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The Massless Electron Limit of the Vlasov–Poisson–Landau System

Patrick Flynn[1,2], **Yan Guo**[1]

[1] Division of Applied Mathematics, Brown University, Providence, USA
[2] Department of Mathematics, University of California, Los Angeles, Los Angeles, USA.
E-mail: pflynn@math.ucla.edu

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Abstract: Due to ion–electron collisions, it is impossible to derive any two-fluid model for plasma as a direct hydrodynamic limit of the Vlasov–Poisson–Landau system for ions and electrons. At the same time, electrons are much lighter than their ion counterparts. In this work, we derive the massless electron limit of the Vlasov–Poisson–Landau system. This is done via a re-scaling of the electron velocity, leading to multiple velocity scales. Importantly, we demonstrate that ion–electron collisions vanish in this limit, due to special structure of the Landau collisions. We also show that this is invalid for the classical Boltzmann kernel with hard sphere interaction. This mechanism serves as the first step for the derivation of the two-fluid model for ions from a two-species kinetic equation.

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1. Introduction

A plasma is a collection of fast-moving, charged particles. It is believed that more than 95 percent of matter in the universe takes the form of a plasma. Besides this, a major impetus for the study of plasmas is to obtain nuclear fusion as a means of energy production. Due to its complex and rich nature, there are three main distinct families of models (kinetic, two-fluid and magnetohydrodynamic) for describing a plasma in different physical regimes. The two-fluid models (Euler–Poisson and Euler–Maxwell systems) describe dynamics of two *distinct* and *separate* compressible ion and electron fluids, interacting with their self-consistent electromagnetic field. Such two-fluid models are important both from physical as well as mathematical standpoint, serving as an origin for most well-known dispersive PDEs, for instance KdV [4,36], NLS [34], Zakharov [37], etc.

It is an important question to derive and justify two-fluid theory from more basic kinetic models for plasma. Consider a classical kinetic model for ions and electrons, the Vlasov–Poisson–Landau system, which models the distribution functions $F_+(t,x,v)$ and $F_-(t,x,v)$ for ions (+) and electrons (−) respectively:

$$\begin{aligned}\{\partial_t + v\cdot\nabla_x + \frac{Z\mathsf{e}}{m_+}E\cdot\nabla_v\}F_+ &= 2\pi\mathsf{e}^4\ln(\Lambda)\{Z^4Q_{++}(F_+,F_+) + Z^2Q_{-+}(F_-,F_+)\},\\ \{\partial_t + v\cdot\nabla_x - \frac{\mathsf{e}}{m_-}E\cdot\nabla_v\}F_- &= 2\pi\mathsf{e}^4\ln(\Lambda)\{Q_{--}(F_-,F_-) + Z^2Q_{+-}(F_+,F_-)\},\\ -\Delta_x\phi &= 4\pi\mathsf{e}(Zn_+ - n_-).\end{aligned} \tag{1.1}$$

In the above, e is the electron charge,[1] and Z is the atomic number of the ion species. The ion and electron masses are m_+ and m_- respectively. We also have the densities $n_\pm = \int_{\mathbf{R}^3} F_\pm dv$, the electric potential ϕ, the electric field $E := -\nabla_x\phi$, and the Landau collision operators Q_{++}, Q_{-+}, Q_{--} and Q_{+-} (tabulated in Sect. 2.1 below). We refer the reader to [30], and Chapter 2 of [2] for the physical justification of these models.

It is well known that any straightforward fluid limit must lead to

$$Z^4Q_{++}(F_+,F_+) + Z^2Q_{-+}(F_-,F_+) = 0, \tag{1.2}$$

$$Q_{+-}(F_+,F_+) + Z^2Q_{--}(F_-,F_+) = 0. \tag{1.3}$$

However, due to the presence of ion and electron interactions Q_{+-} and Q_{-+}, the only possible solutions to the above are of the form[2]

$$F_+ = n_+\left(\frac{m_+}{2\pi T}\right)^{3/2}\exp\left(-\frac{m_+|v-u|^2}{2T}\right), \tag{1.4}$$

[1] Not to be confused with the mathematical constant $e \approx 2.718$.

[2] See equation (40) in [30].

$$F_- = n_-(\frac{m_-}{2\pi T})^{3/2} \exp\left(-\frac{m_-|v-u|^2}{2T}\right). \tag{1.5}$$

Here, the electron and ion fluids exhibit the *same* velocity $u(t,x)$ and temperature $T(t,x)$, which *excludes* the possibility of justification of any two-fluid models from (1.1). It is well-known, however, that it is possible to derive two-fluid models from two-species kinetic equations *if ion–electron collisions* Q_{+-} *and* Q_{-+} *are disregarded*. This was done in the context of the Vlasov–Poisson–Boltzmann system [23–25], and more recently for single-species Landau-type equations [12,33]. We highlight the work [3], which derives the fluid limit for the Vlasov–Poisson–Boltzmann system, after first deriving the massless electron limit. This work in parituclar served as a major inspiration for our work. Importantly, however, none of these works consider the inter-species collisions.[3]

In all physical situations, the electron mass m_- is much smaller than the ion mass m_+. For instance, $\frac{m_-}{m_+} \approx 0.005$ for a hydrogen plasma. This small parameter has been exploited in different contexts in plasma studies both from physical as well as mathematical standpoints. For instance, one can derive the Euler–Poisson system for ions in this limit [16]. By sending the electron mass to zero, the density of the electrons (and, in turn, the electrostatic potential) becomes wholly determined by the density of the ions. This eliminates the need to solve a fluid or kinetic equation for the electrons separately from the ions, leading to a simpler model.

When the ions and electrons are near thermal equilibrium (say, at temperature 1), a typical ion speed will be comparable to $(\frac{1}{m_+})^{\frac{1}{2}}$, while a typical electron speed will be comparable to $(\frac{1}{m_-})^{\frac{1}{2}}$. Because of these divergent velocity scales, it is necessary to introduce the parameter $\varepsilon := (\frac{m_-}{m_+})^{\frac{1}{2}}$, and the rescaled electron velocity

$$\xi = \varepsilon v. \tag{1.6}$$

Applying this rescaling the system (1.1), we get the rescaled system (1.7) below. The ion–electron collisions then exhibit *two velocity scales*. Our main result is that by taking the limit $\varepsilon \to 0$ in the two-scale system, we are able derive the massless electron system (1.18) below. A salient feature of this derived equation is that the *ion–electron collisions are no longer present*. In the future, we hope this work will clear the way for the derivation of the Euler–Poisson system for ions. This would be accomplished by first by taking $\varepsilon \to 0$, and then taking hydrodynamic limit $\kappa \to 0$, where $\kappa > 0$ is the analogue of the Knudsen number for our system.

Besides the relevance to the fluid limit, the massless electron limit in kinetic theory is of independent interest. There are a number of works that have handled some version of this limit. For instance, Degond and Lucquin-Desreux [10,11] give formal expansions for the kinetic equations with Boltzmann and Landau collisions for systems of particles with disparate masses, including the cross collision terms. The works [6,29], give derivations the massless electron limit for Vlasov–Poisson system with linear Fokker–Planck collisions (and in the latter case, with an magnetic field). The aforementioned work [3] derives the analogue of (1.18) for the Vlasov–Poisson–Boltzmann system. We note that they require a regularity assumption, and do not include ion–electron collisions.

[3] In the case of diffusive limits, the work [31] derives the single fluid Vlasov–Navier–Stokes–Fourier system from the two-species Vlasov–Maxwell–Boltzmann system.

There is also a growing literature on the Vlasov–Poisson system with massless electrons. We refer the reader to [5,7,15,17–20,26–28] and the references therein. In particular, the overview [18] has a formal derivation of the system with a Landau collision term. We also mention the recent work [8] for the Vlasov–Poisson–Landau system with weak collisions.

1.1. The rescaled Vlasov–Poisson–Landau system. In Sect. 2.1, we apply a rescaling procedure to the system (1.1). In particular, we use the rescaled electron velocity $\xi = \varepsilon v$, as aforementioned. This system depends on three positive parameters: $\kappa > 0$ (defined in Sect. 2.1), the atomic number Z, and $\varepsilon = (\frac{m_-}{m_+})^{\frac{1}{2}}$. The rescaled versions of the distribution functions $F_+^\varepsilon(t, x, v)$ and $F_-^\varepsilon(t, x, \xi)$ then solve the system

$$\begin{aligned} \{\partial_t + v \cdot \nabla_x + Z E^\varepsilon \cdot \nabla_v\} F_+^\varepsilon &= \kappa^{-1}\{Z^3 Q(F_+^\varepsilon, F_+^\varepsilon) + Z^2 Q_{-+}^\varepsilon(F_-^\varepsilon, F_+^\varepsilon)\}, \\ \{\varepsilon\partial_t + \xi \cdot \nabla_x - E^\varepsilon \cdot \nabla_\xi\} F_-^\varepsilon &= \kappa^{-1}\{Q(F_-^\varepsilon, F_-^\varepsilon) + Z Q_{+-}^\varepsilon(F_+^\varepsilon, F_-^\varepsilon)\}, \\ -\Delta_x \phi^\varepsilon &= 4\pi(n_+^\varepsilon - n_-^\varepsilon) \end{aligned} \tag{1.7}$$

Above, ϕ^ε and $E^\varepsilon = -\nabla_x \phi^\varepsilon$ are the electric potential and field respectively. The functions $n_\pm(t, x)$ are the charge densities, defined by

$$n_+^\varepsilon = \int_{\mathbf{R}^3} F_+^\varepsilon dv, \quad n_-^\varepsilon = \int_{\mathbf{R}^3} F_-^\varepsilon d\xi. \tag{1.8}$$

The collision operators are given by

$$Q(G_1, G_2) = \nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v - v')\left\{G_1(v')\nabla_v G_2(v) - G_2(v)\nabla_{v'} G_1(v')\right\} dv', \tag{1.9}$$

$$Q_{-+}^\varepsilon(G_1, G_2) = \nabla_v \cdot \int_{\mathbf{R}^3} \Phi(\varepsilon v - \xi')\left\{\varepsilon G_1(\xi')\nabla_v G_2(v) - G_2(v)\nabla_{\xi'} G_1(\xi')\right\} d\xi', \tag{1.10}$$

$$Q_{+-}^\varepsilon(G_1, G_2) = \nabla_\xi \cdot \int_{\mathbf{R}^3} \Phi(\xi - \varepsilon v')\left\{G_1(v')\nabla_\xi G_2(\xi) - \varepsilon G_2(\xi)\nabla_{v'} G_1(v')\right\} dv'. \tag{1.11}$$

Here, Φ gives the Landau collision kernel: for each $z \in \mathbf{R}^3$,

$$\Phi(z) := \frac{1}{|z|}\left(I - \frac{z \otimes z}{|z|^2}\right). \tag{1.12}$$

From here on this paper, we only consider the case where $Z = \kappa = 1$, and fix the domain $(x, v) \in \mathbf{T}^3 \times \mathbf{R}^3$, where $\mathbf{T}^3 = (\mathbf{R}/\mathbf{Z})^3$ denotes the 3D torus.

We recall that the system (1.7) comes equipped with conservation of mass, momentum and energy:

$$\frac{d}{dt}\int_{\mathbf{T}^3\times\mathbf{R}^3} F_+^\varepsilon dx dv = \frac{d}{dt}\int_{\mathbf{T}^3\times\mathbf{R}^3} F_-^\varepsilon dx d\xi = 0, \tag{1.13}$$

$$\frac{d}{dt}\left(\int_{\mathbf{T}^3\times\mathbf{R}^3} v F_+^\varepsilon dx dv + \varepsilon \int_{\mathbf{T}^3\times\mathbf{R}^3} \xi F_-^\varepsilon dx d\xi\right) = 0, \tag{1.14}$$

$$\frac{d}{dt}\left(\int_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}F_+^\varepsilon dxdv+\int_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|\xi|^2}{2}F_-^\varepsilon dxd\xi+\frac{1}{8\pi}\int_{\mathbf{T}^3}|E^\varepsilon|^2dx\right)=0. \quad (1.15)$$

From (1.7), it is clear that as $\varepsilon \to 0$, the the term involving ∂_t in the electron equation vanishes. This is a singular limit. Under appropriate circumstances, the electron distribution formally converges to an equilibrium solution at every time, evolving quasi-statically according to the ions. The ξ-dependence of these equilibrium solutions are Maxwellians (i.e. Gaussians). We denote the Maxwellians by

$$\mu_q(\xi) := \left(\frac{q}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{q}{2}|\xi|^2}, \quad q > 0. \quad (1.16)$$

On the other hand, the x dependence is determined by the electric field, specifically $\lim_{\varepsilon\to 0}\ln(n_-^\varepsilon)$ is proportional to $\phi^0 = \lim_{\varepsilon\to 0}\phi^\varepsilon$ (up to an additive constant). The main claim of this paper is that for some $\beta(t) > 0$, we have

$$\lim_{\varepsilon\to 0} F_-^\varepsilon(t,x,\xi) = \mu_{\beta(t)}(\xi)e^{\beta(t)\phi^0(t,x)} \quad (1.17)$$

Here, β^0 is the inverse temperature. This is known as the Maxwell–Boltzmann approximation. Here, we shift ϕ^0 by an additive constant to ensure $\int_{\mathbf{T}^3} e^{\beta(t)\phi^0(t,x)}dx = 1$.

The above claim is imprecise, because we have not yet made clear our assumptions, nor the sense in which this limit holds. Nevertheless, when the above holds, then the formal limit of (1.7), combined with conservation of energy, leads to the following equation, which we refer to as Vlasov–Poisson–Landau for ions:

$$\begin{aligned}
&\{\partial_t + v\cdot\nabla_x - E^0\cdot\nabla_v\}F_+^0 = Q(F_+^0, F_+^0)\\
&\frac{d}{dt}\left\{\frac{3}{2\beta}+\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}F_+^0dxdv+\frac{1}{8\pi}\int_{\mathbf{T}^3}|E^0|^2dx\right\}=0,\\
&-\Delta_x\phi^0 = 4\pi(n_+^0 - e^{\beta\phi^0}).
\end{aligned} \quad (1.18)$$

A formal derivation of this system is given in Sect. 2.2.

1.2. Main result. The goal of this paper is to show that for appropriate initial data, this formal limit holds on a small time interval. We first provide local well-posedness theorems for the systems (1.7) and (1.18) for general initial data. To do this, we must first define the function spaces which capture the dissipation rate from the collision kernels. We define the matrix $\sigma_{ij}(v) = \Phi_{ij} * \mu$, with $i, j \in \{1,2,3\}$. Given $u \in \mathbf{R}^3$, we have that

$$u^T\sigma(v)u \sim \frac{1}{\langle v\rangle^3}|P_v u|^2 + \frac{1}{\langle v\rangle}|P_{v^\perp}u|^2 \quad (1.19)$$

Here, $\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$ is the Japanese bracket. The matrices P_v and $P_{v^\perp}$ denote the orthogonal projections onto span$\{v\}$ and $\{v\}^\perp$ respectively. Using this matrix, we define the following spaces which will capture the dissipation produced by the various collision operators. Let $\mathcal{H}_\sigma$ denote the Hilbert space with inner product (with $\psi = \psi(v)$, $\zeta = \zeta(v)$)

$$\|\psi\|_{\mathcal{H}_\sigma}^2 = \langle\sigma_{ij}\partial_{v_i}\psi, \partial_{v_j}\psi\rangle_{L^2} + \langle \mathrm{tr}(\sigma)\psi,\psi\rangle_{L^2} \quad (1.20)$$

Here and throughout the paper, we use repeated index notation, i.e. $u_k u'_k = \sum_{k=1}^3 u_k u'_k$ given any two vectors $u, u' \in \mathbf{R}^3$. The above inner product captures the dissipation associated with self-collisions when linearized near Maxwellian [9,21,22]. We also define the semi-norms

$$\|\psi\|^2_{\dot{\mathcal{H}}_\sigma} = \langle \sigma_{ij}(v)\partial_{v_i}\psi, \partial_{v_j}\psi\rangle_{L^2_v} \tag{1.21}$$

$$\|\psi\|^2_{\mathcal{H}^+_{\sigma;\varepsilon}} = \varepsilon\langle \sigma_{ij}(\varepsilon v)\partial_{v_i}\psi, \partial_{v_j}\psi\rangle_{L^2_v} \tag{1.22}$$

$$\|\psi\|^2_{\mathcal{H}^-_{\sigma;\varepsilon}} = \frac{1}{\varepsilon}\langle \sigma_{ij}(\frac{v}{\varepsilon})\partial_{\xi_i}\psi, \partial_{\xi_j}\psi\rangle_{L^2_\xi}. \tag{1.23}$$

Using these norms, we now define the spaces $\mathfrak{E}$, $\mathfrak{D}$ and $\mathfrak{D}^\pm_\varepsilon$ by their (semi-)norms which we use to control $F_\pm$. Fix the parameters $m_k = 5(k+1)$ for $k \in \{0, 1, 2\}$, and $s = 3$.[4] Given $u = u(x, v)$ (or $u = u(x, \xi)$), we define

$$\|u\|^2_{\mathfrak{E}} := \|\langle v\rangle^{m_2} u\|^2_{L^2_{x,v}} + \|\langle v\rangle^{m_1}\langle\nabla_x\rangle^s u\|^2_{L^2_{x,v}}, \tag{1.24}$$

$$\|u\|^2_{\mathfrak{D}} := \|\langle v\rangle^{m_2} u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|\langle v\rangle^{m_1}\langle\nabla_x\rangle^s u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}, \tag{1.25}$$

$$\|u\|^2_{\mathfrak{D}^\pm_\varepsilon} := \|\langle v\rangle^{m_2} u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}^\pm_{\sigma;\varepsilon})_v} + \|\langle v\rangle^{m_1}\langle\nabla_x\rangle^s u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}^\pm_{\sigma;\varepsilon})_v}. \tag{1.26}$$

We note that as spaces, $\mathfrak{D}$ and $\mathfrak{D}^\pm_\varepsilon$ are the same. However, the norms are only similar up to a constant which goes to infinity as $\varepsilon \to 0$.

To state the main theorem, we will also make use of the following spaces to measure the error in the ion distribution:

$$\|u\|^2_{\mathfrak{E}'} := \|\langle v\rangle^{m_1} u\|^2_{L^2_{x,v}} + \|\langle v\rangle^{m_0}\langle\nabla_x\rangle^s u\|^2_{L^2_{x,v}}, \tag{1.27}$$

$$\|u\|^2_{\mathfrak{D}'} := \|\langle v\rangle^{m_1} u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|\langle v\rangle^{m_0}\langle\nabla_x\rangle^s u\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{1.28}$$

Finally, to measure the distance of F^ε_- from its limit, we let $\eta > 0$ be a constant to be determined later, and define $q_\alpha = e^{\alpha\eta}\beta_{in}$ for each $\alpha \in \{1, 2, 3\}$, where $\beta_{in} = \beta(0)$. Then, for $\alpha \in \{1, 2\}$, we define

$$\begin{aligned}\mathscr{E}^\varepsilon_{-,\alpha} := \;&\|e^{q_\alpha|\xi|^2/4}\langle\nabla_x\rangle^s\{F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\}\|^2_{L^2_{x,\xi}}\\ &+ \|e^{q_{\alpha+1}|\xi|^2/4}\{F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\}\|^2_{L^2_{x,\xi}}\end{aligned} \tag{1.29}$$

and

$$\begin{aligned}\mathscr{D}^\varepsilon_{-,\alpha} := \;&\|e^{q_\alpha|\xi|^2/4}\langle\nabla_x\rangle^s\{F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\\ &+ \|e^{q_{\alpha+1}|\xi|^2/4}\{F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi}.\end{aligned} \tag{1.30}$$

Also, throughout the rest of the paper, we will use the subscript "in" to denote the initial value of some quantity, for instance, $\phi^0_{in}(x) = \phi^0(0, x)$.

We can now state our main theorem:

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[4] Although these parameters may take on a range of values, it is simpler to fix their values to exact constants.

Theorem 1.1. *Suppose for some $T_0 > 0$ and $M > 0$, there exists a weak solution (F_+^0, β, ϕ^0) to (1.7) with $0 \le F_+^0 \in C([0,T_0]; \mathfrak{E}) \cap L^2([0,T_0]; \mathfrak{D})$, $\beta \in C([0,T_0])$ and $\phi^0 \in C([0,T]; H^{s+2})$, with*

$$\sup_{t\in[0,T_0]} \{\|\frac{1}{n_+^0}\|_{L^\infty([0,T]\times\mathbf{T}^3)} + \|F_+^0\|_{\mathfrak{E}}\} + \|\ln(\beta(t))\|_{L^\infty([0,T_0])} \le M. \tag{1.31}$$

(i) *There exist positive constants η, and $\overline{\varepsilon}$ depending only on M, as well as $\hat{T} \in [0,T_0]$ depending on (F_+^0, β, ϕ^0) such that the following holds. For each $\varepsilon \in (0,\overline{\varepsilon}]$, take $(F_{+,in}^\varepsilon, F_{-,in}^\varepsilon)$, with $0 \le F_{\pm,in}^\varepsilon \in \mathfrak{E}$, satisfying the estimates*

$$\sup_{\varepsilon\in(0,\overline{\varepsilon}]} \|\frac{1}{n_{+,in}^\varepsilon}\|_{L_x^\infty} + \|F_{+,in}^\varepsilon\|_{\mathfrak{E}} \le M, \tag{1.32}$$

$$\|F_{+,in}^\varepsilon - F_{+,in}^0\|_{\mathfrak{E}'} + \mathscr{E}_{-,2,in}^\varepsilon \le M\varepsilon. \tag{1.33}$$

Then, for each $\varepsilon \in (0,\overline{\varepsilon}]$, there exists a solution $(F_+^\varepsilon, F_-^\varepsilon)$ to (1.7) with $0 \le F_\pm^\varepsilon \in C([0,\hat{T}]; \mathfrak{E}) \cap L^2([0,\hat{T}] \cap \mathfrak{D})$. This satisfies

$$\sup_{t\in[0,\hat{T}]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'} \lesssim_M \varepsilon \tag{1.34}$$

$$\sup_{t\in[0,\hat{T}]} \varepsilon\mathscr{E}_{-,2}^\varepsilon(t) + \int_0^{\hat{T}} \mathscr{D}_{-,2}^\varepsilon(t)dt \lesssim_M \varepsilon^2 \tag{1.35}$$

Moreover, for all $t \in [0,\hat{T}]$,

$$\mathscr{E}_{-,1}^\varepsilon(t) \lesssim_M \varepsilon^{\frac{1}{2}} e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}} + \varepsilon^{\frac{5}{3}} t^{\frac{1}{3}} + \varepsilon^2. \tag{1.36}$$

(ii) *There exist positive constants η, δ and $\overline{\varepsilon}$ depending on M, and $\hat{T} \in (0,T_0]$ depending on (F_+^0, β, ϕ^0) and M such that the following holds. For all $\varepsilon \in (0,\overline{\varepsilon}]$, we set $F_{+,in}^\varepsilon = F_{+,in}^0$ and $F_{-,in}^\varepsilon = F_{-,in}$ (where $F_{-,in}^\varepsilon$ is independent of ε) satisfying*

$$\mathscr{E}_{-,2,in}^\varepsilon \le \delta. \tag{1.37}$$

We note the expression on the left is independent of ε. We also impose the condition on the kinetic energy of $F_{-,in}$:

$$\frac{1}{2}\iint_{\mathbf{T}^3\times\mathbf{R}^3} |\xi|^2 F_{-,in}dxd\xi - \frac{3}{2\beta_{in}} + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x\phi_{in}|^2 - |\nabla_x\phi_{in}^0|^2 dx = 0. \tag{1.38}$$

Then, for each $\varepsilon \in (0,\overline{\varepsilon}]$, there exists a solution $(F_+^\varepsilon, F_-^\varepsilon)$ to (1.7) with $0 \le F_\pm^\varepsilon \in C([0,\hat{T}]; \mathfrak{E}) \cap L^2([0,\hat{T}] \cap \mathfrak{D})$. This satisfies

$$\sup_{t\in[0,\hat{T}]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'} \lesssim_M \sqrt{\varepsilon}\delta + \varepsilon \tag{1.39}$$

$$\sup_{t\in[0,\hat{T}]} \varepsilon\mathscr{E}_{-,2}^\varepsilon(t) + \int_0^{\hat{T}} \mathscr{D}_{-,2}^\varepsilon(t)dt \lesssim_M \varepsilon(\delta^2 + \varepsilon) \tag{1.40}$$

Moreover, for all $t \in [0,\hat{T}]$,

$$\mathscr{E}_{-,1}^\varepsilon(t) \lesssim_M \delta e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}} + \varepsilon^{\frac{5}{3}} t^{\frac{1}{3}} + \varepsilon\delta. \tag{1.41}$$

Remark 1.2. This theorem depends on the existence of solutions to the system (1.18). The local well-posedness theory for this system will be given in a forthcoming paper [14].

1.3. Methodology and outline. Due to the nature of the rescaled velocity ξ in the two-scale system (1.7), the Landau kernel Q^{ε}_{+-} exhibits a severe singularity in ξ as $\varepsilon \to 0$. This is mirrored by the degeneracy of the Q^{ε}_{-+} in this limit. This makes it very difficult to propagate v derivatives of F^{ε}_{+} and ξ derivatives of F^{ε}_{-}. The main technical achievement of this paper is that we obtain uniform bound in ε in weighted Sobolev spaces, *without any ξ or v derivatives*. In other words, all regularity in v and ξ comes from the diffusive control from the top order part of the collision operators.

We highlight some of the techniques that go into this. In Lemma 3.2, we obtain a *non-perturbative* lower bound for $\Phi * G$ (where $G \geq 0$), which allows us to access diffusive control from collisions. One challenge we encounter is that we are unable to close a bootstrap estimate on F^{ε}_{+} solely in the $\mathfrak{E}$ norm. Proposition 4.1 roughly states that if F^{ε}_{+} is of size M on $[0, T]$, then we can only say that F^{ε}_{+} will be of size C_M on the same interval. In general, C_M may grow exponentially in M. However, we can close a bootstrap estimate by also assuming $\|F^{\varepsilon}_{+}\|_{\mathfrak{D}'}$ is small on this interval. This can be done uniformly in ε by taking T small enough that $\|F^{0}_{+}\|_{\mathfrak{D}'}$ is as small as desired, and then controlling the difference $\|F^{\varepsilon}_{+} - F^{0}_{+}\|_{\mathfrak{D}'}$. We do this through the error estimate in Proposition 4.2.

The next challenge is to control the difference $F^{\varepsilon}_{-} - \mu_{\beta} e^{\beta\phi^0}$. This is roughly the content of Proposition 5.1, although we work with what we call the "intermediary quantities" $(\gamma^{\varepsilon}, \psi^{\varepsilon})$ rather than (β, ϕ^0). This proposition utilizes the linear decay estimates of the linearized collision operator, via the framework developed in [21,22]. Because of the small parameter in front of the ∂_t term in the second line of (1.7), the question of convergence of F^{ε}_{-} to the Maxwellian $\mu_{\gamma^{\varepsilon}} e^{\gamma^{\varepsilon}\psi^{\varepsilon}}$ is in some sense equivalent to the question of asymptotic stability of the Maxwellian. However, there are a number of complications in our context. First, the underlying Maxwellian has a varying temperature. To deal with this, we take a window of time such that $\gamma^{\varepsilon} \approx \beta_{in}$, and choose $q_1 < q_2 < q_3$ (constant in time) close enough to β_{in}, and work with these constants instead. This results in a perturbed version of the linearized collision operator found in [21,22]. Showing that this operator retains the desired coercivity properties requires some care. There is also the ion–electron collision operator Q^{ε}_{+-}, which acquires a singularity at $\xi = 0$ as $\varepsilon \to 0$. To control this, we use Lemma 3.2 to extract a dissipative, albeit singular and degenerate, top order part. We then use the dissipation from the linearization of Q to control the singular lower order terms.

A key feature of the linearized collision operator is that it has a five dimensional kernel, corresponding to the macroscopic quantities of mass, momentum and energy density. Besides having to work with a perturbation of this operator, there are some additional challenges in adapting the macroscopic estimates from previous works. One issue is the effect of Q^{ε}_{+-} on the macroscopic quantities. Surprisingly, this term has a perturbative effect on the mass and energy density, and even contributes an extra dampening effect on the momentum density (see Lemma 5.6). A second challenge is how one controls the mean energy of the perturbation. The mean kinetic energy density of $F^{\varepsilon}_{-} - \mu_{\gamma^{\varepsilon}} e^{\gamma^{\varepsilon}\psi^{\varepsilon}}$ is not conserved, and corresponds to neutral mode at the level of the linearization of the equation for $F^{\varepsilon}_{-} - \mu_{\gamma^{\varepsilon}} e^{\gamma^{\varepsilon}\psi^{\varepsilon}}$. In contrast, all other components of the macroscopic quantities are either conserved, or dissipated through hypo-coercive effects. In part (i) of Theorem 1.1, we can control the mean energy because it is initially small. In part (ii), we must impose (1.38) and use some nonlinear identities to control the average energy density, as in the first step of Proposition 5.7. This necessitates the use of the the intermediary quantities $(\gamma^\varepsilon, \psi^\varepsilon)$ instead of (β, ϕ^0), since $\frac{3}{2\gamma^\varepsilon}$ serves as a better approximation of the mean kinetic energy of F_-^ε than $\frac{3}{2\beta}$.

The structure of the paper is as follows. Section 2 contains all of the formal analysis in this paper. In Sect. 2.1 we derive the rescaled system (1.7). In Sect. 2.2, we give a formal derivation of the system (1.18) from (1.7). In Sect. 2.2, we show how the analogous derivation does not work in the context of the Vlasov–Poisson–Boltzmann system, and we highlight the difficulties in defining the limiting system as $\varepsilon \to 0$.

Section 3 contains two preliminary results that will be used in the paper. The first is Lemma 3.1. This is a statement of local well-posedness that is compatible with the main theorem. We do not prove this result, as it follows from straightforward modifications of the a priori estimates shown in this work. The second result is Lemma 3.2, which gives upper and lower bounds on $\Phi * G$. This is particularly important for extracting diffusive control via the norms $\dot{\mathcal{H}}_\sigma$, and $\mathcal{H}^{\pm}_{\sigma;\varepsilon}$.

In Sect. 4, we prove a priori estimates on the ion distribution F_+^ε. First, we have Proposition 4.1, which gives control of $F_+^\varepsilon \in L^\infty(\mathfrak{E}) \cap L^2(\mathfrak{D})$ uniform in ε. Then, we have Proposition 4.2, which gives uniform control of $F_+^\varepsilon - F_+^0 \in L^\infty(\mathfrak{E}') \cap L^\infty(\mathfrak{D}')$. Finally, with Lemma 4.3, we construct the intermediary quantities $(\gamma^\varepsilon, \psi^\varepsilon)$, which solve the to the system (4.138) below. These are defined in the same way as (β, ϕ^0), except using F_+^ε in place of F_+^0.

In Sect. 5, we prove Proposition 5.1, which gives the error estimate for the electrons under certain assumptions on the solutions. The proof of this is broken up into two components: energy estimates for $F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}$ (as in Proposition 5.4), and macroscopic estimates (as in Lemma 5.6 and Proposition 5.7).

Finally, in Sect. 6, we prove Theorem 1.1. For both parts (i) and (ii), we make five bootstrap assumptions, which allow us to satisfy the assumptions made by the various propositions throughout this paper. We then show show that these assumptions can be propagated over a time interval independent of $\varepsilon > 0$. The error estimates then follow from Propositions 4.2 and 5.1.

2. Formal Analysis

2.1. Non-dimensionalization of the Vlasov–Poisson–Landau system. Here, we show how to rescale the Vlasov–Poisson–Landau system (1.1) to the non-dimensionalized form (1.7). The collision operators in (1.1) are

$$Q_{++}(G_1, G_2)(v) = \frac{1}{m_+^2}\nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v-v')\left\{G_1(v')\nabla_v G_2(v) - G_2(v)\nabla_v G_1(v')\right\} dv', \tag{2.1}$$

$$\begin{aligned} Q_{-+}(G_1, G_2)(v) = \frac{1}{m_+}\nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v-v') \\ \left\{\frac{1}{m_+}G_1(v')\nabla_v G_2(v) - \frac{1}{m_-}G_2(v)\nabla_v G_1(v')\right\} dv', \end{aligned} \tag{2.2}$$

$$Q_{+-}(G_1, G_2)(v) = \frac{1}{m_-}\nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v-v')$$

$$\left\{\frac{1}{m_-}G_1(v')\nabla_v G_2(v) - \frac{1}{m_+}G_2(v)\nabla_v G_1(v')\right\} dv', \tag{2.3}$$

$$Q_{--}(G_1, G_2)(v) = \frac{1}{m_-^2}\nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v - v')\left\{G_1(v')\nabla_v G_2(v) - G_2(v)\nabla_v G_1(v')\right\} dv', \tag{2.4}$$

In order to define the Coulomb logarithm, we assume F_+ and F_- have approximately a common temperature $\theta > 0$ (with the Boltzmann constant set to 1). To have a charge balance, the electrons will have on average N total number of particles per unit volume, and the ions will have total number N/Z per unit volume. For simplicity, one might take F_+ and F_- to be perturbations of such equilibrium, i.e.

$$F_+ \approx \frac{N}{Z}(\frac{m_+}{\theta})^{3/2} e^{-\frac{m_+|v|^2}{2\theta}}, \tag{2.5}$$

$$F_- \approx N(\frac{m_-}{\theta})^{3/2} e^{-\frac{m_-|v|^2}{2\theta}}. \tag{2.6}$$

When F_+ and F_- are equal to the Maxwellians, they solve (1.1) with $\phi = 0$.

Next, the Coulomb logarithm $\ln(\Lambda)$ arises as logarithmically divergent integral in the derivation of the Vlasov–Poisson–Landau system from the BBGKY hierarchy [2], which is then cutoff at length scales where the assumptions of the model break down. Several choices of cutoffs exist, the most common pair being the Debye length at large scales

$$\lambda_D = \left(\frac{\theta}{4\pi \mathsf{e}^2 N}\right)^{1/2} \tag{2.7}$$

and the distance of closest approach at small scales

$$b_0 = \frac{3\theta}{Z\mathsf{e}^2}. \tag{2.8}$$

This gives the expression

$$\ln(\Lambda) = \ln\left(\frac{\lambda_D}{b_0}\right). \tag{2.9}$$

Importantly, $\ln(\Lambda)$ does not depend on the m_- or m_+. Unfortunately, this choice fails in contexts where $\lambda_D \le b_0$, for instance when N is large or θ is small, and another definition of $\ln(\Lambda)$ is needed. For our purposes, we shall simply take $\ln(\Lambda)$ to be some positive parameter independent of the masses. We note here that this logarithmic divergence can be resolved by using the Lenard-Balescu model for Coulombic collisions. Regarding this, see chapter 2 of [2], as well as the works [13,35].

