Strategic Quantization with Quadratic Distortion Measures

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Abstract— **This paper studies the problem of strategic quantization, where an encoder and a decoder with misaligned objectives communicate over a rate-constrained noiseless channel. Specifically, we focus on a 2-dimensional source, state and bias variables, and quadratic distortion measures. We provide a design algorithm for this particular case of strategic quantization, as well as an upper and lower bounds on the encoder distortion via employing linear communication strategies. Finally, we present comparative numerical results obtained via the proposed method. We provide our numerical results and the code for research purposes at `https://github.com/strategic-quantization/quadratic-distortion-measures`.**

I. Introduction

Consider the following problem: Two smart cars by competing manufacturers, e.g., Tesla and Honda, are communicating, without sample delay, over a noiseless fixed bit rate channel. Tesla (the decoder) asks for traffic congestion information from Honda (the encoder), which is ahead in traffic, to decide on its route. Honda's objective might be to make Tesla take a specific action, e.g., change its current route, while Tesla wants to estimate the traffic conditions accurately. Since Honda's objective is different from Tesla's, Honda needs an incentive to convey a truthful congestion estimation. Tesla is aware of Honda's motives but would still like to use Honda's information. How would these cars communicate? Problems of this nature can be handled using the strategic quantization model (coarse persuasion) given in [1], [2], or more broadly, strategic communication models [3], [4]. Note that here, Honda has three different behavioral choices: it can choose not to communicate (non-revealing strategy), can precisely communicate what Tesla wants (fully-revealing strategy), or can craft a message that would make Tesla change its route (partially revealing strategy). Tesla can choose not to use Honda's message if it is statistically too far from the truth. Hence, crafting an optimal message for Honda that would serve its objective, knowing that Tesla's objective differs from it, is a formidable research challenge.

This research area has been well studied in Economics literature without the quantization cardinality constraint as the information design or the Bayesian persuasion problem [3], [5]. Such problems explore the use of information by a communication system designer (sender) to influence the action taken by a receiver [6], [7]. In a related but distinctly different class of signaling games called cheap talk [8], authors noted that quantizers can arise as equilibrium strategies endogenously, without an external constraint. In [8], the encoder chooses the mapping from the realization of the source X to message Z *after* observing it, *ex-post*, as different source realizations indicate optimality of different mappings for the encoder. This results in a Nash equilibrium since both agents form a strategy that best responds to each other's mapping because of the encoder's lack of commitment power in the cheap talk setting. In contrast, in the strategic quantization problem (and the information design problems in general as in [3], [5]), the encoder designs Q *ex-ante*, *before* seeing the source realization, and is committed to the designed Q afterward. This difference in commitment manifests in the notion of equilibrium since the encoder may not necessarily form the best response to the decoder due to its commitment to Q.

Strategic quantization was analyzed from a computational perspective in [2]. Aybaş and Türkel [1] studied the same problem via an information Economics lens, employing the mathematical tools developed in the Economics prior work, e.g., [3] and derived several theoretical properties of optimal strategic quantizers in general probability spaces. In [9], authors consider a Bayesian persuasion problem with an imperfect channel and a limited number of messages and provide an upper bound on the pay-off of the sender. In [10], authors study Bayesian signaling games and characterize the minimum number of distinct source symbols that can be correctly recovered by a receiver in any equilibrium of this game, which they call the informativeness of the sender. In [11], authors study dynamic variations of the cheap talk signaling games.

In our prior work [12]–[15], we used the rich collection of prior quantizer design and optimization work to study this practically significant problem via an engineering lens. More specifically, in [13], we derived several properties of strategic quantization and proposed a straightforward gradient descent-based design strategy that yields a locally optimal strategic quantizer. We showed that the encoder behaves in one of three ways:

- Non-revealing: the encoder does not send any information.
- Fully-revealing: the strategic quantizer is identical to the one in non-strategic quantization of X.
- Partially-revealing: the encoder sends some information, but the quantizer differs from a nonstrategic quantizer.

