{1}{2}} + \frac{\varepsilon^{\frac{1}{4}}}{L} \left\| \frac{q_x}{\langle y \rangle} w \right\|.$$

To conclude, we apply the Hardy inequality in (3.13).

Step 7:

Proof of (3.23) Again using (3.3), the fact that $q|_{x=0} = 0$, and the Fundamental Theorem of Calculus:

$$\begin{aligned} \|\sqrt{u_s} \partial_{yy} \left\{ \frac{u_{sx}}{u_s} q \right\} w\|_{x=L} &= \|\sqrt{u_s} \left[\frac{u_{sx}}{u_s} q_{yy} + 2\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} q_y + \left(\frac{u_{sx}}{u_s} \right)_{yy} q \right] w\|_{x=L} \\ &\lesssim \sqrt{L} \|\sqrt{u_s} q_{xyy} w\| + \sqrt{L} \|q_{xy}\| + \sqrt{L} \left\| \left(\frac{u_{sx}}{u_s} \right)_{yy} y\|_{\infty} \left\| \frac{q_x}{y} \right\|. \end{aligned}$$

□

We must now collect some blow-up rates near $y = 0$ of various quantities according to the H_0^4 norm. We emphasize that these are *qualitative* estimates (and thus, any ε dependence on the right-hand side is acceptable):

Lemma 3.8. *Let $v \in H_0^4$. Then the following are valid for $j = 0, 1, 2$ and $k = 0, 1, 2, 3$:*

$$\sup_{y_0 \leq 1} [\|\nabla^k v\|_{y=y_0} + \|\nabla^j q\|_{y=y_0} + \sqrt{y_0} \|\nabla^3 q\|_{y=y_0}] \leq C_\varepsilon, \quad (3.28)$$

for some constant $C_\varepsilon < \infty$ that may depend poorly on small ε .

Proof. First, that $\sup_y |\nabla^k v|_{L_x^2} < \infty$, for $k = 0, 1, 2, 3$, follows immediately from $\|v\|_{H^4} < \infty$. We now use the elementary formula $\frac{1}{a+b} = \frac{1}{a} - \frac{b}{a(a+b)}$ to write:

$$q = \frac{v}{u_s} = \frac{v}{u_{sy}(0)y + [u_s - u_{sy}(0)y]} = \frac{1}{u_{sy}(0)} \frac{v}{y} - v \frac{u_s - u_{sy}(0)y}{yu_{sy}(0)u_s}.$$

Using the estimates $u_s \gtrsim y$ as $y \downarrow 0$ and $|u_s - u_{sy}(0)y| \lesssim y^2$ as $y \downarrow 0$, it is easy to see that the second quotient above is bounded and in fact C^k . We may thus limit our study to $q_0 := \frac{v}{y}$. We let $k_1 + k_2 = 3$ and differentiate the formula:

$$q_0(x, y) = \frac{1}{y} \int_0^y v_y(x, y') dy' = \int_0^1 v_y(x, ty) dt,$$

where we changed variables via $ty = y'$, to obtain:

$$\sqrt{y_0} \partial_x^{k_1} \partial_y^{k_2} q_0(x, y_0) = \int_0^1 \partial_x^{k_1} \partial_y^{k_2} v_y(x, ty_0) t^{k_2} \sqrt{y_0} dt.$$

We take L_x^2 and use Cauchy-Schwartz in y to majorize:

$$\begin{aligned} \sqrt{y_0} \|\partial_x^{k_1} \partial_y^{k_2} q_0\|_{y=y_0} &\leq \left(\int_0^1 \|\partial_x^{k_1} \partial_y^{k_2+1} v\|_{y=ty_0}^2 y_0 t^{2k_2} dt \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^1 \|\partial_x^{k_1} \partial_y^{k_2+1} v\|_{y=ty_0}^2 y_0 dt \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^{y_0} \|\nabla^4 v\|^2 \right)^{\frac{1}{2}} \end{aligned}$$

This establishes the $\nabla^3 q$ estimate. For $\nabla^2 q$, a similar calculation produces:

$$\begin{aligned}
 \|\partial_x^{j_1} \partial_y^{j_2} q_0\|_{y=y_0} &= \left\| \int_0^1 \partial_x^{j_1} \partial_y^{j_2+1} v(x, ty_0) t^{j_2} dt \right\|_{y=y_0} \\
 &\leq \int_0^1 \|\nabla^3 v\|_{y=ty_0} t^{j_2} dt \\
 &\leq \left(\int_0^y \|\nabla^3 v\|_{y=s}^2 \frac{s^{j_2}}{y^{j_2}} \frac{ds}{y} \right)^{\frac{1}{2}} \\
 &\leq \left(\frac{1}{y} \int_0^y \|\nabla^3 v\|_{y=s}^2 ds \right)^{\frac{1}{2}} \\
 &\leq \left(\frac{1}{y} y \sup_{s \leq 1} \|\nabla^3 v\|_{y=s} \right)^{\frac{1}{2}} < \infty.
 \end{aligned}$$

Above, we have used that $\sup_{s \leq 1} \|\nabla^3 v\|_{y=s} < \infty$ has already been established. This concludes the proof of the lemma. $\square$

Corollary 3.9. *Let $v \in H_0^4$. The trace $\nabla^2 q|_{y=0}$ is well defined as an element of L_x^2 , and moreover the following continuity is satisfied: $\nabla^2 q(\cdot, y) \xrightarrow{y \downarrow 0} \nabla^2 q(\cdot, 0)$ in $L^2(0, L)$.*

Proof. $(\nabla^2 q|_{y=0})^2$ is realized as the boundary trace of a $W^{1,1}$ function $|\nabla^2 q|^2$. Indeed, this follows from estimating the product $\nabla^2 q \cdot \partial_y \nabla^2 q \in L^1$:

$$\|\nabla^2 q \cdot \partial_y \nabla^2 q\|_1 \leq \|\nabla^2 q\|_{L_y^\infty(L_x^2)} \|\nabla^3 q\|_{L_x^2 L_y^1} < \infty,$$

The continuity statement in the lemma is a consequence of the above estimate and the Lebesgue Differentiation Theorem. $\square$

Corollary 3.10. *Let $v \in H_0^4$. Then all quantities appearing in $\|\cdot\|_{X_1}$ are finite.*

Proof. All $\nabla^3 q$ terms, upon taking $|\cdot|_{L_x^2}$ scale like $y^{-1/2}$, and so clearly $\|\sqrt{u_s} \nabla^3 q\| < \infty$. The second derivatives, upon taking $|\cdot|_{L_x^2}$ are bounded, and so clearly $\|\nabla^2 q\| < \infty$. The boundary terms are well-defined from the above corollary. $\square$

4 | A-PRIORI ESTIMATES FOR DNS

In light of Proposition 3.2, it suffices to control $\|v\|_{X_1}$ in order to solve the DNS system (A.27). This is achieved in this section via a cascade of estimates on $|||q|||_w$ (Quotient Estimates, Subsection 4.1) and $|||v|||_w$ (Trace Estimates, Subsection 4.2).

4.1 | Quotient estimates

Lemma 4.1. *Let v be a solution to (3.1), let w satisfy $|\partial_y^k w| \lesssim w$, and let $L \ll 1$. Then*

$$\begin{aligned} & \|\sqrt{u_s}\{q_{xyy}, \sqrt{\varepsilon}q_{xxy}, \varepsilon q_{xxx}\} \cdot w\|^2 + \|u_s q_{xy} \cdot w\|_{x=0}^2 + \|q_{xy} \cdot w\|_{y=0}^2 \\ & + \|\sqrt{\varepsilon}u_s q_{xx} \cdot w\|_{x=L}^2 \\ & \leq o_L(1)[\|q\|_w^2 + \|v\|_w^2] + L^{-\frac{1}{8}}\|\nabla_\varepsilon q_x \cdot u_s w\|^2 + L^{\frac{1}{8}}\|q_{xx} w_y\|^2 \\ & + |(F, q_{xx} w^2)| \end{aligned} \quad (4.1)$$

Proof. We will compute (Equation (3.1), $q_{xx} w^2$).

Step 1: Rayleigh Terms

$$\begin{aligned} (-\partial_x R[q], q_{xx} w^2) & \lesssim -\|u_s q_{xy} w\|_{x=0}^2 - \|u_s \sqrt{\varepsilon} q_{xx} w\|_{x=L}^2 + L\|q\|_w^2 \\ & + L^{\frac{1}{8}}\|q_{xx} w_y\|^2 + L^{-\frac{1}{8}}\|u_s \nabla_\varepsilon q_x w\|^2. \end{aligned} \quad (4.2)$$

We first integrate by parts in y , distribute the ∂_x , and then integrate by parts in x :

$$\begin{aligned} (-\partial_{xy}\{u_s^2 q_y\}, q_{xx} w^2) & = (\partial_x\{u_s^2 q_y\}, q_{xxy} w^2) + (\partial_x\{u_s^2 q_y\}, q_{xx} 2w w_y) \\ & = (2u_s u_{sx} q_y, q_{xxy} w^2) + (u_s^2 q_{xy}, q_{xxy} w^2) \\ & \quad + (4u_s u_{sx} q_y, q_{xx} w w_y) + (2u_s^2 q_{xy}, q_{xx} w w_y) \\ & = -(2u_s u_{sx}, q_{xy}^2 w^2) - (2\partial_x\{u_s u_{sx}\} q_y, q_{xy} w^2) \\ & \quad - (u_s u_{sx}, q_{xy}^2 w^2) + (4u_s u_{sx} q_y, q_{xx} w w_y) \\ & \quad + (2u_s^2 q_{xy}, q_{xx} w w_y) + 2(u_s u_{sx} q_y, q_{xy} w^2)_{x=L} \\ & \quad + \frac{1}{2}\|u_s q_{xy} w\|_{x=L}^2 - \frac{1}{2}\|u_s q_{xy} w\|_{x=0}^2. \end{aligned} \quad (4.3)$$

The term (4.3.8) is a favorable contribution. The cross terms, (4.3.{4, 5}), are the most dangerous terms:

$$\begin{aligned} |(4.3.\{4, 5\})| & \lesssim \|u_s q_{xy} \cdot w\| \|u_s q_{xx} w_y\| \lesssim L^{-\frac{1}{8}}\|u_s q_{xy} w\|^2 + L^{\frac{1}{8}}\|u_s q_{xx} w_y\|^2, \\ |(4.3.\{1, 2, 3\})| & \lesssim \|u_s q_{xy} \cdot w\|^2, \\ |(4.3.6)| + |(4.3.7)| & \lesssim o_L(1)\|u_s q_{xy} \cdot w\|^2 \end{aligned}$$

To estimate (4.3.2) we have used (3.14) because $q|_{x=0} = 0$. For the two boundary terms, (4.3.{6,7}), we have used (3.21).

We will move to the next Rayleigh term, which upon expanding reads:

$$-(\varepsilon \partial_{xx} \{u_s^2 q_x\}, q_{xx} w^2) = -\varepsilon (u_s^2 q_{xxx} + 4u_s u_{sx} q_{xx} + 2[u_s u_{sxx} + u_{sx}^2] q_x, q_{xx} w^2). \quad (4.4)$$

We integrate the first term by parts in x :

$$\begin{aligned} (4.4.1) &= (\varepsilon u_s u_{sx} q_{xx}, q_{xx} w^2) - \frac{1}{2} \|\sqrt{\varepsilon} u_s q_{xx} w\|_{x=L}^2 + \frac{1}{2} \|\sqrt{\varepsilon} u_s q_{xx} w\|_{x=0}^2 \\ &\lesssim -\|\sqrt{\varepsilon} u_s q_{xx} w\|_{x=L}^2 + \|\sqrt{\varepsilon} u_s q_{xx} w\|_{x=0}^2 + \|\sqrt{\varepsilon} u_s q_{xx} w\|^2, \end{aligned}$$

where we appeal to (3.21). The remaining two terms in (4.4) are also directly majorized by $\|\sqrt{\varepsilon} u_s q_{xx} w\|^2$ upon using (3.21) and the Fundamental Theorem of Calculus.

Step 2: Δ_ε^2 Terms

$$\begin{aligned} (\Delta_\varepsilon^2 v, q_{xx} w^2) &\lesssim -\|\sqrt{u_s} \{q_{xyy}, \sqrt{\varepsilon} q_{xxy}, \varepsilon q_{xxx}\} w\|^2 - \|q_{xy} w\|_{y=0}^2 \\ &\quad + o_L(1) \|q\|_w^2 + \sqrt{L} \|v\|_w^2 + L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q_x w\|^2 \\ &\quad + L^2 \|\sqrt{\varepsilon} q_{xx} w_y\|^2. \end{aligned} \quad (4.5)$$

We now treat the contributions arising from $\Delta_\varepsilon^2 v$, starting with ∂_y^4 ¹:

$$\begin{aligned} (v_{yyyy}, q_{xx} w^2) &= -(v_{yyy}, q_{xxy} w^2) - 2(v_{yyy}, q_{xx} w w_y) \\ &= (v_{yyyy}, q_{xy} w^2) - (v_{yyy}, q_{xy} w^2)_{x=L} \\ &\quad + 2(v_{yy}, q_{xxy} w w_y) + (v_{yy}, q_{xx} (w^2)_{yy}) \\ &= -(v_{xyy}, q_{xyy} w^2) - 2(v_{xyy}, q_{xy} w w_y) - (v_{xyy}, q_{xy} w^2)_{y=0} \\ &\quad - (v_{yyy}, q_{xy} w^2)_{x=L} - 2(v_{xyy}, q_{xy} w w_y) \\ &\quad + 2(v_{yy}, q_{xy} w w_y)_{x=L} + (v_{yy}, q_{xx} (w^2)_{yy}). \end{aligned} \quad (4.6)$$

The main terms are (4.6.1) and (4.6.3), so we begin with these. First, an expansion of:

$$v_{xyy} = u_s q_{xyy} + u_{sxyy} q + u_{syy} q_x + u_{sx} q_{yy} + 2u_{sxy} q_y + 2u_{sy} q_{xy},$$

shows:

$$\begin{aligned} (4.6.1) &= -([u_s q_{xyy} + u_{sxyy} q + u_{syy} q_x + u_{sx} q_{yy} \\ &\quad + 2u_{sxy} q_y + 2u_{sy} q_{xy}], q_{xyy} w^2). \end{aligned}$$

¹ Note that all integrations by parts are justified rigorously by Lemma 3.8 and its corollaries.

First, (4.6.1.1) is a favorable contribution to the left-hand side. We estimate immediately using Poincaré estimate (3.14), |(4.6.1.4)| $\lesssim L|||q|||_w^2$. Using the Hardy inequality in (3.17), the fact that $q|_{y=0} = q_x|_{y=0} = 0$, and the interpolation inequality (3.20) with appropriate selections of α :

$$\begin{aligned} |(4.6.1.3)| &\lesssim \|u_{syy}\langle y \rangle\|_\infty \left\| \frac{q_x}{y} w \right\| \|\sqrt{u_s} q_{xyy} w\| \\ &\lesssim \{\|q_{xy} w\| + L\|\sqrt{\varepsilon} q_{xx} w\|\} \|\sqrt{u_s} q_{xyy} w\| \\ &\lesssim L^{\frac{1}{64}} |||q|||_w^2 + L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q_x w\|^2. \end{aligned}$$

Let us explain the computation above, as it will be used repeatedly. We simply apply (3.20) twice with different choices of α :

$$\begin{aligned} \|q_{xy} w\| |||q|||_w &\lesssim \left\{ L^{\frac{1}{64}} |||q|||_w + L^{-\frac{1}{32}} \|u_s q_{xy} w\| \right\} |||q|||_w \\ &\lesssim L^{\frac{1}{64}} |||q|||_w^2 + L^{-\frac{1}{32}} \left\{ L^{-\frac{3}{32}} \|u_s q_{xy} w\|^2 + L^{\frac{3}{32}} \|\sqrt{u_s} q_{xyy} w\|^2 \right\} \quad (4.7) \\ &\lesssim L^{\frac{1}{64}} |||q|||_w^2 + L^{-\frac{1}{8}} \|u_s q_{xy} w\|^2. \end{aligned}$$

For (4.6.1.2) we may first use Poincaré in x as $q|_{x=0} = 0$ to majorize in the same way as above. Integration by parts in y and use of the assumption that $|w_y| \lesssim |w|$ yields:

$$\begin{aligned} (4.6.1.5) &= (2u_{sxyy} q_{xy}, q_y w^2) + (2u_{sxy} q_{xy}, q_{yy} w^2) \\ &\quad + (4u_{sxy} q_{xy}, q_y w w_y) + (2u_{sxy} q_y, q_{xy} w^2)_{y=0} \\ &\lesssim \|q_{xy}, q_{yy} \cdot w\|^2 + L\|q_{xy} w\|^2 + L\|q_{xy} w\|_{y=0}^2. \end{aligned}$$

We use above that q_{yy} comes with a factor of $\sqrt{L}$ according to estimate (3.18).

Integrate by parts in y :

$$\begin{aligned} (4.6.1.6) &= (q_{xy}^2, u_{syy} w^2) + (q_{xy}^2, u_{sy} 2w w_y) + (q_{xy}^2, u_{sy} w^2)_{y=0} \\ &\leq C \|\sqrt{u_s} q_{xy} \cdot w\|^2 + C \|q_{xy} \cdot \sqrt{w w_y}\|^2 + |\sqrt{|u_{sy}|} q_{xy} w|_{L^2(y=0)}^2 \\ &\leq L^{\frac{1}{16}} |||q|||_w^2 + L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q_x w\|^2 + \|\sqrt{u_{sy}} q_{xy} w\|_{y=0}^2 \quad (4.8) \end{aligned}$$

Above, we have used $|w_y| \lesssim |w|$ and the interpolation inequality (3.20). Let us emphasize the $\{y = 0\}$ boundary term from (4.6.1.6) arises with a pre-factor of $+1$, which is of bad sign. We postpone the estimation of this boundary term until (4.10).

We move to (4.6.3) for which an expansion shows:

$$\begin{aligned} (4.6.3) &= -(\{u_s q_{xyy} + u_{sxyy} q + u_{syy} q_x + u_{sx} q_{yy} \\ &\quad + 2u_{sxy} q_y + 2u_{sy} q_{xy}\}, q_{xy} w^2)_{y=0} \\ &\leq -(2 - C_0 L) \|\sqrt{u_y} q_{xy}\|_{y=0}^2, \end{aligned}$$

for some $C_0 < \infty$, independent of small L, ε . Let us provide some details regarding the above estimate. For (4.6.3.1), we use (3.28) and the fact that $|u_s| \lesssim y$ near $y = 0$ to conclude that (4.6.3.1) vanishes. Using that $q|_{y=0} = 0$ shows that (4.6.3.2, 3) vanishes. Using (3.28) together with $|u_{sx}| \lesssim y$ for $y \sim 0$ shows that (4.6.4) vanishes. This leaves only (4.6.3.5) and (4.6.3.6). The main favorable term is (4.6.3.6). For this, we have used that:

$$\begin{aligned} u_{sy}|_{y=0} &= \bar{u}|_{y=0} + \sum_{i=1}^n \sqrt{\varepsilon}^{n+1} u_{eY}^i|_{y=0} + \sum_{i=1}^n \sqrt{\varepsilon}^n u_{py}^i|_{y=0} \\ &\geq (1 - C_1 \varepsilon) u_{sy}|_{y=0}, \end{aligned} \quad (4.9)$$

for some $C_1 < \infty$ independent of L, ε . Note that $u_{sy}|_{y=0}$ is bounded below according to the first line of (A.33), which ensures that (4.6.3.6) is, in fact, a favorable contribution. For (4.6.3.5), we use that $q|_{x=0} = 0$ to invoke the Poincaré inequality:

$$|(4.6.3.5)| \leq L \left\| \frac{u_{sxy}}{\bar{u}_y} \right\|_{y=0} \|\sqrt{\bar{u}_y} q_{xy} w\|_{y=0} \|\sqrt{\bar{u}_y} q_{xy} w\|_{y=0}.$$

This concludes the estimate of (4.6.3).

We apply the same calculation as in (4.9) to conclude:

$$\begin{aligned} (4.6.3) + (4.8.3) &\leq -(2 - C_0 L) \|\sqrt{\bar{u}_y} q_{xy}\|_{y=0}^2 + (1 + C_1 \varepsilon) \|\sqrt{\bar{u}_y} q_{xy}\|_{y=0}^2 \\ &\leq -\frac{1}{2} \|\sqrt{\bar{u}_y} q_{xy}\|_{y=0}^2. \end{aligned} \quad (4.10)$$

Using (3.3) and the Fundamental Theorem of Calculus to integrate from $x = 0$ produces the identity:

$$\begin{aligned} (4.6.4) &= \left(v_{yyy}, \partial_y \left\{ \frac{u_{sx}}{u_s} q \right\} w^2 \right)_{x=L} \\ &= \left(v_{xyyy}, \partial_y \left\{ \frac{u_{sx}}{u_s} q \right\} w^2 \right) + \left(v_{yyy}, \partial_{xy} \left\{ \frac{u_{sx}}{u_s} q \right\} w^2 \right) \\ &= - \left(v_{xyy}, \partial_{yy} \left\{ \frac{u_{sx}}{u_s} q \right\} w^2 \right) - \left(v_{xyy}, \partial_y \left\{ \frac{u_{sx}}{u_s} q \right\} 2w w_y \right) \\ &\quad - \left(v_{xyy}, \frac{u_{sx}}{u_s} q_y w^2 \right)_{y=0} + \left(v_{yyy}, \partial_{xy} \left\{ \frac{u_{sx}}{u_s} q \right\} w^2 \right) \end{aligned}$$

For the first term, we distribute the ∂_{yy} and subsequently use (3.27), Poincaré in x , and (3.18) to obtain:

$$\begin{aligned} |(4.6.4.1)| &= \left| - \left(v_{xyy}, \left[\partial_{yy} \left\{ \frac{u_{sx}}{u_s} \right\} q + 2\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} q_y + \frac{u_{sx}}{u_s} q_{yy} \right] w^2 \right) \right| \\ &\lesssim \|v_{xyy} w\| \left[\|\partial_{yy} \left\{ \frac{u_{sx}}{u_s} \right\} y\|_\infty \left\| \frac{q}{y} w \right\| + L \|\partial_y \left\{ \frac{u_{sx}}{u_s} \right\}\|_\infty \|q_{xy} w\| \right. \\ &\quad \left. + \left\| \frac{u_{sx}}{u_s} \right\|_\infty \|q_{yy} w\| \right] \end{aligned}$$

$$\begin{aligned} &\lesssim \|v_{xyy}w\| \left[\|q_y w\| + \|\sqrt{\varepsilon} q w\| + L \|q_{xy} w\| + \|q_{yy} w\| \right] \\ &\lesssim o_L(1) \|q\|_w^2. \end{aligned}$$

For the second term, we again distribute the ∂_y and use that $|w_y| \lesssim |w|$:

$$\begin{aligned} |(4.6.4.2)| &\leq \left| \left(v_{xyy}, \partial_y \left\{ \frac{u_{sx}}{u_s} \right\} q 2w w_y \right) \right| + \left| \left(v_{xyy}, \frac{u_{sx}}{u_s} q_y 2w w_y \right) \right| \\ &\lesssim L \|\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} y\|_\infty \|v_{xyy} w\| \frac{q_x}{y} w + L \left\| \frac{u_{sx}}{u_s} \right\|_\infty \|v_{xyy} w\| \|q_{xy} w\| \\ &\lesssim L \|\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} y\|_\infty \|v_{xyy} w\| \{ \|q_{xy} w\| \\ &\quad + L \|\sqrt{\varepsilon} q w\| \} + L \left\| \frac{u_{sx}}{u_s} \right\|_\infty \|v_{xyy} w\| \|q_{xy} w\| \\ &\lesssim L \|q\|_w^2. \end{aligned}$$

For the third term, we expand the expression for v_{xyy} via:

$$\begin{aligned} (4.6.4.3) &= - \left(u_s q_{xyy} + u_{sx} q_{yy} + 2u_{sxy} q_y + 2u_{sy} q_{xy} \right. \\ &\quad \left. + u_{sxyy} q + u_{sy} q_x, \frac{u_{sx}}{u_s} q_y w^2 \right)_{y=0}. \end{aligned}$$

(4.6.4.3.1) and (4.6.4.3.2) vanish by combining (3.28) with $|\partial_x^j u_s| \lesssim y$ for y small, and (4.6.4.3.5), (4.6.4.3.6) vanish by using that $q|_{y=0} = q_x|_{y=0} = 0$. This then leaves:

$$|(4.6.4.3.3)| + |(4.6.4.3.4)| \lesssim L \|q_{xy} w\|_{y=0}^2 \lesssim L \|q\|_w^2,$$

where we have used the Poincaré inequality, which is available as $q|_{x=0} = 0$.

For the fourth term, we use the interpolation inequality, (3.20), and then Young's inequality for products to establish:

$$\begin{aligned} |(4.6.4.4)| &\lesssim \|v_{yyy} w\| \|q_{xy} w\| \lesssim \|v_{yyy}\| (\delta \|q\|_w + N_\delta \|u_s \nabla_\varepsilon q_x w\|^2) \\ &\lesssim o_L(1) \|q\|_w^2 + L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q_x w\|^2. \end{aligned}$$

We now move to (4.6.6). Again using (3.3) and that $v|_{x=0} = q|_{x=0} = 0$:

$$\begin{aligned} (4.6.6) &= -2 \left(v_{yy}, \partial_y \left\{ \frac{u_{sx}}{u_s} q \right\} w w_y \right)_{x=L} \\ &\lesssim L \|v_{xyy} w\| \left\| \partial_{xy} \left\{ \frac{u_{sx}}{u_s} q \right\} w \right\| \lesssim L \|q\|_w^2. \end{aligned}$$

For (4.6.{2, 5}) we use $|w_y| \lesssim |w|$ and the interpolation inequality (3.20), whereas for (4.6.7) we use Poincaré in x , (3.16), and the assumption that $|(w^2)_{yy}| \lesssim |w'|^2$:

$$|(4.6.\{2, 5\})| \leq L^{\frac{1}{16}} \|q\|_w^2 + L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q_x w\|^2,$$

$$|(4.6.7)| \lesssim L \|v_{xxy} w\| \|q_{xx} w_y\|.$$

This concludes the treatment of ∂_y^4 contributions.

