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Hydrodynamics, anomaly inflow and bosonic effective field theory

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Abstract: Euler hydrodynamics of perfect fluids can be viewed as an effective bosonic field theory. In cases when the underlying microscopic system involves Dirac fermions, the quantum anomalies should be properly described. In 1+1 dimensions the action formulation of hydrodynamics at zero temperature is reconsidered and shown to be equal to standard field-theory bosonization. Furthermore, it can be derived from a topological gauge theory in one extra dimension, which identifies the fluid variables through the anomaly inflow relations. Extending this framework to 3+1 dimensions yields an effective field theory/hydrodynamics model, capable of elucidating the mixed axial-vector and axial-gravitational anomalies of Dirac fermions. This formulation provides a platform for bosonization in higher dimensions. Moreover, the connection with 4+1 dimensional topological theories suggests some generalizations of fluid dynamics involving additional degrees of freedom.

Keywords: Anomalies in Field and String Theories, Field Theory Hydrodynamics, Topological Field Theories, Topological States of Matter

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1 Introduction

In this study we aim to discuss the effective dynamics of Dirac fermions and their anomalies in 1+1 and 3+1 dimensions. In particular, we investigate the connections between the methods of bosonic field theory and hydrodynamics.

On one hand, bosonization, the relation between bosonic and fermionic field theories, is now attracting much interest beyond the well-understood 1+1 dimensional case [1–8]. On the other hand, hydrodynamic theories incorporating anomalous currents are actively developed [9–19]. Both approaches can be viewed as constructions of effective theories for interacting fermionic systems.

In order to understand the relations between these themes we take advantage of the studies on topological phases of matter, which can be described in terms of both fermionic and bosonic degrees of freedom [20–23]. Physical systems in these phases usually possess a large bulk gap and massless boundary excitations. Within the low-energy effective field theory approach, the bulk is described by a topological gauge theory and the boundary by a relativistic field theory which is anomalous. Bulk and boundary theories are related by anomaly inflow and anomaly matching, enforcing gauge symmetry in the combined bulk-edge system [24].

Topological states are primarily (interacting) fermionic systems, yet there exists a well-developed bosonic description for both bulk and boundary excitations. A notable example is the quantum Hall effect, where the bulk is 2+1 dimensional and the boundary 1+1 dimensional [25, 26]. The respective degrees of freedom are interconnected: the effective gauge field within the Chern-Simons theory, often referred to as the 'hydrodynamic' field, dictates the behavior of the boundary scalar, which in turn represents the chiral fermion and expresses its anomalous current [24]. It is worth to emphasize that within the effective bosonic theory, anomalies arise as consequences of the equations of motion, thereby allowing for a classical description, with additional considerations required for global effects and quantization conditions.

The study of topological phases of matter is progressing towards finding the general topological theories in any dimension, along with identifying the corresponding anomalies

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and effective boundary theories [27–32]. This program has led to considerable advances in understanding non-perturbative physics. For instance, it has revealed the concept of fermionic particle-vortex duality in 2+1 dimensions [33, 34], and has elucidated the universal, exact consequences arising from chiral and gravitational anomalies [24, 35].

We want to stress that the physical setting of topological states, involving gapped bulk and massless boundary, is ideal for understanding bosonization in higher dimensions, where topological theories hold significant importance.

In this work, we combine field theory techniques with the hydrodynamic approach recently introduced in references [19, 36]. Specifically, we describe Euler barotropic perfect fluids, where local energy depends only on density, using a variational method based on an action functional. Because viscosity and heat currents are absent in this hydrodynamic formulation, it parallels bosonic effective field theory. This connection enables us to unify results across seemingly different domains of study.

To begin with, we demonstrate a precise correspondence between Euler hydrodynamics in 1+1 dimensions and the conventional bosonization of Dirac fermions, encompassing both vector and axial anomalies. Additionally, we clarify the use of 2+1 dimensional BF topological theory and the inflow mechanism for identifying the hydrodynamic fields in 1+1 dimensions. It is worth noting that in this dimension, the axial charge corresponds to the kink topological charge of the scalar theory.

These results set the stage for the following analysis in 3+1 dimensions. In investigations of anomalous hydrodynamics, the 3+1 dimensional axial charge has been associated with fluid helicity — the scalar product of vorticity and velocity [10]. We reformulate and extend the variational formulation of Euler hydrodynamics and obtain a bosonic 'effective field theory'. This theory accommodates general chiral and mixed axial-gravitational anomalies of interacting Dirac fermions in fluid phases.

Furthermore, we find the corresponding 4+1 dimensional 'bulk' topological theory which realizes the anomaly inflow and, more importantly, identifies the needed 'boundary' fluid variables. Beside the fluid momentum, a new pseudoscalar degree of freedom is introduced, as a minimum requirement for describing all anomalies of Dirac fermions, in particular the mixed axial-gravitational anomaly.

The derived 3+1 dimensional hydrodynamics/effective field theory is applicable to both relativistic and non-relativistic systems. Importantly, it demonstrates explicitly that anomalies remain unaffected by the dynamic terms in the Lagrangian, which specify the hydrodynamic equation of state. This finding affirms that anomalies encapsulate the system's universal 'geometric' response to background deformations.

The general picture that emerges is that the 4+1d topological theory serves as a key input for building the effective theory/hydrodynamics. Additionally, we identify other 4+1 dimensional theories that could potentially lead to extended hydrodynamics. We discuss one such theory involving independent vector and axial fluid variables, and mention another one incorporating two-form fields as degrees of freedom.

The plan of the paper is the following. In section 2, we review standard bosonization of 1+1d Dirac fermions and its anomalies. We present the corresponding 2+1d BF theory originating from studies of the quantum spin Hall effect, and show how the degrees of freedom describing bulk and boundary currents can be matched via the anomaly inflow

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mechanism. Moving forward to section 3, we introduce the action formulation of Euler hydrodynamics, which is further elaborated on in appendix A. We emphasize its equivalence with the bosonic theory.

In section 4 we discuss the hydrodynamics in 3+1 dimensions. We identify the corresponding 4+1 dimensional topological theory and introduce an additional pseudoscalar field necessary for a complete description of anomalous currents. In section 5, we incorporate the gravitational background, obtain the spin current, and discuss the mixed axial-gravitational anomaly. In section 6, we discuss the physical interpretation of the new pseudoscalar field. The Conclusion outlines potential extensions of our hydrodynamic approach and presents some open questions regarding 3+1 dimensional bosonization. Appendices A, B, and C respectively cover the variational formulation of Euler hydrodynamics, the general form of anomaly coefficients in 3+1 dimensions, and the behavior of fluids in chiral backgrounds in both 1+1 and 3+1 dimensions.

2 1+1 dimensional anomalies in bosonic field theory and anomaly inflow

2.1 The scalar theory

We shall start by reminding some known facts about bosonization in 1+1d, and later discuss the anomaly inflow from a 2+1d topological theory. The bosonic action is

$$S = -\int d^2x\, (\partial_\mu\theta)^2\,, \tag{2.1}$$

within our conventions for Minkowskian signature $\eta_{\mu\nu} = diag(-1,1)$. The theory possesses the U(1) global symmetry $\theta \to \theta + \text{const}$, but two conserved currents

$$J^\mu = 2\partial_\mu\theta, \qquad\qquad \text{(Noether current)}, \tag{2.2}$$
$$\tilde{J}^\mu = 2\varepsilon^{\mu\nu}\partial_\nu\theta\,, \qquad\qquad \text{(topological axial current)}, \tag{2.3}$$

where hereafter tilded quantities represent pseudo-scalar, pseudo-vectors etc. Note that the axial current is called topological because it is conserved without using the equations of motion. It expresses the kink charge when a periodic potential is added to the action (2.1). As is known, $J, \tilde{J}$ correspond to vector and axial currents of Dirac fermions, respectively.

The scalar field can be coupled to both vector and axial backgrounds $A_\mu, \tilde{A}_\mu$ as follows:

$$S = \int d^2x - (\partial_\mu\theta - A_\mu)^2 + 2\tilde{A}_\mu\varepsilon^{\mu\nu}(\partial_\nu\theta - A_\nu)\,, \tag{2.4}$$

leading to currents

$$J^\mu_{\text{cons}} = \frac{\delta S}{\delta A_\mu} = 2(\partial^\mu\theta - A^\mu) + 2\varepsilon^{\mu\nu}\tilde{A}_\nu, \tag{2.5}$$
$$\tilde{J}^\mu_{\text{cons}} = \frac{\delta S}{\delta \tilde{A}_\mu} = 2\varepsilon^{\mu\nu}(\partial_\nu\theta - A_\nu), \tag{2.6}$$

where the subscript stands for 'consistent' current which will be explained momentarily. The equation of motion obtained by varying (2.4) over θ is

$$\partial_\mu(\partial^\mu\theta - A^\mu) + \partial_\mu\varepsilon^{\mu\nu}\tilde{A}_\nu = 0\,. \tag{2.7}$$

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The relations

$$\partial_\mu J^\mu_{\rm cons} = 0, \tag{2.8}$$

$$\partial_\mu \tilde{J}^\mu_{\rm cons} = -2\varepsilon^{\alpha\beta}\partial_a A_\beta\,, \qquad \left(\frac{e}{2\pi}\to 1\right), \tag{2.9}$$

show that the vector current is conserved (using the equations of motion) and the axial current is anomalous. Actually, the form of the action (2.4) is manifestly invariant under the vector gauge transformations

$$A_\mu(x)\to A_\mu(x)+\partial_\mu\lambda(x)\,, \qquad \theta(x)\to\theta(x)+\lambda(x)\,, \tag{2.10}$$

in which the scalar field compensates. Instead, axial gauge transformations

$$\tilde{A}_\mu(x)\to \tilde{A}_\mu(x)+\partial_\mu\tilde{\lambda}(x)\,, \tag{2.11}$$

change the action. Note that the axial anomaly (2.9) is normalized to the Dirac fermion value, in units of flux quantum, $\Phi_0 = e/2\pi$, hereafter set to one.

2.2 Anomaly inflow, Chern-Simons and Wess-Zumino-Witten actions

The 1+1d bosonic theory is assumed to be at the boundary $\partial\mathcal{M}_3$ of a 'bulk' region $\mathcal{M}_3$ where a gapped topological phase of matter is present. At energies below the gap, the response to background variations is given by a topological effective action. The paradigmatic example is given by the quantum Hall effect (QHE), described by the Chern-Simons (CS) action,

$$S_{\rm CS}[A] = -\frac{1}{2}\int_{\mathcal{M}_3} AdA\,, \qquad \left(\frac{e}{2\pi}\to 1\right). \tag{2.12}$$

In this expression, we use the notation of differential forms, e.g. $A = A_\mu dx^\mu$, omitting the wedge-product symbol ($\wedge$) for simplicity. It is convenient to write formulas involving both tensor components and forms; we also represent components of dual tensors with square brackets, e.g. $[\omega_r]^{\mu\nu\cdots}$, which are defined as follows:

$$\omega_r = \frac{1}{r!}\omega_{\mu_1\ldots\mu_r}dx^{\mu_1}\cdots dx^{\mu_r}, \qquad \Omega_{d-r} = *\omega_r = \frac{\sqrt{|g|}}{(d-r)!r!}\varepsilon_{\nu_1\ldots\nu_{d-r}}{}^{\mu_1\ldots\mu_r}\,\omega_{\mu_1\ldots\mu_r}dx^{\nu_1}\cdots dx^{\nu_r},$$
$$[\omega_r]^{\nu_1\ldots\nu_{d-r}} = \frac{1}{r!}\varepsilon^{\nu_1\ldots\nu_{d-r}\mu_1\ldots\mu_r}\,\omega_{\mu_1\ldots\mu_r}\,. \tag{2.13}$$

Quantities without indices can be easily recognized as being differential forms; finally, the integral of the top form is defined as $\int\omega_d = \int d^dx\;\omega_{1\ldots d}(x)$.

The variation of the Chern-Simons action (2.12) gives the non-dissipative classical Hall current[1] j^μ in 2+1d,

$$j^\mu = \frac{\delta S_{\rm CS}}{\delta A_\mu} = -\varepsilon^{\mu\alpha\beta}\partial_\alpha A_\beta, \qquad \mu,\alpha,\beta = 1,2,3. \tag{2.14}$$

This bulk current is evaluated in the direction $\hat{3}$ orthogonal to the boundary $\partial\mathcal{M}_3$ located at $x^3=0$, for points $x^\alpha, \alpha = 1,2$ on the boundary, where it reads

$$j^3 = -\varepsilon^{3\alpha\beta}\partial_\alpha A_\beta\Big|_{\partial\mathcal{M}_3} = \partial_\alpha J^\alpha, \qquad \alpha,\beta = 1,2; \tag{2.15}$$

[1]Note that the QHE filling fraction is $\nu = 1$, corresponding to free fermions on the boundary.

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namely, it is equal to the divergence of the 1+1d current J. Upon integration along the one-dimensional space boundary, it implies that the bulk flux equals the time variation of the charge in the 1+1d theory. This is the anomaly inflow, the compensation of the quantum anomaly by a classical current in one extra dimension.

The $\nu = 1$ quantum Hall system gives rise to a chiral boundary fermion. To describe a time-reversal symmetric system, we consider chiral spin-up and antichiral spin-down fermions at the boundary. The resulting time-reversal invariant system is realized in the quantum spin Hall effect and topological insulators [20, 24, 37].

Neglecting the spin degree of freedom, this setting can describe 1+1d Dirac fermions. The bulk action is obtained by writing Chern-Simons actions (2.12) for each chiral background $A_\pm = A \pm \tilde{A}$, that exchange under time reversal, leading to

$$S_{\mathrm{BF}}[A, \tilde{A}] = -2 \int_{\mathcal{M}_3} \tilde{A} dA \,. \tag{2.16}$$

The corresponding bulk currents are

$$j^3 = \frac{\delta S_{\mathrm{BF}}}{\delta A_3} = -2\varepsilon^{3\alpha\beta}\partial_\alpha \tilde{A}_\beta = \partial_\alpha J^\beta_{\mathrm{cov}} \,, \tag{2.17}$$

$$\tilde{j}^3 = \frac{\delta S_{\mathrm{BF}}}{\delta \tilde{A}_3} = -2\varepsilon^{3\alpha\beta}\partial_\alpha A_\beta = \partial_\alpha \tilde{J}^\beta_{\mathrm{cov}} \,. \tag{2.18}$$

The 1+1d currents obtained by anomaly inflow $J_{\mathrm{cov}}, \tilde{J}_{\mathrm{cov}}$ are called 'covariant' and are different from the 'consistent' currents (2.5) (2.6), obtained by variation of the boundary action (2.4), and their respective anomalies are different. The two definitions of currents will often appear in this work: thus, it is important to clarify their technical and physical properties.[2] Covariant and consistent currents are best known in non-Abelian gauge theories [38], but they also occur in presence of both Abelian backgrounds $A, \tilde{A}$, where 'covariant' actually stands for invariant.

The 2+1 bulk currents $j, \tilde{j}$ are manifestly gauge invariant, and their reduction in 1+1d, $J_{\mathrm{cov}}, \tilde{J}_{\mathrm{cov}}$ are also invariant, as explicitly shown in the following section. Gauge invariant currents should be used when coupling to any physical quantity, either in 1+1d or 2+1d. The 2+1 bulk here plays the role of a charge reservoir for the 1+1 boundary. It is known that covariant anomalies can be obtained by heat-kernel regularization of the fermion path-integral, they obey the Index theorem and describe the spectral flow.

Consistent currents $J_{\mathrm{cons}}, \tilde{J}_{\mathrm{cons}}$ are not gauge invariant in general; they are obtained by a variation of the 1+1d effective action, as seen before, or by fermion loop calculations. They are also derived from the Wess-Zumino-Witten (WZW) action, obeying the Wess-Zumino consistency conditions, thus explaining their name [39]. These currents describe the physics of isolated 1+1d systems, with some provisos.

The relation between the Chern-Simons and WZW action,

$$S_{\mathrm{CS}}[A + d\lambda, \tilde{A} + d\tilde{\lambda}] = S_{\mathrm{CS}}[A, \tilde{A}] + S_{\mathrm{WZW}}[A, \lambda, \tilde{A}, \tilde{\lambda}] \,, \tag{2.19}$$

[2] A full analysis of this subject can be found in section 4 and appendix D of [24].

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shows some properties of the two kinds of currents. Inserting the action (2.16), we obtain from (2.19)[3]

$$S_{\rm WZW} = -2 \int_{\partial \mathcal{M}_3} \tilde{\lambda} dA \,, \tag{2.20}$$

whose variations over $\lambda, \tilde{\lambda}$ correctly reproduce the consistent anomalies (2.8), (2.9).

Note that another form of the bulk action (2.16), $S_{\rm BF} = -2\int A d\tilde{A}$, differing by a boundary term, would give the same covariant currents (2.17), (2.18), but would lead to different WZW action and consistent anomalies,

$$\hat{S}_{\rm WZW} = -2 \int_{\partial \mathcal{M}_3} \lambda d\tilde{A}, \qquad \partial_\mu \hat{J}^\mu_{\rm cons} = -2\varepsilon^{\alpha\beta}\partial_\alpha \tilde{A}_\beta \,, \qquad \partial_\mu \hat{\tilde{J}}^\mu_{\rm cons} = 0 \,. \tag{2.21}$$

The difference between $J_{\rm cons}$ and $\hat{J}_{\rm cons}$ (and their axial companions) is related to the freedom of changing the 1+1d effective action by terms polynomial in the backgrounds. This can be used, e.g. to switch anomaly from one symmetry to the other. However, the relation between $J_{\rm cons}$ and $J_{\rm cov}$ is not of this kind. The differences,

$$J_{\rm cov} = J_{\rm cons} + \Delta J, \qquad \tilde{J}_{\rm cov} = \tilde{J}_{\rm cons} + \Delta \tilde{J}, \tag{2.22}$$

are called Bardeen-Zumino terms [38]; they can be computed by paying attention to boundary terms when varying the bulk action (2.16). We find

$$\delta S_{\rm BF} = -2 \int_{\mathcal{M}_3} \delta \tilde{A} dA + \delta A d\tilde{A} - 2 \int_{\partial \mathcal{M}_3} \delta A \tilde{A} \,. \tag{2.23}$$

From the variation of the bulk term we recover the covariant currents (2.17), (2.18), while from the boundary term we get the Bardeen-Zumino terms

$$\Delta J^\alpha = -2\varepsilon^{\alpha\beta}\tilde{A}^\beta, \qquad \Delta \tilde{J}^\alpha = 0 \,, \tag{2.24}$$

which match the differences between the two kinds of anomalies (2.8)(2.9) and (2.17), (2.18).

In summary, the covariant currents are obtained by anomaly inflow and appear in gauge invariant expressions; the consistent currents follow by variation of the 1+1d effective action. The relation between the two kinds of currents can be unambiguously determined from the 2+1d topological theory (2.16).

2.3 'Hydrodynamic' topological theory and bosonic fields

Topological phases of matter are characterized by global effects and excitations which are described by topological actions written in terms of matter currents. Borrowing the reasoning from hydrodynamics, after integrating out high-energy excitations, the low-energy states should be described in terms of conserved currents [40]. Furthermore, these currents can be parameterized by dual gauge fields: in 2+1d we can write

$$j^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu \tilde{q}_\rho \,, \qquad \tilde{j}^\mu = \varepsilon^{\mu\nu\rho}\partial_\nu p_\rho \,, \tag{2.25}$$

where $p, \tilde{q}$ are one-form dynamic gauge fields expressing matter degrees of freedom. The parameterization (2.25) makes conservation of currents automatic ('topological currents').

[3]The WZW action is 1+1 dimensional in the Abelian case.

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Regarding the interaction between these fields, we can introduce a BF term leading to the following 'hydrodynamic' topological theory,

$$S[p,\tilde{q};A,\tilde{A}] = 2\int_{\mathcal{M}_3} \tilde{q}dp + \tilde{q}dA + pd\tilde{A}\,, \tag{2.26}$$

which has been used for describing the quantum spin Hall effect [24].

