



On the Fourier Coefficients of High-Dimensional Random Geometric Graphs

Kiril Bangachev

kirilb@mit.edu

Department of EECS, MIT
Cambridge, Massachusetts, USA

Guy Bresler

guy@mit.edu

Department of EECS, MIT
Cambridge, Massachusetts, USA

ABSTRACT

The random geometric graph $\text{RGG}(n, \mathbb{S}^{d-1}, p)$ is formed by sampling n i.i.d. vectors $\{V_i\}_{i=1}^n$ uniformly on $\mathbb{S}^{d-1}$ and placing an edge between pairs of vertices i and j for which $\langle V_i, V_j \rangle \geq \tau_d^p$, where τ_d^p is such that the expected density is p . We study the low-degree Fourier coefficients of the distribution $\text{RGG}(n, \mathbb{S}^{d-1}, p)$ and its Gaussian analogue.

Our main conceptual contribution is a novel two-step strategy for bounding Fourier coefficients which we believe is more widely applicable to studying latent space distributions. First, we *localize the dependence among edges* to few *fragile edges*. Second, we partition the space of latent vector configurations $(\mathbb{S}^{d-1})^{\otimes n}$ based on the set of fragile edges and on each subset of configurations, we define a *noise operator acting independently* on edges not incident (in an appropriate sense) to fragile edges.

We apply the resulting bounds to: 1) Settle the low-degree polynomial complexity of distinguishing spherical and Gaussian random geometric graphs from Erdős-Rényi both in the case of observing a complete set of edges and in the non-adaptively chosen mask $\mathcal{M}$ model recently introduced by Mardia, Verchand, and Wein; 2) Exhibit a statistical-computational gap for distinguishing RGG and a planted coloring model in a regime when RGG is distinguishable from Erdős-Rényi; 3) Reprove known bounds on the second eigenvalue of random geometric graphs.

CCS CONCEPTS

• Theory of computation $\rightarrow$ Random network models.

KEYWORDS

Random Geometric Graphs, Fourier Coefficients, Low-Degree Polynomial Hardness

ACM Reference Format:

Kiril Bangachev and Guy Bresler. 2024. On the Fourier Coefficients of High-Dimensional Random Geometric Graphs. In *Proceedings of the 56th Annual ACM Symposium on Theory of Computing (STOC '24)*, June 24–28, 2024, Vancouver, BC, Canada. ACM, New York, NY, USA, 12 pages. <https://doi.org/10.1145/3618260.3649676>



This work is licensed under a Creative Commons Attribution 4.0 International License.

STOC '24, June 24–28, 2024, Vancouver, BC, Canada

© 2024 Copyright held by the owner/author(s).

ACM ISBN 979-8-4007-0383-6/24/06

<https://doi.org/10.1145/3618260.3649676>

1 INTRODUCTION

Random graphs with a latent high-dimensional geometric structure are increasingly relevant in an era of massive networks over complex computer, social, or biological populations. Such graphs provide a fruitful, even if idealized, model in which to study algorithmic and statistical questions. For these reasons, in the last 15 years random geometric graphs have seen a surge of attention in the combinatorics, statistics, and computer science communities. Tasks addressed in the literature include: 1) *Detecting* the presence of a latent geometric structure [4, 5, 11, 12, 14, 16, 18, 26, 28, 29], 2) *Estimating* the dimension of the latent geometry [4, 14, 20], 3) *Embedding* the graph in a geometric space and *clustering* [24, 30, 35], 4) *Matching* unlabelled noisy copies of the same geometric graph [25, 39]. In a different direction of study, high-dimensional random geometric graphs exhibit an intricate and useful combinatorial structure. Most notably, in [27], the authors show that in certain regimes spherical random geometric graphs are efficient *2-dimensional expanders*, objects for which no other simple randomized constructions are known as of now.

Two of the most common models, studied since the early works [14, 16], are spherical and Gaussian (hard thresholds) random geometric graphs.

Definition 1 (Spherical and Gaussian Random Geometric Graphs). The spherical random geometric graph $\text{RGG}(n, \mathbb{S}^{d-1}, p)$ on n vertices $[n] = \{1, 2, \dots, n\}$ of dimension d with expected density p is defined as follows. First, n independent vectors $V_1, \dots, V_n$ are drawn iid from the uniform distribution on the sphere $\mathbb{S}^{d-1}$. Then, an edge between i and j is formed if and only if $\langle V_i, V_j \rangle \geq \tau_d^p$. Here, τ_d^p is chosen so that $p = \mathbb{P}[\langle V_i, V_j \rangle \geq \tau_d^p]$.

Similarly, in the Gaussian case $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$, one samples $Z_1, \dots, Z_n \sim \mathcal{N}(0, \frac{1}{d}I_d)$ and forms an edge (ji) whenever $\langle Z_i, Z_j \rangle \geq \rho_d^p$ with ρ_d^p chosen such that the expected density is p .

The main goal of the current paper is to analyse the low-degree Fourier coefficients of the probability mass functions of those two distributions. The Fourier coefficients of an n -vertex random graph distribution $\mathbf{R}$ are parametrized by edge-subgraphs H . The p -biased Fourier coefficient corresponding to H is defined by

$$\begin{aligned} \Phi_{\mathbf{R}}^p(H) &:= (p(1-p))^{-|E(H)|/2} \times \mathbb{E}_{\mathbf{G} \sim \mathbf{R}} \left[\prod_{(ji) \in E(H)} (\mathbf{G}_{ji} - p) \right] \\ &= (p(1-p))^{-|E(H)|/2} \times \mathbb{E}_{\mathbf{G} \sim \mathbf{R}} \left[\text{SW}_H^p(\mathbf{G}) \right]. \end{aligned} \quad (1)$$

$\text{SW}_H^p(\mathbf{G})$ is the signed weight of H defined by the above equation. Fourier coefficients are (signed) expectations of subgraphs.