In [12], we proposed a dynamic programming solution that achieves global optimality at the cost of increased complexity. In [14]–[16], we explored strategic quantization

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in noisy scenarios.

This paper focuses on the setting where both communicating agents use quadratic distortion measures. Particularly, the encoder observes a two-dimensional source $X, \theta \sim f(x, \theta)$ with a known joint distribution over X and θ, where X and θ can be interpreted as the state and bias variables. The decoder's objective is to estimate the state in the minimum mean squared error (MMSE) sense, i.e., the decoder minimizes $\mathbb{E}\{(X-\hat{X})^2\}$ by choosing an action $\hat{X}$ which is the optimal MMSE estimate of x given the quantization index from the encoder $y = Q(x, \theta)$, hence $\hat{X} = \mathbb{E}\{X|Y = y\}$. In sharp contrast with the conventional quantization problem where the encoder chooses Q that minimizes $\mathbb{E}\{(X-\hat{X})^2\}$, in this setting the encoder's choice of quantization mapping Q minimizes a biased estimate, i.e., $\mathbb{E}\{(X+\theta-\hat{X})^2\}$. The objectives and the source distribution are common knowledge available for both agents. Similar signaling problems with quadratic measures have been analyzed in the Economics literature [8], [17], [18].

We then provide a lower and an upper bound on the encoder's distortion. The lower bound simply follows from the observation that the encoder cannot outperform the performance of the encoder in the perfect channel (without any rate constraints) case, which we refer to as the "strategic communication" setting. The tractable expressions for setting reported in [4] yield a lower bound on the encoder distortion. The upper bound is obtained by quantizing a particular linear function of the bias and source variables.

The basic design problem, as studied in [13], focuses on the scalar settings. The optimal strategic quantization of a two-dimensional source considered in this paper poses a challenge in developing an algorithmic solution similar to the one in [13]. We circumvent this issue by designing a separate quantizer for each realization of θ for the encoder[1].

This paper is organized as follows: In Section II we present the problem formulation. In Section III, we present a gradient-descent based algorithm to compute the strategic quantizer, and an upper bound on the encoder distortion. We provide numerical results in Section IV, and conclude in Section V.

II. PRELIMINARIES

A. Notation

In this paper, random variables are denoted using capital letters (say X), their sample values with respective lowercase letters (x), and their alphabet with respective calligraphic letters ($\mathcal{X}$). The set of real numbers is denoted by $\mathbb{R}$. The uniform distribution over an interval $[t_1, t_2]$, and the 2-dimensional jointly Gaussian distribution with mean $\begin{bmatrix} t_1 & t_2 \end{bmatrix}'$ and respective variances σ_1^2, σ_2^2 with a correlation ρ are denoted by $U[t_1, t_2]$, and

$$\mathcal{N}\left(\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix}\right), \ -1 \leq \rho < 1,$$

$t_1, t_2 \in \mathbb{R}$, respectively. The expectation operator is written as $\mathbb{E}\{\cdot\}$. The operator $|\cdot|$ denotes the absolute value if the argument is a scalar real number and the cardinality if the argument is a set.

B. Problem Formulation

An encoder observes realizations of the two stochastic sources $X \in \mathcal{X} \subseteq [a_X, b_X], \theta \in \mathcal{T} \subseteq [a_\theta, b_\theta]$, $a_X, b_X, a_\theta, b_\theta \in \mathbb{R}$ with joint probability distribution $(X, \theta) \sim f(x, \theta)$, and maps (X, θ) to a message $Z \in \mathcal{Z}$, where $\mathcal{Z}$ is a set of discrete messages with a cardinality constraint $|\mathcal{Z}| \leq M$ using a non-injective mapping parameterized by θ, $q_\theta : \mathcal{X} \rightarrow \mathcal{Z}$. After receiving the message Z, the decoder applies a mapping $\phi : \mathcal{Z} \rightarrow \mathcal{Y}$ on the message Z and takes an action $Y = \phi(Z)$.