Upon integration in $p,\tilde{q}$, one correctly obtains the 'response' action (2.16) $S_{\rm BF}[A,\tilde{A}]$ involving the backgrounds only. The action (2.26) for general coupling constant $2k$ (here $k=1$) describes anyonic braiding of excitations coupled to the $p,\tilde{q}$ fields and the ground state degeneracy on topologically non-trivial manifolds, the so-called topological order [25].

In the following we show how the bulk $p,\tilde{q}$ fields are related to the boundary degrees of freedom which define the scalar theory of section 2.1. This identification follows by matching the covariant currents in the anomaly inflow. We first solve the equations of motion of the bulk theory (2.26), with the result

$$dp + dA = 0 \quad \to \quad p = d\theta - A, \qquad\qquad d\tilde{q} + d\tilde{A} = 0 \quad \to \quad \tilde{q} = d\psi - \tilde{A}, \tag{2.27}$$

where θ,ψ are respectively scalar and pseudoscalar gauge degrees of freedom, which are free variables at this level. Then, we reconsider the inflow relation (2.18)

$$\tilde{j}^3 = 2\varepsilon^{3\alpha\beta}\partial_\alpha p_\beta = \partial_\alpha \tilde{J}^\alpha_{\rm cov} \quad \to \quad \tilde{J}^\alpha_{\rm cov} = 2\varepsilon^{\alpha\beta}p_\beta \underset{eq.m.}{=} 2\varepsilon^{\alpha\beta}(\partial_\beta\theta - A_\beta)\,. \tag{2.28}$$

In the identification of the 1+1d current $\tilde{J}_{\rm cov}$, we used the equations of motion and the freedom in choosing θ. An analogous expression is found for the current $J_{\rm cov}$,

$$J^\alpha_{\rm cov} = 2\varepsilon^{\alpha\beta}\tilde{q}_\beta \underset{eq.m.}{=} 2\varepsilon^{\alpha\beta}(\partial_\beta\psi - \tilde{A}_\beta)\,, \tag{2.29}$$

in terms of the pseudoscalar field ψ.

We now remark that 2+1d currents are manifestly gauge invariant, thus the 1+1d covariant currents should also be so. This requirement implies a boundary gauge condition for the bulk fields, namely:

$$p = d\theta - A\,, \qquad \tilde{q} = d\psi - \tilde{A}\,, \qquad\qquad \text{gauge invariant on } \partial\mathcal{M}_3\,. \tag{2.30}$$

It follows that the fields θ,ψ must compensate the gauge transformations of background fields: $(A \to A + d\lambda, \theta \to \theta + \lambda)$ and $(\tilde{A} \to \tilde{A} + d\tilde{\lambda}, \psi \to \psi + \tilde{\lambda})$.

We see that the inflow relation (2.28) reproduces the axial current (2.6) as a function of previously introduced scalar field θ with correct gauge transformations. Indeed, this field is the gauge degree of freedom that becomes physical, as it occurs in σ-models of spontaneous symmetry breaking and the Higgs phenomenon; being a consequence of gauge symmetry only, the same mechanism also occurs in other phases of matter, having different dynamics.

The complete form of the effective action including bulk and boundary terms can be written as follows[4]

$$S = \int_{\mathcal{M}_2} -p_\mu^2 + 2\int_{\mathcal{M}_3} \tilde{q}dp + \tilde{q}dA + pd\tilde{A}\,. \tag{2.31}$$

[4] In this and following integrals we use a mixed notation involving both vectors (with indices) and differential forms (without indices), with rules specified earlier in (2.13).

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In this expression, the first term encodes the dynamics of boundary excitations, while the second term is the topological action (2.26).

It is also convenient to introduce the notations $\pi = p + A$ and $\tilde{\pi} = \tilde{q} + A$ and rewrite (2.31) as

$$S = \int_{\mathcal{M}_2} -(\pi_\mu - A_\mu)^2 + 2\tilde{A}(\pi - A) + 2\int_{\mathcal{M}_3} \tilde{\pi} d\pi - \tilde{A} dA\,, \tag{2.32}$$

$$\pi = p + A, \qquad \tilde{\pi} = \tilde{q} + A\,, \tag{2.33}$$

paying attention to boundary terms.

Upon using the bulk equations of motion (2.27) for $p, \tilde{q}$, which now read $\pi = d\theta$ and $\tilde{\pi} = d\psi$, one is left with an action which depends on the yet undetermined 1+1d fields θ, ψ at the boundary, as follows

$$S = \int_{\mathcal{M}_2} -(\partial_\mu \theta - A_\mu)^2 + 2\tilde{A}\,(d\theta - A) - 2\int_{\mathcal{M}_3} \tilde{A} dA\,. \tag{2.34}$$

The expressions (2.34) and (2.32) of combined bulk-boundary theory are manifestly gauge-invariant, according to (2.30): the 1+1d part reproduces the bosonic theory (2.4) described in section 2.1. Thus, we succeeded in re-deriving the boundary bosonic theory from bulk topological data and anomaly inflow.

Let us add some remarks:

- The form of the bulk-boundary action (2.32) is uniquely obtained by gauge invariance from the topological theory (2.26), apart from the addition of the gauge invariant dynamic term p_μ^2. The ambiguities on boundary terms which might occur in defining the 2+1d theory (2.26) have been fixed by earlier assumptions (2.21), (2.16), determining the form of consistent currents on the boundary.

- Bulk and boundary variables have been matched by comparing currents via inflow (2.28). (2.29). These observable quantities are free of ambiguities. The boundary degrees of freedom can also be obtained by other methods, but we find that the matching of currents provides the most reliable procedure. Note that ψ has disappeared from (2.34) but is relevant for other forms of the action, as shown in the next section.

- The bulk equations of motion (2.27) for $p, \tilde{q}$ were obtained by assuming vanishing variations on the boundary. Their solution is extended by continuity to the boundary. Next, the remaining undetermined degrees of freedom θ, ψ are subjected to the boundary equations of motion and actually acquire a dynamics due to the first term in (2.34).

- Finally, a general remark on using classical actions: they may have some arbitrariness because different functional forms can share the same extrema. For example, it is possible to solve a subset of equations of motion and substitute them into the action, obtaining another, equivalent functional, to be extremized on the remaining variables.

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2.3.1 Duality of 1+1 dimensional theory

The scalar fields ψ, θ play a rather symmetric role in the 2+1d action (2.26) and covariant currents (2.28), (2.29): actually, the theory is manifestly invariant under the following duality that exchanges fields, currents and anomalies,

$$(\theta, J_{\text{cov}}, A) \leftrightarrow (\psi, \tilde{J}_{\text{cov}}, \tilde{A}). \tag{2.35}$$

The boundary theory (2.34) is not explicitly self-dual. The field ψ cancels out and the consistent currents (2.5), (2.6) are of different nature: $\tilde{J}_{\text{cov}} = \tilde{J}_{\text{cons}}$ is topological, while J_{cons} is of Noether type, obeying $\partial_\mu J^\mu_{\text{cons}} = 0$, Actually, the duality is also present in 1+1d: it is expressed by mapping the action (2.34) into another one with the same functional from in terms of dual variables.

We start again from the bulk-boundary expressions (2.28), (2.29), and substitute the following partial solution of the bulk equations of motion: $p = \pi - A$ and $\tilde{q} = d\psi - \tilde{A}$, where π transforming as A for respecting the gauge condition (2.30).

The action (2.32) becomes (neglecting the inert background part $\int \tilde{A}\, dA$)

$$\tilde{S} = 2 \int_{\mathcal{M}_2} -\frac{1}{2}(\pi_\mu - A_\mu)^2 + \tilde{A}\,(\pi - A) + \psi d\pi\,. \tag{2.36}$$

The pseudoscalar ψ is now present and it enforces the constraint $d\pi = 0$. Solving it by $\pi = d\theta$, the action (2.34) is reobtained. Thus, (2.36) is an equivalent form of (2.34).

The equations of motion can be equivalently obtained by varying with respect to π, leading to

$$\pi^\mu - A^\mu = \varepsilon^{\mu\nu}(\partial_\nu \psi - \tilde{A}_\nu)\,. \tag{2.37}$$

This equation can be used to eliminate π in (2.36), with the result

$$\tilde{S} = 2 \int_{\mathcal{M}_2} -\frac{1}{2}(\partial_\mu \psi - \tilde{A}_\mu)^2 + A\left(d\psi - \tilde{A}\right) + A\tilde{A}\,. \tag{2.38}$$

This is the dual theory of (2.34), having the same form under the exchanges (2.35). It is expressed in terms of the pseudoscalar field and the consistent currents (2.5), (2.6), exchange their nature: $\tilde{J}$ is of Noether type, obeying $\partial_\nu \tilde{J}^\nu_{\text{cons}} = 0$, and $J_{\text{cons}} = J_{\text{cov}}$ is topological.

The equations of motion (2.37) for $A = \tilde{A} = 0$ and $\pi = d\theta$ become

$$\partial^\mu \theta = \varepsilon^{\mu\nu}\partial_\nu \psi\,, \tag{2.39}$$

showing that ψ is the dual field of θ. In conclusion, the covariant currents obtained by inflow (2.29), (2.28) are both of topological kind, with ψ the dual field of θ. The consistent currents are expressed in terms of a single field, one is topological and the other of Noether type.

In the appendix C we consider the bosonic theory (2.34) in a chiral background, i.e. $A = \tilde{A}$ and show that boundary excitation split into chiral components, one interacting with the background and the other one remaining inert.

Let us finally recall the description of interacting fermions by the bosonic theory, which is the main virtue of 1+1d bosonization.[5] The analysis in previous sections has shown how the

[5]A complete account can be found in [24, 41].

boson maps into the free fermion and reproduces its anomalies. The bosonic theory admits a simple generalization. Starting from the topological action (2.26), a coupling constant k can be added to the first term, $2\int \tilde{q}dp \to 2k\int \tilde{q}dp$, leading to anomalies (2.28), (2.29) multiplied by the factor $1/k$, as well as the corresponding WZW action. As a consequence, the 1+1d bosonic theory (2.34) is also modified by the coupling k, which parameterizes a critical line describing massless fermions with current-current interaction. The duality transformation of the previous section also extends to a map between theories with $k \leftrightarrow 1/k$.

At the quantum level, the bosonic field becomes a compact variable, $\theta(x) \equiv \theta(x) + 2\pi$, for the proper definition of large gauge transformations on physical states (cf. (2.30)). The coupling constant k can be included in θ by a field redefinition, thus changing the compactification radius: within this convention, the critical line is equivalently parameterized by the compactification radius.

3 Hydrodynamics with anomalies

3.1 Variational formulation of Euler hydrodynamics

In this section we recall the description of perfect barotropic fluids (in any dimension) developed in the refs. [19, 36]. In these systems, the temperature vanishes, entropy is constant and there is no heat conduction, viscosity and shear. Furthermore, the pressure depends on density only. The subject has a very long history and some introductory material can be found in appendix A and the review [42]. The same formulation applies to both non-relativistic and relativistic fluids: the first case is briefly discussed here for simplicity.[6]

The Lagrangian variational problem is obtained starting from the Hamiltonian description of the Euler equation in terms of the basic variables of density ρ and velocity v^i, obeying Poisson brackets. The 'particle current' is defined as $\mathcal{J}^\mu = (\rho, \rho v^i)$ and satisfies $\partial_\mu \mathcal{J}^\mu{=}0$. It is possible to Legendre transform from energy $\varepsilon(\rho)$ to pressure $P(\mu)$, which is a function of the chemical potential μ, to obtain the action,

$$S = \int \rho\mu - \varepsilon(\rho) = \int P(\mu), \qquad \frac{dP}{d\mu} = \rho, \quad \frac{d\varepsilon}{d\rho} = \mu\,. \tag{3.1}$$

The chemical potential itself depends on the particle momenta, $\mu = \mu(p_\alpha)$, with dependence specific for nonrelativistic (Galilean) and relativistic (Lorentzian) fluids (See appendix A). Thus, the variational approach is defined for the following functional of momenta p_α

$$S[p_\alpha] = \int P(p_\alpha)\,. \tag{3.2}$$

This formulation of hydrodynamics is very similar to an effective field theory for the 'field' p_α. Both are low-energy descriptions, and we are not dealing with entropy and dissipation, which would require more advanced quantum field theory settings, such as the Schwinger-Keldysh approach [40, 43].

However, there is still a difference, because the variational problem for the hydrodynamic action (3.2) is constrained. As explained in the review [42], in going from the Lagrangian

[6]The relativistic case is discussed in the appendix A.

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fluid description[7] to the Eulerian formulation, the motion of each fluid element is lost, leaving a reduced set of degrees of freedom that obey constraints. Due to this reduction, the Hamiltonian Poisson algebra obeyed by (ρ, v^i) possesses extra Casimirs which prevents the writing of a standard Lagrangian variational principle (see also appendix A).

As a consequence, the variations δp_α extremizing (3.2) are constrained: one possibility is to use a reduced functional dependence for p_α given by the Clebsch parameterization [42]. Another formulation uses restricted, 'admissible' variations [44] which correspond to the diffeomorphisms, expressed by the Lie derivative $\delta_\varepsilon p = \mathcal{L}_\varepsilon p$

$$\delta S[p] = 0\,, \qquad \text{for} \qquad \delta_\varepsilon p_\nu = \varepsilon^\mu \partial_\mu p_\nu + p_\mu \partial_\nu \varepsilon^\mu, \tag{3.3}$$

where ε^μ is an arbitrary vector field.

The variation of the action (3.2) leads to the following two results:

I. We identify the particle current $\mathcal{J}$, and obtain after one integration by parts

$$\mathcal{J}^\mu = -\frac{\delta S}{\delta p_\mu} = -\frac{\partial P}{\partial p_\mu}, \qquad \mathcal{J}^\nu(\partial_\mu p_\nu - \partial_\nu p_\mu) + p_\nu \partial_\mu \mathcal{J}^\mu = 0\,. \tag{3.4}$$

Upon contracting the second relation with $\mathcal{J}^\mu$, we get two equations of motion

$$\partial_\mu \mathcal{J}^\mu = 0, \qquad \mathcal{J}^\nu(\partial_\mu p_\nu - \partial_\nu p_\mu) = 0, \tag{3.5}$$

corresponding to current conservation and the so-called Carter-Lichnerowicz (CL) equation [45]. As shown in appendix A, the latter is equal to the Euler equation after expressing the variables $\mathcal{J}^\mu, p_\mu$ in terms of ρ, v^i and the chemical potential μ.

II. Given that the Lie derivative expresses reparameterization invariance, the variation (3.3) defines the stress tensor of the fluid and establishes its conservation: we find,

$$T^\nu_\mu = -\frac{\partial P}{\partial p_\nu} p_\mu + \delta^\nu_\mu P\,, \qquad \partial_\nu T^\nu_\mu = 0. \tag{3.6}$$

These equations reproduce again the Euler equation (for spatial μ) and the energy conservation (for μ the time direction). The relativistic fluid dynamics is also obtained for specific forms of pressure $P(p_\alpha)$ and variables p_μ, as shown in appendix A.

3.1.1 Topological currents in 1+1 and 3+1 dimensions

The Carter-Lichnerowicz equation (3.5) can be rewritten in differential-form notation using the inner product i_X,

$$i_{\mathcal{J}} dp = 0, \qquad \mathcal{J}^\mu = -\frac{\partial P}{\partial p_\mu}\,. \tag{3.7}$$

Note that this is not a diffeomorphism-invariant equation, because the current $\mathcal{J}$ depends itself on p_α and contains dynamic information through the form of pressure P.

However, even spacetime dimensions are special. In 1+1d we observe that dp is a top form, dual to a scalar quantity: the CL equation is actually a scalar condition. Thus, if

[7] Not to be confused with the Lagrangian functional.

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this is satisfied for the particle current $\mathcal{J}$, it also holds for arbitrary currents: it amounts to the constraint

$$dp = 0\,, \qquad\qquad (1+1\text{ d}). \tag{3.8}$$

In 3+1d, by multiplying the CL equation by dp, it gives the condition $i_{\mathcal{J}}(dpdp) = 0$ for a top form, which similarly implies the constraint $dpdp = 0$.

Therefore, in even spacetime dimensions the CL equation results in a geometric condition independent of dynamics. It leads to the definition of conserved topological currents, as follows

$$1+1d: \qquad dp = 0 \qquad \to \qquad \partial_\mu \tilde{J}^\mu = 0\,, \qquad \tilde{J}^\mu = \varepsilon^{\mu\nu} p_\nu\,, \tag{3.9}$$

$$3+1d: \qquad dpdp = 0 \qquad \to \qquad \partial_\mu \tilde{J}^\mu = 0\,, \qquad \tilde{J}^\mu = \varepsilon^{\mu\nu\rho\sigma} p_\nu \partial_\rho p_\sigma\,. \tag{3.10}$$

In 1+1d, one recovers the axial current (2.3) in the bosonic form; in 3+1d a new axial current is conserved by the flow, whose charge is the so-called fluid helicity

$$\tilde{Q} = \int d^3x\, \tilde{J}^0 \propto \int d^3x\, \vec{v}\cdot\vec{\omega}, \qquad \vec{\omega} = \nabla\times\vec{v}\,, \tag{3.11}$$

which is the integral of the scalar product of vorticity and velocity.

In conclusion, the hydrodynamic approach in even spacetime dimension possesses both vector and axial currents. This rather remarkable fact allows for the description of fermionic systems and their anomalies, which will be analyzed in detail in the following discussion.

3.2 Equivalence of hydrodynamics and bosonization in 1+1 dimensions

3.2.1 Admissible variations

The coupling to backgrounds $A, \tilde{A}$ is introduced in the hydrodynamic action as follows. The kinetic momentum is replaced by the canonical momentum π,

$$p = \pi - A\,, \tag{3.12}$$

where π varies as A under gauge transformations for keeping p gauge invariant (cf. (2.30)). Next, $\tilde{A}$ is coupled to the axial current found in the previous section. The action becomes

$$S[\pi] = \int d^2x\, P(\pi - A) + 2\int_{\mathcal{M}_2} \tilde{A}(\pi - A)\,. \tag{3.13}$$

As discussed in the previous section, the variational problem is given by diffeomorphism variations $\delta\pi = \mathcal{L}_\varepsilon \pi$, and the result is the CL equation, implying a constraint

$$\mathcal{J}^\nu(\partial_\mu \pi_\nu - \partial_\nu \pi_\mu) = 0\,, \qquad \to \qquad d\pi = 0\,, \tag{3.14}$$

together with the conservation of the particle current

$$\partial_\nu \mathcal{J}^\nu = 0, \qquad \mathcal{J}^\nu = -\frac{\partial P}{\partial \pi_\nu} + 2\epsilon^{\nu\mu}\tilde{A}_\mu\,, \tag{3.15}$$

which now includes a background term.

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The consistent currents are obtained by variation of the action

$$J^\nu_{\text{cons}} = \frac{\delta S}{\delta A_\nu} = -\frac{\partial P}{\partial \pi_\nu} + 2\varepsilon^{\nu\mu}\tilde{A}_\mu = \mathcal{J}^\nu \,, \tag{3.16}$$

$$\tilde{J}^\nu_{\text{cons}} = \frac{\delta S}{\delta \tilde{A}_\nu} = 2\varepsilon^{\nu\mu}(\pi_\mu - A_\mu)\,. \tag{3.17}$$

Upon using the equations of motion, their divergences reproduce the expected expressions of section 2.1,

$$\partial_\nu J^\nu_{\text{cons}} = 0, \qquad\qquad \partial_\nu \tilde{J}^\nu_{\text{cons}} = -2\varepsilon^{\nu\mu}\partial_\nu A_\mu\,. \tag{3.18}$$

We conclude that the hydrodynamic theory is able to describe Dirac fluids in 1+1 dimensions.