Low-degree Fourier coefficients of distributions (and, more generally, Boolean functions) are at the core of many milestone results in theoretical computer science and combinatorics such as constructing succinct nearly k -wise independent distributions [2], learning various classes of Boolean functions [13, 19], the Margulis-Russo formula on sharp-thresholds [33, 38] and many more (see [36]). More recently, low-degree Fourier coefficients have become central to the design of efficient algorithms for problems in high-dimensional statistics, as well as providing evidence for computational hardness, via the *low-degree polynomial* framework [21, 22].

Unfortunately, estimating Fourier coefficients is a highly non-trivial task for complex distributions with dependencies among variables. We introduce a conceptually novel approach (described shortly in Section 3.2) for bounding the Fourier coefficients of distributions with random latent structure and use it for $\text{RGG}(n, \mathbb{S}^{d-1}, p)$ and $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$. This unlocks the powerful methods mentioned above which leads to several applications, described next.

1. Testing. Testing against Erdős-Rényi is one of the most natural and well-studied questions on high-dimensional random geometric graphs, starting with [16]. Testing is a prerequisite for more sophisticated tasks: if one cannot even distinguish a graph from pure noise, one can hardly hope to do any other meaningful inference about its structure.

In the spherical case, one observes a graph G and the goal is to test between the two hypotheses

$$H_0 : G \sim G(n, p) \quad \text{and} \quad H_1 : G \sim \text{RGG}(n, \mathbb{S}^{d-1}, p).$$

The state-of-the-art results for $p \leq 1/2$ and $p = \Theta(1/n)$ are as follows. By counting signed triangles, one succeeds with high probability whenever $d \leq (np)^3 (\log 1/p)^C$ for some constant C [14, 26]. Counting signed triangles is conjectured to be *information theoretically optimal*, i.e., for $d \gg (np)^3 (\log 1/p)^C$ it is believed to be impossible to test between the two graph distributions [5, 12, 26]. The best bounds on when $\text{RGG}(n, \mathbb{S}^{d-1}, p)$ and $G(n, p)$ are indistinguishable, due to [26], are: 1) $d \geq n^3 p^2 (\log 1/p)^C$ for all $p \leq 1/2$; 2) $d \geq (np)^3 (\log 1/p)^C = \text{polylog}(n)$ for $p = \Theta(1/n)$. In particular, the threshold in dimension d at which testing becomes possible is only known (up to lower order terms) when $p = \tilde{\Theta}(1)$ or $p = \Theta(1/n)$.

We make progress in the intermediate regime $1/n \ll p \ll 1/2$ by showing that the signed triangle statistic is *computationally optimal with respect to low-degree polynomial tests* at all densities, even in a stronger non-adaptive edge query model recently introduced by [32]. Surprisingly, we show that this is not the case for Gaussian random geometric graphs. For small p , low-degree tests other than the signed-triangle statistics are much more powerful: when $p = \Theta(1/n)$, one can distinguish $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$ and $G(n, p)$ for dimensions as large as $\sqrt{n}(\log n)^{C'}$, in sharp contrast to the $d = \text{polylog}(n)$ threshold in the spherical case [26].

We additionally prove low-degree indistinguishability between $\text{RGG}(n, \mathbb{S}^{d-1}, \frac{1}{2})$ and a planted coloring model [23] in a regime when both are distinguishable from $G(n, 1/2)$ via simple low-degree tests. The two models can be easily distinguished from one another by determining the largest clique, a computationally *inefficient* test, which shows a computation-information gap for this testing problem. To the best of our knowledge, this is the first negative

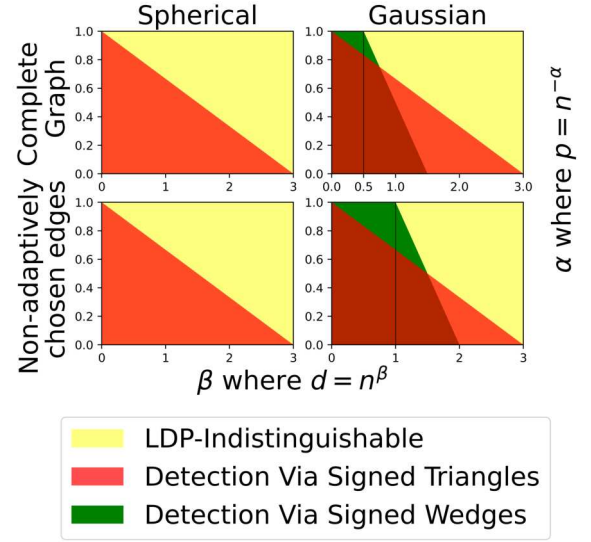


Figure 1: Detecting d -dimensional geometry via low-degree polynomials. In the model of non-adaptively queried edges $\mathcal{M}$, $n := \sqrt{|E(\mathcal{M})|}$. A wedge is a path on 3 vertices.

result on testing between $\text{RGG}(n, \mathbb{S}^{d-1}, \frac{1}{2})$ and a non-geometric distribution when $d \ll n^3$.

2. Spectral properties. The second eigenvalue λ_2 of $G \sim \text{RGG}$ is captured by low-degree polynomials via the trace method. λ_2 naturally plays an important role in the expansion properties of $\text{RGG}(n, \mathbb{S}^{d-1}, p)$ [27]. The top eigenvalues are also used in embedding and clustering random geometric graphs via the top eigenvectors [24]. These works have characterized the behavior of λ_2 : when $d \ll np$, $\lambda_2 = \tilde{\Theta}(np/\sqrt{d})$ and when $d \gg np$, the behaviour is similar to Erdős-Rényi and $\lambda_2 = \tilde{\Theta}(\sqrt{np})$.¹ We reprove this bound in the case $p = 1/2$ using our estimates on the Fourier coefficients. While our approach yields the same quantitative bounds, its methodology is rather different and much more combinatorial.