We define relevant length, velocity and time scales for our problem. This will allow us to redefine a non-dimensionalized equation depending on three parameters: the (square root of the) mass ratio, Z, and a dissipation rate. The relevant velocity scales for each species are

$$V_\pm = (\frac{\theta}{m_\pm})^{1/2}. \tag{2.10}$$

These are the the thermal speeds as in [1]. Next, we define the length scale

$$X = \left(\frac{\theta}{N\mathsf{e}^2}\right)^{1/2}, \tag{2.11}$$

which can be thought of the smallest length scale at which the electrostatic force is significant for a thermal particle. This gives us natural time scales for each species

$$T^{\pm} = \frac{X}{V^{\pm}} = \left(\frac{m_{\pm}}{N\mathsf{e}^2}\right)^{1/2} \tag{2.12}$$

While F_+ and F_- will be re-scaled in v differently, according to V_+ and V_- respectively, we will re-scale time by the ion-time scale for both species. The electrons then evolve according to a faster time-scale than the ions. The ratio of these time-scales is the same as the square-root of the mass ratio,

$$\varepsilon = \frac{T_-}{T_+} = \left(\frac{m_-}{m_+}\right)^{1/2} \tag{2.13}$$

The final non-dimensional parameter is the dissipation rate, which gives the size of the collisional effects:

$$\kappa^{-1} = 2\pi \ln(\Lambda) \frac{N^{1/2}\mathsf{e}^3}{\theta^{3/2}}. \tag{2.14}$$

We now perform the non-dimensionalization. Define the re-scaled coordinates

$$\bar{t} = \frac{t}{T_+}, \tag{2.15}$$

$$\bar{x} = \frac{x}{X}, \tag{2.16}$$

$$\bar{v}_{\pm} = \frac{v}{V_{\pm}}, \tag{2.17}$$

and define re-scaled versions of (F_-, F_+):

$$\tilde{F}_+(\bar{t}, \bar{x}, \bar{v}_+) = \frac{ZV_+^3}{N} F_+(t, x, v), \tag{2.18}$$

$$\tilde{F}_-(\bar{t}, \bar{x}, \bar{v}_-) = \frac{V_-^3}{N} F_-(t, x, v). \tag{2.19}$$

Moreover, define

$$\tilde{n}_{\pm}(\bar{t}, \bar{x}) = \int \tilde{F}_+(\bar{t}, \bar{x}, \bar{v}_{\pm}) d\bar{v}_{\pm}, \tag{2.20}$$

$$-\Delta_{\bar{x}}\tilde{\phi} = 4\pi(\tilde{n}_+(\bar{t}, \bar{x}) - \tilde{n}_-(\bar{t}, \bar{x})), \tag{2.21}$$

$$\tilde{E}(\bar{t}, \bar{x}) = -\nabla_{\bar{x}}\tilde{\phi}(\bar{t}, \bar{x}). \tag{2.22}$$

In particular,

$$\tilde{E}(\bar{t}, \bar{x}) = \frac{1}{NX\mathsf{e}} E(t, x) \tag{2.23}$$

The "Vlasov–Poisson" parts of the Eq. (1.1) transform like

$$\{\partial_t + v \cdot \nabla_x + \frac{Z\mathsf{e}}{m_+} E \cdot \nabla_v\} F_+ = \frac{N}{ZT_+V_+^3} \{\partial_{\bar{t}} + \bar{v}_+ \cdot \nabla_{\bar{x}} + Z\tilde{E} \cdot \nabla_{\bar{v}_+}\} \tilde{F}_+ \tag{2.24}$$

$$\{\partial_t + v \cdot \nabla_x - \frac{\mathsf{e}}{m_-} E \cdot \nabla_v\} F_- = \frac{N}{T_-V_-^3} \{\varepsilon\partial_{\bar{t}} + \bar{v}_- \cdot \nabla_{\bar{x}} - \tilde{E} \cdot \nabla_{\bar{v}_-}\} \tilde{F}_- \tag{2.25}$$

We now compute the transformations of the collision kernels:

$$Q_{++}(F_+, F_+)(v) \tag{2.26}$$

$$= \frac{N^2}{Z^2 m_+^2 V_+^6} \nabla_{\overline{v}_+} \cdot \int_{\mathbf{R}^3} \Phi(\overline{v}_+ - \overline{v}'_+) \left\{ \tilde{F}_+(\overline{v}'_+) \nabla_{\overline{v}_+} \tilde{F}_+(v_+) - \tilde{F}_+(\overline{v}_+) \nabla_{\overline{v}_+} \tilde{F}_+(\overline{v}'_+) \right\} d\overline{v}'_+, \tag{2.27}$$

$$Q_{-+}(F_-, F_+)(v) \tag{2.28}$$

$$= \frac{N^2}{Z m_+^2 V_+^6} \nabla_{\overline{v}_+} \cdot \int_{\mathbf{R}^3} \Phi(\varepsilon \overline{v}_+ - \overline{v}'_-) \left\{ \varepsilon \tilde{F}_-(\overline{v}'_-) \nabla_{\overline{v}_+} \tilde{F}_+(v_+) - \tilde{F}_+(\overline{v}_+) \nabla_{\overline{v}_-} \tilde{F}_-(\overline{v}'_-) \right\} d\overline{v}'_-, \tag{2.29}$$

$$Q_{--}(F_-, F_-)(v) \tag{2.30}$$

$$= \frac{N^2}{m_-^2 V_-^6} \nabla_{\overline{v}_-} \cdot \int_{\mathbf{R}^3} \Phi(\overline{v}_- - \overline{v}'_-) \left\{ \tilde{F}_-(\overline{v}'_-) \nabla_{\overline{v}_-} \tilde{F}_-(v_-) - \tilde{F}_-(\overline{v}_-) \nabla_{\overline{v}_-} \tilde{F}_-(\overline{v}'_-) \right\} d\overline{v}'_-, \tag{2.31}$$

$$Q_{+-}(F_+, F_-)(v) \tag{2.32}$$

$$= \frac{N^2}{Z m_-^2 V_-^6} \nabla_{\overline{v}_-} \cdot \int_{\mathbf{R}^3} \Phi(\overline{v}_- - \varepsilon \overline{v}'_+) \left\{ \tilde{F}_+(\overline{v}'_+) \nabla_{\overline{v}_-} \tilde{F}_-(\overline{v}_-) - \varepsilon \tilde{F}_-(\overline{v}_-) \nabla_{\overline{v}_+} \tilde{F}_+(\overline{v}'_+) \right\} d\overline{v}'_+ \tag{2.33}$$

Ignoring Z, the ratios of the pre-factors appearing in the expressions above for the collisions, over that of the transport terms for each species are

$$\frac{N^2}{m_\pm^2 V_\pm^6} \left(\frac{N}{T_\pm V_\pm^3} \right)^{-1} = \frac{NX}{m_\pm^2 V_\pm^4} = \frac{N^{1/2}}{\theta^{3/2} \mathsf{e}} \tag{2.34}$$

Multiplying the above by $2\pi \mathsf{e}^4 \ln(\Lambda)$ gives κ^{-1}.

We now give the non-dimensionalized version of (1.1). By abuse of notation, we shall use the symbols (t, x, v) to denote the re-scaled coordinates for the ions; we will use (t, x, ξ) to denote the re-scaled coordinates of the electrons (i.e., we replace $\overline{t} \to t$, $\overline{x} \to x$, $\overline{v}_+ \to v$, and $\overline{v}_- \to \xi$). Moreover, we write $F_\pm^\varepsilon = \tilde{F}_\pm$, and similarly for the potential and electric field. Then, these solve (1.7).

2.2. Formal derivation of the ion equation. We give a formal derivation of (1.18) from (1.7) by setting $\varepsilon = 0$. Then,

$$\{\partial_t + v \cdot \nabla_x + E^0 \cdot \nabla_v\} F_+^0 = Q(F_+^0, F_+^0) + Q_{-+}^0(F_-^0, F_+^0), \tag{2.35}$$

$$\{\xi \cdot \nabla_x - E^0 \cdot \nabla_\xi\} F_-^0 = Q(F_-^0, F_-^0) + Q_{+-}^0(F_+^0, F_-^0). \tag{2.36}$$

where

$$Q_{-+}^0(F_-^0, F_+^0)(v) = -\nabla_v \cdot \left(\int_{\mathbf{R}^3} \Phi(\xi') \nabla_\xi F_-^0(\xi') d\xi' F_+^0(v) \right), \tag{2.37}$$

$$Q_{+-}^0(F_+^0, F_-^0)(\xi) = \left(\int_{\mathbf{R}^3} F_+^0(v') dv' \right) \nabla_\xi \cdot (\Phi(\xi) \nabla_\xi F_-^0(\xi)). \tag{2.38}$$

We assume F_- is positive at each (t, x, ξ), smooth in (x, ξ), and decays sufficiently fast in ξ. We can then use the entropy identity to deduce possible solutions. Multiplying (2.36) by $\ln(F_-)$, we have

$$0 = \iint_{\mathbf{T}^3\times\mathbf{R}^3} (Q(F_-^0, F_-^0) + Q_{-+}^0(F_+^0, F_-^0)) \ln(F_-) dx dv \tag{2.39}$$

$$= -\frac{1}{2} \iiint_{\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{R}^3} F_-^0(t, x, \xi) F_-^0(t, x, \xi') \tag{2.40}$$

$$\mathrm{tr}(\Phi(\xi - \xi')\{\nabla_\xi \ln(F_-^0(t, x, \xi)) - \nabla_{\xi'} \ln(F_-^0(t, x, \xi'))\}^{\otimes 2}) dx d\xi d\xi' \tag{2.41}$$

$$- n_+^0(t, x) \int_{\mathbf{T}^3\times\mathbf{R}^3} F_-^0(t, x, \xi) \mathrm{tr}(\Phi(\xi)\{\nabla_\xi \ln(F_-^0(t, x, \xi))\}^{\otimes 2}) dx d\xi. \tag{2.42}$$

Note that both terms are nonpositive, and therefore zero. The null space of $\Phi(\xi)$ is $\mathrm{span}\{\xi\}$, so we deduce that $\nabla_\xi \ln(F_-^0(t, x)(t, x, \xi)) \in \mathrm{span}\{\xi\}$ for all (t, x, ξ). Thus $F_-(t, x, \xi)$ is radial in ξ for each (t, x). Taking $F_-^0(t, x, \xi) = \tilde{F}_-^0(t, x, |\xi|)$, we have that

$$0 = \frac{1}{2} \iiint_{\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{R}^3} \tilde{F}_-^0(t, x, |\xi|) \tilde{F}_-^0(t, x, |\xi'|) \tag{2.43}$$

$$\mathrm{tr}(\Phi(\xi - \xi')\{\frac{\xi}{|\xi|} \partial_r \ln(\tilde{F}_-^0(t, x, r))|_{r=|\xi|} - \frac{\xi'}{|\xi'|} \partial_r \ln(\tilde{F}_-^0(t, x, r))|_{r=|\xi'|}\}^{\otimes 2}) dx d\xi d\xi' \tag{2.44}$$

Then, for every ξ, ξ' so that $\xi \neq 0$, $\xi' \neq 0$ and $\xi - \xi' \neq 0$, we have

$$\frac{\xi}{|\xi|} \partial_r \ln(\tilde{F}_-^0(t, x, r))|_{r=|\xi|} - \frac{\xi'}{|\xi'|} \partial_r \ln(\tilde{F}_-^0(t, x, r))|_{r=|\xi'|} \in \mathrm{span}\{\xi - \xi'\} \tag{2.45}$$

The above implies that on the set where ξ and ξ' are linearly independent, we have

$$\frac{\xi}{|\xi|} \partial_r \ln(\tilde{F}_-^0(t, x, r))|_{r=|\xi|} = \frac{\xi'}{|\xi'|} \partial_r \ln(\tilde{F}_-^0(t, x, r))|_{r=|\xi'|} \tag{2.46}$$

For any given values of $|\xi|, |\xi'| > 0$, we can find ξ, ξ' which are linearly independent. Thus, for all $r, r' > 0$, we have

$$\frac{1}{r} \partial_r \ln(\tilde{F}_-^0(t, x, r)) = \frac{1}{r'} \partial_{r'} \ln(\tilde{F}_-^0(t, x, r')) \tag{2.47}$$

In particular, $\partial_r(\frac{1}{r}\partial_r \ln(\tilde{F}_-^0(t, x, r))) = 0$. Thus, there exist functions $\beta(t, x)$ and $\psi(t, x)$ such that

$$\ln(\tilde{F}_-^0(t, x, r)) = -\frac{\beta(t, x)}{2} r^2 + \psi(t, x) \tag{2.48}$$

In other words, $F_-^0(t, x, \xi)$ is a local Maxwellian centered at $\xi = 0$, i.e.

$$F_-^0(t, x, \xi) = e^{-\frac{\beta(t,x)|\xi|^2}{2} + \psi(t,x)} \tag{2.49}$$

Observe that $Q(F_-^0, F_-^0) = Q_{+-}^0(F_+^0, F_-^0) = 0$. Therefore,

$$\{\xi \cdot \nabla_x - E \cdot \nabla_\xi\} F_-^0 = 0. \tag{2.50}$$

Thus,

$$0 = \{\xi \cdot \nabla_x - E \cdot \nabla_\xi\} \ln(F_-^0) = \xi \cdot \nabla_x \psi + \xi \cdot \nabla_x \beta \frac{|\xi|^2}{2} + \xi \cdot E^0 \beta. \tag{2.51}$$

Taking $|\xi| \to \infty$, we deduce that $\nabla_x \beta = 0$, i.e. $\beta(t, x) = \beta(t)$. Finally, we conclude $\nabla_x \psi = -\beta E^0 = \nabla_x(\beta \phi^0)$. Thus, $\psi - \beta \phi^0$ is independent of x and ξ. We integrate in ξ, use conservation of mass (1.13) to determine this function, and add an appropriate constant to ϕ^0 to deduce

$$F_-^0(t, x, \xi) = \left(\frac{\beta(t)}{2\pi}\right)^{\frac{3}{2}} e^{-\beta(t)(\frac{|\xi|^2}{2} - \phi^0(t,x))} \tag{2.52}$$

This implies $n_-^0(t, x) = \frac{e^{\beta(t)\phi^0(t,x)}}{\int_{\mathbf{T}^3} e^{\beta(t)\phi^0(t,x')} dx'}$, giving the third line of (1.18). As for the first line, since $\nabla_\xi F_-^0 = -\beta w F_-^0$, we have the important vanishing property

$$Q_{-+}^0(F_-^0, F_+^0) = \beta \nabla_\xi \cdot \{\int_{\mathbf{R}^3} \Phi(\xi) \xi F_-^0(t, x, \xi) d\xi F_+^0(v)\} = 0, \tag{2.53}$$

as $\Phi(\xi)\xi = (0, 0, 0)^T$. Finally, for the second line, we invoke (1.15) and the identity

$$\iint_{\mathbf{T}^3 \times \mathbf{R}^3} \frac{|\xi|^2}{2} F_-^0 dx d\xi = \frac{3}{2\beta}. \tag{2.54}$$

2.3. The Boltzmann case. We now apply the analogous re-scaling to the Vlasov–Poisson–Boltzmann system with hard spheres as was done in Sect. 2.1. Unlike the Vlasov–Poisson–Landau system, the ion–electron collisions do not seem to vanish in the limit $\varepsilon \downarrow 0$. Moreover, we show that the formal expansion in ε seems to yield a system of equations that seems difficult, if not impossible, to solve. This is because the the $O(1)$ terms in the expansion depend on the $O(\varepsilon)$ terms in a nontrivial way, which in turn depend on the $O(\varepsilon^2)$ terms, and so on.

For simplicity, we let $Z = \mathsf{e} = 1$. Following [38], the Vlasov–Poisson–Boltzmann system reads

$$\begin{aligned} &\{\partial_t + v \cdot \nabla_x + \frac{1}{m_-} E \cdot \nabla_v\} F_+ = Q_{++}(F_+, F_+) + Q_{-+}(F_-, F_+)\}, \\ &\{\partial_t + v \cdot \nabla_x - \frac{1}{m_+} E \cdot \nabla_v\} F_- = Q_{--}(F_-, F_-) + Q_{+-}(F_+, F_-), \\ &\qquad -\Delta_x \phi = 4\pi(n_+ - n_-) \end{aligned} \tag{2.55}$$

Once again, $E = -\nabla_x \phi$. The Boltzmann collision operators are given as follows: with $\alpha, \beta \in \{-, +\}$, we have

$$Q_{\alpha\beta}(G_\alpha, G_\beta) = \frac{(\sigma_\alpha + \sigma_\beta)^2}{4} \iint_{\mathbf{R}^3\times\mathbf{S}^2} |(u-v)\cdot\omega|\{G_\alpha(u')G_\beta(v') \\ - G_\alpha(u)G_\beta(v)\}du d\omega, \tag{2.56}$$

where $\sigma_\pm$ are the diameters of the particles, and

$$u' = u + \frac{2m_\beta}{m_\alpha + m_\beta}((v-u)\cdot\omega)\omega, \tag{2.57}$$

$$v' = v - \frac{2m_\alpha}{m_\alpha + m_\beta}((v-u)\cdot\omega)\omega. \tag{2.58}$$

For simplicity, we set $\sigma_\pm = 1$ as well. We now set $\xi = \varepsilon v$ (and $\zeta = \varepsilon u$), and $\tilde{F}_-(\xi) = \varepsilon^{-3}F_-(\xi/\varepsilon)$.
We now rewrite the first two lines of (2.55),

$$\{\partial_t + v\cdot\nabla_x + E\cdot\nabla_v\}F_+ = Q(F_+, F_+) + \tilde{Q}^\varepsilon_{-+}(\tilde{F}_-, F_+) \tag{2.59}$$

$$\{\varepsilon\partial_t + \xi\cdot\nabla_x - E\cdot\nabla_\xi\}\tilde{F}_- = Q(\tilde{F}_-, \tilde{F}_-) + \tilde{Q}^\varepsilon_{+-}(F_+, \tilde{F}_-). \tag{2.60}$$

Here, $Q = Q_{++} = Q_{--}$. Before we define the cross-collision terms, it is worthwhile to define the reflection matrix

$$R_\omega z = z - 2(z\cdot\omega)\omega, \tag{2.61}$$

where $\omega \in \mathbf{S}^2$. The ion–electron collisions have the following (singular!) effect on the ions:

$$\tilde{Q}^\varepsilon_{-+}(\tilde{F}_-, F_+) \tag{2.62}$$

$$= \frac{1}{\varepsilon}\iint_{\mathbf{R}^3\times\mathbf{S}^2} |(\zeta - \varepsilon v)\cdot\omega|\{\tilde{F}_-(\zeta')F_+(v') - \tilde{F}_-(\zeta)\tilde{F}_+(v)\}d\zeta d\omega, \tag{2.63}$$

where

$$\zeta' = \zeta + \frac{2}{1+\varepsilon^2}((\varepsilon v - \zeta)\cdot\omega)\omega = R_\omega\zeta + 2\varepsilon(v\cdot\omega)\omega + O(\varepsilon^2), \tag{2.64}$$

$$v' = v - \frac{2\varepsilon}{1+\varepsilon^2}((\varepsilon v - \zeta)\cdot\omega)\omega = v + 2\varepsilon(\zeta\cdot\omega)\omega + O(\varepsilon^2). \tag{2.65}$$

On the other hand, the effect of ion–electron collisions on the electrons are the following:

$$\tilde{Q}^\varepsilon_{+-}(F_+, \tilde{F}_-) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} |(\varepsilon u - \xi)\cdot\omega|\{F_+(u')\tilde{F}_-(\xi') - F_+(u)\tilde{F}_-(\xi)\}du d\omega, \tag{2.66}$$

where in the above,

$$u' = u + \frac{2\varepsilon}{1+\varepsilon^2}((\xi - \varepsilon u)\cdot\omega)\omega = u - 2\varepsilon(\xi\cdot\omega)\omega + O(\varepsilon^2), \tag{2.67}$$

$$\xi' = \xi - \frac{2}{1+\varepsilon^2}((\xi - \varepsilon u)\cdot\omega)\omega = R_\omega\xi + 2\varepsilon(u\cdot\omega)\omega + O(\varepsilon^2). \tag{2.68}$$

The ε^{-1} term in $\tilde{Q}^\varepsilon_{+-}$ is the source of the difficulty in describing the formal analysis as $\varepsilon \downarrow 0$. In particular, we have no reason to expect that the effect of $\tilde{Q}^\varepsilon_{-+}$ will vanish in this limit. To better understand this limit, we use a formal asymptotic expansion. First, we expand the collisons as follows:

$$\tilde{Q}^{\varepsilon}_{-+}(G_1, G_2) = \sum_{j=-1}^{\infty} \varepsilon^j q^j_{-+}(G_1, G_2), \tag{2.69}$$

$$\tilde{Q}^{\varepsilon}_{+-}(G_1, G_2) = \sum_{j=0}^{\infty} \varepsilon^j q^j_{+-}(G_1, G_2). \tag{2.70}$$

We now give the first two terms in these expansions. The first two terms in (2.69) are

$$q^{-1}_{-+}(G_1, G_2). = \left(\int_{\mathbf{R}^3 \times \mathbf{S}^2} |\zeta \cdot \omega| \{G_1(R_\omega \zeta) - G_1(\zeta)\} d\zeta d\omega \right) G_2(v) \tag{2.71}$$

and

$$q^0_{-+}(G_1, G_2) = -\left(\int_{\mathbf{R}^3 \times \mathbf{S}^2} \operatorname{sgn}(\zeta \cdot \omega)(v \cdot \omega) \{G_1(R_\omega \zeta) - G_1(\zeta)\} d\zeta d\omega \right) G_2(v) \tag{2.72}$$

$$+ 2 \left(\int_{\mathbf{R}^3 \times \mathbf{S}^2} |\zeta \cdot \omega| (v \cdot \omega) \omega \cdot \nabla_\zeta G_1(R_\omega \zeta) d\zeta d\omega \right) G_2(v) \tag{2.73}$$

$$- 2 \left(\int_{\mathbf{R}^3 \times \mathbf{S}^2} |\zeta \cdot \omega| (\zeta \cdot \omega) G_1(R_\omega \zeta) \omega_i d\zeta d\omega \right) \partial_{v_i} G_2(v). \tag{2.74}$$

On the other hand, the first two terms in (2.70) are

$$q^0_{+-}(G_1, G_2) = \left(\int_{\mathbf{R}^3} G_1(u) du \right) \int_{\mathbf{S}^2} |\xi \cdot \omega| \{G_1(R_\omega \xi) - G_1(\xi)\} d\omega. \tag{2.75}$$

and

$$q^1_{+-}(G_1, G_2) = -\left(\int_{\mathbf{R}^3} u_i G_1(u) du \right) \int_{\mathbf{S}^2} \operatorname{sgn}(\xi \cdot \omega) \omega_i \{G_2(R_\omega \xi) - G_2(\xi)\} d\omega \tag{2.76}$$

$$+ 2 \left(\int_{\mathbf{R}^3} u_i G_1(u) du \right) \int_{\mathbf{S}^2} \operatorname{sgn}(\xi \cdot \omega) \omega_i \omega \cdot \nabla_\xi G_2(R_\omega \xi) d\omega. \tag{2.77}$$

In the above, there is another $O(\varepsilon)$ term arising from the expansion $G_1(u') = G_2(u) + 2\varepsilon(\xi \cdot \omega)\omega \cdot \nabla_u G_1(u) + O(\varepsilon^2)$, but this integrates to zero.

Next, we expand F_+ and $\tilde{F}_-$ into an formal series in ε:

$$F_+ \sim \sum_{j=0}^{\infty} \varepsilon^j F_{+,j}, \quad \tilde{F}_- \sim \sum_{j=0}^{\infty} \varepsilon^j \tilde{F}_{-,j}, \tag{2.78}$$

where each $F_{\pm,j}$ is independent of ε. Similarly, we expand $\phi \sim \sum_{j=0}^{\infty} \varepsilon^j \phi_j$ and $E \sim \sum_{j=0}^{\infty} \varepsilon^j E_j$. Collecting the $O(1)$ terms of (2.60), we see that $F^0_{-,j}$ solves

$$\{\xi \cdot \nabla_x - E^0 \cdot \nabla_\xi\} \tilde{F}_{-,0} = Q(\tilde{F}_{-,0}, \tilde{F}_{-,0}) + q^0_{-+}(F_{+,0}, \tilde{F}_{-,0}). \tag{2.79}$$

It is straightforward to check that

$$\tilde{F}_{-,0} = \left(\frac{\beta(t)}{2\pi}\right)^{\frac{3}{2}} e^{-\beta(t)(\frac{|v|^2}{2}-\phi^0)} \tag{2.80}$$

solves the above, with (β, ϕ^0) solving the same equation as in (1.7). Using a similar entropy identity as was used in Sect. 2.2, we expect that this is the unique such solution. With this ansatz, we have $q^j_{-,+}(\tilde{F}_{-,0}, G) = 0$ for both $j = -1$ and $j = 0$, and any G, due to the radial symmetry of $\tilde{F}_{-,0}$. In particular, the $O(\frac{1}{\varepsilon})$ term in (2.59) vanishes. Collecting all the $O(1)$ terms in (2.59), we then have

$$\partial_t F_{+,0} + \{v \cdot \nabla_x + E^0 \cdot \nabla_v\} F_{+,0} = Q(F_{+,0}, F_{+,0}) + q^{-1}_{-+}(\tilde{F}_{-,1}, F_{+,0}). \tag{2.81}$$

Thus, in order to solve for $F_{+,0}$, we must solve for $\tilde{F}_{-,1}$ as well. Then, collecting all the $O(\varepsilon)$ terms in (2.60), we have

$$\{\xi \cdot \nabla_x - E_0 \cdot \nabla_\xi\} \tilde{F}_{-,1} - E_1 \cdot \nabla_\xi F_{-,0} \tag{2.82}$$

$$- Q(\tilde{F}_{-,1}, \tilde{F}_{-,0}) - Q(\tilde{F}_{-,0}, \tilde{F}_{-,1}) - q^0_{+-}(F_{+,0}, \tilde{F}_{-,1}) \tag{2.83}$$

$$= \partial_t \tilde{F}_{-,0} + q^1_{-+}(F_{+,0}, \tilde{F}_{-,0}) \tag{2.84}$$

In the above, we used the fact that $q^0(F_{+,1}, \tilde{F}_{-,0}) = 0$. Now we arrive at the issue: in order to solve for $F_{+,0}$, we must solve for $\tilde{F}_{-,1}$ as well. However, the equation for $F_{-,1}$ depends on E_1, which in turn depends on $F_{+,1}$, which depends on $\tilde{F}_{-,2}$, and so on. In summary, it seems difficult to solve for the solve the terms in the expansion (2.78), in order to get a well-defined limiting equation.

3. Preliminaries

3.1. Well-posedness. In this section, we state a prerequisite local well-posedness result that guarantees solutions to (1.7) in our context. The proof follows by the methods in this paper, so we omit it.

Lemma 3.1. *Fix $M, T > 0$, and suppose (F^0_+, β, ϕ^0) is a solution to* (1.18) *satisfying the hypothesis of Theorem* 1.1. *Fix $\varepsilon > 0$. Then there exists $T_\varepsilon \in (0, T_0]$ (depending on (F^0_+, β, ϕ^0), $\varepsilon > 0$ and M), and $\varrho = \varrho(M)$ (depending only on M) such that the following holds. If we take $(F^\varepsilon_{+,in}, F^\varepsilon_{+,in})$ satisfying*

$$\|\frac{1}{n^\varepsilon_{+,in}}\|_{L^\infty_x} + \|F^\varepsilon_{+,in}\|_{\mathfrak{E}} \le M, \tag{3.1}$$

$$\|F^\varepsilon_{+,in} - F^0_{+,in}\|_{\mathfrak{E}'} + \sqrt{\mathscr{E}^\varepsilon_{-,2,in}} \le \varrho. \tag{3.2}$$

then there exists a unique weak solution $(F^\varepsilon_+, F^\varepsilon_-)$ with $0 \le F^\varepsilon_\pm \in C([0, T_\varepsilon]; \mathfrak{E}) \cap L^2([0, T_\varepsilon]; \mathfrak{D})$ to (1.7) *with the initial data, satisfying the estimates*

$$\sup_{t\in[0,T_\varepsilon]} \left(\|\frac{1}{n^\varepsilon_+(t)}\|_{L^\infty_x} + \|F^\varepsilon_+(t)\|_{\mathfrak{E}}\right) \le 2M, \tag{3.3}$$

$$\sup_{t\in[0,T_\varepsilon]} \left(\|F^\varepsilon_+(t) - F^0_+(t)\|_{\mathfrak{E}'} + \sqrt{\mathscr{E}^\varepsilon_{-,2}(t)}\right) \lesssim_M \|F^\varepsilon_{+,in} - F^0_{+,in}\|_{\mathfrak{E}'} + \sqrt{\mathscr{E}^\varepsilon_{-,2,in}}. \tag{3.4}$$

3.2. Lower and upper bounds on the diffusion matrix. In order prove local well-posedness for the system (1.7) and (1.18), we first prove the following lemma, which gives upper and lower bounds on the diffusion matrix appearing in the collision kernels:

Lemma 3.2. *Let $G(v) = G : \mathbf{R}^3 \to \mathbf{R}$, $\nu \in \mathbf{S}^2$. Then for all $v \in \mathbf{R}^3$,*

$$|\Phi_{ij} * G(v)\nu_i\nu_j| \lesssim \|\langle v\rangle^5 G\|_{L^2}\sigma_{ij}(v)\nu_i\nu_j. \tag{3.5}$$

Assuming $G \geq 0$, we have the lower bound

$$\frac{\|G\|_{L^1}}{\langle \|\langle v\rangle^2 G\|_{L^2}/\|G\|_{L^1}\rangle^{17}}\sigma_{ij}(v)\nu_i\nu_j \lesssim \Phi_{ij} * G(v)\nu_i\nu_j. \tag{3.6}$$

Proof. We first prove (3.5). Observe that $|\Phi_{ij} * G(v)\nu_i\nu_j| \leq \Phi_{ij} * |G|(v)\nu_i\nu_j$. Thus, it suffices to consider the case when $G \geq 0$. First we have uniform boundedness,

$$\|\Phi_{ij} * G(v)\nu_i\nu_j\|_{L^\infty} \lesssim \|(-\Delta_v)^{-1}G\|_{L^\infty} \lesssim \|\langle v\rangle^2 G\|_{L^2} \tag{3.7}$$

Alternatively,

$$\Phi_{ij} * G(v)\nu_i\nu_j \leq \int_{2|v'|<|v|} \frac{|v-v'|^2 - \langle v-v', \nu\rangle^2}{|v-v'|^3} G(v')dv' \tag{3.8}$$

$$+ \int_{2|v'|\geq|v|} \frac{|v-v'|^2 - \langle v-v', \nu\rangle^2}{|v-v'|^3} G(v')dv'. \tag{3.9}$$

Now,

$$\sqrt{|v-v'|^2 - \langle v-v', \nu\rangle^2} = |P_{\nu^\perp}(v-v')| \leq |P_{\nu^\perp}v| + |P_{\nu^\perp}v'|, \tag{3.10}$$

so

$$(3.8) \lesssim \int_{2|v'|<|v|} \frac{|P_{\nu^\perp}v|^2 + |P_{\nu^\perp}v|^2}{|v-v'|^3} G(v')dv' \tag{3.11}$$

$$\lesssim \frac{|P_{\nu^\perp}v|^2}{|v|^3}\|G\|_{L^1} + \frac{1}{|v|^3}\||v'|^2 G\|_{L^1} \tag{3.12}$$

$$\lesssim (\frac{|P_{\nu^\perp}\nu|^2}{|v|} + \frac{1}{|v|^3})\|\langle v'\rangle^5 G(v')\|_{L^2_{v'}} \tag{3.13}$$

On the other hand, we have

$$(3.9) \leq \int_{2|v'|\geq|v|} \frac{1}{|v-v'|} G(v')dv' \tag{3.14}$$

$$\leq \frac{1}{|v|^3}\int_{2|v'|\geq|v|} \frac{1}{|v-v'|}|v'|^3 G(v')dv' \tag{3.15}$$

$$\leq \frac{1}{|v|^3}\|(-\Delta_{v'})^{-1}(\langle v'\rangle^3 G(v'))\|_{L^\infty_{v'}} \tag{3.16}$$

$$\leq \frac{1}{|v|^3}\|\langle v'\rangle^5 G(v')\|_{L^2_{v'}}. \tag{3.17}$$

Thus,

$$\Phi_{ij} * G(v)\nu_i\nu_j \lesssim \min\{1, \frac{|P_{v^\perp}\nu|^2}{|v|} + \frac{1}{|v|^3}\}\|\langle v'\rangle^5 G(v')\|_{L^2_{v'}}, \tag{3.18}$$

which implies (3.5).