The encoder and decoder minimize their respective objectives $D_E = \mathbb{E}\{\eta_E(X, \theta, Y)\} = \mathbb{E}\{(X+\theta-Y)^2\}$ and $D_D = \mathbb{E}\{\eta_E(X, Y)\} = \mathbb{E}\{(X-Y)^2\}$, which are misaligned ($\eta_E \neq \eta_D$). The encoder designs $Q = \{q_\theta, \theta \in \mathcal{T}\}$ *ex-ante*, i.e., without the knowledge of the realization of (X, θ), using only the objectives D_E and D_D, and the statistics of the source $f(\cdot, \cdot)$. The objectives (D_E and D_D), the shared prior (f), and the mapping (Q) are known to the encoder and the decoder. The problem is to design Q for the equilibrium, i.e., the encoder minimizes its distortion if used with a corresponding decoder that minimizes its own distortion. This communication setting is given in Fig. 1, and the problem is summarized in the box. Since the encoder chooses the quantization decision levels Q first, followed by the decoder choosing the quantization representative levels ($\mathbf{y}$), we look for a Stackelberg equilibrium.

The set $\mathcal{X}$ is divided into mutually exclusive and exhaustive sets parameterized by the realization of θ as $\mathcal{V}_{\theta,1}, \mathcal{V}_{\theta,2}, \ldots, \mathcal{V}_{\theta,M}$. The $m-$th quantization region is denoted by $\mathcal{V}_{:,m} = \{\mathcal{V}_{\theta,m}, \forall\theta \in \mathcal{T}\}$. The encoder chooses the set of quantizers $Q = \{q_\theta, \theta \in \mathcal{T}\}$ to minimize its distortion,

$$D_E = \sum_{m=1}^{M} \int_{\theta\in\mathcal{T}} \int_{x\in\mathcal{V}_{\theta,m}} (x+\theta-y_m^*(Q))^2 \mathrm{d}f(x,\theta) \quad (1)$$

where the optimal reconstruction points y_m^* are determined by the decoder as a best response to Q to minimize its distortion,

$$y_m^* = \arg\min_{y\in\mathcal{Y}} \sum_{m=1}^{M} \mathbb{E}\{(X-y)^2 | x \in \mathcal{V}_{:,m}\} = \frac{\int_{\theta\in\mathcal{T}} \int_{x\in\mathcal{V}_{\theta,m}} x \mathrm{d}f(x,\theta)}{\int_{\theta\in\mathcal{T}} \int_{x\in\mathcal{V}_{\theta,m}} \mathrm{d}f(x,\theta)}. \quad (2)$$

The decoder determines a single set of actions $\mathbf{y}$ since it is unaware of the realization of θ.

Throughout this paper, we make the following "monotonicity" assumption on the sets $\{\mathcal{V}_{\theta,m}\}$.

Assumption 1. *$\mathcal{V}_{\theta,m}$ is convex for all $\theta \in \mathcal{T}, m \in [1 : M]$,.*

[1]In cases where θ is not purely discrete, we discretize it over a uniform grid to facilitate the numerical analysis.

Problem. *For a given rate R, 2−dimensional source (X, θ) with a probability distribution function $f(x, \theta)$ find the decision boundaries $Q \in \mathcal{Q}$, $\mathcal{Q} \in \mathbb{R}^{|\mathcal{T}| \times (M+1)}$, $Q = \{q_\theta, \theta \in \mathcal{T}\}$, $q_\theta = [x_{\theta,0}, x_{\theta,1}, \ldots, x_{\theta,M}], \forall\theta \in \mathcal{T}$ as a function of boundaries that satisfy:*

$$Q^* = \arg\min_{Q \in \mathcal{Q}} \sum_{m=1}^{M} \mathbb{E}\{(X + \theta - y_m^*(Q))^2 | X \in \bigcup_{\theta \in \mathcal{T}} [x_{\theta,m-1}, x_{\theta,m}]\},$$

where actions $\mathbf{y}(Q)$ are given as

$$y_m^*(Q) = \arg\min_{y \in \mathcal{Y}} \mathbb{E}\left\{(X - y)^2 | X \in \bigcup_{\theta \in \mathcal{T}} [x_{\theta,m-1}, x_{\theta,m}]\right\} \quad \forall m \in [1 : M],$$

and the rate satisfies $\log M \le R$.