1.1 Organization of Paper

Our main contribution is a new methodology for deriving strong bounds on Fourier coefficients which we use to argue about the random geometric graph distributions. In Section 3.1 we describe the challenges in bounding low-degree Fourier coefficients followed by the main ideas used to overcome them in Section 3.2. Our main theorem followed by applications to testing and the second eigenvalue are stated in Section 4. In Section 5 we give the full proof of our main theorem. The different applications follow by variations of what are by now well-known techniques and are given in the arXiv version [6]. For testing, we use the χ^2 low-degree advantage formula (when testing against the planted coloring model, we need a more subtle version of it from [23]). For the second eigenvalue, we use the trace method.

¹To be fully precise, [27] considers the normalized adjacency matrix and [24] considers a Gaussian rather than spherical random geometric graph

2 PRELIMINARIES

2.1 Graph Notation

Our graph notation is mostly standard. We denote by $V(H), E(H)$ the vertex and edge sets of a graph. For $i \in V(H)$, we define $N_H(i) := \{j \in V(H) : (ji) \in E(H)\}$ and for $S \subseteq V(H)$, $N_H(S) := \cup_{i \in S} N_H(i)$. $\delta(H) := \max_{S \subseteq V(H)} |S| - |\{j \in V(H) : \exists i \in S \text{ s.t. } (ji) \in E(H)\}|$. We denote by K_n the complete graph on n vertices, and by Star_n the star graph which has one central vertex v and n leaves $i_1, i_2, \dots, i_n$ adjacent to v .

We will frequently define graphs by the edges that induce them. That is, for $A \subseteq E(K_n)$, we also denote by A the graph on vertex set $\{i \in V(K_n) : \exists j \in V(K_n) \text{ s.t. } (ji) \in A\}$ and edge set A . Identifying graphs by edges that span them is convenient as edges are the variables of polynomials that we consider.

2.2 Low-Degree Polynomials

Our results in Sections 4.2 to 4.4 are based on the low-degree polynomial framework introduced in [21, 22]. One way to motivate it is the following. When testing between graph distributions H_0 and H_1 (say, $H_0 = G(n, p), H_1 = \text{RGG}(n, \mathbb{S}^{d-1}, p)$), one observes a single graph G and needs to output 0 or 1. The graph G is simply a bit sequence in $\{0, 1\}^{E(K_n)}$. Hence, the output is a function $f : \{0, 1\}^{E(K_n)} \rightarrow \{0, 1\}$. All Boolean functions are polynomials [36]. Therefore, one simply needs to compute a polynomial in the edges. Importantly, one can write polynomials over $\{0, 1\}^m$ in their Fourier expansion. In the p -biased case over graphs, one represents $f : \{0, 1\}^{E(K_n)} \rightarrow \mathbb{R}$ as

$$f(G) = \sum_{H : H \subseteq E(K_n)} \hat{f}(H) \times (\sqrt{p(1-p)})^{-|E(H)|} \prod_{(ji) \in E(H)} (G_{ji} - p). \quad (2)$$

Here, $\hat{f}(H)$ is just a constant (the Fourier coefficient corresponding to H) and $\{\sqrt{p(1-p)}^{-|E(H)|} \prod_{(ji) \in E(H)} (G_{ji} - p)\}_{H \subseteq E(K_n)}$ is a basis of polynomials. Conveniently, as can be seen from Eq. (1), this basis is composed of signed-subgraphs SW_H^p . What makes it useful is the following fact [36]:

$$\text{The polynomials } \left\{ \sqrt{p(1-p)}^{-|E(H)|} \prod_{(ji) \in E(H)} (G_{ji} - p) \right\}_{H \subseteq E(K_n)} \text{ are orthonormal with respect to } G \sim G(n, p). \quad (3)$$

When computationally restricted, a tester needs to apply a poly-time computable polynomial f . What are classes of poly-time computable polynomials? One such class is of sufficiently low-degree polynomials (where degree refers to the largest number of edges in a monomial corresponding to some H for which $\hat{f}(H)$ is non-zero). Since those are usually not $\{0, 1\}$ -valued, one needs to threshold after computing the polynomial, which leads to the following definition, motivated by Chebyshev's inequality.

Definition 2 (Success of a Low-Degree Polynomial, e.g. [21]). We say that a polynomial $f : \{0, 1\}^{E(K_n)} \rightarrow \{0, 1\}$ distinguishes H_0 and H_1 with high probability if

$$\left| \mathbb{E}_{G \sim H_0} [f(G)] - \mathbb{E}_{G \sim H_1} [f(G)] \right| = \omega \left(\sqrt{\text{Var}_{G \sim H_0} [f(G)] + \text{Var}_{G \sim H_1} [f(G)]} \right).$$

If f is poly-time computable, this leads to the poly-time algorithm which compares $f(G)$ to $(\mathbb{E}_{G \sim H_0} [f(G)] + \mathbb{E}_{G \sim H_1} [f(G)]) / 2$.

Very commonly, one takes f to be a signed subgraph count [14]. That is, for some small graph A (e.g. triangle or wedge), one computes

$$\text{SC}_A^p(G) := \sum_{H \subseteq K_n : H \sim A} \text{SW}_H^p(G), \quad (4)$$

where $H \sim A$ denotes graph isomorphism. I.e., one computes the total number of signed weights.

Importantly, the framework of [21] allows one to refute the existence of low-degree polynomials which distinguish with high probability H_0 and H_1 . Namely, the condition in Definition 2 fails for all low-degree polynomials. Of course, one needs to quantify “low-degree”. Typically, this means degree $O(\log n)$. While not all $O(\log n)$ -degree polynomials are necessarily poly-time computable, the class of $O(\log n)$ -degree polynomials captures a broad class of algorithms including subgraph counting algorithms [21], spectral algorithms [3], SQ algorithms (subject to certain conditions) [9], approximate message passing algorithms (with constant number of rounds) [34], and are in general conjectured to capture all poly-time algorithms for statistical tasks in sufficiently noisy high-dimensional regimes [21].