We now move to contributions from $2\varepsilon \partial_{xxyy}$. We first integrate by parts in y , second expand the expression for v_{xxy} , and third perform a further y -integration by parts for the $2u_{sy}q_{xx}$ contribution. This produces:

$$\begin{aligned} (2\varepsilon v_{xxyy}, q_{xx} w^2) &= (-2\varepsilon v_{xxy}, q_{xxy} w^2) - 4(\varepsilon v_{xxy}, q_{xx} w w_y) \\ &= (-2\varepsilon [u_{sxxxy} q + 2u_{sxy} q_x + u_{sxx} q_y + u_s q_{xxy} + 2u_{sx} q_{xy} \\ &\quad + 2u_{sy} q_{xx}], q_{xxy} w^2) - 4(\varepsilon v_{xxy}, q_{xx} w w_y) \\ &= -(2\varepsilon [u_{sxxxy} q + 2u_{sxy} q_x + u_{sxx} q_y + u_s q_{xxy} \\ &\quad + 2u_{sx} q_{xy}], q_{xxy} w^2) + (\varepsilon u_{syy}, q_{xx}^2 w^2) \\ &\quad + (2\varepsilon u_{sy}, q_{xx}^2 w w_y) - (4\varepsilon v_{xxy}, q_{xx} w w_y) \end{aligned} \quad (4.11)$$

Term (4.11.4) contributes favorably. Terms (4.11.{1, 2}) are estimated through the weighted Hardy's inequality (3.27), terms (4.11.{3,5}) are estimated via Poincaré's inequality (3.14) and Cauchy-Schwartz, and terms (4.11.{6,7,8}) are estimated through the use of the assumption that $|w_y| \lesssim |w|$:

$$|(4.11.\{1, 2\})| \leq \sqrt{\varepsilon} \|\{u_{sxy}, u_{sxxxy}\} \langle y \rangle\|_\infty \frac{q, q_x}{y} w \| \sqrt{u_s} \sqrt{\varepsilon} q_{xxy} w \|,$$

$$|(4.11.\{3, 5\})| \leq \sqrt{\varepsilon} \|q_{xy} w\| \| \sqrt{u_s} \sqrt{\varepsilon} q_{xxy} w \|$$

$$|(4.11.\{6, 7, 8\})| \lesssim L^{-\frac{1}{10}} \| \sqrt{\varepsilon} q_{xx} w \|^2 + o_L(1) \| \sqrt{\varepsilon} v_{xxy} w \|^2$$

$$\lesssim L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q w\|^2 + o_L(1) \|q\|_w^2.$$

We next get to the contributions from $\varepsilon^2 v_{xxxx}$. We first integrate by parts in x , use that $v_{xxx}|_{x=L} = 0$, and then expand $\partial_x^3 v$ in terms of q to obtain:

$$\begin{aligned} (\varepsilon^2 v_{xxxx}, q_{xx} w^2) &= -(\varepsilon^2 [u_{sxxxx} q + 3u_{sxxx} q_x + 3u_{sxx} q_{xx} \\ &\quad + u_s q_{xxx}], q_{xxx} w^2) - (\varepsilon^2 v_{xxx}, q_{xx} w^2)_{x=0}. \end{aligned} \quad (4.12)$$

We first estimate the first three terms with the use of (3.17) – (3.18):

$$|(4.12.1, 2, 3)| \leq L \| \sqrt{u_s} \varepsilon q_{xxx} \cdot w \|^2 + \sqrt{\varepsilon} L \| \sqrt{\varepsilon} q_{xx} \cdot w \|^2.$$

For the boundary term, (4.12.5), we use the identity (3.3) to simplify and (3.21) to estimate:

$$\begin{aligned}
 (4.12.5) &= \left(2\varepsilon^2 v_{xxx}, \frac{u_{sx}}{u_s} q_x w^2 \right)_{x=0} \\
 &\lesssim \|\varepsilon^{\frac{3}{2}} v_{xxx} \cdot w \left(\frac{u_{sx}}{u_s} \right)\|_{x=0} \|\sqrt{\varepsilon} q_x \cdot w\|_{x=0} \\
 &\lesssim \|\varepsilon^{\frac{3}{2}} v_{xxx} \cdot w \left(\frac{u_{sx}}{u_s} \right)\|_{x=0} \sqrt{L} \|q\| \|w\| \\
 &\lesssim \sqrt{L} \|v\| \|q\| \|w\|^{\frac{1}{2}} \|q\| \|w\|^{\frac{3}{2}}.
 \end{aligned}$$

Note we have invoked the fourth-order norm, $\|v\|_w$, due to the boundary contribution at $\{x = 0\}$, through the following trace inequality:

$$\|\varepsilon^{\frac{3}{2}} v_{xxx} w\|_{x=0} \lesssim \|\varepsilon v_{xxx} w\|^{\frac{1}{2}} \|\varepsilon^2 v_{xxx} w\|^{\frac{1}{2}} \lesssim \|q\|_w^{\frac{1}{2}} \|v\|_w^{\frac{1}{2}}. \quad (4.13)$$

Step 3: $J(v)$ Terms

$$|(J, q_{xx} w^2)| \lesssim o_L(1) \|q\|_w^2 + L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q w\|^2 + L \|q_{xx} w_y\|^2. \quad (4.14)$$

Recalling the definition of J in (1.29), we expand $(J, q_{xx} w^2)$ via:

$$\begin{aligned}
 &(-\varepsilon v_{sx} v_{xy} - v_s v_{yyy} - \varepsilon v_s v_{yxx} \\
 &\quad + \Delta_\varepsilon v_s v_y - v_{sx} I_x[v_{yy}] + \Delta_\varepsilon v_{sx} I_x[v_y], q_{xx} w^2)
 \end{aligned} \quad (4.15)$$

An integration by parts first in x and then in y shows:

$$\begin{aligned}
 (4.15.5) &= (v_{sxx} I_x[v_{yy}], q_x w^2) + (v_{sx} v_{yyy}, q_x w^2) - (v_{sx} I_x[v_{yy}], q_x w^2)_{x=L} \\
 &= - (v_{sxx} I_x[v_{yy}], q_x w^2) - (v_{sxx} I_x[v_{yy}], q_{xy} w^2) - (v_{sxx} I_x[v_{yy}], q_x 2w w_y) \\
 &\quad + (v_{sxy} v_{yy}, q w^2) + (v_{sx} v_{yy}, q_y w^2) + (v_{sx} v_{yy}, q_x 2w w_y) \\
 &\quad + (I_x[v_{yy}], \partial_y \{v_{sx} q w^2\})_{x=L} \\
 &\lesssim L [\|q\|_w^2 + \|q_{xx} \cdot w_y\|^2]
 \end{aligned}$$

We have used the Hardy inequality (3.27) and Poincare in x , (3.16).

The estimates for (4.15.2) follow along the same lines. Again, integration by parts in y then in x and an appeal to the boundary condition (3.3) produces the identity:

$$\begin{aligned}
 (4.15.2) &= - (v_{sxy} v_{yy}, q_x w^2) - (v_{sx} v_{yy}, q_{xy} w^2) - (v_{sx} v_{yy}, q_x 2w w_y) \\
 &\quad - (v_{sy} v_{xyy}, q_x w^2) - (v_s v_{xyy}, q_{xy} w^2) - (v_s v_{xyy}, q_x 2w w_y)
 \end{aligned}$$

$$\begin{aligned}
& + \left(v_{yy} w^2, \partial_y \left\{ v_s \left(\frac{u_{sx}}{u_s} \right) q \right\} \right)_{x=L} + \left(v_{yy} v_s, \frac{u_{sx}}{u_s} q 2w w_y \right)_{x=L} \\
& \lesssim o_L(1) \|q\|_w^2 + L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q_x \cdot w\|^2 + L \|q_{xx} \cdot w_y\|^2.
\end{aligned}$$

The above estimate relies on the Hardy type inequality (3.27) for (4.15.2.{1,4}), the interpolation inequality (3.19) for (4.15.2.5), and Poincaré in x as $v|_{x=0} = 0$ for the boundary terms (4.15.2.{7,8}).

Next, we trivially obtain:

$$\begin{aligned}
|(4.15.1)| & \lesssim \sqrt{\varepsilon} \|v_{xy} w\| \|\sqrt{\varepsilon} q_{xx} w\| \lesssim \sqrt{\varepsilon} \|q\|_w^2 \\
|(4.15.3)| & \lesssim \|\sqrt{\varepsilon} v_{xxy} w\| \|\sqrt{\varepsilon} u_s q_{xx} w\| \lesssim L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q w\|^2 + o_L(1) \|q\|_w^2.
\end{aligned}$$

For (4.15.4), we integrate by parts in x and appeal to the boundary condition (3.3) and $v|_{x=0} = 0$:

$$\begin{aligned}
(4.15.4) & = -(\Delta_\varepsilon v_{sx} v_y, q_x w^2) - (\Delta_\varepsilon v_s v_{xy}, q_x w^2) - (\Delta_\varepsilon v_s v_y, \frac{u_{sx}}{u_s} q w^2)_{x=L} \\
& \leq L \|\Delta_\varepsilon v_{sx} y\|_\infty \|v_{xy} w\| \|\frac{q_x}{y} w\| + \|\Delta_\varepsilon v_s \langle y \rangle\|_\infty \|v_{xy} w\| \|\frac{q_x}{y} w\| \\
& \quad + L \|\frac{u_{sx}}{u_s} \langle y \rangle\|_\infty \|\Delta_\varepsilon v_s\|_\infty \|v_{xy} w\| \|\frac{q_x}{y} w\| \\
& \lesssim o_L(1) \|q\|_w^2 + L^{-\frac{1}{8}} \|u_s \nabla_\varepsilon q_x w\|^2.
\end{aligned}$$

Above we have used the Hardy type inequality (3.27), and the interpolation inequality (3.20) to conclude.

Finally, for the final term (4.15.6) we first split the coefficient via:

$$\begin{aligned}
(4.15.6) & = (\Delta_\varepsilon v_{px}^0 I_x[v_y], q_{xx} w^2) + \sum_{i=1}^n (\sqrt{\varepsilon}^i \Delta_\varepsilon v_{px}^i I_x[v_y], q_{xx} w^2) \\
& \quad + \sum_{i=1}^n (\sqrt{\varepsilon}^{i+1} \Delta v_{ex}^i I_x[v_y], q_{xx} w^2).
\end{aligned}$$

The higher order contributions are easily estimated using the extra power of $\sqrt{\varepsilon}$ by:

$$|(4.15.6.2)| + |(4.15.6.3)| \lesssim L \|v_y w\| \|\sqrt{\varepsilon} q_{xx} w\| \lesssim L \|q\|_w^2.$$

For the leading order Prandtl contribution, we integrate by parts in x , use that $I_x|_{x=0} = 0$, and estimate the resulting quantity using the rapid decay of v_p^0 :

$$\begin{aligned}
(4.15.6.1) & = -(\Delta_\varepsilon v_{pxx}^0 I_x[v_y], q_x w^2) - (\Delta_\varepsilon v_{px}^0 v_y, q_x w^2) \\
& \quad + \left(\Delta_\varepsilon v_{px}^0 \frac{u_{sx}}{u_s} q, I_x[v_y] w^2 \right)_{x=L} \\
& \lesssim L \|q\|_w^2
\end{aligned}$$

This concludes the treatment of the $J(v)$ contributions.

We estimate directly: $|(F, q_{xx}w^2)|$ is placed on the right-hand side of the desired estimate. This concludes the proof. $\square$

Lemma 4.2. *Let v be a solution to (3.1). Let w satisfy $|\partial_y^k w| \lesssim |w|$:*

$$\begin{aligned} & \|q_{yy} \cdot w\|_{y=0}^2 + \|\sqrt{u_s}\{q_{yyy}, \sqrt{\varepsilon}q_{xyy}, \varepsilon q_{xxy}\} \cdot w\|^2 \\ & \lesssim \sqrt{L} \|q\|_w^2 + |(F, q_{yy}w^2)|. \end{aligned} \quad (4.16)$$

Proof. We will compute the inner-product (Equation(3.1), $q_{yy}w^2$).

Step 1: Estimate of Rayleigh terms

$$(-\partial_x R[q], q_{yy}w^2) \lesssim -\|u_s q_{yy}w\|_{x=L}^2 + L \|q\|_w^2. \quad (4.17)$$

First, we will expand the term:

$$\partial_{xy}\{u_s^2 q_y\} = u_s^2 q_{xyy} + 2u_s u_{sx} q_{yy} + 2u_s u_{sy} q_{xy} + 2[u_s u_{sxy} + u_{sx} u_{sy}]q_y,$$

and upon doing so we will integrate by parts the highest order contribution, that is $(u_s^2 q_{xyy}, q_{yy}w^2)$ in x :

$$\begin{aligned} (-\partial_{xy}\{u_s^2 q_y\}, q_{yy}w^2) &= -(2u_{sx} u_{sy} q_y, q_{yy}w^2) - (2u_s u_{sxy} q_y, q_{yy}w^2) \\ &\quad - (2u_s u_{sx} q_{yy}w^2, q_{yy}) - (2u_s u_{sy} q_{xy}, q_{yy}w^2) \\ &\quad + (u_s u_{sx} q_{yy}w^2, q_{yy}) - \left(\frac{u_s^2}{2} q_{yy}w^2, q_{yy} \right)_{x=L} \\ &\lesssim L \|q\|_w^2 - \|u_s q_{yy}w\|_{x=L}^2. \end{aligned}$$

Second, we expand the term:

$$\partial_{xx}\{u_s^2 q_x\} = u_s^2 q_{xxx} + 4u_s u_{sx} q_{xx} + [2u_s u_{sxx} + 2u_{sx}^2]q_x.$$

We subsequently use the Poincare inequality (3.14) followed by (3.15),

$$\begin{aligned} (-\varepsilon \partial_{xx}\{u_s^2 q_x\}, q_{yy}w^2) &= -(\varepsilon[(u_s^2)_{xx} q_x + 2\partial_x\{u_s^2\}q_{xx} + u_s^2 q_{xxx}], q_{yy}w^2) \\ &\lesssim L \|q\|_w^2. \end{aligned}$$

Step 2: Estimate of Δ_ε^2 terms

$$(\Delta_\varepsilon^2 v, q_{yy}w^2) \lesssim -\|\sqrt{u_s}\{q_{yyy}, \sqrt{\varepsilon}q_{xyy}, \varepsilon q_{xxy}\}w\|^2 - \|q_{yy}w\|_{y=0}^2 + L \|q\|_w^2 \quad (4.18)$$

We begin with ∂_{yyy} . First, we integrate by parts in y , and then we expand the term v_{yyy} :

$$\begin{aligned} (v_{yyyy}, q_{yy} w^2) &= -(v_{yyy}, q_{yyy} w^2) - 2(v_{yyy}, q_{yy} w w_y) - (v_{yyy}, q_{yy} w^2)_{y=0} \\ &= -([u_s q_{yyy} + 3u_{syy} q_y + u_{sy} q + 3u_{sy} q_{yy}], q_{yy} w^2) \\ &\quad - 2(v_{yyy}, q_{yy} w w_y) - (v_{yyy}, q_{yy} w^2)_{y=0}. \end{aligned} \quad (4.19)$$

We first handle the important boundary contribution from above. We integrate by parts, expand the boundary term to obtain:

$$\begin{aligned} (4.19.\{4+6\}) &= \left(\frac{3}{2} u_{syy} q_{yy} w^2, q_{yy} \right) + \left(\frac{3}{2} u_{sy} q_{yy} w^2, q_{yy} \right)_{y=0} \\ &\quad - (3u_{sy} q_{yy} w^2, q_{yy})_{y=0} + (3q_{yy}^2, u_{sy} w w_y) \\ &\leq -\frac{3}{2} \|\sqrt{u_{sy}} q_{yy} \cdot w\|_{y=0}^2 + \sqrt{L} \|q\|_w^2. \end{aligned}$$

Above, we have used $|w_y| \lesssim |w|$, and the estimate (3.18) to estimate the q_{yy} term. We have also used the expansion for (4.19.6): $v_{yyy} = u_{syyy} q + 3u_{syy} q_y + 3u_{sy} q_{yy} + u_s q_{yyy}$, and subsequently that $q|_{y=0} = u_s|_{y=0} = 0$, $\|q_y\|_{y=0} \lesssim L \|q_{xy}\|_{y=0}$. We emphasize the importance of the precise prefactors of -3 and $+\frac{3}{2}$ in the above boundary terms, which enable us to generate the required positivity.

The first term, (4.19.1) is a favorable contribution which contributes $\|\sqrt{u_s} q_{yyy} w\|^2$. The third term is controlled by the Hardy-type inequality, (3.27):

$$|(4.19.3)| \leq \|u_{syy} \langle y \rangle\|_\infty \left\| \frac{q}{y} w \right\| \|\sqrt{u_s} q_{yyy} w\| \lesssim L \|q\|_w^2.$$

The second term, (4.19.2), is controlled via an integration by parts in y , Poincare in x , (3.14) which is available since $q|_{x=0} = 0$, and finally (3.18) to estimate the q_{yy} contribution:

$$\begin{aligned} (4.19.2) &= (3u_{syyy} q_y + 3u_{syy} q_{yy}, q_{yy} w^2) + 6(u_{syy} q_y, q_{yy} w w_y) \\ &\quad + 3(u_{syy} q_y, q_{yy} w^2)_{y=0} \\ &\lesssim \|q_{yy} w\|^2 + L \|q_{xy}\|_{y=0} \|q_{yy}\|_{y=0} \lesssim L \|q\|_w^2. \end{aligned}$$

Finally, we move to (4.19.5), for which an expansion of $v_{yyy} = u_s q_{yyy} + u_{syyy} q + 3u_{syy} q_y + 3u_{sy} q_{yy}$ and $|w_y| \lesssim |w|$ gives $|(4.19.5)| \lesssim L \|q\|_w^2$ upon invoking (3.18). This concludes the contributions of ∂_y^4 .

We next move to $2\varepsilon \partial_{xxyy}$. We integrate by parts the following term upon using that $v_x|_{x=L} = 0$ and $q|_{x=0} = 0$:

$$\begin{aligned} (2\varepsilon v_{xxyy}, q_{yy} w^2) &= -(2\varepsilon v_{xyy}, q_{xxy} w^2) \\ &= -(2\varepsilon \partial_{xyy} \{u_s q\}, q_{xxy} w^2) \\ &= -(2\varepsilon [u_{sxyy} q + 2u_{sxy} q_y + u_{sx} q_{yy} \\ &\quad + u_{syy} q_x + 2u_{sy} q_{xy} + u_s q_{xxy}], q_{xxy} w^2). \end{aligned} \quad (4.20)$$

While (4.20.6) is a favorable contribution, straightforward computations using the Poincare inequalities, (3.14), show that

$$|(4.20.1)| + |(4.20.3)| + |(4.20.4)| \lesssim L \sqrt{\varepsilon} \|q\|_w^2.$$

We must treat (4.20.{2, 5}) via integration by parts in y because their coefficients do not vanish as $y \downarrow 0$. For (4.20.2), integrate by parts in y , and use $|w_y| \lesssim |w|$ to obtain:

$$\begin{aligned} (4.20.2) &= (4\varepsilon q_{xy}, \partial_y \{u_{sxy} q_y w^2\}) + (4\varepsilon q_{xy}, u_{sxy} q_y w^2)_{y=0} \\ &= (4\varepsilon u_{sxyy} q_{xy}, q_y w^2) + (4\varepsilon q_{xy}, u_{sxy} q_{yy} w^2) \\ &\quad + (8\varepsilon q_{xy}, u_{sxy} q_y w w_y) + (4\varepsilon q_{xy}, u_{sxy} q_y w^2)_{y=0} \\ &\lesssim \varepsilon \|q\|_w^2 + \varepsilon L \|q_{xy} w\|^2 + L \varepsilon \|q_{xy}\|_{y=0}^2. \end{aligned}$$

For (4.20.5), integration by parts in y produces the expression

$$(4.20.5) = (2\varepsilon q_{xy} \partial_y \{u_{sy} w^2\}, q_{xy}) + (2\varepsilon q_{xy}, q_{xy} u_{sy} w^2)_{y=0} \lesssim \varepsilon \|q\|_w^2.$$

We next move to $\varepsilon^2 v_{xxxx}$. For this, we integrate by parts twice in x , use the boundary conditions $v_{xxx}|_{x=L} = 0$ and $q_{yy}|_{x=0} = 0$ from (3.3), subsequently integrate by parts in y , and finally expand the term v_{xxy} . We show this below:

$$\begin{aligned} (\varepsilon^2 v_{xxxx}, q_{yy} w^2) &= (\varepsilon^2 v_{xx}, q_{xxyy} w^2) - (\varepsilon^2 v_{xx}, q_{xyy})_{x=L} \\ &= -(\varepsilon^2 v_{xxy}, q_{xxy} w^2) - 2(\varepsilon^2 v_{xx}, q_{xxy} w w_y) - (\varepsilon^2 v_{xx}, q_{xyy})_{x=L} \\ &= (-\varepsilon^2 [u_{sxxxy} q + 2u_{sxy} q_x + u_{sy} q_{xx} + u_s q_{xxy} \\ &\quad + u_{sxx} q_y + 2u_{sx} q_{xy}], q_{xxy} w^2) - (\varepsilon^2 v_{xx}, q_{xxy} 2w w_y) \\ &\quad + (\varepsilon^2 v_{xx}, \partial_{yy} \left\{ \frac{u_{sx}}{u_s} q \right\} w^2)_{x=L}. \end{aligned} \quad (4.21)$$

The term (4.21.4) is favorable. The terms with coefficients that vanish as $y \downarrow 0$ are (4.21.5) and (4.21.6), and so these may be estimated directly via

$$|(4.21.5)| + |(4.21.6)| \lesssim L \varepsilon^{\frac{3}{2}} \|v\|_w^2.$$

The remaining interior terms require integration by parts in y :

$$\begin{aligned} (4.21.1) &= (\varepsilon^2 q_{xx}, u_{sxxxy} q w^2) + (\varepsilon^2 q_{xx}, u_{sxxxy} q_y w^2) \\ &\quad + (\varepsilon^2 q_{xx}, u_{sxxxy} q 2w w_y) \\ &\lesssim \varepsilon^{\frac{3}{2}} \|v\|_w^2. \end{aligned}$$

Above, we have used the weighted Hardy inequality from (3.27). Next, in a similar fashion:

$$\begin{aligned} (4.21.2) &= (2\varepsilon^2 q_{xx}, u_{sxyy} q_x w^2) + (2\varepsilon^2 q_{xx}, u_{sxyy} q_{xy} w^2) \\ &\quad + (4\varepsilon^2 q_{xx}, u_{sxy} q_x w w_y) \\ &\lesssim \varepsilon^{\frac{3}{2}} \|q\|_w^2. \end{aligned}$$

We have used (3.27) for the q_x term appearing in (4.21.2.1), (4.21.2.3). The term (4.21.3) can be handled analogously using that $q_{xx}|_{y=0} = 0$:

$$(4.21.3) = \frac{\varepsilon^2}{2}(u_{syy}q_{xx}, q_{xx}w^2) + (\varepsilon^2u_{sy}q_{xx}, q_{xx}ww_y) \lesssim \varepsilon\|\sqrt{\varepsilon}q_{xx}w\|^2.$$

Next, we use that $|w_y| \lesssim |w|$ and split the term (4.21.7) into $y \leq 1$ and $y \geq 1$. On the $y \leq 1$ piece, we use $|w| \lesssim 1$, whereas in the far-field piece we use that $|w_y| \lesssim |w|$. Recall also that $u_s \gtrsim y$ for $y \lesssim 1$. Thus,

$$\begin{aligned} |(4.21.7)| &\lesssim |(\varepsilon^2v_{xx}, q_{xxy}ww_y\chi(y))| + |(\varepsilon^2v_{xx}, q_{xxy}ww_y[1-\chi(y)])| \\ &\lesssim \varepsilon\|\sqrt{\varepsilon}\frac{v_{xx}}{y}\| \|\sqrt{u_s}\sqrt{\varepsilon}q_{xxy}\| + \varepsilon\|\sqrt{\varepsilon}v_{xx}w\| \|\sqrt{u_s}\sqrt{\varepsilon}q_{xxy}w\|. \end{aligned}$$

For the boundary term we distribute the ∂_{yy} and estimate using the Fundamental Theorem of Calculus since both $v_{xx}|_{x=0} = q|_{x=0} = 0$:

$$\begin{aligned} |(4.21.8)| &= \left(\varepsilon^2v_{xx}, \left[\partial_{yy} \left\{ \frac{u_{sx}}{u_s} \right\} q + 2\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} q_y + \frac{u_{sx}}{u_s} q_{yy} \right] w^2 \right)_{x=L} \\ &= \left(\varepsilon^2v_{xxx}, \left[\partial_{yy} \left\{ \frac{u_{sx}}{u_s} \right\} q + 2\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} q_y + \frac{u_{sx}}{u_s} q_{yy} \right] w^2 \right) \\ &\quad + \left(\varepsilon^2v_{xx}, \partial_x \left[\partial_{yy} \left\{ \frac{u_{sx}}{u_s} \right\} q + 2\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} q_y + \frac{u_{sx}}{u_s} q_{yy} \right] w^2 \right) \\ &\lesssim \sqrt{\varepsilon} \frac{1}{\sqrt{u_s}} \varepsilon v_{xxx} w \| \sqrt{\varepsilon} \sqrt{u_s} \{q + q_y + q_{yy} + q_x + q_{xy} + q_{xyy}\} w \| \\ &\lesssim \sqrt{\varepsilon} \|q\|_{|w|}^2. \end{aligned}$$

Above, we have expanded:

$$\begin{aligned} \left\| \frac{1}{\sqrt{u_s}} \varepsilon v_{xxx} w \right\| &= \left\| \frac{1}{\sqrt{u_s}} \varepsilon \partial_{xxx} \{u_s q\} w \right\| \\ &\leq \left\| \frac{1}{\sqrt{u_s}} \varepsilon \{u_{sxxx} q + 3u_{sxx} q_x + 3u_{sx} q_{xx} + u_s q_{xxx}\} \right\| \\ &\lesssim \|q\|_{|w|}, \end{aligned}$$

where we have used that $|\partial_x^j u_s| \lesssim y$ near $\{y = 0\}$.