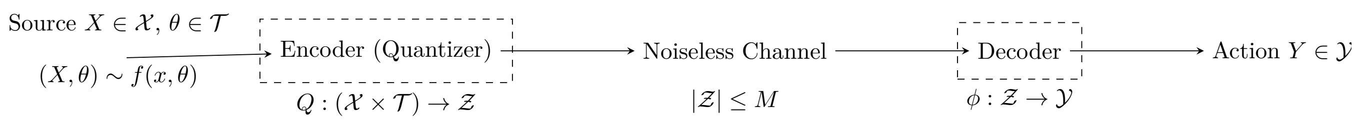

Fig. 1. Communication diagram: (X, θ) over a noiseless channel

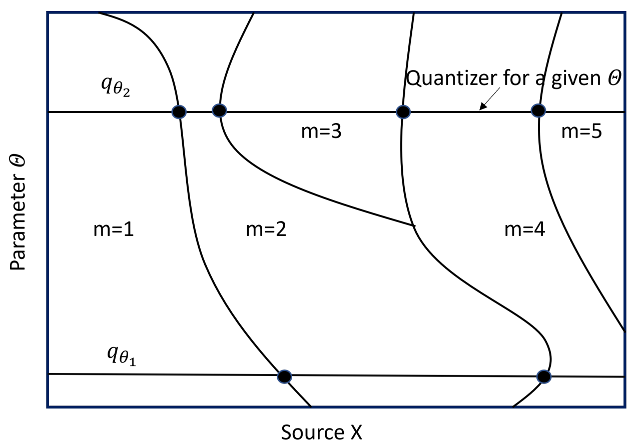

Fig. 2. Quantization of X parameterized by θ for $M = 5$ illustrated.

Remark 1. *Assumption 1 is the first of the two regularity conditions commonly employed in the classical quantization literature, cf. [19]. Note that the second regularity condition, $y_m \in \mathcal{V}_m$, is not included in Assumption 1.*

Note that implementing a quantizer $Q : (\mathcal{X}, \mathcal{T}) \to \mathcal{Z}$ can be simplified to computing a set of quantizers corresponding to each $\theta \in \mathcal{T}$ as in Fig. 2 without loss of generality. If the quantizer does not include a region m for some realization of θ, the encoder never sends the message m i.e., the encoder chooses a lower rate and is less revealing for that value of θ. In Fig. 2, we see that the quantizer q_{θ_1} only includes $m = 1, 2, 4$ regions, while the quantizer q_{θ_2} contains all five regions.

III. Main Results

In this section, we first present the gradient-descent based optimization algorithm. We also present a straightforward method of quantization along with linear estimation that results in an upper bound on the encoder distortion.

A. Proposed algorithm

In [13], we proposed a gradient-descent-based algorithm to solve the problem of quantization of a scalar source with misaligned encoder and decoder objectives communicating over a fixed rate noiseless channel. We extend this algorithm to a 2-dimensional source (X, θ) by a simple method of computing quantizers for each value of θ as $Q = \{q_\theta, \theta \in \mathcal{T}\}$. The gradient descent optimization is performed with the objective as the encoder distortion optimized over the encoder's choice of quantizer decision levels $Q = \{q_\theta, \theta \in \mathcal{T}\}$. Although the encoder distortion depends on decoder reconstruction levels $\mathbf{y}$, since $\mathbf{y}$ is a function of Q, the optimization can be implemented as a function of solely Q.

