LIMIT THEOREMS FOR THE VOLUMES OF SMALL CODIMENSIONAL RANDOM SECTIONS OF ℓ_p^n-BALLS

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We establish central limit theorems for the volumes of intersections of B_p^n (the unit ball of ℓ_p^n) with uniform random subspaces of codimension d for fixed d and $n \to \infty$. As a corollary we obtain higher-order approximations for expected volumes, refining previous results by Koldobsky and Lifschitz and approximations obtained from the Eldan–Klartag version of CLT for convex bodies. We also obtain a central limit theorem for the Minkowski functional of the intersection body of B_p^n, evaluated on a random vector distributed uniformly on the unit sphere.

1. Introduction. An important aspect of stochastic geometry is the investigation of volumes of random sets. They have been studied in a variety of contexts, including, for instance, volumes of convex hulls of i.i.d. Gaussian vectors in $\mathbb{R}^m$ (e.g., Bárány and Vu [10], Calka and Yukich [17]), points selected from the boundary of a fixed convex set (e.g., Reitzner [73], Schütt and Werner [80], Vu [87], Thäle [83], Turchi and Wespi [84]), projections of high-dimensional cubes onto a random subspace of fixed dimension (Paouris–Pivovarov–Zinn [71], Kabluchko–Prochno–Thäle [38]). In the high-dimensional setting, asymptotics and phase transitions for the expected volume of i.i.d. random points selected from the vertices of the unit cube or more general product distributions were investigated by Dyer–Füredi–McDiarmid [22] and Gatzouras and Giannopoulos [29]. Recently, the case of points drawn from a simplex and from nonproduct convex measures has been studied by Frieze–Pegden–Tkocz [27] and Chakraborti–Tkocz–Vritsiou [19]. Another important line of research is devoted to sets obtained from point processes defined on spaces of geometric objects, for example, random tesselations (Gusakova and Thäle [31]) and random cylinder processes (Baci et al. [8]). Let us finally mention work concerning unions and Minkowski sums of random sets (see the monograph by Molchanov [64]). While far from exhaustive, the lines of research above give an indication of the rich and diverse perspectives on volumetric questions in stochastic geometry.

Most of the early results focused on first-order asymptotics in the sense of expectation or convergence (almost sure or in probability). More recent developments also treat concentration inequalities, small ball probabilities, large deviations or weak limit theorems. In particular, central limit theorems for the volume or log-volume in the respective models were established in [3, 10, 30, 31, 71, 83]. The three former references treat convex polytopes in a fixed dimension, whereas the other ones random simplices for the dimension tending to infinity.

In this article we focus on another model of random sets, namely, on sections of high-dimensional origin-symmetric bodies by random subspaces (i.e., subspaces drawn from the Haar measure on the corresponding Grassmann manifold). When the bodies in question are convex, this model has played a central role in geometric functional analysis, especially via

probabilistic methods put forth by Vitali Milman in his proof of Dvoretzky's theorem [61]. Over the years this has grown into the whole new area of asymptotic geometric analysis (see the classical book [62] and recent monographs [7, 16]). Convexity is often used by invoking duality, especially between sections and projections, but some key results actually extend to star-shaped sets (e.g., Litvak et al. [54, 55]). Moreover, star-shaped sets also furnish deeper dualites in convex geometry, especially in dual Brunn–Minkowski theory (e.g., Lutwak et al. [34, 56–58]). However, investigation of the asymptotic distributions of the volumes of random sections of star bodies from a stochastic geometry perspective is a less explored path. Our focus here is on asymptotic properties of random sections of ℓ_p^n-balls, including the star-shaped case when $p \in (0, 1)$.

Geometric properties of random sections depend strongly on the relation between the dimensions of the ambient space and the section. For convex bodies in special positions, low-dimensional sections are generically spherical, and their approximate radia can be calculated in terms of certain geometric characteristics of the body (as stated in Milman's version of Dvoretzky's theorem). Accurate volume approximation in this regime is just one of the important consequences. At the other extreme, one has sections of small codimension. The asymptotic behaviour of their volumes has been obtained more recently and is directly related to the celebrated Klartag's central limit theorem for convex bodies [40, 41] and subsequent results by Eldan and Klartag [23], which we will now recall briefly (and somewhat informally). We will also restrict attention to convex bodies, even though these results hold in the more general setting of log-concave measures. Below by $|\cdot|$ we denote the standard Euclidean norm on $\mathbb{R}^n$.

Assume thus that K is a convex body in $\mathbb{R}^n$ in isotropic position, that is, a random vector X distributed uniformly in K has mean zero and covariance matrix equal to the identity. Let E be a random k-dimensional subspace of $\mathbb{R}^n$ distributed according to the Haar measure on the Grassmanian $G_{n,k}$. The central limit theorem, due to Klartag, asserts that there exists a universal constant $c > 0$ such that with probability at least $1 - e^{-cn^c}$ on the Grassmanian, the total variation between γ_E – the standard Gaussian measure on E (i.e., the measure with density $g_E(x) = (2\pi)^{-k/2} \exp(-|x|^2/2)$ with respect to the k-dimensional Lebesgue measure on E) and the law $\mathcal{L}(P_E X)$ of $P_E X$, where P_E is the orthogonal projection onto E, satisfies

$$\|\gamma_E - \mathcal{L}(P_E X)\|_{\mathrm{TV}} \leq \frac{1}{cn^c}.$$

Thus, informally, for $k \leq n^c$ and n tending to infinity, almost all k-dimensional marginals of X are almost Gaussian.

The total variation estimate given by Klartag was subsequently complemented by Eldan and Klartag [23] with pointwise approximation of density. It turns out that with probability $1 - e^{-cn^c}$, where c is again a positive universal constant, for $x \in E$ with $|x| < n^c$, the density f of $P_E X$ satisfies

$$\left| \frac{f(x)}{g_E(x)} - 1 \right| \leq \frac{1}{cn^c}.$$

This result is of particular importance from the point of view of volumes, since $f(x)$ equals to the ratio $\mathrm{Vol}_{n-k}(K \cap (x + E^\perp))/\mathrm{Vol}_n(K)$, where Vol_i stands for the i-dimensional Lebesgue measure and $E^\perp$ is the orthogonal complement of E. The above approximation holds for isotropic convex bodies. By an appropriate scaling, one obtains that if $K_n \subseteq \mathbb{R}^n$ is a sequence of convex bodies such that the random vectors uniformly distributed on K_n satisfy $\mathbb{E}X_n = 0$, $\mathrm{Cov}(X_n) = c_n \mathrm{Id}$ and H_n is a random subspace of K_n of *codimension* $d \leq n^c$, then with probability tending to one as $n \to \infty$,

$$(1) \qquad \mathrm{Vol}_{n-d}(K_n \cap H_n) = \frac{1}{(2\pi c_n)^{d/2}} \mathrm{Vol}_n(K_n)(1 + o(1)).$$

The result by Eldan and Klartag provides us thus, in particular, with the *first-order* approximations for the volumes of random section of *fixed codimension d* of high-dimensional convex bodies.

The goal of this article is to complement (1) with central limit theorems in the special case of ℓ_p^n-balls. Recall that, for $p \in (0, \infty)$, the ball B_p^n (the unit ball in the space ℓ_p^n) is defined as

$$B_p^n = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i|^p \leq 1 \right\},$$

whereas

$$B_\infty^n = \left\{ x \in \mathbb{R}^n : \max_{i \leq n} |x_i| \leq 1 \right\}.$$

B_p^n is a symmetric star-shaped set, and for $p \geq 1$, it is convex. One can verify that

$$(2) \qquad \mathrm{Vol}_n(B_p^n) = \frac{(2\Gamma(1 + 1/p))^n}{\Gamma(1 + n/p)},$$

which via Stirling's approximation shows, in particular, that for fixed p and large n, $\mathrm{Vol}_n(B_p^n)^{1/n}$ behaves like $c(p)/n^{1/p}$ for some constant $c(p)$.

Finite-dimensional ℓ_p-balls are of interest from various points of view, including the functional analytic, geometric and probabilistic. The interplay between these viewpoints is apparent in early research on sections of ℓ_p^n-balls (e.g., Ball [9], Meyer and Pajor [60], Koldobsky [43]). In fact, questions about low codimensional sections of convex bodies have been a major driving force in convex geometry in the last 30 years [16, 44]. From a probabilistic perspective, uniform distributions on B_p^n provide examples of natural nonproduct distributions that nevertheless exhibit properties that arise in the classical theory of independent random variables. In recent years many strong results in this direction have been established. In particular, Schechtman–Zinn [77], Sodin [81] and Latała–Wojtaszczyk [51] studied concentration and isoperimetric properties. Alonso-Gutiérrez–Prochno–Thäle [4, 5], Gantert–Kim–Ramanan [28] and Kabluchko–Prochno–Thäle [36, 37] investigated limit theorems together with large and moderate deviations for various norms of vectors drawn at random from B_p^n as well as their random projections. Naor and Romik [66, 67] studied proximity of the normalized cone and surface measures on the boundary of B_p^n and its consequences for concentration. Eskenazis–Nayar–Tkocz [24, 25] established optimal constants in Khintchine inequalities for vectors distributed uniformly on B_p^n. Another reason for investigating probabilistic aspects of B_p^n is that probabilistic tools can be used in the study of other geometric aspects, seemingly purely deterministic. This approach has been initiated in the work by Schechtman and Zinn [76] and further continued in the seminal paper [11] by Barthe et al. and, more recently, for example, by Chasapis, Eskenazis, Nayar, Tkocz (see [20, 24–26, 68]). Geometric results obtained this way include among others monotonicity of various geometric quantities and identification of subspaces of maximal/minimal volume. Recently, the interplay between probabilistic and geometric aspects of sections of ℓ_p^n balls was also used to establish sharp isoperimetric inequalities in Brunn–Minkowski theory. In particular, fundamental inequalities of Lutwak and Zhang for L_p-centroid bodies [59] were extended to the star-shaped range $p < 1$ using a certain model of random sections of ℓ_p^n balls [2]. This further develops a probabilistic approach to proving stochastic versions of isoperimetric inequalities; see, for example, [70].

Let us mention that some of the results mentioned above are special cases of statements conjectured for general log-concave measures. In fact, also the CLT for convex bodies was first established by Anttila–Ball–Perissinaki [6] for B_p^n rescaled to the isotropic position.

The first-order asymptotics for the expected volume of $B_p^n \cap H_n$, where H_n is a random subspace of fixed codimension d, asymptotically equivalent to (1) (which formally covers the case of $p \geq 1$), were obtained before the result by Eldan–Klartag by Koldobsky and Lifschitz [45] through a Fourier-analytic approach (in fact, the analysis in [45] covers not only the convex case but the full range of $p > 0$). These authors obtained asymptotics also in the cases of sections of proportional and fixed *dimension*. The equation (1) specialized to B_p^n, after some calculations concerning the covariance matrix, gives (see [11], p. 490, for an explicit formula for moments of coordinates) that, with probability tending to one as $n \to \infty$,

$$(3) \qquad \mathrm{Vol}_{n-d}\big(B_p^n \cap H_n\big) = \left(\frac{3\Gamma(1 + \frac{1}{p})\Gamma(1 + \frac{n+2}{p})}{2\pi\,\Gamma(1 + \frac{3}{p})\Gamma(1 + \frac{n}{p})} \right)^{d/2} \mathrm{Vol}_n\big(B_p^n\big)\big(1 + o(1)\big)$$

(with $\mathrm{Vol}_n(B_p^n)$ given by (2)). The results by Koldobsky–Lifschitz [45] give the same asymptotic behaviour also for the expected volume. We remark that different approaches to the first-order asymptotics yield different explicit expressions, which turn out to be equivalent thanks to Stirling's formula. In particular, it can be read from the above asymptotics that

$$(4) \qquad \frac{\mathrm{Vol}_{n-d}(B_p^n \cap H_n)}{\mathrm{Vol}_{n-d}(B_p^{n-d})} \xrightarrow{n \to \infty} a_{p,d} := \left(\frac{\Gamma(1/p)}{\Gamma(3/p)} \right)^{d/2} \frac{2^{d/2}\Gamma(1 + 1/p)^d}{\pi^{d/2}}$$

in probability. This formulation is more convenient for higher-order analysis than (3), as it allows to absorb certain normalization factors.

Our main results allow to complement this approximation with central limit theorems. We postpone the precise formulation, which involves additional quite complicated formulas, to Section 3, and here we just state them in a simplified form:

- For $p \in (0, 2)$ and any fixed $d > 0$, we show that, for some explicit constants $b_{p,d}$, $\Sigma_{p,d}^2$, the random variable

$$n^{3/2}\left(\frac{\mathrm{Vol}_{n-d}(H_n \cap B_p^n)}{\mathrm{Vol}_{n-d}(B_p^{n-d})} - a_{p,d} - \frac{1}{n}b_{p,d} \right)$$

converges weakly and in all Wasserstein distances $\mathcal{W}_q$ for $q \geq 1$ to a Gaussian random variable with mean zero and variance $\Sigma_{p,d}^2$ (recall that $a_{p,d}$ is defined in (4)). This is the content of Theorem 3.1.
- As a corollary to the CLT in Wasserstein distance, we obtain higher-order approximations of $\mathbb{E}\,\mathrm{Vol}_{n-d}(B_p^n \cap H_n)$ for $p \in (0, 2)$ of the form

$$\mathbb{E}\,\mathrm{Vol}_{n-d}\big(B_p^n \cap H_n\big) = \frac{(2\Gamma(1 + \frac{1}{p}))^{n-d}}{\Gamma(1 + \frac{n-d}{p})}\left(a_{p,d} + \frac{b_{p,d}}{n} + o\big(n^{-3/2}\big) \right)$$

(Corollary 3.2).
- For $d = 1$, we extend the weak convergence to arbitrary $p > 0$ (Theorem 3.4).
- For $p = \infty$ and $d = 1$, we establish similar weak convergence also for noncentral sections parallel to H_n (Theorem 3.5).
- Reinterpreting the result for $d = 1$ as a CLT for the radial function of the *intersection body* of B_p^n evaluated on a random vector from the sphere, we infer a central limit theorem for the norm induced by the intersection body evaluated on such a random vector (Corollary 3.6).

Our approach relies on probabilistic formulas for volumes of sections developed by Nayar and Tkocz [68] (for the case $p \in (0, 2)$) and Chasapis, Nayar and Tkocz [20] (for $p > 2$). The former one allows us to represent the volume in terms of expected determinants involving some auxiliary random variables, which after applying certain reverse Hölder inequalities for

polynomials together with some geometric analysis allow to reduce the problem to the central limit theorem for U-statistics. The latter formula relates the volume of a random section to the value of the density at zero for a randomly weighted sum of independent random variables, which allows to apply conditionally an appropriate version of the Edgeworth expansion for non-i.i.d. sequences. We note that it seems that the lack of a version of Edgeworth expansion in higher dimension suitable for our randomized setting is the main obstacle in extending our results for $p > 2$ to general d. Since the investigation of expansions of this form is rather distant from the main tools used in this article, we postpone it to future research. We refer the reader to Remarks 6.2 and 6.3 for detailed comments concerning the difficulties in using Edgeworth-type results present in the literature and description of recent developments on randomized central limit theorems and Edgeworth expansions.

The organization of the article is as follows. After introducing the basic notation (Section 2), we state our main results (Section 3). Section 4 is devoted to introduction of various auxiliary results used in the main arguments. Finally, Section 5 is devoted to the proof of results for $p \in (0, 2)$ and general d, Section 6 to general $p < \infty$ and $d = 1$, while in Section 7 we sketch the proof for $p = \infty$ and $d = 1$, and in Section 8 we prove the result on the intersection body of B_p^n.