Step 3: $J(v)$ terms

$$|(J, q_{yy}w^2)| \lesssim \sqrt{L} \|q\|_{|w|}^2.$$

Recalling (1.29), we expand and estimate immediately

$$\begin{aligned} &(-v_s v_{yyy} - \varepsilon v_{sx} v_{xy} - \varepsilon v_s v_{xxy} \\ &\quad + \Delta_\varepsilon v_s v_y - v_{sx} I_x[v_{yyy}] + I_x[v_y], q_{yy}w^2) \lesssim L \|q\|_{|w|}^2. \end{aligned}$$

The forcing term clearly contributes $|(F, q_{yy}w^2)|$ to the right-hand side, which concludes the proof. $\square$

Lemma 4.3. *Let v be a solution to (3.1). Let w satisfy $|\partial_y^k w| \lesssim |w|$, $|(w^2)_{yy}| \lesssim |w_y|^2$ and $|w_y| \lesssim |w|$. Then:*

$$\|\nabla_\varepsilon q_x \cdot u_s w\|^2 \lesssim L^{\frac{1}{2}} \{ \|q\|_w^2 + \|v\|_w^2 \} + L \|q_{xx} w_y\|^2 + |(F, q_x w^2)|. \quad (4.22)$$

Proof. We compute the following inner product: (Equation (3.1), $q_x w(y)^2$).

Step 1: Rayleigh Terms estimates

$$(-\partial_x R[q], q_x w^2) \gtrsim \|u_s q_{xy} w\|^2 - L \|q_{xx} w_y\|^2 - L \|q\|_w^2. \quad (4.23)$$

First, integrate by parts in y and expand via the product rule:

$$\begin{aligned} (-\partial_{xy} \{u_s^2 q_y\}, q_x w^2) &= (\partial_x \{u_s^2 q_y\}, q_{xy} w^2) + (\partial_x \{u_s^2 q_y\}, q_x 2w w_y) \\ &= \|u_s q_{xy} w\|^2 + (2u_s u_{sx} q_y, q_{xy} w^2) + (4u_s u_{sx} q_y, q_x w w_y) \\ &\quad + 2(u_s^2 q_{xy}, q_x w w_y). \end{aligned} \quad (4.24)$$

The second and third terms are majorized by $L \|u_s q_{xy} w\|^2 + L^2 \|u_s q_{xy} w\| \|q_x w_y\|$ upon using Poincare in x as in (3.14). For the fourth, integrate by parts in y to produce:

$$-(q_x, q_x [2u_s u_{sy} w w_y + u_s^2 (w w_y)_y]) \leq L \|u_{sy} y\|_\infty \left\| \frac{q_x}{y} w \right\| \|q_{xx} w_y\| + L^2 \|q_{xx} w_y\|^2$$

In the above estimate, we have used Poincare in x , (3.14), Hardy in y , and the estimate that $|(w^2)_{yy}| \lesssim |w_y|^2$.

The second Rayleigh contribution is as follows, upon integrating by parts in x and then expanding:

$$\begin{aligned} (-\varepsilon \partial_{xx} \{u_s^2 q_x\}, q_x w^2) &= (\varepsilon \partial_x \{u_s^2 q_x\}, q_{xx} w^2) - (\varepsilon q_x, \partial_x \{u_s^2 q_x\} w^2)|_{x=0}^{x=L} \\ &= \|\sqrt{\varepsilon} u_s q_{xx} w\|^2 + (2\varepsilon u_s u_{sx} q_x, q_{xx} w^2) \\ &\quad - (\varepsilon q_x, \partial_x \{u_s^2 q_x\} w^2)|_{x=0}^{x=L} \\ &= \|\sqrt{\varepsilon} u_s q_{xx} w\|^2 + (2\varepsilon u_s u_{sx} q_x, q_{xx} w^2) \\ &\quad - (\varepsilon u_s^2 q_x, q_{xx} w^2)|_{x=0}^{x=L} - 2|\sqrt{\varepsilon u_s u_{sx} q_x} w|_{x=0}^{x=L}|^2 \\ &\gtrsim \|\sqrt{\varepsilon} u_s q_{xx} w\|^2 - L \|q\|_w^2, \end{aligned} \quad (4.25)$$

where we have used (3.14)–(3.16). The boundary terms follow from (3.21).

Step 2: Estimate for Δ_ε^2 terms

$$|(\Delta_\varepsilon^2 v, q_x w^2)| \lesssim \sqrt{L} [|||q|||_w^2 + |||v|||_w^2] + L \|q_{xx} w_y\|^2. \quad (4.26)$$

We begin with the ∂_y^4 contribution. A series of integration by parts in y gives:

$$\begin{aligned} (v_{yyyy}, q_x w^2) &= (v_{yy}, q_{xyy} w^2) + (v_{yy}, q_{xy} 2w w_y) \\ &\quad + (v_{yy}, q_{xy} 2w w_y) - (v_y, q_{xy} 2(w w_y)_y) \\ &\quad - (v_y, q_x 2(w w_y)_{yy}) + (v_{yy}, q_{xy} w^2)_{y=0} \\ &\quad - (v_{yyy}, q_x w^2)_{y=0}. \end{aligned} \quad (4.27)$$

Specifically, we have integrated by parts twice in y , expanded the resulting quantity, $\partial_{yy}\{q_x w^2\} = q_{xyy} w^2 + 4q_{xy} w w_y + q_x \partial_{yy}\{w^2\}$, and further integrated by parts the final term in y .

We will first treat the boundary terms from (4.27). First, $(4.27.7) = 0$ due to the boundary condition $q_x|_{y=0} = 0$ coupled with the asymptotic estimate (3.28) for v_{yyy} . Next, an expansion shows:

$$(4.27.6) = ([u_{syy} q + 2u_{sy} q_y + u_s q_{yy}], q_{xy} w^2)_{y=0}.$$

The first term vanishes as $q|_{y=0} = 0$, whereas the third term vanishes according to the asymptotics in (3.28). The only contribution is thus the middle term for which we use that $q|_{x=0} = 0$ to estimate $|(q_y, q_{xy})|_{y=0}| \leq L \|q_{xy}\|_{y=0}^2$, which is an acceptable contribution to the right-hand side of (4.26) due to the inclusion $\|q_{xy}\|_{y=0}$ in $|||q|||_w$.

We now turn to the bulk terms from (4.27). An expansion shows

$$(4.27.1) = ([u_{syy} q + 2u_{sy} q_y + u_s q_{yy}], q_{xyy} w^2).$$

For the first term, we estimate via Hardy in y , (3.13), and Poincaré in x :

$$\begin{aligned} |(4.27.1.1)| &\leq \|u_{syy} \langle y \rangle\|_\infty \left\| \frac{q}{y} w \right\| \|\sqrt{u_s} q_{xyy} w\| \\ &\lesssim \{ \|q_y w\| + \|\sqrt{\varepsilon} q w\| \} \|\sqrt{u_s} q_{xyy} w\| \\ &\lesssim L \{ \|q_{xy} w\| + \|\sqrt{\varepsilon} q_x w\| \} \|\sqrt{u_s} q_{xyy} w\|. \end{aligned}$$

The middle term requires an integration by parts in y via:

$$\begin{aligned} (4.27.1.2) &= -(2q_{xy}, \partial_y \{u_{sy} q_y w^2\}) - (2q_{xy}, u_{sy} q_y w^2)_{y=0} \\ &= -(2u_{syy} q_{xy}, q_y w^2) - (2q_{xy}, u_{sy} q_y w^2) \\ &\quad - (4u_{sy} q_{xy}, q_y 2w w_y) - (2q_{xy}, q_y u_{sy} w^2)_{y=0} \\ &\lesssim L \|q_{xy} \cdot w\|^2 + \|q_{xy} \cdot w\| \|q_{yy} \cdot w\| \\ &\quad + L \|q_{xy} \cdot \sqrt{w w_y}\|^2 + L \|q_{xy} \cdot w\|_{y=0}^2 \\ &\lesssim \sqrt{L} |||q|||_w^2. \end{aligned}$$

Above, we have used (3.14) for the q_y terms, the assumption that $|w_y| \lesssim |w|$, and most importantly the estimate (3.18) to obtain $\sqrt{L}$ control of $\|q_{yy}w\|$. The final term can be estimated via Poincare in x : $|(4.27.1.3)| \lesssim L\|\sqrt{u_s}q_{xy}w\|^2$.

We continue with the bulk contributions from (4.27), for which straightforward bounds using (3.14) and the inequalities $|w_y| \lesssim |w|$, $(w^2)_{yy} \lesssim |w_y|^2$ show:

$$|(4.27.2, 3)| \leq L\|v_{xy}w\|\|\sqrt{u_s}q_{xy} \cdot w\|,$$

$$|(4.27.4)| \leq L\|v_{xy}w\|\|q_{xy}w\|,$$

$$|(4.27.5)| \leq L^2\|v_{xy}w\|\|q_{xx}w_y\|,$$

all of which are acceptable contributions to the right-hand side of (4.26). This concludes our treatment of (4.27).

We move on to contributions from εv_{xxyy} . We begin with one integration by parts in y and an expansion of $v_{xxy} = \partial_{xxy}\{u_s q\}$:

$$\begin{aligned} (2\varepsilon v_{xxyy}, q_x w^2) &= (-2\varepsilon v_{xxy}, q_{xy} w^2) - (4\varepsilon v_{xxy}, q_x w w_y) - (2\varepsilon v_{xxy}, q_x w^2)_{y=0} \\ &= -(2\varepsilon[u_{sxxxy}q + u_{sy}q_{xx} + 2u_{sxy}q_x + u_{sxx}q_y \\ &\quad + u_s q_{xxy} + 2u_{sx}q_{xy}], q_{xy} w^2) - (4\varepsilon q_x, v_{xxy} w w_y). \end{aligned} \quad (4.28)$$

We have used (3.28) to conclude that the $\{y=0\}$ boundary contribution vanishes. It is straightforward to estimate using (3.16) and that $|w_y| \lesssim |w|$:

$$|(4.28.1)| + \dots + |(4.28.6)| \lesssim \sqrt{\varepsilon}\|q\|_w^2$$

$$|(4.28.7)| \lesssim L\|\sqrt{\varepsilon}v_{xxy}w\|\|\sqrt{\varepsilon}q_{xx}w\|.$$

We now move to ∂_x^4 contributions, for which an integration by parts in x and expansion gives:

$$\begin{aligned} (\varepsilon^2 v_{xxxx}, q_x w^2) &= -(\varepsilon^2 v_{xxx}, q_{xx} w^2) - (\varepsilon^2 v_{xxx}, q_x w^2)_{x=0} \\ &= -(\varepsilon^2[u_{sxxxx}q + 3u_{sxxx}q_x + 3u_{sxx}q_{xx} + u_s q_{xxx}], q_{xx} w^2) \\ &\quad - (\varepsilon^2 v_{xxx}, q_x w^2)_{x=0} \\ &\lesssim \sqrt{\varepsilon}\|q\|_w^2 + \|\varepsilon^{\frac{3}{2}}v_{xxx}w\|_{x=0}\|\sqrt{\varepsilon}q_x w\|_{x=0} \\ &\lesssim \sqrt{\varepsilon}\|q\|_w^2 + \sqrt{L}\|v\|_w^{\frac{1}{2}}\|q\|_w^{\frac{3}{2}}, \end{aligned} \quad (4.29)$$

where we have used estimate (3.21) for the $q_x|_{x=0}$ boundary term, and the trace inequality (4.13).

Step 3: $J(v)$ terms

$$|(J, q_x w^2)| \lesssim \sqrt{L}\|q\|_w + L^2\|q_{xx}w_y\|^2.$$

Recalling the definition of J in (1.29), we have

$$\begin{aligned} &(-\varepsilon v_{sx}v_{xy} - v_s v_{yyy} - \varepsilon v_s v_{yxx} + \Delta_\varepsilon v_s v_y \\ &\quad - v_{sx}I_x[v_{yyy}] + \Delta_\varepsilon v_{sx}I_x[v_y], q_x w^2). \end{aligned} \quad (4.30)$$

We first record the elementary inequality which will be in repeated use:

$$\|I_x[f]\| \leq \sqrt{L} \|I_x[f]\|_{L_x^\infty L_y^2} \leq L \|f\|. \quad (4.31)$$

We integrate by parts in y :

$$(4.30.5) = (I_x[v_{yy}]v_{sxy}, q_x w^2) + (I_x[v_{yy}]v_{sx}, q_{xy} w^2) + (I_x[v_{yy}]v_{sx}, q_x 2w w_y).$$

Using (4.31), we immediately estimate (4.30.5.{2, 3}). The first term, (4.30.5.1), is controlled upon using that $\|v_{sxy}y\|_\infty < \infty$ and an appeal to the Hardy inequality, (3.27):

$$|(4.30.5.1)| \lesssim L \|v_{sxy}\langle y \rangle\|_\infty \|v_{yy}w\| \left\| \frac{q_x}{\langle y \rangle} w \right\|,$$

$$|(4.30.5.2)| \lesssim L \|v_{yy}w\| \|q_{xy}w\|,$$

$$|(4.30.5.3)| \lesssim L^2 \|v_{yy}w\| q_{xx} w_y.$$

Integration by parts in y for the term (4.30.2) produces:

$$(4.30.2) = (v_{yy}v_{sy}, q_x w^2) + (v_{yy}v_s, q_{xy} w^2) + 2(v_{yy}v_s, q_x w w_y)$$

From here an analogous set of estimates to (4.30.5) produces the desired estimate upon using one further Poincare inequality, $\|v_{yy}w\| \leq L \|v_{xyy}w\|$, which is valid as $v|_{x=0} = 0$. Direct Poincare inequality in x using (3.14) yields $|(4.30.1)| + |(4.30.3)| \lesssim L \|q\| \|q\|_w^2$. Terms (4.30.4) and (4.30.6) are estimated identically so we focus on (4.30.4). We estimate the Δ_ε term using (3.27) and Poincare in x as $v|_{x=0} = 0$:

$$\begin{aligned} |(4.30.4)| &\leq \|\Delta_\varepsilon v_s y\|_\infty \|v_y w\| \left\| \frac{q_x}{y} w \right\| \\ &\lesssim L \|v_{xy}w\| \{ \|q_{xy}w\| + \|\sqrt{\varepsilon} q_x w\| \} \\ &\lesssim L \|q\| \|q\|_w^2. \end{aligned}$$

This concludes the terms in J .

We put directly the contribution $|(F, q_x w^2)|$ on the right-hand side of the desired estimate, which concludes the proof. $\square$

4.2 | Trace estimates

For the first fourth order bound, we will perform a weighted estimate for a weight $w(y)$. Let us make the following definition:

$$\bar{B}(w) := \|\sqrt{\varepsilon} u_s v_{xyy} \cdot w\|_{x=0}^2 + \|\varepsilon u_s v_{xxy} \cdot w\|_{x=L}^2. \quad (4.32)$$

Lemma 4.4. *Let v be a solution to (3.1). Then the following estimate is valid:*

$$\begin{aligned} \tilde{B}(w) + \left\| \left\{ \sqrt{\varepsilon} v_{xyy}, \varepsilon v_{xxy}, \varepsilon^{\frac{3}{2}} v_{xxxy} \right\} \sqrt{u_s} w \right\|^2 \\ \lesssim \sqrt{\varepsilon} \|q\|_1^2 + \|q\|_{\sqrt{\varepsilon} w}^2 + \|q\|_{\sqrt{\varepsilon} w} \|q\|_{w_y} + |(F, \varepsilon u_s v_{xxy} w^2)|. \end{aligned} \quad (4.33)$$

Proof. We compute the inner-product (Equation (3.1), $\varepsilon v_{xxy} u_s w^2$).

Step 1: Rayleigh terms

The main estimate in this step is:

$$\begin{aligned} (-\partial_x R[q], \varepsilon u_s v_{xxy} w^2) &\gtrsim \|\sqrt{\varepsilon} u_s v_{xxy} w\|_{x=0}^2 + \|\varepsilon u_s v_{xxy} w\|_{x=L}^2 \\ &\quad - \|q\|_{\sqrt{\varepsilon} w}^2 - \sqrt{\varepsilon} \|q\|_1^2. \end{aligned} \quad (4.34)$$

First, we rewrite the Rayleigh operator via

$$-\partial_x R[q] = -\partial_{xy} \{u_s v_y\} - \varepsilon \partial_{xx} \{u_s v_x\} + \partial_{xy} \{u_s v\} + \varepsilon \partial_{xx} \{u_{sx} v\}. \quad (4.35)$$

A series of integration by parts shows:

$$\begin{aligned} &(-\partial_{xy} \{u_s v_y\}, \varepsilon v_{xxy} w^2 u_s) \\ &= -(\varepsilon [u_{sx} v_{yy} + u_{sy} v_{xy} + u_{sxy} v_y + u_s v_{xxy}], v_{xxy} w^2 u_s) \\ &= (\varepsilon v_{xyy}, \partial_x \{u_s u_{sx} v_{yy}\} w^2) + (\varepsilon v_{xxy}, \partial_y \{u_s u_{sy} v_{xy}\} w^2) \\ &\quad + (\varepsilon v_{xyy} w^2, \partial_x \{u_s u_{sxy} v_y\}) + (\varepsilon u_s u_{sx} v_{xxy}, v_{xxy} w^2) \\ &\quad + \frac{1}{2} \|\sqrt{\varepsilon} v_{xyy} w u_s\|_{x=0}^2 \\ &\gtrsim \|\sqrt{\varepsilon} v_{xyy} w u_s\|_{x=0}^2 - \|q\|_{\sqrt{\varepsilon} w}^2. \end{aligned}$$

Again, we expand and perform a series of integrations by parts which produces:

$$\begin{aligned} &-(\varepsilon \partial_{xx} \{u_s v_x\}, \varepsilon v_{xxy} u_s w^2) \\ &= -(\varepsilon^2 [u_s v_{xxx} + 2u_{sx} v_{xx} + u_{sxx} v_x], v_{xxy} u_s w^2) \\ &= (\varepsilon^2 v_{xxy}, \partial_y \{u_s^2 v_{xxx} w^2\}) + (2\varepsilon^2 v_{xxy} w^2, \partial_x \{u_s u_{sx} v_{xx}\}) \\ &\quad + (\varepsilon^2 v_{xxy}, \partial_y \{u_{sxx} u_s v_x w^2\}) \\ &= (\varepsilon^2 v_{xxy}, \partial_y \{u_s^2 w^2\} v_{xxx}) + (\varepsilon^2 v_{xxy}, u_s^2 v_{xxxy} w^2) \\ &\quad + 2(\varepsilon^2 v_{xxy} w^2, \partial_x \{u_s u_{sx} v_{xx}\}) + (\varepsilon^2 v_{xxy}, \partial_y \{u_{sxx} u_s v_x w^2\}). \end{aligned} \quad (4.36)$$

First, let us deal with (4.36.1). Using (A.5) we split $u_s = u_s^P + u_s^E$, where u_s^P decays rapidly as $y \rightarrow \infty$, which produces

$$|\nabla^k u_s| \leq |\nabla^k u_s^P| + \sqrt{\varepsilon} \text{ for } k \geq 1, \quad (4.37)$$

and so:

$$\begin{aligned} |(4.36.1)| &\lesssim \|\sqrt{\varepsilon} v_{xxy} \sqrt{\varepsilon} w\| \|\varepsilon v_{xxx} w_y\| + \sqrt{\varepsilon} \|\sqrt{\varepsilon} v_{xxy} \|\varepsilon v_{xxx}\| \\ &\quad + \sqrt{\varepsilon} \|\sqrt{\varepsilon} v_{xxy} \sqrt{\varepsilon} w\| \|\varepsilon v_{xxx} \sqrt{\varepsilon} w\| \\ &\lesssim \|q\|_{\sqrt{\varepsilon} w} \|q\|_{w_y} + \sqrt{\varepsilon} \|q\|_1^2 + \sqrt{\varepsilon} \|q\|_{\sqrt{\varepsilon} w}^2. \end{aligned}$$

The term (4.36.2) produces a positive boundary contribution via integration by parts in x :

$$\begin{aligned} (4.36.2) &= \frac{1}{2} \|\varepsilon u_s v_{xxy} w\|_{x=L}^2 - (u_s u_{sx} \varepsilon^2 v_{xxy}, v_{xxy} w^2) \\ &\gtrsim \|\varepsilon u_s v_{xxy} w\|_{x=L}^2 - \|q\|_{\sqrt{\varepsilon} w}^2 \end{aligned}$$

We estimate (4.36.3) directly:

$$|(4.36.3)| \lesssim \|v_{xyy} \sqrt{\varepsilon} w\| \|\varepsilon v_{xxx} \sqrt{\varepsilon} w\| \lesssim \|q\|_{\sqrt{\varepsilon} w}^2.$$

Finally, for (4.36.4), we distribute the ∂_y :

$$\begin{aligned} |(4.36.4)| &= |(\varepsilon^2 v_{xxy}, u_{sxy} u_s v_x w^2 + u_{sxx} u_{sy} v_x w^2 + u_{sxx} u_s v_{xy} w^2 \\ &\quad + u_{sxx} u_s v_x 2w w_y)| \\ &\lesssim \|\sqrt{\varepsilon} v_{xxy} \sqrt{\varepsilon} w\| \{ \|\sqrt{\varepsilon} v_x \sqrt{\varepsilon} w\| + \|v_{xy} \sqrt{\varepsilon} w\| \} \end{aligned}$$

We now have the lower order Rayleigh contributions. Here, the main mechanism is the pointwise inequality (4.37). We simply expand the product, integrate by parts once, expand further the resulting expression, and estimate using this pointwise inequality:

$$\begin{aligned} (\partial_{xy} \{u_{sy} v\}, \varepsilon v_{xxy} u_s w^2) &= (u_{sxy} v + u_{sxy} v_y + u_{syy} v_x + u_{sy} v_{xy}, \varepsilon v_{xxy} u_s w^2) \\ &= -(\varepsilon v_{xyy} w^2, \partial_x \{u_s u_{sxy} v\}) - (\varepsilon v_{xyy} w^2, \partial_x \{u_s u_{sxy} v_y\}) \\ &\quad - (\varepsilon v_{xxy}, \partial_y \{u_{syy} u_s v_x w^2\}) - (\varepsilon v_{xxy}, \partial_y \{u_{sy} u_s v_{xy} w^2\}) \\ &= -(\varepsilon u_{sx} u_{sxy} v_{xyy}, v w^2) - (\varepsilon u_s u_{sxx} v_{xyy}, v w^2) \\ &\quad - (\varepsilon u_s u_{sxy} v_x, v_{xyy} w^2) - (\varepsilon u_{sx} u_{sxy} v_y, v_{xyy} w^2) \\ &\quad - (\varepsilon u_s u_{sxx} v_{xyy}, v_y w^2) - (\varepsilon u_s u_{sxy} v_{xyy}, v_{xy} w^2) \\ &\quad - (\varepsilon u_s u_{syy} v_{xxy}, v_x w^2) - (\varepsilon u_s u_{syy} v_{xy}, v_{xxy} w^2) \\ &\quad - (\varepsilon u_{syy} u_{sy} v_{xxy}, v_x w^2) - (2\varepsilon u_{syy} u_s v_x, v_{xxy} w w_y) \\ &\quad - (\varepsilon u_s u_{syy} v_{xxy}, v_{xy} w^2) - (\varepsilon u_s u_{sy} v_{xxy}, v_{xyy} w^2) \end{aligned}$$

$$\begin{aligned}
 & -(\varepsilon u_{sy}^2 v_{xxy}, v_{xy} w^2) - (2\varepsilon u_s u_{sy} v_{xxy}, v_{xy} w w_y) \\
 & \lesssim \sqrt{\varepsilon} |||q|||_1^2 + \sqrt{\varepsilon} |||q|||_{\sqrt{\varepsilon} w}^2.
 \end{aligned}$$

Above, we have used that $|w_y| \lesssim |w|$. We have the final lower-order Rayleigh terms, for which a nearly identical argument to above is carried out:

$$\begin{aligned}
 (\varepsilon \partial_{xx} \{u_{sx} v\}, \varepsilon v_{xxyy} u_s w^2) &= (\varepsilon^2 [u_{sxxx} v + 2u_{sxx} v_x + u_{sx} v_{xx}], v_{xxyy} u_s w^2) \\
 &= -(\varepsilon^2 v_{xyy}, \partial_x \{u_{sxxx} u_s v\} w^2) - (2\varepsilon^2 v_{xxy}, \partial_y \{u_{sxxx} v_x u_s w^2\}) \\
 &\quad - (\varepsilon^2 v_{xyy}, \partial_x \{u_{sxx} u_s v_{xx}\} w^2) \\
 &= -(\varepsilon^2 u_{sxxx} u_s v_{xyy}, v w^2) - (\varepsilon^2 u_{sxxx} u_s v_{xyy}, v_x w^2) \\
 &\quad - (\varepsilon^2 u_{sxxx} u_{sx} v_{xyy}, v w^2) - (2\varepsilon^2 u_s u_{sxxx} v_{xxy}, v_x w^2) \\
 &\quad - (2\varepsilon^2 u_s u_{sxx} v_{xy}, v_{xxy} w^2) - (4\varepsilon^2 u_s u_{sxx} v_x, v_{xxy} w w_y) \\
 &\quad - (2\varepsilon^2 u_{sxx} u_{sy} v_{xxy}, v_x w^2) - (\varepsilon^2 u_{sxx} u_s v_{xyy}, v_{xx} w^2) \\
 &\quad - (\varepsilon^2 u_{sx} u_s v_{xyy}, v_{xxx} w^2) - (\varepsilon^2 u_{sx}^2 v_{xyy}, v_{xx} w^2) \\
 &\lesssim \sqrt{\varepsilon} |||q|||_1^2 + \sqrt{\varepsilon} |||q|||_{\sqrt{\varepsilon} w}^2.
 \end{aligned}$$

Step 2: Estimate of Δ_ε^2 terms

This is done in (3.9).