We now prove (3.6). By re-scaling $G \mapsto G/\|G\|_{L^1}$, it suffices to show (3.6) in the case $\|G\|_{L^1} = 1$. For convenience, we define

$$k_G = \langle \|\langle v\rangle^2 G\|_{L^2_v}\rangle. \tag{3.19}$$

Let $\nu \in \mathbf{S}^2$,

$$\Phi_{ij} * G(v)\nu_i\nu_j = \int_{\mathbf{R}^3} \frac{\sin^2\theta}{|v'|} G(v - v')dv' \tag{3.20}$$

where $\theta = \theta(v')$ is the angle between v' and ν. Taking $\theta_0 > 0$ to be a constant to be chosen later, we bound the above from below by

$$\Phi_{ij} * G(v)\nu_i\nu_j \geq \sin^2\theta_0 \left(\int_{\mathbf{R}^3} \frac{1}{|v'|} G(v - v')dv' - \int_{\sin^2\theta < \sin^2\theta_0} \frac{1}{|v'|} G(v - v')dv'\right) \tag{3.21}$$

Now,

$$\begin{aligned}
\int_{\mathbf{R}^3} \frac{1}{|v'|} G(v - v')dv' &= \int_{|v|\geq 10|v'|} \frac{1}{|v - v'|} G(v')dv' && (3.22)\\
&\geq \frac{9}{10|v|} \int_{|v|\geq 10|v'|} G(v')dv' && (3.23)\\
&\geq \frac{9}{10|v|}(1 - \int_{|v|\leq 10|v'|} G(v')dv') && (3.24)\\
&\geq \frac{9}{10|v|}(1 - \||v'|^{-2} 1_{|v|\leq 10|v'|}\|_{L^2_{v'}} \||v'|^2 G(v')\|_{L^2_{v'}}) && (3.25)\\
&\geq \frac{9}{10|v|}(1 - \frac{k_G}{|v|}) && (3.26)
\end{aligned}$$

On the other hand, for any $\lambda > 0$,

$$\begin{aligned}
\int_{\mathbf{R}^3} \frac{1}{|v'|} G(v - v')dv' &\geq \frac{1}{\lambda}\left(1 - \int_{|v'|>\lambda} G(v - v')dv'\right) && (3.27)\\
&\geq \frac{1}{\lambda}\left(1 - \||v'|^{-2} 1_{|v'|>\lambda}\|_{L^2_{v'}} \||v - v'|^2 G(v')\|_{L^2_{v'}}\right) && (3.28)\\
&\geq \frac{1}{\lambda}(1 - \frac{k_G\langle v\rangle^2}{\lambda^{\frac{1}{2}}}) && (3.29)
\end{aligned}$$

Taking $\lambda = 4k_G^2\langle v\rangle^4$, and combining with (3.26),

$$\int_{\mathbf{R}^3} \frac{1}{|v'|} G(v - v')dv' \gtrsim \min\{\frac{1}{|v|}(1 - \frac{k_G}{|v|}), \frac{1}{k_G^2\langle v\rangle^4}\} \gtrsim \frac{1}{k_G^5\langle v\rangle} \tag{3.30}$$

We now consider the second term in (3.21). First,

$$\int_{\sin^2\theta<\sin^2\theta_0} \frac{1}{|v'|} G(v-v')dv' \lesssim k_G \left(\int_{\sin^2\theta<\sin^2\theta_0} \frac{1}{|v'|^2\langle v-v'\rangle^4} dv'\right)^{\frac{1}{2}} \tag{3.31}$$

$$\lesssim k_G \left(\int_{\sin^2\theta<\sin^2\theta_0} \frac{1}{|v'|^2(1+|v|^2+2\langle v, v'\rangle+|v'|^2)^2} dv'\right)^{\frac{1}{2}}. \tag{3.32}$$

We then use spherical coordinates to evaluate the integral above, setting the span of ν to be the z-axis, and the span of $P_{\nu^\perp} v$ to be the x axis. Letting $v_\parallel = \langle v, \nu\rangle$, and $|P_{\nu^\perp} v| = v_\perp$, we get

$$\int_{\sin^2\theta<\sin^2\theta_0} \frac{1}{|v'|^2(1+|v|^2+2\langle v, v'\rangle+|v'|^2)^2} dv' \tag{3.33}$$

$$= \int_0^\infty \int_0^{2\pi} \int_{[0,\theta_0]\cup[\pi-\theta_0,\pi]} \frac{\sin\theta}{(1+v_\parallel^2+v_\perp^2+2r(v_\parallel\cos\theta+v_\perp\cos\alpha\sin\theta)+r^2)^2} d\theta d\alpha dr \tag{3.34}$$

We now rewrite

$$v_\parallel^2+v_\perp^2+2r(v_\parallel\cos\theta+v_\perp\cos\alpha\sin\theta)+r^2 \tag{3.35}$$

$$= v_\parallel^2\sin^2\theta+v_\perp^2(\cos^2\theta+\sin^2\alpha\sin^2\theta)+(r+v_\parallel\cos\theta+v_\perp\cos\alpha\sin\theta)^2 \tag{3.36}$$

$$\geq v_\perp^2\cos^2\theta+(r+v_\parallel\cos\theta+v_\perp\cos\alpha\sin\theta)^2 \tag{3.37}$$

Extending the domain of integration to be $r\in(-\infty,\infty)$, and using shift invariance in r, and then the symmetries of $\sin\theta$, we bound the integral by

$$(3.33) \lesssim \int_{\mathbf{R}} \int_0^{\theta_0} \frac{\sin\theta}{(1+v_\perp^2\cos^2\theta+r^2)^2} d\theta dr \tag{3.38}$$

Taking $\theta_0 \leq \frac{\pi}{4}$, we have $\cos^2\theta \geq \frac{1}{2}$. This ensures that

$$(3.33) \lesssim \int_{\mathbf{R}} \int_0^{\theta_0} \frac{\sin\theta}{(1+v_\perp+|r|)^4} d\theta dr \tag{3.39}$$

$$\lesssim \frac{1}{\langle v_\perp\rangle^3} \int_0^{\theta_0} \sin\theta d\theta \tag{3.40}$$

$$\lesssim \frac{\theta_0^2}{\langle v_\perp\rangle^3}. \tag{3.41}$$

Combining with (3.21), we have

$$\Phi_{ij} * G(v)\nu_i\nu_j \gtrsim \theta_0^2 \left(\frac{1}{k_G^4\langle v\rangle} - k_G\frac{\theta_0}{\langle v_\perp\rangle^{\frac{3}{2}}}\right) \tag{3.42}$$

At this point, we take $\theta_0 = \lambda\frac{\langle v_\perp\rangle}{\langle v\rangle}$, where $\lambda\in(0,\frac{\pi}{4})$ is to be chosen later (not necessarily the same λ from before). In particular, $v_\parallel\theta_0 \lesssim \lambda\langle v_\perp\rangle$, so

$$\Phi_{ij} * G(v)\nu_i\nu_j \gtrsim \frac{\lambda^2\langle v_\perp\rangle^2}{\langle v\rangle^2} \left(\frac{1}{k_G^5\langle v\rangle} - k_G\frac{\lambda}{\langle v_\perp\rangle^{\frac{1}{2}}\langle v\rangle}\right). \tag{3.43}$$

Taking $\lambda = \frac{1}{10k_G^6}$, we deduce

$$\Phi_{ij} * G(v)v_i v_j \gtrsim \frac{\langle v_\perp\rangle^2}{k_G^{17}\langle v\rangle^3} = \frac{1}{k_G^{17}}(\frac{|P_v v|^2}{\langle v\rangle^3} + \frac{|P_{v^\perp} v|^2}{\langle v\rangle}). \tag{3.44}$$

We deduce (3.6). □

4. Estimates on the Ion Distribution

4.1. Boundedness of the ion distribution. We now prove a priori estimates for the ion distribution.

Proposition 4.1. *Let $M, T, \varepsilon > 0$. Suppose $(F_-^\varepsilon, F_+^\varepsilon)$ is a weak solution to (1.7) with $0 \le F_\pm^\varepsilon \in C([0,T];\mathfrak{E}) \cap L^2([0,T];\mathfrak{D})$. Assume the following bootstrap assumptions:*

$$\sup_{\pm\in\{-,+\}, 0\le t\le T} (\|F_\pm^\varepsilon(t)\|_{\mathfrak{E}}^2 + \|\frac{1}{n_\pm^\varepsilon(t)}\|_{L_x^\infty}) \le M \tag{4.1}$$

In particular, there exists $\mathscr{X}_+^\varepsilon(t) \sim_M \|F_+^\varepsilon\|_{\mathfrak{E}}^2$ (depending on M) such that

$$\frac{d}{dt}\mathscr{X}_+^\varepsilon + \|F_+^\varepsilon(t)\|_{\mathfrak{D}_\varepsilon^+}^2 \le C_M \langle \|F_-^\varepsilon\|_{\mathfrak{D}}\rangle^{\frac{3}{2}}. \tag{4.2}$$

Alternatively,

$$\frac{d}{dt}\|F_+^\varepsilon\|_{\mathfrak{E}}^2 + \frac{1}{C_M}\|F_+^\varepsilon(t)\|_{\mathfrak{D}_\varepsilon^+}^2 \le C_M(\langle \|F_-^\varepsilon\|_{\mathfrak{D}}\rangle^{\frac{3}{2}} + \|\langle v\rangle^{m_1} F_+^\varepsilon(t)\|_{L_x^2(\mathcal{H}_\sigma\cap\mathcal{H}_{\sigma;\varepsilon}^+)_v}^2). \tag{4.3}$$

Proof. It is convenient to ignore dependence on M: we write $C = C_M$, and "$\lesssim$" instead of "$\lesssim_M$." We also drop the dependence on ε, for instance, $F_+ = F_+^\varepsilon$. Given $m', s' \in \mathbf{R}$, we define

$$F_+^{(m',s')} := \langle v\rangle^{m'}\langle\nabla_x\rangle^{s'} F_+, \tag{4.4}$$

$$F_-^{(m',s')} := \langle \xi\rangle^{m'}\langle\nabla_x\rangle^{s'} F_-. \tag{4.5}$$

It is also convenient to split the collision operators into their "diffusion" (second order) and "transport" (first order) parts in divergence form:

$$Q(G_1, G_2) = Q_D(G_1, G_2) + Q_T(G_1, G_2), \tag{4.6}$$

where

$$Q_D(G_1, G_2) := \partial_{v_j}((\Phi_{ij} *_v G_1)\partial_{v_i} G_2), \tag{4.7}$$

$$Q_T(G_1, G_2) := \partial_{v_j}(\partial_{v_i}(\Phi_{ij} *_v G_1) G_2). \tag{4.8}$$

We split $Q_{\pm\mp}^\varepsilon = Q_{\pm\mp,D}^\varepsilon + Q_{\pm\mp,T}^\varepsilon$ similarly. We now proceed with the proof, broken into several steps.

Step 1: We have the following estimates:

$$\frac{d}{dt}\|F_+^{(m_2,0)}\|_{L_{x,v}^2}^2 + \frac{1}{C}\|F_+^{(m_2,0)}\|_{L_x^2(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}_{\sigma;\varepsilon}^+)_v}^2 \le C(1 + \|F_-\|_{\mathfrak{D}}^{\frac{3}{2}}), \tag{4.9}$$

and for all $\kappa > 0$,

$$\begin{aligned}&\frac{d}{dt}\|F_+^{(m_1,s)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|F_+^{(m_1,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}^+_{\sigma;\varepsilon})_v}\\ &\quad\le C(1+\|F_+^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}^+_{\sigma;\varepsilon})_v} + \|F_-\|^{\frac{3}{2}}_{\mathfrak{D}}).\end{aligned} \tag{4.10}$$

The proof of (4.9) follows by a similar method as (4.10), so we only show the latter. To prove (4.10), observe that $F_+^{(m_1,s)}$ satisfies

$$(\partial_t + v\cdot\nabla_x + E\cdot\nabla_v)F_+^{(m_1,s)} + [\langle v\rangle^{m_1}\langle\nabla_x\rangle^s, E\cdot\nabla_v]F_+ \tag{4.11}$$
$$= \langle v\rangle^{m_1}\langle\nabla_x\rangle^s\{Q(F_+,F_+) + Q^\varepsilon_{-+}(F_-,F_+)\}. \tag{4.12}$$

Multiplying the above by by $F_+^{(m_1,s)}$ integrating, we have

$$\frac{1}{2}\frac{d}{dt}\|F_+^{(m_1,s)}\|^2_{L^2_{x,v}} \tag{4.13}$$
$$= \langle[E\cdot\nabla_v, \langle v\rangle^m\langle\nabla_x\rangle^s]F_+, F_+^{(m_1,s)}\rangle_{L^2_{x,\xi}} \tag{4.14}$$
$$+ \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q(F_+,F_+), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.15}$$
$$+ \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q^\varepsilon_{-+}(F_-,F_+), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.16}$$

What follows are estimates for each term on the right-hand side.
Step 1.1 (electric field commutator): We now bound (4.14):

$$(4.14) \lesssim 1 + \|F_+^{(m_1,s)}\|_{L^2_x(\mathcal{H}_\sigma)_v}. \tag{4.17}$$

Observe first that

$$[E\cdot\nabla_v, \langle v\rangle^m\langle\nabla_x\rangle^s]F_+ = E\cdot\nabla_v(\langle v\rangle^{m_1}\langle\nabla_x\rangle^s F_+) - \langle v\rangle^{m_1}E\cdot\nabla_v\langle\nabla_x\rangle^s F_+ \tag{4.18}$$
$$\langle v\rangle^{m_1}E\cdot\nabla_v\langle\nabla_x\rangle^s F_+ - \langle v\rangle^{m_1}\langle\nabla_x\rangle^s(E\cdot\nabla_v F_+) \tag{4.19}$$
$$= v\cdot EF_+^{(m_1-2,s)} + \langle v\rangle^{m_1}\nabla_v\cdot(E\langle\nabla_x\rangle^s F_+ - \langle\nabla_x\rangle^s(EF_+)) \tag{4.20}$$

Thus,

$$(4.14) = \langle v\cdot EF_+^{(m-2,s)}, F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.21}$$
$$- \langle E\langle\nabla_x\rangle^s F_+ - \langle\nabla_x\rangle^s(EF_+), \nabla_v(\langle v\rangle^{m_1}F_+^{(m_1,s)})\rangle_{L^2_{x,v}} \tag{4.22}$$
$$\lesssim \|E\|_{L^\infty_x}\|F_+^{(m_1,s)}\|^2_{L^2_{x,v}} \tag{4.23}$$
$$+ (\|\nabla_x E\|_{L^\infty_x}\|F_+^{(m_1+\frac{3}{2},s-1)}\|_{L^2_{x,v}} + \|E\|_{H^s_x}\|F_+^{(m_1+\frac{3}{2},0)}\|_{L^2_{x,v}})\|F_+^{(m_1-\frac{3}{2},s)}\|_{L^2_xH^1_v} \tag{4.24}$$
$$\lesssim (\|F_+^{(m_1+\frac{3}{2},s-1)}\|_{L^2_{x,v}} + \|F_+^{(m_2,0)}\|_{L^2_{x,v}})(1+\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}) \tag{4.25}$$

Next, note that we have the interpolation inequality

$$\|F^{(m_1+\frac{3}{2},s-1)}\|_{L^2_{x,v}} \lesssim \|F_+^{(m_1,s)}\|^{\frac{s-1}{s}}_{L^2_{x,v}}\|F^{(m_1+\frac{3s}{2},0)}\|^{\frac{1}{s}}_{L^2_{x,v}} \lesssim \|F_+\|_{\mathfrak{E}} \lesssim 1, \tag{4.26}$$

since $m_1 + \frac{3s}{2} \le m_2$. We conclude with (4.17).
Step 1.2 (self-collisions): Next, we have the following bound on (4.15):

$$(4.15) \le -\frac{1}{C}\|F_+^{(m_1,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + C(1 + \|F_+^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}). \tag{4.27}$$

For this, we separate

$$(4.15) = \langle Q_D(F_+, F_+^{(m_1,s)}), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.28}$$
$$+ \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q_D(F_+, F_+) - Q_D(F_+, F_+^{(m_1,s)}), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.29}$$
$$+ \langle\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q_T(F_+, F_+), F_+^{(m_1,s)}\rangle_{L^2_{x,v}}. \tag{4.30}$$

Using (3.6), we have the bound

$$(4.28) = -\langle(\Phi_{ij} *_v F_+)\partial_{v_i} F_+^{(m_1,s)}, \partial_{v_j} F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.31}$$
$$\le -\frac{1}{C_{k_{F_+}}}\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.32}$$

where

$$k_{F_+} = \max\{1, \|\langle v\rangle^5 F_+\|_{L^\infty_x L^2_v}, \|n_+^{-1}\|_{L^\infty_x}\}. \tag{4.33}$$

Now,

$$\|\langle v\rangle^5 F_+\|_{L^\infty_x L^2_v} \lesssim \|F_+^{(5,2)}\|_{L^2_{x,v}} \lesssim 1. \tag{4.34}$$

so $k_{F_+} \lesssim 1$. Using these estimates, we get that

$$(4.28) \le -\frac{1}{C}\|F_+\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.35}$$

Next, we have the commutator term (4.29). Let us first write

$$\langle v\rangle^{m_1}\langle\nabla_x\rangle^s Q_D(F_+, F_+) - Q_D(F_+, F_+^{(m_1,s)}) \tag{4.36}$$
$$= \langle\nabla_x\rangle^s Q_D(F_+, F_+^{(m_1,0)}) - Q_D(F_+, F_+^{(m_1,s)}) \tag{4.37}$$
$$+ \langle\nabla_x\rangle^s\{\langle v\rangle^m Q_D(F_+, F_+) - Q_D(F_+, F_+^{(m_1,0)})\}. \tag{4.38}$$

Now, let

$$\mathcal{C}_j := \langle\nabla_x\rangle^s((\Phi_{ij} *_v F_+)\partial_{v_i} F_+^{(m_1,0)}) - (\Phi_{ij} *_v F_+)\partial_{v_i} F_+^{(m_1,s)} \tag{4.39}$$

Then

$$\langle(4.37), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} = \langle\mathcal{C}_i, \partial_{v_i} F_+\rangle_{L^2_{x,v}} \tag{4.40}$$
$$\le \langle(\sigma^{-1})_{ij}\mathcal{C}_i, \mathcal{C}_j\rangle^{\frac{1}{2}}_{L^2_{x,v}}\langle\sigma_{ij}\partial_{v_i} F_+^{(m_1,s)}, \partial_{v_j} F_+^{(m_1,s)}\rangle^{\frac{1}{2}}_{L^2_{x,v}} \tag{4.41}$$

Now, the latter factor is bounded by $\|F_+\|_{L^2_x(\mathcal{H}_\sigma)_v}$. On the other hand, writing

$$A_G = \sigma^{-\frac{1}{2}}\Phi *_v G\sigma^{-\frac{1}{2}}. \tag{4.42}$$

Note that by (3.5), $\|A_G\|_{L^\infty_v} \lesssim \|\langle v\rangle^5 G\|_{L^2_v}$ given a function $G = G(v)$. Therefore, we have

$$\|\sigma^{-\frac{1}{2}}\mathcal{C}\|_{L^2_{x,v}} = \|\langle\nabla_x\rangle^s(A_{F_+}\sigma^{\frac{1}{2}}\nabla_v F_+^{(m_1,0)}) - A_{F_+}\sigma^{\frac{1}{2}}\nabla_v F_+^{(m_1,s)}\|_{L^2_{x,v}} \tag{4.43}$$

$$\lesssim \|A_{F_+^{(0,s)}}\|_{L^2_x L^\infty_v}\|\sigma^{\frac{1}{2}}\nabla_v F_+^{(m_1,s-1)}\|_{L^2_{x,v}} \tag{4.44}$$

$$\lesssim \|F_+^{(m_1,s-1)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.45}$$

$$\lesssim \|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F_+^{(m_1,s)}\|^{\frac{s-1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.46}$$

Thus,

$$\langle(4.37), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \lesssim \|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F_+^{(m_1,s)}\|^{\frac{2s-1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.47}$$

Next, we have

$$(4.38) = -m_1\langle\nabla_x\rangle^s\{v_j(\Phi_{ij} *_v F_+)\partial_{v_i}F_+^{(m_1-2,0)} + \partial_{v_j}(v_i(\Phi_{ij} *_v F_+)F_+^{(m_1-2,0)}) \tag{4.48}$$

$$-(m_1-2)v_iv_j(\Phi_{ij} *_v F_+)F_+^{(m_1-4,0)}\}. \tag{4.49}$$

Thus,

$$\langle(4.38), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} = -m_1\langle\langle\nabla_x\rangle^s\{v_j(\Phi_{ij} *_v F_+)\partial_{v_i}F_+^{(m_1-2,0)}\}, F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.50}$$

$$+ m_1\langle\langle\nabla_x\rangle^s\{\langle v\rangle^{m_1-2}v_i(\Phi_{ij} *_v F_+)F_+^{(m_1-2,0)}\}, \partial_{v_j}F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.51}$$

$$+ m_1(m_1-2)\langle\langle\nabla_x\rangle^s\{v_iv_j(\Phi_{ij} *_v F_+)F_+^{(m_1-4,0)}\}, F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.52}$$

$$\lesssim \left\| |\sigma^{\frac{1}{2}}v|\|A_{F_+^{(0,s)}}\|_{L^2_x}\|\sigma^{\frac{1}{2}}\nabla_v F_+^{(m_1-2,s)}\|_{H^s_x}\right\|_{L^2_v}\|F_+^{(m_1,s)}\|_{L^2_{x,v}} \tag{4.53}$$

$$+ \left\| |\sigma^{\frac{1}{2}}v|\|A_{F_+^{(0,s)}}\|_{L^2_x}\|F_+^{(m_1-2,s)}\|_{H^s_x}\right\|_{L^2_v}\|\sigma^{\frac{1}{2}}\nabla_v F_+^{(m_1,s)}\|_{L^2_{x,v}} \tag{4.54}$$

$$+ \left\|(v\cdot(\sigma^{\frac{1}{2}}v))\|A_{F_+^{(0,s)}}\|_{L^2_x}\|F_+^{(m_1-4,s)}\|_{H^s_x}\right\|_{L^2_v}\|F_+^{(m_1,s)}\|_{L^2_{x,v}} \tag{4.55}$$

$$\lesssim \|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.56}$$

Therefore,

$$(4.29) \lesssim \|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} + \|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F_+^{(m_1,s)}\|^{\frac{2s-1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.57}$$

Next we estimate (4.30). We bound this as follows:

$$(4.30) = \langle\langle\nabla_x\rangle^s\{\partial_{v_i}(\Phi_{ij} *_v F_+)F_+^{(m_1,0)}\}, \partial_{v_j}F^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.58}$$

$$+\langle\langle\nabla_x\rangle^s\{v_j\partial_{v_i}(\Phi_{ij}*_v F_+)F_+^{(m_1-2,0)}\}, F^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.59}$$

$$\lesssim \|\langle\nabla_x\rangle^s\{\sigma^{-\frac{1}{2}}(\nabla_v\cdot\Phi*_v F_+)F_+\}\|_{L^2_{x,v}}\|\sigma^{\frac{1}{2}}\nabla_v F_+^{(m_1,s)}\|_{L^2_{x,v}} \tag{4.60}$$

$$+\|\langle\nabla_x\rangle^s\{v_j\partial_{v_i}(\Phi_{ij}*_v F_+)F_+^{(m_1-2,0)}\}\|_{L^2_{x,v}}\|F_+^{(m_1,s)}\|_{L^2_{x,v}} \tag{4.61}$$

$$\lesssim \|\sigma^{-\frac{1}{2}}\nabla_v(-\Delta_v)^{-1}F_+^{(0,s)}\|_{L^2_xL^\infty_v}\|F_+^{(m_1,s)}\|_{L^2_{x,v}}\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.62}$$

$$+\|\nabla_v(-\Delta_v)^{-1}F_+^{(0,s)}\|_{L^2_xL^\infty_v}\|F_+^{(m_1,s)}\|_{L^2_{x,v}}\|F_+^{(m_1,s)}\|_{L^2_{x,v}} \tag{4.63}$$

$$\lesssim \|\langle v\rangle^{\frac{3}{2}}\nabla_v(-\Delta_v)^{-1}F_+^{(0,s)}\|_{L^2_xL^\infty_v}(1+\|F_+^{(m_1,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}). \tag{4.64}$$

Now, by

$$\langle v\rangle^{\frac{3}{2}}|\nabla_v(-\Delta_v)^{-1}F_+^{(0,s)}| \lesssim \langle v\rangle^{\frac{3}{2}}\int_{\langle v\rangle>2|v'|}\frac{|F_+^{(0,s)}(v')|}{|v-v'|^2}dv' + \langle v\rangle^{\frac{3}{2}}\int_{\langle v\rangle\le 2|v'|}\frac{|F_+^{(0,s)}(v')|}{|v-v'|^2}dv' \tag{4.65}$$

$$\lesssim \langle v\rangle^{-\frac{1}{2}}\|F_+^{(0,s)}\|_{L^1_v} + \int_{\mathbf{R}^3}\frac{|F_+^{(\frac{3}{2},s)}(v')|}{|v-v'|^2}dv' \tag{4.66}$$

Thus,

$$\|\langle v\rangle^{\frac{3}{2}}\nabla_v(-\Delta_v)^{-1}F_+^{(0,s)}\|_{L^2_xL^\infty_v} \lesssim \|F_+^{(0,s)}\|_{L^2_xL^1_v} + \||\nabla_v|^{-1}|F_+^{(\frac{3}{2},s)}|\|_{L^2_xL^\infty_v} \tag{4.67}$$

$$\lesssim \|F_+^{(2,s)}\|_{L^2_{x,v}} + \|\langle\nabla_v\rangle^{\frac{3}{4}}|F_+^{(\frac{3}{2},s)}|\|_{L^2_{x,v}} \tag{4.68}$$

$$\lesssim 1+\||F_+^{(\frac{3}{2},s)}|\|^{\frac{3}{4}}_{L^2_xH^1_v}. \tag{4.69}$$

$$\lesssim 1+\|F_+^{(m_1,s)}\|^{\frac{3}{4}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.70}$$

In the above, we use the interpolation inequality $\|u\|_{H^{3/4}} \lesssim \|u\|^{\frac{1}{4}}_{L^2}\|u\|^{\frac{3}{4}}_{H^1}$, followed by the fact that $\big|\nabla_v|u|\big| = |\nabla_v u|$ a.e., for any function $u \in H^1(\mathbf{R}^3)$. Thus,

$$(4.30) \lesssim 1+\|F_+^{(m_1,s)}\|^{\frac{7}{4}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \tag{4.71}$$

Combining the estimates (4.28), (4.29) and (4.30), we conclude

$$(4.15) \le -\frac{1}{C}\|F_+^{(m_1,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.72}$$

$$+C(1+\|F_+^{(m_1,s)}\|^{\frac{7}{4}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}+\|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\|F_+^{(m_1,s)}\|^{\frac{2s-1}{s}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}). \tag{4.73}$$

We then use Hölder's inequality to get (4.27).

Step 1.3 (control of ion–electron collisions): Now, we turn to (4.16), for which we have the bound

$$\begin{aligned}(4.16) \le &-\frac{1}{C}\{\|F_+^{(m_1,s)}\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} - \lambda\|F_+^{(m_1,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\}\\ &+C\|F_+^{(m_1,0)}\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v}\\ &+C_\lambda(1+\|F_-^{(m_1,s)}\|^{\frac{3}{2}}_{L^2_x(\dot{\mathcal{H}}_\sigma)_\xi}).\end{aligned} \tag{4.74}$$

for all $\lambda > 0$. We decompose this term in the same way as (4.15),

$$(4.16) = \langle Q^{\varepsilon}_{-+,D}(F_-, F_+^{(m_1,s)}), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.75}$$

$$+ \langle \langle v\rangle^{m_1} \langle \nabla_x\rangle^s Q^{\varepsilon}_{-+,D}(F_-, F_+) - Q^{\varepsilon}_{-+,D}(F_-, F_+^{(m_1,s)}), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.76}$$

$$+ \langle \langle v\rangle^{m_1} \langle \nabla_x\rangle^s Q^{\varepsilon}_{-+,T}(F_-, F_+), F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.77}$$

The terms (4.75) and (4.76) using the same approach as (4.28) and (4.29),

$$(4.75) \le -\frac{1}{C}\|F_+^{(m_1,s)}\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v}, \tag{4.78}$$

$$(4.76) \lesssim \|F_+^{(m_1,s)}\|_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} + \|F_+^{(m_1,0)}\|^{\frac{1}{s}}_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} \|F_+^{(m_1,s)}\|^{\frac{2s-1}{s}}_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v}. \tag{4.79}$$

The final term (4.77) requires a different approach: we separate it into a main term and commutators,

$$(4.77) = \langle Q^{\varepsilon}_{-+,T}(F_-, F_+^{(m_1,s)}), F^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.80}$$

$$+ \langle \langle \nabla_x\rangle^s Q^{\varepsilon}_{-+,T}(F_-, F_+^{(m_1,0)}) - Q^{\varepsilon}_{-+,T}(F_-, F_+^{(m_1,s)}), F^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.81}$$

$$+ \langle \langle \nabla_x\rangle^s \{\langle v\rangle^{m_1} Q^{\varepsilon}_{-+,T}(F_-, F_+) - Q^{\varepsilon}_{-+,T}(F_-, F_+^{(m_1,0)})\}, F^{(m_1,s)}\rangle_{L^2_{x,v}}. \tag{4.82}$$

For the first term, we note that $\partial_{v_i}\partial_{v_j}\Phi_{ij}(v) = -8\pi\delta(v)$ in the sense of distributions, where δ is the Dirac mass. Hence,

$$(4.80) = 4\pi\varepsilon\langle F_-|_{\xi=\varepsilon v} F_+^{(m_1,s)}, F_+^{(m_1,s)}\rangle_{L^2_{x,v}} \tag{4.83}$$

$$\lesssim \varepsilon\|\langle\varepsilon v\rangle^{\frac{3}{4}} F_-|_{\xi=\varepsilon v}\|_{L^\infty_x L^6_v} \|\langle\varepsilon v\rangle^{-\frac{3}{4}} F_+^{(m_1,s)}\|_{L^2_x L^3_v} \|F_+^{(m_1,s)}\|_{L^2_{x,v}} \tag{4.84}$$

$$\lesssim \varepsilon^{\frac{1}{2}}\|F_-^{(\frac{3}{4},0)}\|_{L^\infty_x L^6_\xi} \|\langle\varepsilon v\rangle^{-\frac{3}{4}} F_+^{(m_1,s)}\|_{L^2_x L^3_v} \tag{4.85}$$

Now, using the generalized Minkowski inequality and Sobolev embedding,

$$\|F_-^{(\frac{3}{4},0)}\|_{L^\infty_x L^6_\xi} \lesssim \|\nabla_v F_-^{(\frac{3}{4},0)}\|_{L^\infty_x L^2_\xi} \lesssim \|\nabla_v F_-^{(\frac{3}{4},s)}\|_{L^2_{x,\xi}} \lesssim 1 + \|F_-^{(\frac{9}{4},s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{4.86}$$

On the other hand,

$$\|\langle\varepsilon v\rangle^{-\frac{3}{4}} F_+^{(m_1,s)}\|_{L^2_x L^3_v} \lesssim \|F_+^{(m_1,s)}\|^{\frac{1}{2}}_{L^2_{x,v}} \|\langle\varepsilon v\rangle^{\frac{3}{2}} F_+^{(m_1,s)}\|^{\frac{1}{2}}_{L^2_x H^1_v} \tag{4.87}$$

$$\lesssim \varepsilon^{-\frac{1}{4}}\|F_+^{(m_1,s)}\|^{\frac{1}{2}}_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} \tag{4.88}$$

We conclude that

$$(4.80) \le C\varepsilon^{\frac{1}{4}}\|F_+^{(m_1,s)}\|^{\frac{1}{2}}_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} \|F_-^{(m_1,s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{4.89}$$

As for (4.81), we have

$$\begin{aligned}(4.81) &= 8\pi \langle \langle \nabla_x \rangle^s \{\nabla_\xi (-\Delta_\xi)^{-1} F_-|_{\xi=\varepsilon v} F_+^{(m_1,0)}\} \\ &\quad - \nabla_\xi (-\Delta_\xi)^{-1} F_-|_{\xi=\varepsilon v} F_+^{(m_1,s)}, \nabla_v F_+^{(m_1,s)} \rangle_{L^2_{x,v}} \qquad (4.90)\\ &\lesssim \left\| \langle v \rangle^{\frac{3}{2}} \| |\nabla_\xi|^{-1} |F_-^{(0,s)}|_{\xi=\varepsilon v} | \|_{L^2_x} \| F_+^{(m_1,s-1)} \|_{L^2_x} \right\|_{L^2_v} \| F_+^{(m_1,s)} \|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \qquad (4.91)\\ &\lesssim \| |\nabla_\xi|^{-1} |F_-^{(0,s)}| \|_{L^2_x L^\infty_\xi} \| F_+^{(m+\frac{3}{2},s-1)} \|_{L^2_{x,v}} \| F_+^{(m_1,s)} \|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \qquad (4.92)\\ &\lesssim \| |F_-^{(0,s)}| \|_{L^2_x H^{\frac{3}{4}}_\xi} \| F_+^{(m+\frac{3s}{2},0)} \|^{\frac{1}{s}}_{L^2_{x,v}} \| F_+^{(m_1,s)} \|^{\frac{s-1}{s}}_{L^2_{x,v}} \| F_+^{(m_1,s)} \|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \qquad (4.93)\\ &\lesssim \| F_-^{(m_1,s)} \|^{\frac{3}{4}}_{L^2_x(\mathcal{H}_\sigma)_\xi} \| F_+^{(m_1,s)} \|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}. \qquad (4.94)\end{aligned}$$

The second commutator (4.81) is bounded as follows:

$$\begin{aligned}(4.81) &= 8\pi \langle \langle \nabla_x \rangle^s \{v \cdot \nabla_\xi (-\Delta_\xi)^{-1} F_-|_{\xi=\varepsilon v} F_+^{(m_1-2,0)}\}, F_+^{(m_1,s)} \rangle_{L^2_{x,v}} \qquad (4.95)\\ &\lesssim \| |\nabla_v|^{-1} F_-^{(0,s)} \|_{L^2_x L^\infty_\xi} \| F_+^{(m_1,s)} \|^2_{L^2_{x,v}} \qquad (4.96)\\ &\lesssim \| F_-^{(m_1,s)} \|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \qquad (4.97)\end{aligned}$$

Thus,

$$\begin{aligned}(4.77) &\lesssim (\| F_-^{(m_1,s)} \|_{L^2_x(\mathcal{H}_\sigma)_\xi} + \varepsilon^{\frac{1}{4}} \| F_+^{(m_1,s)} \|^{\frac{1}{2}}_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} \| F_-^{(m_1,s)} \|_{L^2_x(\mathcal{H}_\sigma)_\xi} \qquad (4.98)\\ &\quad + \| F_+^{(m_1,s)} \|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \| F_-^{(m_1,s)} \|^{\frac{3}{4}}_{L^2_x(\mathcal{H}_\sigma)_\xi}). \qquad (4.99)\end{aligned}$$

Now, combining the bounds on (4.75), (4.76), (4.77), and Young's inequality, we conclude (4.74).

To conclude step 1, we combine (4.17), (4.27) and (4.74), taking λ sufficiently small, to get (4.10).

Step 2 (combining the estimates): Take $0 < \kappa_1 < \kappa_2$, and set $\mathscr{X}_+ := \kappa_1 \| F_+^{(m_1,s)} \|^2_{L^2_{x,v}} + \kappa_2 \| F_+^{(m_2,0)} \|^2_{L^2_{x,v}}$. Then by (4.9) and (4.10), we have

$$\begin{aligned}&\frac{d}{dt}\mathscr{X}_+ + \frac{\kappa_1}{C} \| F^{(m_1,s)} \|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} + \left(\frac{\kappa_2}{C} - C\kappa_1\right) \| F^{(m_2,0)} \|^2_{L^2_x(\dot{\mathcal{H}}_\sigma \cap \mathcal{H}^+_{\sigma;\varepsilon})_v} \qquad (4.100)\\ &\quad \le C(\kappa_1 + \kappa_2) \langle \| F_- \|_{\mathfrak{D}} \rangle^{\frac{3}{2}} \qquad (4.101)\end{aligned}$$

Taking κ_2 sufficiently large and κ_1 sufficiently small depending on M, we get (4.2).