Definition 3 (Low-Degree Polynomial Hardness). We say that no low-degree polynomial distinguishes H_0 and H_1 with probability $\Omega(1)$ if there exists some $D = \omega(\log n)$ such that

$$\left| \mathbb{E}_{G \sim H_0} [f(G)] - \mathbb{E}_{G \sim H_1} [f(G)] \right| = o \left(\sqrt{\text{Var}_{G \sim H_0} [f(G)] + \text{Var}_{G \sim H_1} [f(G)]} \right)$$

holds for all polynomials of degree at most D . In particular, this holds (e.g., [23]), if $\mathbb{E}_{G \sim H_1} [f(G)] / \mathbb{E}_{G \sim H_0} [f(G)^2] - 1 = o_n(1)$ for all polynomials f of degree at most D .

Using the orthonormality in Eq. (3), this condition simplifies significantly when $H_0 = G(N, p) \odot \mathcal{M}$ for some mask $\mathcal{M}$.² Recall the notation in Eq. (1).

CLAIM 2.1 ([21]). Suppose that $H_0 = G(N, p) \odot \mathcal{M}, H_1 = R \odot \mathcal{M}$ and $D = \omega(\log n)$. If

$$\sum_{H \subseteq E(\mathcal{M}) : 1 \leq |E(H)| \leq D} \left(\Phi_R^p(H) \right)^2 = o_n(1),$$

no low-degree polynomial can distinguish H_0 and H_1 with probability $\Omega(1)$.

For our results on the planted coloring, we need a more sophisticated version of Claim 2.1 due to [23] when $H_0 = \text{PCol}$.

3 CHALLENGES AND MAIN IDEAS

3.1 Challenges in Bounding Low-Degree Fourier Coefficients

The importance of Fourier coefficients of graph distributions has motivated a series of previous works on (hyper)graphs with latent random vectors. Existing methods for computing Fourier coefficients, however, seem not fully adequate towards our goal.

²This fact holds and usually stated for general binary distributions when H_0 is a product distribution [21], but we only state the result in the case of interest.

Approach 0: Direct Integration. The most naive approach to estimating Fourier coefficients is a direct integration (summation) over the latent space. Recalling Eq. (1), one can compute the Fourier coefficient of H by integrating $\prod_{(ji) \in E(H)} (\mathbb{1}[\langle Z_i, Z_j \rangle \geq \rho_d^p] - p)$ against $\mathcal{N}(0, \frac{1}{d}I_d)^{\otimes V(H)}$. Such a calculation, however, seems out of reach due to the complex dependencies between different terms in the product. As latent vectors Z_i, Z_j vary smoothly, so does the distance between Z_i and Z_j and consequently also the probabilities of various events (such as Z_k being a common neighbour). As concrete evidence of the difficulty of this approach, even in the simplest case of triangles for $\text{RGG}(n, \mathbb{S}^{d-1}, p)$, the authors of [14] spend 5 pages of calculations. For a similar random graph model with L_p geometry, the calculation for triangles is open [4].

Approach 1: Vertex Conditioning. Many Fourier computations are for problems defined by planting small dense communities in an ambient Erdős-Rényi graph [17, 22, 23, 32, 37]. In such works, one can use the following simple *vertex conditioning* strategy (exploiting the ambient Erdős-Rényi structure) to overcome the technical difficulty of a direct summation (integration). As a prototypical example, discussed in [7, 21], consider the planted k -clique distribution where each vertex $i \in [n]$ independently receives a label x_i , where $\mathbb{P}[x_i = 1] = k/n$, $\mathbb{P}[x_i = 0] = 1 - k/n$. Conditioned on the labels, each edge G_{ji} appears with probability 1 if $x_i = x_j = 1$ and independently with probability $1/2$ otherwise. Now, consider the Fourier coefficient $\mathbb{E}[\prod_{(ji) \in E(H)} (2G_{ji} - 1)]$ indexed by a graph H without isolated vertices. Again, there are complex correlations between different edges. However, unless all vertices of H have label 1, there is a random (probability $1/2$) edge and this zeros out the Fourier coefficient $\mathbb{E}[\prod_{(ji) \in E(H)} (2G_{ji} - 1)]$. By conditioning on all vertex labels being 1, one shows that the Fourier coefficient equals $(k/n)^{|V(H)|}$. An approach based on *vertex conditioning* seems to not be applicable to hard threshold random geometric graphs: In $\text{RGG}(n, \mathbb{S}^{d-1}, p)$, conditioned on the latent vectors there is no randomness left in $G \sim \text{RGG}(n, \mathbb{S}^{d-1}, p)$. Hence, one cannot exploit cancellations due to left-over randomness in edges once labels are known, which is crucial in models with ambient Erdős-Rényi.

Approach 2: Lifting From a Single Dimension. The work [4] bounds the low-degree Fourier coefficients of (hard threshold) random geometric graphs over the d -dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$ with the L_∞ metric. The insight in [4] is that an L_∞ random geometric graph is the AND of d 1-dimensional random geometric graphs over $\mathbb{S}^1$. They combine the contributions of different coordinates via an analytical approach mimicking the cluster-expansion formula from statistical physics. As explained in [4], in our case of $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$ the edges are closer to MAJORITY over the coordinates. Unfortunately, extending the techniques for the simpler AND combination to the present setting seems to be technically challenging (in particular, because the Fourier expansion of AND is much simpler than that of MAJORITY).