Step 3: Estimate of $J(v)$ terms

$$|(J, \varepsilon v_{xxyy} u_s w^2)| \lesssim o(1) \text{LHS}(4.33) + |||q|||_{\sqrt{\varepsilon} w}^2 \quad (4.38)$$

Recalling the definition of J from (1.29), we expand

$$\begin{aligned}
 & (-v_s v_{yy} - \varepsilon v_s v_{yxx} - \varepsilon v_{sx} v_{xy} - v_y \Delta_\varepsilon v_s \\
 & \quad - v_{sx} I_x[v_{yy}] + I_x[v_y] \Delta_\varepsilon v_{sx}, \varepsilon v_{xxyy} u_s w^2)
 \end{aligned} \quad (4.39)$$

Straightforward estimates give:

$$|(4.39.2)| \lesssim \|\sqrt{\varepsilon} v_{yx} \sqrt{\varepsilon} w\| \|\varepsilon v_{xxyy} u_s w\| \lesssim |||q|||_{\sqrt{\varepsilon} w} \times \text{LHS}(4.33),$$

$$|(4.39.3)| \lesssim \sqrt{\varepsilon} \|v_{xy} \sqrt{\varepsilon} w\| \|\varepsilon v_{xxyy} u_s w\| \lesssim \sqrt{\varepsilon} |||q|||_{\sqrt{\varepsilon} w} \times \text{LHS}(4.33),$$

which upon using Young's inequality for products is clear acceptable to the right-hand side of (4.38).

We now turn to (4.39.1) for which we integrate by parts in x , and subsequently integrate by parts the middle term in y thanks to the boundary condition $v|_{y=0} = v_y|_{y=0} = 0$:

$$\begin{aligned}
 (4.39.1) &= (v_{sx} v_{yyy}, \varepsilon v_{xyy} u_s w^2) + (v_s v_{xyyy}, \varepsilon v_{xyy} u_s w^2) + (v_s v_{yyy}, \varepsilon v_{xyy} u_{sx} w^2) \\
 &= (v_{sx} v_{yyy}, \varepsilon v_{xyy} w^2) - \frac{1}{2} (v_{sy} v_{xyy}, \varepsilon v_{xyy} u_s w^2) - \frac{1}{2} (v_s v_{xyy}, \varepsilon v_{xyy} u_{sy} w^2) \\
 &\quad - (v_s v_{xyy}, \varepsilon v_{xyy} u_s w w_y) + (v_s v_{yyy}, \varepsilon v_{xyy} u_{sx} w^2) \\
 &\lesssim |||q|||_{\sqrt{\varepsilon} w}^2.
 \end{aligned}$$

We next move to (4.39.4) for which we integrate by parts in x using that $v|_{x=0} = 0$ and $v_x|_{x=L} = 0$:

$$\begin{aligned}
 (4.39.4) &= (v_{xy} \Delta_\varepsilon v_s, \varepsilon v_{xyy} u_s w^2) + (v_y \Delta_\varepsilon v_{sx}, \varepsilon v_{xyy} u_s w^2) + (v_y \Delta_\varepsilon v_s, \varepsilon v_{xyy} u_{sx} w^2) \\
 &\lesssim \|\Delta_\varepsilon v_s + \Delta_\varepsilon v_{sx}\|_\infty |||q|||_{\sqrt{\varepsilon} w}^2.
 \end{aligned}$$

Next, we move to (4.39.5) for which we integrate by parts in x and use that $I_x[f]|_{x=0} = 0$ by definition:

$$\begin{aligned}
 (4.39.5) &= (v_{sxx} I_x[v_{yyy}], \varepsilon v_{xyy} u_s w^2) + (v_{sx} v_{yyy}, \varepsilon v_{xyy} u_s w^2) \\
 &\quad + (v_{sx} I_x[v_{yyy}], \varepsilon v_{xyy} u_{sx} w^2) \\
 &\lesssim |||q|||_{\sqrt{\varepsilon} w}^2.
 \end{aligned}$$

Lastly, we move to (4.39.6), for which we again integrate by parts in x and subsequently use the Poincaré inequality in x , (3.14), to produce:

$$\begin{aligned}
 (4.39.6) &= - (v_y \Delta_\varepsilon v_{sx}, \varepsilon v_{xyy} u_s w^2) - (I_x[v_y] \Delta_\varepsilon v_{sxx}, \varepsilon v_{xyy} u_s w^2) \\
 &\quad - (I_x[v_y] \Delta_\varepsilon v_{sx}, \varepsilon v_{xyy} u_{sx} w^2) \\
 &\lesssim \|\partial_x^j \Delta_\varepsilon v_s\|_\infty |||q|||_{\sqrt{\varepsilon} w}^2.
 \end{aligned}$$

This concludes the estimation of the $J(v)$ terms.

To conclude the proof, we simply put the forcing term, $|(F, \varepsilon u_s v_{xxyy} w^2)|$ to the right-hand side of the desired estimate. $\square$

Lemma 4.5. *Let $\zeta > 0$ be arbitrary. Let v be a solution to (3.1), and suppose $|\partial_y^k w| \lesssim |w|$. Then:*

$$\begin{aligned}
 &\|\varepsilon^{\frac{3}{2}} \sqrt{u_s} v_{xxx} \cdot w\|_{x=0}^2 + \left\| \varepsilon v_{xxyy}, \varepsilon^{\frac{3}{2}} v_{xxxy}, \varepsilon^2 v_{xxxx} \right\| \cdot \|w\|^2 \\
 &\lesssim \frac{1}{\zeta} \bar{B}(1) + \bar{B}(w) + \left(\zeta^3 + \sqrt{\varepsilon} \right) |||q|||_1^2 + |||q|||_{\sqrt{\varepsilon} w}^2 \\
 &\quad + |||q|||_{\sqrt{\varepsilon} w} |||q|||_{w_y} + |(F, \varepsilon^2 v_{xxxx} w^2)|,
 \end{aligned} \tag{4.40}$$

where $\bar{B}$ has been defined in (4.32).

Proof. We will compute the inner-product (Equation (3.1), $\varepsilon^2 v_{xxxx} w^2$).

Step 1: Estimate of Rayleigh terms

$$\begin{aligned} (-\partial_x R[q], \varepsilon^2 v_{xxxx} w^2) &\gtrsim \|\sqrt{u_s} \varepsilon^{\frac{3}{2}} v_{xxx} w\|_{x=0}^2 - (\bar{B}(w) + \zeta^{-1} \bar{B}(1)) \\ &\quad + \left(\zeta^3 + \sqrt{\varepsilon} \right) \| |q| \|_1^2 + \| |q| \|_{\sqrt{\varepsilon} w}^2 - o(1) \text{LHS(4.40)} \\ &\quad - \| |q| \|_{\sqrt{\varepsilon} w} \| |q| \|_{w_y}. \end{aligned} \quad (4.41)$$

Recall (4.35). First, we will extract the positive terms:

$$\begin{aligned} (-\varepsilon \partial_x \{u_s v_{xx}\}, \varepsilon^2 v_{xxxx} w^2) &= (\varepsilon^3 u_{sxx} v_{xx}, v_{xxx} w^2) + \frac{3}{2} (\varepsilon^{\frac{3}{2}} u_{sx} v_{xxx}, v_{xxx} w^2) \\ &\quad + \frac{1}{2} \|\sqrt{u_s} \varepsilon^{\frac{3}{2}} v_{xxx} w\|_{x=0}^2 \\ &\gtrsim \frac{1}{2} \|\sqrt{u_s} \varepsilon^{\frac{3}{2}} v_{xxx} w\|_{x=0}^2 - \|\varepsilon v_{xxx} \cdot \sqrt{\varepsilon} w\|^2. \end{aligned} \quad (4.42)$$

The lower order Rayleigh term is treated as follows, using the Poincare inequality paired with $v_x|_{x=L} = 0$:

$$\begin{aligned} |(\varepsilon^3 \partial_x \{u_{sxx} v\}, v_{xxxx} w^2)| &\lesssim \|\varepsilon^2 v_{xxxx} \cdot w\| \|\sqrt{\varepsilon} v_x \cdot \sqrt{\varepsilon} w\| \\ &\lesssim L \times \text{LHS(4.40)} + L \| |q| \|_{\sqrt{\varepsilon} w}^2. \end{aligned}$$

The next Rayleigh contributions are of the following form:

$$-(\partial_x \{u_s v_{yy}\} \cdot \varepsilon^2 v_{xxxx} w^2) = -(\varepsilon^2 [u_{sx} v_{yy} + u_s v_{xyy}], v_{xxxx} w^2). \quad (4.43)$$

For the first term from (4.43), we integrate by parts in x with no boundary contributions according to (3.1):

$$\begin{aligned} (4.43.1) &= (\varepsilon^2 u_{sxx} v_{yy}, v_{xxx} w^2) + (\varepsilon^2 u_{sx} v_{xyy}, v_{xxx} w^2) \\ &\lesssim \|\varepsilon v_{xxx} \cdot \sqrt{\varepsilon} w\| \|v_{xyy} \sqrt{\varepsilon} w\|. \end{aligned}$$

For the latter term, we require a localization. Recall the definition of (1.20) and define:

$$\chi_{\leq \zeta}(y) := \chi\left(\frac{y}{\zeta}\right), \quad \chi_{\zeta \leq y \leq 1}(y) := \chi(y) - \chi\left(\frac{y}{\zeta}\right), \quad \chi_{\geq 1}(y) := 1 - \chi(y).$$

We then decompose:

$$(4.43.2) = -(\varepsilon^2 u_s v_{xyy}, v_{xxxx} w^2 [\chi_{\leq \zeta}(y) + \chi_{\zeta \leq y \leq 1}(y) + \chi_{\geq 1}(y)]).$$

Using that $u_s(0) = 0$ and $|\partial_y u_s| \lesssim 1$ gives that $|u_s| \lesssim y \lesssim \zeta$ in the support of $\chi_{\leq \zeta}$, and so we estimate with Young's inequality for products:

$$\begin{aligned} |(\varepsilon^2 u_s v_{xyy}, v_{xxx} w^2 \chi_{\leq \zeta})| &\lesssim \zeta \|w \chi_{\leq \zeta}\|_{\infty} \|v_{xyy}\| \|\varepsilon^2 v_{xxx} \cdot w\| \\ &\leq o(1) \underbrace{\|\varepsilon^2 v_{xxx} \cdot w\|^2}_{\mathcal{O}(\text{LHS})} + N \zeta^3 \|q\|_1^2 + \sqrt{\varepsilon} \|q\|_1^2, \end{aligned} \quad (4.44)$$

for some large number N . All of these contributions are acceptable to the right-hand side of (4.41). To establish estimate (4.44), we expand $v_{xyy} = u_s q_{xyy} + u_{sx} q_{yy} + 2u_{sy} q_{yx} + 2u_{sxy} q_y + u_{syy} q_x + u_{sxyy} q$. The first, second, fifth, and sixth terms of the expansion provide an extra $\sqrt{\zeta}$ factor due to $u_{sx}|_{y=0} = u_s|_{y=0} = q|_{y=0} = q_x|_{y=0}$. For the fourth term we estimate

$$\|q_y \chi_{\leq \zeta}\| \leq \|q_y|_{y=0} \chi_{\leq \zeta}\| + \left\| \left(\int_0^y q_{xy} \right) \chi_{\leq \zeta} \right\| \lesssim \sqrt{\zeta} \|q\|_1.$$

For the third term, we integrate by parts via

$$\begin{aligned} (\varepsilon^2 u_{sy} u_s q_{xy}, v_{xxx} w^2 \chi_{\leq \zeta}) &= -(\varepsilon^2 u_{sy} u_s q_{xy}, v_{xxx} w^2 \chi_{\leq \zeta})_{x=0} - (\varepsilon^2 u_{sxy} u_s q_{xy}, v_{xxx} w^2 \chi_{\leq \zeta}) \\ &\quad - (\varepsilon^2 u_{sy} u_s q_{xxy}, v_{xxx} w^2 \chi_{\leq \zeta}) - (\varepsilon^2 u_{sx} u_{sy} q_{xy}, v_{xxx} w^2 \chi_{\leq \zeta}) \\ &\lesssim \sqrt{\varepsilon} \|u_s q_{xy}\|_{x=0} \|\varepsilon^{\frac{3}{2}} v_{xxx}\|_{x=0} + \varepsilon \|q_{xy}\| \|\varepsilon v_{xxx}\| \\ &\quad + \sqrt{\varepsilon} \|\sqrt{u_s} q_{xxy}\| \|\varepsilon v_{xxx}\| + \varepsilon \|q_{xy}\| \|\varepsilon v_{xxx}\| \\ &\lesssim \sqrt{\varepsilon} \|q\|_1^2. \end{aligned}$$

Let now $\phi = \chi_{\zeta \leq y \leq 1}$ or $\chi_{y \geq 1}$. We integrate by parts the term in (4.43.2), and use that $v_{xxx}|_{x=L} = 0$ to produce only boundary contributions at $\{x = 0\}$:

$$\begin{aligned} -(\varepsilon^2 u_s v_{xyy}, v_{xxx} w^2 \phi) &= (\varepsilon^2 u_s v_{xxyy}, v_{xxx} w^2 \phi) + (\varepsilon^2 u_{sx} v_{xyy}, v_{xxx} w^2 \phi) \\ &\quad + (\varepsilon^2 u_s v_{xyy}, v_{xxx} w^2 \phi)_{x=0}. \end{aligned} \quad (4.45)$$

We estimate:

$$\begin{aligned} |(4.45.2)| &\lesssim \|v_{xyy}\| \sqrt{\varepsilon} w \|\varepsilon^{\frac{3}{2}} v_{xxx} w\|, \\ |(4.45.3)| &\lesssim \frac{1}{\sqrt{\zeta}} \|\varepsilon^{\frac{3}{2}} \sqrt{u_s} v_{xxx}\|_{x=0} \|u_s \sqrt{\varepsilon} v_{xyy}\|_{x=0} \\ &\quad + \|\varepsilon^{\frac{3}{2}} \sqrt{u_s} v_{xxx} w\|_{x=0} \|u_s \sqrt{\varepsilon} v_{xyy} w\|_{x=0} \\ &\leq \frac{N}{\zeta} \bar{B}(1)^2 + o(1) \text{LHS}(4.40) + N \bar{B}(w)^2, \end{aligned}$$

for a potentially large constant N . Above for (4.45.3), we have split into two cases:

$$\begin{aligned}
 & |(\varepsilon^2 u_s v_{xyy}, v_{xxx} w^2 \chi_{\zeta \leq y \leq 1})_{x=0}| \\
 &= \left| \left(\varepsilon^2 u_s v_{xyy}, v_{xxx} w^2 \frac{\sqrt{u_s}}{\sqrt{u_s}} \chi_{\zeta \leq y \leq 1} \right)_{x=0} \right| \\
 &\leq \frac{1}{\sqrt{\zeta}} \left\| \frac{\sqrt{\zeta}}{\sqrt{u_s}} \chi_{\zeta \leq y \leq 1} \right\|_{\infty} \|\varepsilon^{\frac{3}{2}} \sqrt{u_s} v_{xxx}\|_{x=0} \|u_s \sqrt{\varepsilon} v_{xyy}\|_{x=0},
 \end{aligned} \tag{4.46}$$

whereas in the $\phi = \chi_{\geq 1}$ case, we use that $u_s \gtrsim 1$. For the highest order term, (4.45.1), we integrate by parts in y to get:

$$\begin{aligned}
 (4.45.1) &= -(\varepsilon^2 u_{sy} v_{xxy}, v_{xxx} w^2 \phi) - (\varepsilon^2 u_s v_{xxy}, v_{xxy} w^2 \phi) \\
 &\quad - 2(\varepsilon^2 u_s v_{xxy}, v_{xxx} w w_y \phi) - (\varepsilon^2 u_s v_{xxy}, v_{xxx} w^2 \phi_y).
 \end{aligned}$$

First, we estimate the lower order terms:

$$\begin{aligned}
 |(4.45.1.1)| &\leq \sqrt{\varepsilon} \|u_{sy}^P w^2\|_{\infty} \|\sqrt{\varepsilon} v_{xxy}\| \|\varepsilon v_{xxx}\| \\
 &\quad + \sqrt{\varepsilon} \|u_{sY}^E\|_{\infty} \|\sqrt{\varepsilon} v_{xxy}\| \|\sqrt{\varepsilon} w\| \|\varepsilon v_{xxx}\| \|\sqrt{\varepsilon} w\|, \\
 |(4.45.1.3)| &\lesssim \|\sqrt{\varepsilon} v_{xxy}\| \|\sqrt{\varepsilon} w\| \|\varepsilon v_{xxx} w_y\| \lesssim \|q\| \|\sqrt{\varepsilon} w\| \|q\| \|w_y\|, \\
 |(4.45.1.4)| &\lesssim \|u_s \partial_y \phi w^2\|_{\infty} \sqrt{\varepsilon} \|\sqrt{\varepsilon} v_{xxy}\| \|\varepsilon v_{xxx}\|.
 \end{aligned}$$

For (4.45.1.1), we split $u_{sy} = u_{sy}^P + \sqrt{\varepsilon} u_{sY}^E$ according to (A.5). We highlight above that (4.45.1.3) is an acceptable term into the right-hand side of (4.41). For the term (4.45.1.4), we use that the following quantity is bounded independent of ζ :

$$\begin{aligned}
 \|u_s \partial_y \phi\|_{\infty} &= \|u_s \partial_y \{\chi_{\zeta \leq y \leq 1} + \chi_{y \geq 1}\}\|_{\infty} \\
 &= \|u_s \partial_y \left\{ \chi(y) - \chi\left(\frac{y}{\zeta}\right) + 1 - \chi(y) \right\}\|_{\infty} \\
 &= \left\| -u_s \frac{1}{\zeta} \chi'\left(\frac{y}{\zeta}\right) \right\|_{\infty} \lesssim 1,
 \end{aligned}$$

since $\frac{u_s}{\zeta} \lesssim \frac{y}{\zeta}$.

The highest order term, (4.45.1.2), we integrate by parts in x to produce (recall the definition of $\bar{B}$ in (4.32)):

$$\begin{aligned}
 (4.45.1.2) &= \frac{1}{2} (\varepsilon^2 u_{sx} v_{xxy}, v_{xxy} w^2 \phi) - \frac{1}{2} \|\varepsilon \sqrt{u_s} v_{xxy} w \sqrt{\phi}\|_{x=L}^2 \\
 &\lesssim \|\sqrt{\varepsilon} v_{xxy}\| \|\sqrt{\varepsilon} w\|^2 + \frac{1}{\zeta} \bar{B}(1) + \bar{B}(w).
 \end{aligned}$$

This concludes the treatment of (4.45). Piecing together (4.45) and (4.44), we complete the estimate of (4.43.2). Summarizing the above estimates:

$$\begin{aligned} |(4.43.2)| &\lesssim \zeta^3 |||q|||_1^2 + \sqrt{\varepsilon} |||q|||_1^2 + |||q|||_{\sqrt{\varepsilon}w}^2 + \frac{1}{\zeta} \bar{B}(1) + \bar{B}(w) \\ &\quad + |||q|||_{\sqrt{\varepsilon}w} |||q|||_{w_y} + o(1) \text{LHS}(4.40). \end{aligned}$$

For the next Rayleigh contribution, we integrate by parts in x and expand:

$$\begin{aligned} (\partial_x \{u_{syy} v\}, \varepsilon^2 v_{xxxx} w^2) &= -(\varepsilon^2 v_{xxx} w^2, u_{sxyy} v) - (2\varepsilon^2 v_{xxx} w^2, u_{sxyy} v_x) \\ &\quad - (\varepsilon^2 v_{xxx} w^2, u_{syy} v_{xx}) - (\varepsilon^2 v_{xxx} w^2, u_{syy} v_x)_{x=0}. \end{aligned} \quad (4.47)$$

Upon using the decomposition (A.5) to write:

$$\partial_x^j u_{syy} = \partial_x^j u_{syy}^P + \varepsilon \partial_x^j u_{sYY}^E, \quad (4.48)$$

we estimate:

$$|(4.47.\{1, 2, 3\})| \leq \sqrt{\varepsilon} |||q|||_1^2 + \varepsilon |||q|||_{\sqrt{\varepsilon}w}^2.$$

Next, again using (4.48) and (3.22):

$$\begin{aligned} |(4.47.4)| &\lesssim \left\| \frac{u_{syy}^P}{u_s} w \langle y \rangle \right\|_{\infty} \|\varepsilon^{\frac{1}{4}} v_x \langle y \rangle^{-1}\|_{x=0} \|\varepsilon^{\frac{3}{2}} \sqrt{u_s} v_{xxx} w\|_{x=0} \varepsilon^{1/4} \\ &\quad + \varepsilon \|\varepsilon^{\frac{3}{2}} \sqrt{u_s} v_{xxx} w\|_{x=0} \|\sqrt{\varepsilon} v_x \sqrt{\varepsilon} w\|_{x=0} \\ &\lesssim \varepsilon^{\frac{1}{4}} |||q|||_w |||v|||_w + \varepsilon |||v|||_w |||q|||_1, \end{aligned}$$

where we have invoked the crucial fact that $u_{syy}^P|_{y=0} = 0$.

Step 2: Δ_ε^2 terms
This is done in (3.8).

Step 3: $J(v)$ terms

$$|(J, \varepsilon^2 v_{xxxx} w^2)| \lesssim |||q|||_{\sqrt{\varepsilon}w}^2 + \varepsilon |||q|||_1^2.$$

Recalling the definition of J from (1.29), we expand

$$\begin{aligned} (J, \varepsilon^2 v_{xxxx} w^2) &= -(\varepsilon^3 v_{sx} v_{xy}, v_{xxxx} w^2) - (v_s v_{yyy}, \varepsilon^2 v_{xxxx} w^2) \\ &\quad - (\varepsilon^3 v_s v_{xxy}, v_{xxxx} w^2) + (\varepsilon^2 v_y, v_{xxxx} \Delta_\varepsilon v_s w^2) \\ &\quad - (v_{sx} I_x[v_{yyy}], \varepsilon^2 v_{xxxx} w^2) + (\Delta_\varepsilon v_{sx} I_x[v_y], \varepsilon^2 v_{xxxx} w^2). \end{aligned} \quad (4.49)$$

Next, we integrate by parts in x , and there are no boundary contributions at $x = 0$ due to $I_x[f]|_{x=0} = 0$ by definition:

$$(4.49.5) = -(v_{sxx} I_x[v_{yyy}], \varepsilon^2 v_{xxx} w^2) - (v_{sx} v_{yyy}, \varepsilon^2 v_{xxx} w^2) \\ \lesssim \|v_{yyy} \sqrt{\varepsilon} w\| \|\varepsilon v_{xxx} \sqrt{\varepsilon} w\| \lesssim \|q\|_{\sqrt{\varepsilon} w}^2.$$

Similarly, an integration by parts in x produces:

$$(4.49.6) = -(\Delta_\varepsilon v_{sxx} I_x[v_y] + \Delta_\varepsilon v_{sx} v_y, \varepsilon^2 v_{xxx} w^2) \lesssim L \|v_{xy} \sqrt{\varepsilon} w\| \|\varepsilon v_{xxx} \sqrt{\varepsilon} w\|.$$

For (4.49.1), we perform Young's inequality for products:

$$|(4.49.1)| \lesssim \sqrt{\varepsilon} \|\varepsilon^2 v_{xxxx} \cdot w\| \|v_{xy} \cdot \sqrt{\varepsilon} w\| \lesssim \delta \|\varepsilon^2 v_{xxxx} w\|^2 + N_\delta \varepsilon \|v_{xy} \sqrt{\varepsilon} w\|^2.$$

We will now integrate by parts in x to produce:

$$(4.49.2) = (\varepsilon^2 v_{sx} v_{yyy}, v_{xxx} w^2) + (\varepsilon^2 v_s v_{xyy}, v_{xxx} w^2).$$

The first term can be majorized by $\|v_{yyy} \sqrt{\varepsilon} w\| \|\varepsilon v_{xxx} \sqrt{\varepsilon} w\|$, which is clearly admissible. For the latter term, we integrate by parts in y :

$$(4.49.2.2) = -(\varepsilon^2 v_{sy} v_{xyy}, v_{xxx} w^2) - (\varepsilon^2 v_s v_{xyy}, v_{xxx} w^2) - 2(\varepsilon^2 v_s v_{xyy}, v_{xxx} w w_y).$$

The first and third are evidently majorized by $\|v_{xyy} \sqrt{\varepsilon} w\| \|\varepsilon v_{xxx} \sqrt{\varepsilon} w\|$ upon using $|w_y| \lesssim |w|$. The middle term can be majorized $\|v_{xyy} \sqrt{\varepsilon} w\| \|\varepsilon^{\frac{3}{2}} v_{xxxy} w\|$ upon which we use Young's inequality for products. This concludes the bound for (4.49.2).