Alternatively, by simply adding (4.9) and (4.10), we have (4.3). □

4.2. Error estimate for the ion distribution.

Proposition 4.2. *Let $M, T > 0$, and let $(F_+^\varepsilon, F_-^\varepsilon)$ be a solution to (1.7) satisfying the same hypotheses as in 4.1, and (F_+^0, β, ϕ^0) be a solution to (1.18), satisfying the hypothesis of Theorem 1.1. Then, there is a function $\mathscr{Y}_+^\varepsilon : [0, T^*) \to \mathbf{R}_+$ such that $\mathscr{Y}_+^\varepsilon \sim_M \|F_+^\varepsilon - F_+^0\|_{\mathfrak{E}'}$ and*

$$\begin{aligned}&\frac{d}{dt}\mathscr{Y}_+^\varepsilon + \|F_+^\varepsilon - F_+^0\|_{\mathfrak{D}'}^2 \\ &\quad\lesssim_M \langle \|(F_+^0, F_-^\varepsilon)\|_{\mathfrak{D}}\rangle^2(\varepsilon^2 + \mathscr{Y}_+^\varepsilon) + \mathscr{D}_{-,2}^\varepsilon.\end{aligned} \tag{4.102}$$

Proof. Let $G = F_+^\varepsilon - F_+^0$. Then,

$$\partial_t G + \{v\cdot\nabla_x + E^\varepsilon\cdot\nabla_v\}G - (E^\varepsilon - E^0)\cdot\nabla_v F_+^0 - Q(F_+^\varepsilon, G) - Q(G, F_+^0) \tag{4.103}$$
$$= Q_{-+}(F_-^\varepsilon, F_+^\varepsilon) \tag{4.104}$$

Similarly to the proof of the previous proposition, given $m', s' \in \mathbf{R}$, we denote $G^{(m',s')} = \langle v\rangle^{m'}\langle\nabla_x\rangle^s G$, $(F_\pm^\varepsilon)^{(m',s')} = \langle v\rangle^{m'}\langle\nabla_x\rangle^s F_\pm^\varepsilon$, etc. The above gives

$$\frac{1}{2}\frac{d}{dt}\|G^{(m_0,s)}\|_{L^2_{x,v}}^2 = \langle[\langle v\rangle^{m_0}\langle\nabla_x\rangle^s, E^\varepsilon\cdot\nabla_v]G, G^{(m_0,s)}\rangle_{L^2_x} \tag{4.105}$$
$$+ \langle\langle v\rangle^{m_0}\langle\nabla_x\rangle^s\{(E^0 - E^\varepsilon)\cdot\nabla_v F_+^0\}, G^{(m_0,s)}\rangle_{L^2_{x,\xi}} \tag{4.106}$$
$$+ \langle\langle v\rangle^{m_0}\langle\nabla_x\rangle^s Q(F_+^\varepsilon, G), G^{(m_0,s)}\rangle_{L^2_{x,v}} \tag{4.107}$$
$$+ \langle\langle v\rangle^{m_0}\langle\nabla_x\rangle^s Q(G, F_+^0), G^{(m_0,s)}\rangle_{L^2_{x,v}} \tag{4.108}$$
$$+ \langle\langle v\rangle^{m_0}\langle\nabla_x\rangle^s Q_{-+}^\varepsilon(F_-^\varepsilon, F_+^\varepsilon), G^{(m_0,s)}\rangle_{L^2_{x,v}}. \tag{4.109}$$

We modify the estimates from Proposition 4.1 to get

$$|(4.105)| \lesssim \|G\|_{\mathfrak{E}'}(\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}), \tag{4.110}$$
$$|(4.107)| \le -\frac{1}{C}\|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}^2 + C(\langle\|F_+^\varepsilon\|_{\mathfrak{D}}\rangle^2\|G\|_{\mathfrak{E}'}^2 + \|G^{(m_1,0)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}^2), \tag{4.111}$$
$$|(4.108)| \lesssim \|G\|_{\mathfrak{E}'}^2 + \|G\|_{\mathfrak{E}'}^{\frac{1}{4}}\|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}^{\frac{7}{4}} + \langle\|F_+^0\|_{\mathfrak{D}}\rangle\|G\|_{\mathfrak{E}'}\|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.112}$$

What's left is to bound (4.106) and (4.109). For the former, we have

$$|(4.106)| \lesssim \|E^0 - E^\varepsilon\|_{H_x^s}(\|(F_+^0)^{(m_0,s)}\|_{L^2_{x,\xi}} + \|(F_+^0)^{(m_0+\frac{3}{2},s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v})\|G^{(m_0,s)}\|_{L^2_{x,v}} \tag{4.113}$$
$$\lesssim (\|G\|_{L^1_v H_x^{s-1}} + \|F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\|_{L^1_\xi H_x^{s-1}})\langle\|F_+^0\|_{\mathfrak{D}}\rangle \tag{4.114}$$
$$\lesssim (\|G\|_{\mathfrak{E}'} + \|F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\|_{\mathfrak{E}'})\langle\|F_+^0\|_{\mathfrak{D}}\rangle \tag{4.115}$$

In the above, we used the fact that $m_0 + \frac{3}{2} \le m_1$, and

$$-\Delta_x(\phi^\varepsilon - \phi^0) = \int_{\mathbf{R}^3} G dv - \int_{\mathbf{R}^3} F_-^\varepsilon - \mu_\beta e^{\beta\phi^0} d\xi. \tag{4.116}$$

As for (4.109), we split the term as follows:

$$(4.109) = \langle \langle v\rangle^{m_0} \langle \nabla_x\rangle^s Q^\varepsilon_{-+}(F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}, F^\varepsilon_+), G^{(m_0,s)}\rangle_{L^2_{x,v}} \tag{4.117}$$

$$+ \langle \langle v\rangle^{m_0} \langle \nabla_x\rangle^s Q^\varepsilon_{-+}(\mu_\beta e^{\beta\phi^0}, F^\varepsilon_+), G^{(m_0,s)}\rangle_{L^2_{x,v}} \tag{4.118}$$

Now,

$$|(4.117)| \lesssim \varepsilon \|\Delta_\xi^{-1}(F^\varepsilon_- - \mu_\beta e^{\beta\phi^0})\|_{L^\infty_\xi H^s_x} \|(F^\varepsilon_+)^{(m_0+\frac{3}{2},s)}\|_{L^2_x H^1_v} \|G^{(m_0-\frac{3}{2},s)}\|_{L^2_x H^1_v} \tag{4.119}$$

$$+ \|\nabla_\xi \Delta_\xi^{-1}(F^\varepsilon_- - \mu_\beta e^{\beta\phi^0})\|_{L^\infty_\xi H^s_x} \|(F^\varepsilon_+)^{(m_0+\frac{3}{2},s)}\|_{L^2_{x,v}} \|G^{(m_0-\frac{3}{2},s)}\|_{L^2_x H^1_v} \tag{4.120}$$

$$\lesssim \varepsilon \langle \|F^\varepsilon_+\|_{\mathfrak{D}}\rangle (\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}) \tag{4.121}$$

$$+ \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|^{\frac{1}{4}}_{\mathfrak{E}'} \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|^{\frac{3}{4}}_{\mathfrak{D}'} (\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}) \tag{4.122}$$

In the above, we use $m_0 + 3 \le m_1$. Next, in order to bound (4.118), we compute directly from (1.10) that

$$Q^\varepsilon_{-+}(\mu_\beta e^{\beta\phi^0}, F^\varepsilon_+) \tag{4.123}$$

$$= e^{\beta\phi^0} \nabla_v \cdot \int_{\mathbf{R}^3} \Phi(\xi - \varepsilon v)\{\varepsilon \mu_\beta(\xi)\nabla_v F^\varepsilon_+(v) + \beta\xi\mu_\beta(\xi) F^\varepsilon_+(v)\} d\xi \tag{4.124}$$

$$= \varepsilon e^{\beta\phi^0} \nabla_v \cdot \{\Phi * \mu_\beta(\varepsilon v)(\beta v + \nabla_v) F^\varepsilon_+(v)\}. \tag{4.125}$$

In the above, we use $\Phi(\xi - \varepsilon v)\xi = \varepsilon \Phi(\xi - \varepsilon v) v$. Now, $\|\Phi_{ij} * \mu_\beta(v)\|_{L^\infty} \lesssim 1$, so

$$|(4.118)| \lesssim \varepsilon (\|(F^\varepsilon_+)^{(m_0+\frac{5}{2},s)}\|_{L^2_{x,v}} \|G^{(m_0-\frac{3}{2},s)}\|_{L^2_x H^1_v}$$

$$+ \|(F^\varepsilon_+)^{(m_0+\frac{3}{2},s)}\|_{L^2_x H^1_v} \|G^{(m_0-\frac{3}{2},s)}\|_{L^2_x H^1_v}) \tag{4.126}$$

$$\lesssim \varepsilon \langle \|F^\varepsilon_+\|_{\mathfrak{D}}\rangle (\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\mathcal{H}_\sigma)_v}). \tag{4.127}$$

Thus,

$$|(4.109)| \lesssim (\varepsilon \langle \|F^\varepsilon_+\|_{\mathfrak{D}}\rangle + \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|^{\frac{1}{4}}_{\mathfrak{E}'} \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|^{\frac{3}{4}}_{\mathfrak{D}'}) \tag{4.128}$$

$$\cdot (\|G\|_{\mathfrak{E}'} + \|G^{(m_0,s)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}). \tag{4.129}$$

Combining the bounds on (4.105) through (4.109), we have

$$\frac{d}{dt}\|G^{(m_0,s)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|G^{(m_0,s)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.130}$$

$$\le C\{\langle \|(F^\varepsilon_+, F^0_+)\|_{\mathfrak{D}}\rangle^2 (\varepsilon^2 + \|G\|^2_{\mathfrak{E}'}) + \|G^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.131}$$

$$+ \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|^{\frac{1}{2}}_{\mathfrak{E}'} \|F^\varepsilon_- - \mu_\beta e^{\beta\phi^0}\|^{\frac{3}{2}}_{\mathfrak{D}'}\} \tag{4.132}$$

Following a similar argument, we can show the following upper bound on $G^{(m_1,0)}$:

$$\frac{d}{dt}\|G^{(m_1,0)}\|^2_{L^2_{x,v}} + \frac{1}{C}\|G^{(m_1,0)}\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma)_v} \tag{4.133}$$

$$\leq C(\langle \|(F_+^\varepsilon, F_+^0)\|_{\mathfrak{D}}\rangle^2(\varepsilon^2 + \|G\|_{\mathfrak{E}'}^2)$$
$$+ \|F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\|_{\mathfrak{E}'}^{\frac{1}{2}} \|F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\|_{\mathfrak{D}'}^{\frac{3}{2}}). \tag{4.134}$$

Now, choosing $0 < \kappa_1 < \kappa_2$ appropriately, we have that

$$\mathscr{Y}_+^\varepsilon = \kappa_2 \|G^{(m_1,0)}(t)\|_{L^2_{x,v}}^2 + \kappa_1 \|G^{(m_0,s)}(t)\|_{L^2_{x,v}}^2 \tag{4.135}$$

saqtisfies

$$\frac{d}{dt}\mathscr{Y}_+^\varepsilon + \|G\|_{\mathfrak{D}'}^2 \tag{4.136}$$
$$\lesssim \langle \|(F_+^\varepsilon, F_+^0)\|_{\mathfrak{D}}\rangle^2(\varepsilon^2 + \|G\|_{\mathfrak{E}'}^2) + \|F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\|_{\mathfrak{E}'}^{\frac{1}{2}} \|F_-^\varepsilon - \mu_\beta e^{\beta\phi^0}\|_{\mathfrak{D}'}^{\frac{3}{2}}. \tag{4.137}$$

From this, we deduce (4.102). □

4.3. The intermediary quantities. In preparation for Sect. 5, we introduce the what we call the intermediary potential ψ^ε and intermediary inverse temperature γ^ε. Fixing an initial inverse temperature β_{in}, the pair $(\gamma^\varepsilon, \psi^\varepsilon)$ solve the Poincaré-Poisson system for each $\varepsilon > 0$:

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{3}{2\gamma^\varepsilon} + \iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2} F_+^\varepsilon dx dv + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \psi^\varepsilon|^2 dx\right\} = 0,\\ &\gamma^\varepsilon(0) = \beta_{in}\\ &-\Delta_x \psi^\varepsilon = 4\pi(n_+^\varepsilon - e^{\gamma^\varepsilon\psi^\varepsilon}) \end{aligned} \tag{4.138}$$

We call ψ^ε the intermediary potential because like ϕ^0, it solves a Poincaré-Poisson system; however, unlike ϕ^0, we use F_+^ε instead of F_+^0. Thus ψ serves as a better approximation to ϕ^ε than ϕ^0. Similarly, $\frac{3}{2\gamma^\varepsilon}$ serves as a better approximation of $\iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|\xi|^2}{2} F_-^\varepsilon dx dv$ than $\frac{3}{2\beta}$. The lemma below gives existence and uniqueness for the pair $(\gamma^\varepsilon, \psi^\varepsilon)$, for a given F_+^ε, along with bounds that will be used in the coming section.

Lemma 4.3. *Let $M, T, \eta \in (0, 1]$ and $\beta_{in} \in (e^{-M}, e^M)$. Assume also that $(F_+^\varepsilon, F_-^\varepsilon)$ is a weak solution to (1.7) with $0 \leq F_\pm^\varepsilon \in C([0, T]; \mathfrak{E}) \cap L^2([0, T]; \mathfrak{D})$ satisfying*

$$\sup_{\pm\in\{-,+\}, t\in[0,T]} \left\|\frac{1}{n_\pm^\varepsilon(t)}\right\|_{L_x^\infty} + \|F_\pm^\varepsilon(t)\|_{\mathfrak{E}} \leq M. \tag{4.139}$$

Then there exists $0 < T^*$ such that there exists a unique solution $(\gamma^\varepsilon, \psi^\varepsilon) \in C([0, T] \cap [0, T^*); \mathbf{R}_+ \times H^{s+2})$ to (4.138). Moreover, if $T^* \leq T$, then $\beta(t) \uparrow +\infty$ as $t \uparrow T^*$.

Moreover, assume that for some $T' \in (0, T]$, that

$$\sup_{t\in[0,T']} |\ln(\frac{\gamma^\varepsilon(t)}{\beta_{in}})| \leq \eta, . \tag{4.140}$$

Note that $T < T^*$ necessarily. Then, we have the estimates

$$\sup_{t\in[0,T']} \|\psi^\varepsilon\|_{H_x^{s+2}} \lesssim_M 1, \tag{4.141}$$

$$\sup_{t\in[0,T']} \|\partial_t \psi^\varepsilon\|_{H_x^{s+1}} + |\dot{\gamma}^\varepsilon| \lesssim_M \langle \|\langle\xi\rangle^5 F_-^\varepsilon\|_{L_x^2(\dot{\mathcal{H}}_\sigma)_\xi}\rangle, \tag{4.142}$$

Next, assume (F_+^0, β, ϕ^0) *is a weak solution to* (1.18) *and satisfies the hypotheses of Theorem* 1.1 *(with* $T_0 = T$*), with the given* β_{in}*. Then, we also control the difference of* $(\gamma^\varepsilon, \psi^\varepsilon)$ *and* (β, ϕ^0)*:*

$$\sup_{t\in[0,T']} \{|\gamma^\varepsilon(t) - \beta(t)| + \|\psi^\varepsilon(t) - \phi^0(t)\|_{H_x^{s+1}}\} \lesssim \sup_{t\in[0,T]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'}. \tag{4.143}$$

Proof. We break the proof up into 3 parts: first, the bounds (4.141), and (4.142); second, the bound (4.143); and third, we sketch how to construct the solutions. Once again, all bounds involved may depend on M.

Step 1: We now show (4.141), and (4.142). For this step, it is convenient to drop the superscripts of ε, i.e. $F_+^\varepsilon = F_+$, $F_-^\varepsilon = F_-$, etc. We now prove (4.141), from the third line of (4.138), and the maximum principle, we have the estimate

$$\|\psi\|_{L_x^\infty} \lesssim \frac{1}{\gamma}\|\ln(n_+)\|_{L_x^\infty} \lesssim 1. \tag{4.144}$$

Next, applying $|\nabla_x|^s$ to both sides of the third line of (4.138),

$$\frac{1}{4\pi}|\nabla_x|^{s'+2}\psi = |\nabla_x|^{s'}\{n_+ - e^{\gamma\psi}\}. \tag{4.145}$$

Using Theorem 5.2.6 in [32],

$$\|e^{\gamma\psi}\|_{H_x^s} \le C_{\|\psi\|_{L_x^\infty}}(\|\psi\|_{H_x^s} + 1) \lesssim \|\psi\|_{H_x^s} + 1. \tag{4.146}$$

Hence,

$$\|\psi\|_{H_x^{s+2}} \lesssim 1 + \|n_+\|_{H_x^s} + \|\psi\|_{H_x^s} \lesssim 1 + \|\psi\|_{H_x^{s+2}}^{\frac{s}{s+2}}\|\psi\|_{L_x^2}^{\frac{2}{s+2}}. \tag{4.147}$$

Now, using $\|\psi\|_{L_x^2} \lesssim \|\psi\|_{L_x^\infty} \lesssim 1$ and Young's inequality, we deduce $\|\psi\|_{H_x^{s+2}} \lesssim 1$.

Next, we have

$$(\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)\partial_t\psi = \partial_t n_+ - \dot{\gamma}e^{\gamma\psi}\psi. \tag{4.148}$$

The maximum principle implies

$$e^{-C\|\psi\|_{L_x^\infty}}\|\partial_t\psi\|_{L_x^\infty} \le C\|\partial_t n_+\|_{L_x^\infty} + |\dot{\gamma}|e^{C\|\psi\|_{L_x^\infty}}\|\psi\|_{L_x^\infty}, \tag{4.149}$$

From the continuity equation, we know $\|\partial_t n_+\|_{L_x^\infty} \lesssim \|\partial_t n_+\|_{H_x^{s-1}} \lesssim 1$; in combination with the estimates on ψ, this implies

$$\|\partial_t\psi\|_{L_x^\infty} \lesssim 1 + |\dot{\gamma}|. \tag{4.150}$$

Similarly as was done to get the bound on $\|\psi\|_{H_x^{s+2}}$, we deduce

$$\|\partial_t\psi\|_{H_x^{s+1}} \lesssim 1 + |\dot{\gamma}|. \tag{4.151}$$

We now estimate $\dot{\gamma}$: first,

$$\frac{3\dot{\gamma}}{2\gamma^2} = \frac{1}{2}\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 \partial_t F_+ dx dv + \frac{1}{4\pi}\int_{\mathbf{T}^3} \nabla_x\psi \cdot \partial_t \nabla_x \psi dx. \tag{4.152}$$

We first analyze the last term. From (4.148),

$$\frac{1}{4\pi}\int_{\mathbf{T}^3} \nabla_x\psi \cdot \partial_t\nabla_x\psi dx = -\frac{1}{4\pi}\langle \Delta_x\psi, \partial_t\psi\rangle_{L^2_x} \tag{4.153}$$

$$= -\frac{1}{4\pi}\langle \Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}\partial_t n_+\rangle_{L^2_x} \tag{4.154}$$

$$+ \frac{\dot{\gamma}}{4\pi}\langle \Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} \tag{4.155}$$

Looking more closely at the latter term, we observe

$$\frac{1}{4\pi}\langle \Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} \tag{4.156}$$

$$= -\langle(\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} \tag{4.157}$$

$$+ \gamma\langle e^{\gamma\psi}\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} \tag{4.158}$$

$$= -\langle \psi, e^{\gamma\psi}\psi\rangle_{L^2_x} - \gamma\langle e^{\gamma\psi}\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x} \leq 0. \tag{4.159}$$

Thus, we have that

$$\dot{\gamma} = \frac{\frac{1}{2}\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2\partial_t F_+ dx dv - \frac{1}{4\pi}\langle \Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}\partial_t n_+\rangle_{L^2_x}}{\frac{3}{2\gamma^2} - \frac{1}{4\pi}\langle \Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma\psi}\psi)\rangle_{L^2_x}}, \tag{4.160}$$

with the denominator being strictly bounded from below by $\frac{3}{2\gamma^2}$. On the other hand, by the continuity equation $\partial_t n_+ + \int_{\mathbf{R}^3} v \cdot \nabla_x F_+ dx dv = 0$, we have

$$|\langle \Delta_x\psi, (\gamma e^{\gamma\psi} - \frac{1}{4\pi}\Delta_x)^{-1}\partial_t n_+\rangle_{L^2_x}| \lesssim 1. \tag{4.161}$$

Thus,

$$|\dot{\gamma}| \lesssim 1 + |\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2\partial_t F_+ dx dv|. \tag{4.162}$$

Now,

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2\partial_t F_+ dx dv = 2\iint_{\mathbf{T}^3\times\mathbf{R}^3} v\cdot E F_+ dx dv + \iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 Q^{\varepsilon}_{-+}(F_-, F_+) dx dv \tag{4.163}$$

The first term has the bound

$$2\iint_{\mathbf{T}^3\times\mathbf{R}^3} v\cdot E F_+ dx dv \lesssim \|E\|_{L^2_x} \||v|F_+\|_{L^2_x L^1_v} \tag{4.164}$$

$$\lesssim (\|\langle\xi\rangle^2 F_-\|_{L^2_{x,v}} + \|\langle v\rangle^2 F_+\|_{L^2_{x,v}})\|\langle v\rangle^3 F_+\|_{L^2_{x,v}} \tag{4.165}$$

$$\lesssim 1. \tag{4.166}$$

Next,

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 Q^\varepsilon_{-+}(F_-, F_+)dxdv \tag{4.167}$$

$$= -2\iint_{\mathbf{T}^3\times\mathbf{R}^3} v_j\{\varepsilon\Phi_{ij} *_\xi F_-|_{\xi=\varepsilon v}\partial_{v_i}F_+ - \partial_{\xi_i}\Phi_{ij} *_\xi F_-|_{\xi=\varepsilon v}F_+\}dxdv \tag{4.168}$$

$$= 2\iint_{\mathbf{T}^3\times\mathbf{R}^3} \{\varepsilon\,\mathrm{tr}(\Phi *_\xi F_-|_{\xi=\varepsilon v}) + (1+\varepsilon^2)v_j\partial_{\xi_i}\Phi_{ij} *_\xi F_-|_{\xi=\varepsilon v}\}F_+dxdv \tag{4.169}$$

Therefore,

$$|\iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 Q^\varepsilon_{-+}(F_-, F_+)dxdv| \tag{4.170}$$

$$\lesssim (\|(-\Delta_\xi)^{-1}F_-\|_{L^2_xL^\infty_\xi} + \|\nabla_\xi(-\Delta_\xi)^{-1}F_-\|_{L^2_xL^\infty_\xi})\|vF_+\|_{L^2_xL^1_v} \tag{4.171}$$

$$\lesssim \|\langle\xi\rangle^2\langle\nabla_v\rangle F_-\|_{L^2_{x,v}}\|\langle v\rangle^3F_+\|_{L^2_{x,v}} \tag{4.172}$$

$$\lesssim \langle\|\langle\xi\rangle^5F_-\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_\xi}\rangle\|\langle v\rangle^3F_+\|_{L^2_{x,v}} \tag{4.173}$$

$$\lesssim \langle\|\langle\xi\rangle^5F_-\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_\xi}\rangle. \tag{4.174}$$

From this, we conclude (4.142).

Part 2: We now control the difference (4.143). reintroduce the superscripts of ε, so as to distinguish F^ε_+ from F^0_+, etc, although we still ignore the dependence of constants on M. We first bound the difference $\psi^\varepsilon_{in} - \phi^0_{in}$. It suffices to show this in the case

$$\sup_{t\in[0,T]} \|F^\varepsilon_+(t) - F^0_+(t)\|_{\mathfrak{E}'} \le \alpha, \tag{4.175}$$

where $\alpha = \alpha(M)$ is sufficiently small. If this bound does not hold, then we use the bound

$$\sup_{t\in[0,T]} |\gamma^\varepsilon(t) - \beta(t)| + \|\psi^\varepsilon(t) - \phi^0(t)\|_{H^{s+1}_x} \le C_M \tag{4.176}$$

$$\le \frac{C_M}{\alpha}\sup_{t\in[0,T]}\|F^\varepsilon_+(t) - F^0_+(t)\|_{\mathfrak{E}'}. \tag{4.177}$$

We first control the difference at time zero:

$$-\frac{1}{4\pi}\Delta_x(\psi^\varepsilon_{in} - \phi^0_{in}) = n^\varepsilon_{+,in} - n^0_{+,in} + e^{\beta_{in}\phi^0_{in}} - e^{\beta_{in}\psi^\varepsilon_{in}}. \tag{4.178}$$

Using the maximum principle, we have

$$\|e^{\beta_{in}\psi^\varepsilon_{in}} - e^{\beta_{in}\phi^0_{in}}\|_{L^\infty_x} \lesssim \|n^\varepsilon_{+,in} - n^0_{+,in}\|_{L^\infty_x} \lesssim \sup_{t\in[0,T]}\|F^\varepsilon_+(t) - F^0_+(t)\|_{\mathfrak{E}'}. \tag{4.179}$$

By the mean-value theorem, this implies $\|\psi^\varepsilon_{in} - \phi^0_{in}\|_{L^\infty_x} \lesssim \sup_{t\in[0,T]}\|F^\varepsilon_+(t) - F^0_+(t)\|_{\mathfrak{E}'}$. In fact, by writing

$$e^{\beta_{in}\phi^0} - e^{\beta_{in}\psi^\varepsilon_{in}} = \beta_{in}\int_0^1 e^{\lambda\phi^0 + (1-\lambda)\psi^\varepsilon}(\phi^0 - \psi^\varepsilon)d\lambda, \tag{4.180}$$

it is easy to show that after applying $\langle \nabla_x \rangle^s$ to (4.178), we get

$$\|\psi_{in}^{\varepsilon} - \phi_{in}^{0}\|_{H_x^{s+2}} \lesssim \|n_{+,in}^{\varepsilon} - n_{+,in}^{0}\|_{H_x^s} + \|\psi_{in}^{\varepsilon} - \phi_{in}^{0}\|_{H_x^s}. \tag{4.181}$$

Thus, by interpolation, we get

$$\|\psi_{in}^{\varepsilon} - \phi_{in}^{0}\|_{H_x^{s+2}} \lesssim \sup_{t\in[0,T]} \|F_+^{\varepsilon}(t) - F_+^{0}(t)\|_{\mathfrak{E}'}. \tag{4.182}$$

Now, we show how to bound the differences for positive times. Take $\kappa \in (0, 1]$ to be a constant to be determined later, and let $\tilde{T}$ be the longest time such that $\tilde{T} \leq T$ and

$$\sup_{t\in[0,\tilde{T}]} |\beta - \gamma^{\varepsilon}| + \|\psi^{\varepsilon} - \phi^{0}\|_{H_x^{s+2}} \leq \kappa. \tag{4.183}$$

The equation we have for the difference $\psi^{\varepsilon} - \phi^0$ is

$$-\Delta_x(\psi^{\varepsilon} - \phi^0) = 4\pi(n_+^{\varepsilon} - n_+^0 + e^{\beta\phi^0} - e^{\gamma^{\varepsilon}\psi^{\varepsilon}}). \tag{4.184}$$

Rearranging, we have

$$(\gamma^{\varepsilon}e^{\gamma^{\varepsilon}\psi^{\varepsilon}} - \frac{1}{4\pi}\Delta_x)(\psi^{\varepsilon} - \phi^0) \tag{4.185}$$

$$= n_+^{\varepsilon} - n_+^0 + e^{\gamma^{\varepsilon}\psi^{\varepsilon}}(e^{\beta\phi^0 - \gamma^{\varepsilon}\psi^{\varepsilon}} - 1 - (\beta\phi^0 - \gamma\psi^{\varepsilon}) + (\beta - \gamma^{\varepsilon})(\phi^0 - \psi^{\varepsilon})) \tag{4.186}$$

$$+ (\beta - \gamma^{\varepsilon})e^{\gamma^{\varepsilon}\psi^{\varepsilon}}\psi^{\varepsilon}. \tag{4.187}$$

Then, on the interval $t \in [0, \tilde{T}_{\varepsilon}]$, we have

$$\begin{aligned} &\|\psi^{\varepsilon} - \phi^0 - (\beta - \gamma^{\varepsilon})(\gamma^{\varepsilon}e^{\gamma^{\varepsilon}\psi^{\varepsilon}} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^{\varepsilon}\psi^{\varepsilon}}\psi^{\varepsilon})\|_{H_x^{s+2}} \\ &\quad\lesssim \|F_+^{\varepsilon} - F_+^0\|_{\mathfrak{E}'} + \kappa(|\gamma^{\varepsilon} - \beta| + \|\psi^{\varepsilon} - \phi^0\|_{H_x^{s+2}}). \end{aligned} \tag{4.188}$$

In particular, taking κ small enough,

$$\|\psi^{\varepsilon} - \phi^0\|_{H_x^{s+2}} \lesssim \sup_{t\in[0,T]} \|F_+^{\varepsilon}(t) - F_+^0(t)\|_{\mathfrak{E}'} + |\gamma^{\varepsilon} - \beta|. \tag{4.189}$$

Now,

$$\Big|\frac{1}{8\pi}\int_{\mathbf{T}^3} \nabla_x(\psi^{\varepsilon} - \phi^0)\cdot\nabla_x(\psi^{\varepsilon} + \phi^0)dx \tag{4.190}$$

$$+ \frac{1}{4\pi}\int_{\mathbf{T}^3} (\beta - \gamma^{\varepsilon})(\gamma^{\varepsilon}e^{\gamma^{\varepsilon}\psi^{\varepsilon}} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^{\varepsilon}\psi^{\varepsilon}}\psi^{\varepsilon})\Delta_x\psi^{\varepsilon}dx\Big| \tag{4.191}$$

$$\lesssim \Big|\int_{\mathbf{T}^3} \{(\psi^{\varepsilon} - \phi^0) - (\beta - \gamma^{\varepsilon})(\gamma^{\varepsilon}e^{\gamma^{\varepsilon}\psi^{\varepsilon}} - \frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^{\varepsilon}\psi^{\varepsilon}}\psi^{\varepsilon})\}\Delta_x\psi^{\varepsilon}dx\Big| \tag{4.192}$$

$$+ \int_{\mathbf{T}^3} |\nabla_x(\psi^{\varepsilon} - \phi^0)|^2 dx \tag{4.193}$$

$$\lesssim \sup_{t\in[0,T]} \|F_+^{\varepsilon}(t) - F_+^0(t)\|_{\mathfrak{E}'} + |\gamma^{\varepsilon} - \beta| \tag{4.194}$$

Next,

$$\frac{d}{dt}\left(\frac{3}{2\beta}-\frac{3}{2\gamma^{\varepsilon}}\right) \tag{4.195}$$

$$=\frac{d}{dt}\left(\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(F_+^{\varepsilon}-F_+^0)dxdv+\frac{1}{8\pi}\int_{\mathbf{T}^3}\nabla_x(\psi^{\varepsilon}-\phi^0)\cdot\nabla_x(\psi^{\varepsilon}+\phi^0)dx\right) \tag{4.196}$$

Then, using $\beta_{in}=\gamma(t=0)^{\varepsilon}$ and integrating the above, we have

$$\frac{3(\gamma^{\varepsilon}-\beta)}{2\beta\gamma^{\varepsilon}}=\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(F_+^{\varepsilon}-F_+^0)dxdv \tag{4.197}$$

$$+\frac{1}{8\pi}\int_{\mathbf{T}^3}\nabla_x(\psi^{\varepsilon}-\phi^0)\cdot\nabla_x(\psi^{\varepsilon}+\phi^0)dx \tag{4.198}$$

$$-\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(F_{+,in}^{\varepsilon}-F_{+,in}^0)dxdv \tag{4.199}$$

$$-\frac{1}{8\pi}\int_{\mathbf{T}^3}\nabla_x(\psi_{in}^{\varepsilon}-\phi_{in}^0)\cdot\nabla_x(\psi_{in}^{\varepsilon}+\phi_{in}^0)dx. \tag{4.200}$$

Using the previous bounds, plus (4.182), we have

$$|\gamma^{\varepsilon}-\beta|\left|\frac{3}{2\beta\gamma^{\varepsilon}}-\frac{1}{4\pi}\int_{\mathbf{T}^3}(\gamma^{\varepsilon}e^{\gamma^{\varepsilon}\psi^{\varepsilon}}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^{\varepsilon}\psi^{\varepsilon}}\psi^{\varepsilon})\Delta_x\psi^{\varepsilon}dx\right| \tag{4.201}$$

$$\lesssim|\gamma^{\varepsilon}-\beta|^2+\sup_{t\in[0,T]}\|F_+^{\varepsilon}(t)-F_+^0(t)\|_{\mathfrak{E}'}. \tag{4.202}$$

By a similar argument as was used to bound $\dot{\gamma}$, we have that

$$\int_{\mathbf{T}^3}(\gamma^{\varepsilon}e^{\gamma^{\varepsilon}\psi^{\varepsilon}}-\frac{1}{4\pi}\Delta_x)^{-1}(e^{\gamma^{\varepsilon}\psi^{\varepsilon}}\psi^{\varepsilon})\Delta_x\psi^{\varepsilon}dx\le 0. \tag{4.203}$$

We conclude that for all $t\in[0,\tilde{T}]$,

$$|\gamma^{\varepsilon}(t)-\beta(t)|\lesssim\kappa|\gamma^{\varepsilon}(t)-\beta(t)|+\sup_{t\in[0,T]}\|F_+^{\varepsilon}(t)-F_+^0(t)\|_{\mathfrak{E}'} \tag{4.204}$$

Taking κ sufficiently small, we have $|\gamma^{\varepsilon}(t)-\beta(t)|\le C_0\alpha$ for some C_0. So, by taking $\alpha=\frac{\kappa}{2C_0}$, we have $\tilde{T}=T$.

Step 3: Finally, we sketch how to construct solutions $(\gamma^{\varepsilon},\psi^{\varepsilon})$ satisfying (4.138). We refer the reader (ii) of Theorem 1.4 in [3] and its proof. Using standard elliptic theory, there exists a unique $\psi_{in}^{\varepsilon}\in H^{s+2}$ solving the system

$$-\Delta_x\psi_{in}^{\varepsilon}=4\pi(n_{+,in}^{\varepsilon}-e^{\beta_{in}\psi_{in}^{\varepsilon}}). \tag{4.205}$$

Now, for $t>0$, define

$$E(t):=\frac{3}{2\beta_{in}}+\iint_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|v|^2}{2}(F_{+,in}^{\varepsilon}-F_+^{\varepsilon}(t))dxdv+\frac{1}{8\pi}\int_{\mathbf{T}^3}|\nabla_x\psi_{in}^{\varepsilon}|^2dx. \tag{4.206}$$

Then, for all $t > 0$ such that $E(t) > 0$, there exists $(\gamma(t), \psi(t)) \in \mathbf{R}_+ \times H^{s+2}$ solving third line of (4.138), and the first line integrated on $[0, t]$. Since $t \mapsto \iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2} F_+^\varepsilon(t)\, dxdv$ is continuous, and the above condition holds at $t = 0$, we take $T^* > 0$ to be the first time less that T such that the above condition does not hold, or take $T^* = T$ if no such time exists. Then there exists a unique (γ, ψ) satisfying (4.138), with $\ln(\gamma) \in L^\infty_{loc}([0, T^*))$ and $\psi \in L^\infty_{loc}([0, T^*); H^{s+2})$. Moreover, $\frac{3}{2\gamma^\varepsilon(t)} \leq E(t)$, so $\gamma^\varepsilon(t) \to \infty$ as $t \to T^*$ whenever $T^* \leq T$.

Next, we note that γ is continuous in time. This follows by the identity (4.160), and the fact that

$$\|\partial_t n_+^\varepsilon(t)\|_{L_x^2}, \iint_{\mathbf{T}^3\times\mathbf{R}^3} \frac{|v|^2}{2} \partial_t F_+^\varepsilon(t, x, v)dxdv \in L^1([0, T]), \tag{4.207}$$

also shown above. It is straightforward to show $\psi \in C([0, T^*); H^{s+2})$ from this, and (4.148).

□

5. Estimates on the Electron Distribution

5.1. Stability of the Maxwellian. The main result of this section is Proposition 5.1, which, roughly speaking, shows that the Maxwellian $\mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon}$ is stable; that is, if $F^\varepsilon_{-,in}$ is close to $\mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon}$, then it will remain close. We also utilize the stretched exponential decay to acquire some form of asymptotic stability.