2. Notation. By C we will denote universal constants, whereas C_a will stand for constants depending only on the parameter a. In both cases the values of constants may change between occurrences (even within the same line).

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $p \in (0, \infty)$, we will denote $|x|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$. We set $|x|_\infty = \max_{i \le n} |x_i|$. As a function on $\mathbb{R}^n$, $|\cdot|_p$ is a norm for $p \ge 1$ and a quasi-norm for $p \in (0, 1)$. We will also write $|\cdot|$ for $|\cdot|_2$.

For a random variable X and $p \in \mathbb{R}$, by $\|X\|_p$, we will denote the pth absolute moment of X, that is, for $p \ne 0$, $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$, $\|X\|_0 = \exp(\mathbb{E} \log |X|)$ (in fact, we will use $\|X\|_p$ for $p \ne 0$ only).

For a sequence of random variables X_n and a sequence of positive real numbers a_n, we will write $X_n = o_{\mathbb{P}}(a_n)$ if X_n/a_n converges in probability to zero and $X_n = \mathcal{O}_{\mathbb{P}}(a_n)$ if X_n/a_n is bounded in probability. Let us also introduce similar notation, which despite being less standard, will allow us to shorten some formulas. We will write $X_n = o_m(a_n)$ (resp., $X_n = \mathcal{O}_m(a_n)$) if X_n/a_n converges to zero (resp., is bounded) in *all* spaces L_q for $q > 0$ (with the subscript m corresponding to *moments*). We note that the speed of convergence (resp., implicit constants) may and usually will depend on q.

When dealing with independent random variables, for example, X, Y, we will denote by $\mathbb{E}_X$, $\mathbb{E}_Y$ expectation with respect to just one of them (conditional expectation with respect to the other one). Variants of this standard convention will be used for larger families of independent random variables; the exact meaning will be either explicitly introduced or clear from the context.

By $\mathcal{L}(X)$ we will denote the law of a random variable X.

We will often work with multiindices $\mathbf{i} = (i_1, \ldots, i_d) \in [n]^d$. By $[n]^{\underline{d}}$ we will denote the set of multiindices with pairwise distinct coordinates. Similarly, for a set I by $[n]^I$ (resp., $[n]^{\underline{I}}$) we will denote the set of all (resp., all one-to-one) functions from I to $[n]$. For $I \subset [d]$ and a multiindex $\mathbf{i} \in [n]^d$ by $\mathbf{i}_I$, we will denote $(i_\ell)_{\ell \in I} \in [n]^I$. Sometimes, we will also use the notation $\mathbf{i}_I$ to denote a stand-alone multiindex, writing, for example, $\sum_{\mathbf{i}_I \in [n]^{\underline{I}}} a_{\mathbf{i}_I}$. For instance, if $I = \{2, 3\}$, this notation should be understood as $\sum_{1 \le i_2 \ne i_3 \le n} a_{i_2 i_3}$.

3. Main results. Recall that for $q \ge 1$, the Wasserstein distance $\mathcal{W}_q(P, Q)$ between two probability measures P, Q on $\mathbb{R}^m$ is defined via the formula

$$\mathcal{W}_q(P, Q)^q = \inf_{\substack{(X,Y): \\ \mathcal{L}(X)=P, \mathcal{L}(Y)=Q}} \mathbb{E}|X - Y|^q.$$

In dimension $m = 1$, the Wasserstein distance admits a more explicit representation as

$$\mathcal{W}_q(P, Q)^q = \int_0^1 |F_P^{-1}(t) - F_Q^{-1}(t)|^q \, dt,$$

where F_P^{-1}, F_Q^{-1} are generalized inverses of the cumulative distribution functions of P, Q, respectively (see, e.g., [85], p. 75).

It is well-known (see, e.g., [85], Theorem 7.12) that a sequence of probability measures (P_n) converges in $\mathcal{W}_q$ to some measure P if and only if it converges weakly and the qth absolute moments of P_n converge to the qth absolute moment of P.

We are now ready to formulate the first result.

THEOREM 3.1.   *Let d be a positive integer, and let H_n be a random Haar distributed subspace of $\mathbb{R}^n$ of codimension d. For $p \in (0, 2]$, define*

(5)
$$a_{p,d} = \left(\frac{\Gamma(1/p)}{\Gamma(3/p)}\right)^{d/2} \frac{2^{d/2}\Gamma(1 + 1/p)^d}{\pi^{d/2}},$$

$$b_{p,d} = (d + 2)d\left(\frac{\Gamma(5/p)}{\Gamma(1/p)} - 3\left(\frac{\Gamma(3/p)}{\Gamma(1/p)}\right)^2\right)\left(\frac{\Gamma(1/p)}{\Gamma(3/p)}\right)^{d/2+2} \frac{2^{d/2-3}\Gamma(1 + 1/p)^d}{\pi^{d/2}}.$$

Then the sequence of random variables

$$n^{3/2}\left(\frac{\mathrm{Vol}_{n-d}(H_n \cap B_p^n)}{\mathrm{Vol}_{n-d}(B_p^{n-d})} - a_{p,d} - \frac{1}{n}b_{p,d}\right)$$

converges in distribution as $n \to \infty$ to a mean zero Gaussian random variable with variance

(6) $$\Sigma_{p,d}^2 = \frac{2^{d-4}d(d+5)\Gamma(1 + 1/p)^{2d}}{\pi^d} \cdot \left(\frac{\Gamma(5/p)}{\Gamma(1/p)} - 3\left(\frac{\Gamma(3/p)}{\Gamma(1/p)}\right)^2\right)^2\left(\frac{\Gamma(1/p)}{\Gamma(3/p)}\right)^{d+4}.$$

Moreover, for each $q \geq 1$, the convergence holds in the Wasserstein distance $\mathcal{W}_q$.

As a corollary to the convergence in Wasserstein distances, we immediately obtain the following refinement of the asymptotic expansion (4) of $\mathbb{E} \, \mathrm{Vol}_{n-d}(B_p^n \cap H_n)$.

COROLLARY 3.2.   *In the setting of Theorem 3.1,*

$$\mathbb{E} \, \mathrm{Vol}_{n-d}(B_p^n \cap H_n) = \frac{(2\Gamma(1 + \frac{1}{p}))^{n-d}}{\Gamma(1 + \frac{n-d}{p})}\left(a_{p,d} + \frac{b_{p,d}}{n} + o(n^{-3/2})\right).$$

REMARK 3.3.   The implicit constant in $o(n^{-3/2})$ in the above theorem depends on p and d. In principle, our proofs allow for obtaining estimates with explicit dependence on these parameters, also for moments of higher order, which would lead to a concentration of measure type result. However, we do not pursue this direction.

The method of proof of Theorem 3.1 is restricted to $p \in (0, 2)$. In the special case of $d = 1$, we can, however, extend the central limit theorem to arbitrary $p \in (0, \infty)$.

THEOREM 3.4.   *Let H_n be a random Haar distributed subspace of $\mathbb{R}^n$ of codimension one. For $p \in (0, \infty)$, let $a_{p,1}, b_{p,1}, \Sigma_{p,1}^2$ be defined by (5) and (6). Then the sequence of random variables*

$$n^{3/2}\left(\frac{\mathrm{Vol}_{n-1}(B_p^n \cap H_n)}{\mathrm{Vol}_{n-1}(B_p^{n-1})} - a_{p,1} - \frac{1}{n}b_{p,1}\right)$$

converges in distribution to a mean zero Gaussian random variable with variance $\Sigma_{p,1}^2$.

Using the same method as in the proof of Theorem 3.4, we can obtain a result concerning more general sections of the cube B_∞^n by random hyperplanes. Note that, while for fixed $p < \infty$, $\mathrm{Vol}_n(B_p^n) \to 0$ as $n \to \infty$, in the case of $p = \infty$, we have $\mathrm{Vol}_n(B_p^n) = 2^n$. For this reason in the limit theorem, it is more convenient to normalize the volume of the section by 2^n.

THEOREM 3.5. *Let H_n be a random Haar distributed subspace of $\mathbb{R}^n$ of codimension one, and let u be a unit vector normal to H_n. For $x \in \mathbb{R}$, set*

$$a(x) = \sqrt{\frac{3}{2\pi}} e^{-\frac{3x^2}{2}},$$

$$b(x) = -\frac{9\sqrt{3}}{20\sqrt{2\pi}}(3x^4 - 6x^2 + 1)e^{-\frac{3x^2}{2}}.$$

For every $x \in \mathbb{R}$, the sequence

$$n^{3/2}\left(2^{-n}\,\mathrm{Vol}_{n-1}\big(B_\infty^n \cap (xu + H_n)\big) - a(x) - \frac{b(x)}{n}\right)$$

converges in distribution to a centered Gaussian variable with variance

$$\Sigma(x)^2 = \frac{81}{100\pi} e^{-3x^2}(3x^4 - 6x^2 + 1)^2.$$

Let us conclude the presentation of our results with a corollary concerning the intersection body of B_p^n. Recall that the radial function of a star-shaped body $K \subset \mathbb{R}^n$ in direction $u \in S^{n-1}$ is defined as $\rho_K(u) = \sup\{t > 0 \colon tu \in K\}$. The intersection body of a body K, introduced by Lutwak [57], is a star-shaped body $\mathcal{I}K$ whose radial function is given by $\rho_{\mathcal{I}K}(u) = \mathrm{Vol}_{n-1}(K \cap u^\perp)$, where $u^\perp$ is the hyperplane orthogonal to u. Thus, our results for sections of codimension $d = 1$ can be rephrased as a limit theorem for $\rho_{\mathcal{I}B_p^n}(\eta)$ for a random vector η distributed uniformly on S^{n-1}. With some additional work one can infer from it a corollary concerning $\|\eta\|_{\mathcal{I}B_p^n} := \rho_{\mathcal{I}B_p^n}(\eta)^{-1}$ – the Minkowski functional associated to $\mathcal{I}B_p^n$. We state it only for $p < \infty$ but clearly a similar result can be obtained for the cube.

COROLLARY 3.6. *Assume that $p \in (0, \infty)$, and let $\|\cdot\|_{\mathcal{I}B_p^n}$ be the Minkowski functional of the intersection body $\mathcal{I}B_p^n$. Let also η be a random vector distributed uniformly on the sphere S^{n-1}, and let $a_{p,1}, b_{p,1}$ and $\Sigma_{p,1}^2$ be as in Theorem 3.4. Then as $n \to \infty$, the sequence of random variables*

$$n^{3/2}\left(\|\eta\|_{\mathcal{I}B_p^n}\,\mathrm{Vol}_{n-1}(B_p^{n-1}) - \frac{1}{a_{p,1}} + \frac{b_{p,1}}{na_{p,1}^2}\right)$$

converges in distribution to a Gaussian random variable with mean zero and variance

$$a_{p,1}^{-4}\Sigma_{p,1}^2 = \frac{3\pi}{16}\left(\frac{\Gamma(1/p)}{\Gamma(3/p)}\right)^3 \Gamma(1+1/p)^{-2}\left(\frac{\Gamma(5/p)}{\Gamma(1/p)} - 3\frac{\Gamma(3/p)^2}{\Gamma(1/p)^2}\right)^2.$$

4. Tools.

4.1. *Stable random variables and the first volume formula.*   Recall that, for $\alpha \in (0, 1)$, a positive random variable Y is called a standard positive α-stable random variable if $\mathbb{E}e^{-tY} = e^{-t^{\alpha}}$ for $t \geq 0$. In this case for $q < \alpha$,

$$(7) \qquad \mathbb{E}Y^q = \frac{\Gamma(-q/\alpha)}{\alpha\Gamma(-q)}.$$

A positive α-stable variable has a density, which we will henceforth denote by g_α. We refer to [39, 75] for basic information concerning stable random variables.

Our main tool will be the following formula for volumes of sections of B_p^n due to Nayar and Tkocz [68]. The article [68] provides the proof in the case of $p = 1$ with a remark that the same method works for general p. For reader's convenience and to provide the constants in the case $p \neq 1$, in Appendix 8 we sketch the argument from [68] in full generality.

THEOREM 4.1.   *Let* $p \in (0, 2)$, *and let* H *be a subspace of* $\mathbb{R}^n$ *of codimension* d. *Let* $u_1, \ldots, u_d$ *be an orthornormal basis in* $H^\perp$, *and let* $v_1, \ldots, v_n$ *be the columns of the matrix with rows* $u_1, \ldots, u_d$. *Then*

$$(8) \qquad \mathrm{Vol}_{n-d}\big(B_p^n \cap H\big) = \frac{2^n}{\pi^{d/2}} \frac{\Gamma(1 + \frac{1}{p})^n}{\Gamma(1 + \frac{n-d}{p})} \mathbb{E}\left(\det\left(\sum_{j=1}^n \frac{1}{W_j} v_j v_j^T\right)\right)^{-1/2},$$

where W_j*'s are i.i.d. random variables with density proportional to* $t \mapsto \frac{1}{\sqrt{t}} g_{p/2}(t)$.

COROLLARY 4.2.   *In the setting of Theorem* 4.1,

$$\frac{\mathrm{Vol}_{n-d}(B_p^n \cap H)}{\mathrm{Vol}_{n-d}(B_p^{n-d})} = \frac{2^d \Gamma(1 + \frac{1}{p})^d}{\pi^{d/2}} \mathbb{E}\left(\det\left(\sum_{j=1}^n \frac{1}{W_j} v_j v_j^T\right)\right)^{-1/2}.$$

In the sequel we will need a formula for moments given in the next lemma.

LEMMA 4.3.   *For* $\alpha \in (0, 1)$, *let* W *be a random variable with density proportional to* $t \mapsto \frac{1}{\sqrt{t}} g_\alpha(t)$. *Then for* $q < \alpha + 1/2$,

$$\mathbb{E}W^q = \frac{\Gamma(\frac{1-2q}{2\alpha})\Gamma(\frac{1}{2})}{\Gamma(\frac{1-2q}{2})\Gamma(\frac{1}{2\alpha})} = \frac{\Gamma(\frac{1-2q}{2\alpha})\sqrt{\pi}}{\Gamma(\frac{1-2q}{2})\Gamma(\frac{1}{2\alpha})}.$$

PROOF.   Let Y be a standard positive α-stable random variable. Noting that the density of W equals to $\frac{1}{\sqrt{t}\mathbb{E}Y^{-1/2}} g_\alpha(t)$, we get

$$\mathbb{E}W^q = \frac{\mathbb{E}Y^{q-1/2}}{\mathbb{E}Y^{-1/2}} = \frac{\Gamma(\frac{1-2q}{2\alpha})\Gamma(\frac{1}{2})}{\Gamma(\frac{1-2q}{2})\Gamma(\frac{1}{2\alpha})},$$

where in the last equality we used (7).   □

4.2. *The second volume formula.*   In the proof of Theorem 3.4, we will use another formula for volumes of sections of B_p^n. We will use only the special case for codimension one, but we formulate it in full generality. The case $d = 1$ is stated in [20], Corollary 4.8; the general case also follows from arguments presented therein.