Next, for (4.49.3), an integration by parts first in x , using that $v_{xx}|_{x=0} = v_{xxx}|_{x=L} = 0$, and then in y for the highest order term produces:

$$(4.49.3) = (\varepsilon^3 v_{sx} v_{xxy}, v_{xxx} w^2) + (\varepsilon^3 v_s v_{xxxy}, v_{xxx} w^2) \\ = (\varepsilon^3 v_{sx} v_{xxy}, v_{xxx} w^2) - \frac{1}{2} (\varepsilon^3 v_{sy} v_{xxx}, v_{xxx} w^2) - (\varepsilon^3 v_s v_{xxx}, v_{xxx} w w_y) \\ \lesssim \|\sqrt{\varepsilon} v_{xxy} \cdot \sqrt{\varepsilon} w\|^2 + \|\varepsilon v_{xxx} \cdot \sqrt{\varepsilon} w\|^2 \lesssim \|q\|_{\sqrt{\varepsilon} w}^2.$$

Finally, for (4.49.4), we again integrate by parts in x using that $v_y|_{x=0} = v_{xxx}|_{x=L} = 0$, and use that:

$$\Delta_\varepsilon v_s = \Delta_\varepsilon v_s^P + \varepsilon \Delta v_s^E, \quad (4.50)$$

we estimate

$$(4.49.4) = -(\varepsilon^2 v_{xy}, \Delta_\varepsilon v_s v_{xxx} w^2) - (\varepsilon^2 v_y, \Delta_\varepsilon v_{sx} v_{xxx} w^2) \\ \lesssim \varepsilon \|\Delta_\varepsilon v_s^P w^2\|_\infty \|\varepsilon v_{xxx}\| \|v_{xy}\| + \sqrt{\varepsilon} \|\Delta v_s^E\|_\infty \|\varepsilon v_{xxx} \sqrt{\varepsilon} w\| \|v_{xy} \sqrt{\varepsilon} w\| \\ \lesssim \varepsilon \|q\|_1^2 + \sqrt{\varepsilon} \|q\|_{\sqrt{\varepsilon} w}^2.$$

This concludes the treatment of $J(v)$ terms.

To conclude the proof, we simply place $|(F, \varepsilon^2 v_{xxxx} w^2)|$ to the right-hand side of the desired estimate. $\square$

We next establish the *a-priori* estimate for Proposition 3.2:

Proposition 4.6. *Let $v \in X_1 \cap Y_{w_0}$ be a solution (3.1). Then the following estimate holds:*

$$\|v\|_{Y_{w_0}} \lesssim \|Fw_0\| \text{ and } \|v\|_{X_1} \lesssim \|F\|, \quad (4.51)$$

and

$$\|v_{yyyy} w\| \lesssim \text{RHS of Estimates (4.22), (4.1), (4.16), (4.40), (4.33)} + \|Fw\|^2.$$

Proof. We use the equation (3.1) to write the identity:

$$v_{yyyy} = -2\varepsilon v_{xxyy} - \varepsilon^2 v_{xxxx} - \partial_{xy}\{u_s^2 \partial_y q\} - \varepsilon \partial_{xx}\{u_s^2 \partial_x q\} + J(v) + F. \quad (4.52)$$

We will place each term in $L^2(w)$. It is easy to see that all of the terms are controlled by the left-hand sides of estimates (4.22), (4.1), (4.16), (4.40), (4.33).

From here, we take the linear combination $\varepsilon^{-\frac{1}{8}}(4.33) + (4.40)$ and $(4.1) + (4.16) + (4.22)$, which corresponds to a selection of $\zeta = \varepsilon^{\frac{1}{8}}$ in estimate (4.40) to obtain the X_1 bound.

Next, we take the combination $\varepsilon^{-\frac{1}{8}}(4.33) + (4.40)$ and $(4.1) + (4.16) + (4.22)$ for $L \ll 1$ and $w = w_0$, which produces the Y_{w_0} bound. $\square$

5 | SOLUTION TO DNS AND NS

The aim in this section is to bring together the estimates of the prior sections. Recall our ultimate aim is the nonlinear problem defined by (A.23) and (A.27). Motivated by these, we define the problem of interest in this section:

$$-\partial_x R[q] + \Delta_\varepsilon^2 v + J(v) \quad (5.1)$$

$$= -B_{(v^0)}(\bar{v}^0) + \varepsilon^{N_0} \mathcal{N}(\bar{u}^0, \bar{v}^0, \bar{v}) + F_{(q)}(\bar{u}^0, \bar{v}^0, \bar{v}),$$

$$\mathcal{L}v^0 = F_{(v)}(\bar{v}) + Q(\bar{u}^0, \bar{v}^0, \bar{v}) + \mathcal{H} + F_R^a. \quad (5.2)$$

Recall the definition of $F_{(q)}$ from (A.27). While $\partial_x F_R$, $\partial_x b_{(u)}(a^\varepsilon)$, and the h -dependent terms are pure forcing terms, $H[a^\varepsilon]$ is linear. We thus take $H[a^\varepsilon] = H[a^\varepsilon][\bar{v}, \bar{u}^0, \bar{v}^0]$.

We build the following linear combinations:

$$\begin{aligned} B_{X_1} &:= (B_{(v^0)}, q_x + q_{xx} + q_{yy} \\ &\quad + \varepsilon^{-\frac{3}{8}} \varepsilon^2 v_{xxxx} + \varepsilon^{-\frac{3}{8}} \varepsilon^{-\frac{1}{8}} \varepsilon u_s v_{xxyy}), \end{aligned} \quad (5.3)$$

$$\begin{aligned} B_{Y_w} &:= (B_{(v^0)}, \{\varepsilon q_x + \varepsilon q_{xx} + \varepsilon q_{yy} + \varepsilon^2 v_{xxxx} + \varepsilon u_s v_{xxyy}\} w^2 \\ &\quad + \{\varepsilon^2 v_{xxxx} + \varepsilon^{-\frac{1}{8}} \varepsilon u_s v_{xxyy}\}). \end{aligned} \quad (5.4)$$

The quantities $\mathcal{N}_{X_1}, \mathcal{N}_{Y_w}$ are defined as above, with $\varepsilon^{N_0} \mathcal{N}(\bar{u}^0, \bar{v}^0, \bar{v})$ taking the place of $B_{(v^0)}$. Similarly, the quantities $\mathcal{F}_{X_1}, \mathcal{F}_{Y_w}$ are defined as above, with $F_{(q)}$ taking the place of $B_{(v^0)}$. As a notational point, we will sometimes need to think of $\mathcal{F}_{X_1}, \mathcal{F}_{Y_w}$ as a bilinear term. In this case, we introduce the notation $\mathcal{F}_{X_1}(F_{(q)}(\bar{v}, \bar{u}^0, \bar{v}^0), q)$ (and same with $\mathcal{F}_{Y_w}$, and $\mathcal{F}_B$ below).

We also define the quantities:

$$\begin{aligned} \mathcal{B}_B &:= |(F_{(v)}(\bar{v}), q^0)| + \|F_{(v)}(\bar{v})w_0\|^2, \\ \mathcal{N}_B &:= |(\mathcal{Q}(\bar{u}^0, \bar{v}^0, \bar{v}) + \mathcal{H}, q^0)| + \|\{\mathcal{Q}(\bar{u}^0, \bar{v}^0, \bar{v}) + \mathcal{H}\}w_0\|^2, \\ \mathcal{F}_B &:= |(F_R^a, q^0)| + \|F_R^a w_0\|^2. \end{aligned} \quad (5.5)$$

One sees from the specification of F_R^a in (A.23) that F_R^a is a pure forcing term.

The purpose of all of these definitions is:

Lemma 5.1. *Let v be a solution to (5.1) and $[u^0, v^0]$ a solution to (5.2), and $\mathbf{u} \in \mathcal{X}$ as in (1.22). Then the following estimates are valid:*

$$\begin{aligned} \|v\|_{X_1}^2 &\lesssim \mathcal{B}_{X_1} + \mathcal{N}_{X_1} + \mathcal{F}_{X_1} + C(h), \\ [u^0, v^0]_B^2 &\lesssim \mathcal{B}_B + \mathcal{N}_B + \mathcal{F}_B, \\ \|v\|_{Y_{w_0}}^2 &\lesssim \|v\|_{X_1}^2 + \mathcal{B}_{Y_{w_0}} + \mathcal{N}_{Y_{w_0}} + \mathcal{F}_{Y_{w_0}} + C(h), \end{aligned} \quad (5.6)$$

where $\mathcal{F}_B$ has been defined in (5.5) and $\mathcal{F}_{X_1}, \mathcal{F}_{Y_{w_0}}$ have been defined analogously to (5.3), (5.4) as explained above.

Proof. The $[u^0, v^0]_B$ bound follows immediately from (2.6) upon replacing the abstract forcing, F , in (2.6), by the right-hand side of (5.2). We refer to the definition (1.22), where there is a gain of $\varepsilon^{-\frac{1}{8}}$ when $w = 1$ due to the disparity in scaling.

We take the combination $\varepsilon^{-\frac{1}{8}}(4.33) + (4.40)$ and $(4.1) + (4.16) + (4.22)$, which corresponds to a selection of $\zeta = \varepsilon^{\frac{1}{8}}$ in estimate (4.40), for $L \ll 1$ and $w = 1$, which produces

$$\begin{aligned} |||v|||_1^2 &\lesssim \varepsilon^{\frac{3}{8}} |||q|||_1^2 + |(F, \varepsilon^2 v_{xxxx} + \varepsilon^{-\frac{1}{8}} \varepsilon v_{xxyy})|, \\ |||q|||_1^2 &\lesssim o_L(1) |||q|||_1^2 + o_L(1) |||v|||_1^2 + |(F, q_{yy} + q_{xx} + q_x)|. \end{aligned}$$

The above F stands for an abstract forcing. In place of this, we insert the right-hand side of (5.1). From here, we conclude the X_1 estimate.

Next, we take the combination $\varepsilon^{-\frac{1}{8}}(4.33) + (4.40)$ and $(4.1) + (4.16) + (4.22)$ for $L \ll 1$ and $w = w_0$, which produces

$$\begin{aligned} |||v|||_w^2 &\lesssim \varepsilon^{\frac{3}{8}} |||q|||_1^2 + |||q|||_{\sqrt{\varepsilon}w}^2 + |||q|||_{w_y}^2 \\ &\quad + |(F, \varepsilon^2 v_{xxxx} w^2 + \varepsilon v_{xxyy} u_s w^2 + \varepsilon^{-\frac{1}{8}} \varepsilon u_s v_{xxyy})| + C(h), \\ |||q|||_w^2 &\lesssim o_L(1) |||v|||_w^2 + o_L(1) |||q_{xx}|||_{w_y}^2 + |(F, [q_{xx} + q_{yy} + q_x] w^2)| \end{aligned}$$

From here, using the inequality $|w_{0y}| \lesssim 1 + \sqrt{\varepsilon}|w_0|$, and again replacing the abstract forcing F by the right-hand side of (5.1), we conclude the Y_{w_0} estimate. $\square$

Our aim now is to estimate each of the quantities appearing on the right-hand sides of (5.6). We do this in a sequence of lemmas.

Lemma 5.2. *Let $\mathbf{u} \in \mathcal{X}$ as in (1.22). Let $C(h)$ denote a constant that is $\mathcal{O}(\|h\|_{C^{M_0}(e^y)})$ for a large M_0 . Then for $B_{X_1}, B_{Y_{w_0}}$ defined as in (5.3), (5.4), the following estimates are valid*

$$|B_{X_1}| \lesssim o(1)\|v\|_{X_1}^2 + \varepsilon^{-\frac{1}{2}}[\bar{u}^0, \bar{v}^0]_B^2, \quad (5.7)$$

$$|B_{Y_{w_0}}| \lesssim o(1)\|v\|_{Y_{w_0}}^2 + [\bar{u}^0, \bar{v}^0]_B^2. \quad (5.8)$$

Proof. Recall the specification of $B_{(v^0)}(\bar{v}^0)$ given in (1.30). Recall also the specification of the norms (1.22) and (2.4). The inequality (3.17) will be in constant use throughout the proof of this lemma.

Step 1: q_{xx} Multiplier

$$|(B_{(v^0)}, q_{xx}w^2)| \lesssim \begin{cases} \varepsilon^{-1/2}[\bar{u}^0, \bar{v}^0]_B^2 + o(1)\|q\|_1^2 & \text{if } w = 1 \\ \varepsilon^{-1}[\bar{u}^0, \bar{v}^0]_B^2 + o(1)\|q\|_w^2 & \text{if } w = w_0 \end{cases}. \quad (5.9)$$

Recall the specification of $B_{(v^0)}$ given in (1.30). We compute

$$(\bar{v}_{yyyy}^0, q_{xx}w^2) = (\bar{v}_{yyyy}^0, q_{xx}w^2\chi(y)) + (\bar{v}_{yyyy}^0, q_{xx}w^2\{1 - \chi(y)\}). \quad (5.10)$$

For ease of notation, denote $\chi^C(y) := 1 - \chi(y)$. For the localized quantity, we estimate

$$\begin{aligned} (5.10.1) &= (\bar{v}_{yyyy}^0, q_x w^2 \chi)|_{x=L} - (\bar{v}_{yyyy}^0, q_x w^2 \chi)_{x=0} \\ &= -(\bar{v}_{yyyy}^0, \frac{u_{sx}}{u_s} q w^2 \chi)|_{x=L} - (\bar{v}_{yyyy}^0, q_x w^2 \chi)_{x=0} \\ &\lesssim \sqrt{L} \|\bar{v}_{yyyy}^0 w_0\| \|q_{xy}\| + \|\bar{v}_{yyyy}^0 w_0\| (\|q_x \chi\|^{1/2} \|q_{xx}\|^{1/2}) \\ &\lesssim \sqrt{L} \|\bar{v}_{yyyy}^0 w_0\| \|q_{xy}\| + \varepsilon^{-\frac{1}{4}} \|\bar{v}_{yyyy}^0 w_0\| \|\frac{q_x}{y}\|^{1/2} \|\sqrt{\varepsilon} q_{xx}\|^{1/2} \\ &\lesssim \varepsilon^{-\frac{1}{4}} [\bar{u}^0, \bar{v}^0]_B \|q\|_1, \end{aligned}$$

where we have used $q_x^2|_{x=0} = q_x^2|_{x=L} + 2I_L[q_x q_{xx}]$, and (1.22), (2.4), and (3.17).

For the far field quantity, we integrate by parts to produce

$$\begin{aligned} (5.10.2) &= -(\bar{v}_{yyy}^0, q_{xxy} w^2 \chi^C) - (\bar{v}_{yyy}^0, q_{xx} 2w_y \chi^C) \\ &\quad - (\bar{v}_{yyy}^0, q_{xx} w^2 (\chi^C)') \\ &= -(\bar{v}_{yyy}^0, q_{xy} w^2 \chi^C)_{x=0} + (\bar{v}_{yyy}^0, q_{xy} w^2 \chi^C)_{x=L} \\ &\quad - (\bar{v}_{yyy}^0, q_{xx} 2w_y \chi^C) - (\bar{v}_{yyy}^0, q_{xx} w^2 (\chi^C)') \\ &= -(\bar{v}_{yyy}^0, q_{xy} w^2 \chi^C)_{x=0} - (\bar{v}_{yyy}^0, \partial_y \left\{ \frac{u_{sx}}{u_s} q \right\} w^2 \chi^C)_{x=L} \end{aligned}$$

$$\begin{aligned}
 & -(\bar{v}_{yyy}^0, q_{xx}2ww_y\chi^C) - (\bar{v}_{yyy}^0, q_{xx}w^2(\chi^C)') \\
 & = -(\bar{v}_{yyy}^0, q_{xy}w^2\chi^C)_{x=0} - (\bar{v}_{yyy}^0, \partial_y \left\{ \frac{u_{sx}}{u_s} q \right\} w^2\chi^C)_{x=L} \\
 & \quad - (\bar{v}_{yyy}^0, q_{xx}2ww_y\chi^C) - (\bar{v}_{yyy}^0, q_x w^2(\chi^C)')_{x=L} \\
 & \quad + (\bar{v}_{yyy}^0, q_x w^2(\chi^C)')_{x=0} \\
 & \lesssim \|\bar{v}_{yyy}^0 w_0\| \|u_s q_{xy} w\|_{x=0} + \sqrt{L} \|\bar{v}_{yyy}^0 w_0\| \|q_{xy} w\| \\
 & \quad + \|\bar{v}_{yyy}^0 w_0\| \|q_{xx} w_y\| + \varepsilon^{-1/4} \|\bar{v}_{yyy}^0 w_0\| \|q_{xy}\|^{1/2} \|\sqrt{\varepsilon} q_{xx}\|^{1/2} \\
 & \lesssim [\bar{u}^0, \bar{v}^0]_B \left(\varepsilon^{-\frac{1}{2}} \|q\|_{w_y} + \varepsilon^{-\frac{1}{4}} \|q\|_w \right).
 \end{aligned}$$

Above, we have used that $|\frac{1}{u_s} \chi^C| \lesssim \frac{1}{\bar{u}_p^0} \chi^C \lesssim 1$. We have also used the same estimates as in (5.10.1) for $|(\bar{v}_{yyy}^0, q_x w^2(\chi^C)')_{x=L} - (\bar{v}_{yyy}^0, q_x w^2(\chi^C)')_{x=0}|$.

We next compute

$$\begin{aligned}
 -2(\partial_y \{u_s u_{sx} \bar{q}_y^0\}, q_{xx} w^2) &= 2(u_s u_{sx} \bar{q}_y^0, q_{xxy} w^2) + 4(u_s u_{sx} \bar{q}_y^0, q_{xx} w w_y) \\
 &= -2(\partial_x \{u_s u_{sx}\} \bar{q}_y^0, q_{xy} w^2) + 2(u_s u_{sx} \bar{q}_y^0, q_{xy} w^2)_{x=L} \\
 &\quad - 2(u_s u_{sx} \bar{q}_y^0, q_{xy} w^2)_{x=0} + 4(u_s u_{sx} \bar{q}_y^0, q_{xx} w w_y) \\
 &= -2(\partial_x \{u_s u_{sx}\} \bar{q}_y^0, q_{xy} w^2) - 2 \left(u_s u_{sx} \bar{q}_y^0, \partial_y \left\{ \frac{u_{sx}}{u_s} q \right\} w^2 \right)_{x=L} \\
 &\quad - 2(u_s u_{sx} \bar{q}_y^0, q_{xy} w^2)_{x=0} + 4(u_s u_{sx} \bar{q}_y^0, q_{xx} w w_y) \\
 &\lesssim \sqrt{L} \|u_s \bar{q}_y^0 w_0\| \|q_{xy} w\| + L \|\sqrt{u_s} \bar{q}_y^0 w_0\| \|q_{xy} w\| \\
 &\quad + \|u_s \bar{q}_y^0\| \|u_s q_{xy} w\|_{x=0} + \sqrt{L} \|u_s \bar{q}_y^0\| \|q_{xx} w_y\| \\
 &\lesssim [\bar{u}^0, \bar{v}^0]_B \left(\|q\|_w + \varepsilon^{-1/2} \|q\|_{w_y} \right)
 \end{aligned}$$

We compute

$$\begin{aligned}
 & (\partial_x \{(x+1)v_s\} \bar{v}_{yyy}^0 - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0, q_{xx} w^2) \\
 &= -(\partial_x \{(x+1)v_s\} \bar{v}_{yy}^0 - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0, q_{xxy} w^2 + 2q_{xx} w w_y) \\
 &= (\partial_{xx} \{(x+1)v_s\} \bar{v}_{yy}^0 - \partial_{xx} \{(x+1)v_{sy}\} \bar{v}_y^0, q_{xy} w^2) - (\partial_x \{(x+1)v_s\} \bar{v}_{yy}^0 \\
 &\quad - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0, q_{xy} w^2)_{x=L} + (\partial_x \{(x+1)v_s\} \bar{v}_{yy}^0 \\
 &\quad - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0, q_{xy} w^2)_{x=0} - (\partial_x \{(x+1)v_s\} \bar{v}_y^0 \\
 &\quad - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0, 2q_{xx} w w_y).
 \end{aligned} \tag{5.11}$$

Above, we have used the identity

$$\partial_x \{(x+1)v_s\} \bar{v}_{yy}^0 - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0 = \partial_y \{\partial_x \{(x+1)v_s\} \bar{v}_{yy}^0 - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0\}.$$

We estimate the first term in the $w = 1$ case:

$$\begin{aligned} |(5.11.1)| &\leq \|(x+1)v_{sxx} + 2v_{sx}\|_\infty \|\bar{v}_{yy}^0\| \|q_{xy}\| + \|(x+1)u_{sxx} + 2u_{sx}\|_\infty \|\bar{v}_y^0\| \|q_{xy}\| \\ &\lesssim [\bar{u}^0, \bar{v}^0]_B \|q\|_1 \end{aligned}$$

We next expand

$$\begin{aligned} (5.11.2) &= \left(\partial_x \{(x+1)v_s\} \bar{v}_{yy}^0 - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0, \partial_y \left\{ \frac{u_{sx}}{u_s} q \right\} w^2 \right)_{x=L} \\ &= \left(\partial_x \{(x+1)v_s\} \bar{v}_{yy}^0 - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0, \partial_y \left\{ \frac{u_{sx}}{u_s} \right\} q w^2 \right)_{x=L} \\ &\quad + \left(\partial_x \{(x+1)v_s\} \bar{v}_{yy}^0 - \partial_x \{(x+1)v_{sy}\} \bar{v}_y^0, \frac{u_{sx}}{u_s} q_y w^2 \right)_{x=L} \end{aligned}$$

Thus, in the $w = 1$ case, we estimate

$$\begin{aligned} &\|(x+1)v_{sx} + v_s\|_\infty \|y \partial_y \left\{ \frac{u_{sx}}{u_s} \right\} + \frac{u_{sx}}{u_s}\|_\infty \|\bar{v}_{yy}^0\| \|q_{xy}\| \sqrt{L} \\ &\quad + \|(x+1)u_{sxx} + u_{sx}\|_\infty \|y \partial_y \left\{ \frac{u_{sx}}{u_s} \right\} + \frac{u_{sx}}{u_s}\|_\infty \|\bar{v}_y^0\| \|q_{xy}\| \sqrt{L} \\ &\lesssim [\bar{u}^0, \bar{v}^0]_B \|q\|_1. \end{aligned}$$

We next continue with $w = 1$ to estimate

$$\begin{aligned} |(5.11.3)| &\lesssim \left\| \frac{(x+1)v_{sx} + v_s}{u_s} \right\|_\infty \|\bar{v}_{yy}^0\| \|u_s q_{xy}\|_{x=0} + \left\| \frac{v_{sy} + (x+1)v_{sxy}}{u_s} \right\|_\infty \|\bar{v}_y^0\| \|u_s q_{xy}\|_{x=0} \\ &\lesssim [\bar{u}^0, \bar{v}^0]_B \|q\|_1 \end{aligned}$$

This concludes the $w = 1$ case, and we move on to the $w = w_0$ case. We first record using (1.6), the following estimate (using $Y = \sqrt{\varepsilon}y$ and $v_e^1 = v_e^1(Y)$)

$$\|\partial_x^k v_s w_0\|_\infty \lesssim \|\partial_x^k \{v_p^0 + v_e^1\} \langle y \rangle \langle Y \rangle^m\|_\infty + \mathcal{O}(1) \lesssim \varepsilon^{-\frac{1}{2}}. \quad (5.12)$$

We begin with the following, using (5.12):

$$\begin{aligned} |(5.11.1)| &\lesssim \|\partial_{xx} \{(x+1)v_s\} w_0\|_\infty \|\bar{v}_{yy}^0\| \|q_{xy} w_0\| + \|\partial_{xx} \{(x+1)v_{sy}\} w_0\|_\infty \|\bar{v}_y^0\| \|q_{xy} w\| \\ &\lesssim \varepsilon^{-1/2} \|\bar{v}_{yy}^0\| \|q_{xy} w_0\| + \|\bar{v}_y^0\| \|q_{xy} w\| \\ &\lesssim \varepsilon^{-1/2} [\bar{u}^0, \bar{v}^0]_B \|q\|_1. \end{aligned}$$

We move to the (5.11.2) for which

$$\begin{aligned} |(5.11.2)| &\lesssim \|(x+1)v_{sx} + v_s\|_\infty \|wy\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} + w \frac{u_{sx}}{u_s}\|_\infty \|\bar{v}_{yy}^0\| \|q_{xy}w_0\| \sqrt{L} \\ &\quad + \|(x+1)u_{sxx} + u_{sx}\|_\infty \|wy\partial_y \left\{ \frac{u_{sx}}{u_s} \right\} + w \frac{u_{sx}}{u_s}\|_\infty \|\bar{v}_y^0\| \|q_{xy}\| \sqrt{L} \\ &\lesssim \sqrt{L}[\bar{u}^0, \bar{v}^0]_B \|q\|_w \end{aligned}$$

Next, again recalling (5.12),

$$\begin{aligned} |(5.11.3)| &\lesssim \left\| \frac{\partial_x \{(x+1)v_s\}}{u_s} w \right\|_\infty \|\bar{v}_{yy}^0\| \|u_s q_{xy} w\|_{x=0} \\ &\quad + \left\| \frac{\partial_x \{(x+1)v_{sy}\}}{u_s} w \right\|_\infty \|\bar{v}_y^0\| \|u_s q_{xy} w\|_{x=0} \\ &\lesssim \varepsilon^{-1/2} [\bar{u}^0, \bar{v}^0]_B \|q\|_{w_0}. \end{aligned}$$

Last, again using (5.12),

$$\begin{aligned} |(5.11.4)| &\lesssim \varepsilon^{-1/2} \|\partial_x \{(x+1)v_s\} w_y\|_\infty \|\bar{v}_{yy}^0\| \sqrt{\varepsilon} q_{xx} w \\ &\quad + \varepsilon^{-1/2} \|\partial_x \{(x+1)v_{sy}\} w_y\|_\infty \|\bar{v}_y^0\| \sqrt{\varepsilon} q_{xx} w \\ &\lesssim \varepsilon^{-1/2} [\bar{u}^0, \bar{v}^0]_B \|q\|_w. \end{aligned}$$

Finally, upon using again (5.12),

$$|(\varepsilon \bar{v}_y^0 \partial_x \{(x+1)v_{sxx}\}, q_{xx} w^2)| \lesssim \sqrt{L} \|\bar{v}_{yy}^0\| \sqrt{\varepsilon} q_{xx} w \lesssim \sqrt{L} [\bar{u}^0, \bar{v}^0]_B \|q\|_w.$$

This concludes the $B_{(v^0)}$ terms for this multiplier.