3.2 Main Ideas

We focus on $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), \frac{1}{2})$ for concreteness as this case captures most of the main ideas. The argument for other densities is similar, but requires some modification (most notably, it additionally exploits a novel energy-entropy trade-off of the RGG

distribution; see Remark 5). Modifying to the sphere can be done via the observation that when $Z \sim \mathcal{N}(0, \frac{1}{d}I_d)$, $V = Z/\|Z\|_2 \sim \text{Unif}(\mathbb{S}^{d-1})$, the variables $V, \|Z\|_2$ are independent and $\|Z\|_2$ concentrates strongly around 1. Recall that $\rho_d^{1/2} = 0$. Our goal is to bound $\mathbb{E}_{G \sim \text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), \frac{1}{2})} [\text{SW}_H^{1/2}(G)]$. We assume that $|V(H)| = \text{polylog}(d)$ as this is most relevant to our applications.

Motivation: A Noise-Operator View. The following *noise-operator* interpretation from [9] of the calculation for planted clique will turn out to be useful. Consider first the standard noise operator T_γ for functions over $\{\pm 1\}^n$ [36]. It acts on functions $f : \{\pm 1\}^n \rightarrow \mathbb{R}$ by $T_\gamma f(x) = \mathbb{E}_{y \sim N_\gamma(x)} [f(y)]$, where $N_\gamma(x)$ is the distribution in which each coordinate y_i independently equals x_i with probability γ and otherwise with probability $1 - \gamma$ it is re-randomized. This noise operator contracts Fourier coefficients as $\widehat{T_\gamma f}(S) = \gamma^{|S|} \widehat{f}(S)$.

OBSERVATION 3.1 (NOISE OPERATOR VIEW OF PLANTED CLIQUE). *The planted clique distribution can itself be viewed as arising from application of a different noise operator. Given a function on graphs $f : \{\pm 1\}^{\binom{[n]}{2}} \rightarrow \mathbb{R}$, let $T_\gamma f(x) = \mathbb{E}_{y \sim N_\gamma(x)} [f(y)]$, where now $N_\gamma(x)$ is the distribution obtained by including each vertex in A with probability γ and then rerandomizing all edges in x except those with both endpoints in A . This operator again contracts Fourier coefficients, $\widehat{T_\gamma f}(H) = \gamma^{|V(H)|} \widehat{f}(H)$. If we start with the point mass distribution δ_{K_n} on the complete graph, then the planted clique probability mass function is obtained by applying $T_{k/n}$.*

Our goal will be to derive such a noise operator perspective for RGG as well and use it to bound Fourier coefficients. We formally give such a view in Observation 5.1, but due to its more complicated nature we gradually build towards it. Remarkably, our noise-operator also implies that (at least on a small enough scale), RGG can also be represented as a small planted subgraph in an ambient Erdős-Rényi!

1. Strategy: Localizing Edges That Create Dependencies. We solve the challenges outlined in the previous subsection with the following high-level idea. We will *localize the dependence among edges* to a small set of edges $\mathcal{F} = \mathcal{F}(Z)$ (depending on the latent vectors). The other edges, in $\mathcal{F}^c$, will be close to uniformly random. Edges in $\mathcal{F}$ will in general depend also on edges in $\mathcal{F}^c$, and we write $\partial\mathcal{F}$ as the set of edges upon which those in $\mathcal{F}$ depend. Letting $U \sim \text{Unif}(\{0, 1\}^{\otimes E(H)})$,

$$G = (G_{\mathcal{F}}, G_{\mathcal{F}^c}) \approx (G_{\mathcal{F}}, U_{\mathcal{F}^c}) = (G_{\mathcal{F}}, U_{\partial\mathcal{F} \cap \mathcal{F}^c}, U_{\mathcal{F}^c \setminus \partial\mathcal{F}}).$$

Note that by definition, edges in $\mathcal{F}^c \setminus \partial\mathcal{F}$ are independent of all other edges. Hence, conditioning on $\mathcal{F}$, we can re-randomize $\mathcal{F}^c \setminus \partial\mathcal{F}$ (i.e., apply the noise operator T_0 on $\mathcal{F}^c \setminus \partial\mathcal{F}$). With this idea, we solve both difficulties arising when one attempts the first two approaches outlined before: randomness ensures cancellations and independence makes calculations easy!

2. Key Idea: An Edge-Independent Basis of Latent Vectors. Localizing dependence to a small set of edges $\mathcal{F}$ means that most of the edges are independent. This may seem impossible at first. When we add even a small amount of noise to Z_i , this will likely affect all inner products $\{\langle Z_i, Z_j \rangle\}_{j \in V(H)}$ and, hence, all edges incident to i , in a complicated, correlated, fashion.

To overcome this issue, we define a convenient basis for the latent vectors. Namely, for each edge $(ji) \in E(H)$, we construct a random variable Z_{ji} (depending on latent vectors) such that the collection of random variables $\{Z_{ji}\}_{(ji) \in E(H)}$ is independent and Z_{ji} nearly determines the edge ji . We exploit the fact that independent Gaussian vectors in high dimension are nearly orthonormal. This suggests that the Gram-Schmidt operation on the latent vectors will produce an orthonormal basis close to the original vectors and, hence, projections on the Gram-Schmidt basis will approximate the inner products. Applying Gram-Schmidt to the $k = |V(H)|$ Gaussian vectors $Z_1, \dots, Z_k$ corresponding to vertices of H , we obtain the Bartlett decomposition [8]:

$$\begin{aligned} Z_1 &= (Z_{11}, 0, 0, \dots, 0) \\ Z_2 &= (Z_{21}, Z_{22}, 0, \dots, 0) \\ Z_3 &= (Z_{31}, Z_{32}, Z_{33}, \dots, 0) \\ &\vdots \\ Z_k &= (Z_{k1}, \dots, Z_{k,k-1}, Z_{kk}, 0, \dots, 0). \end{aligned} \quad (5)$$