The conservation law for the stress tensor is obtained by applying a diffeomorphism to all fields in the action (3.13):

$$\begin{aligned}
\delta S &= \int d^4x \left(\frac{\delta S}{\delta \pi_\nu}\delta_\varepsilon \pi_\nu + \frac{\delta S}{\delta A_\nu}\delta_\varepsilon A_\nu + \frac{\delta S}{\delta \tilde{A}_\nu}\delta_\varepsilon \tilde{A}_\nu \right) \\
&= \int d^4x\, \epsilon^\mu \left(-\partial_\nu T^\nu_\mu + J^\nu_{\text{cons}}\partial_\mu A_\nu - \partial_\nu (A_\mu J^\nu_{\text{cons}}) + \tilde{J}^\nu_{\text{cons}}\partial_\mu \tilde{A}_\nu - \partial_\nu (\tilde{A}_\mu \tilde{J}^\nu_{\text{cons}}) \right) \\
&= \int d^4x\, \epsilon^\mu \left(-\partial_\nu T^\nu_\mu + F_{\mu\nu} J^\nu_{\text{cons}} - A_\mu \partial_\nu J^\nu_{\text{cons}} + \tilde{F}_{\mu\nu}\tilde{J}^\nu_{\text{cons}} - \tilde{A}_\mu \partial_\nu \tilde{J}^\nu_{\text{cons}} \right)\,.
\end{aligned} \tag{3.19}$$

In these equations, all variations are Lie derivatives (3.3) with parameter ε, the first term giving the stress tensor divergence, with

$$T^\nu_\mu = -\frac{\partial P}{\partial \pi_\nu}(\pi_\mu - A_\mu) + \delta^\nu_\mu P\,. \tag{3.20}$$

After rewriting the consistent currents in terms of covariant ones, one obtains[8] [36]

$$\partial_\nu T^\nu_\mu = F_{\mu\nu} J^\nu_{\text{cov}} + \tilde{F}_{\mu\nu}\tilde{J}^\nu_{\text{cov}}\,. \tag{3.21}$$

Note that the currents appearing in the stress tensor equation are of the covariant type, in agreement with gauge invariance.

The geometric condition $d\pi = 0$ can be solved in terms of a scalar variable $\pi = d\theta$; upon this substitution, it is clear that the hydrodynamic action and the currents exactly corresponds to expressions of the bosonic theory of section 2.1. In this correspondence, the relativistic expression of the pressure $P(p_\alpha) \sim p_\alpha^2$ maps in the dynamic term of the bosonic action $P(d\theta - A) \leftrightarrow -(\partial_\mu\theta - A_\mu)^2$. Actually, in the hydrodynamic as well as the bosonic theory, one can check that the anomalies are completely independent of the dynamics, provided that gauge invariance is respected. In particular, they also occur for non-relativistic and vanishing dynamics ($P = 0$).

Of course, it is well known that anomalies are geometric, i.e. largely independent of interactions, but the present approach makes this property manifest. This is one interesting aspect of it.

[8] The result can also be checked from definitions of $T, J, \tilde{J}$, using the equation of motion (3.15) and the 1+1d identity $F_{\mu\nu} = F\varepsilon_{\mu\nu}$, $F = \partial_1 A_2 - \partial_2 A_1$.

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3.2.2 Free variations with explicit constraint

One difference remains to be understood before establishing the complete correspondence between bosonic field theory and hydrodynamics in 1+1d: the field-theory Lagrangian is extremized under general variations, while they are restricted in hydrodynamics.

Recall that admissible variations (3.3) imply the simple constraint $d\pi = 0$ (3.14) in 1+1d. Thus, we can freely vary $\delta\pi$ by adding the constraint via a (pseudoscalar) Lagrange multiplier ψ.

The action (3.13) becomes

$$S[\pi, \psi] = \int d^2x\, P(\pi - A) + 2\int_{\mathcal{M}_2} \tilde{A}(\pi - A) + \pi d\psi\,. \tag{3.22}$$

The expressions of the stress tensor (3.20) and currents (3.17) are clearly unchanged. The equations of motion for the unconstrained variables π and ψ are

$$\psi: \qquad d\pi = 0 \qquad \to \qquad \pi_\mu = \partial_\mu\theta\,, \tag{3.23}$$

$$\pi: \qquad \frac{\partial P}{\partial \pi_\mu} = 2\varepsilon^{\mu\nu}\left(\tilde{A}_\nu - \partial_\nu\psi\right)\,. \tag{3.24}$$

Using them in the (unchanged) expressions of the currents (3.17) and stress tensor (3.20), one verifies that the same anomalies and conservation equations (3.18), (3.21) are reobtained.

More precisely, the new equations of motion (3.23), (3.24) can be compared with those obtained by admissible variations (3.14), (3.15):

$$\text{admissible}: \qquad d\pi = 0, \qquad \partial_\mu J^\mu_{\text{cons}} = \partial_\mu\left(-\frac{\partial P}{\partial \pi_\mu} + 2\varepsilon^{\mu\nu}\tilde{A}_\nu\right), \tag{3.25}$$

$$\text{free}: \qquad d\pi = 0, \qquad J^\mu_{\text{cons}} = \left(-\frac{\partial P}{\partial \pi_\mu} + 2\varepsilon^{\mu\nu}\tilde{A}_\nu\right) \underset{eq.m.}{=} 2\varepsilon^{\mu\nu}\partial_\nu\psi\,. \tag{3.26}$$

It is apparent that current conservation has been replaced by its 'first integral', namely the general parameterization of a conserved current in terms of the unconstrained parameter ψ. Note that in classical mechanics, freely varying the action with a time-dependent constraint imposed by a Lagrange multiplier is generally not equivalent to taking restricted variations. However, in the present case we checked that the same result is obtained.

In conclusion, we have shown that the hydrodynamic theory can be reformulated as the unconstrained Lagrangian variational problem (3.22), and as such it is completely equivalent to standard field-theory bosonization in 1+1 dimensions.

3.3 Summary of sections two and three

In section 2 we recalled some properties of bosonization in 1+1d and got acquainted with different forms of currents and their relation with Chern-Simons and WZW actions. We then showed that the bosonic theory can be inferred from the 'hydrodynamic' topological theory in one extra dimension, which not only determines the anomalies but also suggests the needed boundary degrees of freedom.

In section 3, we reviewed the hydrodynamic approach developed in ref. [19], relying on an action functional. We observed that the equations of motion imply dynamics-independent

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constraints in 1+1 and 3+1d, thus allowing free variational problems as those considered in quantum field theory. In 1+1d we established the complete correspondence between this formulation of hydrodynamics and the well-known bosonic field theory describing fermions.

Based on this background material, in the next section we are going to develop the hydrodynamics with anomalies in 3+1d, extending the results of [36]. The input from the 'hydrodynamic' topological theory in 4+1d will suggest new variables which are necessary for a complete description of anomalous fluids in 3+1 dimensions.

4 Hydrodynamics and inflow in 3+1 dimensions

4.1 4+1 dimensional Chern-Simons action and general form of anomalies

The 'bulk' response action is well known in the literature [24, 46]: it is given by the 4+1d Chern-Simons action written in terms of background fields.

We start from the expression describing the chiral anomaly of a 3+1d Weyl fermion[9]

$$S[A_+] = -\frac{1}{6}\int_{\mathcal{M}_5} A_+ dA_+ dA_+ \,, \qquad \left(\frac{e}{2\pi} \to 1\right), \tag{4.1}$$

where the chiral decomposition is defined by

$$A_\pm = A \pm \tilde{A}, \qquad J_\pm = \frac{1}{2}(J \pm \tilde{J}) \qquad \to \qquad J^\mu A_\mu + \tilde{J}^\mu \tilde{A}_\mu = J^\mu_+ A_{+\mu} + \tilde{J}^\mu_- \tilde{A}_{-\mu}\,. \tag{4.2}$$

Note that this action is time reversal invariant but not parity invariant, as it should. Then one subtracts to it another copy $S[A_-]$, thus obtaining the parity invariant expression describing Dirac fermions,

$$S[A,\tilde{A}] = -\int_{\mathcal{M}_5} \tilde{A} dA dA + \alpha \tilde{A} d\tilde{A} d\tilde{A}\,, \qquad \alpha = \frac{1}{3}\,, \tag{4.3}$$

with $A_\pm = A \pm \tilde{A}$. The action (4.3) shows two independent parity and time reversal invariant expressions, build out of an odd number of pseudovectors $\tilde{A}$. Therefore, anomalies of general fermionic particles involve two parameters depending on their vector and axial couplings.[10] We fix the first one to the value for the Dirac particle and let the parameter α vary off the Dirac value $\alpha = 1/3$ for later convenience.

As explained in section 2.2, the variation of response actions gives the bulk currents and then the 3+1d covariant anomalies by inflow. One finds

$$j^5 = \frac{\delta S}{\delta A_5} = -2\varepsilon^{5\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_r \tilde{A}_\sigma \qquad \to \qquad \partial_\mu J^\mu_{\rm cov} = -2[dAd\tilde{A}]\,, \tag{4.4}$$

$$\tilde{j}^5 = \frac{\delta S}{\delta \tilde{A}_5} \qquad \to \qquad \partial_\mu \tilde{J}^\mu_{\rm cov} = -[dAdA] - 3\alpha[d\tilde{A}d\tilde{A}]\,, \tag{4.5}$$

within our normalization $(e/2\pi)^2 \to 1$.

The gauge transformation ($A \to A + d\lambda, \tilde{A} \to \tilde{A} + d\tilde{\lambda}$) of the bulk action (4.3) determines the WZW action

$$S_{\rm WZW} = -\int_{\mathcal{M}_4} \tilde{\lambda}(dAdA + \alpha d\tilde{A}d\tilde{A})\,, \tag{4.6}$$

[9]See appendix B for the normalization of fields.
[10]In appendix B we collect anomaly formulas for general matter content.

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and its variation provides the anomalies of consistent currents,

$$\partial_\mu J^\mu_{\text{cons}} = 0\,, \tag{4.7}$$
$$\partial_\mu \tilde{J}^\mu_{\text{cons}} = -[dAdA] - \alpha[d\tilde{A}d\tilde{A}]\,. \tag{4.8}$$

Note that the order of $(A, \tilde{A})$ factors in the first term of the action (4.3) can be changed by adding 3+1d boundary terms; the present form is vector gauge invariant, leading to a conserved consistent vector current in agreement with the hydrodynamic approach.

Finally, following section 2.2, the relation between covariant and consistent currents is obtained by close inspection of boundary terms in the variation of the bulk action as done in the 1+1d case [24]:

$$J^\mu_{\text{cov}} = J^\mu_{\text{cons}} - 2[\tilde{A}dA]^\mu\,, \tag{4.9}$$
$$\tilde{J}^\mu_{\text{cov}} = \tilde{J}^\mu_{\text{cons}} - 2\alpha[\tilde{A}d\tilde{A}]^\mu\,. \tag{4.10}$$

These expressions match the anomalies just found.

4.2 Hydrodynamics with axial current

The hydrodynamic description of 3+1d anomalies developed in ref. [36] follows the same steps of the earlier 1+1d case. The axial current is identified as the 3+1d helicity current (3.10), which is coupled to its background as follows[11]

$$S[\pi] = \int d^4x\, P(\pi - A) + \int_{\mathcal{M}_4} \tilde{A}(\pi - A)d(\pi + A)\,. \tag{4.11}$$

The helicity current (3.10) is rewritten by replacing $p = \pi - A$ and an additional term is introduced which was physically motivated in [36] and will be confirmed by the following analysis.

The admissible (diffeomorphism) variations of this action involves the particle current,

$$\mathcal{J}^\mu = -\frac{\delta S}{\delta \pi_\mu} = -\frac{\partial P}{\partial \pi_\mu} + \left[2\tilde{A}d\pi - (\pi - A)d\tilde{A}\right]^\mu\,, \tag{4.12}$$

which obeys the Carter-Lichnerowicz equation (3.14). Following the discussion of section 3.1.1, in 3+1 dimensions this reduces to the constraint $d\pi d\pi = 0$; thus, the equations of motion are summarized by

$$\partial_\mu \mathcal{J}^\mu = 0, \qquad d\pi d\pi = 0\,. \tag{4.13}$$

A standard approach in hydrodynamics is that of introducing the so-called Clebsch parameterization[12] for π in terms of three scalar functions (θ, α, β), as follows:

$$\pi = d\theta + \alpha d\beta, \qquad \pi d\pi = d\theta d\alpha d\beta\,, \tag{4.14}$$

which provide a local solution to the constraint $d\pi d\pi = 0$. However, contrary to 1+1d, this non-linear map complicates expressions and cannot be extended globally because the

[11] Note that the coupling to the topological current is expressed by a top form.

[12] See appendix A for details.

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parameters (θ, α, β) develops singularities [42]. We shall, therefore, continue using the momentum variable π, keeping in mind the condition $d\pi d\pi = 0$.

The consistent currents are readily evaluated

$$J^\mu_{\text{cons}} = \frac{\delta S}{\delta A_\mu} = -\frac{\partial P}{\partial \pi_\mu} + [(\pi - A)d\tilde{A} + 2\tilde{A}dA]^\mu = \mathcal{J}^\mu + \left[d\Big(2\tilde{A}(\pi - A)\Big)\right]^\mu , \tag{4.15}$$

$$\tilde{J}^\mu_{\text{cons}} = \frac{\delta S}{\delta \tilde{A}_\mu} = [(\pi - A)d(\pi + A)]^\mu . \tag{4.16}$$

Using equations of motion, we find

$$\partial_\mu J^\mu_{\text{cons}} = 0 , \tag{4.17}$$

$$\partial_\mu \tilde{J}^\mu_{\text{cons}} = -[dAdA] . \tag{4.18}$$

These results were obtained in the analysis of ref. [36]; however, comparison with the general expressions (4.7), (4.8) in the previous section, shows that a term is missing in the axial current, proportional to $[d\tilde{A}d\tilde{A}]$. Namely, the hydrodynamic theory (4.11) is able to reproduce the anomalies correctly only for the parameter $\alpha = 0$. In the following sections, we shall see how to obtain this missing piece, as well as the contribution coming from curved metric backgrounds.

4.2.1 Free variations with generalized constraint

Paralleling the 1+1d analysis in section 3.2.2, we now consider the standard variational problem for the action (4.11), by adding the constraint via a Lagrange multiplier. The action is now

$$S[\pi] = \int_{\mathcal{M}_4} P(\pi - A) + \tilde{A}(\pi - A)d(\pi + A) + \psi(d\pi d\pi + \alpha d\tilde{A}d\tilde{A}) , \tag{4.19}$$

where ψ is pseudoscalar.

In this action, we also considered an extension of the constraint that is motivated by the form of the WZW action (4.6), with parameter α: this modification will be fully justified in the next section by matching the anomaly inflow from the 4+1d topological theory.

The equations of motion obtained by varying freely over π and ψ are:

$$\pi : \qquad -\frac{\partial P}{\partial \pi_\mu} + \left[2(\tilde{A} - d\psi)d\pi - (\pi - A)d\tilde{A}\right]^\mu = 0 , \tag{4.20}$$

$$\psi : \qquad d\pi d\pi + \alpha d\tilde{A}d\tilde{A} = 0 . \tag{4.21}$$

Comparing with those obtained by admissible variations (4.13) (for $\alpha = 0$), one sees that free variations again imply a ‘first integral’ of the particle current conservation:

$$\text{admissible :} \qquad \partial_\mu \mathcal{J}^\mu = \partial_\mu \left(-\frac{\partial P}{\partial \pi_\mu} + \left[2\tilde{A}d\pi - (\pi - A)d\tilde{A}\right]^\mu \right) = 0 , \tag{4.22}$$

$$\text{free :} \qquad \mathcal{J}^\mu \underset{eq.m.}{=} 2[d\psi d\pi]^\mu . \tag{4.23}$$

The parameterization (4.23) of the current in terms of ψ is general enough when $d\pi \neq 0$, i.e. in presence of fluid vorticity. As fully described in appendix A, it also holds at points

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where $d\pi \to 0$ by considering a limiting procedure where $d\psi \to \infty$, such that the current stays finite. In conclusion, free variation of the action (4.19) is equivalent to admissible variation for $\alpha = 0$, and extends to the case $\alpha \neq 0$.

Let us show that the correct anomalies and conservation laws are obtained. The vector current $J_{\rm cons}$ has the same form as the one of the previous section (4.15) and is conserved. The consistent current $\tilde{J}^\mu_{\rm cons}$ acquires a contribution from the term parameterized by α,

$$\tilde{J}^\mu_{\rm cons} = [(\pi - A)d(\pi + A) + 2\alpha d\psi d\tilde{A}]^\mu \,. \tag{4.24}$$

Its conservation law follows from (4.21)

$$\partial_\mu \tilde{J}^\mu_{\rm cons} = -[dAdA] - \alpha[d\tilde{A}d\tilde{A}] \,. \tag{4.25}$$

The equation satisfied by the energy-momentum tensor is obtained by considering a diffeomorphism transformation of the action. Since the top form $\psi(d\pi d\pi + \alpha d\tilde{A}d\tilde{A})$ is diffeomorphism-invariant, it does not contribute to the energy-momentum tensor, and the result is the same as (3.19):

$$\partial_\nu T^\nu_\mu = F_{\mu\nu} J^\nu_{\rm cons} - A_\mu \partial_\nu J^\nu_{\rm cons} + \tilde{F}_{\mu\nu}\tilde{J}^\nu_{\rm cons} - \tilde{A}_\mu \partial_\nu \tilde{J}^\nu_{\rm cons} \,. \tag{4.26}$$

Using (4.25) and the 3+1d algebraic identity[13] $\tilde{A}_\mu[dAdA] = -2F_{\mu\nu}[\tilde{A}dA]^\nu$ we find the following expressions

$$\begin{aligned} \partial_\mu T^\mu_\nu &= F_{\nu\mu} J^\mu_{\rm cov} + \tilde{F}_{\nu\mu}\tilde{J}^\mu_{\rm cov} \,, && (4.27)\\ J^\mu_{\rm cov} &= J^\mu_{\rm cons} - 2[\tilde{A}dA]^\mu \,, && (4.28)\\ \tilde{J}^\mu_{\rm cov} &= \tilde{J}^\mu_{\rm cons} - 2\alpha[\tilde{A}d\tilde{A}]^\mu \,, && (4.29) \end{aligned}$$

involving the covariant currents (cf. (4.10)). Their anomalies have the correct form (4.4), (4.5) which we reproduce here for convenience:

$$\begin{aligned} \partial_\mu J^\mu_{\rm cov} &= -2[d\tilde{A}dA] \,, && (4.30)\\ \partial_\mu \tilde{J}^\mu_{\rm cov} &= -[dAdA] - 3\alpha[d\tilde{A}d\tilde{A}] \,. && (4.31) \end{aligned}$$

Therefore, we have shown that the hydrodynamic action (4.19) provides a bosonic effective description of 3+1d anomalies. The field variables are the fluid momentum π and the new pseudoscalar ψ, obeying the gauge conditions

$$p = \pi - A, \qquad \tilde{q} = d\psi - \tilde{A}, \qquad \text{gauge invariant.} \tag{4.32}$$

4.3 4+1 dimensional 'hydrodynamic' topological theories

The derivation of the 2+1d bulk theory in section 2.3 started from the introduction of 'hydrodynamic' gauge fields that are dual to conserved currents, such as $j = *da$. In 4+1 dimensions, the corresponding expressions, $j = *dc$, involve 3-form fields c, which could be independent degrees of freedom or polynomial of lower-order forms. The anomaly inflow relations (2.29), (2.28) then imply 3+1d currents parameterized as $J_{\rm cov} = *c$.

[13] See (A.31) in appendix A.

The hydrodynamic currents obtained in previous sections, (4.15), (4.24), once evaluated on the 3+1d equations of motion, have the form

$$* J_{\rm cov} \underset{eq.m.}{=} 2(d\psi - \tilde{A})d\pi + 2(\pi - A)d\tilde{A}, \tag{4.33}$$

$$* \tilde{J}_{\rm cov} \underset{eq.m.}{=} (\pi - A)d(\pi + A) + 2\alpha(d\psi - \tilde{A})d\tilde{A}\,, \tag{4.34}$$

where the fluid momentum π is constrained by $d\pi d\pi + \alpha d\tilde{A}d\tilde{A} = 0$. These expressions[14] are polynomials of the field $p = \pi - A$ and its axial counterpart $\tilde{q} = d\psi - \tilde{A}$.

Therefore, we take a conservative approach and search for 4+1d topological actions which reproduce the hydrodynamic currents and thus are polynomials in one-form fields; we also expect cubic expressions implying quadratic equations of motion (4.21). We shall find one such theory that can be considered as the minimal one, together with other instances describing more general dynamics and additional degrees of freedom.