THEOREM 4.4. *Let $p \in (0, \infty)$, and let H be a subspace of $\mathbb{R}^n$ of codimension d. Let $u_1, \ldots, u_d$ be an orthonormal basis in $H^\perp$, and let $v_1, \ldots, v_n$ be the columns of the matrix with rows $u_1, \ldots, u_d$. Let moreover $Y_1, \ldots, Y_n$ be independent random variables with density $e^{-\beta_p^p |x|^p}$, where $\beta_p = 2\Gamma(1 + 1/p)$. Then*

$$\frac{\mathrm{Vol}_{n-d}(B_p^n \cap H)}{\mathrm{Vol}_{n-d}(B_p^{n-d})} = f(0),$$

where $f : \mathbb{R}^d \to [0, \infty)$ is the continuous version of the density of the $\mathbb{R}^d$-valued random vector $\sum_{i=1}^n v_i Y_i$.

4.3. *Uniform random subspaces and the uniform spherical distribution.* Let us now introduce basic facts concerning uniform random subspaces of $\mathbb{R}^n$ and uniform distribution on S^{n-1}, which we will need in the proofs. We refer the reader, for example, to the classical monograph [62] by Milman and Schechtman for a more detailed account.

By $G_{n,k}$ we will denote the Grassmanian, that is, the space of all k-dimensional subspaces of $\mathbb{R}^n$. It is endowed with the unique normalized Haar measure inherited from the natural action of the orthogonal group. Whenever we speak about a random k-dimensional subspace of $\mathbb{R}^n$, we mean a random element of $G_{n,k}$ distributed according to this measure. It follows from the uniqueness of Haar measure that H is a random k-dimensional subspace if and only if its orthogonal complement $H^\perp$ is a random $(n - k)$-dimensional subspace.

Basic properties of Gaussian random variables imply that if $G_1, \ldots, G_k$ are i.i.d. standard Gaussian vectors in $\mathbb{R}^n$, then span$(G_1, \ldots, G_k)$ is a random k-dimensional subspace of $\mathbb{R}^n$. Moreover, if $u_1, \ldots, u_k$ are obtained from $G_1, \ldots, G_k$ as a result of the Gram–Schmidt orthogonalization, then each of the random vectors u_i is distributed uniformly on the sphere S^{n-1}.

An important role in our arguments will be played by integrability properties of the random vectors u_i, which will follow from the classical concentration inequality. It is a consequence of the isoperimetic inequality on the sphere due to Lévy [53] and Schmidt [78, 79]. Its crucial role in high-dimensional geometry and probability was first observed by V. Milman in his proof of the Dvoretzky theorem [61].

THEOREM 4.5. *Let u be a random vector distributed uniformly on the sphere S^{n-1}. Then for any 1-Lipschitz function $f : S^{n-1} \to \mathbb{R}$, and any $t > 0$,*

$$\mathbb{P}(|f(u) - \mathbb{E}f(u)| \geq t) \leq 2 \exp(-cnt^2),$$

where c is a universal constant.

4.4. *Reverse Hölder inequalities for polynomials.* We will also need comparison of moments for polynomials in independent random variables. Historically, results of this type (for positive moments) appeared first for Gaussian and Rademacher variables in the work by Nelson [69] and Bonami [15] in the context of hypercontractivity of Markov semigroups. Later, they were extended to more general variables in particular in [46, 48]. We refer to the monographs [21, 49] for a detailed account.

Apart from comparison of positive moments, we will need a result allowing us to compare negative moments with the first moment for tetrahedral polynomials with nonnegative coefficients (recall that a multivariate polynomial is called tetrahedral if it is affine in each of the variables). We have not been able to find results of this spirit in the literature; we provide them in Lemma 4.10 and Corollary 4.11 below. We remark that certain reverse Hölder inequalities for negative moments of positive functions in the discrete setting have been considered in

[65]. Inequalities for general polynomials and log-concave measures were obtained in [18]. For our purposes we need, however, stronger estimates in a simpler setting of independent, positive random variables.

Let $\kappa > \rho > 1$. We say that a random variable X is (κ, ρ) hypercontractive with constant γ if, for every $a, b \in \mathbb{R}$,

$$\|a + b\gamma X\|_\kappa \le \|a + bX\|_\rho. \tag{9}$$

The following theorem is the real valued case of [46], Proposition 3.2. We remark that the original formulation is more general, as it involves hypercontractivity for Hilbert space valued coefficients. Moreover, it provides explicit formulas for the value of the constant γ, which we will omit, as it will not be needed in the sequel.

PROPOSITION 4.6. *If $1 < \rho \le 2 \le \kappa$ and X is a centered random variable such that $\|X\|_\kappa < \infty$, then X is (κ, ρ)-hypercontractive.*

As is well-known (see, e.g., [46], Theorem 2.5, or [48], formula (1.4)), hypercontractivity implies comparison of moments for tetrahedral polynomials.

PROPOSITION 4.7. *Let $1 < \rho \le \kappa$. If $X_1, \ldots, X_n$ are independent random variables, (κ, ρ)-hypercontractive with the same constant γ, then for all tetrahedral polynomials Q in n variables*

$$\big\| Q(\gamma X_1, \ldots, \gamma X_n) \big\|_\kappa \le \big\| Q(X_1, \ldots, X_n) \big\|_\rho.$$

COROLLARY 4.8. *Let $\kappa \ge 2$, and let $X_1, \ldots, X_n$ be i.i.d. copies of a random variable X such that $\mathbb{E}|X|^\kappa < \infty$. Then for every $\rho > 0$, there exists a constant K, depending only on d, κ, ρ and the law of X, such that, for every tetrahedral polynomial of degree d in n variables,*

$$\big\| Q(X_1, \ldots, X_n) \big\|_\kappa \le K \big\| Q(X_1, \ldots, X_n) \big\|_\rho.$$

PROOF.   Since $Q(X_1, \ldots, X_n)$ can be written as a tetrahedral polynomial in the variables $X_1 - \mathbb{E}X, \ldots, X_n - \mathbb{E}X$, we may and will assume that X is centered. Moreover, we can assume that $\rho \in (1, 2)$. Indeed, let Z be a random variable such that, for some $\kappa > \rho > 0$ and a constant D,

$$\|Z\|_\kappa \le D\|Z\|_\rho,$$

and let $\rho' \in (0, \rho)$. Then by Hölder's inequality applied with exponents $t, t/(t-1)$ for $t = (\kappa - \rho')/(\kappa - \rho) > 1$, we get, for $r = \rho'/(\rho t) < 1$,

$$\mathbb{E}Z^\rho = \mathbb{E}Z^{\rho r} Z^{\rho(1-r)} \le \big(\mathbb{E}Z^{\rho r t}\big)^{1/t} \big(\mathbb{E}Z^{\rho(1-r)t/(t-1)}\big)^{1-1/t} = \big(\mathbb{E}Z^{\rho'}\big)^{1/t} \big(\mathbb{E}Z^\kappa\big)^{1-1/t}$$

$$\le \big(\mathbb{E}Z^{\rho'}\big)^{1/t} D^{\kappa(1-1/t)} \big(\mathbb{E}Z^\rho\big)^{\kappa(1-1/t)/\rho} = \big(\mathbb{E}Z^{\rho'}\big)^{1/t} D^{\rho(1-r)} \big(\mathbb{E}Z^\rho\big)^{1-r},$$

from which one obtains $\|Z\|_\rho \le D^{1/r-1}\|Z\|_{\rho'}$ and as a consequence $\|Z\|_\kappa \le D^{1/r}\|Z\|_{\rho'}$. In what follows, we will thus assume that $\rho \in (1, 2)$.

Write $Q = Q_0 + \cdots + Q_d$, where Q_i is the homogeneous part of Q of degree i. By Propositions 4.6 and 4.7,

$$\big\| Q(X_1, \ldots, X_n) \big\|_\kappa \le \sum_{i=0}^{d} \big\| Q_i(X_1, \ldots, X_n) \big\|_\kappa \le K \sum_{i=0}^{d} \big\| Q_i(X_1, \ldots, X_n) \big\|_\rho$$

for some constant K, as in the statement of the corollary.

By [1], Proposition 1.2, (see also [47], Lemma 2) the right-hand side above is bounded from above by $C_d K \|Q(X_1, \ldots, X_n)\|_\rho$, which ends the proof. $\square$

Above we have considered only tetrahedral polynomials; however, in the Gaussian case, thanks to infinite divisibility, each polynomial can be approximated by a tetrahedral one. Hypercontractivity of Gaussian variables in the sense of (9) goes back to the seminal work of Nelson, Bonami and Gross related to hypercontractivity for Markov semigroups. As a consequence of their results, we have the following theorem (we remark that explicit constants are known, but as they are not needed for our proofs, we state the result in a simplified version).

THEOREM 4.9. *Let G be a standard Gaussian vector in $\mathbb{R}^n$. For each $\kappa, \rho > 0$ and every polynomial $Q \colon \mathbb{R}^n \to \mathbb{R}$ of degree at most d,*

$$\|Q(G)\|_\kappa \leq C_{\kappa,\rho,d} \|Q(G)\|_\rho.$$

So far we have dealt only with positive moments; however, a crucial step in the proof of Theorem 3.1 will be based on comparison of the first moment with negative moments for positive polynomials in independent random variables. Its proof relies on a similar idea as the one presented above.

LEMMA 4.10. *Let X be a positive random variable such that $\mathbb{E}X < \infty$ and $\mathbb{E}X^{-\rho} < \infty$ for some $\rho > 0$. Then for any $\kappa \in (0, \rho]$ and any $a \geq 0, b \geq 0$,*

$$\mathbb{E}\frac{1}{(a + bX)^\kappa} \leq \frac{1}{(a + \sigma b \mathbb{E}X)^\kappa},$$

where $\sigma = \|X\|_{-\rho}/\|X\|_1$.

PROOF. We may assume without loss of generality that $a, b > 0$. Consider the function $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$, given by

$$\varphi(z) = \frac{1}{(1 + z^{-1/\rho})^\kappa}.$$

We have

$$\varphi''(z) = -\frac{\kappa (z^{-1/\rho} + 1)^{-\kappa}(-\kappa + \rho z^{1/\rho} + z^{1/\rho} + \rho)}{\rho^2 z^2 (z^{1/\rho} + 1)^2} \leq 0$$

for $z > 0$. Thus, φ is concave on $\mathbb{R}_+$. Setting $Z = X^{-\rho}$, by Jensen's inequality we get

$$
\begin{aligned}
\mathbb{E}\frac{1}{(a + bX)^\kappa} &= \frac{1}{a^\kappa}\mathbb{E}\frac{1}{(1 + ((a/b)^\rho Z)^{-1/\rho})^\kappa} \\
&= \frac{1}{a^\kappa}\mathbb{E}\varphi((a/b)^\rho Z) \leq \frac{1}{a^\kappa}\varphi((a/b)^\rho \mathbb{E}Z) \\
&= \frac{1}{(a + b\|X\|_{-\rho})^\kappa} = \frac{1}{(a + \sigma b \mathbb{E}X)^\kappa}. \qquad \square
\end{aligned}
$$

COROLLARY 4.11. *In the setting of Lemma 4.10, let $X_1, \ldots, X_n$ be i.i.d. copies of the random variable X and let $(a_{i_1,\ldots,i_d})_{i_1,\ldots,i_d \in [n]}$ be an array of nonnegative numbers such that $a_{i_1,\ldots,i_d} = 0$ whenever there exists $l \neq m$ such that $i_l = i_m$. Let $Y = \sum_{i_1,\ldots,i_d=1}^n a_{i_1,\ldots,i_d} X_{i_1} \cdots X_{i_d}$. Then*

$$\|Y\|_1 \leq \sigma^{-d}\|Y\|_{-\kappa}.$$

PROOF. We will prove by induction on n that for any tetrahedral polynomial Q in n variables with nonnegative coefficients,

$$(10) \qquad \mathbb{E} \frac{1}{Q(X_1, \ldots, X_n)^\kappa} \leq \frac{1}{(\mathbb{E}Q(\sigma X_1, \ldots, \sigma X_n))^\kappa}$$

from which the corollary follows by homogeneity of the d-linear form.

For $n = 0$, (10) is trivial, while for $n = 1$ it reduces to the assertion of Lemma 4.10. Assume thus that (10) holds for $0, 1, \ldots, n-1$. Any tetrahedral polynomial in n variables with nonnegative coefficients is of the form $Q(x_1, \ldots, x_n) = x_n Q_1(x_1, \ldots, x_{n-1}) + Q_2(x_1, \ldots, x_{n-1})$ where Q_1, Q_2 are tetrahedral and also have nonnegative coefficients. Thus, by Lemma 4.10 applied conditionally on $X_1, \ldots, X_{n-1}$ and the induction assumption we obtain

$$\mathbb{E} \frac{1}{Q(X_1, \ldots, X_n)^\kappa}$$

$$= \mathbb{E}_{X_1, \ldots, X_{n-1}} \mathbb{E}_{X_n} \frac{1}{(X_n Q_1(X_1, \ldots, X_{n-1}) + Q_2(X_1, \ldots, X_{n-1}))^\kappa}$$

$$\leq \mathbb{E}_{X_1, \ldots, X_{n-1}} \frac{1}{((\mathbb{E}_{X_n} \sigma X_n) Q_1(X_1, \ldots, X_{n-1}) + Q_2(X_1, \ldots, X_{n-1}))^\kappa}$$

$$\leq \frac{1}{(\mathbb{E}_{X_1, \ldots, X_{n-1}}((\mathbb{E}_{X_n} \sigma X_n) Q_1(\sigma X_1, \ldots, \sigma X_{n-1}) + Q_2(\sigma X_1, \ldots, \sigma X_{n-1})))^\kappa}$$

$$= \frac{1}{(\mathbb{E}Q(\sigma X_1, \ldots, \sigma X_n))^\kappa},$$

where we used the fact that $(\sigma \mathbb{E}_{X_n} X_n) Q_1(x_1, \ldots, x_{n-1}) + Q_2(x_1, \ldots, x_{n-1})$ is tetrahedral, which allows us to use the induction assumption. $\square$

4.5. *U-Statistics.* Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with values in some measurable space $(\mathcal{S}, \mathcal{F})$, and let $h: \mathcal{S}^d \to \mathbb{R}$ be a measurable function. The U-statistics of order d and kernel h, based on the sequence $\{X_i\}_{i \geq 1}$, are random variables defined as

$$U_n(h) = U_n^{(d)}(h) = \frac{(n-d)!}{n!} \sum_{\mathbf{i} \in [n]^{\underline{d}}} h(X_{i_1}, \ldots, X_{i_d}), \quad n \geq d.$$

Define also the unnormalized sum

$$S_n(h) = S_n^{(d)} = \sum_{\mathbf{i} \in [n]^{\underline{d}}} h(X_{i_1}, \ldots, X_{i_d}) = \frac{n!}{(n-d)!} U_n(h).$$

We will also use the above definitions for $d = 0$ interpreting $\mathcal{S}^0$ as a one element set and identifying functions from $\mathcal{S}^0$ into $\mathbb{R}$ with constants. Thus, according to this convention for any constant h, $S_n^{(0)}(h) = U_n^{(0)}(h) = h$.

Going back to general d, note that, by modifying the kernel h, we may assume without loss of generality that it is invariant under permutation of the arguments. In the subsequent part, we will work under this assumption.