To state the result, let $\eta > 0$, and recall $q_\alpha = e^{\alpha\eta}\beta_{in}$ for each $\alpha \in \{1, 2, 3\}$. Then, similarly to $\mathscr{E}_{-,\alpha}$ and $\mathscr{D}_{-,\alpha}$, we define for each $\alpha \in \{1, 2\}$,

$$\begin{aligned} \widetilde{\mathscr{E}}^\varepsilon_{-,\alpha} &:= \|e^{q_\alpha|\xi|^2/4}\langle\nabla_x\rangle^s\{F_-^\varepsilon - \mu_{\gamma^\varepsilon\psi^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon}\}\|^2_{L^2_{x,\xi}} \\ &\quad + \|e^{q_{\alpha+1}|\xi|^2/4}\{F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon}\}\|^2_{L^2_{x,\xi}} \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \widetilde{\mathscr{D}}^\varepsilon_{-,\alpha} &:= \|e^{q_\alpha|\xi|^2/4}\langle\nabla_x\rangle^s\{F_-^\varepsilon - \mu_{\gamma^\varepsilon\psi} e^{\gamma^\varepsilon\psi^\varepsilon}\}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \\ &\quad + \|e^{q_{\alpha+1}|\xi|^2/4}\{F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon}\}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi}. \end{aligned} \tag{5.2}$$

Proposition 5.1. *Fix $M > 0$. Then there exists positive constants η, ς and $\overline{\varepsilon}$ such that the following holds. Assume $0 < \varepsilon \leq \overline{\varepsilon}$. For each such ε, let $(F_+^\varepsilon, F_-^\varepsilon)$ be solution to the system (1.7) with $0 \leq F_\pm^\varepsilon \in C([0, T]; \mathfrak{E}) \cap L^2([0, T]; \mathfrak{D})$. Fixing $\beta_{in} \in (e^{-M}, e^M)$, let $(\psi^\varepsilon, \gamma^\varepsilon)$ be the corresponding solution to the system (4.138), assuming that*

$$\sup_{t\in[0,T]} (\|\frac{1}{n_+^\varepsilon(t)}\|_{L^\infty_x} + \|F_+^\varepsilon(t)\|_{\mathfrak{E}}) \leq M, \tag{5.3}$$

$$\sup_{t\in[0,T]} \sqrt{\widetilde{\mathscr{E}}^\varepsilon_{-,2}(t)} \leq \varsigma, \tag{5.4}$$

$$\sup_{t\in[0,T]} |\ln(\frac{\gamma^\varepsilon(t)}{\beta_{in}})| \leq \eta, \tag{5.5}$$

$$\int_0^T \|\partial_t(n_-^\varepsilon - e^{\gamma^\varepsilon\psi^\varepsilon})\|_{L_x^2} dt \le 1. \tag{5.6}$$

We also impose the following assumption on the initial data:

$$\left|\int_{\mathbf{R}^3\times\mathbf{T}^3} \frac{|\xi|^2}{2}(F_{-,in}^\varepsilon - \mu_{\beta_{in}} e^{\beta_{in}\psi_{in}^\varepsilon})dxd\xi + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x\phi_{in}^\varepsilon|^2 - |\nabla_x\psi_{in}^\varepsilon|^2 dx\right| \le M\varepsilon. \tag{5.7}$$

Then, we have

$$\varepsilon \sup_{t\in[0,T]} \widetilde{\mathscr{E}}_{-,2}^\varepsilon(t) + \int_0^T \widetilde{\mathscr{D}}_{-,2}^\varepsilon(t)dt \le C_M(\varepsilon\widetilde{\mathscr{E}}_{-,2,in}^\varepsilon + \varepsilon^2) \tag{5.8}$$

and for all $t \in [0,T]$, we have

$$\mathscr{E}_{-,1}^\varepsilon(t) \le C_M(e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}} \sup_{t'\in[0,T]} \mathscr{E}_{-,2}^\varepsilon(t') + \varepsilon^{\frac{5}{3}}t^{\frac{1}{3}}) \tag{5.9}$$

Throughout the rest of this section, we will not make reference to (F_+^0, β, ϕ^0) and its derived quantities, and instead only work with $(F_-^\varepsilon, F_+^\varepsilon)$. With this understood, throughout this section, we will write $F_+^\varepsilon = F_+$, $F_-^\varepsilon = F_-$, and so on, with the dependence on ε being implicit.

5.2. Setup. We introduce $a = n_- - e^{\gamma\psi}$ and note that

$$4\pi a = -\Delta_x(\phi - \psi), \tag{5.10}$$

Using (5.5), we note

$$\gamma(t) \le q_1 < q_2 < q_3. \tag{5.11}$$

Next, we separate

$$F_- - \mu_\gamma e^{\gamma\psi} = \sqrt{\mu_\gamma} f = \sqrt{\mu_{q_\alpha} e^{q_\alpha\phi}} g_\alpha \tag{5.12}$$

where $\alpha \in \{1,2,3\}$.

The equation for f reads as follows

$$\begin{aligned} &\varepsilon\mu_\gamma^{-\frac{1}{2}}\partial_t\{\mu_\gamma^{\frac{1}{2}} f\} + \{\xi\cdot\nabla_x + E\cdot(\frac{\xi}{2} - \nabla_\xi)\}f + e^{\gamma\psi}\mathcal{L}_\gamma f + \mathcal{M}_{\gamma,F_+} f \\ &\quad - 4\pi\gamma\xi\cdot\nabla_x\Delta_x^{-1} a\mu_\gamma^{\frac{1}{2}} e^{\gamma\psi} \\ &= -\varepsilon\mu_\gamma^{\frac{1}{2}}\partial_t\{\mu_\gamma e^{\gamma\psi}\} - e^{\gamma\psi}\mathcal{M}_{\gamma,F_+}\mu_\gamma^{\frac{1}{2}} + \Gamma_\gamma(f,f) \end{aligned} \tag{5.13}$$

where

$$\mathcal{L}_\gamma h = \mu_\gamma^{-\frac{1}{2}}\{Q(\mu_\gamma^{\frac{1}{2}}h, \mu_\gamma) + Q(\mu_\gamma, \mu_\gamma^{\frac{1}{2}}h)\}, \tag{5.14}$$

$$\mathcal{M}_{\gamma,F_+} h = \mu_\gamma^{-\frac{1}{2}} Q_{+-}^\varepsilon(F_+, \mu_\gamma^{\frac{1}{2}}h), \tag{5.15}$$

$$\Gamma_\gamma(h_1,h_2) = -\mu_\gamma^{-\frac{1}{2}} Q(\mu_\gamma^{\frac{1}{2}}h_1, \mu_\gamma^{\frac{1}{2}}h_2). \tag{5.16}$$

On the other hand, g_α, $\alpha \in \{1,2,3\}$ solve the equation

$$\begin{aligned}
&\varepsilon\{\partial_t + q\partial_t\phi\}g_\alpha + \{v\cdot\nabla_x - E\cdot\nabla_\xi\}g_\alpha + e^{\gamma\psi}\widetilde{\mathcal{L}}_{\gamma,q}g_\alpha + \mathcal{M}_{q_\alpha,F_+}g_\alpha \\
&\quad - 4\pi\gamma\xi\cdot\nabla_x\Delta_x^{-1}a\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}e^{\gamma\psi-\frac{q_\alpha}{2}\phi} \\
&= -\varepsilon\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\psi}}} - e^{\gamma\psi-\frac{q_\alpha}{2}\phi}\mathcal{M}_{q_\alpha,F_+}(\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma) + e^{\frac{q_\alpha}{2}\phi}\Gamma_{q_\alpha}(g_\alpha,g_\alpha)
\end{aligned} \tag{5.17}$$

where $\mathcal{M}_{q_\alpha,F_+}$ and Γ_{q_α} are defined the same way as $\mathcal{M}_{\gamma,F_+}$ and Γ_γ respectively, with γ substituted with q_α. The operator $\widetilde{\mathcal{L}}_{\gamma,q}$ is defined as follows.

$$\widetilde{\mathcal{L}}_{\gamma,q}\psi = \mu_q^{-\frac{1}{2}}\{Q(\mu_q^{\frac{1}{2}}\psi,\mu_\gamma) + Q(\mu_\gamma,\mu_q^{\frac{1}{2}}\psi)\} \tag{5.18}$$

Now, in addition, we define the operator $\mathcal{P}_\gamma$ to be the L^2-projection (in the ξ variable) to the kernel of $\mathcal{L}_\gamma$, and we take $\mathcal{P}_\gamma^\perp = \mathrm{Id}_{L^2(\mathbf{R}^3)} - \mathcal{P}_\gamma$. From [21], we have that this kernel is

$$N(\mathcal{L}_\gamma) = \mathrm{span}\{\mu_\gamma^{\frac{1}{2}}(\xi), w_1\mu_\gamma^{\frac{1}{2}}, \xi_2\mu_\gamma^{\frac{1}{2}}, \xi_3\mu_\gamma^{\frac{1}{2}}, |\xi|^2\mu_\gamma^{\frac{1}{2}}\}. \tag{5.19}$$

In addition to the density of the perturbation a already defined, we define the macroscopic variables $c : \mathbf{T}^3 \to \mathbf{R}$ and $b : \mathbf{T}^3 \to \mathbf{R}^3$ as the coefficients of this projection on f:

$$\mathcal{P}_\gamma f = a\mu_\gamma^{\frac{1}{2}} + \gamma^{\frac{1}{2}}b\cdot w\mu_\gamma^{\frac{1}{2}} + c\frac{\gamma|\xi|^2-3}{\sqrt{6}}\mu_\gamma^{\frac{1}{2}} \tag{5.20}$$

We note that the representation above is in terms of an orthonormal basis for $N(\mathcal{L}_\gamma)$. Moreover,

$$a = \int_{\mathbf{R}^3}\{F_- - \mu_\gamma e^{\gamma\psi}\}d\xi = n_- - e^{\gamma\psi} \tag{5.21}$$

$$b = \int_{\mathbf{R}^3}\gamma^{\frac{1}{2}}\xi\{F_- - \mu_\gamma e^{\gamma\psi}\}d\xi = \gamma^{\frac{1}{2}}\int_{\mathbf{R}^3}\xi F_- d\xi \tag{5.22}$$

$$c = \int_{\mathbf{R}^3}\frac{(\gamma|\xi|^2-3)}{\sqrt{6}}\{F_- - \mu_\gamma e^{\gamma\psi}\}d\xi = \frac{1}{\sqrt{6}}\left(\gamma\int_{\mathbf{R}^3}|\xi|^2F_-d\xi - 3n_-\right). \tag{5.23}$$

Thus, (a,b,c) form a linear transformation of the physical macroscopic variables of density, current and local kinetic energy of the electrons. Given $r \geq 0$, we shall denote $a^{(r)} = \langle\nabla_x\rangle^r a$, $g_\alpha^{(r)} = \langle\nabla_x\rangle^r g_\alpha$, and so forth.

Finally, we close this section by mentioning some commonly occurring estimates that will occur throughout this section. We note that under the bootstrap assumptions, $\|\phi\|_{H_x^{s+2}} + \|\psi\|_{H_x^{s+2}} \lesssim_M 1$. Therefore,

$$\|f\|_{L^2_{x,\xi}} \lesssim_M \|g_1\|_{L^2_{x,\xi}} \lesssim_M \|g_2\|_{L^2_{x,\xi}} \lesssim_M \|g_3\|_{L^2_{x,\xi}} \tag{5.24}$$

$$\|f^{(s)}\|_{L^2_{x,\xi}} \lesssim_M \|g_1^{(s)}\|_{L^2_{x,\xi}} \lesssim_M \|g_2^{(s)}\|_{L^2_{x,\xi}} \lesssim_M \|g_3^{(s)}\|_{L^2_{x,\xi}} \tag{5.25}$$

The same inequalities hold over the spaces $L^2_x(\mathcal{H}_\sigma)_\xi$ and $L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi$. Moreover,

$$\widetilde{\mathscr{E}}_{-,\alpha} \sim_M \|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} + \|g_{\alpha+1}\|^2_{L^2_{x,\xi}} \tag{5.26}$$

$$\widetilde{\mathscr{D}}_{-,\alpha} \sim_M \|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{5.27}$$

5.3. Preliminary estimates. Next, we have upper and lower bounds on the linearized collision operators $\widetilde{\mathcal{L}}_{q,\gamma}$.

Lemma 5.2. *Assume* (5.5). *Let* $h_1, h_2, h_3 : \mathbf{R}^3 \to \mathbf{R}$. *We have the following:*

(i) *Then*

$$\|\mathcal{P}_\gamma^\perp h_1\|^2_{\mathcal{H}_\sigma} \lesssim \langle \mathcal{L}_\gamma h_1, h_1\rangle_{L^2} \lesssim \|\mathcal{P}_\gamma^\perp h_1\|^2_{\mathcal{H}_\sigma}. \tag{5.28}$$

(ii) *Taking* $q \in \{q_1, q_2, q_3\}$, *and* $h_1, h_2 : \mathbf{R}^3 \to \mathbf{R}$,

$$\langle \widetilde{\mathcal{L}}_{\gamma,q} h_1, h_2\rangle_{L^2} \lesssim \|\mathcal{P}_\gamma^\perp h_1\|_{\mathcal{H}_\sigma}\|\mathcal{P}_\gamma^\perp h_2\|_{\mathcal{H}_\sigma} + \eta\|h_1\|_{\mathcal{H}_\sigma}\|h_2\|_{\mathcal{H}_\sigma}, \tag{5.29}$$

$$\|\mathcal{P}_\gamma^\perp h_1\|^2_{\mathcal{H}_\sigma} \lesssim \langle \widetilde{\mathcal{L}}_{\gamma,q} h_1, h_1\rangle_{L^2} + \eta^2\|\mathcal{P}_\gamma h_1\|^2_{L^2}. \tag{5.30}$$

(iii) *Take* $\eta > 0$ *sufficiently small. Suppose* $h_1 = \sqrt{\frac{\mu_q}{\mu_\gamma}} h_2$ *where* $q \in \{q_1, q_2, q_3\}$. *Then,*

$$\|\mathcal{P}_\gamma^\perp h_1\|^2_{\mathcal{H}_\sigma} \lesssim \eta\|\mathcal{P}_\gamma h_1\|^2_{L^2} + \|\mathcal{P}_\gamma^\perp h_2\|^2_{\mathcal{H}_\sigma}. \tag{5.31}$$

Also,

$$\|h_1\|_{\mathcal{H}_\sigma} \lesssim \|\mathcal{P}_\gamma h_1\|_{L^2} + \|\mathcal{P}_\gamma^\perp h_1\|_{\mathcal{H}_\sigma}. \tag{5.32}$$

(iv) *If* $q \in \{q_1, q_2\}$, *then, taking* $q' \in [0, q)$, *we have*

$$\langle \Gamma_q(h_1, h_2), h_3\rangle_{L^2} \lesssim_{q'} (\|e^{\frac{q'|\xi|^2}{4}} h_1\|_{L^2}\|h_2\|_{\mathcal{H}_\sigma} + \|e^{\frac{q'|\xi|^2}{4}} h_1\|_{\mathcal{H}_\sigma}\|h_2\|_{L^2})\|h_3\|_{\mathcal{H}_\sigma}. \tag{5.33}$$

Proof. For (i), the equivalence (5.28) in the case $\gamma = 2$ can be found in [21]. The case of general γ follows by re-scaling in ξ.

For (ii), we compute

$$\langle \widetilde{\mathcal{L}}_{\gamma,q} h_1, h_2\rangle_{L^2} = \langle \mathcal{L}_\gamma h_1, h_2\rangle_{L^2} + \frac{\gamma - q}{2}\{\langle \sigma^\gamma_{ij}\xi_i h_1, \partial_{\xi_j} h_2\rangle_{L^2} - \langle \sigma^\gamma_{ij}\xi_j\partial_{\xi_i} h_1, h_2\rangle_{L^2}\} \tag{5.34}$$

$$- \frac{(\gamma - q)^2}{2}\langle \sigma^\gamma_{ij}\xi_i\xi_j h_1, h_2\rangle_{L^2} \tag{5.35}$$

It is clear from the above that we have (5.29). In the case $h_2 = h_1$, we have by (5.28) and the computation above that

$$\langle \widetilde{\mathcal{L}}_{\gamma,q} h_1, h_1\rangle_{L^2} \geq \langle \mathcal{L}_\gamma h_1, h_1\rangle_{L^2_\xi} - C\eta^2\|h_1\|^2_{\mathcal{H}_\sigma} \tag{5.36}$$

$$\geq (\frac{1}{C} - C\eta^2)\|\mathcal{P}_\gamma^\perp h_1\|^2_{\mathcal{H}_\sigma} - C\eta^2|\mathcal{P}_\gamma h_1|^2_{\mathcal{H}_\sigma}. \tag{5.37}$$

Taking η sufficiently small, we get (5.30).

We now prove (iii). We write $\zeta = \sqrt{\frac{\mu_q}{\mu_\gamma}}$. Now,

$$\|\mathcal{P}_\gamma^\perp h_1\|_{\mathcal{H}_\sigma} = \|\mathcal{P}_\gamma^\perp(\zeta h_2)\|_{\mathcal{H}_\sigma} \leq \|\mathcal{P}_\gamma^\perp(\zeta\mathcal{P}_\gamma h_2)\|_{\mathcal{H}_\sigma} + \|\mathcal{P}_\gamma^\perp(\zeta\mathcal{P}_\gamma^\perp h_2)\|_{\mathcal{H}_\sigma} \tag{5.38}$$

$$\lesssim \|\mathcal{P}_\gamma^\perp((\zeta-1)\mathcal{P}_\gamma h_2)\|_{\mathcal{H}_\sigma} + \|\mathcal{P}_\gamma^\perp h_2\|_{\mathcal{H}_\sigma} \tag{5.39}$$

$$\lesssim \|(\zeta-1)\mathcal{P}_\gamma h_2\|_{\mathcal{H}_\sigma} + \|\mathcal{P}_\gamma^\perp h_2\|_{\mathcal{H}_\sigma} \tag{5.40}$$

Now, $|\zeta - 1| \lesssim \eta|\xi|^2$, and $|\mathcal{P}_\gamma h_2(\xi)| \lesssim \langle\xi\rangle^2 e^{-\frac{\gamma}{2}|\xi|^2}\|\mathcal{P}_\gamma h_2\|_{L^2}$, $\|(\zeta-1)\mathcal{P}_\gamma h_2\|_{\mathcal{H}_\sigma} \lesssim \eta\|\mathcal{P}_\gamma h_2\|_{L^2}$. Now,

$$\|\mathcal{P}_\gamma h_2\|_{L^2} \leq \|\mathcal{P}_\gamma(\zeta^{-1}\mathcal{P}_\gamma h_1)\|_{L^2} + \|\mathcal{P}_\gamma((\zeta^{-1}-1)\mathcal{P}_\gamma^\perp h_1)\|_{L^2} \tag{5.41}$$

$$\lesssim \|\mathcal{P}_\gamma h_1\|_{L^2} + \eta\|\mathcal{P}_\gamma^\perp h_1\|_{L^2}. \tag{5.42}$$

Combining these bounds, we deduce (5.31) after taking η sufficiently small. From the latter estimate, we deduce (5.32).

Finally, we prove (iv). The bound (5.33) in the case $q = 2$ follows from the proof of Theorem 3 in [21]. The case of general q follows by re-scaling in ξ. □

Lemma 5.3. *Suppose $q \in \{q_1, q_2, q_3\}$ and assume (5.5) with η taken sufficiently small. Let $G, h_1, h_2 : \mathbf{R}^3 \to \mathbf{R}$, $G = G(v)$, $h_1 = h_1(\xi)$, $h_2 = h_2(\xi)$.*

(i) *We have the upper bound*

$$\begin{aligned}\langle\mathcal{M}_{q,G}h_1, h_2\rangle_{L^2} \lesssim \|\langle v\rangle^3 G\|_{L^2}(\|h_1\|_{\mathcal{H}^-_{\sigma;\varepsilon}} &\\ + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_1\|_{L^2})(\|h_2\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_2\|_{L^2}) &\\ + \varepsilon^{\frac{1}{2}}\|\langle v\rangle^{\frac{7}{2}}G\|_{\mathcal{H}_\sigma}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_1\|_{L^2}\|h_2\|_{\mathcal{H}^-_{\sigma;\varepsilon}}.&\end{aligned} \tag{5.43}$$

(ii) *Assume $G \geq 0$, $\|\langle v\rangle^3 G\|_{L^2}, \|\langle v\rangle^{\frac{7}{2}}G\|_{\mathcal{H}_\sigma} < \infty$ and $\|G\|_{L^1_v} > 0$. Let*

$$k_G := \sup\{1, \|\langle v\rangle^3 G\|_{L^2_v}, \frac{1}{\|G\|_{L^1_v}}\}, \tag{5.44}$$

Then,

$$\langle\mathcal{M}_{q,G}h_1, h_1\rangle_{L^2_\xi} \geq \frac{1}{C_{k_G}}\|h_1\|^2_{\mathcal{H}^-_{\sigma;\varepsilon}} - C_{k_G}\varepsilon\langle\|\langle v\rangle^{\frac{7}{2}}G\|_{\mathcal{H}_\sigma}\rangle^2\|h_1\|^2_{L^2}. \tag{5.45}$$

Proof. Recall that $\mathcal{M}_{q,G}h = \frac{1}{\sqrt{\mu_q}}Q^\varepsilon_{+-}(\mu_q, \sqrt{\mu_q}h)$, with Q^ε_{+-} defined in (1.11). Then,

$$\langle\mathcal{M}_{q,G}h_1, h_2\rangle_{L^2_\xi} \tag{5.46}$$

$$= \frac{1}{\varepsilon}\langle(\Phi_{ij} * G)|_{v=\frac{\xi}{\varepsilon}}(\partial_{\xi_i} - \frac{q\xi_i}{2})h_1, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_1\rangle_{L^2_\xi} \tag{5.47}$$

$$- \langle(\partial_{v_i}\Phi_{ij} * G)|_{v=\frac{\xi}{\varepsilon}}h_1, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_1\rangle_{L^2_\xi}. \tag{5.48}$$

For the first term, we have

$$(5.47) \lesssim \frac{1}{\varepsilon}\langle(\Phi_{ij} * |G|)|_{v=\frac{\xi}{\varepsilon}}(\partial_{\xi_i} - \frac{q\xi_i}{2})h_1, (\partial_{\xi_j} - \frac{q\xi_i}{2})h_1\rangle^{\frac{1}{2}}_{L^2_\xi} \tag{5.49}$$

$$\cdot \langle(\Phi_{ij} * |G|)|_{v=\frac{\xi}{\varepsilon}}(\partial_{\xi_i} + \frac{q\xi_i}{2})h_2, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_2\rangle^{\frac{1}{2}}_{L^2_\xi} \tag{5.50}$$

$$\lesssim \|\langle v\rangle^3 G\|_{L^2_v}(\|h_1\|^2_{\mathcal{H}^-_{\sigma;\varepsilon}} + \frac{1}{\varepsilon}\langle\sigma_{ij}|_{v=\frac{\xi}{\varepsilon}}\xi_i\xi_j h_1, h_1\rangle_{L^2_\xi})^{\frac{1}{2}} \tag{5.51}$$

$$\cdot (\|h_2\|^2_{\mathcal{H}^-_{\sigma;\varepsilon}} + \frac{1}{\varepsilon}\langle\sigma_{ij}|_{v=\frac{\xi}{\varepsilon}}\xi_i\xi_j h_2, h_2\rangle_{L^2_\xi})^{\frac{1}{2}}. \tag{5.52}$$

Above we used the upper bound (3.5). Now, recall that by (1.19), $\sigma_{ij}(z)z_i z_j \sim \frac{|z|^2}{\langle z\rangle^3}$. Hence,

$$\frac{1}{\varepsilon}\sigma_{ij}(\frac{\xi}{\varepsilon})\xi_i\xi_j \sim \frac{|\xi|^2}{\varepsilon\langle\xi/\varepsilon\rangle^3} = \frac{\varepsilon|\xi/\varepsilon|^2}{\langle\xi/\varepsilon\rangle^3} \leq \frac{\varepsilon}{\langle\xi/\varepsilon\rangle} \leq \min\{\varepsilon, \frac{1}{|\xi|}\}. \tag{5.53}$$

Thus,

$$(5.47) \lesssim \|\langle v\rangle^3 G\|_{L^2_v}(\|h_1\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|h_1\|_{L^2_\xi})(\|h_2\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|h_2\|_{L^2_\xi}). \tag{5.54}$$

Next, we consider the second term: using (1.19) again, and recalling the definition of $\mathcal{H}^\varepsilon_{\sigma,-}$ in (1.23), we have

$$|\langle(\partial_{v_i}\Phi_{ij} * G)|_{v=\frac{\xi}{\varepsilon}}h_1, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_2\rangle_{L^2_\xi}| \tag{5.55}$$

$$\lesssim \left(\int_{\mathbf{R}^3}(\sigma^{-1})_{ij}(\frac{\xi}{\varepsilon})\partial_{v_k}\Phi_{ki} * G|_{v=\frac{\xi}{\varepsilon}}\partial_{v_l}\Phi_{lj} * G|_{v=\frac{\xi}{\varepsilon}}h_1^2(\xi)d\xi\right)^{\frac{1}{2}} \tag{5.56}$$

$$\cdot \langle\sigma_{ij}(\frac{\xi}{\varepsilon})(\partial_{\xi_i} + \frac{q\xi_i}{2})h_2, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_2\rangle^{\frac{1}{2}}_{L^2_\xi} \tag{5.57}$$

$$\lesssim \left(\int_{\mathbf{R}^3}\langle\frac{\xi}{\varepsilon}\rangle^3|(\nabla_v(-\Delta_v)^{-1}G)(\frac{\xi}{\varepsilon})|^2 h_1^2(\xi)d\xi\right)^{\frac{1}{2}} \tag{5.58}$$

$$\lesssim \varepsilon^{\frac{1}{2}}\sup_{v\in\mathbf{R}^3, i\in\{1,2,3\}}\{\langle v\rangle^2|\nabla_v(-\Delta_v)^{-1}G(v)|\}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_1\|_{L^2_\xi}(\|h_2\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_2\|_{L^2_\xi}) \tag{5.59}$$

Now, we have

$$\sup_{v\in\mathbf{R}^3}\|\langle v\rangle^2|\partial_{v_j}\Phi_{ij} * G(v)\| \lesssim \|\langle v\rangle^2 G\|_{H^1_v} \lesssim \|\langle v\rangle^{\frac{7}{2}}G\|_{\mathcal{H}_\sigma}. \tag{5.60}$$

Proof of (ii): We write

$$\langle\mathcal{M}_{q,G}h_1, h_1\rangle_{L^2_\xi} = \frac{1}{\varepsilon}\langle(\Phi_{ij} * G)|_{v=\frac{\xi}{\varepsilon}}(\partial_{\xi_i} - \frac{q\xi_i}{2})h_1, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_1\rangle_{L^2_\xi} \tag{5.61}$$

$$- \langle(\partial_{v_i}\Phi_{ij} * G)|_{v=\frac{\xi}{\varepsilon}}h_1, (\partial_{\xi_j} + \frac{q\xi_i}{2})h_1\rangle_{L^2_\xi} \tag{5.62}$$

$$= \frac{1}{\varepsilon}\langle(\Phi_{ij} * G)|_{v=\frac{\xi}{\varepsilon}}\partial_{\xi_i}h_1, \partial_{\xi_j}h_1\rangle_{L^2_\xi} \tag{5.63}$$

$$-\frac{1}{\varepsilon}\langle(\Phi_{ij}*G)|_{v=\frac{\xi}{\varepsilon}}\xi_i\xi_j h_1, h_1\rangle_{L^2_\xi} \tag{5.64}$$

$$-\langle(\partial_{v_i}\Phi_{ij}*G)|_{v=\frac{\xi}{\varepsilon}}h_1, (\partial_{\xi_j}+\frac{q\xi_i}{2})h_1\rangle_{L^2_\xi} \tag{5.65}$$

Now, applying (3.6), we have

$$(5.63) \geq \frac{1}{C_{k_G}\varepsilon}\langle\sigma_{ij}|_{v=\frac{\xi}{\varepsilon}}\partial_{\xi_i}h_1, \partial_{\xi_j}h_1\rangle_{L^2_\xi}. \tag{5.66}$$

Once again, we apply (3.5) to get

$$\frac{1}{\varepsilon}\sup_{\xi\in\mathbf{R}^3}\left|(\Phi_{ij}*G)(\frac{\xi}{\varepsilon})\xi_i\xi_j\right| \lesssim \frac{k_G}{\varepsilon}\sigma_{ij}(\frac{\xi}{\varepsilon})\xi_i\xi_j \lesssim \frac{k_G|\xi|^2}{\varepsilon\langle\xi/\varepsilon\rangle^3} \leq \frac{k_G\varepsilon}{\langle\xi/\varepsilon\rangle}. \tag{5.67}$$

Hence,

$$|(5.64)| \leq C_{k_G}\varepsilon\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_1\|^2_{L^2} \tag{5.68}$$

Finally, we reuse the bound from part (i) to get for all $\lambda > 0$,

$$|(5.65)| \lesssim \varepsilon^{\frac{1}{2}}\|\langle v\rangle^{\frac{7}{2}}G\|_{\mathcal{H}_\sigma}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_1\|_{L^2_\xi}\|h_1\|_{\mathcal{H}^-_{\sigma;\varepsilon}} \tag{5.69}$$

$$\leq \frac{\lambda}{C_{k_G}}\|h_1\|^2_{\mathcal{H}^-_{\sigma;\varepsilon}} + \frac{C_{k_G}\varepsilon}{\lambda}\|\langle v\rangle^{\frac{7}{2}}G\|^2_{\mathcal{H}_\sigma}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}h_1\|^2_{L^2}. \tag{5.70}$$

Taking λ sufficiently small so that the first term in the above is dominated by (5.63), we deduce (5.45). □

5.4. Energy estimates.

Proposition 5.4. *We assume that* (5.3), (5.4) *and* (5.5) *hold. Then, for all* $\kappa > 0$, $t \in [0, T]$*:*

(i) *for* $\alpha \in \{1, 2, 3\}$*, we have*

$$\begin{aligned}
&\varepsilon\frac{d}{dt}\{\|g_\alpha\|^2_{L^2_{x,\xi}} + 4\pi\sqrt{\gamma}\||\Delta_x|^{-1}a\|^2_{L^2_x}\} + \frac{1}{C}\{\|\mathcal{P}^\perp_\gamma g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}\} \\
&\leq C(\varepsilon+\varsigma+\eta+\kappa)\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \\
&\quad + C\varepsilon\langle\|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle\|g_\alpha\|^2_{L^2_{x,\xi}} \\
&\quad + C_\kappa\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2
\end{aligned} \tag{5.71}$$

(ii) *For* $\alpha \in \{1, 2\}$,

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}} + \frac{1}{C}\{\|\mathcal{P}^\perp_\gamma g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}\} \\
&\leq C(\varepsilon+\varsigma+\eta+\kappa)\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + C_{\eta,\kappa}\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \\
&\quad + C\varepsilon\langle\|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}} \\
&\quad + C_\kappa\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2
\end{aligned} \tag{5.72}$$

Proof. We separate the proof into each part:
Proof of (i): We multiply (5.17) by g_α and integrate:

$$\frac{\varepsilon}{2}\frac{d}{dt}\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{5.73}$$

$$+\frac{q_\alpha \varepsilon}{2}\langle \partial_t \phi g_\alpha, g_\alpha\rangle_{L^2_{x,\xi}} \tag{5.74}$$

$$+\langle e^{\gamma\psi}\widetilde{\mathcal{L}}_{\gamma,q_\alpha} g_\alpha, g_\alpha\rangle_{L^2_{x,\xi}} \tag{5.75}$$

$$+\langle \mathcal{M}_{q_\alpha,F_+} g_\alpha, g_\alpha\rangle_{L^2_{x,\xi}} \tag{5.76}$$

$$-4\pi\gamma\langle \xi\cdot\nabla_x\Delta_x^{-1}a\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}e^{\gamma\psi-\frac{q_\alpha}{2}\phi}, g_\alpha\rangle_{L^2_{x,\xi}} \tag{5.77}$$

$$=-\varepsilon\langle\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}}, g_\alpha\rangle_{L^2_{x,\xi}} \tag{5.78}$$

$$-\langle e^{\gamma\psi-\frac{q_\alpha}{2}\phi}\mathcal{M}_{q_\alpha,F_+}(\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}), g_\alpha\rangle_{L^2_{x,\xi}} \tag{5.79}$$

$$+\langle e^{\frac{q_\alpha}{2}\phi}\Gamma_{q_\alpha}(g_\alpha,g_\alpha), g_\alpha\rangle_{L^2_{x,\xi}}. \tag{5.80}$$

By (4.142),

$$|(5.74)| \lesssim \varepsilon(\|\partial_t\psi\|_{L^\infty_x}+\|\Delta_x^{-1}\partial_t a\|_{L^\infty_x})\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{5.81}$$

$$\lesssim \varepsilon\langle\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi}+\|\partial_t a\|_{L^2_x}\rangle\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{5.82}$$

$$\lesssim \varepsilon^2+\varepsilon\langle\|\partial_t a\|_{L^2_x}\rangle\|g_\alpha\|^2_{L^2_{x,\xi}}+\varsigma\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.83}$$

Next, we apply (5.30) to get

$$(5.75) \geq \frac{1}{C}e^{-C\|\psi\|_{L^\infty_x}}\|\mathcal{P}_\gamma^\perp g_\alpha\|^2_{L^2_{x,\xi}} - Ce^{C\|\psi\|_{L^\infty_x}}\eta^2\|\mathcal{P}_\gamma g_\alpha\|^2_{L^2_{x,\xi}} \tag{5.84}$$

$$\geq \frac{1}{C}\|\mathcal{P}_\gamma^\perp g_\alpha\|^2_{L^2_{x,\xi}} - C\eta^2\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.85}$$

In the second line, we use $\|\phi\|_{L^\infty_x}\lesssim M+\varsigma\lesssim 1$. Next, we apply (5.45) to get

$$(5.76) \geq \frac{1}{C_{k_{F_+}}}\|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} - C_{K_{F_+}}\varepsilon\langle\|\langle v\rangle^{\frac{7}{2}}F_+\|_{L^\infty_x(\mathcal{H}_\sigma)_\xi}\rangle^2\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{5.86}$$

where

$$k_{F_+} = \max\{1, \|\langle v\rangle^3 F_+\|_{L^\infty_x L^2_v}, \|n_+^{-1}\|_{L^\infty_x}\} \lesssim 1. \tag{5.87}$$

Using this, we have

$$(5.76) \geq \frac{1}{C}\|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} - C\varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle^2\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{5.88}$$

For the next term, we write

$$\frac{e^{\frac{q_\alpha}{2}\phi}g_\alpha}{\sqrt{\mu_{q_\alpha}}} = \frac{\sqrt{\mu_\gamma}f}{\mu_{q_\alpha}} = \frac{f}{\sqrt{\mu_\gamma}} + \left(\frac{1}{\sqrt{\mu_\gamma}} - \frac{\sqrt{\mu_\gamma}}{\mu_{q_\alpha}}\right)f = \frac{f}{\sqrt{\mu_\gamma}} + \left(\frac{\mu_{q_\alpha}}{\mu_\gamma} - 1\right)\frac{e^{\frac{q_\alpha}{2}\phi}g_\alpha}{\sqrt{\mu_{q_\alpha}}} \tag{5.89}$$

Then,

$$(5.77) = -4\pi\gamma\langle\xi\cdot\nabla_x\Delta_x^{-1}a\sqrt{\mu_\gamma}e^{\gamma\psi-q_\alpha\phi}, f\rangle_{L^2_{x,\xi}} \tag{5.90}$$

$$-4\pi\gamma\langle\xi\cdot\nabla_x\Delta_x^{-1}a\frac{\mu_{q_\alpha}-\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}e^{\gamma\psi-\frac{q_\alpha}{2}\phi}, g_\alpha\rangle_{L^2_{x,\xi}} \tag{5.91}$$

$$= -4\pi\gamma\langle\nabla_x\Delta_x^{-1}a, b\rangle_{L^2_{x,\xi}} \tag{5.92}$$

$$-4\pi\gamma\langle(e^{\gamma\psi-q_\alpha\phi}-1)\nabla_x\Delta_x^{-1}a, b\rangle_{L^2_{x,\xi}} \tag{5.93}$$

$$-4\pi\gamma\langle\xi\cdot\nabla_x\Delta_x^{-1}a\frac{\mu_{q_\alpha}-\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}e^{\gamma\psi-\frac{q_\alpha}{2}\phi}, g_\alpha\rangle_{L^2_{x,\xi}} \tag{5.94}$$

For term (5.92), we use the continuity equation (see (5.201) below)

$$\varepsilon\partial_t a + \sqrt{\gamma}\nabla_x\cdot b = -\varepsilon\partial_t(e^{\gamma\psi}), \tag{5.95}$$

which gives

$$(5.92) = 2\pi\sqrt{\gamma}\varepsilon\frac{d}{dt}\||\nabla_x|^{-1}a\|^2_{L^2_x} - 4\pi\sqrt{\gamma}\varepsilon\langle\partial_t(e^{\gamma\psi}), a\rangle_{L^2_x} \tag{5.96}$$

$$= 2\pi\varepsilon\frac{d}{dt}\{\sqrt{\gamma}\||\nabla_x|^{-1}a\|^2_{L^2_x}\} \tag{5.97}$$

$$-\frac{\varepsilon\pi\dot{\gamma}}{\sqrt{\gamma}}\||\nabla_x|^{-1}a\|^2_{L^2_x} - 4\pi\sqrt{\gamma}\varepsilon\langle\partial_t(e^{\gamma\psi}), a\rangle_{L^2_x} \tag{5.98}$$