Step 2:

q_{yy} Multiplier

$$|(B_{(v^0)}, q_{yy} w^2)| \lesssim o(1) \|q\|_w^2 + \sqrt{L} [\bar{u}^0, \bar{v}^0]_B^2. \quad (5.13)$$

Recall again the specification of $B_{(v^0)}$ given in (1.30). We begin with

$$|(\bar{v}_{yyyy}^0, q_{yy} w^2)| \leq \sqrt{L} \|\bar{v}_{yyyy}^0\| \|q_{yy} w\|.$$

Second,

$$\begin{aligned} &-2(\partial_y \{u_s u_{sx} \bar{q}_y^0\}, q_{yy} w^2) \\ &= -2(\partial_y \{u_s u_{sx}\} \bar{q}_y^0, q_{yy} w^2) - 2(u_s u_{sx} \bar{q}_{yy}^0, q_{yy} w^2) \\ &\lesssim L \|u_s \bar{q}_y^0\| \|q_{yy} w\| + L \|u_s \bar{q}_{yy}^0\| \|q_{yy} w\| \\ &\lesssim [\bar{u}^0, \bar{v}^0] \|q\|_w. \end{aligned}$$

Next,

$$\begin{aligned}
 & |(\partial_x \{(x+1)v_s\} \bar{v}_{yyy}^0 - \partial_x \{(x+1)v_{syy}\} \bar{v}_y^0, q_{yy} w^2)| \\
 & \lesssim \sqrt{L} [\|\bar{v}_{yyy}^0 w_0\| + \|\bar{v}_y^0\|] \|q_{yy} w\| \\
 & \lesssim \sqrt{L} [\bar{u}^0, \bar{v}^0]_B \|q\|_w,
 \end{aligned} \tag{5.14}$$

Finally, $|(\varepsilon \partial_x \{(x+1)v_{sxx}\} \bar{v}_y^0, q_{yy} w^2)| \lesssim \sqrt{L} \sqrt{\varepsilon} \|\bar{v}_y^0\| \|q_{yy} w\|$, using the bound $\|\varepsilon y \{v_{sxxx} x + v_{sxx}\}\| \lesssim \sqrt{\varepsilon}$.

Step 3: q_x Multiplier

$$|B_{(v^0)}, q_x w^2| \lesssim o_L(1) \|q\|_w^2 + o_L(1) \begin{cases} [\bar{u}^0, \bar{v}^0]_B^2 & \text{if } w = 1, \\ \varepsilon^{-1} [\bar{u}^0, \bar{v}^0]_B^2 & \text{if } w = w_0 \end{cases}. \tag{5.15}$$

Recall again the specification of $B_{(v^0)}$ given in (1.30). We begin with (letting w be either w_0 or 1 for this calculation)

$$\begin{aligned}
 |(\bar{v}_{yyyy}, q_x w^2)| &= |-(\bar{v}_{yyy}, q_{xy} w^2) - 2(\bar{v}_{yyy}, q_x w w_y)| \\
 &\lesssim \sqrt{L} \|\bar{v}_{yyy} w_0\| \|q_{xy} w\| + \sqrt{L} \|\bar{v}_{yyy} w\| \{\sqrt{L} \|\sqrt{\varepsilon} q_{xx} w\| + \|\frac{q_x}{y} w\|\},
 \end{aligned}$$

where we have used that $|w_y| \lesssim \sqrt{\varepsilon} |w| + 1$, which is true for both choices of w .

Next,

$$\begin{aligned}
 & (\partial_x \{(x+1)v_s\} \bar{v}_{yyy}^0 - \partial_x \{(x+1)v_{syy}\} \bar{v}_y^0, q_x w^2) \\
 &= (\partial_x \{(x+1)v_{sy}\} \bar{v}_y^0 - \partial_x \{(x+1)v_s\} \bar{v}_{yy}^0, q_{xy} w^2 + 2q_x w w_y)
 \end{aligned} \tag{5.16}$$

We must now distinguish the weights for $w = 1$ and $w = w_0$. In the case $w = 1$, we majorize the above quantity by

$$\begin{aligned}
 |(5.16)| &\lesssim (\|v_{sy} + (x+1)v_{sxy}\|_\infty \|\bar{v}_y^0\| + \|v_s + (x+1)v_{sx}\|_\infty \|\bar{v}_{yy}^0\|) \|q_{xy}\| \\
 &\lesssim [\bar{u}^0, \bar{v}^0]_B \|q\|.
 \end{aligned}$$

In the case of $w = w_0$, recalling (5.12)

$$\begin{aligned}
 |(5.16)| &\lesssim [\|\{(x+1)v_{sxy} + v_{sy}\} w_y\|_\infty \|\bar{v}_y^0\| + \|\{v_s + (x+1)v_{sx}\} w_y\|_\infty \|\bar{v}_{yy}^0\|] \\
 &\quad \times [\|q_{xy} w\| + \varepsilon^{-1/2} L \|\sqrt{\varepsilon} q_{xx} w\|] \\
 &\lesssim \varepsilon^{-1/2} L [\bar{u}^0, \bar{v}^0]_B \|q\|_w
 \end{aligned}$$

Next, $(\partial_y \{u_s u_{sx} \bar{q}_y^0\}, q_x w^2) = -(u_s u_{sx} \bar{q}_y^0, q_{xy} w^2) - (u_s u_{sx} \bar{q}_y^0, q_x 2w w_y)$. We again distinguish between the case of $w = 1$ and $w = w_0$. In the $w = 1$ case, we estimate by $\sqrt{L} \|\bar{q}_y^0 u_s\| \|q_{xy}\| \lesssim \sqrt{L} [\bar{u}^0, \bar{v}^0]_B \|q\|_1$. In the $w = w_0$ case, we majorize by

$$\begin{aligned}
 & \sqrt{L} \|\bar{q}_y^0 u_s\| \|q_{xy} w\| + \sqrt{L} \|u_s \bar{q}_y^0\| \{L \|\sqrt{\varepsilon} q_{xx} w\| + \|\frac{q_x}{y} w\|\} \\
 & \lesssim (\sqrt{L} + L^{3/2} + \sqrt{L}) [\bar{u}^0, \bar{v}^0]_B \|q\|_w
 \end{aligned}$$

We move to the final term. In the case $w = 1$,

$$|(\varepsilon \bar{v}_y^0 \partial_x \{(x+1)v_{sxx}\}, q_x)| \lesssim \sqrt{\varepsilon} \|\bar{q}_{xy}\| \|v_y^0\| \lesssim \sqrt{\varepsilon} [\bar{u}^0, \bar{v}^0]_B \|q\|_1$$

upon using that $|\sqrt{\varepsilon} y \partial_x \{(x+1)v_{sxx}\}| \lesssim 1$. In the case $w = w_0$,

$$|(\varepsilon \bar{v}_y^0 \partial_x \{(x+1)v_{sxx}\}, q_x)| \lesssim \left\| \frac{q_x}{y} w \right\| \|v_y^0\| \lesssim [\bar{u}^0, \bar{v}^0]_B \|q\|_w.$$

upon using that $|\varepsilon y w_0 \partial_x \{(x+1)v_{sxx}\}| \lesssim 1$. For these profile estimates, we have used (A.33).

Step 4: $\varepsilon v_{xxyy} u_s$ Multiplier

$$|(B_{(v^0)}, \varepsilon u_s v_{xxyy} w^2)| \leq C \sqrt{\varepsilon} [\bar{u}^0, \bar{v}^0]_B^2 + o(1) \sqrt{\varepsilon} \|q\|^2_{\sqrt{\varepsilon} w} + C \sqrt{\varepsilon} \|v\|^2_w. \quad (5.17)$$

Recall again the specification of $B_{(v^0)}$ given in (1.30). We compute

$$\begin{aligned} & (\bar{v}_{yyy}^0, \varepsilon u_s v_{xxyy} w^2) \\ &= -(\bar{v}_{yyy}^0, \varepsilon u_{sx} v_{xyy} w^2) - (\bar{v}_{yyy}^0, \varepsilon u_s v_{xyy} w^2)_{x=0} \\ &\leq \sqrt{L} \sqrt{\varepsilon} \|\bar{v}_{yyy}^0 w\| \|v_{xyy}\| \sqrt{\varepsilon} w + \sqrt{\varepsilon} \|\bar{v}_{yyy}^0 w\| \|u_s v_{xyy}\| \sqrt{\varepsilon} w_{x=0} \\ &\lesssim \sqrt{L} \sqrt{\varepsilon} [\bar{u}^0, \bar{v}^0]_B \|q\|_{\sqrt{\varepsilon} w} + \sqrt{\varepsilon} [\bar{u}^0, \bar{v}^0]_B \|v\|_w \end{aligned}$$

Next,

$$\begin{aligned} & (\partial_x \{(x+1)v_s\} \bar{v}_{yyy}^0 - \partial_x \{(x+1)v_{syy}\} \bar{v}_y^0, \varepsilon u_s v_{xxyy} w^2) \\ &= -(\partial_x \{(x+1)v_s\} \bar{v}_{yyy}^0 - \partial_x \{(x+1)v_{syy}\} \bar{v}_y^0, \varepsilon u_s v_{xyy} w^2)_{x=0} \\ &\quad - (\partial_x \{(x+1)v_s\} \bar{v}_{yyy}^0 - \partial_x \{(x+1)v_{syy}\} \bar{v}_y^0, \varepsilon u_{sx} v_{xyy} w^2) \\ &\quad - (\partial_{xx} \{(x+1)v_s\} \bar{v}_{yyy}^0 - \partial_{xx} \{(x+1)v_{syy}\} \bar{v}_y^0, \varepsilon u_s v_{xyy} w^2). \end{aligned} \quad (5.18)$$

First,

$$\begin{aligned} |(5.18.1)| &\lesssim (\sqrt{\varepsilon} \|\sqrt{u_s} \bar{v}_{yyy}^0 w_0\| + \sqrt{\varepsilon} \|\bar{v}_y^0\|) \|\sqrt{\varepsilon} u_s v_{xyy} w\|_{x=0} \\ &\lesssim \sqrt{\varepsilon} [\bar{u}^0, \bar{v}^0]_B \|v\|_w + \sqrt{\varepsilon} [\bar{u}^0, \bar{v}^0]_B \|v\|_w. \end{aligned}$$

Next,

$$\begin{aligned} & |(5.18.2)| + |(5.18.3)| \lesssim \sqrt{L} \sqrt{\varepsilon} (\|u_s \bar{v}_{yyy}^0 w_0\| + \|\bar{v}_y^0\|) \|v_{xyy}\| \sqrt{\varepsilon} w \\ & (-2\partial_y \{u_s u_{sx} \bar{q}_y^0\}, \varepsilon u_s v_{xxyy} w^2) \\ &= (2\partial_y \{u_s u_{sx} \bar{q}_y^0\}, \varepsilon u_{sx} v_{xyy} w^2) + (2\partial_y \{(u_s u_{sx})_x \bar{q}_y^0\}, \varepsilon u_s v_{xyy} w^2) \\ &\quad + 2(\partial_y \{u_s u_{sx} \bar{q}_y^0\}, \varepsilon u_s v_{xyy} w^2)_{x=0} \\ &= (2\partial_y \{u_s u_{sx}\} \bar{q}_y^0, \varepsilon u_{sx} v_{xyy} w^2) + (2u_s u_{sx} \bar{q}_{yy}^0, \varepsilon u_{sx} v_{xyy} w^2) \end{aligned}$$

$$\begin{aligned}
& + (2\{u_s u_{sx}\}_{xy} \bar{q}_y^0, \varepsilon u_s v_{xyy} w^2) + (2(u_s u_{sx})_x \bar{q}_{yy}^0, \varepsilon u_s v_{xyy} w^2) \\
& + 2(\partial_y \{u_s u_{sx}\} \bar{q}_y^0, \varepsilon u_s v_{xyy} w^2)_{x=0} + 2(u_s u_{sx} \bar{q}_{yy}^0, \varepsilon u_s v_{xyy} w^2)_{x=0}
\end{aligned} \quad (5.19)$$

We begin with the first two terms. Since u_{sx} decays at $y = \infty$,

$$|(5.19.1)| \lesssim \sqrt{\varepsilon} \sqrt{L} \|u_s \bar{q}_y^0\| \|v_{xyy} \sqrt{\varepsilon} w\|,$$

$$|(5.19.2)| \lesssim \sqrt{\varepsilon} \sqrt{L} \|\sqrt{u_s} \bar{q}_{yy}^0\| \|v_{xyy} \sqrt{\varepsilon} w\|,$$

Next, we estimate the third and fourth terms

$$|(5.19.3)| + |(5.19.4)| \lesssim \sqrt{\varepsilon} \sqrt{L} (\|u_s \bar{q}_y^0\| + \|u_s \bar{q}_{yy}^0\|) \|v_{xyy} \sqrt{\varepsilon} w\|,$$

The last two terms follow very similarly from the first two, yielding

$$|(5.19.5)| \lesssim \sqrt{\varepsilon} \|u_s \bar{q}_y^0\| \|\sqrt{\varepsilon} u_s v_{xyy} w\|_{x=0},$$

$$|(5.19.6)| \lesssim \sqrt{\varepsilon} \|\sqrt{u_s} \bar{q}_{yy}^0\| \|\sqrt{\varepsilon} u_s v_{xyy} w\|_{x=0}$$

We finally move to

$$\begin{aligned}
& (\varepsilon \partial_x \{(x+1)v_{sxx}\} \bar{v}_y^0, \varepsilon u_s v_{xyy} w^2) \\
& = -(\varepsilon \partial_{xx} \{(x+1)v_{sxx}\} \bar{v}_y^0, \varepsilon u_s v_{xyy} w^2) - (\varepsilon \partial_x \{(x+1)v_{sxx}\} \bar{v}_y^0, \varepsilon u_{sx} v_{xyy} w^2) \\
& \quad - (\varepsilon \partial_x \{(x+1)v_{sxx}\} \bar{v}_y^0, \varepsilon u_s v_{xyy} w^2)_{x=0} \\
& \lesssim \varepsilon \sqrt{L} \|\bar{v}_y^0\| \|\sqrt{\varepsilon} u_s v_{xyy} w\|_{x=L} + \sqrt{L} \varepsilon \|\bar{v}_y^0\| \|\sqrt{\varepsilon} v_{xyy} w\| \\
& \quad + \varepsilon \|\bar{v}_y^0\| \|\sqrt{\varepsilon} u_s v_{xyy} w\|_{x=0}
\end{aligned}$$

Step 5: $\varepsilon^2 v_{xxx}$ Multiplier

$$|(B_{(v^0)}, \varepsilon^2 v_{xxx} w^2)| \leq C \varepsilon [\bar{u}^0, \bar{v}^0]_B^2 + \varepsilon \|q\|_{\sqrt{\varepsilon} w}^2 + C \varepsilon \|v\|_{w^0}^2. \quad (5.20)$$

This follows in the same manner as the previous multiplier. Putting together estimates (5.9), (5.13), (5.15), (5.17), (5.20) according to the linear combinations in (5.3) and (5.4) gives the desired bound and completes the proof of the lemma. $\square$

Lemma 5.3. *Let $u \in \mathcal{X}$ as in (1.22). For B_B defined as in (5.5), and for any $\delta > 0$, the following estimates are valid*

$$|B_B| \leq \delta [u^0, v^0]_B + C_\delta \varepsilon \|\bar{v}\|_{Y_{w_0}}^2 + C_\delta \varepsilon^{\frac{1}{2} + \frac{3-}{16}} \|\bar{v}\|_{X_1}^2. \quad (5.21)$$

Proof. We estimate each term in $F_{(v)}(\bar{v})$ which are defined in (A.23) and we write here for convenience:

$$F_{(v)} := -2\varepsilon u_s u_{sx} \bar{q}_x|_{x=0} - 2\varepsilon \bar{v}_{xyy}|_{x=0} - \varepsilon^2 \bar{v}_{xxx}|_{x=0} + \varepsilon v_s \bar{v}_{xy}|_{x=0}. \quad (5.22)$$

Starting with the higher order terms,

$$\begin{aligned} \|\varepsilon^2 \bar{v}_{xxx} w\|_{x=0} &\leq \|\varepsilon^2 \bar{v}_{xxx} w\{1 - \chi\}\|_{x=0} + \|\varepsilon^2 \bar{v}_{xxx} w \chi\|_{x=0} \\ &\leq \sqrt{\varepsilon} \|\varepsilon^{\frac{3}{2}} u_s \bar{v}_{xxx} w\|_{x=0} + \varepsilon^{\frac{1}{2}} \|\varepsilon \bar{v}_{xxx}\|_{\frac{1}{2}} \|\varepsilon^2 \bar{v}_{xxx} w\|_{\frac{1}{2}} \\ &\lesssim \sqrt{\varepsilon} \|\bar{v}\|_{Y_w} + \varepsilon^{\frac{1}{2} + \frac{3}{32}} \|\bar{v}\|_{X_1}. \end{aligned}$$

The identical argument is performed for (5.22.2).

For the fourth term, we expand $\bar{v}_{xy}|_{x=0} = u_s \bar{q}_{xy}|_{x=0} + u_{sy} \bar{q}_x|_{x=0}$, perform a Hardy type inequality for the $\bar{q}_x$ term, and use (3.22) to obtain

$$\begin{aligned} \|\varepsilon v_s \bar{v}_{xy} w\|_{x=0} &\leq \|\varepsilon v_s u_s \bar{q}_{xy} w\|_{x=0} + \|\varepsilon v_s u_{sy} \bar{q}_x w\|_{x=0} \\ &\leq \sqrt{\varepsilon} \|\bar{v}\|_{Y_w} + \varepsilon \|v_s u_{sy} \bar{q}_x \chi\|_{x=0} \\ &\leq \sqrt{\varepsilon} \|\bar{v}\|_{Y_w} + \varepsilon^{\frac{3}{4}} \|\varepsilon^{\frac{1}{4}} \frac{\bar{q}_x}{\langle y \rangle}\|_{x=0} \\ &\leq \sqrt{\varepsilon} \|\bar{v}\|_{Y_w} + \varepsilon^{\frac{3}{4}} \|\bar{v}\|_{X_1}. \end{aligned}$$

To estimate the first term from (5.22), we split into Euler and Prandtl:

$$\begin{aligned} \|\varepsilon u_s u_{sx} \bar{q}_x w\|_{x=0} &\leq \|\varepsilon u_s u_{sx}^P \bar{q}_x w\|_{x=0} + \varepsilon^{\frac{3}{2}} \|u_s u_{sx}^E \bar{q}_x w\|_{x=0} \\ &\leq \|u_{sx}^P w \langle y \rangle\|_{\infty} \varepsilon \|\frac{\bar{q}_x}{\langle y \rangle}\|_{x=0} + \sqrt{\varepsilon} [\|\sqrt{\varepsilon} \bar{q}_x \sqrt{\varepsilon} w\| + \|\sqrt{\varepsilon} \bar{q}_{xx} \sqrt{\varepsilon} w\|] \\ &\lesssim \varepsilon^{\frac{3}{4}} \|\bar{v}\|_{X_1} + \sqrt{\varepsilon} \|\bar{v}\|_{Y_w}. \end{aligned}$$

We have thus established: $\|F_{(v)} w_0\| \lesssim \sqrt{\varepsilon} \|\bar{v}\|_{Y_{w_0}} + \varepsilon^{\frac{1}{2} + \frac{3}{32}} \|\bar{v}\|_{X_1}$. This concludes the proof. $\square$

Lemma 5.4. *Let $\mathbf{u} \in \mathcal{X}$ as in (1.22). The following estimates are valid*

$$\begin{aligned} |\mathcal{N}_{X_1}| &\lesssim \varepsilon^{N_0-1} [\bar{u}^0, \bar{v}^0]_B \|\bar{v}\|_{X_1} \|v\|_{X_1} + \varepsilon^{N_0-\frac{3}{4}} \|\bar{v}\|_{X_1} \|\bar{v}\|_{X_1} \|v\|_{X_1}, \\ &\quad + \varepsilon^{N_0-1} [\bar{u}^0, \bar{v}^0]_B^2 \|v\|_{X_1}, \end{aligned} \quad (5.23)$$

$$\begin{aligned} |\mathcal{N}_{Y_w}| &\lesssim \varepsilon^{N_0-1} [\bar{u}^0, \bar{v}^0]_B \|\bar{v}\|_{Y_w} \|v\|_{Y_w} + \varepsilon^{N_0-\frac{3}{4}} \|\bar{v}\|_{X_1} \|\bar{v}\|_{Y_w} \|v\|_{Y_w} \\ &\quad + \varepsilon^{N_0-1} [\bar{u}^0, \bar{v}^0]_B^2 \|v\|_{X_1}, \end{aligned} \quad (5.24)$$

$$|\mathcal{N}_B| \lesssim \varepsilon^{N_0-1} [\bar{u}^0, \bar{v}^0]_B^4 + \varepsilon^{N_0-1} [\bar{u}^0, \bar{v}^0]_B^2 + C(h, a_2^\varepsilon) + \varepsilon^{N_0-1} \|\bar{v}\|_{Y_w}^2. \quad (5.25)$$

Proof. *Proof of (5.23), (5.24):*

We begin with the immediate estimates:

$$|\mathcal{N}_{X_1}| \lesssim \varepsilon^{N_0-\frac{1}{2}} \|\mathcal{N}\| \|v\|_{X_1} \quad |\mathcal{N}_{Y_w}| \lesssim \varepsilon^{N_0} \|\mathcal{N} w\| \|v\|_{Y_w}.$$

First, recall the specification of $\mathcal{N} = Q_{11} + Q_{12} + Q_{22}$ given in (A.27). We now establish the following bound:

$$\|\mathcal{N} \cdot w\| \lesssim \{\varepsilon^{-1/2}[\bar{u}^0, \bar{v}^0]_B + \varepsilon^{-\frac{1}{4}}\|\bar{v}\|_{X_1}\}||\bar{q}||_w.$$

To establish this, we go term by term through Q_{11} :

$$\begin{aligned} \|\bar{v}_y \Delta_\varepsilon \bar{v} w\| &\leq \|\bar{v}_y\|_\infty \|\Delta_\varepsilon \bar{v} w\| \\ \|I_x[\bar{v}_y] \Delta_\varepsilon \bar{v}_x w\| &\leq \|\bar{v}_y\|_\infty \|\Delta_\varepsilon \bar{v}_x w\| \\ \|\bar{v}_x I_x[\bar{v}_{yyy}] w\| &\leq \varepsilon^{-\frac{1}{4}} \|\varepsilon^{\frac{1}{4}} \bar{v}_x\|_{L_x^2 L_y^\infty} \|\bar{v}_{yyy} w\| \\ \|\bar{v} \bar{v}_{yyy} w\| &\leq \varepsilon^{-\frac{1}{4}} \|\varepsilon^{\frac{1}{4}} \bar{v}\|_{L_x^2 L_y^\infty} \|\bar{v}_{yyy} w\| \\ \|\varepsilon \bar{v}_x \bar{v}_{xy} w\| &\leq \sqrt{\varepsilon} \|\sqrt{\varepsilon} \bar{v}_x\|_\infty \|\bar{v}_{xy} w\| \\ \|\bar{v} \Delta_\varepsilon \bar{v}_y w\| &\leq \varepsilon^{-\frac{1}{4}} \|\varepsilon^{\frac{1}{4}} \bar{v}\|_\infty \|\Delta_\varepsilon \bar{v}_y w\| \\ \|\bar{u}^0 \Delta_\varepsilon \bar{v}_x w\| &\leq |\bar{u}^0|_\infty \|\Delta_\varepsilon \bar{v}_x w\| \lesssim [\bar{u}^0, \bar{v}^0]_B ||\bar{q}||_w, \\ \|\bar{u}_{yy}^0 \bar{v}_x w\| &\leq \|\bar{u}_{yy}^0 \langle y \rangle\|_{L_x^\infty L_y^2} \|\bar{v}_x \langle y \rangle^{-1} w\|_{L_x^2 L_y^\infty} \lesssim [\bar{u}^0, \bar{v}^0]_B ||\bar{q}||_w. \end{aligned}$$

Above, we have used the following interpolation:

$$\|\bar{v}_x \langle y \rangle^{-\frac{1}{2}} w\|_{L_x^\infty L_y^2} \leq \|\bar{v}_x \langle y \rangle^{-1} w\|^{\frac{1}{2}} \|\bar{v}_{xy} w\|^{\frac{1}{2}},$$

and the weighted Hardy's inequality (3.27). The result follows upon remarking the following basic fact. For any function $g(x, y)$ such that $g_{x=0}$ OR $g_{x=L} = 0$ and $g|_{y=0} = 0$: $|g|^2 \leq \|g_x\| \|g_y\| + \|g\| \|g_{xy}\|$. This immediately gives: $\|\varepsilon^{\frac{1}{4}} \bar{v}\|_\infty + \|\bar{v} \langle y \rangle^{-1/2}\|_\infty + \|\nabla_\varepsilon \bar{v}\|_\infty \lesssim ||\bar{q}||_1$. A basic interpolation also gives $\|\varepsilon^{\frac{1}{4}} \bar{v}_x\|_{L_y^\infty} \leq \|\sqrt{\varepsilon} \bar{v}_x\|_{L_y^2}^{\frac{1}{2}} \|\bar{v}_{xy}\|_{L_y^2}^{\frac{1}{2}}$.