U-statistics appeared for the first time in the 1940s in the work by Halmos [32] and Hoeffding [33] in the context of unbiased estimation. Since then, they have found applications, for example, as higher-order terms in Taylor expansions of smooth statistics [21] or in random graph theory [35]. In a geometric context, they were used in [86] to prove a CLT for volumes of Minkowski sums of random convex sets (see also the monograph [64]) and in [71] in the

proof of the CLT for volumes of low dimensional random projections of the cube. The reason behind the appearance of U-statistics in these references and our proof are quite different. In our case they show up as some conditional variances in the one-dimensional Taylor expansion of functions appearing in the volume formula of Theorem 4.1, whereas in the cited references they are directly related to mixed volumes and the zonotope volume formula. Yet another line of research involving a somewhat related but different notion of Poisson U-statistics has been pursued, for example, in [50, 74] where the authors among other examples consider applications to intrinsic volumes of sets arising from the Poisson flat process. These results are of a different flavour than ours, they are related to stochastic analysis on the Poisson space and concern asymptotic behaviour of random sets in a fixed dimension, when the intensity of the underlying Poisson process tends to infinity.

Below we describe basic properties of U-statistics, referring for a more detailed description of their asymptotic theory to the monographs [21, 35, 52].

The kernel h (and the corresponding U-statistic) is called canonical if $\mathbb{E}h(X, x_2, \ldots, x_d) = 0$ for all $x_2, \ldots, x_d \in \mathcal{S}$. One of the basic tools in the theory of U-statistics is the Hoeffding decomposition, which allows to represent any U-statistic based on a square integrable kernel of mean zero as a sum of uncorrelated canonical ones. Let us now briefly describe how it works. We will use the notation

$$\mu_1 \otimes \cdots \otimes \mu_d f = \int_{\mathcal{S}^d} f(x_1, \ldots, x_d)\mu_1(dx_1)\cdots\mu_d(dx_d),$$

where $\mu_1, \ldots, \mu_d$ are signed measures on $\mathcal{S}$. For $k = 0, \ldots, d$, we define the kth Hoeffding projection of h as

$$\pi_k h(x_1, \ldots, x_k) = (\delta_{x_1} - P) \otimes \cdots \otimes (\delta_{x_k} - P) \otimes P^{\otimes(d-k)}h,$$

where P is the law of X and δ_x stands for the Dirac mass at x. For instance,

$$\pi_0 h = \mathbb{E}h(X_1, \ldots, X_d),$$

$$\pi_1 h(x_1) = \mathbb{E}h(x_1, X_2, \ldots, X_d) - \mathbb{E}h(X_1, \ldots, X_d),$$

$$\pi_2 h(x_1, x_2) = \mathbb{E}h(x_1, x_2, X_3, \ldots, X_d) - \mathbb{E}h(x_1, X_2, \ldots, X_d)$$
$$- \mathbb{E}h(x_2, X_2, \ldots, X_d) + \mathbb{E}h(X_1, \ldots, X_d).$$

One checks that for $k > 0$ the kernel $\pi_k h$ is canonical. Moreover,

$$U_n^{(d)}(h) = \sum_{k=0}^{d} \binom{d}{k} U_n^{(k)}(\pi_k h)$$

or in terms of $S_n^{(k)}$,

$$(11) \qquad S_n^{(d)}(h) = \sum_{k=0}^{d} \binom{d}{k} \frac{(n-k)!}{(n-d)!} S_n^{(k)}(\pi_k h).$$

Canonical U-statistics can be thus considered building blocks for the general case. Their advantage stems from the following elementary estimate for their second moment. If h is canonical, then

$$\mathbb{E}\big|S_n^{(d)}(h)\big|^2 = \frac{n!d!}{(n-d)!}\mathbb{E}h(X_1, \ldots, X_d)^2.$$

Indeed, the left-hand side above equals

$$\sum_{\mathbf{i}\in[n]^{\underline{d}}} \sum_{\mathbf{j}\in[n]^{\underline{d}}} \mathbb{E}h(X_{i_1}, \ldots, X_{i_d})h(X_{j_1}, \ldots, X_{j_d}),$$

and using the canonicity and symmetry, one can easily see that the expectations do not vanish only if $\mathbf{i}$ and $\mathbf{j}$ differ just by a permutation of coordinates in which case they equal $\mathbb{E}h^2(X_1, \ldots, X_d)$. In particular, since $\pi_k h$ is a canonical kernel of order k, we obtain that for $k > 0$,

$$(12) \qquad \mathbb{E}|S_n^{(k)}(\pi_k h)|^2 = \frac{n!k!}{(n-k)!}\mathbb{E}(\pi_k h(X_1, \ldots, X_k))^2.$$

It is also straightforward to check that $S_n^{(k)}(\pi_k h)$, $k = 0, \ldots, n$ are uncorrelated. In particular (11), for $n = d$ together with (12) give

$$(13) \qquad \mathbb{E}h(X_1, \ldots, X_d)^2 = \sum_{k=0}^{d} \binom{d}{k} \mathbb{E}(\pi_k h(X_1, \ldots, X_k))^2.$$

4.6. *Hermite polynomials.* Recall that Hermite polynomials are orthogonal polynomials with respect to the standard Gaussian measure. There are many conventions concerning their normalization, we will use the probabilistic one and define for $n \in \mathbb{N}$,

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

Thus, in particular,

$$(14) \qquad \begin{aligned} H_0(x) &= 1, \\ H_1(x) &= x, \\ H_2(x) &= x^2 - 1, \\ H_3(x) &= x^3 - 3x, \\ H_4(x) &= x^4 - 6x^2 + 3. \end{aligned}$$

We refer, for example, to the monograph [35] for the presentation of various probabilistic properties and applications of Hermite polynomials. In the sequel only low degree polynomials will appear, in two different settings, to simplify calculation of expectations of Gaussian polynomials and as ingredients in the Edgeworth expansion.

4.7. *Cumulants.* Let us now briefly recall the basic properties of cumulants of random variables.

Let X be a real valued random variable. For $n \geq 1$, the cumulants $\mathfrak{c}_n = \mathfrak{c}_n(X)$ are defined as

$$\mathfrak{c}_n = \mathrm{i}^{-n} \frac{d^n}{dt^n} \log \mathbb{E}e^{\mathrm{i}tX}\Big|_{t=0}.$$

Clearly, $\mathfrak{c}_n$ are well-defined if X has all moments. Moreover, we have the identity (see, e.g., [63], equation (1.2))

$$(15) \qquad \mathfrak{c}_n = \sum_{\substack{1 \cdot r_1 + \cdots + n \cdot r_n = n \\ r_1, \ldots, r_n \in \mathbb{N}}} \frac{(-1)^{r_1 + \cdots + r_n - 1}(r_1 + \cdots + r_n - 1)!n!}{(1!)^{r_1}(2!)^{r_2} \cdots (n!)^{r_n} r_1! \cdots r_n!}(\mathbb{E}X)^{r_1}(\mathbb{E}X^2)^{r_2} \cdots (\mathbb{E}X^n)^{r_n}.$$

In particular,

$$(16) \qquad \begin{aligned} \mathfrak{c}_1 &= \mathbb{E}X, \\ \mathfrak{c}_2 &= \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathrm{Var}(X), \\ \mathfrak{c}_3 &= 2(\mathbb{E}X)^3 - 3\mathbb{E}X\mathbb{E}X^2 + \mathbb{E}X^3 = \mathbb{E}(X - \mathbb{E}X)^3, \\ \mathfrak{c}_4 &= \mathbb{E}(X - \mathbb{E}X)^4 - 3(\mathbb{E}(X - \mathbb{E}X)^2)^2. \end{aligned}$$

The other important properties of cumulants are

$$\mathfrak{c}_n(X + t) = \mathfrak{c}_n(X) \quad \text{for } n \geq 2, t \in \mathbb{R},$$
$$\mathfrak{c}_n(t X) = t^n \mathfrak{c}_n(X) \quad \text{for } t \in \mathbb{R}$$

and

$$\mathfrak{c}_n(X_1 + \cdots + X_m) = \mathfrak{c}_n(X_1) + \cdots + \mathfrak{c}_n(X_m)$$

if $X_1, \ldots, X_m$ are independent random variables.

It is also easy to see that odd cumulants of a symmetric random variable vanish.

4.8. *Edgeworth expansion.* We are also going to need basic results on Edgeworth expansion of the density for sums of independent nonidentically distributed random variables, which provides the correction to the local central limit theorem. As references for this topic, we propose the classical monographs [12, 72].

Consider a sequence $X_1, X_2, \ldots$ of independent centered random variables. Let $B_n = \sum_{i=1}^n \mathbb{E}X_i^2$, and for integers $k \geq 2$ and $n \geq 1$, define the number

$$(17) \qquad \lambda_{k,n} = \frac{n^{(k-2)/2}}{B_n^{k/2}} \sum_{i=1}^n \mathfrak{c}_k(X_i)$$

and a function $q_{k,n} \colon \mathbb{R} \to \mathbb{R}$,

$$q_{k,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{\substack{1 \cdot r_1 + \cdots + k \cdot r_k = k \\ r_1, \ldots, r_k \in \mathbb{N}}} H_{k+2(r_1 + \cdots + r_k)}(x) \prod_{i=1}^k \frac{1}{r_i!} \left(\frac{\lambda_{i+2,n}}{(i+2)!} \right)^{r_i}.$$

In particular,

$$q_{0,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

$$q_{1,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} H_3(x) \frac{\lambda_{3,n}}{6},$$

$$(18) \qquad q_{2,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left(H_4(x) \frac{\lambda_{4,n}}{24} + H_6(x) \frac{\lambda_{3,n}^2}{72} \right),$$

$$q_{3,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left(H_5(x) \frac{\lambda_{5,n}}{120} + H_7(x) \frac{\lambda_{3,n} \lambda_{4,n}}{144} + H_9(x) \frac{\lambda_{3,n}^3}{1296} \right).$$

The Edgeworth-type theorem we are going to use is the following special case ($l = 1$) of [72], Theorem 7, p. 175.

THEOREM 4.12. *Let $X_1, X_2, \ldots$ be independent random variables with mean zero. Denote $B_n = \sum_{i=1}^n \mathbb{E}X_i^2$. Let $K \geq 3$ be an integer. Assume that:*

(i) $\liminf_{n\to\infty} \frac{B_n}{n} > 0$, $\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i|^K < \infty$,

(ii) $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i|^K \mathbf{1}_{\{|X_i|>n^\tau\}} = 0$ *for some positive $\tau < 1/2$,*

(iii) *for $\alpha = (K-1)/2$ and every fixed $\varepsilon > 0$,*

$$(19) \qquad \lim_{n\to\infty} n^\alpha \int_{|t|>\varepsilon} \prod_{i=1}^n \mathbb{E}e^{\mathrm{i}t X_i} \, dt = 0.$$

Then for all n sufficiently large the random variable $S_n = B_n^{-1/2} \sum_{i=1}^{n} X_i$ *admits a density* f_n *such that*

$$(20) \qquad f_n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \sum_{k=1}^{K-2} \frac{q_{k,n}(x)}{n^{k/2}} + o\left(\frac{1}{n^{(K-2)/2}}\right)$$

uniformly in $x \in \mathbb{R}$.

To apply the above theorem, we will need the following lemma (a corollary to [72], Lemma 10, and the remark following its proof (p. 174)).

LEMMA 4.13.   *Let* $X_1, X_2, \ldots$ *be a sequence of independent centered random variables. Assume that there exists a set* $\mathcal{I}$ *of positive integers and positive constants* λ, δ, C, R *such that*

$$\liminf_{n \to \infty} \frac{|\mathcal{I} \cap [n]|}{n^\lambda} > 0,$$

and for all $n \in \mathcal{I}$ *and* $|t| > R$,

$$(21) \qquad \left| \mathbb{E} e^{it X_n} \right| \le \frac{C}{|t|^\delta}.$$

Then the condition (19) *is satisfied for any* $\alpha > 0$. *Moreover, the condition* (21) *is satisfied for some* $C, \delta, R > 0$ *if the variables* $X_n, n \in \mathcal{I}$, *have densities of uniformly* (*with respect to* $n \in \mathcal{I}$) *bounded variation.*

5. Proof of Theorem 3.1.   Let us now give the proof of Theorem 3.1. It will be split into several steps presented in separate sections. To simplify the notation, we will often suppress the dependence of random variables on the dimension n. It should be remembered that the asymptotic notation such as $o(1)$, $O(n)$ etc. concerns $n \to \infty$. The implicit constants may depend on d, p and some other parameters (independent of n), for example, the order of moments considered. Recall also the notation $\mathcal{O}_m(\cdot)$, $o_m(\cdot)$ introduced in Section 2.

5.1. *Preliminary steps.*   A random subspace of $\mathbb{R}^n$ of dimension d may be obtained as span$(G_1, \ldots, G_d)$, where $G_i = G_i^{(n)} = (g_{i1}, \ldots, g_{in})$, $i = 1, \ldots, d$ are i.i.d. standard Gaussian vectors in $\mathbb{R}^n$ (i.e., g_{ij}, $i \in [d]$, $j \in [n]$ are i.i.d. Gaussian variables of mean zero and variance one). If we define $u_1, \ldots, u_d$ as the result of the Gram–Schmidt orthogonalization applied to $G_1, \ldots, G_d$, then $H_n = $ span$(G_1, \ldots, G_d)^\perp = $ span$(u_1, \ldots, u_d)^\perp$ is a uniform subspace of codimension d in $\mathbb{R}^n$. More explicitly, we have

$$u_i = \frac{G_i - P_{i-1}G_i}{|G_i - P_{i-1}G_i|},$$

where for $i = 0, \ldots, d-1$, $P_i \colon \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection on the random subspace span$(G_1, \ldots, G_i)$. Note that with probability one this subspace is of dimension i. Moreover, the random operator P_{i-1} is stochastically independent of the random vectors $G_i, \ldots, G_d$.

Observe also that each u_i is distributed uniformly on the sphere S^{n-1}. Note that the second moment of a single coordinate of u_i is equal to $1/\sqrt{n}$. Therefore, by the concentration inequality on the sphere (Theorem 4.5) and the union bound, for any $t \ge 1$,

$$(22) \qquad \mathbb{P}\left(\max_{i \le d} |u_i|_\infty \ge Ct \frac{\sqrt{\log(n)}}{\sqrt{n}} \right) \le 2nd \exp(-ct^2 \log n).$$

As a consequence, we obtain

$$
\text{(23)} \qquad \mathbb{P}\left(\max_{i \leq d} |u_i|_\infty \geq C \frac{\sqrt{\log n}}{\sqrt{n}}\right) = o(1).
$$

Moreover, for any $q > 0$, (22) and integration by parts give

$$
\text{(24)} \qquad \left\| \max_{i \leq d} |u_i|_\infty \right\|_q \leq C_{d,q} \frac{\sqrt{\log n}}{\sqrt{n}}
$$

in other words $\max_{i \leq d} |u_i|_\infty = \mathcal{O}_m(\sqrt{(\log n)/n})$ (recall the notation $\mathcal{O}_m, o_m$ introduced in Section 2).

Let $W_1, \ldots, W_n$ be i.i.d. random variables with density proportional to $\frac{1}{\sqrt{x}} g_{p/2}(x)$ and $X_i = W_i^{-1}$. We will assume that the family $(X_i)_i$ is stochastically independent of $(g_{ij})_{ij}$, and we will denote integration with respect to them by $\mathbb{E}_X$ and $\mathbb{E}_G$, respectively.