The derivation starts by reconsidering the 2+1d theory, in the chiral case for simplicity. We rewrite the Chern-Simons action (2.26), up to a boundary term (cf. (2.32))), as

$$S = \int_{\mathcal{M}_3} \frac{1}{2} ada + adA = \frac{1}{2}\int_{\mathcal{M}_3} \pi d\pi - AdA\,, \qquad\qquad (\pi = a + A). \tag{4.35}$$

It is apparent that the stationary point is quadratic in π and thus the equation of motion is linear, $d\pi = 0$. The generalization to 4+1 dimensions is the cubic expression

$$S = \frac{1}{6}\int_{\mathcal{M}_5} \pi d\pi d\pi - AdAdA, \qquad\qquad (\pi = a + A), \tag{4.36}$$

leading to the desired quadratic condition $d\pi d\pi = 0$.

Note that variations of these actions can be equivalently taken with respect to a and π since A is inert. Furthermore, the current can be defined by varying the action at fixed a or π: the two resulting expressions become equal once evaluated on the equations of motion (as they should). In the following, it is more convenient to focus on the dynamics of π in both bulk and boundary actions.

Note also that the actions (4.35), (4.36) are differences of two Chern-Simons theories for gauge fields π and A which transform in the same way,[15] $A \to A + d\lambda, \pi \to \pi + d\lambda$. These expressions are called 'transgression forms' in the mathematical literature [47], where they are made gauge invariant by adding a boundary term, in the same spirit of our approach. Transgression forms have already been introduced in hydrodynamics [14, 15], within an approach using different fluid variables as fundamental fields.

4.3.1 Two-fluid theory

The topological action which realizes the precise doubling of the chiral action (4.36) is:

$$\begin{aligned} S_{II} = &\int_{\mathcal{M}_5} \tilde{\pi} d\pi d\pi - \tilde{A}dAdA + \alpha\tilde{\pi}d\tilde{\pi}d\tilde{\pi} - \alpha\tilde{A}d\tilde{A}d\tilde{A} \\ &+ \int_{\mathcal{M}_4} P(\pi - A, \tilde{\pi} - \tilde{A}) + \tilde{A}\left[(\pi - A)d(\pi + A) + \alpha(\tilde{\pi} - \tilde{A})d(\tilde{\pi} + \tilde{A})\right]\,, \end{aligned} \tag{4.37}$$

$$\pi = p + A, \qquad \tilde{\pi} = \tilde{q} + \tilde{A}\,, \tag{4.38}$$

[14] Note that they are explicitly gauge invariant, being functions of the quantities $(\pi - A)$ and $(d\psi - \tilde{A})$, see (4.32).

[15] This gauge condition has been found before, see (4.32).

involving two transgression forms and respecting parity and time reversal invariances. In this action the symmetry between the two fields $p, \tilde{q}$ and corresponding fluids is manifest. For this reason, it will be called the 'two-fluid' theory. The 3+1d boundary action is obtained following the same strategy of section 2.3: it involves the pressure dynamic term that can depend on both fluid variables, and the axial coupling whose expression is consistent with overall bulk-boundary gauge invariance (having established that the combinations $\pi - A$ and $\tilde{\pi} - \tilde{A}$ are gauge invariant).

The bulk equations of motion following from (4.37) are

$$d\pi d\tilde{\pi} = 0\,, \qquad d\pi d\pi + 3\alpha d\tilde{\pi} d\tilde{\pi} = 0\,. \tag{4.39}$$

The 3+1d covariant currents determined by anomaly inflow are

$$* J_{\rm cov} = 2\tilde{\pi} d\pi - 2\tilde{A} dA + d\tilde{b}\,, \tag{4.40}$$

$$* \tilde{J}_{\rm cov} = (\pi - A) d(\pi + A) + 3\alpha(\tilde{\pi} - \tilde{A}) d(\tilde{\pi} + \tilde{A}) + db\,, \tag{4.41}$$

where $b, \tilde{b}$ are undetermined two-form fields. The conservation of these currents evaluated on the bulk equations of motion lead to the expected anomalies of section 4.1, now realized by the two fluids on equal footing. It can also be shown that the boundary equations of motion lead to the correct conservation equations for consistent currents and stress tensor.

We note that the expressions of currents (4.40), (4.41) differ from those found earlier in hydrodynamics (4.33), (4.34), in particular for the presence of two independent fluid momenta (4.38).

Actually, the topological action (4.37) describes systems where chirality is conserved, which are physically different from those considered so far. In this work, we have been assuming a single Dirac fluid described by π, where the other variable $\tilde{\pi} = d\psi$ does not have a proper dynamics, i.e. it is 'dragged', and can describe an irrotational fluid component, at most. We imagine that fermion interactions lead to a dynamical effective mass and non-conservation of fermion chirality. One check of this fact is given in appendix C, where we discuss the behavior of Dirac fluids in a chiral background, e.g. $A = \tilde{A}$: we find that the two chiral currents (4.2)), $J_\pm = J \pm \tilde{J}$, do not decouple from each other. The properties of two-fluid hydrodynamics (4.37) as well as its chiral half will be analyzed in future investigations.

4.3.2 Single-fluid theory and Dirac hydrodynamics

We now consider the reduction of the two-fluid theory (4.37) assuming that the axial momentum is irrotational and that the pressure is only function of vector momentum $\pi - A$,

$$\tilde{\pi} = d\psi, \qquad P = P(\pi - A), \tag{4.42}$$

which will be referred to as the 'single-fluid' theory.

In this case, part of the bulk action in (4.37) is a total derivative and reduces to a boundary term so that one is left with the expression

$$\begin{aligned} S_I = &- \int_{\mathcal{M}_5} \tilde{A} dA dA + \alpha \tilde{A} d\tilde{A} d\tilde{A} \\ &+ \int_{\mathcal{M}_4} P(\pi - A) + \tilde{A}(\pi - A) d(\pi + A) + \psi(d\pi d\pi + \alpha d\tilde{A} d\tilde{A})\,. \end{aligned} \tag{4.43}$$

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We recognize that the 3+1d term is equal to the hydrodynamic action (4.19) introduced in section 4.2.1, where ψ is the Lagrange multiplier enforcing the equation of motion (4.21). The 4+1d piece is the response action (4.3) not participating to the boundary dynamics.

Therefore, we have obtained the 3+1d hydrodynamic theory of section 4.2 from the study of bulk topological theories and their degrees of freedom, under the assumption of a single dynamic fluid. This nonetheless includes the additional pseudoscalar variable ψ with respect to the 3+1d Euler hydrodynamics described in ref. [36]. This additional degree of freedom is crucial for obtaining the most general anomaly characterized by $\alpha \neq 0$.

We remark that in this theory the anomaly inflow relations only involve the 4+1d response action (4.3) and determine the anomalies of covariant currents (4.4), (4.5). The correspondence between bulk and boundary hydrodynamic fields is rather clear and can be partially checked through the form of currents.

4.3.3 Theory involving higher-form fields

Earlier in this section we pointed out that the description of matter currents in 4+1d and 3+1d naturally suggests the inclusion of 2- and 3-form hydrodynamic fields b, c into the topological action. They were not considered in the two previous theories, which only involve the one-form fields expressing fluid momenta: in particular, the single-fluid case correctly matches the expected hydrodynamic theory, in which vorticity and helicity are polynomial of fluid momentum/velocity.

Nonetheless, we now present another 4+1d topological theory which is consistent with Dirac anomalies and makes use of 3-form fields for parameterizing currents. In this theory, fluid velocity and vorticity are described by two independent degrees of freedom. Let us consider the following hydrodynamic topological action

$$S_{IIb} = \int_{\mathcal{M}_5} \tilde{c}d(p + A) + cd(\tilde{q} + \tilde{A}) + \tilde{q}dpdp + \alpha\tilde{q}d\tilde{q}d\tilde{q} \,, \tag{4.44}$$

where the $A, \tilde{A}$ backgrounds couple to dual 3-form currents $c, \tilde{c}$, while the $p, \tilde{q}$ 1-forms realize 4+1d Chern-Simons terms. This is the natural generalization of the 2+1d theory (2.26), and is similarly characterized by quadratic stationary points.

The solution of the bulk equations of motion are

$$c: \quad \tilde{q} = -\tilde{A} + d\psi, \qquad i.e. \qquad \tilde{\pi} = d\psi \,, \tag{4.45}$$

$$\tilde{c}: \quad p = -A + d\theta, \qquad i.e. \qquad \pi = d\theta \,, \tag{4.46}$$

$$p: \quad \tilde{c} = -2pd\tilde{q} + db = 2(d\theta - A)d\tilde{A} + db, \tag{4.47}$$

$$\tilde{q}: \quad c = -pdp - 3\alpha\tilde{q}d\tilde{q} + db = (d\theta - A)dA + 3\alpha(d\psi - \tilde{A})d\tilde{A} + d\tilde{b} \,, \tag{4.48}$$

where $b, \tilde{b}$ are undetermined 2-form fields. The hydrodynamic action (4.44) evaluated on the solution of equations of motion gives the response action (4.3), as needed.

The anomaly inflow relations determine the following functional form of covariant 3+1d currents

$$* J_{\text{cov}} = \tilde{c} \underset{eq.m.}{=} = 2(d\theta - A)d\tilde{A} + db \,, \tag{4.49}$$

$$* \tilde{J}_{\text{cov}} = c \underset{eq.m.}{=} (d\theta - A)dA + 3\alpha(d\psi - \tilde{A})d\tilde{A} + d\tilde{b} \,. \tag{4.50}$$

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Gauge invariance requires

$$c, \qquad \tilde{c}, \qquad p = d\theta - A, \qquad \tilde{q} = d\psi - \tilde{A}, \qquad \text{gauge invariant on } \mathcal{M}_4 \tag{4.51}$$

which generalize earlier gauge conditions. These establish that θ, ψ are compensating scalar/pseudoscalar gauge field.

The expressions of currents (4.49), (4.50) have rather interesting physical consequences:

- They identify a 3+1d hydrodynamics characterized by a pair of scalar fields and a pair of two-form fields $b, \tilde{b}$. The latter parameterize the 3+1d currents in absence of backgrounds.

- The currents (4.49), (4.50) match the expressions found for the two-fluid theory, (4.40), (4.41), evaluated for $d\pi = d\tilde{\pi} = 0$. In that case, the contribution by the two-form fields was not actually necessary, but here is rather important.

- The Dirac anomalies are reproduced by the expressions (4.49), (4.50) irrespectively of the value of $b, \tilde{b}$ fields. The hydrodynamics with $b = \tilde{b} = 0$ describes a system with two irrotational fluids, owing to $d\pi = d\tilde{\pi} = 0$, which is static in absence of backgrounds. This minimal theory shows that the requirement of reproducing 3+1d anomalies is not very strong. The response is entirely anomalous and quadratic in the background fields: e.g. in a static magnetic field causing the dimensional reduction, it reproduces the physics of 1+1d systems [24].

We conclude that the topological theory (4.44) identifies an interesting non-standard hydrodynamics. The 3+1d action is obtained as in earlier cases by adding a dynamic pressure term, $P = P(d\theta - A, d\psi - \tilde{A}, db, \tilde{db})$, in general depending on all fields, and the coupling to vector and axial backgrounds for reproducing the consistent currents. Again anomalies are independent of the form of the pressure. The minimal fluid solutions with $b = \tilde{b} = 0$ do not involve any vorticity, while the general one include vorticities which are parameterized by independent degrees of freedom. The analysis of this theory as well as that in section 4.3.1 requires an independent investigation.

We remark that the list of 4+1d topological theories presented here and corresponding generalized hydrodynamics is not meant to be exhaustive. Among further possibilities, it would be interesting to study the hydrodynamics of chiral-only (Weyl) fermions, which can be obtained from the two-fluid theory by identifying the two halves.

Interacting phases of matter in 3+1 and 4+1d can give rise to monodromy phases between point-particles and extended excitations that are also described by topological theories involving higher forms [7, 28]. For example, the addition of a BF term $\int \tilde{q} db$ in the 3+1d action can describe anyonic monodromies of vortex lines and particles. Finally, associated to extended excitations there are higher-form extended symmetries that allow for further possible terms in the topological actions [48].

The complete description of higher-form excitations requires the introduction of corresponding higher form backgrounds B, C, leading to higher anomalies. The response of systems with respect to these new external fields may help in distinguishing among the various topological theories compatible with Dirac anomalies in 4+1 dimensions.

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These generalizations of the hydrodynamic/effective bosonic theory will be considered in future works. The correspondence between hydrodynamic, bosonization of fermions and topological theories in one extra dimension is the main result of this work. It is apparent that insight from topological theories in 4+1d can lead to very interesting generalizations of hydrodynamics and corresponding bosonic effective descriptions of anomalies.

5 3+1 dimensional axial-gravitational anomaly

5.1 Axial anomaly on Riemannian geometries

The 3+1d hydrodynamic theory is now considered in the presence of a gravitational background, expressed by vierbein and spin connection $(e^a_\mu, \omega^{ab}_\mu)$. We shall first consider Riemannian geometries (vanishing torsion) and later discuss the general case.

Fermions do not have a purely gravitational anomaly ($\partial_\mu T^{\mu\nu} \neq 0$ only occurs in dimensions $d = 2, 6, 10, \dots$), but show a gravitational contribution to the axial anomaly [49]

$$D_\mu \tilde{J}^\mu_{\rm cov} = -\frac{\beta}{4}\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}{}^a{}_b\, R_{\rho\sigma}{}^b{}_a = -\beta[\mathrm{Tr}\,(R^2)]\,. \tag{5.1}$$

This expression contains the Riemann 2-form $R^a{}_b = \frac{1}{2}R_{\mu\nu}{}^a{}_b dx^\mu \wedge dx^\nu$, expressed as $R^a{}_b = d\omega^a{}_b + \omega^a{}_c\omega^c{}_b$ in terms of the spin-connection 1-form, $\omega^\alpha{}_b = \omega^a_{\mu b}dx^\mu$. The trace is over the local Lorentz indices a, b, and D_μ is the covariant derivative w.r.t. the gravitational background.[16] The integral of the mixed axial-gravitational anomaly (5.1) is proportional to the pseudoscalar 4d Pontryagin topological invariant.[17] The anomaly coefficient for a Dirac fermion is $\beta = 1/96\pi^2 \to 1/24$, in our conventions.

The anomaly (5.1) can be described by the following 4+1d addition to the hydrodynamic theory (4.43)

$$\Delta S_I = \beta \int_{\mathcal{M}_5} (d\psi - \tilde{A})\,\mathrm{Tr}\,(R^2)\,. \tag{5.2}$$

Since the curvature 4-form obeys $\mathrm{Tr}\,(R^2) = d\Omega_{\rm CS,3}(\omega) = d\,\mathrm{Tr}\,(\omega d\omega + 2\omega^3/3)$, the mixed Chern-Simons 5-form $\Omega_{\rm CS,5}(A,\omega) = A\;\; \mathrm{Tr}\,(R^2)$ verifies $d\Omega_5 = F\,\mathrm{Tr}\,(R^2)$, which is the topological invariant 6-form appearing in the 5+1d index theorem for mixed gauge-gravitational anomalies. Thus, the expression (5.2) is consistent with the literature on anomalies and their relations in various dimensions [46, 50]. Earlier hydrodynamic approaches including gravitational anomalies can be found in [13, 51].

The 4+1d current obtained by varying the action (5.2) w.r.t. $\tilde{A}$ reproduces the contribution (5.1) by anomaly inflow, following the same steps an in section 4.3.1. Furthermore, the WZW action associated to ΔS_I is,

$$\Delta S_{\rm WZW} = -\beta \int_{\mathcal{M}_4} \tilde{\lambda}\,\mathrm{Tr}\,(R^2)\,, \tag{5.3}$$

which determines a consistent anomaly equal to (5.1) — the gravitational contributions to $D_\mu \tilde{J}^\mu_{\rm cov}$ and $D_\mu \tilde{J}^\mu_{\rm cons}$ are equal.

[16] Note the covariant form of the antisymmetric tensor $\epsilon^{\mu\nu\cdots\lambda} = \varepsilon^{\mu\nu\cdots\lambda}/|e|$, where the metric determinant is $\sqrt{-g} = |e|$.

[17] There is no analog in 1+1d since the curvature is a scalar quantity.

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The 3+1d hydrodynamic action (4.43) takes the following form in presence of vector, axial and gravity backgrounds

$$
\begin{aligned}
S = \int_{\mathcal{M}_4} d^4x\, |e| P(p) + \int_{\mathcal{M}_4} \tilde{A}(\pi - A)d(\pi + A) + \psi\left(d\pi d\pi + \alpha d\tilde{A}d\tilde{A} + \beta\,\mathrm{Tr}\,(R^2)\right) \\
- \int_{\mathcal{M}_5} \tilde{A}\left(dAdA + \alpha d\tilde{A}d\tilde{A} + \beta\,\mathrm{Tr}\,(R^2)\right) .
\end{aligned}
\tag{5.4}
$$

One recognizes the additional $\mathrm{Tr}\,(R^2)$ terms and the metric determinant multiplying the pressure term, while exterior derivatives are unaffected by the Levi-Civita connection, owing to $\Gamma^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\beta\alpha}$. Similarly, Lie derivatives can be expressed in terms of both ordinary and covariant derivatives.

It follows that earlier expressions of $J, \tilde{J}$ and T^{ν}_{μ} are unchanged. We rewrite them for convenience

$$
T^{\nu}_{\mu} = -\frac{\partial P}{\partial \pi_\nu}(\pi_\mu - A_\mu) + P\delta^{\nu}_{\mu} , \tag{5.5}
$$

$$
J^{\nu}_{\mathrm{cov}} = -\frac{\partial P}{\partial \pi_\nu} + \left[(\pi - A)d\tilde{A}\right]^{\nu} , \tag{5.6}
$$

$$
\tilde{J}^{\nu}_{\mathrm{cov}} = \left[(\pi - A)d(\pi + A) - 2\alpha(\tilde{A} - d\psi)d\tilde{A}\right]^{\nu} . \tag{5.7}
$$

The π equation of motion is also unchanged, while that of ψ acquires the R^2 term

$$
\pi : \qquad -\frac{\partial P}{\partial \pi_\mu} + \left[2(\tilde{A} - d\psi)d\pi - (\pi - A)d\tilde{A}\right]^{\mu} = 0 , \tag{5.8}
$$

$$
\psi : \qquad d\pi d\pi + \alpha d\tilde{A}d\tilde{A} + \beta\,\mathrm{Tr}\,(R^2) = 0 . \tag{5.9}
$$

It follows that currents now obey the expected anomalous equations

$$
D_\nu J^{\nu}_{\mathrm{cov}} = -\left[2d\tilde{A}dA\right] , \tag{5.10}
$$

$$
D_\nu \tilde{J}^{\nu}_{\mathrm{cov}} = -\left[dAdA + 3\alpha d\tilde{A}d\tilde{A} + \beta\,\mathrm{tr}\,(R^2)\right] , \tag{5.11}
$$

in presence of all three backgrounds.

5.2 The spin current

The stress tensor (5.5) has been obtained by diffeomorphism invariance and thus is defined in terms of metric variations of the action

$$
\delta S = \frac{1}{2}\int |e| T_{\mu\nu}\delta g^{\mu\nu} . \tag{5.12}
$$

When using the variables (e_μ, ω_μ) it is possible to consider independent variations with respect to each background, as follows

$$
\delta S = \int |e| \left(t^{\mu}_{a}\delta e^{a}_{\mu} + \frac{1}{2}\mathcal{S}^{\mu}_{ab}\delta\omega^{ab}_{\mu}\right) , \tag{5.13}
$$

which define a new response given by the spin current $\mathcal{S}^{\mu}_{ab}$ and another definition of the stress tensor t^{μ}_{a}. The backgrounds (e, ω) and corresponding currents $(t, \mathcal{S})$ are independent variables

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only for geometries with torsion. In absence of it, the spin connection can be expressed in terms of vierbeins, $\omega_\mu^{ab} = \omega_\mu^{ab}(e)$, using the metric compatibility condition $D_\lambda e_\mu^a = 0$. Therefore, for Riemannian geometries the stress tensor T in (5.12) is well defined while $t, \mathcal{S}$ can be redefined by shifting terms among them. As explained in ref. [52], these redefinitions can be seen as improvements of the stress tensor t by derivatives of $\mathcal{S}$.