Now, by (4.142),

$$\left|\frac{\varepsilon\pi\dot{\gamma}}{\sqrt{\gamma}}\||\nabla_x|^{-1}a\|^2_{L^2_x} + 4\pi\sqrt{\gamma}\varepsilon\langle\partial_t(e^{\gamma\psi}), a\rangle_{L^2_x}\right| \lesssim \varepsilon(1+|\dot{\gamma}|+\|\partial_t\psi\|_{L^\infty_x})\|a\|_{L^2_x} \tag{5.99}$$

$$\lesssim \varepsilon(1+\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi})\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.100}$$

$$\lesssim \varepsilon^2 + \varepsilon\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.101}$$

Hence,

$$\left|(5.92) - 2\pi\varepsilon\frac{d}{dt}\{\sqrt{\gamma}\||\nabla_x|^{-1}a\|^2_{L^2_x}\}\right| \lesssim \varepsilon^2 + \varepsilon\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.102}$$

For (5.93), we have

$$\|e^{\gamma\psi-q_\alpha\phi}-1\|_{L^\infty_x} = \|e^{(\gamma-q_\alpha)\psi-4\pi q_\alpha\Delta_x^{-1}a}-1\|_{L^\infty_x} \lesssim \varsigma+\eta. \tag{5.103}$$

Thus,

$$|(5.93)| \lesssim (\varsigma+\eta)\|a\|_{L^2_x}\|b\|_{L^2_x} \tag{5.104}$$

$$\lesssim (\varsigma + \eta)\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.105}$$

Third, we have (5.94). For this, we note that when η is taken sufficiently small, we can guarantee that

$$|\frac{\mu_{q_\alpha} - \mu_\gamma}{\sqrt{\mu_{q_\alpha}}}| \lesssim \eta e^{-\frac{\beta_{in}}{8}|\xi|^2} \tag{5.106}$$

Hence,

$$|(5.94)| \lesssim \eta\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.107}$$

Combining these bounds, we find

$$|(5.77) - 2\pi\varepsilon\frac{d}{dt}\{\sqrt{\gamma}\||\nabla_x|^{-1}a\|_{L^2_x}\}| \lesssim \varepsilon^2 + (\varepsilon + \varsigma + \eta)\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.108}$$

We move on to our next term:

$$(5.78) \lesssim \varepsilon\|\langle\xi\rangle^{\frac{1}{2}}\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}}\|_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha\|_{L^2_{x,\xi}} \tag{5.109}$$

$$\lesssim \varepsilon\|\langle\xi\rangle^{\frac{1}{2}}\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}}\|_{L^2_{x,\xi}}\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.110}$$

Now,

$$\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}} = (\frac{\dot{\gamma}}{\gamma}(\frac{3}{2} - \gamma(|\xi|^2 - \psi)) + \gamma\partial_t\psi)\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}e^{\gamma\psi - \frac{q_\alpha}{2}\phi}. \tag{5.111}$$

Taking η sufficiently small, we ensure that $\langle\xi\rangle^{\frac{1}{2}}\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma \lesssim e^{-\frac{\beta_{in}|\xi|^2}{8}}$. Thus, combining this with (4.142),

$$(5.78) \lesssim \varepsilon(|\dot{\gamma}| + \|\partial_t\psi\|_{L^\infty_x})\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.112}$$

$$\leq C_\kappa\varepsilon^2 + (\varepsilon + \kappa)\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.113}$$

Next, we bound (5.79) using (5.43):

$$(5.79) \lesssim (\|\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma\|_{\mathcal{H}^-_{\sigma;\varepsilon}} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma\|_{L^2_\xi})(\|g_\alpha\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}g_\alpha\|_{L^2_{x,\xi}}) \tag{5.114}$$

$$+ \varepsilon^{\frac{1}{2}}\|F_+\|_{\mathfrak{D}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma\|_{L^2_\xi}\|g_\alpha\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{5.115}$$

Now, by (1.19) again, we have

$$\|\mu_{q_\alpha}^{-\frac{1}{2}}\mu_\gamma\|_{\mathcal{H}^-_{\sigma;\varepsilon}} \lesssim \frac{1}{\varepsilon}\int_{\mathbf{R}^3}\sigma_{ij}(\frac{\xi}{\varepsilon})\xi_i\xi_j e^{-\frac{\beta_{in}}{10}|\xi|^2}d\xi \tag{5.116}$$

$$\lesssim \frac{1}{\varepsilon}\int_{\mathbf{R}^3}\frac{|\xi|^2}{\langle\xi/\varepsilon\rangle^3}e^{-\frac{\beta_{in}}{10}|\xi|^2}d\xi \tag{5.117}$$

$$\le \varepsilon^2 \int_{\mathbf{R}^3} \frac{1}{|\xi|} e^{-\frac{\beta_{in}}{10}|\xi|^2} d\xi \tag{5.118}$$

$$\lesssim \varepsilon^2. \tag{5.119}$$

since $\frac{1}{|\xi|}$ is locally integrable. Similarly,

$$\|\langle \frac{\xi}{\varepsilon} \rangle^{-\frac{1}{2}} \mu_{q_\alpha}^{-\frac{1}{2}} \mu_\gamma \|_{L^2_{x,\xi}} \lesssim \varepsilon. \tag{5.120}$$

Moreover, by Sobolev embedding

$$\|\langle \frac{\xi}{\varepsilon} \rangle^{-\frac{1}{2}} g_\alpha \|_{L^2_{x,\xi}} \lesssim \varepsilon^{\frac{1}{2}} \| 1_{B_1}(\xi) |\xi|^{-\frac{1}{2}} g_\alpha \|_{L^2_{x,\xi}} + \| \langle \xi \rangle^{-\frac{1}{2}} g_\alpha \|_{L^2_{x,\xi}} \lesssim \|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.121}$$

Hence,

$$|(5.79)| \lesssim \varepsilon \langle \|F_+\|_{\mathfrak{D}} \rangle \|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma \cap \mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{5.122}$$

Thus, taking $\lambda > 0$ to be chosen later,

$$|(5.79)| \le \kappa \|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma \cap \mathcal{H}^-_{\sigma;\varepsilon})_\xi} + C_\kappa \varepsilon^2 \langle \|F_+\|_{\mathfrak{D}} \rangle^2. \tag{5.123}$$

Finally, applying (5.33), we have

$$(5.80) \lesssim \|g_1\|_{L^\infty_x L^2_\xi} \|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.124}$$

$$\lesssim \|g_1^{(s)}\|_{L^2_{x,\xi}} \|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.125}$$

$$\lesssim \varsigma \|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.126}$$

Combining the upper bounds for (5.74) through (5.80), we conclude

$$\frac{\varepsilon}{2}\frac{d}{dt}\{\|g_\alpha\|^2_{L^2_{x,\xi}} + 4\pi\sqrt{\gamma}\| |\Delta_x|^{-1} a\|^2_{L^2_x}\} + \frac{1}{C}\{\|\mathcal{P}^\perp_\gamma g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}\} \tag{5.127}$$

$$\le C(\varepsilon + \varsigma + \eta + \kappa) \|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma \cap \mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{5.128}$$

$$+ C\varepsilon \langle \|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}} \rangle \|g_\alpha\|^2_{L^2_{x,\xi}} \tag{5.129}$$

$$+ C_\kappa \varepsilon^2 \langle \|F_+\|_{\mathfrak{D}} \rangle^2. \tag{5.130}$$

Using (5.32) and taking ε, ς, η and κ sufficiently small, conclude (5.71).

We now prove (ii). Let $g_j^{(s)} := \langle \nabla_x \rangle^s g_\alpha$. Then,

$$\frac{\varepsilon}{2}\frac{d}{dt}\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{5.131}$$

$$+ q_\alpha \varepsilon \langle \langle \nabla_x \rangle^s \{\partial_t \phi g_\alpha\}, g_\alpha^{(s)} \rangle_{L^2_{x,\xi}} \tag{5.132}$$

$$+ \langle \langle \nabla_x \rangle^s \{E g_\alpha\} - E g_\alpha^{(s)}, \nabla_v g_\alpha^{(s)} \rangle_{L^2_{x,\xi}} \tag{5.133}$$

$$+ \langle \langle \nabla_x \rangle^s (e^{\gamma\psi} \widetilde{\mathcal{L}}_{\gamma, q_\alpha} g_\alpha), g_j^{(s)} \rangle_{L^2_{x,\xi}} \tag{5.134}$$

$$-4\pi\gamma\langle\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}}\langle\nabla_x\rangle^s\{\xi\cdot\nabla_x\Delta_x^{-1}ae^{\gamma\psi-\frac{q_\alpha}{2}\phi}\},g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.135}$$

$$+\langle\langle\nabla_x\rangle^s(\mathcal{M}_{q_\alpha,F_+}g_\alpha),g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.136}$$

$$=-\varepsilon\langle\langle\nabla_x\rangle^s\left(\frac{\partial_t\{\mu_\gamma e^{\gamma\psi}\}}{\sqrt{\mu_{q_\alpha}e^{q_\alpha\phi}}}\right),g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.137}$$

$$-\langle\langle\nabla_x\rangle^s\{e^{\gamma\psi-\frac{q_\alpha}{2}\phi}\mathcal{M}_{q_\alpha,F_+}(\frac{\mu_\gamma}{\sqrt{\mu_{q_\alpha}}})\},g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.138}$$

$$+\langle\langle\nabla_x\rangle^s\{e^{\frac{q_\alpha}{2}\phi}\Gamma_{q_\alpha}(g_\alpha,g_\alpha)\},g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.139}$$

First,

$$|(5.132)|=q_\alpha\varepsilon|\langle\langle\nabla_x\rangle^s\{(\partial_t\psi+4\pi\Delta_x^{-1}\partial_t a)g_\alpha\},g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.140}$$

$$\lesssim\varepsilon\int_{\mathbf{R}^3}\big(\|\partial_t\psi\|_{H^s_x}\|g_\alpha^{(s)}\|^2_{L^2_x}+\|\partial_t a\|_{H^{s-2}_x}\|g_\alpha\|_{L^\infty_x}\|g_\alpha^{(s)}\|_{L^2_x} \tag{5.141}$$

$$+\|\Delta_x^{-1}\partial_t a\|_{L^\infty_x}\|g_\alpha^{(s)}\|_{L^2_x}\big)d\xi \tag{5.142}$$

Applying (5.95) to the second term, we get

$$\varepsilon\int_{\mathbf{R}^3}\|\partial_t a\|_{H^{s-2}_x}\|g_\alpha\|_{L^\infty_x}\|g_\alpha^{(s)}\|_{L^2_x}d\xi \tag{5.143}$$

$$\lesssim\int_{\mathbf{R}^3}\|b\|_{H^{s-1}_x}\|g_\alpha^{(s-1)}\|_{L^2_x}\|g_\alpha^{(s)}\|_{L^2_x}d\xi+\varepsilon(1+|\dot\gamma|+\|\partial_t\psi\|_{H^s_x})\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{5.144}$$

$$\lesssim\|b\|_{H^{s-1}_x}\|\langle\xi\rangle^{\frac12}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac12}g_\alpha^{(s)}\|_{L^2_{x,\xi}}+\varepsilon(1+|\dot\gamma|+\|\partial_t\psi\|_{H^s_x})\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{5.145}$$

Collecting terms,

$$|(5.132)|\lesssim\|b\|_{H^{s-1}_x}\|\langle\xi\rangle^{\frac12}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}} \tag{5.146}$$

$$+\varepsilon(1+\|\partial_t a\|_{L^\infty_x}+|\dot\gamma|+\|\partial_t\psi\|_{H^s_x})\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{5.147}$$

Applying (4.142), we get

$$\varepsilon(|\dot\gamma|+\|\partial_t\psi\|_{H^s_x})\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}}\lesssim\varepsilon^2+\varepsilon\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}}+\varsigma\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.148}$$

On the other hand, using interpolation, we have

$$\|\langle\xi\rangle^{\frac12}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac12}g_\alpha^{(s)}\|_{L^2_{x,\xi}}\le(C_\eta\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}+\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}). \tag{5.149}$$

Thus,

$$|(5.132)|\lesssim\varepsilon\langle\|\partial_t a\|_{L^2_x}\rangle\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}}+C\varsigma\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}+C_\eta\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.150}$$

Next,

$$|(5.133)|\lesssim\|\langle\xi\rangle^{\frac12}(\langle\nabla_x\rangle^s\{Eg_\alpha\}-Eg_\alpha^{(s)})\|_{L^2_{x,\xi}}\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.151}$$

$$\lesssim (\|E\|_{H^s_x}\|\langle\xi\rangle^{\frac{1}{2}}g_\alpha\|_{L^\infty_x L^2_\xi} + \|\nabla_x E\|_{L^\infty_x}\|\langle\xi\rangle^{\frac{1}{2}}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}})\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.152}$$

Above we use the usual commutator estimate for $[\langle\nabla_x\rangle^s, E_j]$. Next we interpolate

$$\|\langle\xi\rangle^{\frac{1}{2}}g_\alpha^{(s-1)}\|_{L^2_{x,\xi}} \lesssim \|\langle\xi\rangle^{s-\frac{1}{2}}g_\alpha\|^{\frac{1}{s}}_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha^{(s)}\|^{\frac{s-1}{s}}_{L^2_{x,\xi}} \tag{5.153}$$

$$\lesssim C_\eta\|\langle\xi\rangle^{-\frac{1}{2}}g_{\alpha+1}\|^{\frac{1}{s}}_{L^2_{x,\xi}}\|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha^{(s)}\|^{\frac{s-1}{s}}_{L^2_{x,\xi}} \tag{5.154}$$

$$\lesssim C_\eta\|\langle\xi\rangle^{-\frac{1}{2}}g_{\alpha+1}\|^{\frac{1}{s}}_{L^2_x(\mathcal{H}_\sigma)_\xi}\|\langle\xi\rangle^{-\frac{1}{2}}g_\alpha^{(s)}\|^{\frac{s-1}{s}}_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.155}$$

This implies

$$|(5.133)| \le \kappa\|g_\alpha^{(s)}\|^2_{L^2_x} + C_{\eta,\kappa}\|\langle\xi\rangle^{-\frac{1}{2}}g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.156}$$

Next, applying (5.29) and (5.30), we have

$$(5.134) = \langle e^{\gamma\psi}\widetilde{\mathcal{L}}_{\gamma,q}g_\alpha^{(s)}, g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.157}$$

$$+ \langle\widetilde{\mathcal{L}}_{\gamma,q}\{\langle\nabla_x\rangle^s(e^{\gamma\psi}g_\alpha) - e^{\gamma\psi}g_\alpha^{(s)}\}, g_1^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.158}$$

$$\ge \frac{1}{C}\|\mathcal{P}_\gamma^\perp g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} - C\varsigma^2\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.159}$$

$$- C\|\langle\nabla_x\rangle^s(e^{\gamma\psi}g_\alpha) - e^{\gamma\psi}g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.160}$$

Now,

$$\|\langle\nabla_x\rangle^s(e^{\gamma\psi}g_\alpha) - e^{\gamma\psi}g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \lesssim \|\langle\nabla_x\rangle^s e^{\gamma\psi}\|_{L^2_x}\|g_\alpha^{(s-1)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.161}$$

$$\lesssim \|g_\alpha^{(s-1)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.162}$$

Hence, we can interpolate to get

$$(5.134) \ge \frac{1}{C}\|\mathcal{P}_\gamma^\perp g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} - \kappa\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.163}$$

$$- C_\kappa\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.164}$$

For the next term, we have

$$|(5.135)| \lesssim \|e^{\gamma\psi-\frac{q\alpha}{2}\phi}\nabla_x\Delta_x^{-1}a\|_{H^s}\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.165}$$

$$\lesssim \|g_\alpha^{(s-1)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.166}$$

$$\le \kappa\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + C_\kappa\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.167}$$

For our next term, we have

$$(5.136) = \langle\mathcal{M}_{q_1,F_+}g_\alpha^{(s)}, g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.168}$$

$$+ \langle\langle\nabla_x\rangle^s(\mathcal{M}_{q_1,F_+}g_\alpha) - \mathcal{M}_{q_1,F_+}g_\alpha^{(s)}, g_\alpha^{(s)}\rangle_{L^2_{x,\xi}} \tag{5.169}$$

For the first term, similarly to (5.134), we have

$$(5.168) \geq \frac{1}{C}\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} - C\varepsilon\langle\|F_+^{(s,m)}\|_{L^2_x(\dot{\mathcal{H}}_\sigma)_v}\rangle^2\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} \tag{5.170}$$

Next, combining the commutator estimate with the proof of (5.43), we have

$$\begin{aligned}
|(5.169)| &\lesssim \|F_+^{(m_1,s)}\|_{L^2_{x,v}}(\|g_1^{(s-1)}\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}g_1^{(s-1)}\|_{L^2_{x,\xi}}) && (5.171)\\
&\quad \cdot(\|g_1^{(s)}\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon^{\frac{1}{2}}\|\langle\frac{\xi}{\varepsilon}\rangle^{-\frac{1}{2}}g_1^{(s)}\|_{L^2_{x,\xi}}) && (5.172)\\
&\quad + \varepsilon^{\frac{1}{2}}\|F_+\|_{\mathfrak{D}}\|g_1^{(s-1)}\|_{L^2_{x,\xi}}\|g_1^{(s)}\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} && (5.173)\\
&\lesssim (\|g_\alpha\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi})^{\frac{1}{s}}(\|g_\alpha\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon\|g_\alpha\|_{L^2_x(\mathcal{H}_\sigma)_\xi})^{\frac{2s-1}{s}} && (5.174)\\
&\quad + \varepsilon^{\frac{1}{2}}\|F_+\|_{\mathfrak{D}}\|g_1^{(s)}\|_{L^2_{x,\xi}}\|g_1^{(s)}\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}. && (5.175)
\end{aligned}$$

Thus, we get

$$\begin{aligned}
(5.136) \geq \frac{1}{C}\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} &- C\{\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}\} && (5.176)\\
&- C\varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle^2\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} && (5.177)
\end{aligned}$$

Next, similarly to (5.78),

$$(5.137) \leq C_\kappa\varepsilon + \kappa\|g_\alpha^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.178}$$

and, as with (5.79), we use (5.43) to get

$$(5.138) \leq \kappa\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + C_\kappa\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2. \tag{5.179}$$

Finally, using (5.33),

$$\begin{aligned}
(5.139) &\lesssim \|g_\alpha^{(s)}\|_{L^2_{x,\xi}}\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} && (5.180)\\
&\lesssim \varsigma\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} && (5.181)
\end{aligned}$$

We now combine the estimates for the terms (5.132) through (5.139):

$$\begin{aligned}
&\frac{\varepsilon}{2}\frac{d}{dt}\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} + \frac{1}{C}\{\|\mathcal{P}_\gamma^\perp g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\} && (5.182)\\
&\quad \leq C(\varepsilon+\varsigma+\eta+\kappa)\|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + C_{\eta,\kappa}\{\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}\} && (5.183)\\
&\quad + C\varepsilon\langle\|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle\|g_\alpha^{(s)}\|^2_{L^2_{x,\xi}} && (5.184)\\
&\quad + C_\kappa\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2. && (5.185)
\end{aligned}$$

Using (5.32) again, we conclude with (5.72). □

5.5. Macroscopic estimates. What remains is to get bounds on $\mathcal{P}_\gamma f$. This requires analysis of the local conservation laws. In order to derive these equations efficiently, and, in particular, see how the transport part of (5.13) couples (a, b, c), we introduce the ladder operators: the lowering and raising operators are respectively

$$\mathcal{A}_j = \gamma^{\frac{1}{2}}\frac{\xi_j}{2} + \gamma^{-\frac{1}{2}}\partial_{\xi_j}, \quad \mathcal{A}_j^* = \gamma^{\frac{1}{2}}\frac{\xi_j}{2} - \gamma^{-\frac{1}{2}}\partial_{\xi_j}, \quad j \in \{1, 2, 3\}. \tag{5.186}$$

We recall the identities

$$\mathcal{A}_j\mu^{\frac{1}{2}} = 0, \quad [\mathcal{A}_j, \mathcal{A}_k^*] = \varsigma_{jk}, \quad j, k \in \{1, 2, 3\}. \tag{5.187}$$

Recall also that the family of Hermite functions

$$\{\frac{1}{\sqrt{n_1!n_2!n_3!}}(\mathcal{A}_1^*)^{n_1}(\mathcal{A}_2^*)^{n_2}(\mathcal{A}_3^*)^{n_3}\mu_\gamma^{\frac{1}{2}}\}_{n_1,n_2,n_3\in\mathbf{N}_0^3} \tag{5.188}$$

gives a complete orthonormal basis for $L^2_\xi(\mathbf{R}^3)$. We shall denote the (unnormalized) Hermite functions

$$\mathfrak{h} = \mu_\gamma^{\frac{1}{2}}, \quad \mathfrak{h}_{j_1\ldots j_m} = \mathcal{A}_{j_1}\cdots\mathcal{A}_{j_n}\mu_\gamma^{\frac{1}{2}}, \quad j_1, \ldots, j_m \in \{1, 2, 3\}. \tag{5.189}$$

We can represent the kernel of $\mathcal{L}_\gamma$ using the hermite functions as follows:

$$N(\mathcal{L}) = \text{span}\{\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \frac{1}{\sqrt{6}}\mathfrak{h}_{jj}\}. \tag{5.190}$$

We can thus rewrite (5.20) as

$$\mathcal{P}_\gamma f = a\mathfrak{h} + b_j\mathfrak{h}_j + c\frac{1}{\sqrt{6}}\mathfrak{h}_{jj} \tag{5.191}$$

It is also convenient to define the following projection of h, involving third-order hermite functions:

$$d_j = \frac{1}{\sqrt{10}}\langle\mathfrak{h}_{jkk}, f\rangle_{L^2_\xi}, \quad j \in \{1, 2, 3\} \tag{5.192}$$

We now rewrite the transport part of (5.13) in terms of $\mathcal{A}$, $\mathcal{A}^*$

$$\xi\cdot\nabla_x + E\cdot(\nabla_\xi - \frac{\xi}{2}) = (\gamma^{-\frac{1}{2}}\partial_{x_j} + \gamma^{\frac{1}{2}}E_j)\mathcal{A}_j^* + \gamma^{-\frac{1}{2}}\partial_{x_j}\mathcal{A}_j. \tag{5.193}$$

We also will need to evaluate projections of $\mu^{-\frac{1}{2}}\partial_t(\mu^{\frac{1}{2}}f)$. Let $\zeta(\xi)$ be some linear combination of the hermite functions in $\gamma^{\frac{1}{2}}\xi$, i.e. there exists a polynomial $p : \mathbf{R}^3 \to \mathbf{R}$ (with coefficients independent of γ) such that

$$\zeta(\xi) = p(\gamma^{\frac{1}{2}}\xi)\mu(\xi)^{\frac{1}{2}}. \tag{5.194}$$

Then,

$$\langle\zeta, \mu^{-\frac{1}{2}}\partial_t(\mu^{\frac{1}{2}}f)\rangle_{L^2_\xi} = \partial_t\langle\zeta, f\rangle_{L^2_\xi} - \langle\partial_t(\zeta\mu^{-\frac{1}{2}})\mu^{\frac{1}{2}}, f\rangle_{L^2_\xi} \tag{5.195}$$

Now,

$$\partial_t\{\zeta\mu^{-\frac{1}{2}}\} = \partial_t\{p(\gamma^{\frac{1}{2}}\xi)\} = \frac{\dot{\gamma}}{2\gamma}\xi\cdot\nabla_\xi\{p(\gamma^{\frac{1}{2}}\xi)\} = \frac{\dot{\gamma}}{2\gamma}\xi\cdot\nabla_\xi\{\zeta\mu^{-\frac{1}{2}}\}. \tag{5.196}$$

Therefore,

$$\partial_t\{\zeta\mu^{-\frac{1}{2}}\}\mu^{\frac{1}{2}} = \frac{\dot{\gamma}}{2\gamma}(\xi\cdot\nabla_\xi + \frac{\gamma|\xi|^2}{2})\zeta \tag{5.197}$$

$$= \frac{\dot{\gamma}}{2\gamma}\xi\cdot(\nabla_\xi + \frac{\gamma\xi}{2})\zeta \tag{5.198}$$

$$= \frac{\dot{\gamma}}{2\gamma}(\mathcal{A}_j + \mathcal{A}_j^*)\mathcal{A}_j\zeta \tag{5.199}$$

In summary,

$$\langle\zeta, \mu^{-\frac{1}{2}}\partial_t(\mu^{\frac{1}{2}}f)\rangle_{L^2_\xi} = \partial_t\langle\zeta, f\rangle_{L^2_\xi} - \frac{\dot{\gamma}}{2\gamma}\langle(\mathcal{A}_j + \mathcal{A}_j^*)\mathcal{A}_j\zeta, f\rangle_{L^2_\xi}. \tag{5.200}$$

The following lemma contains formulas for relevant projections of the equation (5.13).

Lemma 5.5. *The following formulas hold:*

$$\varepsilon\partial_t a + \gamma^{-\frac{1}{2}}\partial_{x_j}b_j = -\varepsilon\partial_t(e^{\gamma\psi}), \tag{5.201}$$

$$\begin{aligned}&\varepsilon(\partial_t - \frac{\dot{\gamma}}{2\gamma})b_j + (\gamma^{-\frac{1}{2}}\partial_{x_j} + \gamma^{\frac{1}{2}}E_j)a - 4\pi\gamma^{\frac{1}{2}}e^{\gamma\psi}\partial_{x_j}\Delta_x^{-1}a + \sqrt{\frac{2}{3}}\partial_{x_j}c\\ &+\partial_{x_k}\langle\mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L^2_\xi} + \langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi} = -e^{\gamma\psi}\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L^2_\xi},\end{aligned} \tag{5.202}$$

$$\begin{aligned}&\varepsilon(\partial_t c - \frac{\dot{\gamma}}{\gamma}(c + \sqrt{\frac{3}{2}}a)) + \sqrt{\frac{2}{3}}(\gamma^{-\frac{1}{2}}\partial_{x_j} + \gamma^{\frac{1}{2}}E_j)b_j + \sqrt{\frac{5}{3}}\gamma^{-\frac{1}{2}}\partial_{x_j}d_j\\ &+\frac{1}{\sqrt{6}}\langle\mathfrak{h}_{jj}, \mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi} = \varepsilon\sqrt{\frac{3}{2}}\frac{\dot{\gamma}}{\gamma}e^{\gamma\psi} + \frac{e^{\gamma\psi}}{\sqrt{6}}\langle\mathfrak{h}_{jj}, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L^2_\xi},\end{aligned} \tag{5.203}$$

$$\begin{aligned}&\varepsilon(\partial_t d_j - \frac{\dot{\gamma}}{2\gamma}(3d_j + \frac{3\sqrt{2}}{\sqrt{5}}b_j)) + \frac{3\sqrt{3}}{\sqrt{5}}(\gamma^{-\frac{1}{2}}\partial_{x^j} + \gamma^{\frac{1}{2}}E_j)c\\ &+\sqrt{\frac{2}{5}}(\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k)\langle\mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L^2_\xi} + \gamma^{-\frac{1}{2}}\frac{1}{\sqrt{10}}\partial_{x_k}\langle\mathfrak{h}_{jkll}, f\rangle_{L^2_\xi}\\ &+\frac{1}{\sqrt{10}}\langle\mathfrak{h}_{jll}, \mathcal{L}_\gamma f + \mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi} = \langle\mathfrak{h}_{jll}, -e^{\gamma\psi}\mathcal{M}_{\gamma,F_+}\mathfrak{h} + \Gamma_\gamma(f,f)\rangle.\end{aligned} \tag{5.204}$$

Proof. These formulas follow from direct computation of the ξ integral of (5.13) multiplied by $\mathfrak{h}$, $\mathfrak{h}_j$, $\frac{1}{\sqrt{6}}\mathfrak{h}_{jj}$ and $\frac{1}{\sqrt{10}}\mathfrak{h}_{jll}$. We only give the details for (5.202). The proof of the other formulas are similar. Using (5.193) and (5.200), and the fact that $\mathcal{L}_\gamma f$, $\mu^{-\frac{1}{2}}\partial_t\nu_\gamma$ and $\Gamma_\gamma(f,f)$ are orthogonal to $\mathfrak{h}_j$, we have

$$\varepsilon(\partial_t b_j - \frac{\dot{\gamma}}{2\gamma}\langle(\mathcal{A}_k\mathcal{A}_k + \mathcal{A}_k^*\mathcal{A}_k)\mathfrak{h}_j, f\rangle_{L^2_\xi}). \tag{5.205}$$

$$+ \langle\mathfrak{h}_j, \{(\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k)\mathcal{A}_k^* + \gamma^{-\frac{1}{2}}\partial_{x_k}\mathcal{A}_k\}f\rangle_{L^2_\xi} \tag{5.206}$$

$$+ \langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi} \tag{5.207}$$

$$= -\langle\mathfrak{h}_j, e^{\gamma\psi}\mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L^2_\xi}. \tag{5.208}$$

It remains to evaluate terms involving the ladder operators. Using (5.187), one computes

$$(\mathcal{A}_k + \mathcal{A}_k^*)\mathcal{A}_k \mathfrak{h}_j = (\mathcal{A}_k^* + \mathcal{A}_k)\mathcal{A}_k \mathcal{A}_j^* \mathfrak{h} \tag{5.209}$$

$$= (\mathcal{A}_j^* + \mathcal{A}_j)\mathfrak{h} \tag{5.210}$$

$$= \mathfrak{h}_j \tag{5.211}$$

and

$$\langle \mathfrak{h}_j, \{(\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k)\mathcal{A}_k^* + \gamma^{-\frac{1}{2}}\partial_{x_k}\mathcal{A}_k\} f\rangle_{L^2_\xi} \tag{5.212}$$

$$= \langle \mathcal{A}_k \mathcal{A}_j^* \mathfrak{h}, (\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k) f\rangle_{L^2_\xi} \tag{5.213}$$

$$+ \gamma^{-\frac{1}{2}}\langle \mathcal{A}_k^* \mathcal{A}_j^* \mathfrak{h}, \partial_{x_k} f\rangle_{L^2_\xi} \tag{5.214}$$

$$= (\gamma^{-\frac{1}{2}}\partial_{x_j} + \gamma^{\frac{1}{2}}E_k)a \tag{5.215}$$

$$+ \gamma^{-\frac{1}{2}}\partial_{x_k}\langle \mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L^2_\xi} \tag{5.216}$$

$$+ \frac{1}{\sqrt{6}}\gamma^{-\frac{1}{2}}\langle \mathfrak{h}_{jk}, \mathfrak{h}_{ll}\rangle_{L^2_\xi}\partial_{x_k} c. \tag{5.217}$$

Finally, noting that in the last term,

$$\langle \mathfrak{h}_{jk}, \mathfrak{h}_{ll}\rangle_{L^2_\xi} = \sqrt{2}\delta_{jk}, \tag{5.218}$$

we conclude with (5.202). □

Lemma 5.6. *Assume the hypotheses of Proposition 5.1. Then, for all $r \geq 0$,*

$$\|b\|_{H^r_x} \lesssim_M \|\mathcal{P}_\gamma^\perp f^{(r)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f^{(r)}\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \varepsilon^2\|(a,c)\|^2_{H^r_x}. \tag{5.219}$$

Proof. It suffices to show the case when $r = 0$. Because of radial symmetry of $\mathfrak{h}$ and $\mathfrak{h}_{jj}$, we have

$$\|(\mathfrak{h}, \mathfrak{h}_{jj})\|_{\mathcal{H}^-_{\sigma;\varepsilon}} \lesssim \varepsilon. \tag{5.220}$$

Hence,

$$\|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \geq \frac{1}{2}\|\mathcal{P}_\gamma^\perp f + b_j\mathfrak{h}_j\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} - C\varepsilon^2\|(a,c)\|^2_{L^2_x} \tag{5.221}$$

Now, writing $u = \mathcal{P}_\gamma^\perp f + b_j\mathfrak{h}_j$, we have

$$\|u\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} = \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\partial_{\xi_i}u, \partial_{\xi_j}u\rangle_{L^2_{x,\xi}} \geq \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}u, \partial_{\xi_j}u\rangle_{L^2_{x,\xi}}. \tag{5.222}$$

On the other hand, for all $\nu \in \mathbf{S}^2$, and $\xi \in \mathbf{R}^3\backslash B_1$, we have

$$\sigma_{ij}(\frac{\xi}{\varepsilon})\nu_i\nu_j \lesssim \sigma_{ij}(\xi)\nu_i\nu_j \tag{5.223}$$

Now, we expand u as follows:

$$\frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}u, \partial_{\xi_j}u\rangle_{L^2_{x,\xi}} \tag{5.224}$$

$$= \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}(b_k\mathfrak{h}_k), \partial_{\xi_j}(b_l\mathfrak{h}_l)\rangle_{L^2_{x,\xi}} \tag{5.225}$$

$$+ \frac{2}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}(b_k\mathfrak{h}_k), \partial_{\xi_j}\mathcal{P}_\gamma^\perp f\rangle_{L^2_{x,\xi}} \tag{5.226}$$

$$+ \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}\mathcal{P}_\gamma^\perp f, \partial_{\xi_j}\mathcal{P}_\gamma^\perp f\rangle_{L^2_{x,\xi}}. \tag{5.227}$$

Now $\frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}\cdot, \partial_{\xi_j}\cdot\rangle_{L^2_{x,\xi}}$ defines a semi-inner product. So, we apply Cauchy-Schwarz and Young's inequality to the cross term to get

$$\frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}u, \partial_{\xi_j}u\rangle_{L^2_{x,\xi}} \tag{5.228}$$

$$\begin{aligned} &\geq \frac{1}{2\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}(b_k\mathfrak{h}_k), \partial_{\xi_j}(b_l\mathfrak{h}_l)\rangle_{L^2_{x,\xi}} \\ &\quad - \frac{2}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}\mathcal{P}_\gamma^\perp f, \partial_{\xi_j}\mathcal{P}_\gamma^\perp f\rangle_{L^2_{x,\xi}} \end{aligned} \tag{5.229}$$

$$\geq \frac{1}{C}W_{kl}\langle b_l, b_k\rangle_{L^2_x} - C\|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}, \tag{5.230}$$

where

$$W_{kl} := \frac{1}{\varepsilon}\langle \sigma_{ij}|_{\frac{\xi}{\varepsilon}} \mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}\mathfrak{h}_k, \partial_{\xi_j}\mathfrak{h}_l\rangle_{L^2_\xi}. \tag{5.231}$$

We now show that

$$W_{kl}\nu_k\nu_l \geq \frac{1}{C} \tag{5.232}$$

for all $\nu \in \mathbf{S}^2$. Now,

$$\partial_{\xi_i}\mathfrak{h}_k = \frac{\gamma^{\frac{1}{2}}}{2}(\delta_{ik} - \frac{\gamma}{2}\xi_i\xi_k)\mu_\gamma^{\frac{1}{2}} \tag{5.233}$$

Then, by the reverse triangle inequality,

$$\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\partial_{\xi_i}\mathfrak{h}_k, \partial_{\xi_j}\mathfrak{h}_l = \frac{\gamma}{4\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}(\nu_i - \frac{\gamma}{2}(\nu\cdot\xi)\xi_i)(\nu_j - \frac{\gamma}{2}(\nu\cdot\xi)\xi_j)\mu_\gamma \tag{5.234}$$

$$\geq \{\frac{\gamma}{8\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\nu_i\nu_j - C\frac{(\nu\cdot\xi)^2}{\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\xi_i\xi_j\}\mu_\gamma \tag{5.235}$$

Now, for $|\xi| \geq 1$, we have

$$\frac{1}{\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\xi_i\xi_j \sim \frac{|\xi|^2}{\langle \xi/\varepsilon\rangle^3} \sim \varepsilon^2\frac{1}{|\xi|}. \tag{5.236}$$

On the other hand,

$$\frac{1}{\varepsilon}\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\nu_i\nu_j \gtrsim \frac{1}{\varepsilon}\frac{|P_{\xi^\perp}\nu|^2}{\langle \xi/\varepsilon\rangle} \gtrsim \frac{|P_{\xi^\perp}\nu|^2}{|\xi|}. \tag{5.237}$$

Hence,

$$W_{kl}\nu_k\nu_l \geq \frac{1}{C}\int_{\mathbf{R}^3\setminus B_1}\frac{|P_{\xi^\perp}\nu|^2}{|\xi|}\mu_\gamma d\xi - \varepsilon^2\int_{\mathbf{R}^3\setminus B_1}|\xi|\mu_\gamma d\xi \tag{5.238}$$

$$\geq \frac{1}{C} - C\varepsilon^2. \tag{5.239}$$

Taking ε small enough, we deduce (5.232), and thus

$$\frac{1}{\varepsilon}\langle\sigma_{ij}|_{\frac{\xi}{\varepsilon}}\mathbf{1}_{B_1^c}(\xi)\partial_{\xi_i}u, \partial_{\xi_j}u\rangle_{L^2_{x,\xi}} \geq \frac{1}{C}\|b\|^2_{L^2_x} - C\|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.240}$$

From this, we conclude (5.219) in the case $r = 0$. □

The next lemma allows us to control (a, b, c).