Going back to random subspaces, denote by A the $d \times n$ matrix with rows $u_1, \ldots, u_d$, and let $v_1, \ldots, v_n$ be its columns (thus, v_i's are random vectors in $\mathbb{R}^d$). Let also $\tilde{A}$ be the matrix with columns $\sqrt{X_i} v_i$. Then

$$
\sum_{j=1}^n X_j v_j v_j^T = \tilde{A} \tilde{A}^T,
$$

and using the Cauchy–Binet formula, we get

$$
\text{(25)} \qquad Y_n := \det\left(\sum_{j=1}^n X_j v_j v_j^T\right) = \sum_{\mathbf{i} \in [n]^{\underline{d}}} a_{\mathbf{i}} X_{i_1} \cdots X_{i_d},
$$

where for $\mathbf{i} = (i_1, \ldots, i_d) \in [n]^{\underline{d}}$,

$$
a_{\mathbf{i}} = \frac{1}{d!} (\det A_{\mathbf{i}})^2,
$$

with $A_{\mathbf{i}}$ being the $d \times d$ matrix with columns $v_{i_1}, \ldots, v_{i_d}$ (we choose to write the quantities of interest in terms of summation over the whole set $[n]^{\underline{d}}$ rather than just over increasing sequences to simplify bookkeeping of indices which will appear in the sequel). Another application of the Cauchy–Binet formula gives

$$
\text{(26)} \qquad \sum_{\mathbf{i} \in [n]^{\underline{d}}} a_{\mathbf{i}} = \det\left(\sum_{j=1}^n v_j v_j^T\right) = \det(\mathrm{Id}) = 1.
$$

Moreover, using the multilinearity of the determinant and the fact that it vanishes for matrices which are not of full rank together with the definition of the vectors u_j and v_j, one can expand $\det A_{\mathbf{i}}$ along the steps of the Gram–Schmidt procedure and see that

$$
\text{(27)} \qquad a_{\mathbf{i}} = \frac{1}{d!} \frac{1}{\prod_{\ell=1}^d |G_\ell - P_{\ell-1} G_\ell|^2} \det(\Gamma_{\mathbf{i}})^2,
$$

where $\Gamma_{\mathbf{i}}$ is the $d \times d$ matrix with columns $\Gamma_{i_1}, \ldots, \Gamma_{i_d}$ with $\Gamma_i = (g_{ji})_{j \in [d]}$.

Moreover, by (23), with probability tending to one as $n \to \infty$,

$$
\text{(28)} \qquad \max_{\mathbf{i} \in [n]^{\underline{d}}} a_{\mathbf{i}} \leq C_d \frac{(\log n)^d}{n^d},
$$

whereas (24) and Hölder's inequality give

$$
\text{(29)} \qquad \max_{\mathbf{i} \in [n]^{\underline{d}}} a_{\mathbf{i}} = \mathcal{O}_m\left(\frac{(\log n)^d}{n^d}\right).
$$

Observe that, thanks to Lemma 4.3 (applied with $\alpha = p/2$), we have $\mathbb{E}X_i^\rho < \infty$ for all $\rho > 0$ and $\mathbb{E}X_i^{-\rho} < \infty$ for $\rho < p/2 + 1/2$. Note that $p/2 + 1/2 > 1/2$. Using Corollary 4.11 together with (26), we obtain that there exists $\rho = \rho_p > 1/2$ and a constant $C_{p,d}$, depending only on p and d, such that

$$\tag{30} \mathbb{E}_X \frac{1}{(\sum_{\mathbf{i}\in[n]^{\underline{d}}} a_\mathbf{i} X_{i_1} \cdots X_{i_d})^\rho} \leq C_{p,d} \frac{1}{(\sum_{\mathbf{i}\in[n]^{\underline{d}}} a_\mathbf{i})^\rho} = C_{p,d}.$$

(Note that thanks to (26) the bound on the random quantity on the left-hand side becomes deterministic.)

Similarly, by Corollary 4.8 we get that, for all $r > 1$, there exists $C_{p,d,r}$ such that, for every tetrahedral polynomial Q of degree at most d,

$$\tag{31} \|Q(X_1, \ldots, X_n)\|_r \leq C_{p,d,r} \|Q(X_1, \ldots, X_n)\|_1.$$

By Corollary 4.2,

$$\tag{32} \frac{\mathrm{Vol}_{n-d}(H_n \cap B_p^n)}{\mathrm{Vol}_{n-d}(B_p^{n-d})} = \frac{2^d \Gamma(1 + 1/p)^d}{\pi^{d/2}} \mathbb{E}_X \frac{1}{\sqrt{Y_n}},$$

our goal is thus to obtain a limit theorem for the variable

$$\tag{33} \mathbb{E}_X \frac{1}{\sqrt{Y_n}} - \frac{\pi^{d/2}}{2^d \Gamma(1 + 1/p)^d}\left(a_{p,d} - \frac{1}{n}b_{p,d}\right).$$

Note that in the above expression the random variables X_i are integrated out and it is a function of the random vectors $G_1, \ldots, G_d$.

5.2. Linearization.

We will make use of the identity

$$\frac{1}{\sqrt{y}} = \frac{1}{\sqrt{\mu}} - \frac{y - \mu}{2\mu^{3/2}} + \frac{3(y - \mu)^2}{8\mu^{5/2}} - \frac{5(y - \mu)^3}{16\mu^{7/2}}$$
$$+ \frac{(\sqrt{y} - \sqrt{\mu})^4(16\mu^{3/2} + 5y^{3/2} + 20\sqrt{\mu}y + 29\mu\sqrt{y})}{16\mu^{7/2}\sqrt{y}},$$

valid for $y, \mu > 0$.

Applying it to $y = Y_n = \sum_{\mathbf{i}\in[n]^{\underline{d}}} a_\mathbf{i} X_{i_1} \cdots X_{i_d}$ and

$$\tag{34} \mu = \mathbb{E}_X Y_n = (\mathbb{E}X_1)^d$$

(where we used (26)) and integrating with respect to X_i's, we obtain

$$\mathbb{E}_X \frac{1}{\sqrt{Y_n}} - \frac{1}{\sqrt{\mu}} = \frac{3\mathbb{E}_X(Y_n - \mu)^2}{8\mu^{5/2}} - \frac{5\mathbb{E}_X(Y_n - \mu)^3}{16\mu^{7/2}}$$

$$\tag{35} + \mathbb{E}_X \frac{(\sqrt{Y_n} - \sqrt{\mu})^4(16\mu^{3/2} + 5Y_n^{3/2} + 20\sqrt{\mu}Y_n + 29\mu\sqrt{Y_n})}{16\mu^{7/2}\sqrt{Y_n}}$$

$$=: T_{1,n} + T_{2,n} + T_{3,n}.$$

We will first show that $T_{2,n}$ and $T_{3,n}$ are $o_m(n^{-3/2})$ as $n \to \infty$.

Let us start with $T_{2,n}$. To simplify the notation, for $\mathbf{i} \in [n]^{\underline{d}}$ denote

$$\tag{36} \mathbf{X_i} = X_{i_1} \cdots X_{i_d} - \mathbb{E}_X X_{i_1} \cdots X_{i_d}.$$

For two multiindices $\mathbf{i}, \mathbf{j} \in [n]^{\underline{d}}$, let

$$\tag{37} \mathcal{E}(\mathbf{i}, \mathbf{j}) = \{(k, l) \in [d]^2 : i_k = j_l\}.$$

For $I, J, K \subset [d]^2$ let

$$A(I, J, K) = \big\{(\mathbf{i}, \mathbf{j}, \mathbf{k}) \in ([n]^{\underline{d}})^3 : \mathcal{E}(\mathbf{i}, \mathbf{j}) = I, \mathcal{E}(\mathbf{j}, \mathbf{k}) = J, \mathcal{E}(\mathbf{i}, \mathbf{k}) = K\big\}.$$

Note that if at least two of the sets I, J, K are empty, then by independence of X_i's, we have $\mathbb{E}_X \mathbf{X_i X_j X_k} = 0$ for all $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \in A(I, J, K)$.

Thus,

$$\mathbb{E}_X (Y_n - \mu)^3 = \sum_{I, J, K} \sum_{(\mathbf{i}, \mathbf{j}, \mathbf{k}) \in A(I, J, K)} a_{\mathbf{i}} a_{\mathbf{j}} a_{\mathbf{k}} \mathbb{E}_X \mathbf{X_i X_j X_k},$$

where the outer summation is over all triples I, J, K of subsets of $[d]^2$ in which at least two sets are nonempty. To prove that $T_{2,n} = o_m(n^{-3/2})$, it thus suffices to show that, for each such triple,

$$(38) \qquad \sum_{(\mathbf{i}, \mathbf{j}, \mathbf{k}) \in A(I, J, K)} a_{\mathbf{i}} a_{\mathbf{j}} a_{\mathbf{k}} \mathbb{E}_X |\mathbf{X_i X_j X_k}| = o_m(n^{-3/2}).$$

Observe first that $|A(I, J, K)| \leq n^{3d-2}$. Moreover, by (29) and Hölder's inequality,

$$a_{\mathbf{i}} a_{\mathbf{j}} a_{\mathbf{k}} = \mathcal{O}_m\big((\log n)^{3d} n^{-3d}\big).$$

Since $X_i's$ have moments of all orders, another application of Hölder's inequality gives $\mathbb{E}_X |\mathbf{X_i X_j X_k}| \leq C_{d,p}$. These three observations imply that, for any $q \geq 1$, the left-hand side of (38) is bounded in L_q by $C_{d,p,q} (\log n)^{3d} n^{-2} = o(n^{-3/2})$. Thus indeed, $T_{2,n} = o_m(n^{-3/2})$.

We will now pass to $T_{3,n}$. Recall (30) and (31). Let $\kappa = 4\rho/(2\rho - 1) > 1$. Using Hölder's inequality applied with exponents $\kappa, \kappa, 2\rho$ and taking into account that $\mu = (\mathbb{E}X_1)^d$ depends only on p and d, we obtain that

$$T_{3,n} = \mathbb{E}_X \frac{(\sqrt{Y_n} - \sqrt{\mu})^4 (16\mu^{3/2} + 5Y_n^{3/2} + 20\sqrt{\mu}Y_n + 29\mu\sqrt{Y_n})}{16\mu^{7/2}\sqrt{Y_n}}$$

$$\leq C_{p,d} \big(\mathbb{E}_X |\sqrt{Y_n} - \sqrt{\mu}|^{4\kappa}\big)^{1/\kappa}$$

$$\times \big(\mathbb{E}_X (16\mu^{3/2} + 5Y_n^{3/2} + 20\sqrt{\mu}Y_n + 29\mu\sqrt{Y_n})^\kappa\big)^{1/\kappa} \big(\mathbb{E}_X Y_n^{-\rho}\big)^{1/(2\rho)}.$$

Using (31) and (34), we see that the second factor on the right-hand side above is bounded from above by $C_{p,d}$, whereas an application of (30) shows that the third factor is bounded by $C_{p,d}$; thus, to prove that $T_{3,n} = o_m(n^{-3/2})$, it is enough to show that

$$(39) \qquad \big(\mathbb{E}_X |\sqrt{Y_n} - \sqrt{\mu}|^{4\kappa}\big)^{1/\kappa} = o_m(n^{-3/2}).$$

Using the inequality $|\sqrt{y} - \sqrt{\mu}| \leq \mu^{-1/2} |y - \mu|$, we get

$$\big(\mathbb{E}_X |\sqrt{Y_n} - \sqrt{\mu}|^{4\kappa}\big)^{1/\kappa} \leq C_{p,d} \big(\mathbb{E}_X |Y_n - \mu|^{4\kappa}\big)^{1/\kappa} \leq C_{p,d} \big(\mathbb{E}_X (Y_n - \mu)^2\big)^2,$$

where in the last inequality we used (31).

Thus, it is enough to show that $\mathbb{E}_X (Y_n - \mu)^2 = o_m(n^{-3/4})$. Note that $\mathbb{E}_X (Y_n - \mu)^2$ is proportional to the term $T_{1,n}$ in (35). In the next step of the proof, we will perform a careful analysis of this term from which it will in particular follow that it is $\mathcal{O}_m(n^{-1})$. For transparency, let us, however, provide now a simple argument showing that it is indeed $o_m(n^{-3/4})$. Some of the calculations will be also used in the said more precise analysis.

Recall the notation (37). Similarly as for $T_{2,n}$, for $I \subset [d]^2$, define $A(I) = \{(\mathbf{i}, \mathbf{j}) \in ([n]^{\underline{d}})^2 : \mathcal{E}(\mathbf{i}, \mathbf{j}) = I\}$. By independence of X_i's, we have

$$(40) \qquad \mathbb{E}_X (Y_n - \mu)^2 = \sum_{\varnothing \neq I \subset [d]^2} \sum_{(\mathbf{i}, \mathbf{j}) \in A(I)} a_{\mathbf{i}} a_{\mathbf{j}} \mathbb{E}_X (\mathbf{X_i X_j}).$$

For fixed I and $q \geq 1$, by the triangle inequality in L_q we obtain that, as $n \to \infty$,

$$(41) \quad \left\| \sum_{\substack{\mathbf{i},\mathbf{j}\in[n]^{\underline{d}} \\ \mathcal{E}(\mathbf{i},\mathbf{j})=I}} a_\mathbf{i} a_\mathbf{j} \mathbb{E}_X(\mathbf{X_i X_j}) \right\|_q \leq C_{p,d} |A(I)| \left\| \max_{\mathbf{i}\in[n]^{\underline{d}}} a_\mathbf{i}^2 \right\|_q \leq C_{p,d,q} |A(I)| \frac{(\log n)^{2d}}{n^{2d}},$$

where we again used the fact that X_i have all moments and (29).

For $I \neq \varnothing$, we have $|A(I)| \leq n^{2d-1}$, and so the right-hand side above is bounded by $C_{p,d,q}(\log n)^{2d} n^{-1} = o(n^{-3/4})$.

Summing over I, we get the claimed estimate.

5.3. *Analysis of randomly normalized U-statistics.*   We will now analyze the term $T_{1,n} = \frac{3\mathbb{E}_X(Y_n - \mu)^2}{8\mu^{5/2}}$. We will use the notation introduced at the end of the previous section. Note that if $I \subset [d]^2$ satisfies $|I| \geq 2$, then $|A(I)| \leq n^{2d-2}$. Thus, by (40) and (41), we see that

$$\mathbb{E}_X(Y_n - \mu)^2 = \sum_{\substack{I \subset [d]^2, |I|=1}} \sum_{\substack{\mathbf{i},\mathbf{j}\in[n]^{\underline{d}} \\ \mathcal{E}(\mathbf{i},\mathbf{j})=I}} a_\mathbf{i} a_\mathbf{j} \mathbb{E}_X(\mathbf{X_i X_j}) + o_m(n^{-3/2}).$$

Since X_i's are i.i.d, for all $\mathbf{i}, \mathbf{j}$ such that $|\mathcal{E}(\mathbf{i}, \mathbf{j})| = 1$, we have

$$(42) \quad \mathbb{E}\mathbf{X_i X_j} = \mathbb{E}X_1^2 \prod_{k=2}^{2d-1} X_k - (\mathbb{E}X_1)^{2d} = (\mathbb{E}X_1)^{2d-2} \mathrm{Var}(X_1) =: \nu.$$

Thus,

$$(43) \quad T_{1,n} = \frac{3\nu}{8\mu^{5/2}} \sum_{\substack{I \subset [d]^2, |I|=1}} \sum_{\substack{\mathbf{i},\mathbf{j}\in[n]^{\underline{d}} \\ \mathcal{E}(\mathbf{i},\mathbf{j})=I}} a_\mathbf{i} a_\mathbf{j} + o_m(n^{-3/2}).$$

Recall that $\Gamma_1, \ldots, \Gamma_n$ are the columns of the matrix with rows $G_1, \ldots, G_d$. Recall also the definition (27) of the coefficients $a_\mathbf{i}$, which implies that

$$(44) \quad Z_n := \sum_{\substack{I \subset [d]^2, |I|=1}} \sum_{\substack{\mathbf{i},\mathbf{j}\in[n]^{\underline{d}} \\ \mathcal{E}(\mathbf{i},\mathbf{j})=I}} a_\mathbf{i} a_\mathbf{j} = \frac{S_n}{V_n},$$

where

$$(45) \quad V_n = \prod_{\ell=1}^d |G_\ell - P_{\ell-1} G_\ell|^4$$

and (recall the notation concerning U-statistics from Section 4.5)

$$(46) \quad S_n = S_n(h) = \sum_{\mathbf{i}\in[n]^{\underline{2d-1}}} h(\Gamma_{i_1}, \ldots, \Gamma_{i_{2d-1}})$$

is a U-statistic of order $2d - 1$ with kernel $h : (\mathbb{R}^d)^{2d-1} \to \mathbb{R}$ defined as

(47)
$$h(x_1, \ldots, x_{2d-1})$$
$$= \frac{1}{(2d-1)!} \sum_{k=1}^{2d-1} \sum_{\substack{J \subset [2d-1]\backslash\{k\} \\ |J|=d-1}} \det(\{x_k\} \cup \{x_\ell\}_{\ell \in J})^2 \det(\{x_k\} \cup \{x_\ell\}_{\ell \in [2d-1]\backslash\{k\}\backslash J})^2.$$

We slightly abuse the notation and treat here the determinant squared as a function of a set rather than a sequence of vectors. Note that h is a symmetric function.