Remark 2. *The proposed method inherits the convergence guarantees of gradient-descent-based algorithms. Hence, local optimality is guaranteed, but the resulting quantizer may not necessarily be globally optimal.*

Like any gradient-descent-based algorithm, the proposed method may get stuck at a local optimum. This issue can be mitigated with several methods in the literature [20]–[22]. As a simple remedy, we perform gradient descent with multiple initializations and choose the best among them. A sketch of the proposed method is presented in Algorithm 1. The MATLAB codes are provided at `https://github.com/strategic-quantization/quadratic-distortion-measures` for research purposes.

B. Lower Bound

In this section, we provide a lower bound on D_E for the case of jointly Gaussian (X, θ). We first reproduce the following theorem from [4]:

Theorem 2. *The problem of strategic communication where*

Algorithm 1 Proposed Design Algorithm

Parameters: ϵ, λ
Input: $f(\cdot,\cdot), \mathcal{X}, \mathcal{T}, M, \eta_E, \eta_D$
Output: $\{q_\theta^*\}$, $\{y_m^*\}$, D_E, D_D
Initialization: assign a set of monotone $\{q_{\theta,0}\}$ randomly, compute associated encoder distortion $D_E(0)$, set iteration index $i=1$;
while $\Delta D > \epsilon$ *or until a set amount of iterations* **do**
  compute the gradients $\{\partial D_E / \partial x_{\theta,:}\}_i$,
  compute the updated quantizer $q_{\theta,i+1} \triangleq q_{\theta,i} - \lambda\{\partial D_E/\partial x_{\theta,:}\}_i$ for $\theta \in \mathcal{T}$,
  compute actions $\mathbf{y}(\{q_{\theta,i+1}\})$ via (2),
  compute encoder distortion $D_E(i+1)$ associated with quantizer values $q_{\theta,i+1}$ and actions $\mathbf{y}(\{q_{\theta,i+1}\})$ via (1),
  compute $\Delta D = D_E(i) - D_E(i+1)$.
return quantizer $\{q_\theta^*\} = \{q_{\theta,i+1}\}$, actions $\{y_m^*\} = \mathbf{y}(\{q_\theta^*\})$, encoder and decoder distortions D_E and D_D computed for optimal quantizer and decoder actions $\{q_\theta^*\}, \mathbf{y}(\{q_\theta^*\})$ via (1).

an encoder with distortion measure $\eta_E = (X+\theta-Y)^2$ communicates with a decoder with distortion measure $\eta_D = (X-Y)^2$ over a noiseless channel with no rate constraints is specified by an encoder mapping $g : (\mathcal{X},\theta) \to \mathcal{Z}$, and a decoder mapping $h : \mathcal{Z} \to \mathcal{Y}$. For a jointly Gaussian source $(X,\theta) \sim \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \sigma_X^2\begin{bmatrix}1 & \rho\\ \rho & r\end{bmatrix}\right)$, the encoder and decoder mappings are of the form:

$$g(X,\theta) = X + \alpha\theta, \quad h(Z) = \kappa Z,$$

where α, κ are given by:

$$\alpha = \frac{A-1}{2(r+\rho)}, \quad \kappa = \frac{1+\alpha\rho}{1+\alpha^2 r + 2\alpha\rho}$$

and $A = \sqrt{1+4(r+\rho)}$. The encoder and decoder distortions for this strategic communication setting are given by

$$D_{E_{SC}} = \sigma_X^2\left(1 + \frac{(A-3)(r+\rho)}{A-1}\right), \tag{3}$$

$$D_{D_{SC}} = \sigma_X^2\left(\frac{(r-\rho^2)(A-1)}{A(2r+A\rho+\rho)}\right). \tag{4}$$

In the next theorem, we present a lower bound on the encoder's distortion D_E in our problem, specialized to the Gaussian sources.

Theorem 3. *The encoder distortion in strategic quantization of a jointly Gaussian source (X,θ) with $\eta_E = (X+\theta-Y)^2, \eta_D = (X-Y)^2$ is lower bounded by the distortion in strategic communication of the jointly Gaussian source, i.e., $D_E \geq \underline{D}_E$ where $\underline{D}_E = D_{E_{SC}}$ given in Theorem 2.*

Proof: The mappings $g(X,\theta)$ and $h(Z)$ in Theorem 2 achieve $D_{E_{SC}}$ without any rate constraints. Additional constraints (such as the rate constraint in our problem) can only increase D_E, hence $D_E \geq D_{E_{SC}}$.