Here $Z_{ji} \sim \mathcal{N}(0, \frac{1}{d})$ for each $i < j$ and $dZ_{jj}^2 \sim \chi^2(d-j+1)$, $Z_{jj} \geq 0$. The collection $(Z_{ji})_{i \leq j}$ are jointly independent. These properties can be easily derived from the isotropic nature of $\mathcal{N}(0, \frac{1}{d}I_d)$. With respect to this decomposition,

$$\langle Z_i, Z_j \rangle = Z_{ji}Z_{ii} + \sum_{\ell < i} Z_{i\ell}Z_{j\ell} = Z_{ji} + \left(\sum_{\ell < i} Z_{i\ell}Z_{j\ell} + Z_{ji}(Z_{ii} - 1) \right). \quad (6)$$

Now, each term of the form $Z_{i\ell}Z_{j\ell}$, as well as $Z_{ji}(Z_{ii} - 1)$, is typically on the order of $\tilde{O}(1/d)$, so the entire right-hand side expression above is on the order of $\tilde{O}(|V(H)|/d) = \tilde{O}(1/d)$. In contrast, $Z_{ji} \sim \mathcal{N}(0, 1/d)$, so it is typically on the order of $\tilde{O}(1/\sqrt{d})$. Therefore, the random variable Z_{ji} nearly determines whether ji is an edge ($\mathbb{1}[\langle Z_i, Z_j \rangle \geq 0] \approx \mathbb{1}[Z_{ji} \geq 0]$). This is very promising as the variables $\{Z_{ji}\}$ are also independent, so we can define the noise operator by rerandomizing (a subset of) the variables Z_{ji} independently and, thus, affecting edges (ji) independently.

3. Construction: Fragile Edges Localize Dependencies. So far, we constructed independent variables $\{Z_{ji}\}_{(ji) \in E(H)}$ which nearly determine the edges. The key word here is nearly – it may well be the case that $Z_{ji} = \tilde{O}(1/d)$, in which case (and with high probability, only in this case) $Z_{i\ell}Z_{j\ell}$ or $Z_{ji}(Z_{ii} - 1)$ could be comparable to or even larger than Z_{ji} . In that case, $G_{ji} = \mathbb{1}[Z_{ji} + (\sum_{\ell < i} Z_{i\ell}Z_{j\ell} + Z_{ji}(Z_{ii} - 1)) \geq 0]$ depends on edges of the form $(i\ell)$, $(j\ell)$ via the variables $Z_{i\ell}, Z_{j\ell}$. As edges (ji) for which $Z_{ji} = \tilde{O}(1/d)$ are the only ones that can depend on other edges they *localize dependence*. We call edges (ji) for which $Z_{ji} = \tilde{O}(1/d)$ *fragile pairs* and they form the fragile set $\mathcal{F}$. The rest of the edges are independent, as demonstrated by a noise operator rerandomizing all Z_{uv} for non-fragile (uv) (in a way that Z_{uv} continues to be large enough so that (uv) is not fragile).

Recall that Z_{ji} is distributed as $\mathcal{N}(0, 1/d)$ and, hence, is smaller than $\tilde{O}(1/d)$ only with probability $\tilde{O}(1/\sqrt{d})$. As variables Z_{ji} are independent, edges are fragile independently. Thus, the probability of observing many fragile edges is very low.

4. Analysis: Combinatorics of Edge Incidences. Our construction so far is of a noise operator which acts independently on all non-fragile edges $\mathcal{F}^c$. Hence, even if we condition on all fragile edges, there is some randomness left (unless all edges are fragile, but this happens with very low probability) and, so, we have solved the issue of destroying all randomness by conditioning outlined in Section 3.1. However, it is still difficult to integrate $\prod_{(ji) \in E(H)} (2G_{ji} - 1)$ even conditioned on the set of fragile edges. The reason is that if (ji) is fragile, but $(j\ell)$ is not for some $\ell < i$, applying the noise operator on $(j\ell)$ via $Z_{j\ell}$ may also affect $\mathbb{1}[Z_{ji} + (\sum_{\ell < i} Z_{i\ell}Z_{j\ell} + Z_{ji}(Z_{ii} - 1)) \geq 0]$.

Our approach to this issue is simple – we define the noise operator only over edges not incident to fragile edges in their *lexicographically larger vertex* (which we formalize in Definition 6 as $\partial\mathcal{F}$). If there is even a single such edge, the noise operator re-randomizes the edge and zeroes out the Fourier coefficient (as in planted clique).

This leads us to analyzing the *combinatorics of edge incidences of subgraphs of H* . A crucial step in this analysis is the realization that we have the freedom to choose an optimal ordering (with respect to the graph H) for the Gram-Schmidt process so that incidences with lexicographically larger fragile edges (i.e. $|\partial\mathcal{F}|$) are minimized. Optimizing over orderings leads us to a combinatorial quantity associated to the graph H which we call the *ordered edge independence number* $\text{OEI}(H)$. Altogether, our bound on Fourier coefficients becomes $|\mathbb{E}[\prod_{(ji) \in E(H)} (2G_{ji} - 1)]| \leq (\log^c d / \sqrt{d})^{\text{OEI}(H)}$. Our last step is to understand the growth of $\text{OEI}(H)$. We derive several bounds, simplest and most easily interpretable of which is $\text{OEI}(H) \geq \lceil (|V(H)| - 1)/2 \rceil$. Perhaps more interesting is $\text{OEI}(H) \geq \delta(H) + 1$, where $\delta(H) := \max_{S \subseteq V(H)} |S| - |\{j \in V(H) : \exists i \in S \text{ s.t. } (ji) \in E(H)\}|$.