Let us first establish a law of large numbers-type behaviour of the denominator on the right-hand side of (44). Denote $\widetilde{H}_\ell = \mathrm{span}(G_1, \ldots, G_{\ell-1})$, and note that, conditionally on $G_1, \ldots, G_{\ell-1}$, $P_{\ell-1} G_\ell$ is a standard Gaussian vector on $\widetilde{H}_\ell$. In particular, using Theorem 4.9, we obtain that

$$\mathbb{E}|P_{\ell-1}G_\ell|^q = \mathbb{E}\mathbb{E}(|P_{\ell-1}G_\ell|^q|G_1, \ldots, G_{\ell-1}) \le C_q^q d^{q/2}.$$

Moreover, $\widetilde{H}_\ell$ is almost surely of dimension $\ell - 1 \le d - 1$ and $|G_\ell - P_{\ell-1}G|^2 = |G_\ell|^2 - |P_{\ell-1}G_\ell|^2$. Using the fact that by Chebyshev's inequality and Theorem 4.9, $||G_\ell|^2 - n| = \mathcal{O}_m(\sqrt{n})$, together with Hölder's inequality, one can see that

$$\prod_{\ell=1}^{d} |G_\ell - P_{\ell-1}G_\ell|^2 = \prod_{\ell=1}^{d} |G_\ell|^2 + \mathcal{O}_m(n^{d-1}) = n^d + \mathcal{O}_m(n^{d-1/2}).$$

As a consequence, again by Hölder's inequality,

(48) $$V_n = \prod_{\ell=1}^{d} |G_\ell|^4 + \mathcal{O}_m(n^{2d-1}) = n^{2d} + \mathcal{O}_m(n^{2d-1/2}).$$

Observe also that if G is a standard Gaussian vector in $\mathbb{R}^m$ and $m > q + 3$, then integrating in polar coordinates together with the Stirling formula gives

$$\mathbb{E}|G|^{-q} = \mathbb{E}|G|^2 \frac{2^{(m-q-2)/2}\Gamma(\frac{m-q}{2})}{2^{m/2}\Gamma(\frac{m}{2}+1)} \le C_q m^{-q/2}.$$

Conditionally on $G_1, \ldots, G_{\ell-1}$, the random vector $G_\ell - P_{\ell-1}G_\ell$ is a standard Gaussian vector on $\widetilde{H}_\ell^\perp$, which is of dimension at least $n - d$. Thus, conditioning successively, we obtain that, for $n > C_{d,q}$,

(49) $$\left\|\frac{n^{2d}}{V_n}\right\|_q \le C_{d,q}.$$

Let us now pass to the numerator. To shorten the notation, denote $\mathbb{E}h = \mathbb{E}h(\Gamma_1, \ldots, \Gamma_{2d-1})$. By the Hoeffding decomposition (11), applied with $2d - 1$ instead of d, we have

(50) $$S_n = \frac{n!}{(n-2d+1)!}\mathbb{E}h + \sum_{k=1}^{2d-1} \binom{2d-1}{k} \frac{(n-k)!}{(n-2d+1)!} S_n^{(k)}(\pi_k h).$$

By (12) and (13), we have

$$\mathbb{E}|S_n^{(k)}(\pi_k h)|^2 = \frac{n!k!}{(n-k)!}\mathbb{E}(\pi_k h(\Gamma_1, \ldots, \Gamma_k))^2 \le \frac{n!k!}{(n-k)!}\mathbb{E}h(\Gamma_1, \ldots, \Gamma_{2d-1})^2.$$

The variables $S_n^{(k)}(\pi_k h)$ are polynomials in the Gaussian vectors $G_1, \ldots, G_d$ of degree at most $4d$. Thus, the above estimate in combination with Theorem 4.9 shows that

$$\frac{(n-k)!}{(n-2d+1)!}S_n^{(k)}(\pi_k h) = \mathcal{O}_m(n^{2d-1-k/2}).$$

Hölder's inequality and (49) thus give

$$\frac{(n-k)!}{(n-2d+1)!}\frac{S_n^{(k)}(\pi_k h)}{V_n} = \mathcal{O}_m(n^{-1-k/2}).$$

Combined with (44) and (50), this shows that

$$
\begin{aligned}
\text{(51)} \qquad Z_n &= \frac{n!}{(n - 2d + 1)!} \frac{\mathbb{E}h}{V_n} + (2d - 1) \frac{(n - 1)!}{(n - 2d + 1)!} \frac{S_n^{(1)}(\pi_1 h)}{V_n} + \mathcal{O}_m(n^{-2}) \\
&= n^{2d-1} \frac{\mathbb{E}h}{V_n} + (2d - 1) \frac{(n - 1)!}{(n - 2d + 1)!} \frac{S_n^{(1)}(\pi_1 h)}{V_n} + \mathcal{O}_m(n^{-2}),
\end{aligned}
$$

where in the second equality we again used (49).

Let us now derive the announced more precise asymptotics for the denominator. In what follows, we will repeatedly and without a direct reference use Theorem 4.9 to pass from boundedness of absolute moments of some fixed order for a sequence of Gaussian polynomials to the assertion that this sequence is, in fact, $\mathcal{O}_m(1)$.

Using (48), we obtain

$$
\begin{aligned}
V_n &= \prod_{\ell=1}^{d} |G_\ell|^4 + \mathcal{O}_m(n^{2d-1}) \\
&= \prod_{\ell=1}^{d} \left| \sum_{j=1}^{n} g_{\ell,j}^2 \right|^2 + \mathcal{O}_m(n^{2d-1}) \\
&= \sum_{\mathbf{i} \in [n]^{2d}} \prod_{\ell=1}^{d} g_{\ell,i_{2\ell-1}}^2 g_{\ell,i_{2\ell}}^2 + \sum_{\mathbf{i} \in [n]^{2d} \setminus [n]^{\underline{2d}}} \prod_{\ell=1}^{d} g_{\ell,i_{2\ell-1}}^2 g_{\ell,i_{2\ell}}^2 + \mathcal{O}_m(n^{2d-1}).
\end{aligned}
$$

Using boundedness of moments of $g_{\ell,i}$'s together with Hölder's inequality and the fact that $|[n]^{2d} \setminus [n]^{\underline{2d}}| \le C_d n^{2d-1}$, one obtains that the first moment of the second summand on the right-hand side above is $\mathcal{O}(n^{2d-1})$. Thus,

$$
\begin{aligned}
V_n &= \sum_{\mathbf{i} \in [n]^{\underline{2d}}} \prod_{\ell=1}^{d} g_{\ell,i_{2\ell-1}}^2 g_{\ell,i_{2\ell}}^2 + \mathcal{O}_m(n^{2d-1}) = \sum_{\mathbf{i} \in [n]^{\underline{2d}}} \prod_{j=1}^{2d} g_{\lfloor (j+1)/2 \rfloor, i_j}^2 + \mathcal{O}_m(n^{2d-1}) \\
&= \sum_{I \subseteq [2d]} \frac{(n - |I|)!}{(n - 2d)!} \sum_{\mathbf{i}_I \in [n]^{\underline{I}}} \prod_{j \in I} (g_{\lfloor (j+1)/2 \rfloor, i_j}^2 - 1) + \mathcal{O}_m(n^{2d-1}).
\end{aligned}
$$

For $I \ne \varnothing$, using independence of $g_{\ell,i}$'s and the equality $\mathbb{E}(g_{\ell,i}^2 - 1)^2 = 2$, we get

$$
\mathbb{E} \left(\sum_{\mathbf{i}_I \in [n]^{\underline{I}}} \prod_{j \in I} (g_{\lfloor (j+1)/2 \rfloor, i_j}^2 - 1) \right)^2 = 2^{|I|} \frac{n! |I|!}{(n - |I|)!},
$$

thus

$$
\frac{(n - |I|)!}{(n - 2d)!} \sum_{\mathbf{i}_I \in [n]^{\underline{I}}} \prod_{j \in I} (g_{\lfloor (j+1)/2 \rfloor, i_j}^2 - 1) = \mathcal{O}_m(n^{2d - |I|/2}).
$$

For $|I| \ge 2$, this is $\mathcal{O}_m(n^{2d-1})$, and so we obtain that

$$
\begin{aligned}
V_n &= \frac{n!}{(n - 2d)!} + 2 \frac{(n - 1)!}{(n - 2d)!} \sum_{\ell=1}^{d} \sum_{i=1}^{n} (g_{\ell,i}^2 - 1) + \mathcal{O}_m(n^{2d-1}) \\
&= n^{2d} + 2 \frac{(n - 1)!}{(n - 2d)!} \sum_{\ell=1}^{d} \sum_{i=1}^{n} (g_{\ell,i}^2 - 1) + \mathcal{O}_m(n^{2d-1}) \\
&= n^{2d} + 2 \frac{(n - 1)!}{(n - 2d)!} \sum_{i=1}^{n} (|\Gamma_i|^2 - d) + \mathcal{O}_m(n^{2d-1}).
\end{aligned}
$$

Going back to (51), we can write

$$Z_n - \frac{\mathbb{E}h}{n}$$

$$= (\mathbb{E}h)\frac{n^{2d} - V_n}{nV_n} + (2d-1)\frac{(n-1)!}{(n-2d+1)!}\frac{S_n^{(1)}(\pi_1 h)}{V_n} + \mathcal{O}_m(n^{-2})$$

$$= \frac{n^{2d}}{V_n}\frac{(n-1)!}{(n-2d)!n^{2d}}\left(-2(\mathbb{E}h)\frac{1}{n}\sum_{i=1}^n(|\Gamma_i|^2 - d) + (2d-1)\frac{1}{n-2d+1}\sum_{i=1}^n \pi_1 h(\Gamma_i)\right)$$

$$+ \mathcal{O}_m(n^{-2}),$$

where in the second equality we used once more (49) and Hölder's inequality.

Taking into account that V_n/n^{2d} converges in probability to one and that the random vector

$$\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n(|\Gamma_i|^2 - d), \frac{1}{\sqrt{n}}\sum_{i=1}^n \pi_1 h(\Gamma_i)\right)$$

converges weakly to a Gaussian vector with covariance matrix

$$\begin{pmatrix} 2d & \mathrm{Cov}(|\Gamma_i|^2 - d, \pi_1 h(\Gamma_1)) \\ \mathrm{Cov}(|\Gamma_i|^2 - d, \pi_1 h(\Gamma_1)) & \mathbb{E}(\pi_1 h(\Gamma_1))^2 \end{pmatrix},$$

we obtain that

$$n^{3/2}\left(Z_n - \frac{\mathbb{E}h}{n}\right)$$

converges weakly to a mean zero Gaussian variable with variance

$$8(\mathbb{E}h)^2 d + (2d-1)^2\mathbb{E}(\pi_1 h(\Gamma_1))^2 - 4(2d-1)\mathbb{E}h \cdot \mathrm{Cov}(|\Gamma_1|^2 - d, \pi_1 h(\Gamma_1)).$$

Moreover, another application of (49) together with Theorem 4.9 and Hölder's inequality shows that, for each $q > 0$, the qth absolute moment of $n^{3/2}(Z_n - \frac{\mathbb{E}h}{n})$ is bounded independently of n, which shows that the convergence in fact holds in $\mathcal{W}_q$ for any $q \geq 1$ (see the remark before Theorem 3.1).

Using (43) and (44) together with (35) and the fact that $T_{2,n}$ and $T_{3,n}$ are $o_m(n^{-3/2})$, we obtain that

$$n^{3/2}\left(\mathbb{E}\frac{1}{\sqrt{Y_n}} - \frac{1}{\sqrt{\mu}} - \frac{3\nu\mathbb{E}h}{8\mu^{5/2}n}\right) = n^{3/2}\left(T_{1,n} - \frac{3\nu\mathbb{E}h}{8\mu^{5/2}n}\right) + o_m(1)$$

converges in $\mathcal{W}_q$ to a centered Gaussian variable with variance

$$(52) \qquad \begin{aligned} \widetilde{\Sigma}_{p,d}^2 = \frac{9\nu^2}{64\mu^5}(8(\mathbb{E}h)^2 d + (2d-1)^2\mathbb{E}(\pi_1 h(\Gamma_1))^2 \\ -4(2d-1)\mathbb{E}h \cdot \mathrm{Cov}(|\Gamma_1|^2 - d, \pi_1 h(\Gamma_1))). \end{aligned}$$

5.4. *Calculation of the parameters.* In order to obtain an explicit formula for $\Sigma_{p,d}^2$, we need to calculate

$$\mu, \nu, \mathbb{E}h(\Gamma_1, \dots, \Gamma_{2d-1}), \mathbb{E}(\pi_1 h(\Gamma_1))^2, \mathrm{Cov}(|\Gamma_1|^2 - d, \pi_1 h(\Gamma_1)).$$

The calculation of the first two parameters is straightforward, as they are expressed in terms of moments of the random variable X_1, which are known thanks to (7). The remaining parameters are moments of Gaussian polynomials of fixed degree and their calculation involves additional combinatorial arguments.