C. Upper Bound

We next present an upper bound for the encoder distortion where (X,θ) follows a general distribution (not necessarily jointly Gaussian). We first consider the linear minimum mean squared error (LMMSE) estimate of X given observation T,

$$\hat{X} = LMMSE(X|T) = h(T) = \kappa T, \tag{5}$$

and a linear encoding strategy:

$$T = g(X,\theta) = (X + \alpha\theta). \tag{6}$$

where the parameters α and κ are

$$\alpha = \frac{A-1}{2(r+\rho)}, \quad \kappa = \frac{1+\alpha\rho}{1+\alpha^2 r + 2\alpha\rho}, \tag{7}$$

and $A, r,$ and ρ are given by

$$A = \sqrt{1+4(r+\rho)}, \quad r = \frac{\sigma_\theta^2}{\sigma_X^2}, \quad \rho = \frac{\mathbb{E}\{X\theta\}}{\sigma_X^2}. \tag{8}$$

The encoder distortion D_E can be written as,

$$\begin{aligned}
D_E &= \mathbb{E}\{(X+\theta-Q(\hat{X}(T)))^2\}\\
&= \mathbb{E}\{(X+\theta-\hat{X}(T)+\hat{X}(T)-Q(\hat{X}(T)))^2\}\\
&\overset{a}{=} \mathbb{E}\{(X+\theta-\hat{X}(T))^2\} + \mathbb{E}\{(\hat{X}(T)-Q(\hat{X}(T)))^2\}\\
&\quad + 2\mathbb{E}\{\theta(\hat{X}(T)-Q(\hat{X}(T)))\}.
\end{aligned} \tag{9}$$

Equality a in the above equation is due to the orthogonality of the estimation error $(X - \hat{X}(T))$ to any function of the observation T,

$$\mathbb{E}\{(X-\hat{X}(T))(\hat{X}(T)-Q(\hat{X}(T)))\} = 0.$$

Remark 3. *Similar decompositions were also used in [23], [24], where they exploit the orthogonality of the estimation error.*

Let us define the optimal quantizer that minimizes (1) as Q^*,

$$\begin{aligned}
Q^* &= \arg\min_Q D_E\\
&= \arg\min_Q \Big\{\mathbb{E}\{(X+\theta-\hat{X}(T))^2\} + 2\mathbb{E}\{\theta\hat{X}(T)\}\\
&\quad + \mathbb{E}\{(\hat{X}(T)-Q(\hat{X}(T)))^2\} - 2\mathbb{E}\{\theta Q(\hat{X}(T))\}\Big\}\\
&\overset{b}{=} \arg\min_Q \Big\{\mathbb{E}\{(\hat{X}-Q(\hat{X}))^2\} - 2\mathbb{E}\{\theta Q(\hat{X})\}\Big\}
\end{aligned} \tag{10}$$

where equality b is due to the fact that the first two terms $\mathbb{E}\{(X+\theta-\hat{X}(T))^2\}$ and $2\mathbb{E}\{\theta\hat{X}(T)\}$ do not include Q.

In general, it is hard to compute Q^*. Instead, we consider Q^{**} which we define as

$$Q^{**} = \arg\min_Q \mathbb{E}\{(\hat{X}-Q(\hat{X}))^2\}. \tag{11}$$

In other words, Q^{**} is the mean squared error (MSE) optimal non-strategic quantizer for $\hat{X} = LMMSE(X|T) = \kappa T$, $T = (X + \alpha\theta)$.