4 RESULTS

We now formally describe our results, beginning with the exact bounds on Fourier coefficients we obtain. Throughout, we will make the following assumption:

$$\text{There exist some absolute constants } \gamma, \epsilon > 0 \text{ such that} \quad (A) \\ 1/2 \geq p \geq n^{-1+\epsilon}, d \geq n^\gamma.$$

Admittedly, some non-trivial cases are not covered by this assumption. Specifically, $p = n^{-1+o(1)}$ and $d = \text{polylog}(n)$. Nevertheless, we note that in the case $p = \Theta(1/n)$, $d = \text{polylog}(n)$ the testing problem between $\text{RGG}(n, \mathbb{S}^{d-1}, p)$ and $\text{G}(n, p)$ is fully resolved by [26] and, thus, Eq. (A) captures most of the open regimes at least for the question of testing against Erdős-Rényi.

4.1 Main Result: The Fourier Coefficients of Gaussian and Spherical RGG

Fourier coefficients of RGG factorize over connected components, so we only state our bounds for H connected. We first define the ordered edge independence mentioned in Section 3.2.

Given an ordering π of the vertices (think of π as the Gram-Schmidt ordering), we denote an edge between u and v as (uv) if $u > v$ and (vu) otherwise. We formalize $\partial\mathcal{F}$ as follows.

Definition 4 (Covering Property). An edge $(uv) \in E(H) \setminus \mathcal{F}$ is *covered* by F if there exists an edge in F with endpoint u . Denote with $\partial_H^\pi F$ the set of all edges covered by F .

Going back to Eq. (6), we interpret as follows. If (uv) is covered by fragile edges F , there exists some fragile edge (uw) . Hence, G_{uw} might depend on G_{uv} via Z_{uv} .

Definition 5 (Ordered Edge Independence Number). For a connected graph H on k vertices and a bijective labelling π of the vertices with the numbers $\{1, 2, \dots, k\}$, we say that a subset of edges $F \subseteq E(H)$ *strongly covers* H if $F \cup \partial_H^\pi F = E(H)$. We define the *ordered edge independence number of H with respect to π* and denote by $\text{OEI}_\pi(H)$ as the size of the smallest strongly covering F . Let $\text{OEI}(H) = \max_\pi \text{OEI}_\pi(H)$.

One should think of $E(H) \setminus (F \cup \partial_H^\pi F)$ as the set of edges which the noise operator rerandomizes.

Illustration of (Strong) Ordered Edge Independence Number.

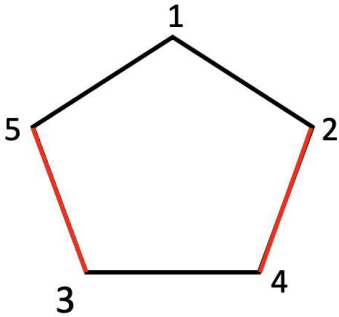


Figure 2: Consider the cycle $H = C_5$ with given labelling π and $F = \{(53), (42)\}$.

Definition 4: Edge (51) is covered by (53) while (21), (43) are covered by (42). Hence, $\partial_H^\pi F = \{(51), (21), (43)\}$. Casework shows that $|F| = \text{OEI}_\pi(H) = \text{OEI}(H) = 2$.

Definition 6: With respect to the strong covering property, $\partial_H^\pi F = \emptyset$. For example, (51) is not strongly covered because $F_{\geq 1, \geq 5}^\pi = \{(53)\}$, but 3 is not a neighbour of 1 in H . However, if we add the edge (43) to F , i.e. $F' = \{(53), (43), (42)\}$, it is the case that both (51) and (21) are strongly covered. Indeed, $(F')_{\geq 1, \geq 5}^\pi = (F')_{\geq 1, \geq 2}^\pi = \{(53), (43), (42)\}$ and 1 is a neighbour to both 2 and 5. Casework shows that $|F'| = \text{SOEI}_\pi(H) = \text{SOEI}(H) = 3$.

THEOREM 4.1. Suppose that Eq. (A) holds and H is connected. Then, there exists some absolute constant C depending only on ϵ, γ in Eq. (A) such that for $G \sim \text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$ and $G \sim \text{RGG}(n, \mathbb{S}^{d-1}, p)$,

$$\begin{aligned} & \left| \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \right] \right| \\ & \leq (8p)^{|E(H)|} \times \left(\frac{C \times |V(H)| \times |E(H)| \times (\log d)^{3/2}}{\sqrt{d}} \right)^{\text{OEI}(H)}. \end{aligned}$$

For applications, we only need $|V(H)|, |E(H)| \leq (\log n)^{1.1}$, in which case the bound is $p^{|E(H)|} (\text{polylog}(n) / \sqrt{d})^{\text{OEI}(H)}$.

To apply Theorem 4.1, we need explicit bounds on $\text{OEI}(H)$.

PROPOSITION 4.2 (BOUNDS ON ORDERED EDGE INDEPENDENCE NUMBER). For a connected graph H ,

- (1) $\text{OEI}(H) \geq \lceil (|V(H)| - 1)/2 \rceil$.
- (2) $\text{OEI}(H) \geq \delta(H) + 1$.

The quantity $\delta(H)$ is useful when discussing the non-adaptive edge query model in which one observes any M edges of a random geometric graph. The reason is that a graph on M edges can have at most $\exp(O(|V(H)|^{1.1})) \times M^{(|V(H)| + \delta(H))/2}$ subgraphs isomorphic to H , as shown in [1].

While the resulting bounds on Fourier coefficients are strong enough for all of our low-degree hardness results, one may still wonder if they are optimal. It turns out that the likely answer is no: In the special case of density $1/2$, the symmetry of the Gaussian distribution around $0 = \rho_d^{1/2} = \tau_d^{1/2}$ allows us to slightly improve the argument outlined in Section 3.2 and define a noise operator that acts also on certain (but not all) edges adjacent to fragile edges. We describe this next.