Proposition 5.7. *Let $M \geq 1$, and let η, ς and ε be sufficiently small depending on M. Assume the bootstrap assumptions (5.3) through (5.6). Then, exists two real valued functions $\mathscr{G} = \mathscr{G}^{(M)}$ and $\mathscr{G}_{reg} = \mathscr{G}_{reg}^{(M)}$ on $[0, T]$ satisfying, for all $t \in [0, T]$*

$$|\mathscr{G}(t)| \lesssim_M \|f(t)\|^2_{L^2_{x,\xi}}, \quad |\mathscr{G}_{reg}(t)| \lesssim_M \|f^{(s)}(t)\|^2_{L^2_{x,\xi}} \tag{5.241}$$

such that

$$\begin{aligned}&\varepsilon\frac{d}{dt}\mathscr{G}(t) + \|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}\\ &\quad\lesssim_M \varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + \min_{\alpha\in\{1,2,3\}}\{\|\mathcal{P}_\gamma^\perp g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\},\end{aligned} \tag{5.242}$$

and

$$\begin{aligned}&\varepsilon\frac{d}{dt}\mathscr{G}_{reg}(t) + \|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}\\ &\quad\lesssim_M \varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + \min_{\alpha\in\{1,2\}}\{\|\mathcal{P}_\gamma^\perp g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha^{(s)}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\}.\end{aligned} \tag{5.243}$$

Proof. We break the proof into a number of steps. Given $u : \mathbf{T}^3 \to \mathbf{R}$, we split $u = u^\times + \overline{u}$, where $\overline{u}(t) = \int_{\mathbf{T}^3} u(t, x)dx$. The first step concerns the bounds on $(\overline{a}, \overline{c})$. In step 2, we bound $\|a\|_{L^2_x}$. In step 3, we bound $(e^{\gamma\psi}c)^\times$. In step 4, we bound $\|(a, c)\|_{H^s_x}$. In step 5, we synthesize these bounds.

Step 1 (estimate on $\overline{a}$ and $\overline{c}$): Regarding $\overline{a}$, from (5.21), we simply have

$$\int_{\mathbf{T}^3} a(t, x)dx \equiv 0 \tag{5.244}$$

for all times. In particular, $\Delta_x^{-1}a$ is well defined.

We now discuss the zero mode of c. Subtracting the first line of (4.138) from (1.15), we have

$$\sqrt{\frac{3}{2}}\frac{d}{dt}\left(\frac{\overline{c}(t)}{\gamma}\right) = \frac{d}{dt}\int_{\mathbf{T}^3\times\mathbf{R}^3}\frac{|\xi|^2}{2}(F_- - \mu_\gamma e^{\gamma\psi})dxd\xi \tag{5.245}$$

Integrating in time, we have

$$\sqrt{\frac{3}{2}}\frac{\overline{c}(t)}{\gamma(t)} = \sqrt{\frac{3}{2}}\frac{\overline{c}_{in}}{\beta_{in}} + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \phi_{in}(x)|^2 - |\nabla_x \psi_{in}(x)|^2 dx \tag{5.246}$$

$$+ \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \psi(t,x)|^2 - |\nabla_x \phi(t,x)|^2 dx \tag{5.247}$$

On the other hand, from (5.10),

$$\int_{\mathbf{T}^3} |\nabla_x \psi(t,x)|^2 - |\nabla_x \phi(t,x)|^2 dx = -2\langle \nabla_x \psi, \nabla_x \Delta_x^{-1} a\rangle_{L^2_x} - \|\nabla_x \Delta_x^{-1} a\|_{L^2_x} \tag{5.248}$$

$$= 2\langle \psi, a\rangle_{L^2_x} - \|\Delta_x^{-\frac{1}{2}} a\|^2_{L^2_x}. \tag{5.249}$$

Combining these with (5.7), we get

$$\left| \sqrt{\frac{3}{2}}\frac{\overline{c}(t)}{\gamma(t)} - \frac{1}{4\pi}\langle \psi, a\rangle_{L^2_x} \right| \lesssim \varepsilon + \|a\|^2_{L^2_x}. \tag{5.250}$$

In what follows, it will become clear that we cannot control $c^\times$ directly. Instead, we can only get a bound on $(ce^{\gamma\psi})^\times$. It is then necessary to compute $c^\times$ in terms of $(ce^{\gamma\psi})^\times$ and $\overline{c}$. Observe that

$$e^{\gamma\psi}(e^{-\gamma\psi}c)^\times + e^{\gamma\psi}\int_{\mathbf{R}^3} e^{-\gamma\psi} c dx' = c^\times + \overline{c}. \tag{5.251}$$

Since $\int_{\mathbf{T}^3} e^{\gamma\psi} dx = 1$, the above integrated in x yields

$$\int_{\mathbf{R}^3} e^{-\gamma\psi} c dx = \overline{c} - \int_{\mathbf{T}^3} e^{\gamma\psi}(e^{-\gamma\psi}c)^\times dx. \tag{5.252}$$

Thus, we have the identity

$$c^\times = (e^{\gamma\psi} - 1)\overline{c} + e^{\gamma\psi}(e^{-\gamma\psi}c)^\times - e^{\gamma\phi_0}\int_{\mathbf{T}^3} e^{\gamma\psi}(e^{-\gamma\psi}c)^\times dx \tag{5.253}$$

In particular, $\|c\|_{L^2_x} \lesssim |\overline{c}| + \|(e^{-\gamma\psi}c)^\times\|_{L^2_x}$.

Step 2 (estimate for a): Now, we turn to bounding a. We claim that the following estimate holds:

$$\begin{aligned} &-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x \Delta_x^{-1} a, b\rangle_{L^2_x} + \frac{1}{C}\|a\|^2_{L^2_x} \\ &\leq C\{\varepsilon^2 \langle \|F_+\|_{\mathfrak{D}}\rangle^2 + \|b\|^2_{L^2_x} + \|(e^{-\gamma\psi}c)^\times\|^2_{L^2_x} + \|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\}. \end{aligned} \tag{5.254}$$

To prove this, first observe

$$(\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}} E)a = \gamma^{-\frac{1}{2}} e^{\gamma\psi}\nabla_x(e^{-\gamma\psi} a) - 4\pi a \nabla_x \Delta_x^{-1} a, \tag{5.255}$$

Therefore,

$$-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, b\rangle_{L_x^2}+\gamma^{-\frac12}\|e^{-\frac{\gamma\psi}{2}}a\|_{L_x^2}^2+4\pi\|e^{\frac{\gamma\psi}{2}}\nabla_x\Delta_x^{-1}a\|_{L_x^2}^2+4\pi\overline{c}^2 \tag{5.256}$$

$$=-\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, \varepsilon\partial_t b+(\gamma^{-\frac12}\nabla_x+\gamma^{\frac12}E-4\pi\gamma^{\frac12}e^{\gamma\psi}\nabla_x\Delta_x^{-1})a+\sqrt{\frac23}\nabla_x c\rangle_{L_x^2} \tag{5.257}$$

$$-\varepsilon\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}\partial_t a, b\rangle_{L_x^2} \tag{5.258}$$

$$+\varepsilon\langle\partial_t(\gamma\psi)e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, b\rangle_{L_x^2} \tag{5.259}$$

$$+4\pi\varepsilon\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, a\nabla_x\Delta_x^{-1}a\rangle_{L_x^2} \tag{5.260}$$

$$+4\pi\overline{c}^2+\sqrt{\frac23}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a, \nabla_x c\rangle_{L_x^2}. \tag{5.261}$$

From (5.202), we have

$$\|\varepsilon\partial_t b_j+(\gamma^{-\frac12}\partial_{x_j}+\gamma^{\frac12}E_j)a-4\pi\gamma^{\frac12}e^{\gamma\psi}\nabla_x\Delta_x^{-1}a\|_{H_x^{-1}} \tag{5.262}$$

$$=\Big\|\varepsilon\frac{\dot\gamma}{2\gamma}b_j+\sqrt{\frac23}\partial_{x_j}c+\partial_{x_k}\langle\mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L_\xi^2}+\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L_\xi^2} \tag{5.263}$$

$$+e^{\gamma\psi}\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L_\xi^2}\Big\|_{H_x^{-1}} \tag{5.264}$$

$$\lesssim\varepsilon|\dot\gamma|\|b\|_{L_x^2}+\|c\|_{L_x^2}+\|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{5.265}$$

$$+\|\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}f\rangle_{L_\xi^2}\|_{H_x^{-1}} \tag{5.266}$$

$$+\|e^{\gamma\psi}\langle\mathfrak{h}_j, \mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L_\xi^2}\|_{H_x^{-1}} \tag{5.267}$$

Applying (5.43) to the latter two terms, we have

$$(5.266)\lesssim\Big\|\|F_+^{(3,0)}\|_{L_v^2}(\|f\|_{(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}+\varepsilon\|f\|_{(\mathcal{H}_\sigma)_\xi})\Big\|_{H_x^{-1}} \tag{5.268}$$

$$+\varepsilon\Big\|\|F_+^{(\frac72,0)}\|_{(\dot{\mathcal{H}}_\sigma)_v}\|f\|_{(\mathcal{H}_\sigma)_\xi}\Big\|_{H_x^{-1}} \tag{5.269}$$

$$\lesssim\|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}+\varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}. \tag{5.270}$$

and

$$(5.267)\lesssim\varepsilon\|F_+^{(3,0)}\|_{H_x^{-1}L_v^2}+\varepsilon\|F_+^{(\frac72,0)}\|_{H_x^{-1}(\dot{\mathcal{H}}_\sigma)_\xi} \tag{5.271}$$

$$\lesssim\varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle. \tag{5.272}$$

Then,

$$|(5.257)|\lesssim\|e^{\gamma\psi}\nabla_x(-\Delta_x)^{-1}a\|_{H_x^1} \tag{5.273}$$

$$\cdot\|\varepsilon\partial_t b+(\gamma^{-\frac12}\nabla_x+\gamma^{\frac12}E)a-4\pi\gamma^{\frac12}e^{\gamma\psi}\nabla_x\Delta_x^{-1}a\|_{H_x^{-1}} \tag{5.274}$$

$$\lesssim\|a\|_{L_x^2}\{\varepsilon\|b\|_{L_x^2}+\|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{5.275}$$

$$+\|f\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}+\varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle\}. \tag{5.276}$$

Now, note that

$$\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\lesssim\|a\|_{L^2_x}+\|b\|_{L^2_x}+|\overline{c}|+\|(e^{\gamma\psi}c)^\times\|_{L^2_x}+\|\mathcal{P}^\perp_\gamma f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.277}$$

Then, taking $\lambda>0$, and using Young's inequality, and the bootstrap assumptions, we have

$$|(5.257)|\le C(\varepsilon+\varsigma+\lambda)(\|a\|^2_{L^2_x}+\overline{c}^2) \tag{5.278}$$

$$+C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2+\|(b,(e^{\gamma\psi}c)^\times)\|^2_{L^2_x}+\|\mathcal{P}^\perp_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}+\|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\} \tag{5.279}$$

We now turn to bounding (5.258) and (5.259). Using (4.142) and (5.201), we have

$$\varepsilon\|\partial_t a\|_{H^{-1}_x}\lesssim\|b\|_{L^2_x}+\varepsilon\|\partial_t e^{\gamma\psi}\|_{H^{-1}_x}\lesssim\|b\|_{L^2_x}+\varepsilon\langle\|f\|_{L^2(\mathcal{H}_\sigma)_\xi}\rangle. \tag{5.280}$$

Hence,

$$|(5.258)|\lesssim(\varepsilon\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle+\|b\|_{L^2_x})\|b\|_{L^2_x} \tag{5.281}$$

$$\lesssim\varepsilon^2\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle^2+\|b\|^2_{L^2_x} \tag{5.282}$$

$$\lesssim\varepsilon^2(\|a\|_{L^2_x}+|\overline{c}|^2)+\|(b,(e^{\gamma\psi}c)^\times)\|^2_{L^2_x}+\|\mathcal{P}^\perp_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.283}$$

Next,

$$|(5.259)|\lesssim\varepsilon(\|\partial_t\psi\|_{L^\infty_x}+|\dot{\gamma}|)\|a\|_{L^2_x}\|b\|_{L^2_x}. \tag{5.284}$$

Applying (4.142) again, the above reduces to

$$|(5.259)|\lesssim\varepsilon\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle\|a\|_{L^2_x}\|b\|_{L^2_x} \tag{5.285}$$

$$\lesssim\varepsilon^2(\|a\|_{L^2_x}+|\overline{c}|^2)+\|(b,(e^{\gamma\psi}c)^\times)\|^2_{L^2_x}+\|\mathcal{P}^\perp_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.286}$$

Thus,

$$|(5.260)|\lesssim\varepsilon\|a\|_{L^\infty}\|a\|^2_{H^{-1}}\lesssim\varepsilon\|a\|^2_{L^2_x}. \tag{5.287}$$

Now to bound (5.261), we use (5.253) to get

$$\|\nabla_x c-\gamma\nabla_x\psi e^{\gamma\psi}\overline{c}\|_{H^{-1}_x}\lesssim\|(e^{-\gamma\psi}c)^\times\|_{L^2_x}. \tag{5.288}$$

Hence,

$$\left|\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a,\nabla_x c\rangle_{L^2_x}-\gamma\langle\nabla_x\Delta_x^{-1}a,\nabla_x\phi_0\rangle_{L^2_x}\overline{c}\right|\lesssim\|a\|_{L^2_x}\|(e^{-\gamma\psi}c)^\times\|_{L^2_x}. \tag{5.289}$$

Combining the above with (5.250), we have

$$(5.261)\le\sqrt{\frac{2}{3}}\left|\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a,\nabla_x c\rangle_{L^2_x}-\gamma\langle\nabla_x\Delta_x^{-1}a,\nabla_x\phi_0\rangle_{L^2_x}\overline{c}\right| \tag{5.290}$$

$$+\left|4\pi\bar{c}-\sqrt{\frac{2}{3}}\gamma\langle\phi_0,a\rangle_{L^2_x}\right||\bar{c}| \tag{5.291}$$

$$\lesssim \varsigma\|a\|^2_{L^2_x}+\|a\|_{L^2_x}\|(e^{-\gamma\psi}c)^\times\|_{L^2_x}+\varepsilon|\bar{c}| \tag{5.292}$$

Now, combining the bounds on (5.257) through (5.261), we now have

$$-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}a,b\rangle_{L^2_x}+\gamma^{-\frac{1}{2}}\|e^{-\frac{\gamma\psi}{2}}a\|^2_{L^2_x}+4\pi\bar{c}^2 \tag{5.293}$$

$$\le C(\varepsilon+\varsigma+\lambda)(\|a\|^2_{L^2_x}+\bar{c}^2) \tag{5.294}$$

$$+C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2+\|b\|^2_{L^2_x}+\|(e^{-\gamma\psi}c)^\times\|^2_{L^2_x}+\|P_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}+\|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\}. \tag{5.295}$$

Thus, taking ε, ς and λ sufficiently small, we have (5.254).
Step 3 (estimate for $a^{(s)}$): Next, we have the following estimate for $a^{(s)}$:

$$-\varepsilon\frac{d}{dt}\langle\nabla_x\Delta_x^{-1}a^{(s)},b^{(s)}\rangle_{L^2_x}+\frac{1}{C}\|a\|^2_{H^s} \tag{5.296}$$

$$\lesssim \varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2+\|a\|^2_{L^2_x}+\|(b,c)\|^2_{H^s_x}+\|\mathcal{P}_\gamma^\perp f^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.297}$$

To show this, we have

$$-\varepsilon\frac{d}{dt}\langle\nabla_x\Delta_x^{-1}a^{(s)},b^{(s)}\rangle_{L^2_x}+\gamma^{-\frac{1}{2}}\|a^{(s)}\|^2_{L^2_x} \tag{5.298}$$

$$=-\langle\nabla_x\Delta_x^{-1}a^{(s)},\varepsilon\partial_t b^{(s)}+\gamma^{-\frac{1}{2}}\nabla_x a^{(s)}\rangle_{L^2_x} \tag{5.299}$$

$$-\varepsilon\langle\nabla_x\Delta_x^{-1}\partial_t a^{(s)},b^{(s)}\rangle_{L^2_x}. \tag{5.300}$$

From (5.202), we have

$$\|\varepsilon\partial_t b_j^{(s)}+\gamma^{-\frac{1}{2}}\partial_{x_j}a^{(s)}\|_{H_x^{-1}} \tag{5.301}$$

$$=\left\|\varepsilon\frac{\dot{\gamma}}{2\gamma}b_j+\gamma^{\frac{1}{2}}E_j a-4\pi\gamma^{\frac{1}{2}}e^{\gamma\psi}\partial_{x_j}\Delta_x^{-1}a+\sqrt{\frac{2}{3}}\partial_{x_j}c+\partial_{x_k}\langle\mathfrak{h}_{jk},\mathcal{P}_\gamma^\perp f\rangle_{L^2_\xi}\right. \tag{5.302}$$

$$\left.+\langle\mathfrak{h}_j,\mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi}+e^{\gamma\psi}\langle\mathfrak{h}_j,\mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L^2_\xi}\right\|_{H_x^{s-1}} \tag{5.303}$$

$$\lesssim \varepsilon|\dot{\gamma}|\|b\|_{H_x^{s-1}}+\langle\|E\|_{H_x^{s-1}}\rangle\|a\|_{H_x^{s-1}}+\|c\|_{H^s}+\|\mathcal{P}_\gamma^\perp f\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.304}$$

$$+\|\langle\mathfrak{h}_j,\mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi}\|_{H_x^{s-1}}+\|e^{\gamma\psi}\langle\mathfrak{h}_j,\mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle_{L^2_\xi}\|_{H_x^{s-1}} \tag{5.305}$$

$$\lesssim \varepsilon\langle\|f\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle\|b\|_{H_x^{s-1}}+\|a\|_{H_x^{s-1}}+\|c\|_{H^s_x}+\|\mathcal{P}_\gamma^\perp f^{(s-1)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.306}$$

$$+\|f^{(s-1)}\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}+\varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f^{(s-1)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle \tag{5.307}$$

In the final line, we use the bootstrap assumptions, plus (4.142) and (5.43) as before, combined with algebra estimates for H^{s-1}. Hence, with the bootstrap assumptions, Young's inequality and interpolation, for any $\lambda>0$ we have

$$|(5.299)|\le C(\varepsilon+\varsigma+\lambda)\|a^{(s)}\|^2_{L^2_x} \tag{5.308}$$

$$+ C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle + \|a\|^2_{L^2_x} + \|(b,c)\|_{H^s_x} + \|P_\gamma^\perp f^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f^{(s)}\|_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\} \tag{5.309}$$

Next, from (4.142) and (5.201),

$$|(5.300)| \lesssim \|b^{(s)}\|^2_{L^2_x} + \varepsilon\|\partial_t(e^{\gamma\psi})\|_{H^{s-1}_x}\|b^{(s)}\|_{L^2_x} \tag{5.310}$$

$$\lesssim (\varepsilon\langle\|f^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}\rangle + \|b^{(s)}\|_{L^2_x})\|b^{(s)}\|_{L^2_x} \tag{5.311}$$

$$\lesssim \varepsilon^2\|a\|_{H^2_x} + \|(b,c)\|^2_{H^s_x} + \|P_\gamma^\perp f^{(s)}\|_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.312}$$

Taking ε, ς and λ sufficiently small, we conclude (5.296).
Step 4 (estimate on $(e^{\gamma\psi}c)^\times$**)**: We now show the following bound,

$$-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^\times, d\rangle_{L^2_x} + \frac{1}{C}\|(e^{-\gamma\psi}c)^\times\|^2_{L^2_x} \tag{5.313}$$

$$\leq C\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon+\varsigma)(\|a\|^2_{L^2_x} + |\overline{c}|^2) \tag{5.314}$$

$$+ \|b\|^2_{L^2_x} + \|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\}. \tag{5.315}$$

By writing

$$(\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}}E)c = \gamma^{-\frac{1}{2}}e^{\gamma\psi}\nabla_x(e^{-\gamma\psi}c) - 4\pi\gamma^{\frac{1}{2}}c\nabla_x\Delta_x^{-1}a, \tag{5.316}$$

we have

$$-\varepsilon\frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^\times, d\rangle_{L^2_x} + \frac{3\sqrt{3}}{5\sqrt{\gamma}}\|(e^{-\gamma\psi}c)^\times\|^2_{L^2_x} \tag{5.317}$$

$$= -\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^\times, \varepsilon\partial_t d + \frac{3\sqrt{3}}{5}(\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}}E)c\rangle_{L^2_x} \tag{5.318}$$

$$-\varepsilon\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}\partial_t c)^\times, d\rangle_{L^2_x} \tag{5.319}$$

$$-\varepsilon\langle\partial_t(e^{-\gamma\psi})\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^\times, d\rangle_{L^2_x} \tag{5.320}$$

$$-\varepsilon\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(\partial_t(e^{-\gamma\psi})c)^\times, d\rangle_{L^2_x} \tag{5.321}$$

$$-4\pi\frac{3\sqrt{3}}{5}\gamma^{\frac{1}{2}}\langle e^{-\gamma\psi}\nabla_x\Delta_x^{-1}(e^{-\gamma\psi}c)^\times, c\nabla_x\Delta_x^{-1}a\rangle_{L^2_x}. \tag{5.322}$$

First, from (5.204), we have

$$\|\varepsilon\partial_t d + \frac{3\sqrt{3}}{5}(\gamma^{-\frac{1}{2}}\nabla_x + \gamma^{\frac{1}{2}}E)c\|_{H^{-1}_x} \tag{5.323}$$

$$\lesssim \varepsilon|\dot{\gamma}|(\|b\|_{H^{-1}_x} + \|d\|_{H^{-1}_x}) \tag{5.324}$$

$$+ \sup_{j,k\in\{1,2,3\}}\{\|(\gamma^{-\frac{1}{2}}\partial_{x_k} + \gamma^{\frac{1}{2}}E_k)\langle\mathfrak{h}_{jk}, \mathcal{P}_\gamma^\perp f\rangle_{L^2_\xi}\|_{H^{-1}_x} \tag{5.325}$$

$$+ \|\partial_{x_k}\langle\mathfrak{h}_{jkll}, f\rangle_{L^2_\xi}\|_{H^{-1}_x} \tag{5.326}$$

$$+ \|\langle\mathcal{L}_\gamma\mathfrak{h}_{jll}, f\rangle_{L^2_\xi}\|_{H^{-1}_x} \tag{5.327}$$

$$+ \|\langle\mathfrak{h}_{jll}, \mathcal{M}_{\gamma,F_+}f\rangle_{L^2_\xi}\|_{H^{-1}_x} \tag{5.328}$$

$$+ \|\langle \mathfrak{h}_{jll}, e^{\gamma\psi}\mathcal{M}_{\gamma,F_+}\mathfrak{h}\rangle\|_{H_x^{-1}} \tag{5.329}$$

$$+ \|\langle \mathfrak{h}_{jll}, \Gamma_\gamma(f, f)\rangle\|_{H_x^{-1}}\} \tag{5.330}$$

Going line by line, we see that

$$(5.324) \lesssim \varepsilon(\|b\|_{L_x^2} + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}) \tag{5.331}$$

Next up, we have

$$(5.325) + (5.326) + (5.327) \leq \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}. \tag{5.332}$$

Next, by (5.43), we have

$$(5.328) \lesssim \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi} + \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{5.333}$$

$$(5.329) \lesssim \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle \tag{5.334}$$

Finally, from (5.33), we have

$$(5.330) \lesssim \varsigma\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} \tag{5.335}$$

Hence,

$$|(5.318)| \lesssim \|(ce^{\gamma\psi})^\times\|_{L_x^2}\{\|b\|_{L_x^2} + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} + \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi} \tag{5.336}$$

$$+ \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}\rangle + \varsigma\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}\}. \tag{5.337}$$

Using (5.277), we deduce that for any $\lambda > 0$,

$$|(5.318)| \leq C(\varepsilon + \varsigma + \lambda)\|(e^{\gamma\psi}c)^\times\|_{L_x^2}^2 \tag{5.338}$$

$$+ C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon + \varsigma)(\|a\|_{L_x^2}^2 + |\bar{c}|^2) \tag{5.339}$$

$$+ \|b\|_{L_x^2}^2 + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 + \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}^2\} \tag{5.340}$$

Next, we have the term (5.319). From (5.203), and (4.142),

$$\varepsilon\|\partial_t c\|_{H_x^{-1}} \lesssim \varepsilon|\dot{\gamma}|\langle\|(a, c)\|_{L_x^2}\rangle + \|(b, d)\|_{L_x^2} + \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}\rangle \tag{5.341}$$

$$\begin{aligned}&\lesssim \varepsilon\langle\|F_+\|_{\mathfrak{D}}\rangle\langle\|f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}\rangle + \varepsilon\|(a, c)\|_{L_x^2} \\ &\quad + \|b\|_{L_x^2} + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi} + \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}\end{aligned} \tag{5.342}$$

Thus

$$|(5.319)| \lesssim \varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + \varepsilon\|(a, c)\|_{L_x^2}^2 + \|b\|_{L_x^2}^2 + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2 + \|f\|_{L_x^2(\mathcal{H}_{\sigma;\varepsilon}^-)_\xi}^2 \tag{5.343}$$

Next,

$$|(5.320)| + |(5.321)| \lesssim \varepsilon\|c\|_{L_x^2}\|d\|_{L_x^2} \lesssim \varepsilon^2 + \|\mathcal{P}_\gamma^\perp f\|_{L_x^2(\mathcal{H}_\sigma)_\xi}^2. \tag{5.344}$$

Finally,

$$|(5.322)| \lesssim \|(e^{-\gamma\psi}c)^\times\|_{L_x^2}(\|c\nabla_x\Delta_x^{-1}a\|_{L_x^2} + \|c\nabla_x\Delta_x^{-1}(n_+^\varepsilon - n_+^0)\|_{L_x^2}) \tag{5.345}$$

$$\lesssim \|a\|_{H^s_x} \|(e^{-\gamma\psi}c)^\times\|_{L^2_x}(\|c\|_{L^2_x} + \|F^\varepsilon_+ - F^0_+\|_{\mathfrak{E}'}) \tag{5.346}$$

$$\lesssim \varsigma \|(e^{-\gamma\psi}c)^\times\|_{L^2_x}(\|c\|_{L^2_x} + \|F^\varepsilon_+ - F^0_+\|_{\mathfrak{E}'}) \tag{5.347}$$

Combining these bounds for (5.318) through (5.322), we conclude that

$$-\varepsilon \frac{d}{dt}\langle e^{-\gamma\psi}\nabla_x \Delta_x^{-1}(e^{-\gamma\psi}c)^\times, d\rangle_{L^2_x} + \frac{3\sqrt{3}}{5\sqrt{\gamma}}\|(e^{-\gamma\psi}c)^\times\|^2_{L^2_x} \tag{5.348}$$

$$\leq C(\varepsilon + \varsigma + \lambda)\|(e^{\gamma\psi}c)^\times\|^2_{L^2_x} \tag{5.349}$$

$$+ C_\lambda\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon+\varsigma)(\|a\|^2_{L^2_x} + |\overline{c}|^2) \tag{5.350}$$

$$+ \|b\|^2_{L^2_x} + \|\mathcal{P}^\perp_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\} \tag{5.351}$$

By taking ε, ς and λ sufficiently small, we conclude (5.313).
Step 5: Estimate on $c^{(s)}$: Following the same method as previous bounds, we have the estimate

$$-\varepsilon \frac{d}{dt}\langle \nabla_x \Delta_x^{-1} c^{(s)}, d^{(s)}\rangle_{L^2_x} + \frac{1}{C}\|c\|^2_{H^s_x} \tag{5.352}$$

$$\leq C\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon+\varsigma)\|a\|^2_{L^2_x} \tag{5.353}$$

$$+ \|c\|^2_{L^2_x} + \|b\|^2_{H^s_x} + \|\mathcal{P}^\perp_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f^{(s)}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\} \tag{5.354}$$

The proof is similar to that of (5.296).
Step 6: Combining the bounds: Combining the upper bounds on (5.254), (5.296), (5.219), (5.313), (5.352) there is a choice of constant $\kappa > 0$ taken sufficiently small, such that the functional

$$\mathscr{G}_{reg}(t) := \kappa\langle e^{-\gamma\psi}\nabla_x \Delta_x^{-1} a, b\rangle_{L^2_x} - \kappa^3\langle \nabla_x \Delta_x^{-1} a^{(s)}, b^{(s)}\rangle_{L^2_x} \tag{5.355}$$

$$- \langle e^{-\gamma\psi}\nabla_x \Delta_x^{-1}(e^{-\gamma\psi}c)^\times, d\rangle_{L^2_x} - \kappa^2\langle \nabla_x \Delta_x^{-1} c^{(s)}, d^{(s)}\rangle_{L^2_x} \tag{5.356}$$

satisfies

$$\varepsilon \frac{d}{dt}\mathscr{G}_{reg}(t) + \frac{1}{C}\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.357}$$

$$\leq C\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon+\varsigma)\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_{x,\xi}} \tag{5.358}$$

$$+ \|\mathcal{P}^\perp_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f^{(s)}\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \|F^\varepsilon_+ - F^0_+\|^2_{\mathfrak{E}'}\} \tag{5.359}$$

On the other hand by (5.31), for each $\alpha \in \{1, 2\}$,

$$\|\mathcal{P}^\perp_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \lesssim \|\mathcal{P}^\perp_\gamma g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{5.360}$$

$$+ \eta\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}. \tag{5.361}$$

Thus, by taking ε, ς and η sufficiently small, and re-scaling $\mathscr{G}$ if necessary, we conclude with (5.243). To get (5.242), we take $\kappa > 0$ and

$$\mathscr{G}(t) := -\kappa\langle e^{-\gamma\psi}\nabla_x \Delta_x^{-1} a, b\rangle_{L^2_x} - \langle e^{-\gamma\psi}\nabla_x \Delta_x^{-1}(e^{-\gamma\psi}c)^\times, d\rangle_{L^2_x} \tag{5.362}$$

so that

$$\varepsilon \frac{d}{dt}\mathscr{G}(t) + \frac{1}{C}\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \leq C\{\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 + (\varepsilon+\varsigma)\|\mathcal{P}_\gamma f\|^2_{L^2_{x,\xi}} \tag{5.363}$$

$$+ \|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi}\}. \tag{5.364}$$

And, once again, by (5.31), for each $\alpha \in \{1, 2, 3\}$,

$$\|\mathcal{P}_\gamma^\perp f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|f\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \lesssim \|\mathcal{P}_\gamma^\perp g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})_\xi} + \eta\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.365}$$

Again, taking ε, ς and η sufficiently small, and re-scaling $\mathscr{G}$ if necessary, we deduce (5.242). □

We conclude this section with a corollary of Propositions 5.4 and 5.7:

Corollary 5.8. *Let $M \geq 1$, and let η be sufficiently small depending on M. We let ς and ε be sufficiently small depending on M and η. Assume the bootstrap assumptions (5.3) through (5.6). Then, for each $M, \eta > 0$, and $\alpha \in \{1, 2, 3\}$ there exists a function $\mathscr{Y}_{-,\alpha} = \mathscr{Y}^{(M,\eta)}_{-,\alpha} : [0, T] \to \mathbf{R}_+$ such that*

$$\mathscr{Y}_{-,\alpha} \sim_{M,\eta} \widetilde{\mathscr{E}}_{-,\alpha} \tag{5.366}$$

and

$$\varepsilon\frac{d}{dt}\mathscr{Y}_{-,\alpha} + \widetilde{\mathscr{D}}_{-,\alpha} \lesssim_{M,\eta} \varepsilon^2. \tag{5.367}$$

Proof. We take $\kappa_0, \kappa_1, \kappa_2$ and $\kappa_3 \in (0, 1)$ be constants to be determined. We combine (5.71) in the case $\alpha = 2$ and (5.242) as follows:

$$\varepsilon\frac{d}{dt}\{\|g_\alpha\|^2_{L^2_{x,\xi}} + 4\pi\sqrt{\gamma}\||\Delta_x|^{-1}a\|^2_{L^2_x} + \kappa_1\mathscr{G}\} \tag{5.368}$$

$$+ (\frac{1}{C} - C\kappa_1)\{\|\mathcal{P}_\gamma^\perp g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} + \|g_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}\} + \kappa_0\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.369}$$

$$\leq C(\varepsilon + \varsigma + \eta + \kappa_0)\|\mathcal{P}_\gamma f\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.370}$$

$$+ C\varepsilon\langle\|\partial_t a\|_{L^2_x} + \|F_+\|^2_{\mathfrak{D}}\rangle\|g_\alpha\|^2_{L^2_{x,\xi}} \tag{5.371}$$

$$+ C_{\kappa_0}\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2. \tag{5.372}$$

Using (5.32), we first fix κ_1 sufficiently small such that the "$\frac{1}{C} - C\kappa_1$" in the above is strictly positive, and

$$\mathscr{X}_{-,\alpha} := \|g_\alpha\|^2_{L^2_{x,\xi}} + 4\pi\sqrt{\gamma}\||\Delta_x|^{-1}a\|^2_{L^2_x} + \kappa_1\mathscr{G} \tag{5.373}$$

satisfies

$$\mathscr{X}_{-,\alpha} \sim \|g_\alpha\|^2_{L^2_{x,\xi}}. \tag{5.374}$$

Then, we fix κ_0 to be a sufficiently small constant, and also require that ε, ς and η are sufficiently small so that

$$\varepsilon\frac{d}{dt}\mathscr{X}_{-,\alpha}+\frac{1}{C}\|g_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{5.375}$$

$$\leq C\{\varepsilon\langle\|\partial_t a\|_{L^2_x}+\|F_+\|^2_{\mathfrak{D}}\rangle\mathscr{X}_{-,\alpha}+\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2\}. \tag{5.376}$$

Next, we combine this bound with (5.72),

$$\varepsilon\frac{d}{dt}(\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}}+\kappa_3\mathscr{G}_{reg}) \tag{5.377}$$

$$+(\frac{1}{C}-C\kappa_3)\{\|\mathcal{P}^\perp_\gamma g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}+\|g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}^-_{\sigma;\varepsilon})}\}+\kappa_3\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi} \tag{5.378}$$

$$\leq C(\varepsilon+\varsigma+\eta+\kappa_2)\|\mathcal{P}_\gamma f^{(s)}\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}+C_{\eta,\kappa_2}\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{5.379}$$

$$+C\varepsilon\langle\|\partial_t a\|_{H^{s-\frac14}_x}+\|F_+\|^2_{\mathfrak{D}}\rangle\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}} \tag{5.380}$$

$$+C_{\kappa_2}\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2 \tag{5.381}$$

We now fix κ_3 small enough so that the "$\frac{1}{C}-C\kappa_3$" in the above is positive, and

$$\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}}+\kappa_3\mathscr{G}_{reg}\sim\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}}, \tag{5.382}$$

and fix κ_2, and take ε, ς and η small enough that

$$\varepsilon\frac{d}{dt}(\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}}+\kappa_3\mathscr{G}_{reg})+\frac{1}{C}\|g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{5.383}$$

$$\leq C_\eta\|g_{\alpha+1}\|^2_{L^2_x(\mathcal{H}_\sigma\cap\mathcal{H}^-_{\sigma;\varepsilon})_\xi} \tag{5.384}$$

$$+C\{\varepsilon\langle\|\partial_t a\|_{L^2_x}+\|F_+\|^2_{\mathfrak{D}}\rangle\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}}+\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2\}. \tag{5.385}$$

FInally, we take $\kappa_4^{(\eta)}$ sufficiently small depending on η so that

$$\mathscr{X}^{(\eta)}_{-,\alpha,reg}:=\kappa_4^{(\eta)}(\|g^{(s)}_\alpha\|^2_{L^2_{x,\xi}}+\kappa_3\mathscr{G}_{reg})+\mathscr{X}_{-,\alpha+1} \tag{5.386}$$

satisfies

$$\frac{d}{dt}\mathscr{X}^{(\eta)}_{-,reg,\alpha}+\frac{1}{C_\eta}\widetilde{\mathscr{D}}_{-,\alpha} \tag{5.387}$$

$$\leq C_\eta\{\varepsilon\langle\|\partial_t a\|_{L^2_x}+\|F_+\|^2_{\mathfrak{D}}\rangle\mathscr{X}^{(\eta)}_{-,reg,\alpha}+\varepsilon^2\langle\|F_+\|_{\mathfrak{D}}\rangle^2\}. \tag{5.388}$$

By (4.2), we have

$$\frac{1}{C}\langle\|F_+\|_{\mathfrak{D}}\rangle^2\leq-\frac{d}{dt}\mathscr{X}_++C(1+\widetilde{\mathscr{D}}_{-,\alpha}). \tag{5.389}$$

Taking ε and ς sufficiently small depending on η now, we can hide the $\|g^{(s)}_\alpha\|^2_{L^2_x(\mathcal{H}_\sigma)_\xi}$ term and get for some collection of constants $C_1^{(\eta)}$, $C_2^{(\eta)}$ and $C_3^{(\eta)}$,

$$\varepsilon(\frac{d}{dt} + C_1^{(\eta)}\frac{d}{dt}\mathscr{X}_+ - C_2^{(\eta)}\langle\|\partial_t a\|_{L_x^2}\rangle)\mathscr{X}_{-,reg,\alpha}^{(\eta)} + \frac{1}{C_3^{(\eta)}}\widetilde{\mathscr{D}}_{-,\alpha} \tag{5.390}$$

$$\leq C_3^{(\eta)}\varepsilon^2. \tag{5.391}$$

We now define

$$\Omega^{(\eta)}(t) := C_1^{(\eta)}\mathscr{X}_+(t) - C_2^{(\eta)}\int_0^t \langle\|\partial_t a(t')\|_{L_x^2}\rangle dt' \tag{5.392}$$

By the (5.3) and (5.6), we have that $|\Omega^{(\eta)}|$ is less than or equal to some constant depending on η, say $C_4^{(\eta)} > 0$. Thus, if we set $\mathscr{Y}_j^{(\eta)} = C_3^{(\eta)}e^{\Omega + C_4^{(\eta)}}\mathscr{X}_{-,reg,\alpha}^{(\eta)}$, we have a function which satisfies (5.366) and (5.367). □

5.6. Proof of Proposition 5.1. We now conclude this section with a proof of Proposition 5.1.

Proof of Proposition 5.1: Throughout this proof, we take η sufficiently small so as to satisfy the hypothesis of Corollary 5.8. Any of the constants appearing in the estimates will implicitly depend on the constants $M, \eta > 0$ i.e. $C = C_{M,\eta}$.