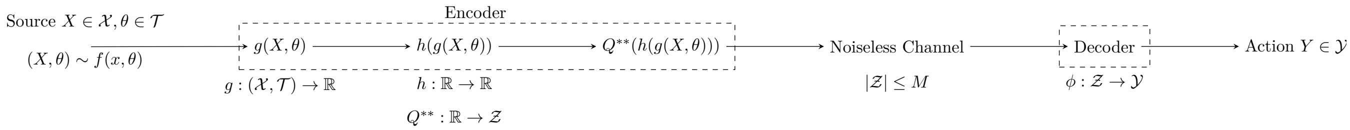

Fig. 3. Computation of an upper bound for the encoder distortion.

We note that Q^{**} is relatively straightforward to compute, e.g., if (X,θ) is jointly Gaussian, $\hat{X}$ is also Gaussian for which the optimal quantizer is well-known, e.g., [25].

Since $Q^{**} \neq Q^{*}$ in general, the resulting distortion of Q^{**}, denoted by $\overline{D_E}$ is an upper bound, i.e., $\overline{D_E} \geq D_E$.

We formalize the preceding discussion in the following theorem:

Theorem 4. *$\overline{D_E} \geq D_E$, where*

$$\overline{D_E} = \mathbb{E}\{(X+\theta-\hat{X}(T))^2\} + \mathbb{E}\{(\hat{X}-Q^{**}(\hat{X}))^2\} + 2\mathbb{E}\{\theta(\hat{X}-Q^{**}(\hat{X}))\},$$

and $T = X+\alpha\theta$, $\hat{X} = LMMSE(X|T) = \kappa T$,

$$\alpha = \frac{A-1}{2(r+\rho)}, \quad \kappa = \frac{1+\alpha\rho}{1+\alpha^2 r + 2\alpha\rho}$$

$$r = \frac{\sigma_\theta^2}{\sigma_X^2}, \quad \rho = \frac{\mathbb{E}\{X\theta\}}{\sigma_X^2}, \quad A = \sqrt{1+4(r+\rho)},$$

and Q^{**} is given in (11).

Proof: The set of quantizers over which the encoder optimizes its distortion includes this specific scheme, hence $\overline{D_E} \geq D_E$.

In summary, for the computation of an upper bound, we consider a system where the encoder computes $\hat{X} = h(g(X,\theta)) = \kappa(X+\alpha\theta)$, quantizes $\hat{X}$ as $Z = Q^{**}(\hat{X})$ and sends the message Z to the decoder, as depicted in Fig. 3. The upper bound is computed as the sum of the estimation error, quantization error, and the term $2\mathbb{E}\{\theta(\hat{X}-Q^{**}(\hat{X}))\}$, as in (9). A sketch of the computation of this upper bound is in Algorithm 2 below.

Algorithm 2 Computation of an upper bound of encoder distortion

Input: $f(\cdot,\cdot), \mathcal{X}, \mathcal{T}, M, \eta_E, \eta_D$
Output: Q^{**}, $\overline{D_E}$
Compute r, ρ, A from (8).
$\alpha \leftarrow (A-1)/(2(r+\rho))$
$\kappa \leftarrow (1+\alpha\rho)/(1+\alpha^2 r+2\alpha\rho)$
Compute probability distribution function of $\hat{X} = \kappa(X+\alpha\theta)$, $f_{\hat{X}}$.
Compute non-strategic quantizer Q^{**} with MSE encoder and decoder objectives $\mathbb{E}\{(\hat{X}-Q^{**}(\hat{X}))^2\}$, and $\hat{X} \sim f_{\hat{X}}$.
Compute the upper bound as $\overline{D_E}$ in Theorem 4.
return quantizer Q^{**}, upper bound $\overline{D_E}$

IV. Numerical Results

We present results for a jointly Gaussian setting with encoder and decoder distortions $\eta_E(x,\theta,y) = (x+\theta-y)^2$, $\eta_D(x,y) = (x-y)^2$ for correlation $\rho \in \{-0.5, 0, 0.5\}$ in Fig.4. The support of θ is discretized by sampling for computational feasibility for the jointly Gaussian source.

We first note that the encoder distortion at zero rate is greater than at other rates for each correlation value, that is, the encoder is not non-revealing. The associated decoder distortion is also greater than that of other rates for each correlation value, i.e., the decoder does not ignore the encoder's message.