Recalling that $\mu = \mathbb{E}Y_n = (\mathbb{E}X_1)^d$ where $X_1 = W_1^{-1}$ and W_1 has density proportional to $x^{-1/2}g_{p/2}(x)$ and using Lemma 4.3 with $\alpha = p/2$ and $q = -1$, we obtain

$$(53) \qquad \mu = (\mathbb{E}W_1^{-1})^d = \left(\frac{\Gamma(3/p)\Gamma(1/2)}{\Gamma(3/2)\Gamma(1/p)}\right)^d = \left(2\frac{\Gamma(3/p)}{\Gamma(1/p)}\right)^d.$$

Similarly,

$$(54) \qquad \nu = (\mathbb{E}X_1)^{2d-2}\,\mathrm{Var}(X_1) = (\mathbb{E}W_1^{-1})^{2d-2}(\mathbb{E}W_1^{-2} - (\mathbb{E}W_1^{-1})^2)$$

$$= \left(\frac{\Gamma(3/p)\Gamma(1/2)}{\Gamma(3/2)\Gamma(1/p)}\right)^{2d-2}\left(\frac{\Gamma(5/p)\Gamma(1/2)}{\Gamma(5/2)\Gamma(1/p)} - \left(\frac{\Gamma(3/p)\Gamma(1/2)}{\Gamma(3/2)\Gamma(1/p)}\right)^2\right)$$

$$= \left(2\frac{\Gamma(3/p)}{\Gamma(1/p)}\right)^{2d-2}\left(\frac{4\Gamma(5/p)}{3\Gamma(1/p)} - \left(2\frac{\Gamma(3/p)}{\Gamma(1/p)}\right)^2\right).$$

Let us now pass to the calculation of $\mathbb{E}h$ and $\pi_1 h(\Gamma_1)$. Recall that

$$\pi_1 h(\Gamma_1) = \mathbb{E}_{\Gamma_2,\dots,\Gamma_{2d-1}} h(\Gamma_1,\dots,\Gamma_{2d-1}) - \mathbb{E}h.$$

We will calculate the first summand and then integrate it to get the other one. To simplify the notation, let us denote $\mathbb{E}' := \mathbb{E}_{\Gamma_2,\dots,\Gamma_{2d-1}}$

Recalling the definition of h, given in (47), one can see that

$$\mathbb{E}'h(\Gamma_1,\dots,\Gamma_{2d-1}) = \frac{1}{(2d-1)!}\binom{2d-2}{d-1}((2d-2)D_1 + D_2),$$

where

$$D_1 = \mathbb{E}'\det(\{\Gamma_1,\dots,\Gamma_d\})^2 \det(\{\Gamma_2,\Gamma_{d+1},\dots,\Gamma_{2d-1}\})^2,$$

$$D_2 = \mathbb{E}'\det(\{\Gamma_1,\dots,\Gamma_d\})^2 \det(\{\Gamma_1,\Gamma_{d+1},\dots,\Gamma_{2d-1}\})^2.$$

Let Q_i, $i = 0,\dots,d-1$, be the orthogonal projection onto $\mathrm{span}(\Gamma_1,\dots,\Gamma_i)^\perp \subset \mathbb{R}^d$, Q_1' be the orthogonal projection onto $\mathrm{span}(\Gamma_2)^\perp$ and Q_i', $i = 2,\dots,d$ be the orthogonal projection onto $\mathrm{span}(\Gamma_2,\Gamma_{d+1},\dots,\Gamma_{d+i-1})^\perp$. Then using the interpretation of the determinant as the volume of the paralellopiped we get

$$D_1 = |\Gamma_1|^2\mathbb{E}'|Q_1\Gamma_2|^2\cdots|Q_{d-1}\Gamma_d|^2 \cdot |\Gamma_2|^2|Q_1'\Gamma_{d+1}|^2 \cdot |Q_2'\Gamma_{d+2}|^2\cdots|Q_{d-1}'\Gamma_{2d-1}|^2.$$

Since conditionally on $\Gamma_2,\dots,\Gamma_i$, $Q_i\Gamma_{i+1}$ is a standard Gaussian vector on a certain subspace of dimension $d-i$ and an analogous property holds for $Q_i'\Gamma_{d+i}$, we have

$$D_1 = |\Gamma_1|^2\mathbb{E}'|Q_1\Gamma_2|^2|\Gamma_2|^2(d-2)!(d-1)!$$

$$= (d-2)!(d-1)!|\Gamma_1|^2\mathbb{E}'(|Q_1\Gamma_2|^4 + |Q_1\Gamma_2|^2|\Gamma_2 - Q_1\Gamma_2|^2).$$

Using the fact that, conditionally on Γ_1, the random vectors $Q_1\Gamma_2$ and $\Gamma_2 - Q_1\Gamma_2$ are independent and have standard Gaussian distributions on spaces of dimension $d-1$ and 1 respectively, we obtain

$$D_1 = (d-2)!(d-1)!|\Gamma_1|^2\left(\mathbb{E}\left(\sum_{i=1}^{d-1}g_i^2\right)^2 + \mathbb{E}\left(\sum_{i=1}^{d-1}g_i^2\right)g_d^2\right),$$

where g_i's are i.i.d. $\mathcal{N}(0,1)$ variables. Thus,

$$D_1 = (d-2)!(d-1)!|\Gamma_1|^2(3(d-1) + (d-1)(d-2) + d-1) = (d-1)!^2(d+2)|\Gamma_1|^2.$$

Similarly (the calculations are simpler),

$$D_2 = |\Gamma_1|^4 (d-1)!^2,$$

so we get

$$\mathbb{E}' h(\Gamma_1, \dots, \Gamma_{2d-1})$$

$$= \frac{1}{(2d-1)!} \binom{2d-2}{d-1} ((2d-2)(d-1)!^2(d+2)|\Gamma_1|^2 + (d-1)!^2|\Gamma_1|^4)$$

$$= \frac{1}{2d-1} ((2d-2)(d+2)|\Gamma_1|^2 + |\Gamma_1|^4).$$

Integrating $\mathbb{E}' h(\Gamma_1, \dots, \Gamma_{2d-1})$, we get

(55) $$\mathbb{E} h = \frac{1}{2d-1} ((2d-2)(d+2)d + 3d + d(d-1)) = (d+2)d,$$

and finally,

(56) $$\pi_1 h(\Gamma_1) = \frac{1}{2d-1} ((2d-2)(d+2)(|\Gamma_1|^2 - d) + (|\Gamma_1|^4 - 3d - d(d-1))).$$

To calculate the variance of $\pi_1 h(\Gamma_1)$ and its covariance with $|\Gamma_1|^2 - d$, it will be convenient to express it in terms of Hermite polynomials of the variables $g_{\ell,1}$. To simplify the notation, let us denote from now on $g_\ell = g_{\ell,1}$. We have

$$\pi_1 h(\Gamma_1) = \frac{1}{2d-1} \left(2d(d+2) \sum_{\ell=1}^{d} (g_\ell^2 - 1) + \sum_{\ell=1}^{d} (g_\ell^4 - 6g_\ell^2 + 3) \right.$$

$$\left. + \sum_{1 \le i \ne j \le d} (g_i^2 - 1)(g_j^2 - 1) \right).$$

Taking into account that the summands above are uncorrelated and that the second moment of the kth Hermite polynomial equals $k!$, we get

(57) $$\mathrm{Var}(\pi_1 h(\Gamma_1)) = \frac{8d^3(d+2)^2 + 24d + 4d(d-1)}{(2d-1)^2}$$

and

(58) $$\mathrm{Cov}(\pi_1 h(\Gamma_1), |\Gamma_1|^2 - d) = \frac{4d^2(d+2)}{2d-1}.$$

Combining (52)–(58), we finally obtain

$$\widetilde{\Sigma}_{p,d}^2 = \frac{9v^2}{64\mu^5} (8(\mathbb{E} h)^2 d + (2d-1)^2 \mathbb{E}(\pi_1 h(\Gamma_1))^2$$

$$- 4(2d-1)\mathbb{E} h \cdot \mathrm{Cov}(|\Gamma_1|^2 - d, \pi_1 h(\Gamma_1)))$$

$$= \frac{9(\frac{4\Gamma(5/p)}{3\Gamma(1/p)} - 4(\frac{\Gamma(3/p)}{\Gamma(1/p)})^2)^2}{64(2\frac{\Gamma(3/p)}{\Gamma(1/p)})^{d+4}} \times$$

$$\times \left(8(d+2)^2 d^2 \cdot d + (2d-1)^2 \frac{8d^3(d+2)^2 + 24d + 4d(d-1)}{(2d-1)^2} \right.$$

$$\left. - 4(2d-1)(d+2)d \cdot \frac{4d^2(d+2)}{2d-1} \right)$$

$$= \frac{(\frac{\Gamma(5/p)}{\Gamma(1/p)} - 3(\frac{\Gamma(3/p)}{\Gamma(1/p)})^2)^2}{2^{d+6}(\frac{\Gamma(3/p)}{\Gamma(1/p)})^{d+4}} (8(d+2)^2 d^3 + 8d^3(d+2)^2$$

$$+ 24d + 4d(d-1) - 16(d+2)^2 d^3)$$

$$= \frac{(\frac{\Gamma(5/p)}{\Gamma(1/p)} - 3(\frac{\Gamma(3/p)}{\Gamma(1/p)})^2)^2}{2^{d+4}(\frac{\Gamma(3/p)}{\Gamma(1/p)})^{d+4}} d(d+5).$$

Recalling that

$$n^{3/2}\left(\mathbb{E}\frac{1}{\sqrt{Y_n}} - \frac{1}{\sqrt{\mu}} - \frac{3\nu\mathbb{E}h}{8\mu^{5/2}n}\right)$$

converges to a centered Gaussian variable with variance $\widetilde{\Sigma}_{p,d}^2$ and going back to (32) and (33) allows us to conclude the proof.

6. Proof of Theorem 3.4. The proof of Theorem 3.4 will be based on the volume formula of Theorem 4.4 and the Edgeworth expansion given in Theorem 4.12.

PROOF OF THEOREM 3.4. Let $g_1, g_2, \ldots$ be a sequence of i.i.d. standard Gaussian random variables, and let $u_n = \frac{(g_1,\ldots,g_n)}{\sqrt{g_1^2+\cdots+g_n^2}}$. Then $H_n = \mathrm{span}\{u_n\}^\perp$ is a random Haar distributed subspace of $\mathbb{R}^n$ of codimension one, and it is clearly enough to prove the theorem for this choice of H_n.

Let $Y_1, Y_2, \ldots$ be a sequence of independent random variables with density $e^{-\beta_p^p |x|^p}$ where $\beta_p = 2\Gamma(1+1/p)$. According to Theorem 4.4,

$$(59) \qquad \frac{\mathrm{Vol}_{n-1}(B_p^n \cap H_n)}{\mathrm{Vol}_{n-1}(B_p^{n-1})} = f_{g_1,\ldots,g_n}(0),$$

where for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $f_{\alpha_1,\ldots,\alpha_n} : \mathbb{R} \to [0, \infty)$ is the continuous density of the linear combination $\frac{\alpha_1 Y_1 + \cdots + \alpha_n Y_n}{\sqrt{\alpha_1^2 + \cdots + \alpha_n^2}}$. To shorten the notation, let us suppress the dependence on the sequence (g_i) and write simply f_n instead of $f_{g_1,\ldots,g_n}$ (this is a slight abuse of notation which, however, should not lead to misunderstanding). We may assume that the probability space we consider is of the form $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ and that the variables g_i depend only on the first coordinate while the variables Y_i on the second one. With some abuse of notation, we will thus sometime think of g_i's as random variables defined on Ω_1 and Y_i's as random variables defined on Ω_2. Denote also $X_i = g_i Y_i$. We will treat X_i's as random variables on the space Ω_2, for the moment fixing the sequence g_i. Let us also denote $G_n = (g_1, \ldots, g_n)$. Thus, f_n can be also interpreted as conditional density of $\frac{1}{|G_n|}\sum_{i=1}^n g_i Y_i$ with respect to the σ-field generated by g_i's.

In what follows, we will write $\mathbb{E}_G$ and $\mathbb{E}_Y$ to denote integration with respect to G and Y.

The variables Y_n are symmetric, so their odd moments and cumulants vanish. Moreover, all moments of Y_n are finite and a simple calculation shows that

$$\mathbb{E}Y_n^2 = \frac{\Gamma(3/p)}{4\Gamma(1/p)\Gamma(1+1/p)^2}, \qquad \mathbb{E}Y_n^4 = \frac{\Gamma(5/p)}{16\Gamma(1/p)\Gamma(1+1/p)^4}.$$

Thus, by (16) we have

$$(60) \qquad \begin{aligned} \mathfrak{c}_2(Y_n) &= \frac{\Gamma(3/p)}{4\Gamma(1/p)\Gamma(1+1/p)^2}, \\[2mm] \mathfrak{c}_4(Y_n) &= \frac{1}{16\Gamma(1+1/p)^4}\left(\frac{\Gamma(5/p)}{\Gamma(1/p)} - 3\frac{\Gamma(3/p)^2}{\Gamma(1/p)^2}\right). \end{aligned}$$

We will now apply the Edgeworth expansion given in Theorem 4.12 to f_n. Let us verify that the sequence (X_i) satisfies the assumptions of this theorem $\mathbb{P}_G$-almost surely. We will use the notation introduced in the formulation of the theorem.

We have

$$B_n = (\mathbb{E}Y_1^2) \sum_{i=1}^{n} g_i^2,$$

and thus by the strong law of large numbers $\liminf_{n\to\infty} \frac{1}{n} B_n = \mathbb{E}Y_1^2$, $\mathbb{P}_G$-a.s. Similarly,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_Y |X_i|^K = (\mathbb{E}|Y_1|^K) \frac{1}{n} \sum_{i=1}^{n} |g_i|^K < \infty$$

$\mathbb{P}_G$-a.s. for any $K > 0$. Thus, the condition (i) holds with $\mathbb{P}_G$-a.s.

We have $\mathbb{E}_Y |X_i|^K \mathbf{1}_{\{|X_i|>n^\tau\}} \leq \mathbb{E}_Y |X_i|^{K+1/\tau}/n = \frac{1}{n} |g_i|^{K+1/\tau} \mathbb{E}|Y_1|^{K+1/\tau}$. Thus,

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_Y |X_i|^K \mathbf{1}_{\{|X_i|>n^\tau\}} \leq (\mathbb{E}|Y_1|^{K+1/\tau}) \frac{1}{n^2} \sum_{i=1}^{n} |g_i|^{K+1/\tau}.$$

Again, by the strong law of large numbers, for every $\tau > 0$, the right-hand side converges $\mathbb{P}_G$-a.s. to zero, which shows validity of the condition (ii).

To verify the condition (iii), we will use Lemma 4.13. Let h be the density of Y_n, note that h' is integrable and so h has finite variation, say V. Moreover, the density of $g_i Y_i$ is of the form $|g_i|^{-1} h(g_i^{-1} \cdot)$, and thus its variation equals $|g_i|^{-1} V \leq V$ for $i \in \mathcal{I} = \{j : |g_j| > 1\}$. Using one more time the strong law of large numbers, we see that $\mathbb{P}_G$-a.s.

$$\lim_{n\to\infty} \frac{|\mathcal{I} \cap [n]|}{n} = \mathbb{P}(|g_1| > 1) > 0.$$

Therefore, the condition (iii) of Theorem 4.12 is satisfied $\mathbb{P}_G$-a.s. by Lemma 4.13.