We treat now the quantity $\|Q_{12} w\|$

$$\begin{aligned} \|v_y v_{yy}^0 w\| &\leq \|v_{xy} w\| \|v_{yy}^0\|_\infty \lesssim [u^0, v^0]_B ||\bar{q}||_w, \\ \|v_y^0 \Delta_\varepsilon v w\| &\leq \|v_y^0\|_\infty \|\Delta_\varepsilon v w\| \lesssim [u^0, v^0]_B ||\bar{q}||_w, \\ \|x v^0 \Delta_\varepsilon v_x w\| &\lesssim \|v^0\|_\infty \|\Delta_\varepsilon v_x w\| \lesssim \varepsilon^{-\frac{1}{2}} [u^0, v^0]_B ||\bar{q}||_w \\ \|x v_x v_{yy}^0 w\| &\lesssim \varepsilon^{-1/2} \|\sqrt{\varepsilon} v_{xx} w\| \|v_{yy}^0\|_\infty \lesssim \varepsilon^{-1/2} [u^0, v^0]_B ||\bar{q}||_w, \\ \|v v_{yyy}^0 w\| &\leq \varepsilon^{-\frac{1}{4}} \|\varepsilon^{\frac{1}{4}} v\|_{L_x^2 L_y^\infty} \|v_{yyy}^0 w\| \lesssim \varepsilon^{-\frac{1}{4}} [u^0, v^0]_B ||\bar{q}||_w, \\ \|v^0 \Delta_\varepsilon v_y w\| &\leq \|v^0\|_\infty \|\Delta_\varepsilon v_y w\| \lesssim \varepsilon^{-\frac{1}{2}} [u^0, v^0]_B ||\bar{q}||_w. \end{aligned}$$

To conclude, we note that the Q_{22} terms have already been treated in Lemmas 2.7 and 2.8.

Proof of (5.25)

Recall the specification of Q from (2.1). We begin with the multiplier of q^0 . First,

$$\begin{aligned} \varepsilon^{N_0}(v^0 v_{yyy}^0 - v_y^0 v_{yy}^0, q^0) &= \varepsilon^{N_0}(u_s q^0 \partial_{yyy} \{u_s q^0\} - \partial_y \{u_s q^0\} \partial_{yy} \{u_s q^0\}, q^0) \\ &= \varepsilon^{N_0}(u_s^2 q^0 q_{yyy}^0 - u_s^2 q_y^0 q_{yy}^0, q^0) + \mathcal{J}_2. \end{aligned}$$

Here,

$$\begin{aligned} \mathcal{J}_2 := & \varepsilon^{N_0}(u_s q^0 [u_{sy} q^0 + 3u_{sy} q_y^0 + 3u_{sy} q_{yy}^0] \\ & - u_{sy} q^0 [u_{sy} q^0 + 2u_{sy} q_y^0 + u_s q_{yy}^0] - u_s q_y^0 [u_{sy} q^0 + 2u_{sy} q_y^0], q^0) \end{aligned}$$

Thus, $\mathcal{J}_2$ contains harmless commutator terms which are easily seen to be size $\varepsilon^{N_0}[[[v^0]]][[q^0]]^2$ upon using (2.8), (2.11), and the rapid decay of $\partial_y^k u_s$ ($k \geq 1$) which is present in each term above. We estimate

$$\begin{aligned} \varepsilon^{N_0} |(u_s^2 q^0 q_{yyy}^0, q^0)| &\lesssim \varepsilon^{N_0} [[q^0]]^2 (u_s^2 |q_{yyy}^0|, \langle y \rangle) \\ &\lesssim \varepsilon^{N_0 - (\frac{1}{2} +)} \left(q_{yyy}^0 \langle y \rangle Y^{\frac{1+}{2}}, \langle y \rangle^{-\frac{1+}{2}} \right) [[q^0]]^2 \\ &\lesssim \varepsilon^{N_0 - (\frac{1}{2} +)} \|q_{yyy}^0 \langle y \rangle Y^{\frac{1+}{2}}\| [[q^0]]^2 \\ &\lesssim \varepsilon^{N_0 - (\frac{1}{2} +)} [[v^0]] [[q^0]]^2. \end{aligned}$$

Next, recalling (2.15)

$$\begin{aligned} \varepsilon^{N_0} |(u_s^2 q_y^0 q_{yy}^0, q^0)| &\lesssim \varepsilon^{N_0} \|q^0\|_\infty \|\sqrt{u_s} q_y^0\| \|u_s q_{yy}^0\| \\ &\lesssim \varepsilon^{N_0 - \frac{1}{2}} C_{\sigma, \lambda} [[q^0]]^3. \end{aligned}$$

The next nonlinear terms are

$$\begin{aligned} & \varepsilon^{N_0+1} (u^0 v_{xx}|_{x=0} + v^0 v_{xy}|_{x=0}, q^0) \\ & \lesssim \varepsilon^{N_0+1} \|u^0\|_\infty \|v_{xx}|_{x=0} \langle y \rangle\| \|q_y^0\| + \varepsilon^{N_0+1} \|v^0\|_\infty \|v_{xy}|_{x=0} w\| \|q_y^0\| \\ & \lesssim \varepsilon^{N_0+1} [u^0, v^0]_B^2 \|a_2^\varepsilon \langle y \rangle\| + \varepsilon^{N_0 + \frac{1}{4}} (\sqrt{\varepsilon} \|v^0\|_\infty) \|v_{xy} w\|^{\frac{1}{2}} \|\sqrt{\varepsilon} v_{xxy} w\|^{\frac{1}{2}} \|q_y^0\| \\ & \lesssim \varepsilon^{N_0+1} [u^0, v^0]_B^2 \|a_2^\varepsilon \langle y \rangle\| + \varepsilon^{N_0 - \frac{1}{4}} [u^0, v^0]_B^2 \|v\|_{Y_w}. \end{aligned}$$

To conclude, we treat the contribution of the h terms:

$$\begin{aligned} & |(\varepsilon^{N_0} \{h v_{yy}^0 - v^0 h_{yy}\}, q^0)| \\ & \lesssim \varepsilon^{N_0} \|h \langle y \rangle\|_\infty \|v_{yy}^0\| \|q_y^0\| + \varepsilon^{N_0} \|h_{yy} y^2\|_\infty \|v_y^0\| \|q_y^0\| \\ & \lesssim \varepsilon^{N_0} \{\|h \langle y \rangle\|_\infty + \|h_{yy} y^2\|_\infty\} [u^0, v^0]_B^2. \end{aligned}$$

Next,

$$\begin{aligned} |\langle \mathcal{H}, q^0 \rangle| &\lesssim [[q^0]] \|\mathcal{H}\langle y \rangle^{1/2}\|_1 \\ &\lesssim [u^0, v^0]_B \|\{h''' - v_s h'' - h \Delta_\varepsilon u_s\} \langle y \rangle^{1/2}\|_1. \end{aligned}$$

We now move to the contribution of $\|Qw_0\|$. We estimate the first term directly upon using (2.15):

$$\|\varepsilon^{N_0} v^0 v_{yyy}^0 w_0\| \leq \varepsilon^{N_0 - \frac{1}{2}} \|\sqrt{\varepsilon} v^0\|_\infty \|v_{yyy}^0 w\| \lesssim \varepsilon^{N_0 - \frac{1}{2}} [u^0, v^0]_B^2.$$

For the second nonlinearity, we have

$$\begin{aligned} &\|\varepsilon^{N_0} \partial_y \{u_s q^0\} \partial_{yy} \{u_s q^0\} w_0\| \\ &= \|\varepsilon^{N_0} (u_s u_{syy} q^0 q_y^0 + 2u_s u_{sy} |q_y^0|^2 + u_s^2 q_y^0 q_{yy}^0 + u_{sy} u_{syy} |q^0|^2 \\ &\quad + 2u_{sy}^2 q^0 q_y^0 + u_s u_{sy} q^0 q_{yy}^0) w_0\| \\ &\leq \varepsilon^{N_0} \|\{u_s u_{syy} q^0 q_y^0 + 2u_s u_{sy} |q_y^0|^2 + u_s^2 q_y^0 q_{yy}^0 + u_{sy} u_{syy} |q^0|^2 \\ &\quad + 2u_{sy}^2 q^0 q_y^0 + u_s u_{sy} q^0 q_{yy}^0\} w\| \\ &\lesssim \varepsilon^{N_0 - 1} [u^0, v^0]_B^2. \end{aligned}$$

Above, we have used

$$\|u_s^2 q_y^0 q_{yy}^0 w\| \lesssim \|q_y^0 \langle y \rangle\|_\infty \|q_{yy}^0 \frac{w}{\langle y \rangle}\| \lesssim \varepsilon^{-1} \|\varepsilon \langle y \rangle q_y^0\|_\infty [u^0, v^0]_B \lesssim \varepsilon^{-1} [u^0, v^0]_B^2.$$

We next move to

$$\begin{aligned} &\|\varepsilon^{N_0+1} u^0 v_{xx}^{(\varepsilon)}|_{x=0} w_0\| + \varepsilon^{N_0+1} \|v^0 v_{xy}^{(\varepsilon)}|_{x=0} w_0\| \\ &\lesssim \varepsilon^{N_0+1} \left(\|u^0\|_\infty \|v_{xx}^{(\varepsilon)}|_{x=0} w\| + \|v^0\|_\infty \|v_{xy}^{(\varepsilon)}|_{x=0} w\| \right) \\ &\lesssim \varepsilon^{N_0+1} [u^0, v^0]_B^2 \|a_2^\varepsilon w\| + \varepsilon^{N_0+\frac{1}{4}} \|\sqrt{\varepsilon} v^0\|_\infty \|v_{xy} w\|^{\frac{1}{2}} \|\sqrt{\varepsilon} v_{xxy} w\|^{\frac{1}{2}} \\ &\lesssim \varepsilon^{N_0+1} [u^0, v^0]_B^2 \|a_2^\varepsilon w\| + \varepsilon^{N_0-\frac{1}{4}} [u^0, v^0]_B^2 \|v\|_{Y_w}. \end{aligned}$$

To conclude, we estimate the contributions of h , starting with

$$\begin{aligned} \|\varepsilon^{N_0} \{h v_{yy}^0 - v^0 h_{yy}\} w_0\| &\lesssim \varepsilon^{N_0} \|h w\|_\infty \|v_{yy}^0\| + \varepsilon^{N_0} \left\| \frac{v^0}{y} \right\|_\infty \|h_{yy} y w\|_2 \\ &\lesssim \varepsilon^{N_0} [[q^0]] C(h) \lesssim \varepsilon^{N_0} [u^0, v^0]_B^2 + C(h). \end{aligned}$$

We next move to the $\mathcal{H}$ terms:

$$\|\mathcal{H} w_0\| \leq \|[-h''' + v_s h'' - h \Delta_\varepsilon v_s] w\| \leq C(h).$$



The remaining terms from the right-hand sides of (5.6) are the $\mathcal{F}$ terms, for which we estimate

Lemma 5.5. *Let $\mathbf{u} \in \mathcal{X}$ as in (1.22). Assume (1.16) and $h \in C^\infty(e^y)$ as in (1.15). Let $n > 1 + 2N_0$ in Theorem A.3. Then the forcing terms satisfy*

$$\varepsilon^{\frac{1}{2}} |\mathcal{F}_{X_1}| + |\mathcal{F}_B| + \varepsilon^{\frac{1}{2}} |\mathcal{F}_{Y_{w_0}}| \leq o(1) + o(1) \|\mathbf{u}\|_{\mathcal{X}}^2 + o(1) \|\bar{\mathbf{u}}\|_{\mathcal{X}}^2. \quad (5.26)$$

Proof. Recalling the definition of $F_{(q)}, F_R^a$ from (A.23), (A.27):

$$\begin{aligned} F_{(q)} &= \partial_x F_R + \partial_x b_{(u)}(a^\varepsilon) + H[a^\varepsilon](\bar{v}, \bar{u}^0, \bar{v}^0) + \{v_{sx} h_{yy} - h \Delta_\varepsilon v_{sx}\}, \\ F_R^a &= F_R|_{x=0} + b_{(u)}(a^\varepsilon). \end{aligned}$$

Examining the definition of $\mathcal{F}_{X_1}, \mathcal{F}_{Y_{w_0}}, \mathcal{F}_B$ (from (5.3), (5.4), (5.5)), we may estimate

$$\begin{aligned} \varepsilon^{\frac{1}{2}} \left(\mathcal{F}_{X_1}(\partial_x F_R, q) + \mathcal{F}_{Y_{w_0}}(\partial_x F_R, q) \right) &\lesssim \varepsilon^{\frac{1}{2}} \left\| \frac{1}{\sqrt{\varepsilon}} \partial_x F_R w_0 \right\| [\|v\|_{Y_{w_0}} + \|v\|_{X_1}] \\ &\lesssim o(1) + o(1) \|\mathbf{u}\|_{\mathcal{X}}^2, \end{aligned}$$

upon recalling (A.34). Next,

$$\begin{aligned} \mathcal{F}_B(F_R|_{x=0}, q) &\leq |(F_R, q^0)| + \|F_R w_0\|^2 \\ &\lesssim \|F_R w_0\| \|q_y^0\| + \|F_R w_0\|^2 \\ &\leq o(1) + o(1) [u^0, v^0]_B^2 \\ &\leq o(1) + o(1) \|\mathbf{u}\|_{\mathcal{X}}^2. \end{aligned}$$

Repeating the above estimates for the $\partial_x b_{(u)}(a^\varepsilon), b_{(u)}(a^\varepsilon)$ terms, we obtain that these contributions to (5.26) are bounded by

$$C \left\| \frac{1}{\sqrt{\varepsilon}} \partial_x b_{(u)}(a^\varepsilon) \right\|^2 + \|b_{(u)}(a^\varepsilon) w_0\|^2 + o(1) \|\mathbf{u}\|_{\mathcal{X}}^2 \lesssim o(1) + o(1) \|\mathbf{u}\|_{\mathcal{X}}^2,$$

upon invoking assumption (1.16) and consulting the definitions (A.25).

A similar computation, consulting the definition of $H[a^\varepsilon](\bar{u}^0, \bar{v}^0, \bar{v})$ given in (A.27), produces a bound

$$\|H[a^\varepsilon](\bar{u}^0, \bar{v}^0, \bar{v}) \frac{w_0}{\sqrt{\varepsilon}}\| \leq o(1) + o(1) \left([\bar{u}^0, \bar{v}^0]_B + \|\bar{v}\|_{X_1} + \|\bar{v}\|_{Y_{w_0}} \right),$$

upon invoking again assumption (1.16). A similar estimate holds for the h terms in $F_{(q)}$ using (1.15). This thus concludes the proof of (5.26). $\square$

We are now ready to insert all of these estimates into (5.6), which gives the following

Proposition 5.6. *For $\sigma \ll 1$ then $L \ll 1$, solutions to (5.1), (5.2) satisfy the following set of estimates:*

$$\begin{aligned} \|v\|_{X_1}^2 &\lesssim o(1)\|v\|_{X_1}^2 + \varepsilon^{-\frac{1}{2}}[\bar{u}^0, \bar{v}^0]_B^2 \\ &\quad + \varepsilon^{N_0-1} \left(\|\bar{v}\|_{X_1}^4 + [\bar{u}^0, \bar{v}^0]_B^4 \right) + C(h) + \mathcal{F}_{X_1} \end{aligned} \quad (5.27)$$

$$\begin{aligned} [u^0, v^0]_B^2 &\lesssim \varepsilon \|\bar{v}\|_{Y_{w_0}}^2 + \varepsilon^{\frac{1}{2} + \frac{3-}{16}} \|\bar{v}\|_{X_1}^2 \\ &\quad + \varepsilon^{N_0-1} [\bar{u}^0, \bar{v}^0]_B^4 + C(h, a_2^\varepsilon) + \mathcal{F}_B \end{aligned} \quad (5.28)$$

$$\begin{aligned} \|v\|_{Y_{w_0}}^2 &\lesssim \|v\|_{X_1}^2 + [\bar{u}^0, \bar{v}^0]_B^2 \\ &\quad + \varepsilon^{N_0-1} \left(\|\bar{v}\|_{X_1}^4 + \|\bar{v}\|_{Y_{w_0}}^4 + [\bar{u}^0, \bar{v}^0]_B^4 \right) + C(h) + \mathcal{F}_{Y_{w_0}}. \end{aligned} \quad (5.29)$$

Above, $C(h) = \mathcal{O}(\|h\|_{C^{M_0}(e^{\mathcal{V}})})$ for a large M_0 .

From here, we may immediately prove the main result:

Proof of Theorem 1.1. We apply a standard contraction mapping theorem to the map Ψ which sends $[\bar{v}, \bar{u}^0, \bar{v}^0]$ to $[v, u^0, v^0]$ via the equations (5.1), (5.2). Such a map is well-defined according to Proposition 2.1 and Proposition 3.2.

Recall the definition of $\|\cdot\|_{\mathcal{X}}$ from (1.22). Motivated by this, we define for a large number $K \gg 1$, the equivalent norm

$$\|\mathbf{u}\|_{\mathcal{X}_K} := \varepsilon^{\frac{1}{4}} \|v\|_{X_1} + \varepsilon^{\frac{1}{4}} \|v\|_{Y_{w_0}} + K[u^0, v^0]_B,$$

and we appropriately modify definition (1.23) to define the space $\mathcal{X}_K$. We now take the linear combination $\varepsilon^{\frac{1}{2}}(5.27) + K^2(5.28) + \varepsilon^{\frac{1}{2}}(5.29)$ to obtain

$$\begin{aligned} \|\mathbf{u}\|_{\mathcal{X}_K}^2 &\lesssim o(1)\varepsilon^{\frac{1}{2}}\|v\|_{X_1}^2 + [\bar{u}^0, \bar{v}^0]_B^2 + \varepsilon^{N_0-\frac{1}{2}} \left(\|\bar{v}\|_{X_1}^4 + [\bar{u}^0, \bar{v}^0]_B^4 \right) + C(h) \\ &\quad + \left(\varepsilon^{\frac{1}{2}} \mathcal{F}_{X_1} + K^2 \mathcal{F}_B + \varepsilon^{\frac{1}{2}} \mathcal{F}_{Y_{w_0}} \right) + \varepsilon K^2 \|\bar{v}\|_{Y_{w_0}}^2 + \varepsilon^{\frac{1}{2} + \frac{3-}{16}} \|\bar{v}\|_{X_1}^2 \\ &\quad + K^2 \varepsilon^{N_0-1} [\bar{u}^0, \bar{v}^0]_B^4 + K^2 C(h, a_2^\varepsilon) + \varepsilon^{\frac{1}{2}} [\bar{u}^0, \bar{v}^0]_B^2 \\ &\quad + \varepsilon^{N_0-\frac{1}{2}} \left(\|\bar{v}\|_{X_1}^4 + \|\bar{v}\|_{Y_{w_0}}^4 + [\bar{u}^0, \bar{v}^0]_B^4 \right) \\ &\leq o(1)\|\mathbf{u}\|_{\mathcal{X}_K}^2 + (o(1) + K^{-2})\|\bar{\mathbf{u}}\|_{\mathcal{X}_K}^2 + \varepsilon^{N_0-1}\|\bar{\mathbf{u}}\|_{\mathcal{X}_K}^4 + \mathcal{O}(1). \end{aligned} \quad (5.30)$$

By repeating the above analysis for differences $\mathbf{u}_1 - \mathbf{u}_2$, and $\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2$, (5.30) shows that Ψ is a contraction map on $\mathcal{X}_K$ for $K \gg 1$, and thus has a unique fixed point. Clearly, from (5.1) and (5.2), such a fixed point solves the nonlinear equations (A.23) and (A.27). The homogenization procedure to derive these two systems (see (1.19)) ensures that this is equivalent to solving:

$$\partial_x \text{LHS Equation (1.12)} = \partial_x \text{RHS Equation (1.12)}, \text{ and}$$

$$\text{LHS Equation (1.12)}|_{x=0} = \text{RHS Equation (1.12)}|_{x=0}.$$

Thus, such a fixed point solves (1.12) itself. To conclude, we note that $\mathcal{X}_K$ is equivalent to $\mathcal{X}$, and thus this fixed point is an element of $\mathcal{X}$. $\square$

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APPENDIX A: ASYMPTOTIC EXPANSIONS

We will assume the expansions:

$$U^\varepsilon = \tilde{u}_s^n + \varepsilon^{N_0} u, \quad V^\varepsilon = \tilde{v}_s^n + \varepsilon^{N_0} v, \quad P^\varepsilon = \tilde{P}_s^n + \varepsilon^{N_0} P. \quad (\text{A.1})$$

We will denote the partial expansions:

$$u_s^i = \sum_{j=0}^i \sqrt{\varepsilon}^j u_e^j + \sum_{j=0}^{i-1} \sqrt{\varepsilon}^j u_p^j, \quad \tilde{u}_s^i = u_s^i + \sqrt{\varepsilon}^i u_p^i, \quad (\text{A.2})$$

$$v_s^i = \sum_{j=1}^i \sqrt{\varepsilon}^{j-1} v_e^j + \sum_{j=0}^{i-1} \sqrt{\varepsilon}^j v_p^j, \quad \tilde{v}_s^i = v_s^i + \sqrt{\varepsilon}^i v_p^i, \quad (\text{A.3})$$

$$P_s^i = \sum_{j=0}^i \sqrt{\varepsilon}^j P_e^j, \quad \tilde{P}_s^i = P_s^i + \sqrt{\varepsilon}^i \left\{ P_p^i + \sqrt{\varepsilon} P_p^{i,a} \right\}. \quad (\text{A.4})$$

We will also define $u_s^{E,i} = \sum_{j=0}^i \sqrt{\varepsilon}^j u_e^j$ to be the “Euler” components of the partial sum. Similar notation will be used for $u_s^{P,i}, v_s^{E,i}, v_s^{P,i}$. The following will also be convenient:

$$\begin{aligned} u_s^E &:= \sum_{i=0}^n \sqrt{\varepsilon}^i u_e^i, \quad v_s^E := \sum_{i=1}^n \sqrt{\varepsilon}^{i-1} v_e^i, \\ u_s^P &:= \sum_{i=0}^n \sqrt{\varepsilon}^i u_p^i, \quad v_s^P := \sum_{i=0}^n \sqrt{\varepsilon}^i v_p^i, \\ u_s &= u_s^P + u_s^E, \quad v_s = v_s^P + v_s^E. \end{aligned} \quad (\text{A.5})$$

The $P_p^{i,a}$ terms are “auxiliary Pressures” in the same sense as those introduced in [27] and [31] and are for convenience. We will also introduce the notation:

$$\tilde{u}_p^i := u_p^i - u_p^i|_{y=0}, \quad \tilde{v}_p^i := v_p^i - v_p^i(x, 0), \quad \tilde{v}_e^i = v_e^i - v_e^i|_{Y=0}. \quad (\text{A.6})$$

We first record the properties of the leading order ($i = 0$) layers. For the outer Euler flow, we will take a shear flow, $[u_e^0(Y), 0, 0]$. The derivatives of u_e^0 decay rapidly in Y and that is bounded below, $|u_e^0| \gtrsim 1$.