In absence of backgrounds, both the stress tensor and the angular momentum current $J^{\mu,\alpha\beta}$ are conserved

$$J^{\mu,\alpha\beta} = x^{[\alpha} T^{\mu\beta]} + \mathcal{S}^{\mu,a\beta}\,, \qquad\qquad \partial_\mu J^{\mu,\alpha\beta} = x^{[\alpha}\partial_\mu T^{\mu\beta]} + T^{[\alpha\beta]} + \partial_\mu \mathcal{S}^{\mu,a\beta} = 0\,. \tag{5.14}$$

Therefore, the spin current is conserved when the stress tensors is both conserved and symmetric. For a Riemannian metric, the stress tensor (5.5), defined by (5.12), is indeed symmetric; the other quantity obtained in (5.13), $t^{\mu\nu} = t_a^\mu E^{a\nu}$, multiplied by the inverse vierbein, can be non-symmetric. Note, however, that the conservation laws of T and $\mathcal{S}$ are modified in presence of external backgrounds and corresponding anomalies, as shown in the following analysis.

Keeping in mind these facts, we now derive the spin current in presence of a Riemannian background. The case of geometries with torsion will be considered later. We first vary the 4+1d topological action (5.2),

$$(\mathcal{S}_{(5)})^\mu_{ab} = \frac{2}{|e|}\frac{\delta S_5}{\delta \omega_\mu^{ba}} = \frac{2\beta}{|e|}\frac{\delta}{\delta\omega_\mu^{ba}} \int_{\mathcal{M}_5} (d\psi - \tilde{A})\,\mathrm{Tr}\,(R(d\omega + \omega^2))\,, \tag{5.15}$$

leading to the following terms ($\hat{A} = \tilde{A} - d\psi$),

$$d(\hat{A}R)_{ab} - \hat{A}(\omega R)_{ab} + \hat{A}(R\omega)_{ab} = d\hat{A}\; R_{ab} - \hat{A}(dR + \omega R - R\omega)_{ab} = d\hat{A}\; R_{ab}\,, \tag{5.16}$$

as due to the Bianchi identity[18] $DR = dR + \omega R - R\omega = 0$.

The anomaly inflow determines the following anomaly for the 3+1d covariant spin current

$$(\mathcal{S}_{(5)})^5_{ab} = (D_\mu \mathcal{S}^\mu_{\mathrm{cov}})_{ab} = -2\beta\epsilon^{5\mu\nu\rho\sigma}\partial_\mu \hat{A}_\nu R_{\rho\sigma,ab} = -2\beta\epsilon^{5\mu\nu\rho\sigma} D_\mu(\hat{A}_\nu R_{\rho\sigma,ab})\,, \tag{5.17}$$

where the last identity follows again from the Bianchi identity. The current itself is[19]

$$\mathcal{S}^\mu_{ab,\mathrm{cov}} = 4\beta\left[(d\psi - \tilde{A})R_{ab}\right]^\mu + \cdots\,, \tag{5.18}$$

where the dots are 'classic' terms that are not determined by the bulk topological theory. Actually, contrary to the currents introduced earlier, the spin current on the boundary might be non-conserved classically (even in the absence of anomalies). Thus, for the time being we should consider the expression (5.18) as only the anomalous part of the spin current

Next, we vary the 3+1d hydro action (5.4) and observe that the spin connection only appears in the anomalous $\mathrm{Tr}\,(R^2)$ term. Actually, terms involving differential forms are independent of the metric. Repeating the same steps of the variation of (5.15), we find

$$\mathcal{S}^\mu_{ab,\mathrm{cons}} = 4\beta[d\psi\; R_{ab}]^\mu\,, \tag{5.19}$$

[18] This Bianchi identity also holds in presence of torsion.

[19] Note that this current is covariant w.r.t. Lorentz indices, and invariant for the Abelian part due to the presence of ψ. Components of dual forms are evaluated with the covariant epsilon tensor.

which obeys $D_\mu \mathcal{S}^\mu_{\rm cons} = 0$ due to the Bianchi identity. This result can also be obtained by varying the WZW action (5.3) under a local Lorentz transformation $\delta\Lambda_{ab}$,

$$\delta_\Lambda S_{\rm WZW} = \int_{\mathcal{M}_4} \delta\Lambda_{ab} \left(D_\mu \mathcal{S}^\mu\right)_{ab,\rm cons} = -\beta\delta_\Lambda \int_{\mathcal{M}_4} \tilde{\lambda}\, {\rm Tr}\, R^2 = 0\,. \tag{5.20}$$

We observe that:

i) The consistent spin current is (covariantly) conserved, i.e. non-anomalous;

ii) There are no additional (classic) terms in (5.18). Namely, the spin current is entirely due to anomaly for Riemannian geometries;

iii) The result for the consistent spin current (5.19) remains valid for vanishing backgrounds $A = \tilde{A} = 0$.

The relation between the two currents, consistent and covariant, is clearly

$$\mathcal{S}^\mu_{ab,\rm cov} = \mathcal{S}^\mu_{ab,\rm cons} - 4\beta \left[\tilde{A} R_{ab}\right]^\mu\,, \tag{5.21}$$

and the anomalous conservation law is given by (5.17).

Let us now discuss the stress-tensor conservation due to the presence of the curvature term in the 3+1d part of the action (5.4). From diffeomorphism invariance, one finds the expression

$$\begin{aligned} D_\nu T^\nu_\mu = {} & F_{\mu\nu} J^\nu_{\rm cons} + \tilde{F}_{\mu\nu} \tilde{J}^\nu_{\rm cons} - A_\mu D_\nu J^\nu_{\rm cons} - \tilde{A}_\mu D_\nu \tilde{J}^\nu_{\rm cons} \\ & + \frac{1}{2}\, {\rm Tr} \left(D_{[\mu}\omega_{\nu]} \mathcal{S}^\nu_{\rm cons} - \omega_\mu D_\nu \mathcal{S}^\nu_{\rm cons} \right)\,, \end{aligned} \tag{5.22}$$

which differs from (4.26) of section 4.2 for the last term and the covariant form of the Lie derivative δ_ε. In the case $\beta = 0$, the equation (5.22) can be rewritten as (4.27),

$$D_\nu T^\nu_\mu = F_{\mu\nu} J^\nu_{\rm cov} + \tilde{F}_{\mu\nu} \tilde{J}^\nu_{\rm cov}\,. \tag{5.23}$$

These terms are also present for $\beta \neq 0$, the expressions of T, J being unchanged. There are two further pieces:

i) the contribution to $-\tilde{A}_\mu \partial_\nu \tilde{J}^\nu = \beta \tilde{A}_\mu [{\rm Tr}\ R^2]$ due to the gravitational anomaly;

ii) the variation of the action w.r.t. the ω_μ background, expressed in terms of the consistent current $\mathcal{S}^\nu_{\rm cons}$.

These additions read,

$$\begin{aligned} \Delta_\mu &= \beta \tilde{A}_\mu [{\rm Tr}\ R^2] + \frac{1}{2}\, {\rm Tr} \left(D_{[\mu}\omega_{\nu]} \mathcal{S}^\nu_{\rm cons} - \omega_\mu D_\nu \mathcal{S}^\nu_{\rm cons} \right) \\ &= \beta \tilde{A}_\mu [{\rm Tr}\ R^2] + 2\beta\, {\rm Tr} \left(R_{\mu\nu} [d\psi\ R]^\nu \right) \\ &= 2\beta\, {\rm Tr} \left(R_{\mu\nu} [(d\psi - \tilde{A}) R]^\nu \right) = \frac{1}{2}\, {\rm Tr} \left(R_{\mu\nu} \mathcal{S}^\nu_{\rm cov} \right)\,. \end{aligned} \tag{5.24}$$

In the second line, we inserted the expression (5.19) for $\mathcal{S}^\nu_{\rm cons}$, that is conserved; in the third line we used the 3+1d identity (cf. (A.31)), $\tilde{A}_\mu\, {\rm Tr}\, [RR] = -2\, {\rm Tr} \left(R_{\mu\nu} [\tilde{A} R]^\nu \right)$.

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In conclusion, the stress tensor equation in presence of the gravitational background reads

$$D_\nu T^\nu_\mu = F_{\mu\nu} J^\nu_{\rm cov} + \tilde{F}_{\mu\nu} \tilde{J}^\nu_{\rm cov} + \frac{1}{2} \,{\rm Tr}\left(R_{\mu\nu} \mathcal{S}^\nu_{\rm cov} \right) , \tag{5.25}$$

where the expressions of stress tensor and covariant currents are given in (5.5), (5.6) and (5.7).

The contribution from the curved background nicely matches the form of other gauge backgrounds, corresponding to (generalized) Lorentz forces. As it was explained above, the spin current (5.18) does not have classic contributions and can be written as

$$\mathcal{S}^\mu_{ab,\rm cov} = 4\beta \left[(d\psi - \tilde{A}) R_{ab} \right]^\mu . \tag{5.26}$$

Note that there is a difference in the fact that electric currents have classical terms (see (5.6), (5.7)), while the spin current is only induced by quantum effects, leading to a force (last term of (5.25)) proportional to the square of the curvature.

5.3 Extension to geometries with torsion

5.3.1 Relation between axial and spin currents for free Dirac fermions

We start by recalling some properties of free Dirac theory and setting the notation. The stress tensor $T^{\mu\nu}$ and the total angular momentum current $J^{\mu,\alpha\beta}$ are given by bilinears of the spinor field Ψ

$$T^{\mu\nu} = \frac{i}{2} \bar{\Psi} \gamma^\mu \overleftrightarrow{\partial^\nu} \Psi, \qquad J^{\mu,\alpha\beta} = x^{[\alpha} T^{\mu\beta]} + \bar{\Psi} \frac{1}{4} \left\{ \gamma^\mu, \sigma^{\alpha\beta} \right\} \Psi = L^{\mu,\alpha\beta} + \mathcal{S}^{\mu,\alpha\beta} , \tag{5.27}$$

where L and $\mathcal{S}$ are the orbital and spin currents, respectively.[20] The matrix $\sigma^{\alpha\beta}$ is the generator of Lorentz rotations on spinors and gamma matrices $(\gamma^\mu)_{ij}$,

$$S(\Omega) \gamma^\mu S^{-1}(\Omega) = (\Omega^{-1})^\mu{}_\nu \gamma^\nu , \tag{5.28}$$

$$\Omega_{\mu\nu} \sim \eta_{\mu\nu} + \Lambda_{\mu\nu} , \qquad S(\Omega)_{ij} \sim I - \Lambda_{ab} \frac{i}{4} (\sigma^{ab})_{ij} , \qquad (\sigma^{ab})_{ij} = \frac{i}{2} [\gamma^a, \gamma^b]_{ij} . \tag{5.29}$$

Our conventions for the gamma matrices can be summarized as follows

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} = 2 \,{\rm diag}(-1,1,1,1), \qquad \gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 . \tag{5.30}$$

They can be used to prove the following identity

$$\frac{i}{8} \left\{ \gamma_\alpha, [\gamma_\beta, \gamma_\sigma] \right\} = \frac{1}{2} \varepsilon_{\alpha\beta\sigma\mu} \gamma^5 \gamma^\mu , \qquad (\varepsilon^{0123} = -\varepsilon^{1230} = 1) . \tag{5.31}$$

Actually, the expression with three gammas in the l.h.s. vanishes for any pair of equal indices, in some cases explicitly, in other cases due to $-(\gamma^0)^2 = (\gamma^i)^2 = 1$. It is also antisymmetric for any exchange of indices, thus mapping into the expression involving γ^5.

The coupling of the spin connection to the spin current is as follows

$$\omega^{ab}_\mu \, \mathcal{S}^\mu_{ab} = \omega^{ab}_\mu \, \bar{\Psi} \frac{i}{8} \left\{ \gamma^\mu, [\gamma_a, \gamma_b] \right\} \Psi . \tag{5.32}$$

[20]Antisymmetrization of indices is indicated by square brackets, e.g. $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$.

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In the case of a flat metric $e^a_\mu = \delta^a_\mu$, we can freely pass from Latin to Greek indices and use the gamma-matrix identity (5.31) to find the relations

$$\mathcal{S}^{\mu,ab} = \frac{1}{2}\varepsilon^{\mu ab\sigma}\tilde{J}_\sigma, \qquad \tilde{J}_\sigma = -\frac{1}{3}\varepsilon_{\sigma\mu ab}\mathcal{S}^{\mu,ab}. \tag{5.33}$$

Therefore, we see that the spin current in the Dirac theory is completely antisymmetric at classical level: it is the dual of the axial current [53].

It follows that the Dirac fermion in flat space only couples to the totally antisymmetric part of the spin connection, that can be expressed in terms of the axial background,

$$\tilde{A}_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\omega^{\nu,\rho\sigma}, \qquad \omega_{\nu,\rho\sigma} = \frac{1}{3}\varepsilon_{\nu\rho\sigma\lambda}\tilde{A}^\lambda + \text{other terms}. \tag{5.34}$$

These relations can be used to 'geometrize' the axial coupling. We shall use this correspondence to find out how the hydrodynamic theory couples to general backgrounds with torsion.

The relation between spin-axial currents (5.33) extends to a curved space by including vierbeins to match the index types, so it is a 'duality' in a weaker sense[21]

$$\mathcal{S}^{\mu,\nu\rho} = \mathcal{S}^{\mu,ab}E^\nu_a E^\rho_b = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\tilde{J}_\sigma\,, \tag{5.35}$$

where E^μ_a is the inverse vierbein. Note, however, that the spin current is modified at the quantum level, since the contribution by the axial-gravitational anomaly (5.19) is not completely antisymmetric.

We also compare the gauge transformations of both backgrounds

$$\delta\omega_{\mu,\rho\sigma} = \left(\partial_\mu\Lambda - i[\Lambda,\omega_\mu]\right)_{\rho\sigma} \sim \varepsilon_{\mu\rho\sigma\beta}\delta\tilde{A}^\beta, \qquad \delta\tilde{A} = d\tilde{\lambda}\,. \tag{5.36}$$

Evaluating the r.h.s. of $\delta\omega$ for antisymmetric spin connections (5.34) one finds the relation

$$\partial^\beta\tilde{\lambda} \sim \varepsilon^{\beta\mu\rho\sigma}\partial_\mu\Lambda_{\rho\sigma} + 2i\Lambda^{\beta\alpha}\tilde{A}_\alpha\,, \tag{5.37}$$

showing that the axial Abelian symmetry is a subgroup of the non-Abelian Lorentz symmetry. Thus, the correspondence between axial and spin connections is one-to-one only within this subspace.

5.3.2 Rewriting the axial background as a flat geometry with torsion

In this section, the relation between axial and spin currents is shown to be one-to-one in the limit of flat metric, $e^a_\mu = \delta^a_\mu$ and non-vanishing spin connection ω^μ_{ab}, which is only possible in Einstein-Cartan geometries with torsion.

In these cases, the metric and spin connection are independent variables, but the connection remains metric compatible

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}\,, \qquad E^\rho_a e^a_\sigma = \delta^\rho_\sigma\,, \qquad D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\rho_{\mu\nu}e^a_\rho + \omega^a_{\mu b}e^b_\nu = 0\,. \tag{5.38}$$

[21] See eq. (2.13) of ref. [53].

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The affine connection acquires an antisymmetric part that identifies the torsion tensor, $\mathcal{T}^a_{\mu\nu}$, a geometric quantity independent of metric,

$$\mathcal{T}^\rho_{\mu\nu} = \Gamma^\rho_{[\mu\nu]} = E^\rho_a \left(\partial_{[\mu} e^a_{\nu]} + \omega^a_{[\mu b} e^b_{\nu]}\right) . \tag{5.39}$$

It turns out that the spin connection can be divided in two parts [54]: the first one is that obtained in absence of torsion, which was considered in section 5.2: it is now denoted by an open dot on top, $\mathring{\omega}^{ab}_\mu$; the second part is called contorsion K^{ab}_μ, a tensor related to torsion, as follows

$$\omega^{ab}_\mu = \mathring{\omega}^{ab}_\mu(e) + K^{ab}_\mu \,, \tag{5.40}$$

$$\partial_{[\mu} e^a_{\nu]} + \mathring{\omega}^a_{[\mu b} e^b_{\nu]} + K^{ab}_{[\mu} e_{b\nu]} = \mathcal{T}^a_{\mu\nu} \,. \tag{5.41}$$

We define the contorsion 1-form, $K^{ab} = K^{ab}_\mu dx^\mu$, and the torsion 2-form, $\mathcal{T}^a = \frac{1}{2}\mathcal{T}^a_{\mu\nu} dx^\mu \wedge dx^\nu$. The Riemann 2-form also splits into

$$R^a_b = \mathring{R}^a_b + (\mathring{D}K)^a_b + K^a_c K^c_b \,, \tag{5.42}$$

where $\mathring{D}_\mu$ is the covariant derivative with respect to the Riemannian metric.

We assume a flat metric, $e^a_\mu = \delta^a_\mu$, $\mathring{\omega}^a_{\mu b} = 0$, and compute the geometry given by the spin connection corresponding to the axial background, using the relations (5.33) and (5.34) (neglecting 'other terms'), i.e.

$$\omega_{\nu,\rho\sigma} = \frac{1}{3}\varepsilon_{\nu\rho\sigma\lambda}\tilde{A}^\lambda \,. \tag{5.43}$$

We thus compute the quantities

$$\omega_{\mu,\nu\rho} \equiv \omega_{\mu,ab} e^a_\nu e^b_\rho = K_{\mu,\nu\rho} = \frac{1}{3}\varepsilon_{\mu\nu\rho\lambda}\tilde{A}^\lambda \,, \tag{5.44}$$

$$\mathcal{T}_{\alpha,\mu\nu} \equiv \mathcal{T}_{a,\mu\nu} e^a_\alpha = 2\omega_{\mu,\alpha\nu} \,, \tag{5.45}$$

$$R_{\mu\nu,\alpha\beta} = (\partial_{[\mu} K_{\nu]} + K_{[\mu} K_{\nu]})_{\alpha\beta} = \frac{1}{3}\partial_{[\mu}\varepsilon_{\nu]\alpha\beta\lambda}\tilde{A}^\lambda \,, \tag{5.46}$$

$$4[\mathrm{Tr}\,(R^2)] = R_{\mu\nu,\alpha\beta} R_{\rho\sigma,\beta\alpha}\varepsilon^{\mu\nu\rho\sigma} = \frac{8}{9}[d\tilde{A}d\tilde{A}] \,, \tag{5.47}$$

$$\mathcal{R}_{\alpha\beta} = R_{\mu\alpha,\mu\beta} = -\frac{1}{3}\varepsilon_{\alpha\beta\mu\lambda}\partial^\mu \tilde{A}^\lambda = -\mathcal{R}_{\beta\alpha} \,. \tag{5.48}$$

We observe that:

— In the evaluation of the Riemann tensor, the term quadratic in K vanishes.

— The Ricci tensor $\mathcal{R}_{\alpha\beta}$ is totally antisymmetric, a characteristic property of geometries with torsion, since $\mathcal{R}$ is symmetric in Riemannian metric.

These results show how the axial field geometrizes into a torsion-only background.