While the distortions decrease with rate, the encoder distortion does not become negligibly small at high rates as in classical quantization. Instead, they are lower bounded by the distortion in the strategic communication setting [4]. In this problem setting, we compute the lower bounds as:

$$\underline{D_E} = \begin{cases} 0.1340, & \text{if } \rho = -0.5 \\ 0.3820, & \text{if } \rho = 0 \\ 0.6771, & \text{if } \rho = 0.5. \end{cases}$$

When $\rho = -1$, the encoder distortion simplifies to $\mathbb{E}\{Y^2\}$, which for this setting is minimized when the encoder is non-revealing, and the decoder's estimate is $y = 0$. When $\rho = 1$, the encoder distortion simplifies to $\mathbb{E}\{(2X-Y)^2\}$, which results in a fully-revealing encoder as shown in [4]. The encoder changes from being non-revealing to fully-revealing as ρ increases. In other words, the encoder's ability to persuade the decoder decreases with increasing ρ. Hence, we observe that the encoder distortion increases, and the decoder distortion decreases with increasing correlation ρ.

V. Conclusions

In this paper, we analyzed the problem of strategic quantization of a 2-dimensional source (X,θ) with the encoder and the decoder objectives $D_E = \mathbb{E}\{(X+\theta-Y)^2\}$ and $D_D = \mathbb{E}\{(X-Y)^2\}$, respectively. We extended our prior work on design, a gradient-descent-based algorithm for scalar sources, to the setting considered in this paper. We then presented a lower bound for jointly Gaussian sources and an upper bound for general distributions based on linear communication strategies. The numerical results obtained via the proposed algorithm suggest several intriguing research problems, which we leave as part of our future work.

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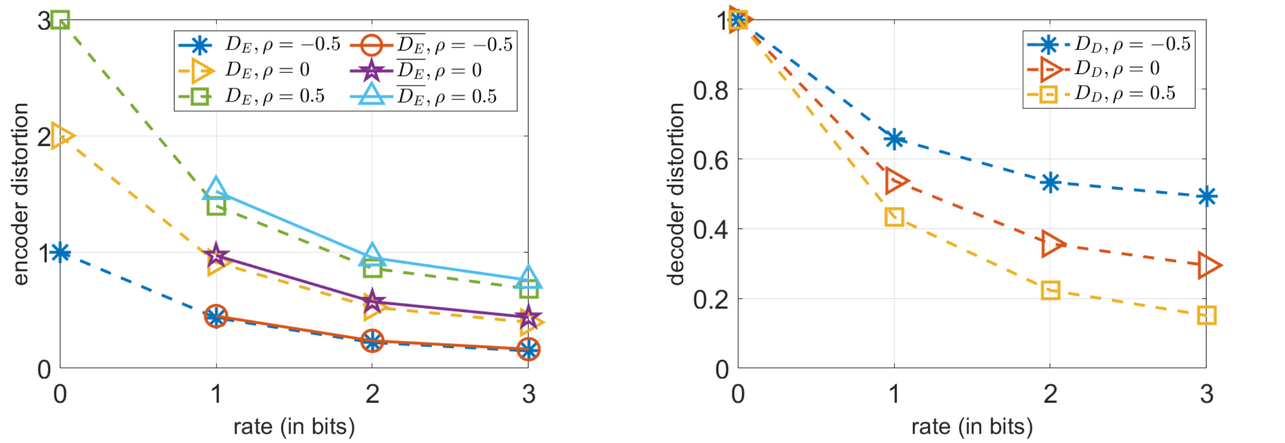

(a) Encoder distortion (b) Decoder distortion

Fig. 4. Encoder distortion, the associated upper bound, and decoder distortion for a jointly Gaussian source $(X, \theta) \sim \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right)$ with $\eta_E(x, \theta, y) = (x + \theta - y)^2, \eta_D(x, \theta, y) = (x - y)^2$.

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