For the leading order Prandtl boundary layer, the equations are:

$$\left. \begin{aligned} \bar{u}_p^0 u_{px}^0 + \bar{v}_p^0 u_{py}^0 - u_{pyy}^0 + P_{px}^0 &= 0, \\ u_{px}^0 + v_{py}^0 &= 0, \quad P_{py}^0 = 0, \quad u_p^0|_{x=0} = U_p^0, \quad u_p^0|_{y=0} = -u_e^0|_{y=0}. \end{aligned} \right\} \quad (\text{A.7})$$

It is convenient to state results in terms of the quantity $\bar{u}_p^0$, whose initial data is simply $\bar{U}_p^0 := u_e^0(0) + U_p^0$. Our starting point is the following result of Oleinik in [46], P. 21, Theorem 2.1.1:

Theorem A.1 (Oleinik). *Assume boundary data is prescribed satisfying $U_p^0 \in C^\infty$ and exponentially decaying $|\partial_y^j \{\bar{U}_p^0 - u_e^0(0)\}|$ for $j \geq 0$ satisfying:*

$$\bar{U}_p^0 > 0 \text{ for } y > 0, \quad \partial_y \bar{U}_p^0(0) > 0, \quad \partial_y^2 \bar{U}_p^0 \sim y^2 \text{ near } y = 0 \quad (\text{A.8})$$

Then for some $L > 0$, there exists a solution, $[\bar{u}_p^0, \bar{v}_p^0]$ to (A.7) satisfying, for some $y_0, m_0 > 0$,

$$\sup_{x \in (0, L)} \sup_{y \in (0, y_0)} |\bar{u}_p^0, \bar{v}_p^0, \partial_y \bar{u}_p^0, \partial_{yy} \bar{u}_p^0, \partial_x \bar{u}_p^0| \lesssim 1, \quad (\text{A.9})$$

$$\sup_{x \in (0, L)} \sup_{y \in (0, y_0)} \partial_y \bar{u}_p^0 > m_0 > 0. \quad (\text{A.10})$$

By evaluating the system (A.7) and ∂_y of (A.7) at $\{y = 0\}$ we conclude:

$$\bar{u}_{pyy}^0|_{y=0} = \bar{u}_{pyyy}^0|_{y=0} = 0.$$

We now list the equations to be satisfied by the i 'th layers, starting with the i 'th Euler layer:

$$\left. \begin{aligned} u_e^0 \partial_x u_e^i + \partial_Y u_e^0 v_e^i + \partial_x P_e^i &=: f_{E,1}^i, \\ u_e^0 \partial_x v_e^i + \partial_Y P_e^i &=: f_{E,2}^i, \\ \partial_x u_e^i + \partial_Y v_e^i &= 0, \\ v_e^i|_{Y=0} = -v_p^0|_{y=0}, \quad v_e^i|_{x=0, L} &= V_{E, \{0, L\}}^i, \quad u_e^i|_{x=0} = U_E^i. \end{aligned} \right\} \quad (\text{A.11})$$

For the i 'th Prandtl layer:

$$\left. \begin{aligned} \bar{u} \partial_x u_p^i + u_p^i \partial_x \bar{u} + \partial_y \bar{u} [v_p^i - v_p^i|_{y=0}] + \bar{v} \partial_y u_p^i + \partial_x P_p^i - \partial_{yy} u_p^i &:= f^{(i)}, \\ \partial_x u_p^i + \partial_y v_p^i &= 0, \quad \partial_y P_p^i = 0 \\ u_p^i|_{y=0} = -u_e^i|_{y=0}, \quad [u_p^i, v_p^i]_{y \rightarrow \infty} &= 0, \quad v_p^i|_{x=0} = \text{prescribed data.} \end{aligned} \right\} \quad (\text{A.12})$$

The relevant definitions of the above forcing terms are given below. Note that as a matter of convention, summations that end with a negative number are empty sums.

Definition A.2 (Forcing Terms).

$$\begin{aligned}
 -f_{E,1}^i &:= u_{ex}^{i-1} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{j-1} \{u_e^j + u_p^j(x, \infty) + u_e^{i-1} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{j-1} u_{ex}^j \\
 &\quad + \sqrt{\varepsilon}^{i-2} [\{u_e^{i-1} + u_p^{i-1}(x, \infty)\} u_{ex}^{i-1} + v_e^{i-1} u_{eY}^{i-1}] \\
 &\quad + u_{eY}^{i-1} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{j-1} v_e^j + v_e^{i-1} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{j-1} u_{eY}^j - \sqrt{\varepsilon} \Delta u_e^{i-1} \\
 -f_{E,2}^i &:= v_{eY}^{i-1} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{j-1} v_e^j + v_e^{i-1} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{j-1} v_{eY}^j + \sqrt{\varepsilon}^{i-2} [v_e^{i-1} v_{eY}^{i-1} + u_e^{i-1} v_{ex}^{i-1}] \\
 &\quad + \{u_e^{i-1} + u_p^{i-1}(x, \infty)\} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{j-1} v_{ex}^j + v_{ex}^{i-1} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{j-1} \{u_e^j + u_p^j(x, \infty)\} \\
 &\quad - \sqrt{\varepsilon} \Delta v_e^{i-1}, \\
 -f^{(i)} &:= \sqrt{\varepsilon} u_{pxx}^{i-1} + \varepsilon^{-\frac{1}{2}} \{v_e^i - v_e^i(x, 0)\} u_{py}^0 + \varepsilon^{-\frac{1}{2}} \{u_e^0 - u_e^0(0)\} u_{px}^{i-1} + \varepsilon^{-\frac{1}{2}} \{u_{sx}^{P,i-1} \\
 &\quad - \bar{u}_{sx}^0\} u_p^{i-1} + \varepsilon^{-\frac{1}{2}} \{u_{sx}^{E,i-1} - \bar{u}_{sx}^0\} \{u_p^{i-1} - u_p^{i-1}(x, \infty)\} + \varepsilon^{-\frac{1}{2}} v_p^{i-1} \{ \bar{u}_{sy}^{i-1} \\
 &\quad - u_{py}^0 \} + u_{px}^{i-1} \sum_{j=1}^{i-1} \sqrt{\varepsilon}^{j-1} (u_e^j + u_p^j) + \varepsilon^{-\frac{1}{2}} (v_s^{i-1} - v_s^1) u_{py}^{i-1} + \varepsilon^{-\frac{1}{2}} (v_e^1 \\
 &\quad - v_e^1(x, 0)) u_{py}^{i-1} + \sqrt{\varepsilon} u_{eY}^i \sum_{j=0}^{i-1} \sqrt{\varepsilon}^j v_p^j + v_e^i \sum_{j=1}^{i-1} \sqrt{\varepsilon}^{j-1} u_{py}^j + u_{ex}^i \sum_{j=0}^{i-1} \sqrt{\varepsilon}^j \{u_p^j \\
 &\quad - u_p^j(x, \infty)\} + u_e^i \sum_{j=0}^{i-1} \sqrt{\varepsilon}^j u_{px}^j + \int_y^\infty \partial_x \{ \sqrt{\varepsilon}^2 u_e^i \sum_{j=0}^{i-1} \sqrt{\varepsilon}^j v_{px}^j + \sqrt{\varepsilon} v_{ex}^i \\
 &\quad \times \sum_{j=0}^{i-1} \sqrt{\varepsilon}^j \{u_p^j - u_p^j(x, \infty)\} + \sqrt{\varepsilon}^2 v_{eY}^i \sum_{j=0}^{i-1} \sqrt{\varepsilon}^j v_p^j + \sqrt{\varepsilon} v_e^i \sum_{j=0}^{i-1} \sqrt{\varepsilon}^j v_{py}^j \\
 &\quad + \sqrt{\varepsilon} v_s^{i-1} v_{py}^{i-1} + \sqrt{\varepsilon} v_{sy}^{i-1} v_p^{i-1} + \sqrt{\varepsilon} v_{sx}^{E,i-1} \{u_p^{i-1} - u_p^{i-1}(x, \infty)\} \\
 &\quad + \sqrt{\varepsilon} v_{sx}^{P,i-1} u_p^{i-1} + \sqrt{\varepsilon} u_s^{i-1} v_{px}^{i-1} + \sqrt{\varepsilon} \Delta_\varepsilon v_p^{i-1} + \sqrt{\varepsilon} \{u_p^{i-1} v_{px}^{i-1} + v_p^{i-1} v_{py}^{i-1}\} \} dz.
 \end{aligned}$$

For $i = 1$ only, we make the following modifications. The aim is to retain only the required order $\sqrt{\varepsilon}$ terms into $f^{(1)}$. $f^{(2)}$ will then be adjusted by including the superfluous terms. Specifically, define:

$$f^{(1)} := -u_p^0 u_{ex}^1|_{Y=0} - u_{px}^0 u_e^1|_{Y=0} - \bar{u}_{eY}^0(0) y u_{px}^0 - v_p^0 u_{eY}^0 - v_{eY}^1(0) y u_{py}^0. \quad (\text{A.13})$$

For the final Prandtl layer, we must enforce the boundary condition $v_p^n|_{y=0} = 0$. Define the quantities $[u_p, v_p, P_p]$ to solve

$$\left. \begin{aligned} \bar{u}\partial_x u_p + u_p\partial_x \bar{u} + \partial_y \bar{u} v_p + \bar{v}\partial_y u_p + \partial_x P_p - \partial_{yy} u_p &:= f^{(n)}, \\ \partial_x u_p + \partial_y v_p &= 0, \quad \partial_y P_p^i = 0 \\ [u_p, v_p]|_{y=0} &= [-u_e^n, 0]|_{y=0}, \quad u_p|_{y \rightarrow \infty} = 0 \quad v_p|_{x=0} = V_P^n. \end{aligned} \right\} \quad (\text{A.14})$$

Note the change in boundary condition of $v_p|_{y=0} = 0$ which contrasts the $i = 1, \dots, n-1$ case. This implies that $v_p = \int_0^y u_{px} dy'$. For this reason, we must cut-off the Prandtl layers:

$$\begin{aligned} u_p^n &:= \chi(\sqrt{\varepsilon}y)u_p + \sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y) \int_0^y u_p(x, y') dy', \\ v_p^n &:= \chi(\sqrt{\varepsilon}y)v_p. \end{aligned}$$

Here $\mathcal{E}^n$ is the error contributed by the cut-off:

$$\mathcal{E}^{(n)} := \bar{u}\partial_x u_p^n + u_p^n\partial_x \bar{u} + \bar{v}\partial_y u_p^n + v_p^n\partial_y \bar{u} - u_{pyy}^n - f^{(n)}.$$

Computing explicitly:

$$\begin{aligned} \mathcal{E}^{(n)} &:= (1 - \chi)f^{(n)} + \bar{u}\sqrt{\varepsilon}\chi'(\sqrt{\varepsilon}y)v_p(x, y) + \bar{u}_x\sqrt{\varepsilon}\chi' \int_0^y u_p \\ &\quad + \bar{v}\sqrt{\varepsilon}\chi'u_p + \varepsilon\bar{v}\chi'' \int_0^y u_p + \sqrt{\varepsilon}\chi'u_p \\ &\quad + \varepsilon^{\frac{3}{2}}\chi''' \int_0^y u_p + 2\varepsilon\chi''u_p + \sqrt{\varepsilon}\chi'u_{py}. \end{aligned} \quad (\text{A.15})$$

We will now define the contributions into the next order, which will serve as the forcing for the remainder term:

$$\begin{aligned} \underline{f}^{(n+1)} &:= \sqrt{\varepsilon}^n [\varepsilon u_{pxx}^n + v_p^n \{\bar{u}_{sy}^n - u_{py}^0\} + \{u_e^0 - u_e^0(0)\} u_{px}^n \\ &\quad + u_{px}^n \sum_{j=1}^n \sqrt{\varepsilon}^j (u_e^j + u_p^j) + \{u_{sx}^n - \bar{u}_{sx}^0\} u_p^n + (v_s^n - v_s^1) u_{py}^n \\ &\quad + \{v_e^1 - v_e^1(x, 0)\} u_{py}^n] + \sqrt{\varepsilon}^n \mathcal{E}^{(n)} + \sqrt{\varepsilon}^{n+2} \Delta u_e^n \\ &\quad + \sqrt{\varepsilon}^n u_{ex}^n \sum_{j=1}^{n-1} \sqrt{\varepsilon}^j u_e^j + \sqrt{\varepsilon}^n u_e^n \sum_{j=1}^{n-1} \sqrt{\varepsilon}^j u_{ex}^j + \sqrt{\varepsilon}^{2n} [u_e^n u_{ex}^n \\ &\quad + u_e^n u_{ey}^n] + \sqrt{\varepsilon}^{n+1} u_{ey}^n \sum_{j=1}^{n-1} \sqrt{\varepsilon}^{j-1} v_e^j + \sqrt{\varepsilon}^{n-1} v_e^n \sum_{j=1}^{n-1} \sqrt{\varepsilon}^{j+1} u_{ey}^j. \end{aligned} \quad (\text{A.16})$$

$$\underline{g}^{(n+1)} := \sqrt{\varepsilon}^n [\varepsilon u_s^n \partial_y u_p^n + \partial_y v_s^n u_p^n + \partial_x u_s^n u_p^n + u_s^n \partial_x v_p^n - \Delta_\varepsilon v_p^n$$

$$\begin{aligned}
 & + \sqrt{\varepsilon}^n \left(u_p^n \partial_x v_p^n + v_p^n \partial_y v_p^n \right) \Big] + (\sqrt{\varepsilon})^n \partial_Y v_e^n \sum_{j=1}^{n-1} (\sqrt{\varepsilon})^{j-1} v_e^j \\
 & + \sqrt{\varepsilon}^{n-1} v_e^n \sum_{j=1}^{i-1} \sqrt{\varepsilon}^j \partial_Y v_e^j + \sqrt{\varepsilon}^{2n-1} [v_e^n v_{eY}^n + u_e^n \partial_x v_e^n] \\
 & + \sqrt{\varepsilon}^n u_e^n \sum_{j=1}^{n-1} (\sqrt{\varepsilon})^{j-1} \partial_x v_e^j + \sqrt{\varepsilon}^{n-1} \partial_x v_e^n \sum_{j=0}^{n-1} \sqrt{\varepsilon}^j u_e^j + \sqrt{\varepsilon}^{n+1} \Delta v_e^n.
 \end{aligned} \tag{A.17}$$

We now move to the remainder system. A straightforward linearization yields:

$$\left. \begin{aligned}
 -\Delta_\varepsilon u^{(\varepsilon)} + S_u + \partial_x P^{(\varepsilon)} &= \varepsilon^{-N_0} \mathbf{f}^{(n+1)} - \varepsilon^{N_0} \left\{ u^{(\varepsilon)} u_x^{(\varepsilon)} + v^{(\varepsilon)} u_y^{(\varepsilon)} \right\} \\
 -\Delta_\varepsilon v^{(\varepsilon)} + S_v + \frac{\partial_y}{\varepsilon} P^{(\varepsilon)} &= \varepsilon^{-N_0} \mathbf{g}^{(n+1)} - \varepsilon^{N_0} \left\{ u^{(\varepsilon)} v_x^{(\varepsilon)} + v^{(\varepsilon)} v_y^{(\varepsilon)} \right\} \\
 \partial_x u^{(\varepsilon)} + \partial_y v^{(\varepsilon)} &= 0.
 \end{aligned} \right\} \tag{A.18}$$

Denote:

$$u_s := \tilde{u}_s^n, \quad v_s := \tilde{v}_s^n. \tag{A.19}$$

Here we have defined:

$$S_u = u_s \partial_x u^{(\varepsilon)} + u_{sx} u^{(\varepsilon)} + v_s \partial_y u^{(\varepsilon)} + u_{sy} v^{(\varepsilon)}, \tag{A.20}$$

$$S_v = u_s \partial_x v^{(\varepsilon)} + v_{sx} u^{(\varepsilon)} + v_s \partial_y v^{(\varepsilon)} + v_{sy} v^{(\varepsilon)}. \tag{A.21}$$

Let us discuss now the boundary conditions. We take

$$\begin{aligned}
 u^\varepsilon|_{x=0} &:= u^0(\text{unknown}), \\
 v^\varepsilon|_{x=0} &:= v^0(\text{unknown}), \\
 v^\varepsilon|_{y=0} &= v_y^\varepsilon|_{y=0} = 0, \\
 v_x^\varepsilon|_{x=L} &:= a_1^\varepsilon(y), v_{xx}^\varepsilon|_{x=0} := a_2^\varepsilon(y), v_{xxx}^\varepsilon|_{x=L} := a_3^\varepsilon(y).
 \end{aligned}$$

Going to the vorticity formulation of (A.18) yields the system (1.12), with

$$F_R := \varepsilon^{-N_0} (\partial_y \mathbf{f}^{(n+1)} - \varepsilon \partial_x \mathbf{g}^{(n+1)}). \tag{A.22}$$

In Section 2, our main object of analysis with the vorticity equation evaluated at the $\{x = 0\}$ boundary, (1.12)| $_{x=0}$, which reads:

$$\begin{aligned}
 \mathcal{L}v^0 &= F_{(v)} + F_R^a + Q(u^0, v^0, v) + \mathcal{H}, \\
 \mathcal{L}v^0 &:= v_{yyy}^0 - \{u_s v_{yy}^0 - u_{syy} v^0\} - \{v_s v_{yy}^0 - v_y^0 v_{syy}\} \\
 &\quad + \varepsilon u_{sxx} v^0 + \varepsilon v_{sxx} v_y^0, \\
 Q(u^0, v^0, v) &:= \varepsilon^{N_0} [v_y^0 v_{yy}^0 - v^0 v_{yyy}^0 + \varepsilon u^0 v_{xx}|_{x=0} + \varepsilon v^0 v_{xy}|_{x=0} \\
 &\quad + h v_{yy}^0 - v^0 h_{yy}] \\
 \mathcal{H} &:= [-h_{yyy} + v_s h_{yy} - h \Delta_\varepsilon u_s], \\
 F_{(v)}(v) &:= \varepsilon u_s v_{xx}^{(\varepsilon)}|_{x=0} - 2\varepsilon v_{xyy}^{(\varepsilon)}|_{x=0} - \varepsilon^2 v_{xxx}^{(\varepsilon)}|_{x=0} + \varepsilon v_s v_{xy}^{(\varepsilon)}|_{x=0}, \\
 F_R^a &:= F_R|_{x=0} + \varepsilon u_s a_{xx}^\varepsilon|_{x=0} - 2\varepsilon a_{xyy}^\varepsilon|_{x=0} - \varepsilon^2 a_{xxx}^\varepsilon|_{x=0} + \varepsilon v_s a_{xy}^\varepsilon|_{x=0} \\
 &:= F_R|_{x=0} + b_{(u)}(a^\varepsilon)|_{x=0}.
 \end{aligned} \tag{A.23}$$

We homogenize the v^ε via (1.19). Define the quotients:

$$q^\varepsilon := \frac{v^\varepsilon}{u_s}, \quad \tilde{q} := \frac{\tilde{v}}{u_s}, \quad q := \frac{v}{u_s}, \quad q^0 := \frac{v^0}{u_s|_{x=0}}.$$

The ∂_x of vorticity equation (DNS) satisfied by $[u^\varepsilon, v^\varepsilon]$ is as follows

$$\begin{aligned}
 &-\partial_x R[q^{(\varepsilon)}] + \Delta_\varepsilon^2 v^{(\varepsilon)} + \partial_x \{v_s \Delta_\varepsilon u^{(\varepsilon)} - u^{(\varepsilon)} \Delta_\varepsilon v_s\} \\
 &= \varepsilon^{N_0} \partial_x \{v^\varepsilon \Delta_\varepsilon u^\varepsilon - u^\varepsilon \Delta_\varepsilon v^\varepsilon\} + \partial_x F_R, \\
 v^{(\varepsilon)}|_{x=0} &= v^0, v_{xx}^{(\varepsilon)}|_{x=0} = a_2^{(\varepsilon)}, v_x^{(\varepsilon)}|_{x=L} = a_1^{(\varepsilon)}, v_{xxx}^{(\varepsilon)}|_{x=L} = a_3^{(\varepsilon)}, \\
 v^{(\varepsilon)}|_{y=0} &= v_y^{(\varepsilon)}|_{y=0} = 0,
 \end{aligned} \tag{A.24}$$

We now homogenize equation (A.24) by writing it in terms of $[u, v]$. First, the linear contributions are given in terms of the following

$$\begin{aligned}
 b_{(u)}(\tilde{v}) &= -R[\tilde{v}] + I_x[\tilde{v}_{yyy}] + 2\varepsilon \tilde{v}_{xyy} + \varepsilon^2 \tilde{v}_{xxx} - \varepsilon \tilde{v}_{xy} \\
 &\quad + v_s I_x[\tilde{v}_{yy}] - \Delta_\varepsilon v_s I_x[\tilde{v}_y],
 \end{aligned} \tag{A.25}$$

We now arrive at the nonlinearity. For this, we will use (1.19) to write

$$\partial_x \{v^\varepsilon \Delta_\varepsilon u^\varepsilon - u^\varepsilon \Delta_\varepsilon v^\varepsilon\} = \varepsilon^{N_0} (Q_{11} + Q_{12} + Q_{13} + Q_{22} + Q_{23} + Q_{33}),$$

where the quadratic terms are

$$\begin{aligned}
 Q_{11} &:= v_y \Delta_\varepsilon v - u \Delta_\varepsilon v_x + v_x \Delta_\varepsilon u - v \Delta_\varepsilon v_y, \\
 Q_{12} &:= v_y v_{yy}^0 + v_y^0 \Delta_\varepsilon v - x v^0 \Delta_\varepsilon v_x - x v_x v_{yy}^0 - v v_{yyy}^0 - v^0 \Delta_\varepsilon v_y, \\
 Q_{22} &:= v_y^0 v_{yy}^0 - v^0 v_{yyy}^0,
 \end{aligned}$$

and the linear terms are

$$\begin{aligned} Q_{13} &:= v_y \Delta_\varepsilon a^\varepsilon + a_y^\varepsilon \Delta_\varepsilon v - u \Delta_\varepsilon a_x^\varepsilon + I_x[a^\varepsilon] \Delta_\varepsilon v_x - v \Delta_\varepsilon a_y^\varepsilon - a^\varepsilon \Delta_\varepsilon v_y \\ Q_{23} &:= v_y^0 \Delta_\varepsilon a^\varepsilon + a_y^\varepsilon v_{yy}^0 - x v^0 \Delta_\varepsilon a_x^\varepsilon - v^0 \Delta_\varepsilon a_y^\varepsilon - a^\varepsilon v_{yy}^0 \end{aligned}$$

and the forcing term is

$$Q_{33} := a_y^\varepsilon \Delta_\varepsilon a^\varepsilon + I_x[a^\varepsilon] \Delta_\varepsilon a_x^\varepsilon - a^\varepsilon \Delta_\varepsilon I_x[a^\varepsilon] - a^\varepsilon \Delta_\varepsilon a_y^\varepsilon.$$

The last step is to use the identity (recalling (A.25), (1.15), and (1.30)):

$$\partial_x b_{(u)}(v^0) + \{v_{sx} u_{yy}^0 - u^0 \Delta_\varepsilon v_{sx}\} = B_{v^0} + \{v_{sx} h_{yy} - h \Delta_\varepsilon v_{sx}\}, \quad (\text{A.26})$$

Piecing together the preceding, we arrive at the homogenized system

$$\begin{aligned} -\partial_x R[q] + \Delta_\varepsilon^2 v + J(v) + B_{v^0} &= \varepsilon^{N_0} \mathcal{N} + F_{(q)}, \\ \mathcal{N} &:= Q_{11} + Q_{12} + Q_{22}, \\ F_{(q)} &:= \partial_x F_R + \partial_x b_{(u)}(a^\varepsilon) + H[a^\varepsilon](v, u^0, u^0) + \{v_{sx} h_{yy} - h \Delta_\varepsilon v_{sx}\}, \\ H[a^\varepsilon](v, u^0, v^0) &:= Q_{13}(u, v) + Q_{23}(v^0) + Q_{33}(a^\varepsilon), \end{aligned} \quad (\text{A.27})$$

where we have defined J, B_{v^0} in (1.29) and (1.30).

The following proposition summarizes the profile constructions from [25]:

Theorem A.3. *Assume the shear flow $u_e^0(Y) \in C^\infty$, whose derivatives decay rapidly. Assume (A.8) regarding $\bar{u}_p^0|_{x=0}$, and the conditions*

$$\bar{v}_{pyyy}^i|_{x=0}(0) = \partial_x g_1|_{x=0, y=0}, \quad (\text{A.28})$$

$$\bar{v}_p^{i''''}|_{x=0}(0) = \partial_{xy} g_1|_{y=0}(x=0), \quad (\text{A.29})$$

$$\bar{u}_{py}^0|_{x=0}(0) u_e^i|_{x=0}(0) - \int_0^\infty \bar{u}_p^0 e^{-\int_1^y \bar{v}_p^0} \{f^{(i)}(y) - r^{(i)}(y)\} dy = 0, \quad (\text{A.30})$$

where $r^{(i)}(y) := \bar{v}_p^i \bar{u}_{py}^0 - \bar{u}_p^0 \bar{v}_{py}^i$. We assume also standard higher order versions of the parabolic compatibility conditions (A.28), (A.29). Let $v_e^i|_{x=0}, v_e^i|_{x=L}, u_e^i|_{x=0}$ be prescribed smooth and rapidly decaying Euler data. We assume on the data standard elliptic compatibility conditions at the corners $(0,0)$ and $(L,0)$ obtained by evaluating the equation at the corners. In addition, assume

$$v_e^1|_{x=0} \sim Y^{-m_1} \text{ or } e^{-m_1 Y} \text{ for some } 0 < m_1 < \infty, \quad (\text{A.31})$$

$$\|\partial_Y^k \{v_e^i|_{x=0} - v_e^i|_{x=L}\} \langle Y \rangle^M\|_\infty \lesssim L \quad (\text{A.32})$$

Then all profiles in $[u_s, v_s]$ exist and are smooth on Ω . The following estimates hold:

$$\begin{aligned} \bar{u}_p^0 &> 0, \bar{u}_{py}^0|_{y=0} > 0, \bar{u}_{pyy}^0|_{y=0} = \bar{u}_{pyyy}^0|_{y=0} = 0 \\ \|\nabla^K \{u_p^0, v_p^0\} e^{My}\|_\infty &\lesssim 1 \text{ for any } K \geq 0, \\ \|u_p^i\|_\infty + \|\nabla^K u_p^i e^{My}\|_\infty + \|\nabla^J v_p^i e^{My}\|_\infty &\lesssim 1 \text{ for any } K \geq 1, M \geq 0, \\ \|\nabla^K \{u_e^1, v_e^1\} w_{m_1}\|_\infty &\lesssim 1 \text{ for some fixed } m_1 > 1 \\ \|\nabla^K \{u_e^i, v_e^i\} w_{m_i}\|_\infty &\lesssim 1 \text{ for some fixed } m_i > 1, \end{aligned} \quad (\text{A.33})$$

where $w_{m_i} \sim e^{m_i Y}$ or $(1 + Y)^{m_i}$.

In addition the following estimate on the remainder forcing holds:

$$\|F_R|_{x=0} w_0\| + \|\partial_x F_R \frac{w_0}{\sqrt{\varepsilon}}\| \lesssim \sqrt{\varepsilon}^{n-1-2N_0}, \quad (\text{A.34})$$

where F_R has been defined in (A.22).