We now consider the hydrodynamic theory with $A = 0$, $\tilde{A} \neq 0$ and flat metric and rewrite it in geometric form, i.e. in terms of the spin current. Recall the expression of the action (4.43) and the consistent axial current:

$$S = \int d^4x \, P(p) + \int_{\mathcal{M}_4} (\tilde{A} - d\psi) p dp + \alpha\psi d\tilde{A} d\tilde{A} \,, \tag{5.49}$$

$$*\tilde{J}_{\mathrm{cons}} = p dp + 2\alpha d\psi d\tilde{A} \,, \qquad \partial_\nu \tilde{J}^\nu_{\mathrm{cons}} = -\alpha[d\tilde{A}d\tilde{A}] \,. \tag{5.50}$$

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In this action, we replace the axial field with the spin connection using previous formulas, and obtain

$$S = \int d^4x P(p) - (3\omega^{\mu,\nu\rho} - \varepsilon^{\mu\nu\rho\sigma}\partial_\sigma\psi)\, p_\mu\partial_\nu p_\rho - \alpha\frac{9}{8}\psi R_{\mu\nu,\alpha\beta}R_{\rho\sigma,\beta\alpha}\varepsilon^{\mu\nu\rho\sigma}\,. \tag{5.51}$$

The spin current is obtained by variation of this action over the spin connection,

$$\mathcal{S}^{\mu,\nu\rho} = -\{p^\mu\partial^\nu p^\rho\} - 9\alpha\varepsilon^{\mu\alpha\beta\gamma}\partial_\alpha\psi R_{\beta\gamma}{}^{\nu\rho}\,, \tag{5.52}$$

where braces represent complete antisymmetrization over the three indices,

$$\{a^\mu b^\nu c^\rho\} = -\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\sigma\mu'\nu'\rho'}a^{\mu'}b^{\nu'}c^{\rho'}\,. \tag{5.53}$$

The result (5.52) shows that the spin current acquires a totally antisymmetric classical term $\{p^\mu\partial^\nu p^\rho\}$ due to the coupling of torsion to the 3-form pdp in the action (5.51). This coupling is not possible in Riemannian backgrounds. There is a notorious ambiguity in the definition of the spin current [52], in which one could add $\{p^\mu\partial^\nu p^\rho\}$ to the spin current with an arbitrary coefficient. Without this addition the spin current is entirely anomalous (the last term in (5.52)), in agreement with the results of section 5.2. The relation (5.33) in the Dirac theory allows us to fix this classical contribution to the spin current: while it is non-minimal, its coefficient vanishes in the absence of torsion.

In the expression (5.52), the anomalous term is also present, which is not fully antisymmetric, as already observed. Actually, it is known that the R^2 anomaly (5.1) keeps the same form in geometries with torsion [55]. The value of the anomaly coefficient β is clearly not reproduced in the present setting which 'Abelianizes' the local Lorentz symmetry.

In conclusion, we have found that it is possible to geometrize the axial background: it maps into torsion and can be described in the setting of Einstein-Cartan gravity. The spin current shows a classical term which is not determined by anomaly inflow and vanishes for Riemannian backgrounds.

5.3.3 Spin current in backgrounds with curvature and torsion

We now discuss the general case of couplings to the A, $\tilde{A}$ backgrounds and gravity with torsion, which is described by the following 3+1d action

$$\begin{aligned} S[\pi,\psi] = &\int d^4x\,\sqrt{-g}\,P(p) \\ &+ \int_{\mathcal{M}_4}\tilde{\mathcal{A}}(\pi - A)d(\pi + A) + \psi\left(d\pi d\pi + \alpha d\tilde{\mathcal{A}}d\tilde{\mathcal{A}} + \beta\,\mathrm{Tr}\,(R^2)\right)\,, \end{aligned} \tag{5.54}$$

with

$$\tilde{\mathcal{A}}_\mu = \tilde{A}_\mu + \lambda\epsilon_{\mu\alpha\beta\gamma}\Gamma^{\alpha,\beta\gamma} = \tilde{A}_\mu + \frac{\lambda}{2}\epsilon_{\mu\alpha\beta\gamma}\mathcal{T}^{\alpha,\beta\gamma}\,. \tag{5.55}$$

The interaction with the completely antisymmetric connection is suggested by the analysis of the previous section, and is expressed in terms of torsion $\mathcal{T}^\alpha_{\beta\gamma}$ (see (5.41)). The 'generalized' axial background $\tilde{\mathcal{A}}$ allows for fermionic excitations with arbitrary coupling to torsion parameterized by the coefficient λ.

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We remark that covariant derivatives of forms can involve the torsion: e.g., for the one-form a we have

$$D_{[\mu}a_{\nu]} = \partial_{[\mu}a_{\nu]} - \Gamma^{\alpha}_{[\mu\nu]}a_{\alpha} = \partial_{[\mu}a_{\nu]} - \mathcal{T}^{\alpha}_{\mu\nu}a_{\alpha}\,. \tag{5.56}$$

The second term in the r.h.s. of this equation is itself a tensor, thus the first term involving ordinary derivatives is also covariant, being the difference of two tensors. This implies that covariantization of derivatives of forms is not compulsory: we can keep such terms in the action (5.54) unchanged, as a minimal choice. Note also that the interaction between axial current and torsion introduced via the $\tilde{\mathcal{A}}$ background (5.55) is a non-canonical coupling, which is physically motivated by the previous study of the Dirac theory.

In the following we determine the currents and conservation laws implied by the action (5.54). Hereafter we use covariant derivatives with respect to the metric only, denoted by $\mathring{D}_\mu$. We consider backgrounds with completely antisymmetric contorsion, as those entering in (5.55) (using $\Gamma_{\alpha,[\beta\gamma]} = \mathcal{T}_{\alpha,\beta\gamma} = K_{[\beta,\alpha\gamma]}$); we also denote quantities in absence of torsion with a dot, as in the previous section.

Variations over π and ψ produce the following equations of motion

$$\mathcal{J}^\nu \equiv -\frac{\partial P}{\partial p_\nu} + \left[2\tilde{\mathcal{A}}d\pi - pd\tilde{\mathcal{A}}\right]^\nu = \left[2d\psi d\pi\right]^\nu, \tag{5.57}$$

$$d\pi d\pi + \alpha d\tilde{\mathcal{A}}d\tilde{\mathcal{A}} + \beta\,\mathrm{tr}\;R^2 = 0\,. \tag{5.58}$$

We obtain from these equations the transport consequences

$$\mathcal{J}^\nu(\partial_\nu\pi_\mu - \partial_\mu\pi_\nu) = \mathring{D}_\mu\psi\left[d\pi d\pi\right], \tag{5.59}$$

$$\mathring{D}_\nu\mathcal{J}^\nu = 0\,, \tag{5.60}$$

$$\mathcal{J}^\nu\mathring{D}_\nu\psi = 0\,. \tag{5.61}$$

The consistent currents are:

$$J^\mu_{\text{cons}} = -\frac{\partial P}{\partial\pi_\mu} + [(\pi - A)d\tilde{A} + 2\tilde{A}dA]^\mu\,, \tag{5.62}$$

$$\tilde{J}^\mu_{\text{cons}} = [(\pi - A)d(\pi + A) + 2\alpha d\psi d\tilde{A}]^\mu\,. \tag{5.63}$$

Using equations of motion, we find the conservation laws

$$\mathring{D}_\mu J^\mu_{\text{cons}} = 0\,, \tag{5.64}$$

$$\mathring{D}_\mu\tilde{J}_{\text{cons}} = -[dAdA] - \alpha[d\tilde{A}d\tilde{A}] - \beta[\mathrm{Tr}\;R^2]\,. \tag{5.65}$$

Note that they have the same form as in absence of torsion.

The consistent spin current is obtained by variation with respect to the spin connection;

$$\mathcal{S}_{\nu,\alpha\beta} = 2\lambda\varepsilon_{\nu\alpha\beta\sigma}[(\pi - A)d(\pi + A)]^\sigma + 2\beta g_{\nu\nu'}\varepsilon^{\nu'\mu\alpha'\beta'}\partial_\mu\psi\;R_{\alpha'\beta',\alpha\beta} \tag{5.66}$$

$$= \mathcal{S}_{\nu,\alpha\beta\;\text{class}} + \mathcal{S}_{\nu,\alpha\beta\;\text{cons}}\,. \tag{5.67}$$

Beside the anomalous term, the completely antisymmetric classic term is also present owing to the linear dependence of the spin connection on contorsion and torsion in (5.40).

We introduce the standard hydrodynamic stress tensor

$$T^{\nu}{}_{\mu} = -\frac{\partial P}{\partial \pi_\nu}(\pi - A)_\mu + P\delta^\nu_\mu\,, \tag{5.68}$$

and derive its conservation law using the equations of motion;[22] after some calculations, we obtain:

$$\begin{aligned}\mathring{D}_\nu T^{\nu}{}_{\mu} = F_{\mu\nu}J^\nu_{\rm cov} + \tilde{F}_{\mu\nu}\tilde{J}^\nu_{\rm cov} + 3\lambda\Big[(\pi - A)d(\pi + A)\Big]^\nu \varepsilon_{\nu\mu\alpha\beta}\mathring{D}_\gamma\Gamma^{\alpha,\beta\gamma} \\ - \beta(\partial_\mu\psi - \tilde{\mathcal{A}}_\mu)\Big[\,{\rm Tr}\,(R^2)\Big]\,.\end{aligned} \tag{5.69}$$

In this expression, there appear the following generalized currents and their anomalies

$$J^\nu_{\rm cov} = -\frac{\partial P}{\partial \pi_\nu} + \Big[(\pi - A)d\tilde{\mathcal{A}}\Big]^\nu\,, \tag{5.70}$$

$$\tilde{J}^\nu_{\rm cov} = \Big[(\pi - A)d(\pi + A) + 2\alpha(d\psi - \tilde{\mathcal{A}})d\tilde{\mathcal{A}}\Big]^\nu\,, \tag{5.71}$$

$$\mathring{D}_\nu J^\nu_{\rm cov} = -\Big[2d\tilde{\mathcal{A}}dA\Big]\,, \tag{5.72}$$

$$\mathring{D}_\nu \tilde{J}^\nu_{\rm cov} = -\Big[dAdA + 2\alpha d\tilde{\mathcal{A}}d\tilde{\mathcal{A}} + \alpha d\tilde{\mathcal{A}}d\tilde{\mathcal{A}} + \beta\,{\rm Tr}\,(R^2)\Big]\,. \tag{5.73}$$

Note that the covariant anomalies are consistent with the inflow from the generalized 4+1d response action

$$S = -\int_{\mathcal{M}_5}\tilde{\mathcal{A}}\left(dAdA + \alpha d\tilde{\mathcal{A}}d\tilde{\mathcal{A}} + \beta\,{\rm Tr}\,(R^2)\right)\,. \tag{5.74}$$

In this expression, the field $\tilde{\mathcal{A}}$ has been generalized in 4+1 dimensions by introducing the fully antisymmetric tensor $\hat{\mathcal{T}}^{a\beta\gamma\sigma}$, whose component $\sigma = 5$ is only nonvanishing at the boundary, where it matches the torsion $\hat{\mathcal{T}}^{a\beta\gamma 5} = \mathcal{T}^{a\beta\gamma}$.

The right-hand side of the stress tensor conservation (5.69) can be rewritten as follows. The fourth term is identified as the contribution due to the anomalous spin current $\mathcal{S}_{\rm cov}$ as in (5.25). Following the same steps as in (5.24), we find,

$$-\beta(\partial_\mu\psi - \tilde{\mathcal{A}}_\mu)\Big[\,{\rm Tr}\,(R^2)\Big] = 2\beta\,{\rm Tr}\,(R_{\mu\sigma}[(d\psi - \tilde{\mathcal{A}})R]^\sigma) = \frac{1}{2}\,{\rm Tr}\,(R_{\mu\sigma}\mathcal{S}^\sigma_{\rm cov})\,. \tag{5.75}$$

We can include the corresponding term with the classical spin current which can be written

$$\frac{1}{2}\,{\rm Tr}\,(R_{\mu\nu}\mathcal{S}^\nu_{cl}) = \frac{1}{2}\,{\rm Tr}\,\left((\mathring{R}_{\mu\nu} + \mathring{D}_{[\mu}K_{\nu]} + K_{[\mu}K_{\nu]})\mathcal{S}^\nu_{cl}\right) = \frac{1}{2}\,{\rm Tr}\,(\mathring{D}_{[\mu}K_{\nu]}\mathcal{S}^\nu_{\rm class})\,, \tag{5.76}$$

by observing that the two of the three terms of the Riemann curvature (cf. (5.42)) vanish, the first once contracted with the totally antisymmetric S_{cl} owing to the second Ricci identity, while the third one by full antisymmetry of contorsion.

The third term in the r.h.s. of (5.69) involves the classical spin current contracted with torsion

$$3\lambda\Big[(\pi - A)d(\pi + A)\Big]^\nu \varepsilon_{\nu\mu\alpha\beta}\mathring{D}_\gamma\Gamma^{\alpha,\beta\gamma} = \frac{3}{2}\mathcal{S}_{\mu,\alpha\beta\ cl}\mathring{D}_\gamma\mathcal{T}^{\gamma,\alpha\beta}\,. \tag{5.77}$$

[22] With the help of the 3+1d identity $p_{[\nu}\varepsilon_{\rho]\alpha\beta\gamma}\Gamma^{\alpha\beta\gamma} = -3\varepsilon_{\nu\rho\alpha\beta}p_\gamma\Gamma^{\alpha\beta\gamma}$, valid for totally antisymmetric affine connection.

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We finally recast the stress-tensor conservation in the following form

$$\mathring{D}_\nu T^\nu{}_\mu = F_{\mu\nu} J^\nu_{\rm cov} + \tilde{F}_{\mu\nu} \tilde{J}^\nu_{\rm cov} + \frac{1}{2}\,{\rm Tr}\,(R_{\mu\sigma} \mathcal{S}^\sigma_{\rm tot}) \\ - 3\mathring{D}_\gamma K^{\gamma,\alpha\beta} \mathcal{S}_{\mu,\alpha\beta\ {\rm class}} + \frac{1}{2}\mathring{D}_{[\mu} K_{\sigma],\alpha\beta} \mathcal{S}^{\sigma,\alpha\beta}_{\rm class}\,, \tag{5.78}$$

where $\mathcal{S}_{\rm tot} = \mathcal{S}_{\rm class} + \mathcal{S}_{\rm cov}$. We observe that for vanishing torsion this conservation law agrees with the expression found in section 5.2, there obtained by the diffeomorphism Ward identity. This provides a consistency check for our approach. Note also that the spin-current term (5.78) is cubic in curvature and torsion, owing to the non-minimal coupling introduced in (5.55).

In conclusion, we obtained the so-called constitutive relations for currents (5.70), (5.71) and stress tensor (5.68) and respective conservation equations (5.72), (5.73) and (5.78) in most general backgrounds.

6 Physical aspects of the variable ψ

The description of anomalous 3+1d hydrodynamics and the matching with 4+1d topological theories has led to the appearance of a new pseudoscalar field ψ, which ensures gauge invariance of the axial quantity $\tilde{q} = d\psi - \tilde{A}$. In the following we suggest some physical effects associated to this field.

6.1 Spinor rotation

The relation between axial and spin currents in the Dirac theory described in section 5.3.1 provides some geometric insight. We have seen that the axial gauge transformation associated to ψ maps into a local Lorentz transformation (cf. (5.37)). Specifically, for a gauge field in the third direction $\tilde{A}_3$, one finds

$$\partial_3 \tilde{\lambda} \sim \dot{\Lambda}_{12}\,, \qquad\qquad \psi \;\to\; \psi + \tilde{\lambda}\,, \tag{6.1}$$

where the dot indicates a time derivative. The gauge transformation Λ_{12} is absorbed by a rotation of Dirac spinors in the direction $\hat{3}$, orthogonal to the (12) plane. This correspondence suggests that the variable ψ is associated to spin degrees of freedom of the Dirac fluid.

6.2 Interpretation as dynamic chiral chemical potential

In some physical settings, the axial background is used to mimic external conditions that cause an imbalance between the two chiralities of fermions [24, 56]. In particular, the axial potential can be seen as the difference of chemical potentials for left and right fermions,

$$\tilde{A}_0 = \mu_R - \mu_L\,. \tag{6.2}$$

In the following, we shall argue that the same interpretation can be given to the companion quantity $\dot{\psi}$, which is a dynamic degree of freedom of the fluid.

Let us illustrate this feature by considering the stress tensor conservation (4.27) in presence of vector background $A \neq 0$ and $\tilde{A} = 0$,

$$\partial_\nu T^\nu_\mu = F_{\mu\alpha} J^\alpha_{\rm cov}\,, \tag{6.3}$$

$$J^\alpha_{\rm cov} = J^\alpha_{\rm cons} = 2[d\psi d\pi]^\alpha\,, \tag{6.4}$$

where we used (4.23) for the expression of the covariant current evaluated on the equations of motion. We write (6.3) for $\mu = 0$, expressing the adiabatic change of energy in the fields $(\vec{E}, \vec{B})$, assumed static or slowly varying,

$$\dot{\mathcal{E}} = 2F_{0i}\varepsilon^{i0jk}\dot{\psi}\partial_j(p_k + A_k) = 2\vec{E}\cdot\vec{B}\,\dot{\psi}\,, \tag{6.5}$$

where the fluid velocity is taken to vanish ($p_k \sim v_k = 0$, $k = 1, 2, 3$).

The result (6.5) should be compared with the expression describing the spectral flow,[23] a characteristic feature of chiral anomalies, which is also realized in the same backgrounds for small $\dot{\vec{E}}$:

$$\dot{Q}_L - \dot{Q}_R = 2\ \mathrm{Vol}\ \vec{E}\cdot\vec{B}\,, \tag{6.6}$$

where Vol is the volume of the system.

The difference between the two physical settings is the following. In the spectral flow (6.6), the charge non-conservation is due to left (resp. right) particles moving above (below) the Fermi level, without energy change. The same effect can be obtained by an imbalance of chemical potentials (6.2).

In the system described by (6.5), there is energy variation $\dot{\mathcal{E}} \propto \dot{\psi}$ in the same gauge background. We interpret this effect as an imbalance of chemical potential between left and right particles, that adds to the spectral flow, causing the energy change. Actually, the equation (6.2) should be modified owing to axial gauge symmetry into

$$\tilde{A}_0 - \dot{\psi} = \mu_R - \mu_L\,, \tag{6.7}$$

supporting the interpretation that ψ, solution of the equations of motion, can realize a dynamic chiral imbalance also in absence of external background $\tilde{A}$.

6.3 Adding dynamics to ψ

We have seen that the action and equations of motion depend on the axial gauge invariant quantity $\tilde{q} = d\psi - \tilde{A}$. While we consider $\tilde{A}$ as a background field, the ψ field corresponds to a physical degree of freedom. Fixing the gauge for $\tilde{A}$ and solving equations of motion for $\tilde{q}$ we can determine the corresponding configuration of ψ. We remark that ψ might be thought of as an axial superfluid phase.

Let us first recall the case of sigma models, low-energy approximation of gauge theories, where gauge degree of freedoms become physical and acquire a dynamics. In the Abelian Higgs model, the gauge parameter for vector gauge symmetry is $\theta(x)$. The action obtained by writing the complex relativistic scalar field as $\phi = \rho\exp(i\theta)$ reads [57]

$$S = \int F(A_\mu - \partial_\mu\theta) + \rho^2(\partial_\mu\theta - A_\mu)^2 + (\partial_\mu\rho)^2 + V(\rho)\,. \tag{6.8}$$

The last three terms express the dynamics for θ: in presence of spontaneous symmetry breaking, $< \rho^2 > \neq 0$, θ acquires a σ-model dynamics, to which we can add WZW terms for anomalies. Note that other phases are possible with $< \rho^2 >= 0$.

[23] See, e.g., section 4.4 of [24] for a detailed discussion of spectral flow in 3+1 dimensions.

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We think that the ψ dependence and its dynamics could be explicitly found by integrating out e.g. the free fermion theory in presence of fixed currents. The proof of this fact requires independent studies and is not given here.

Let us nonetheless check that adding a dynamics for ψ does not affect the form of anomalies. We already know that these are independent of the form of the pressure $P(\pi - A)$, in the action (4.43), including the case $P = 0$. We can add the ψ dependence to pressure in a way that respects gauge invariance, as follows,

$$P = P(\pi - A, d\psi - \tilde{A}), \qquad (p = \pi - A,\ \tilde{q} = d\psi - \tilde{A})\,. \tag{6.9}$$

The equation of motion for ψ (4.21) and the axial current (4.24) are modified as follows:

$$-\partial_\nu \frac{\partial P}{\partial \tilde{q}_\nu} + \left[d\pi d\pi + \alpha d\tilde{A}d\tilde{A}\right] = 0\,, \tag{6.10}$$

$$\tilde{J}^\nu_{\text{cons}} = -\frac{\partial P}{\partial \tilde{q}_\nu} + \left[(\pi - a)d(\pi + A) + 2\alpha d\psi d\tilde{A}\right]^\nu\,. \tag{6.11}$$

One verifies that the anomaly (4.25) is unchanged

$$\partial_\nu \tilde{J}^\nu_{\text{cons}} = -\partial_\nu \frac{\partial Q}{\partial \tilde{q}_\nu} + [d\pi d\pi - dAdA] \underset{eq.m.}{=} -\left[dAdA + \alpha d\tilde{A}d\tilde{A}\right]\,. \tag{6.12}$$

This is clearly a simple consequence of axial gauge invariance, i.e. of the functional form $d\psi - \tilde{A}$.