We have thus proved that the assumptions of Theorem 4.12 hold $\mathbb{P}_G$-a.s. for all $K \geq 3$. We will, however, use it for $K = 5$ and only for $x = 0$. Thanks to the symmetry of X_n (recall that it implies that odd cumulants vanish), the expansion (20) will be actually simplified. Recall also that $\mathfrak{c}_m(X_i) = g_i^m \mathfrak{c}_m(Y_i)$. Using the notation of Section 4.8, we obtain by (17) that

$$\lambda_{3,n} = \lambda_{5,n} = 0$$

and

$$\lambda_{4,n} = n \frac{\mathfrak{c}_4(Y_1)}{(\mathbb{E}Y_1^2)^2} \frac{\sum_{i=1}^{n} g_i^4}{(\sum_{i=1}^{n} g_i^2)^2}.$$

Combining this with (18) and (14), we get

$$q_{0,n}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

$$q_{1,n}(x) = q_{3,n}(x) = 0,$$

$$q_{2,n}(x) = \frac{1}{24\sqrt{2\pi}} e^{-x^2/2} (x^4 - 6x^2 + 3) n \frac{\mathfrak{c}_4(Y_1)}{(\mathbb{E}Y_1^2)^2} \frac{\sum_{i=1}^{n} g_i^4}{(\sum_{i=1}^{n} g_i^2)^2}.$$

Thus, $\mathbb{P}_G$-a.s. we have, as $n \to \infty$,

$$(61) \qquad f_n(0) = \frac{1}{\sqrt{2\pi} (\mathbb{E}Y_1^2)^{1/2}} \left(1 + \frac{1}{8} \frac{\mathfrak{c}_4(Y_1)}{(\mathbb{E}Y_1^2)^2} \frac{\sum_{i=1}^{n} g_i^4}{(\sum_{i=1}^{n} g_i^2)^2} \right) + o(n^{-3/2}).$$

Note that the asymptotic behaviour of the random variable $\frac{\sum_{i=1}^{n} g_i^4}{(\sum_{i=1}^{n} g_i^2)^2}$ has already been analysed as a special case $d = 1$ in the proof of Theorem 3.1. However, since the elementary and easy analysis is hidden there in the rather involved formalism of general U-statistics, let us repeat it here for completeness.

We have

$$n^{3/2}\left(\frac{\sum_{i=1}^{n} g_i^4}{(\sum_{i=1}^{n} g_i^2)^2} - \frac{3}{n}\right)$$

$$= \left(\frac{(n-3)\sum_{i=1}^{n}(g_i^4 - 6g_i^2 + 3)}{n^{3/2}}\right.$$

$$\left.- \frac{12\sum_{i=1}^{n}(g_i^2 - 1) + 3\sum_{1 \le i \ne j \le n}(g_i^2 - 1)(g_j^2 - 1) + 6n}{n^{3/2}}\right)$$

$$\times \frac{n^2}{(\sum_{i=1}^{n} g_i^2)^2}.$$

The last factor on the right-hand side above converges a.s. to 1 by the law of large numbers. The second quotient in parentheses converges in probability to zero, as can be easily seen by calculating the variances (note that the summands in the numerator are multiples of Hermite polynomials of different degrees or products of Hermite polynomials in different variables and are thus uncorrelated). The first quotient converges weakly by the CLT to a mean zero Gaussian variable with variance 24.

Using (61), (59) and (60), we thus obtain that

$$\frac{\mathrm{Vol}_{n-1}(B_p^n \cap H_n)}{\mathrm{Vol}_{n-1}(B_p^{n-1})} - a_{p,1} - \frac{1}{n}b_{p,1}$$

$$= \frac{\mathrm{Vol}_{n-1}(B_p^n \cap H_n)}{\mathrm{Vol}_{n-1}(B_p^{n-1})}$$

$$- \frac{\sqrt{2}\Gamma(1 + 1/p)\Gamma(1/p)^{1/2}}{\sqrt{\pi}\Gamma(3/p)^{1/2}}\left(1 + \frac{3}{8n}\left(\frac{\Gamma(1/p)}{\Gamma(3/p)}\right)^2\left(\frac{\Gamma(5/p)}{\Gamma(1/p)} - 3\frac{\Gamma(3/p)^2}{\Gamma(1/p)^2}\right)\right)$$

$$= \left(f_n(0) - \frac{1}{\sqrt{2\pi}(\mathbb{E}Y_1^2)^{1/2}}\left(1 + \frac{3}{8n}\frac{\mathfrak{c}_4(Y_1)}{(\mathbb{E}Y_1^2)^2}\right)\right)$$

$$= \frac{\mathfrak{c}_4(Y_1)}{8\sqrt{2\pi}(\mathbb{E}Y_1^2)^{5/2}}\left(\frac{\sum_{i=1}^{n} g_i^4}{(\sum_{i=1}^{n} g_i^2)^2} - \frac{3}{n}\right) + o_{\mathbb{P}}(n^{-3/2}),$$

and thus

$$n^{3/2}\left(\frac{\mathrm{Vol}_{n-1}(B_p^n \cap H_n)}{\mathrm{Vol}_{n-1}(B_p^{n-1})} - a_{p,1} - \frac{1}{n}b_{p,1}\right)$$

converges in distribution to a mean zero Gaussian variable with variance

$$\frac{\mathfrak{c}_4(Y_1)^2}{128\pi(\mathbb{E}Y_1^2)^5} \cdot 24$$

$$= \frac{24}{128\pi}\left(\frac{4\Gamma(1/p)\Gamma(1 + 1/p)^2}{\Gamma(3/p)}\right)^5 \frac{1}{256\Gamma(1 + 1/p)^8}\left(\frac{\Gamma(5/p)}{\Gamma(1/p)} - 3\frac{\Gamma(3/p)^2}{\Gamma(1/p)^2}\right)^2$$

$$= \frac{3}{4\pi}\left(\frac{\Gamma(1/p)}{\Gamma(3/p)}\right)^5 \Gamma(1 + 1/p)^2\left(\frac{\Gamma(5/p)}{\Gamma(1/p)} - 3\frac{\Gamma(3/p)^2}{\Gamma(1/p)^2}\right)^2 = \Sigma_{p,1}^2,$$

which ends the proof of Theorem 3.4. □

REMARK 6.1. Note that the term $o(n^{-3/2})$ in (61), in general, depends on the values of the sequence (g_i) and is not given explicitly. For this reason using Theorem 4.12 as a black box will not lead to convergence in Wasserstein distance, contrary to the proof of Theorem 3.1.

REMARK 6.2. In principle, the method of proof of Theorem 3.4 should work for general d. What one would need is a suitable version of multidimensional Edgeworth expansion for the density of sums of independent but nonidentically distributed random vectors (actually only for the value of density at zero). The majority of the literature on Edgeworth expansions in higher dimensions focuses on sums of i.i.d. variables; however, there are several results concerning the non-i.i.d. setting (see, e.g., the monograph [12]). One of the main difficulties in applying such theorems as black boxes for $d > 1$ is that, due to the Gram–Schmidt orthogonalization performed for each n in order to relate a random basis of $H_n^{\perp}$ to Gaussian vectors, one actually would need Edgeworth expansions not for infinite sequences of random variables but for triangular arrays. We are not aware of a result of this type for densities, which would be easily applicable in our setting. It is quite likely that such a result can be obtained by an appropriate adaptation of the proofs of known theorems for sequences of random vectors. Such an extension is, however, beyond the scope of this article.

REMARK 6.3. Let us note that Edgeworth expansions for randomly weighted sums of independent real-valued random variables have been recently investigated in [13]. The results obtained therein concern rather approximations of cumulative distribution functions than densities and the average error in the Edgeworth expansion up to order four. While not directly applicable to our setting, they share some similarities; in particular, they show that the average approximation error for the Edgeworth expansion with deterministic terms (i.e., independent of the direction) in a typical situation is of the order $n^{-3/2}$, which agrees with the normalization in our limit theorems. The results obtained in [13] complement an earlier work [42] in which Berry–Esseen bounds for randomly weighted sums were investigated. Related Berry–Esseen bounds for random vectors in higher dimension were also recently investigated in [14]. Let us mention that actually this line of research was initiated already by V.N. Sudakov in the late 1970s [82].

7. Proof of Theorem 3.5.
The proof is similar and simpler than the proof of Theorem 3.4, so we will just indicate the necessary modifications.

Considering again an i.i.d. sequence $g_1, g_2 \ldots$ of standard Gaussian variables and letting $u_n = \frac{(g_1,\ldots,g_n)}{\sqrt{g_1^2+\cdots+g_n^2}}$, it is now straightforward to see that

$$2^{-n} \operatorname{Vol}_{n-1}\big(B_\infty^n \cap (xu + H_n)\big) = f_n(x),$$

where f_n is the conditional density of $\frac{\sum_{i=1}^n g_i Y_i}{\sqrt{g_1^2+\cdots+g_n^2}}$, where $Y_1, Y_2, \ldots$ is a sequence of i.i.d. random variables, uniform on $[-1, 1]$ and independent of the sequence (g_i) (we condition on (g_i)).

Thus, repeating the steps related to (61), we obtain that

$$2^{-n} \operatorname{Vol}_{n-1}\big(B_\infty^n \cap (xu + H_n)\big)$$

$$= \frac{1}{\sqrt{2\pi}(\mathbb{E}Y_1^2)^{1/2}} \exp\left(-\frac{x^2}{2\mathbb{E}Y_1^2}\right)\left(1 + \frac{1}{24}\left(\frac{x^4}{(\mathbb{E}Y_1^2)^2} - 6\frac{x^2}{\mathbb{E}Y_1^2} + 3\right)\frac{c_4(Y_1)}{(\mathbb{E}Y_1^2)^2}\frac{\sum_{i=1}^n g_i^4}{(\sum_{i=1}^n g_i^2)^2}\right)$$

$$+ o_{\mathbb{P}}(n^{-3/2}).$$

By the analysis from the proof of Theorem 3.4, we thus see that

$$2^{-n} \operatorname{Vol}_{n-1}(B_\infty^n \cap (xu + H_n))$$

$$= \frac{1}{\sqrt{2\pi}(\mathbb{E}Y_1^2)^{1/2}} \exp\left(-\frac{x^2}{2\mathbb{E}Y_1^2}\right)\left(1 + \frac{1}{24}\left(\frac{x^4}{(\mathbb{E}Y_1^2)^2} - 6\frac{x^2}{\mathbb{E}Y_1^2} + 3\right)\frac{\mathfrak{c}_4(Y_1)}{(\mathbb{E}Y_1^2)^2}\left(\frac{3}{n} + U_n\right)\right)$$

for a sequence U_n of random variables such that $n^{3/2}U_n$ converges weakly to a Gaussian variable with mean zero and variance 24. To finish the proof, it is now enough to rearrange the terms and substitute the values of variance and fourth cumulant of the uniform distribution.

8. Proof of Corollary 3.6. The following argument is a simple application of a delta type method in combination with Theorem 3.4.

PROOF OF COROLLARY 3.6.   By Theorem 3.4 we have

$$\frac{\rho_{\mathcal{I}B_p^n}(\eta)}{\operatorname{Vol}_{n-1}(B_p^{n-1})} = a_{p,1} + R_n,$$

where R_n is a random variable such that $n^{3/2}(R_n - b_{p,1}/n)$ converges in distribution to a centered Gaussian variable with variance $\Sigma_{p,1}^2$. In particular, nR_n converges in probability to $b_{p,1}$.

Thus,

$$n^{3/2}\left(\operatorname{Vol}_{n-1}(B_p^{n-1})\|\eta\|_{\mathcal{I}B_p^n} - \frac{1}{a_{p,1}} + \frac{b_{p,1}}{na_{p,1}^2}\right) = n^{3/2}\left(\frac{1}{a_{p,1} + R_n} - \frac{1}{a_{p,1}} + \frac{b_{p,1}}{na_{p,1}^2}\right)$$

$$= \frac{n^{3/2}R_n^2}{a_{p,1}^2(a_{p,1} + R_n)} - \frac{n^{3/2}}{a_{p,1}^2}\left(R_n - \frac{b_{p,1}}{n}\right).$$

To finish the proof, it suffices to note that the first summand on the right-hand side above converges in probability to zero, the second one to $\mathcal{N}(0, \Sigma_{p,1}^2/a_{p,1}^4)$.   □

APPENDIX: PROOF OF THEOREM 4.1

The argument we present below mimics the proof in [68] provided there for $p = 1$; therefore, we will only present a sketch. Our main objective is to derive correct constants on the right-hand side of (8).

PROOF OF THEOREM 4.1.   Let $H(\varepsilon) = \{x \in \mathbb{R}^n : |\langle x, u_j \rangle| \le \varepsilon/2, j = 1, \ldots, d\}$. By a well-known formula for volumes of sections (see [68] for a discussion and references),

(62)
$$\Gamma(1 + (n-d)/p)\operatorname{Vol}_{n-d}(H \cap B_p^n)$$

$$= \lim_{\varepsilon \to 0}\frac{1}{\varepsilon^d}\int_{H(\varepsilon)} e^{-\sum_{j=1}^n |x_i|^p}\, dx$$

$$= (2\Gamma(1 + 1/p))^n \lim_{\varepsilon \to 0}\frac{1}{\varepsilon^d}\mathbb{P}\left(\left\|\sum_{i=1}^n X_i v_i\right\|_\infty \le \varepsilon/2\right),$$

where $X_1, \ldots, X_n$ are i.i.d. random variables with density $\frac{1}{2\Gamma(1+1/p)}\exp(-|x|^p)$. By [24], Lemma 23, X_i's have the same distribution as $(2W_i)^{-1/2}g_i$, where $G = (g_1, \ldots, g_n)$ is a sequence of independent standard Gaussian variables independent of the sequence (W_i). Thus,

$$\mathbb{P}\left(\left\|\sum_{i=1}^n X_i v_i\right\|_\infty \le \varepsilon/2\right) = \mathbb{P}\left(\left\|\sum_{i=1}^n g_i \tilde{v}_i\right\|_\infty \le \varepsilon/2\right) = \mathbb{P}(G \in (\varepsilon/2)K),$$

where $\tilde{v}_i = (2W_i)^{-1/2} v_i$ and $K = A^{-1} B_\infty^d$ with A being the $d \times n$ matrix with columns $\tilde{v}_i$. Let $V := (\operatorname{Ker} A)^\perp$ and $\tilde{G}$ be the orthogonal projection of G onto V. Then $G \in (\varepsilon/2)K$ if and only if $\tilde{G} \in (\varepsilon/2)(K \cap V)$. Conditionally on W_i's, $\tilde{G}$ is a standard Gaussian vector on the d-dimensional subspace V. It has a continuous and bounded density with respect to the Lebesgue measure on V, whose value at zero equals $(2\pi)^{-d/2}$. As a consequence, by the Fubini and Lebesgue dominated convergence theorems

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^d} \mathbb{P}\left(\left| \sum_{i=1}^n g_i \tilde{v}_i \right|_\infty \le \varepsilon/2 \right) = \mathbb{E}_W \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^d} \mathbb{P}_G\big(\tilde{G} \in (\varepsilon/2)(K \cap V) \big)$$

$$= 2^{-d} (2\pi)^{-d/2} \mathbb{E}_W \operatorname{Vol}_d(K \cap V)$$

$$= 2^{-3d/2} \pi^{-d/2} \mathbb{E}_W \operatorname{Vol}_d(K \cap V).$$

It remains to observe that A is a linear isomorphism between V and $\mathbb{R}^d$, and it maps $K \cap V$ onto B_∞^d, which is of volume 2^d. Thus,

$$\operatorname{Vol}_d(K \cap V) = \big(\det(AA^T) \big)^{-1/2} 2^d = 2^{3d/2} \left(\det\left(\sum_{j=1}^n \frac{1}{W_j} v_j v_j^T \right) \right)^{-1/2},$$

which, combined with the previous formula and (62), gives

$$\operatorname{Vol}_{n-d}\big(B_p^n \cap H \big) = \frac{2^n}{\pi^{d/2}} \frac{\Gamma(1+1/p)^n}{\Gamma(1+(n-d)/p)} \mathbb{E}\left(\det\left(\sum_{j=1}^n \frac{1}{W_j} v_j v_j^T \right) \right)^{-1/2}. \qquad \square$$

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