For each $\alpha \in \{1, 2\}$, we integrate the bound in Corollary 5.8 on $t \in [0, T]$ and use (5.366) to get

$$\varepsilon \sup_{t\in[0,\hat{T}]} \widetilde{\mathscr{E}}_{-,\alpha}(t) + \int_0^{\hat{T}} \widetilde{\mathscr{D}}_{-,\alpha}(t)dt \tag{5.393}$$

$$\leq C(\varepsilon^2 + \varepsilon\widetilde{\mathscr{E}}_{-,\alpha}(0)). \tag{5.394}$$

Taking $j = 2$, this implies (5.8).

We now show (5.9). For this, we introduce $\theta > 0$, to be chosen latter. Now, for any $h = h(\xi)$, observe that

$$\|h\|_{\mathcal{H}_\sigma}^2 \gtrsim \int_{\mathbf{R}^3} \frac{1}{\langle\xi\rangle} h(\xi)^2 d\xi \tag{5.395}$$

$$\geq \frac{1}{\theta t^{\frac{1}{3}}} \int_{\langle\xi\rangle\leq\theta t^{\frac{1}{3}}} h(\xi)^2 d\xi \tag{5.396}$$

$$\geq \frac{1}{\theta t^{\frac{1}{3}}}\left(\|h\|_{L^2}^2 - \int_{\langle\xi\rangle\geq\theta t^{\frac{1}{3}}} h(\xi)^2 d\xi\right) \tag{5.397}$$

$$\geq \frac{1}{\theta t^{\frac{1}{3}}}\left(\|h\|_{L^2}^2 - \int_{\langle\xi\rangle\geq\theta t^{\frac{1}{3}}} e^{\frac{1}{4}(e^\eta-1)\beta_{in}(|\xi|^2+1-\theta^2 t^{\frac{2}{3}})} h(\xi)^2 d\xi\right) \tag{5.398}$$

$$\geq \frac{1}{\theta t^{\frac{1}{3}}}\left(\|h\|_{L^2}^2 - e^{-\frac{1}{4}(e^\eta-1)\beta_{in}\theta^2 t^{\frac{2}{3}}}\|e^{\frac{1}{4}(e^\eta-1)\beta_{in}(|\xi|^2+1)}h\|_{L^2}^2\right) \tag{5.399}$$

Applying this to $h = g_1^{(s)}$ and g_2 and integrating in x, and re-scaling θ, we have

$$\widetilde{\mathscr{D}}_{-,1} \gtrsim \frac{1}{\theta t^{\frac{1}{3}}}\left(\mathscr{Y}_{-,1} - e^{-\theta^2 t^{\frac{2}{3}}}\mathscr{Y}_{-,2}\right) \tag{5.400}$$

Thus, using (5.4) with (5.367), we have

$$\varepsilon \frac{d}{dt}\mathscr{Y}_{-,1} + \frac{2}{3C\theta t^{\frac{1}{3}}}\mathscr{Y}_{-,1} \leq C(\varepsilon^2 + \frac{2}{3\theta t^{\frac{1}{3}}} e^{-\theta^2 t^{\frac{2}{3}}} \sup_{t'\in[0,T]} \widetilde{\mathscr{E}}_{-,2}(t'). \tag{5.401}$$

and so

$$\frac{d}{dt}(e^{\frac{1}{C\theta\varepsilon}t^{\frac{2}{3}}}\mathscr{Y}_{-,1}) \leq C(\varepsilon e^{\frac{1}{C\theta\varepsilon}t^{\frac{2}{3}}} + \frac{1}{\varepsilon\theta t^{\frac{1}{3}}} e^{(\frac{1}{C\theta\varepsilon}-\theta^2)t^{\frac{2}{3}}} \sup_{t'\in[0,T]} \widetilde{\mathscr{E}}_{-,2}(t'). \tag{5.402}$$

Hence, by taking $\theta = \left(\frac{2}{C\varepsilon}\right)^{\frac{1}{3}}$ with C as in the above, we get some for some C_0, C_1, C_2,

$$\frac{d}{dt}(e^{\frac{1}{C_0}(\frac{t}{\varepsilon})^{\frac{2}{3}}}\mathscr{Y}_{-,1}) \leq C_2(\varepsilon e^{\frac{1}{C_0}(\frac{t}{\varepsilon})^{\frac{2}{3}}} + \frac{d}{dt}e^{-\frac{1}{C_1}(\frac{t}{\varepsilon})^{\frac{2}{3}}} \sup_{t'\in[0,T]} \widetilde{\mathscr{E}}_{-,2}(t') \tag{5.403}$$

Now,

$$\int_0^t e^{\frac{1}{C_0}(\frac{t'}{\varepsilon})^{\frac{2}{3}}} dt' \leq \varepsilon^{\frac{2}{3}} t^{\frac{1}{3}} \int_0^t \frac{1}{\varepsilon^{\frac{2}{3}}(t')^{\frac{1}{3}}} e^{\frac{1}{C_0}(\frac{t'}{\varepsilon})^{\frac{2}{3}}} dt' \lesssim \varepsilon^{\frac{2}{3}} t^{\frac{1}{3}} e^{\frac{1}{C_0}(\frac{t}{\varepsilon})^{\frac{2}{3}}}. \tag{5.404}$$

Thus, by integrating (5.403), we get

$$\mathscr{Y}_{-,1}(t) \lesssim e^{-\frac{1}{C}(\frac{t}{\varepsilon})^{\frac{2}{3}}} \sup_{t'\in[0,T]} \widetilde{\mathscr{E}}_{-,2}(t') + \varepsilon^{\frac{5}{3}} t^{\frac{1}{3}}. \tag{5.405}$$

for all $t \in [0, T]$. Using (5.366), we conclude (5.9). □

6. Proof of Theorem 1.1

In this section, we conclude with the proof of the main theorem.

Proof of Theorem 1.1: We break the proof of the Theorem into the proofs of parts (i) and (ii).

Proof of part (i): We break the proof of (i) into a number of steps: the setup of the bootstrap argument, combining the estimates of the preceding results, and closing the bootstrap argument.

Step 1 of part (i) (setup & bootstrap assumptions): We take $\kappa = \kappa(M) > 0$ to be a constant to be determined later. We take (F_+^0, β, ϕ^0) to be a weak solution to (1.7) on $[0, T_0]$ satisfying the hypotheses of the theorem. In addition, we may as well take $T_0 > 0$ small enough so that

$$\int_0^{T_0} \|F_+^0(t)\|_{\mathfrak{D}}^2 dt \leq \kappa^2. \tag{6.1}$$

Now, for all ε sufficiently small, there exists a unique weak solution $(F_+^\varepsilon, F_-^\varepsilon)$ to (1.7) on some interval containing zero, in the sense described in Lemma 3.1. Moreover, we let $(\psi^\varepsilon, \gamma^\varepsilon)$ be the solution to (4.138) in the sense of Lemma 4.3, defined on some time interval containing 0. We then take $\eta, \overline{\epsilon}$, and ς as in Proposition 5.1, and take $\varepsilon \in (0, \overline{\epsilon}]$.

By continuity in time of the quantities considered, there exists $\hat{T}_\varepsilon > 0$ such that the following conditions hold:

$$\sup_{t\in[0,\hat{T}_\varepsilon]} (\|\frac{1}{n_+^\varepsilon(t)}\|_{L_x^\infty} + \|F_+^\varepsilon(t)\|_{\mathfrak{E}}) \le 2M, \tag{6.2}$$

$$\sup_{t\in[0,\hat{T}_\varepsilon]} \|F_+^\varepsilon(t) - F_-^0(t)\|_{\mathfrak{E}'} \le \kappa \tag{6.3}$$

$$\sup_{t\in[0,\hat{T}_\varepsilon]} \sqrt{\mathscr{E}_{-,2}^\varepsilon(t)} \le \kappa, \tag{6.4}$$

$$\sup_{t\in[0,\hat{T}_\varepsilon]} |\ln(\frac{\gamma^\varepsilon(t)}{\beta_{in}})| \le \eta, \tag{6.5}$$

$$\int_0^{\hat{T}_\varepsilon} \|\partial_t(n_-^\varepsilon - e^{\gamma^\varepsilon\psi^\varepsilon})\|_{L_x^2} dt \le 1. \tag{6.6}$$

By Lemmas 3.1 (taking $\kappa < \frac{\varrho(2M)}{2}$) and 4.3 (using condition (6.5)), we may take κ small enough so that the interval of existence of $(F_+^\varepsilon, F_-^\varepsilon)$ and $(\gamma^\varepsilon, \psi^\varepsilon)$ is strictly larger than $[0, \hat{T}_\varepsilon]$. Hence, we may take $\hat{T}_\varepsilon > 0$ to be the largest such time, so that either at least one of the above conditions holds with equality, or $\hat{T}_\varepsilon = T_0$.

Next, we note that by condition (6.4) and (6.5), we have

$$\sup_{t\in[0,T]} \{\|\frac{1}{n_-^\varepsilon(t)}\|_{L_x^\infty} + \|F_-^\varepsilon(t)\|_{\mathfrak{E}}\} \le C_M, \tag{6.7}$$

so the consequences of Propositions 4.1 and 4.2 both hold, up to replacing M with some C_M in their hypotheses.

Step 2 of part (i) (combining estimates): We now combine the estimates of the preceding propositions and lemmas. We first show that the hypotheses of Proposition 5.1 are satisfied. By (4.143), for each $\alpha \in \{1, 2\}$, and $T \in [0, \hat{T}_\varepsilon]$, we have

$$\sup_{t\in[0,T]} (\|e^{\frac{q_\alpha|\xi|^2}{2}} \langle\nabla_x\rangle^s \langle\nabla_\xi\rangle(\mu_\beta e^{\beta\phi^0} - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon})\|_{L_{x,\xi}^2} \tag{6.8}$$

$$+ \|e^{\frac{q_{\alpha+1}|\xi|^2}{2}} \langle\nabla_\xi\rangle(\mu_\beta e^{\beta\phi^0} - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon})\|_{L_{x,\xi}^2}) \tag{6.9}$$

$$\lesssim_M \sup_{t\in[0,T]} (|\beta - \gamma^\varepsilon| + \|\phi^0 - \psi^\varepsilon\|_{H_x^s}) \tag{6.10}$$

$$\lesssim_M \sup_{t\in[0,T]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'}. \tag{6.11}$$

Hence, for all such T, and for each $\alpha \in \{1, 2\}$, we have

$$\sup_{t\in[0,T]} |\mathscr{E}_{-,\alpha}^\varepsilon(t) - \widetilde{\mathscr{E}}_{-,\alpha}^\varepsilon(t)| \le C_M \sup_{t\in[0,T]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'}^2, , \tag{6.12}$$

$$\sup_{t\in[0,T]} |\mathscr{D}_{-,\alpha}^\varepsilon(t) - \widetilde{\mathscr{D}}_{-,\alpha}^\varepsilon(t)| \le C_M \sup_{t\in[0,T]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'}^2.. \tag{6.13}$$

Using (6.12) with $T = 0$, we have

$$\widetilde{\mathscr{E}}_{-,\alpha,in}^\varepsilon \lesssim_M \mathscr{E}_{-,\alpha,in}^\varepsilon + C_M\varepsilon^2 \lesssim \varepsilon^2. \tag{6.14}$$

By the condition $\mathscr{E}^{\varepsilon}_{-,2,in} \leq M\varepsilon$, we have

$$\left|\int_{\mathbf{R}^3\times\mathbf{T}^3} \frac{|\xi|^2}{2}(F^{\varepsilon}_{-,in} - \mu_{\gamma^{\varepsilon}_{in}} e^{\beta_{in}\psi^{\varepsilon}_{in}})dxd\xi + \frac{1}{8\pi}\int_{\mathbf{T}^3} |\nabla_x \phi^{\varepsilon}_{in}|^2 - |\nabla_x \psi^{\varepsilon}_{in}|^2 dx\right| \tag{6.15}$$

$$\lesssim \sqrt{\widetilde{\mathscr{E}}_{-,2,in}} + \widetilde{\mathscr{E}}_{-,2,in} \tag{6.16}$$

$$\lesssim C_M(\varepsilon + \varepsilon^2). \tag{6.17}$$

Thus, we have that (5.7) holds in Proposition 5.1. On the other hand, by (6.3), (6.4) and (6.12), we have $\sup_{t\in[0,T]} \sqrt{\widetilde{\mathscr{E}}^{\varepsilon}_{-,\alpha}(t)} \lesssim_M \kappa$. Taking $\kappa \leq \frac{\varsigma}{C_M}$, we have that the conclusions of Proposition 5.1 are hold on $[0, \hat{T}_\varepsilon]$.

From (5.8), we have

$$\varepsilon \sup_{t\in[0,\hat{T}_\varepsilon]} \mathscr{E}^{\varepsilon}_{-,2}(t) + \int_0^{\hat{T}_\varepsilon} \mathscr{D}^{\varepsilon}_{-,2}(t)dt \leq C_M(\varepsilon\mathscr{E}^{\varepsilon}_{-,2,in} + \varepsilon^2). \tag{6.18}$$

Combining the above with (5.9), we have and for all $t \in [0, \hat{T}_\varepsilon]$, we have

$$\mathscr{E}^{\varepsilon}_{-,1}(t) \leq C_M(e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}}\varepsilon + \varepsilon^{\frac{5}{3}} t^{\frac{1}{3}} + \varepsilon^2). \tag{6.19}$$

We now prove the estimate

$$\sup_{t\in[0,\hat{T}_\varepsilon]} \|F^{\varepsilon}_{+}(t) - F^{0}_{+}(t)\|^2_{\mathfrak{E}'} + \int_0^{\hat{T}_\varepsilon} \|F^{\varepsilon}_{+}(t) - F^{0}_{+}(t)\|^2_{\mathfrak{D}'}dt \lesssim_M \varepsilon^2. \tag{6.20}$$

Using the above and (5.8), then for all $T \leq \hat{T}_\varepsilon$, we integrate (4.102) on $t \in [0, T]$ to get

$$\sup_{t\in[0,T]} \|F^{\varepsilon}_{+}(t) - F^{0}_{+}(t)\|^2_{\mathfrak{E}'} + \int_0^{T} \|F^{\varepsilon}_{+}(t) - F^{0}_{+}(t)\|^2_{\mathfrak{D}'}dt \tag{6.21}$$

$$\lesssim_M \|F^{\varepsilon}_{+}(0) - F^{0}_{+}(0)\|^2_{\mathfrak{E}'} \tag{6.22}$$

$$+ \int_0^{T} \langle\|(F^{\varepsilon}_{+}, F^{0}_{+})\|_{\mathfrak{D}}\rangle^2(\varepsilon^2 + \|F^{\varepsilon}_{+}(t) - F^{0}_{+}(t)\|_{\mathfrak{E}'}) + \mathscr{D}_{-,2}(t)dt \tag{6.23}$$

$$\lesssim_M \int_0^{T} \langle\|(F^{\varepsilon}_{+}, F^{0}_{+})\|_{\mathfrak{D}}\rangle^2(\varepsilon^2 + \sup_{t'\in[0,t]} \|F^{\varepsilon}_{+}(t') - F^{0}_{+}(t')\|^2_{\mathfrak{E}'})dt + \varepsilon^2. \tag{6.24}$$

In the final line, we used (6.18). Using the time integrated form of Grönwall's inequality, we deduce (6.20), provided the follow bound holds true:

$$\int_0^{\hat{T}_\varepsilon} \langle\|(F^{0}_{+}(t), F^{\varepsilon}_{+}(t))\|_{\mathfrak{D}}\rangle^2 dt \lesssim_M 1. \tag{6.25}$$

Indeed, by integrating (4.2) in Proposition 4.1, and using (6.18) to control the right-hand side, we have

$$\int_0^{\hat{T}_\varepsilon} \|F^{\varepsilon}_{+}(t)\|^2_{\mathfrak{D}} dt \lesssim_M 1. \tag{6.26}$$

Combined with (6.1), we have (6.25).

Step 3 of part (i) (closing the bootstrap): We now show that $\hat{T} := \inf_{\varepsilon\in(0,\overline{\varepsilon}]} \hat{T}_\varepsilon \geq \kappa$. To show this, we suppose $\hat{T}_\varepsilon < \kappa$ to yield a contradiction. Now, one of (6.2) through (6.6) holds with equality. We now show that in each of these five cases, the condition $\hat{T}_\varepsilon < \kappa$ cannot hold.

First, suppose (6.2) holds with equality. By integrating (4.3), and using (6.18) to control the right-hand side, we have for all $t \in [0, \hat{T}_\varepsilon]$,

$$\|F_+^\varepsilon(t)\|_{\mathfrak{E}}^2 \leq \|F_{+,in}^\varepsilon\|_{\mathfrak{E}}^2 + C_M(t + \|\langle v\rangle^{m_1} F_+^\varepsilon\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}^+_{\sigma;\varepsilon})_v} dt') \tag{6.27}$$

We claim that

$$\int_0^t \|\langle v\rangle^{m_1} F_+^\varepsilon(t')\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}^+_{\sigma;\varepsilon})_v} dt' \lesssim_M \kappa + \varepsilon. \tag{6.28}$$

Indeed,

$$\int_0^t \|\langle v\rangle^{m_1} F_+^\varepsilon(t')\|^2_{L^2_x(\dot{\mathcal{H}}_\sigma\cap\mathcal{H}^+_{\sigma;\varepsilon})_v} dt' \leq \int_0^t \|F_+^\varepsilon(t')\|_{\mathfrak{D}'} dt' + \int_0^t \|\langle v\rangle^{m_1} F_+^\varepsilon(t')\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} dt' \tag{6.29}$$

Now, by (6.1) and (6.20),

$$\int_0^t \|F_+^\varepsilon(t')\|^2_{\mathfrak{D}'} dt' \lesssim \int_0^t \|F_+^\varepsilon\|^2_{\mathfrak{D}'} dt' + \int_0^t \|F_+^\varepsilon(t') - F_+^0(t')\|^2_{\mathfrak{D}'} dt' \lesssim_M \varepsilon^2 + \kappa^2. \tag{6.30}$$

On the other hand, recalling (1.22), we note that for all $\nu \in \mathbf{S}^2$,

$$\varepsilon\sigma_{ij}(\varepsilon v)\nu_i\nu_j \lesssim \varepsilon \lesssim \varepsilon\langle v\rangle^3\sigma_{ij}(v)\nu_i\nu_j. \tag{6.31}$$

Hence, using (6.25),

$$\begin{aligned}\int_0^t \|\langle v\rangle^{m_1} F_+^\varepsilon\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} dt' &\lesssim t + \int_0^t \|\langle v\rangle^{m_2} F_+^\varepsilon\|^2_{L^2_x(\mathcal{H}^+_{\sigma;\varepsilon})_v} dt' &(6.32)\\ &\lesssim_M t + \varepsilon \int_0^t \|F_+^\varepsilon\|^2_{\mathfrak{D}} dt' &(6.33)\\ &\lesssim_M t + \varepsilon. &(6.34)\end{aligned}$$

Using $t \leq \hat{T}_\varepsilon \leq \kappa$, we conclude (6.28) holds true. Taking the supremum of (6.27) over all $t \in [0, \hat{T}_\varepsilon]$, we deduce

$$\sup_{t\in[0,\hat{T}\varepsilon]} \|F_+^\varepsilon(t)\|_{\mathfrak{E}} \leq \|F_{+,in}^\varepsilon\|_{\mathfrak{E}} + C_M(\kappa + \varepsilon). \tag{6.35}$$

Next, by the continuity equation, we have for all $t \in [0, \hat{T}_\varepsilon]$,

$$\frac{1}{n_+^\varepsilon(t,x)} = \frac{1}{n_{+,in}^\varepsilon(x) - \int_0^t \int_{\mathbf{R}^3} v\cdot\nabla_x F_+(t,x,v)dvdt} \tag{6.36}$$

Taking κ smaller if necessary, we get

$$\sup_{t\in[0,\hat{T}_\varepsilon]} \|\frac{1}{n_+^\varepsilon(t)}\|_{L_x^\infty} \le \|\frac{1}{n_{+,in}^\varepsilon}\|_{L_x^\infty} \frac{1}{1 - C\|\frac{1}{n_{+,in}^\varepsilon}\|_{L_x^\infty}\|F_+^\varepsilon\|_{\mathfrak{E}}\kappa} \le \frac{3}{2}\|\frac{1}{n_{+,in}^\varepsilon}\|_{L_x^\infty}. \tag{6.37}$$

Now, taking ε and κ sufficiently small, we have

$$2M = \sup_{t\in[0,\hat{T}_\varepsilon]} (\|\frac{1}{n_+^\varepsilon(t)}\|_{L_x^\infty} + \|F_+^\varepsilon(t)\|_{\mathfrak{E}}) \le \frac{3}{2}\|\frac{1}{n_{+,in}^\varepsilon}\|_{L_x^\infty} + \|F_{+,in}^\varepsilon\|_{\mathfrak{E}} + C_M(\kappa+\varepsilon) \tag{6.38}$$

$$= \frac{3}{2}M. \tag{6.39}$$

Next, assume (6.3) holds with equality. However, by (6.20), we have $\sup_{t\in[0,\hat{T}_\varepsilon]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'} \lesssim_M \varepsilon$, so we may take ε sufficiently small to reach a contradiction.

Next, assume (6.4) holds with equality. However by (6.18), $\sqrt{\mathscr{E}^\varepsilon{}_{-,2}} \lesssim_M \sqrt{\varepsilon}$. Thus, by taking ε small enough, we ensure

$$\kappa = \sqrt{\mathscr{E}^\varepsilon{}_{-,2}(\hat{T}_\varepsilon)} \le \frac{\kappa}{2}, \tag{6.40}$$

a contradiction.

Next, assume (6.5) holds with equality. By (4.142) and (6.18),

$$\eta = |\ln(\frac{\gamma^\varepsilon(\hat{T}_\varepsilon)}{\beta_{in}})| \tag{6.41}$$

$$\le \int_0^{\hat{T}_\varepsilon} |\frac{\dot{\gamma}^\varepsilon(t)}{\gamma^\varepsilon(t)}|dt \tag{6.42}$$

$$\lesssim \int_0^{\hat{T}_\varepsilon} 1 + \sqrt{\mathscr{D}_{-,2}^\varepsilon}dt \tag{6.43}$$

$$\le \hat{T}_\varepsilon + \hat{T}_\varepsilon^{\frac{1}{2}} \left(\int_0^{T_\varepsilon} \mathscr{D}_{-,2}^\varepsilon dt\right)^{\frac{1}{2}}. \tag{6.44}$$

Combining with (6.18), we conclude $\eta \lesssim \hat{T}_\varepsilon$. We conclude $\eta = \eta(M) \lesssim_M \hat{T}_\varepsilon \le \kappa$. Taking κ sufficiently small, we reach a contradiction.

Now, assume (6.6) holds with equality. By (5.201), (4.142), (5.8), we have

$$1 = \int_0^{\hat{T}_\varepsilon} \|\int_{\mathbf{R}^3} (F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon})d\xi\|_{L_x^2}dt \tag{6.45}$$

$$\lesssim_M \int_0^{\hat{T}_\varepsilon} \frac{1}{\varepsilon}\|\int_{\mathbf{R}^3} v\cdot\nabla_x(F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon\psi^\varepsilon})d\xi\|_{L_x^2} + \|\partial_t(\gamma^\varepsilon\psi^\varepsilon)e^{\gamma^\varepsilon\psi^\varepsilon}\|_{L_x^2}dt \tag{6.46}$$

$$\lesssim_M \int_0^{\hat{T}_\varepsilon} 1 + \frac{1}{\varepsilon}\sqrt{\widetilde{\mathscr{D}}_{-,2}^\varepsilon}dt \tag{6.47}$$

$$\lesssim_M \hat{T}_\varepsilon + \hat{T}_\varepsilon^{\frac{1}{2}} \tag{6.48}$$

This implies $1 \lesssim_M \hat{T}_\varepsilon \leq \kappa$. Taking κ small enough, we reach a contradiction.

In conclusion, $\hat{T}_\varepsilon \geq \kappa$ for all ε small enough, so $\hat{T} \geq \kappa$. The bounds (6.18) and (6.19) imply (1.34) and (1.35) respectively.

Proof of part (ii): The proof is similar to part (i), so we omit some details. In the same way as in part (i), we take solutions (F_+, β, ϕ^0) to (1.18) (satisfying (6.1)). We also take the solution $(F_+^\varepsilon, F_-^\varepsilon)$ to (1.7) and $(\gamma^\varepsilon, \psi^\varepsilon)$ satisfying the conditions (6.2) through (6.6) for some $\hat{T}_\varepsilon > 0$. We note that the conclusions of Propositions 4.1 and 4.2 both hold.

By (1.38), and the fact that $\psi_{in}^\varepsilon = \phi_{in}^0$, we have that the expression in (5.7) is exactly zero. The bounds (6.12) and (6.13) both hold in this context as well, so we can guarantee $\sup_{t\in[0,\hat{T}_\varepsilon]} \sqrt{\mathscr{E}_{-,2}^\varepsilon} < \varsigma$ by taking κ small enough. Thus all the hypotheses of Proposition 5.1 are satisfied up to $T = \hat{T}_\varepsilon$. Next, we have that $\mathscr{E}_{-,\alpha,in}^\varepsilon = \widetilde{\mathscr{E}}_{-,\alpha,in}^\varepsilon \lesssim_M \delta^2$ (and this expression is independent of ε). Therefore, by (5.8), we have

$$\varepsilon \sup_{t\in[0,\hat{T}_\varepsilon]} \widetilde{\mathscr{E}}_{-,2}^\varepsilon(t) + \int_0^{\hat{T}_\varepsilon} \widetilde{\mathscr{D}}_{-,2}^\varepsilon(t)dt \leq C_M(\varepsilon\delta^2 + \varepsilon^2). \tag{6.49}$$

On the other hand, the above and (5.9) give the bound

$$\widetilde{\mathscr{E}}_{-,1}^\varepsilon(t) \lesssim_M e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}}\delta^2 + \varepsilon^{\frac{5}{3}} t^{\frac{1}{3}} + \varepsilon^2. \tag{6.50}$$

Next, similarly to (6.20) in part (i), we have

$$\sup_{t\in[0,\hat{T}_\varepsilon]} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{E}'}^2 + \int_0^{\hat{T}_\varepsilon} \|F_+^\varepsilon(t) - F_+^0(t)\|_{\mathfrak{D}'}^2 dt \lesssim \varepsilon(\varepsilon + \delta^2). \tag{6.51}$$

Now, combining (6.51) with (6.12), (6.13), and (6.49) and (6.50), we have

$$\varepsilon \sup_{t\in[0,\hat{T}_\varepsilon]} \mathscr{E}_{-,2}^\varepsilon(t) + \int_0^{\hat{T}_\varepsilon} \mathscr{D}_{-,2}^\varepsilon(t)dt \leq C_M(\varepsilon\delta^2 + \varepsilon^2) \tag{6.52}$$

and for all $t \in [0, \hat{T}_\varepsilon]$, we have

$$\mathscr{E}_{-,1}^\varepsilon(t) \lesssim_M e^{-\frac{1}{C_M}(\frac{t}{\varepsilon})^{\frac{2}{3}}}\delta^2 + \varepsilon^{\frac{5}{3}} t^{\frac{1}{3}} + \varepsilon\delta^2. \tag{6.53}$$

We now show that $\hat{T} = \inf_{\varepsilon\in(0,\bar{\varepsilon}]} \hat{T}_\varepsilon$ is positive. As before, we take $\hat{T}_\varepsilon < \kappa$, and show that each of (6.2) through (6.6) holding with equality yields a contradiction, provided κ is taken small enough. In the cases of (6.2), (6.3), (6.4) and (6.5), the proof is essentially same as in the case of part (i), up to taking δ sufficiently small in addition to κ and ε.

We now address the case of when (6.6) holds with equality at $T = \hat{T}_\varepsilon$. Next, take $0 < t_0 < \hat{T}_\varepsilon$ to be chosen later. Through a trivial modification of the proof of Proposition 5.1, we have that the estimate (5.8) holds with $\alpha = 1$ instead of $\alpha = 2$, and $[t_0, T]$ instead of $[0, T]$. This gives

$$\begin{aligned} \varepsilon \sup_{t\in[t_0,T]} \widetilde{\mathscr{E}}_{-,1}^\varepsilon(t) + \int_{t_0}^{\hat{T}_\varepsilon} \mathscr{D}_{-,1}(t)dt &\lesssim_M \mathscr{E}_{-,1}^\varepsilon(t_0) + \varepsilon^2 && (6.54) \\ &\lesssim_M \varepsilon(e^{-\frac{1}{C_M}(\frac{t_0}{\varepsilon})^{\frac{2}{3}}}\delta^2 + \varepsilon^{\frac{5}{3}} t_0^{\frac{1}{3}}) + \varepsilon^2 && (6.55) \end{aligned}$$

Taking $t_0 = \varepsilon(C_M|\ln(\varepsilon)|)^{\frac{3}{2}}$, we can bound last expression by by ε^2. On the other hand, integrating (6.50) on $[0, t_0]$, we have

$$\int_0^{t_0} \sqrt{\widetilde{\mathscr{E}}^{\varepsilon}_{-,1}(t)}dt \lesssim_M \varepsilon\delta + \varepsilon^{\frac{5}{4}} t_0^{\frac{7}{6}} + \varepsilon t_0 \lesssim_M \varepsilon\delta + \varepsilon^{\frac{3}{2}} \tag{6.56}$$

Therefore, combining these bounds, we deduce

$$\int_0^{\hat{T}_\varepsilon} \| \int_{\mathbf{R}^3} \xi \cdot \nabla_x (F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}) d\xi \|^2_{L^2_x} dt \tag{6.57}$$

$$\lesssim_M \int_0^{t_0} \sqrt{\widetilde{\mathscr{E}}^{\varepsilon}_{-,1}(t)}dt + \left(\int_0^{t_0} \widetilde{\mathscr{D}}^{\varepsilon}_{-,1}(t)dt\right)^{\frac{1}{2}} \lesssim_M \varepsilon. \tag{6.58}$$

Now, combining the above with (5.201), (4.142), and (6.52),

$$1 = \int_0^t \| \int_{\mathbf{R}^3} (F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}) d\xi \|_{L^2_x} dt \tag{6.59}$$

$$\lesssim_M \int_0^{\hat{T}_\varepsilon} \frac{1}{\varepsilon} \| \int_{\mathbf{R}^3} \xi \cdot \nabla_x (F_-^\varepsilon - \mu_{\gamma^\varepsilon} e^{\gamma^\varepsilon \psi^\varepsilon}) d\xi \|_{L^2_x} + \|\partial_t(\gamma^\varepsilon \psi^\varepsilon) e^{\gamma^\varepsilon \psi^\varepsilon}\|_{L^2_x} dt \tag{6.60}$$

$$\lesssim_M \varepsilon\delta + \varepsilon^{\frac{3}{2}} + \hat{T}_\varepsilon + \hat{T}_\varepsilon^{\frac{1}{2}} \tag{6.61}$$

$$\leq \varepsilon\delta + \varepsilon^{\frac{3}{2}} + \kappa^{\frac{1}{2}}. \tag{6.62}$$

Taking ε, δ and κ sufficiently small, we have a contradiction.

In conclusion, we have $\hat{T} \geq \kappa > 0$. Moreover, (1.39), (1.40) and (1.41) follow from (6.51), (6.52), and (6.53) respectively. □

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Declarations

Conflicts of interests The authors declare that there were no conflicts of interest related to this work.

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