Depending on the phase of matter for the fluid under consideration the equation of state might or might not depend on $d\psi - \tilde{A}$. The stiffness with respect to $d\psi - \tilde{A}$ would mean a presence of an axial superfluid condensate. Contrary, if the only dependence on $d\psi - \tilde{A}$ comes in the form of gradients, the corresponding phase of matter would be characterized by a weakly fluctuating axial charge.

6.4 Chiral symmetry breaking

Let us consider a pressure function (6.9) that is not axial invariant and depends explicitly on ψ, namely $P = P(d\theta - A, \psi)$. Repeating the previous steps, we now obtain

$$\partial_\mu \tilde{J}^\mu = -\left[dAdA + \alpha d\tilde{A}d\tilde{A}\right] - \frac{\partial P}{\partial \psi}\,. \tag{6.13}$$

The additional classical term is explicitly breaking the symmetry: due to its definition, it is a pseudoscalar quantity and vector gauge invariant, thus it can be interpreted as the mass term $m\langle\bar{\Psi}\gamma^5\Psi\rangle$ in Dirac theory or its generalization in interacting cases.

7 Conclusions

In this work, we developed the Euler hydrodynamic description of 3+1 dimensional Dirac fermions which is based on the action (4.43), reproduced here for convenience,

$$\begin{aligned} S_I = &-\int_{\mathcal{M}_5} \tilde{A}dAdA + \alpha\tilde{A}d\tilde{A}d\tilde{A} \\ &+ \int_{\mathcal{M}_4} P\,(\pi - A) + \tilde{A}(\pi - A)d(\pi + A) + \psi(d\pi d\pi + \alpha d\tilde{A}d\tilde{A})\,, \end{aligned} \tag{7.1}$$

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and the free variation with respect to the fields (π, ψ). This theory describes the axial anomalies in $A, \tilde{A}$ backgrounds and, with the additions in section 5, the mixed axial-gravitational anomaly in geometries with curvature and torsion. Rather general dynamics, i.e. equations of state, of the fluid are encoded in the form of the pressure function P, which may also depend on the other gauge invariant quantity $d\psi - \tilde{A}$, as explained in section 6.3. This effective theory description demonstrates explicitly that anomalies are independent of dynamics, namely of the form of the pressure.

Our approach makes it clear that the topological theory in one extra dimension provides the main inputs for defining the hydrodynamics: through the anomaly inflow relation between bulk and boundary currents, it establishes the fluid degrees of freedom entering the description and the anomaly content.

In section 4.3 we found two further topological theories which reproduce the same Dirac response action (4.3) and anomalies. These theories can model hydrodynamics featuring additional degrees of freedom: the action (4.37) involves independent vector and axial momenta, and the expression (4.44) presents currents parameterized by two-form fields. Other hydrodynamics might exist involving additional generalized excitations expressed by higher-form fields and their generalized symmetries [48]. The coupling to higher-form background fields should also be considered because it generates corresponding anomalies which help characterizing the theories.

Another possible extension of this work involves the description of fermionic theories with different anomaly content, as e.g. with multi-component Abelian and non-Abelian symmetries [42, 58].

These results provide an interesting starting point for developing effective field theories for bosonization of 3+1 dimensional fermions. The rather simple variational formulation of hydrodynamics described in this work is very close to bosonic effective field theory, actually identical to it in 1+1 dimensions. Both approaches are semiclassical, low-energy descriptions, only valid for local quantities and response functions. Yet, hydrodynamics can capture non-perturbative phenomena in many physical systems [59].

The action (7.1) defines an effective theory for fluid phases of interacting massless fermions. Note that the pressure term in (7.1) takes the form $P = P(\mu^2) = P\left(-(\pi - A)^2_\mu\right)$ in the relativistic case, and it cannot be immediately identified as a field-theory kinetic term. Solving explicitly the constrains proportional to ψ in the action (7.1) might bring its expression into a more familiar form.

The extension of our approach to the chiral case of a single Weyl fermion is also challenging: the two-fluid theory (4.37) is a good starting point, by splitting it into two equal parts. Our current understanding is that the Weyl theory should involve fluid variables with 4+1 dimensional terms in the action, which cannot be reduced to 3+1 dimensions as realized in (7.1). A related proposal is to describe Weyl fluids by strong chiral unbalancing of Dirac fluids [60].

Finally, a natural program is that of introducing temperature and entropy in hydrodynamics. This can be done within the same variational approach.

Our methods are closely related to other variational approaches for so-called adiabatic fluid dynamics with anomalies, as those constructed in [14, 15, 61] for example. The main difference is in the type of independent fields used in the action functional. Here we considered the canonical momentum per particle and scalar fields playing the role of Lagrange multipliers for

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the constraints of the theory. This choice made the discussion closely related to bosonization of one-dimensional fermions. We generally view the obtained actions for fluid dynamics as effective bosonic nonlinear theories of strongly coupled fermionic fluids.

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A Variational principles for perfect barotropic fluids

A.1 Introduction and constrained variations

We start by summarizing some basic facts about Euler hydrodynamics [42]. The fluid is described, in the non-relativistic case, by the density and velocity fields (ρ, v^i) parameterizing the current, $J^\mu = (\rho, \rho v^i)$. These obey the Euler and continuity equations,

$$\partial_0 v_i + v^k \partial_k v_i = -\frac{1}{\rho}\partial_i P = -\partial_i \mu\,, \tag{A.1}$$

$$\partial_\mu J^\mu = \partial_0 \rho + \partial_i(\rho v^i) = 0\,. \tag{A.2}$$

We consider the limit of vanishing temperature, without entropy and heat flows. In barotropic fluids the pressure depends only on density, $P = P(\rho)$, and obeys the thermodynamic relation $dP = \rho d\mu$ in terms of the chemical potential $\mu(\rho)$, which is also equal to the enthalpy h per particle in absence of heat flow.

We now briefly describe the constraints occurring in Euler dynamics. In 1+1 dimensions, the Euler equation can be rewritten as

$$\partial_t v = \partial_x\left(-\mu - \frac{v^2}{2}\right)\,, \qquad \text{i.e.} \qquad \partial_0 p_1 = \partial_1 p_0\,, \qquad (p_0, p_1) = \left(-\mu - \frac{v^2}{2}, v\right)\,. \tag{A.3}$$

This is the constraint[24] $dp = 0$ discussed in section 2: it implies the following invariant of motion

$$C_1 = \int dx\; v\,, \qquad\qquad \partial_0 C_1 = 0\,. \tag{A.4}$$

[24] In this appendix we consider vanishing gauge backgrounds, thus the fluid momentum is $\pi \equiv p$.

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In 3+1 dimension the Euler equation implies the following relation obeyed by the fluid vorticity, $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$,

$$\partial_0 \boldsymbol{\omega} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) , \qquad \boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v} . \tag{A.5}$$

It is also instructive to rewrite the Euler equation (A.1) in four-dimensional covariant form by using the four-current and introducing the fluid four-momentum

$$p_\mu = \left(-\mu - \frac{v^2}{2}, v^i \right) . \tag{A.6}$$

We find,

$$\partial_0 p_i - \partial_i p_0 + v^j (\partial_j p_i - \partial_i p_j) = 0 , \qquad \text{i.e.} \qquad J^\mu (\partial_\mu p_\nu - \partial_\nu p_\mu) = 0 . \tag{A.7}$$

In 3+1 dimension, the constraint $dpdp = 0$ discussed in section 3 amounts to the conservation of the helicity current $\tilde{J} = *(pdp)$. The associated charge is

$$C_3 = \int d^3x \, v^i \omega_i , \qquad \partial_0 C_3 = 0 . \tag{A.8}$$

The time independence of this quantity can be proven by using the Euler and vorticity equations (A.1), (A.5) or more straightforwardly from (A.7).

These results show that the action variational principle for the Euler equation (A.1) cannot be formulated in terms of the original variables $(\rho, \boldsymbol{v})$, because variations should respect the constraints. As discussed in the main text, one solution is to vary with respect to the Clebsch scalar parameters which locally solve the constraints

$$1+1d: \; dp = 0 \; \to \; p = d\theta , \qquad 3+1d: \; dpdp = 0 \; \to \; p = d\theta + \alpha d\beta . \tag{A.9}$$

We refer the reader to [42, 62] for an introduction to the formalism involving Clebsch parameterization of hydrodynamic fields.

Another method described in section 3 is based on the use of admissible (diffeomorphism) variations which leads to the Carter-Lichnerowicz equations of motion reproducing the constraints. This will be further analyzed in the following.

We first give the form of the action. Both relativistic and non-relativistic Lagrangians of a perfect fluid have been considered [63, 64]. In fact, one could use a general variational formulation that is applicable independently of the spacetime symmetry [42]. The action is

$$S[J^\mu, p_\mu] = - \int d^4x \, [J^\mu p_\mu + \varepsilon(J^\mu)] . \tag{A.10}$$

It is a functional of the current J^μ and canonical momentum per particle p_μ. The spacetime symmetries and the dynamics (equation of state) are encoded in the scalar function $\varepsilon(J^\mu)$ which is the energy density.

Even more compact formulation can be obtained by eliminating J^μ by a Legendre transform of the action, leading to

$$S[p_\mu] = \int d^4x \, P(p_\mu) , \tag{A.11}$$

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with

$$J^\mu = -\frac{\delta S}{\delta p_\mu} = -\frac{\partial P}{\partial p_\mu}\,. \tag{A.12}$$

The action is given by the spacetime integral of the pressure $P = P(\mu(p_\mu))$ which is considered as a function of the canonical momentum p_μ. Non-relativistic and relativistic fluids differ in the expression for $\mu(p_\mu)$, respecting the corresponding Galilean and Lorentz symmetries. In the non-relativistic case, we have (cf. (A.6))

$$\mu = -p_0 - \frac{1}{2}p_i p^i, \qquad p^i = v^i\,, \tag{A.13}$$

indeed reproducing the earlier form of the current

$$J^0 = -\frac{\partial P}{\partial p_0} = \frac{\partial P}{\partial \mu} = \rho\,, \qquad J^i = -\frac{\partial P}{\partial p_i} = \frac{\partial P}{\partial \mu}p^i = \rho v^i\,. \tag{A.14}$$

Again μ is understood as a function of ρ and can be obtained from the thermodynamic relations $dP = \rho d\mu$, $\epsilon + P = \rho\mu$, etc. In the main text we considered the action (A.11) and use the admissible (i.e. diffeomorphisms) variations of p_μ to obtained the Carter-Lichnerowicz equations of motion

$$J^\mu(\partial_\mu p_\nu - \partial_\nu p_\mu) = 0\,, \qquad J^\mu \equiv -\frac{\partial P}{\partial p_\mu}\,, \tag{A.15}$$

and current conservation. The first equation in (A.15) is recognized as the Euler equation in covariant form (A.7). This also imply the conservation of the stress tensor

$$\partial_\nu T^\nu_\mu = 0\,, \qquad T^\nu_\mu = J^\nu p_\mu + P\delta^\nu_\mu\,. \tag{A.16}$$

These equations can be rewritten using (A.6) and (A.2) as follows,

$$\partial_t\left(\epsilon + \frac{\rho v^2}{2}\right) + \partial_i\left(\rho v^i\left(\mu + \frac{v^2}{2}\right)\right) = 0\,, \tag{A.17}$$

$$\partial_t(\rho v_i) + \partial_j\left(\rho v_i v^j + P\delta^j_i\right) = 0\,. \tag{A.18}$$

The first and second equations are, respectively, the components $\mu = 0$ and $\mu = j$ of (A.16): they express the local conservations of energy and momentum. The Euler equation (A.1) is also recovered from the momentum (A.18) and current (A.2) conservations. We remark that among all equations described here only four are independent. For example, the conservation of energy and momentum (A.17), (A.18) can be derived from the Euler and continuity equations (A.1), (A.2) and vice versa.

A.2 Relativistic fluids

In this case the pressure depends on the Lorentz invariant combination of momenta p_μ:

$$P = P(\mu)\,, \qquad \mu = \sqrt{-p_\mu p^\mu}\,. \tag{A.19}$$

We introduce the 4-velocity u_μ as

$$p_\mu = \mu u_\mu\,, \qquad u_\mu u^\mu = -1\,, \tag{A.20}$$

and the particle density n as

$$n = \frac{\partial P}{\partial \mu}\,, \tag{A.21}$$

so that the current is

$$J^\mu = -\frac{\partial P}{\partial p_\mu} = n u^\mu\,. \tag{A.22}$$

Upon using these parameterizations, the stress tensor (A.16) and current (A.2) conservations now imply the hydrodynamic equations for perfect relativistic fluids [65]

$$\partial_\nu \Big(\mu n\, u_\mu u^\nu + P \delta^\nu_\mu \Big) = 0\,, \tag{A.23}$$

$$\partial_\mu (n u^\mu) = 0\,. \tag{A.24}$$

For barotropic fluid the chemical potential μ and pressure P can be considered as functions of the particle density n, obeying the thermodynamic relations following from (A.21)

$$dP = n d\mu\,, \qquad d\epsilon = \mu dn\,, \tag{A.25}$$

where $\epsilon(n)$ is the internal energy of the fluid related to the pressure by Gibbs-Durhem formula

$$\epsilon + P = \mu n\,. \tag{A.26}$$

A.3 Free variations with constraint

We now discuss in more detail the variational scheme adopted in this paper, which is based on taking free variations while imposing the constraint into the action via a Lagrange multiplier. We have

$$S[p_\mu, \psi] = \int d^4x\, P(p_\mu) + \int_{\mathcal{M}_4} \psi\, dpdp\,. \tag{A.27}$$

The second term is written as an integral of the 4-form to emphasize its topological nature (independence on the metric). The equations of motion obtained by free variations over p_μ and ψ read:

$$p_\mu : \qquad J^\mu \equiv -\frac{\partial P}{\partial p_\mu} = 2[d\psi dp]^\mu\,, \tag{A.28}$$

$$\psi \;\; : \qquad dpdp = 0\,. \tag{A.29}$$

These expressions form a complete system of equations needed to find the evolution of fields (p_μ, ψ). As fully described in the main text, these imply the stress tensor and current conservations, (A.16) and (A.2), and thus all earlier equations of Euler hydrodynamics in both non-relativistic and relativistic cases.

We now discuss the equivalence between: (F) free-variation equations (A.28), (A.29) and (CL) the Carter-Lichnerowicz and conservation equations, (A.15), (A.2).

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$(F) \rightarrow (CL)$: current conservation clearly follows from (A.28). Upon substituting this equation into (A.15), we find

$$J^\mu(\partial_\mu p_\nu - \partial_\nu p_\mu) = 2[d\psi dp]^\mu(\partial_\mu p_\nu - \partial_\nu p_\mu) = -\partial_\nu\psi[dpdp] = 0\,. \tag{A.30}$$

Here we used the following 4d algebraic identity, also employed in the main text,

$$a_\mu[BC] = [aB]^\nu C_{\nu\mu} + [aC]^\nu B_{\nu\mu}\,, \tag{A.31}$$

valid for any 1-form a and 2-forms B, C. It is proven by checking the vanishing of a 5-form in four dimensions.

$(CL) \rightarrow (F)$: the result depends on the rank of the antisymmetric matrix dp, which can be zero or two, due to (A.15); as explained in section 3.1.1, this implies that (A.29) holds. If the rank of dp is two, we can choose local coordinates in which dp is non-vanishing along directions x^2, x^3, i.e. $dp = \omega_{23}dx^2dx^3$. Then the current $J^\mu = -\partial P/\partial p_\mu$ has components along x^0, x^1; its dual 3-form is

$$*J = (\eta dx^0 + \sigma dx^1)dx^2dx^3\,, \tag{A.32}$$

where η, σ are scalar functions. Current conservation (A.2) implies $\partial_1\eta - \partial_0\sigma = 0$, which can be integrated locally to $*J = (\partial_1\gamma dx^0 - \partial_0\gamma dx^1)dx^2dx^3$. This is equation (A.28) upon identification of $\psi = \gamma/\omega_{23}$.

For vanishing rank $dp = 0$, the constraint (A.29) has a double zero. The CL equation (A.15) is trivially satisfied and (A.28) is not implied. The current J^μ is nonetheless conserved and mildly constrained by $dp = 0$ through the relation between p and J. Equation (A.28) is instead different because it forces the current to vanish. This result is not acceptable, because there can be locally non-vanishing fluid motion ($J \neq 0$) even in the absence of vorticity, e.g., in regions outside of localized vortex lines, where $dp = 0$.

We can resolve this conflict by technically assuming that dp can take very small values but never vanish. When it is small we can allow the quantity $d\psi$ to grow large so as to have a finite value of the current in (A.28). The ψ singularity is 'hidden' because this quantity disappears once equations are rewritten in terms of original variables $(\rho, \boldsymbol{v})$. Thus, this can be viewed as a problem of coordinate singularities of the free variation approach, not a real physical problem.

Furthermore, the issue is of limited interest in the generic case of both non-vanishing backgrounds, because the equations of motion $d\pi d\pi + \alpha d\tilde{A}d\tilde{A} = 0$ implies that $d\pi = 0$ might occur only at points where $d\tilde{A}d\tilde{A} = 0$. This is analyzed in the following subsection. Further related aspects are discussed in two other sections.

A.4 Variational principle in presence of nontrivial backgrounds

Let us consider the following modification of (A.27)

$$S[p_\mu, \psi] = \int d^4x\, P(p_\mu) + \int_{\mathcal{M}_4} \psi\,(dpdp + dC)\,, \tag{A.33}$$

where $C \neq 0$ is a non-trivial background 3-form field. Let us show that the equations obtained from the free variational principle applied to (A.33) are identical to the ones obtained using restricted variations common in the hydrodynamic literature.

The free variations give

$$\mathcal{J}^{\nu} \equiv -\frac{\partial P}{\partial p_{\nu}} = 2\Big[d\psi dp\Big]^{\nu}\,, \tag{A.34}$$

$$dpdp + dC = 0\,. \tag{A.35}$$

On the other hand, we can derive equations of motion using the following set of allowed variations

- $\delta\pi = d\lambda\,, \qquad \delta\psi = 0\,,$
- $\delta\pi = 0\,, \qquad \delta\psi = \tilde{\lambda}\,,$
- $\delta\pi = \mathcal{L}_{\varepsilon}\pi\,, \quad \delta\psi = \mathcal{L}_{\varepsilon}\psi\,,$

respectively corresponding to vector and axial gauge and diffeomorphism transformations. Using these variations parameterized by $\lambda, \tilde{\lambda}, \varepsilon^{\mu}$ after some algebra we derive the following equations

$$\partial_{\mu}\mathcal{J}^{\mu} = 0\,, \tag{A.36}$$

$$dpdp + dC = 0\,, \tag{A.37}$$

$$\left(\mathcal{J}^{\nu} - 2\Big[d\psi dp\Big]^{\nu}\right)(\partial_{\nu}p_{\mu} - \partial_{\mu}p_{\nu}) = 0\,. \tag{A.38}$$

It is obvious that equations (A.34), (A.35) imply (A.36), (A.37), (A.38).

Conversely, having (A.36), (A.37), (A.38) consider first the case $dC \neq 0$. Then we have from (A.37) that $dpdp \neq 0$ and from (A.38) the equation (A.34) follows.[25] We conclude that both the system (A.34), (A.35) and (A.36), (A.37), (A.38) produce the same solutions in the presence of the background $dC \neq 0$ or in the limit $dC \to 0$. In the main text we considered $dC = \alpha d\tilde{A}d\tilde{A} + \beta\,\mathrm{Tr}\,(R^2)$.

A.5 Adding degrees of freedom

One possible way to resolve the coordinate singularity for $dp = 0$ is to add degrees of freedom to the theory. Let us introduce the following gauge invariant term in the action (4.43)

$$\Delta S = \int_{\mathcal{M}_4} \phi dp d\tilde{\tau}\,, \tag{A.39}$$

where the new variables are the gauge invariant scalar ϕ and pseudo-1-form $\tilde{\tau}$. No dynamics is introduced for them. The equation of motion for p_{μ} (A.28) is modified as

$$J^{\mu} = [2d\psi dp + d\phi d\tilde{\tau}]^{\mu}\,, \tag{A.40}$$

and two additional equations follow by variation with respect to the new fields

$$d\tilde{\tau}dp = 0\,, \tag{A.41}$$

$$d\phi dp = 0\,. \tag{A.42}$$

[25] Indeed, assuming that (A.34) is invalid we obtain from (A.38) that dp has a rank two at most and $dpdp = 0$ contradicting (A.37).

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The equations (A.41), (A.42) are trivially satisfied in the case of interest $dp = 0$, leaving completely free the new variables $(\phi, \tilde{\tau})$. Then the second term in the r.h.s. of expression (A.40) can provide a general parameterization of the conserved current J, when the first term vanishes.

The addition (A.39) remains valid in presence of gauge backgrounds, with the replacement $p \to \pi$, and it is found that the corrections introduced in the currents and equations of motion do not have effect on anomalies and stress tensor conservation. Of course the physical consequences of including the term (A.39) into the action remain to be fully understood.

A.6 Equations obeyed by ψ

Specializing (A.28) for nonrelativistic fluids, we obtain

$$\rho = 2[d\psi dp]^0 \,, \tag{A.43}$$

$$\rho v^i = 2[d\psi dp]^i \,, \tag{A.44}$$

$$[dpdp] = 0 \,, \tag{A.45}$$

where $p_i = v_i$ and $p_0 = -\mu - v^2/2$. We rewrite these equations as

$$\rho = 2\boldsymbol{\nabla}\psi \cdot \boldsymbol{\omega} \,, \tag{A.46}$$

$$\rho \boldsymbol{v} = -2\dot{\psi}\boldsymbol{\omega} + 2\boldsymbol{\nabla}\psi \times (\dot{\boldsymbol{v}} - \boldsymbol{\nabla}p_0) \,, \tag{A.47}$$

$$\partial_t(\boldsymbol{v} \cdot \boldsymbol{\omega}) + \boldsymbol{\nabla} \cdot (-p_0\boldsymbol{\omega} + \boldsymbol{v} \times (\dot{\boldsymbol{v}} - \boldsymbol{\nabla}p_0) = 0 \,. \tag{A.48}$$

The last equation can be transformed into

$$\boldsymbol{\omega} \cdot (\dot{\boldsymbol{v}} - \boldsymbol{\nabla}p_0) = 0 \,. \tag{A.49}$$

Vector-multiplying the second equation (A.47) by $\boldsymbol{\omega}$ and assuming that $\rho > 0$, we obtain

$$\dot{\boldsymbol{v}} - \boldsymbol{\nabla}p_0 = \boldsymbol{v} \times \boldsymbol{\omega} \,. \tag{A.50}$$

If we multiply (A.47) by $\boldsymbol{\nabla}\psi$ we obtain

$$\dot{\psi} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\psi = 0 \,. \tag{A.51}$$

Collecting all equations together we write the complete system

$$\rho = 2\boldsymbol{\nabla}\psi \cdot \boldsymbol{\omega} \,, \tag{A.52}$$

$$\dot{\boldsymbol{v}} - \boldsymbol{\nabla}p_0 = \boldsymbol{v} \times \boldsymbol{\omega} \,, \tag{A.53}$$

$$\dot{\psi} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\psi = 0 \,. \tag{A.54}$$

Let us now assume that we are given the configuration $\rho \geq 0$ and generic $\boldsymbol{v}$ at $t = 0$ so that $\boldsymbol{\omega} \neq 0$. Then, we can locally solve (A.52) and find ψ. After that the equations (A.53), (A.54) and continuity equation for ρ (that follows from these equations) gives us the subsequent configurations at any future time (locally).

Taking time derivative of (A.52) and using (A.53), (A.54) we can also derive continuity equation

$$\dot{\rho} + \boldsymbol{\nabla}(\rho\boldsymbol{v}) = 0 \,. \tag{A.55}$$

This argument confirms in detail that the free variational principle with ψ is applicable for generic configuration of $(\rho, \boldsymbol{v})$ and is equivalent to that of restricted variations, provided that vorticity $\boldsymbol{\omega} \neq 0$.

B Anomaly coefficients for Dirac and Weyl fermions

The 4+1d Abelian Chern-Simons action is given by

$$S = -\frac{1}{24\pi^2}\int_{\mathcal{M}_5} AdAdA\,. \tag{B.1}$$

To obtain a conventional normalization we should replace $A \to \frac{e}{\hbar c}A$. In the main text we also put the flux quantum to one, i.e. set $1/(4\pi^2) \to 1$. The action (B.1) accounts for the anomaly inflow of a single right Weyl fermion. For a set of Weyl fermions we have

$$S = -\sum_i \frac{e_i^3\chi_i}{24\pi^2}\int A_i dA_i dA_i\,, \tag{B.2}$$

where e_i, χ_i are, respectively, charges (integer) and chiralities (± 1) of the Weyls and A_i are the gauge fields coupling to them.

From (B.2) we obtain the conservation of corresponding 3+1d covariant currents $J^\mu_{i,\text{cov}}$ by differentiation and inflow correspondence. The consistent currents are also obtained from the associated Wess-Zumino-Witten action. They read

$$\partial_\mu J^\mu_{i,\text{cov}} = -e_i^3\chi_i\frac{1}{8\pi^2}[dA_i dA_i]\,, \tag{B.3}$$

$$\partial_\mu J^\mu_{i,\text{cons}} = -e_i^3\chi_i\frac{1}{24\pi^2}[dA_i dA_i]\,, \tag{B.4}$$

(we used the notation $[abcd] = \varepsilon^{\alpha\beta\mu\nu}a_\alpha b_\beta c_\mu d_\nu$).

Let us now consider a set of Dirac fermions. Using the relation between chiral and vector/axial components (4.2), we can rewrite (B.2) as

$$S = -\sum_i \frac{1}{4\pi^2}\int_{\mathcal{M}_5} \tilde{e}_i e_i^2\ \tilde{A}\ dA\ dA + \frac{\tilde{e}_i^3}{3}\ \tilde{A}d\tilde{A}d\tilde{A}\,. \tag{B.5}$$

Here e_i and $\tilde{e}_i$ are (integer) vector and axial charges of Dirac fermions, respectively. We assumed here and below that there are two common gauge backgrounds, vector A and axial $\tilde{A}$ acting on all Dirac flavors. As a consequence

$$\partial_\mu J^\mu_{\text{cons}} = 0\,, \tag{B.6}$$

$$\partial_\mu J^\mu_{\text{cov}} = -\left(\sum_i \tilde{e}_i e_i^2\right)\frac{1}{2\pi^2}[dAd\tilde{A}]\,, \tag{B.7}$$

$$\partial_\mu \tilde{J}^\mu_{\text{cons}} = -\left(\sum_i \tilde{e}_i e_i^2\right)\frac{1}{4\pi^2}[dAdA] - \left(\sum_i \tilde{e}_i^3\right)\frac{1}{12\pi^2}[d\tilde{A}d\tilde{A}]\,, \tag{B.8}$$

$$\partial_\mu \tilde{J}^\mu_{\text{cov}} = -\left(\sum_i \tilde{e}_i e_i^2\right)\frac{1}{4\pi^2}[dAdA] - \left(\sum_i \tilde{e}_i^3\right)\frac{1}{4\pi^2}[d\tilde{A}d\tilde{A}]\,. \tag{B.9}$$

Let us now describe the mixed axial-gravitational anomaly, starting with a single right Weyl fermion. The mixed Chern-Simons gravity action is

$$\Delta S = -\frac{1}{192\pi^2}\int_{\mathcal{M}_5} \tilde{A}\,\mathrm{Tr}\,(R^2)\,, \tag{B.10}$$

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where $R = d\omega + \omega^2$ is the curvature two-form. We obtain the following contribution to the chiral currents

$$\partial_\mu J^\mu_{\text{cons}} = -\frac{1}{192\pi^2}\Big[\text{Tr}\,(R^2)\Big]\,, \tag{B.11}$$
$$\partial_\mu J^\mu_{\text{cov}} = -\frac{1}{192\pi^2}\Big[\text{Tr}\,(R^2)\Big]\,. \tag{B.12}$$

For a set of Weyls we have

$$\partial_\mu J^\mu_{i,\text{cov}} = -e_i^3\chi_i\frac{1}{8\pi^2}[dA_i dA_i] - e_i\chi_i\frac{1}{192\pi^2}\Big[\text{Tr}\,(R^2)\Big]\,, \tag{B.13}$$
$$\partial_\mu J^\mu_{i,\text{cons}} = -e_i^3\chi_i\frac{1}{24\pi^2}[dA_i dA_i] - e_i\chi_i\frac{1}{192\pi^2}\Big[\text{Tr}\,(R^2)\Big]\,. \tag{B.14}$$

It is easy to get from here the corresponding Dirac anomalies.

C Hydrodynamics in chiral backgrounds

C.1 Chiral topological theory and decoupling of 1+1 dimensional chiral currents

We now consider a chiral background and show that 1+1d excitations split into chiral components, one interacting with the background and the other remaining inert.

The chiral decomposition of backgrounds and currents is defined by

$$\begin{aligned} &J = J_+ + J_-\,, \qquad \tilde{J} = J_+ - J_-\,, \qquad A = \frac{1}{2}(A_+ + A_-)\,, \qquad \tilde{A} = \frac{1}{2}(A_+ - A_-)\,, \\ &p = \frac{1}{2}(b_+ + b_-)\,, \qquad \tilde{q} = \frac{1}{2}(b_+ - b_-)\,, \end{aligned} \tag{C.1}$$

obeying the relation $J^\mu A_\mu + \tilde{J}^\mu \tilde{A}_\mu = J^\mu_+ A_{+\mu} + J^\mu_- A_{-\mu}$.

We consider one chiral part by setting $A_- = b_- = 0$ and redefining $A_+ = A, b_+ = b$. The 2+1d topological theory (2.26) becomes

$$S = \int \frac{1}{2} b db + b dA\,, \tag{C.2}$$

leading to the Chern-Simons response action (2.12).

Following the steps of section 2.2, we use the anomaly inflow to obtain the 1+1d covariant current

$$j^3 = \frac{\delta S}{\delta A_3} = \varepsilon^{3\mu\nu}\partial_\mu b_\nu = \partial_\alpha J^\alpha_{\text{cov}} \underset{eq.m.}{=} \varepsilon^{\mu\nu}\partial_\mu\left(\partial_\nu\chi - A_\nu\right)\,. \tag{C.3}$$

In the last expression, we substituted the solution of the bulk equations of motion, leaving the gauge parameter χ free. The WZW action obtained from the response action (2.12) is,

$$S_{\text{WZW}}[\lambda, A] = -\frac{1}{2}\int_{\mathcal{M}_2} \lambda dA\,, \tag{C.4}$$

giving the anomaly of the consistent current $\partial_\alpha J^\alpha_{\text{cons}} = -\varepsilon^{\mu\nu}\partial_\mu A_\nu/2 = \partial_a J^\alpha_{\text{cov}}/2$.

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The chiral splitting of excitations in the boundary Dirac theory will be obtained on the solutions of the equations of motion. A first indication of this decomposition comes from the form of inflow currents (2.28), (2.29), which we can rewrite using (C.1),

$$J^{\alpha}_{+\mathrm{cov}} \underset{eq.m.}{=} \varepsilon^{\alpha\beta}\left(\partial_\beta\psi+\partial_\beta\theta-A_{+\beta}\right), \tag{C.5}$$

$$J^{\alpha}_{-\mathrm{cov}} \underset{eq.m.}{=} \varepsilon^{\alpha\beta}\left(\partial_\beta\psi-\partial_\beta\theta-A_{-\beta}\right). \tag{C.6}$$

One sees that for $A_-=0$, $J_{-\mathrm{cov}}$ does not couple to the background and vanishes for $\theta=\psi$. The expression for J_+ matches (C.3) by identifying $\chi=\psi+\theta$.

It is convenient to consider the 1+1d theory in the form (2.36) used to establish duality. Paying attention to the factor of two in the chiral splitting (C.1), it can be written

$$S=\frac{1}{2}\int_{\mathcal{M}_2}-\frac{1}{2}(\pi_\mu-A_\mu)^2+(A-d\psi)\pi\,. \tag{C.7}$$

The equations of motion are

$$\begin{aligned}\pi_\mu&=\partial_\mu\theta\,,\\ \pi^\mu-A^\mu&=\varepsilon^{\mu\nu}(\partial_\nu\psi-A_\nu)\,.\end{aligned} \tag{C.8}$$

The consistent current obtained from this action, evaluated on the equations of motion, is

$$J^{\mu}_{\mathrm{cons}} \underset{eq.m.}{=} \frac{1}{2}\epsilon^{\mu\nu}\left(\partial_\nu\theta+\partial_\nu\psi-A_\nu\right), \tag{C.9}$$

whose anomaly agree with that obtained from the above WZW action.

It remains to show that dynamics allows for a single nonvanishing chiral current. Actually, the equations of motion (C.8) in the gauge $A_0=A_1$, became the duality relations (2.39): explicitly, $\partial_0\theta=\partial_1\psi$, $\partial_1\theta=\partial_0\psi$. These admit two solutions

$$\theta=\psi \qquad \to \qquad (\partial_0-\partial_1)\theta=0\,, \tag{C.10}$$

$$\theta=-\psi \qquad \to \qquad (\partial_0+\partial_1)\theta=0\,. \tag{C.11}$$

The first solution gives the expected result $J_{-\mathrm{cov}}=0$ and also makes the other current chiral, $J^0_{\mathrm{cov}}=-J^1_{\mathrm{cov}}$. The solution with opposite chirality ($\theta=-\psi$) is nonetheless present, corresponding to $J_+=0$ and $J_-\neq 0$ but decoupled from the background.

The chiral splitting can be made more explicit by rewriting the action (C.7) in a equivalent form by using its equations of motion (C.8), leading to the expression,

$$S=\frac{1}{2}\int-\frac{1}{4}\partial_x\theta_+(\partial_t-\partial_x)\theta_++\frac{1}{4}\partial_x\theta_-(\partial_t+\partial_x)\theta_-\,, \tag{C.12}$$

where $\theta_+=\theta+\psi$ and $\theta_-=\theta-\psi$ and we set $A=0$ for simplicity. The action has decomposed into two parts, each one describing one chirality only, as it clear from the respective equations of motion. Each piece is known as the chiral boson action, because it bosonizes a single Weyl fermion [24].

In conclusion, we have checked that chiral splitting occurs in 1+1 dimensions for the free massless Dirac fermions, owing to kinematics. In the 3+1 dimensional theory, free massless Dirac fermions also decouple in a pair of Weyls. However, the 3+1d hydrodynamics described in this work does not show this feature, as explained in the next section.

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In the interacting case, there is no reason to expect the chiral splitting of Dirac fermions in both 1+1 and 3+1 dimensions. In the 1+1d case, one can clearly see from previous formulas that non-quadratic Hamiltonians do not split in two independent terms. The fluid made out of a single copy of interacting Weyl fermions can be described by deforming the chiral boson action (C.12) by adding more general Hamiltonians.

C.2 3+1 dimensional hydrodynamics in chiral backgrounds

The 4+1d Chern-Simons action describing the response of the system in a chiral background and the associated 3+1d WZW action have the form

$$S_{\text{ind}} = -\frac{1}{6}\int_{\mathcal{M}_5} AdAdA\,, \qquad\qquad S_{\text{WZW}} = -\frac{1}{6}\int_{\mathcal{M}_4} \psi dAdA\,. \tag{C.13}$$

From the first expression and anomaly inflow follows the 3+1d anomaly of covariant currents, while the WZW action determines the anomaly of consistent currents,

$$* \, dJ_{\text{cov}} = -\frac{1}{2}dAdA\,, \tag{C.14}$$

$$* \, dJ_{\text{cons}} = -\frac{1}{6}dAdA\,, \qquad\qquad *J_{\text{cov}} = *J_{\text{cons}} - \frac{1}{3}AdA\,, \tag{C.15}$$

which have the correct values for a Weyl fermion.

The 3+1d hydrodynamics action (4.19) is now specialized for chiral backgrounds by setting $A = \tilde{A}$, i.e. $A_- = 0$. For $\alpha = 1/3$, it reads

$$S = \frac{1}{8}\int_{\mathcal{M}_4} P(p) + A\pi d(\pi + A) + \psi\left(d\pi d\pi + \frac{1}{3}dAdA\right)\,. \tag{C.16}$$

The overall normalization has been changed as explained in the following. Upon variation, one obtains the equations of motion

$$-\frac{\partial P}{\partial \pi_\nu} = [(\pi - A)dA + 2(d\psi - A)d\pi]^\nu\,, \qquad d\pi d\pi + \frac{1}{3}dAdA = 0\,, \tag{C.17}$$

and the consistent current

$$J^\nu_{\text{cons}} = -\frac{1}{8}\frac{\partial P}{\partial \pi_\nu} + \frac{1}{8}\left[\pi d(\pi + A) + d(A\pi) + \frac{2}{3}d\psi dA\right]^\nu \tag{C.18}$$

$$\underset{eq.m.}{=} \frac{1}{8}\left[(\pi - A)d(\pi + A) + 2d(A\pi) + 2d\psi d(\pi + \frac{1}{3}A)\right]^\nu\,. \tag{C.19}$$

One verifies that this current obeys the anomaly (C.15), thus checking the normalization.

The covariant current can also be found from $J_{\text{cov}} + \tilde{J}_{\text{cov}}$ of the non-chiral theory (4.33), (4.34) evaluated for $A = \tilde{A}$, up to normalization,

$$*J_{\text{cov}} \underset{eq.m.}{=} \frac{1}{8}[(\pi - A)d(\pi + A) + 2(\alpha + 1)(d\psi - A)dA - 2d((d\psi - A)(\pi - A))]\,. \tag{C.20}$$

For $\alpha = 1/3$, this expression is indeed equal to (C.19) plus correction (C.15), as expected; it shows gauge invariance, owing to simultaneous invariance of the $(d\psi - A)$ and $(\pi - A)$ terms.

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C.2.1 Absence of chiral decoupling in 3+1 dimensions

We now ask whether the chiral currents $J_{\pm\mathrm{cov}} = J_{\mathrm{cov}} \pm \tilde{J}_{\mathrm{cov}}$ decouple in the chiral background $A = \tilde{A}$, i.e. $A_- = 0$. In this case, chirality is not a simple kinematic condition.

If e.g. the right $(+)$ Weyl is probed by a gauge background and the left $(-)$ one is not, we would expect that there are solutions of the equations of motion with vanishing $(-)$ current. The current $J_{+\mathrm{cov}} = J_{\mathrm{cov}} + \tilde{J}_{\mathrm{cov}}$ was found before in (C.20). Next we check whether the difference of currents $J_{-\mathrm{cov}} = J_{\mathrm{cov}} - \tilde{J}_{\mathrm{cov}}$ can vanish, thus proving chiral splitting in the 3+1d Dirac hydrodynamics.

From (4.33), (4.34), we find (for $\alpha = 1/3$):

$$\begin{aligned}
*\tilde{J}_{-\mathrm{cov}} \underset{eq.m}{=} & \; \pi d\pi + \frac{1}{3} A dA - d(A\pi) + \frac{2}{3} d\psi dA - 2 d\psi d\pi \\
= & \; (\pi - d\psi) d\pi + \frac{1}{3}(A - d\psi) dA - d[(d\psi - A)(d\psi - \pi)] \\
= & \; dX \neq 0 \,.
\end{aligned} \tag{C.21}$$

In these relations, we used the first equation of motion (C.17) but not the second one $d\pi d\pi + dAdA/3 = 0$.

We remark that in chiral backgrounds the current (C.21) has no anomalies, as it is verified by using the remaining equation of motion. It follows that the (C.21) actually is an exact form dX. However, it does not vanish as a consequence of the equations of motion, not even for $A = 0$. Therefore, the two chiralities are always coupled.

Let us remark that a physical instance of a single Weyl is at the boundary of the 4+1d topological insulators, so it should be possible to obtain its hydrodynamic description by the methods outlined in this work. Presumably one should start from the 'two-fluid' 4+1d topological theory described in section 4.3.1 and decouple it in two symmetric parts.

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