Definition 6 (Strong Covering). Consider an edge $(uv) \in E(H) \setminus F$ and denote by $F_{\geq v}^\pi$ the subset of F formed by edges with both endpoints at least as large as v . $F_{\geq v, \ni u}^{(\pi)}$ is the connected component of $F_{\geq v}^{(\pi)}$ containing u . We say that (uv) is strongly covered by F if v has a neighbour other than u in $V(F_{\geq v, \ni u}^{(\pi)})$ with respect to H . We denote by $\bar{\partial}_H^\pi F$ the set of edges strongly covered by F .

See Fig. 2 for an illustration. The analogue of Definition 5 is:

Definition 7 (Strong Ordered Independence Number). We define $\text{SOEI}_\pi(H)$, the *strong independence number* of H with respect to π , as the minimal cardinality of a set F such that $F \cup \bar{\partial}_H^\pi F = E(H)$. $\text{SOEI}(H) = \max_\pi \text{SOEI}_\pi(H)$.

The analogue of Theorem 4.1 is the following (we state it only for $\text{RGG}(n, \mathbb{S}^{d-1}, \frac{1}{2})$ as the Gaussian and spherical models coincide in the $1/2$ -density case).

PROPOSITION 4.3. Suppose that Eq. (A) holds and H is connected. Then, there exists some absolute constant C depending only on ϵ, γ in Eq. (A) such that for $G \sim \text{RGG}(n, \mathbb{S}^{d-1}, \frac{1}{2})$,

$$\begin{aligned} & \left| \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - 1/2) \right] \right| \\ & \leq (1/2)^{|E(H)|} \times \left(\frac{C \times |V(H)| \times |E(H)| \times (\log d)^{3/2}}{\sqrt{d}} \right)^{\text{SOEI}(H)}. \end{aligned}$$

If an edge (uv) is strongly covered, then the connected component $F_{\geq v, \ni u}^{(\pi)}$ contains a neighbour of u . Hence, there exists a fragile edge with endpoint u and u is also covered according to Definition 6. Thus, $\text{SOEI}(H) \geq \text{OEI}(H)$, so Proposition 4.3 is at least as strong as Theorem 4.1. It turns out that the inequality is strict for many sparse graphs. For example, one can check that when $H = C_k$ is a cycle, $\text{OEI}(C_k) = \lceil (k-1)/2 \rceil$, but $\text{SOEI}(H) = k-2$. The latter follows from the following bound:

PROPOSITION 4.4 (SOEI AND SPARSE GRAPHS). Suppose that H is connected. Then, $\text{SOEI}(H) \geq 2|V(H)| - |E(H)| - 2$.

It turns out that the $\text{OEI}(H)$ bound is too weak for our results on the second eigenvalue of $\text{RGG}(n, \mathbb{S}^{d-1}, \frac{1}{2})$ and we need $\text{SOEI}(H)$ and Proposition 4.4. We leave open the problem of proving Proposition 4.3 for all densities p .

4.2 Application I: Testing Between Spherical RGG and Erdős-Rényi

In the case of spherical random geometric graphs, we not only confirm that the signed triangle statistic is optimal among low-degree polynomial tests, but also show that this is the case even in the non-adaptive edge-query model recently introduced by [32]. For a mask $M \in \{0, 1\}^{N \times N}$ and (adjacency) matrix $A \in \mathbb{R}^{N \times N}$, denote by $A \odot M$ the $N \times N$ array in which $(A \odot M)_{ji} = A_{ji}$ whenever $M_{ji} = 1$ and $(A \odot M)_{ji} = ?$ whenever $M_{ji} = 0$.

Testing between graph distributions with masks corresponds to a non-adaptive edge query model. Instead of viewing a full graph, one can choose to observe a smaller more structured set of edges in order to obtain a more data-efficient algorithm. The number of edges M of M is a natural proxy for “sample complexity” in the case of low-degree polynomials as the input variables of low-degree polynomials are edges rather than vertices. This idea was introduced recently in [32], focusing on the planted clique problem. We obtain the following result for RGG. In it, we use $\tilde{n} := \sqrt{M}$ as a lower-bound on the number of vertices of M . The variable $\tilde{n}$ is useful both in phrasing the assumptions (A) and in comparing with the unmasked case.

THEOREM 4.5. *Consider some M where $\tilde{n} = \sqrt{M}$, d, p satisfy the assumptions in Eq. (A). Let M be any graph on M edges without isolated vertices. Denote by N the number of vertices in M . If $d \geq (M^{1/2}p)^{3+c}$ for any constant $c > 0$, no degree $(\log M)^{1.1}$ polynomial can distinguish with probability $\Omega(1)$ the distributions $G_0 \odot M$ and $G_1 \odot M$ for $G_0 \sim G(N, p)$ and $G_1 \sim \text{RGG}(N, \mathbb{S}^{d-1}, p)$.*

In the case $M = K_n$, we match the conjectured information-theoretic threshold. **Theorem 4.5** is tight in light of the signed triangle statistic [26].

COROLLARY 4.1. *Consider some n, d, p satisfying assumptions in Eq. (A). If $d \geq (np)^{3+c}$ for any positive constant c , no degree $(\log n)^{1.1}$ polynomial can distinguish with probability $\Omega(1)$, the distributions $G(n, p)$ and $\text{RGG}(n, \mathbb{S}^{d-1}, p)$.*

4.3 Application II: Testing Between Gaussian RGG and Erdős-Rényi

We begin with a brief comparison of the Gaussian and spherical models.

Remark 1 (Gaussian vs Spherical Random Geometric Graphs). The Gaussian and spherical models coincide in the case $p = 1/2$. More generally, they are intimately related due to the facts

$$\text{If } Z \sim \mathcal{N}(0, \frac{1}{d}I_d), \text{ then } V := Z/\|Z\|_2 \sim \text{Unif}(\mathbb{S}^{d-1}), \text{ and} \quad (\text{I})$$

$$\text{If } Z \sim \mathcal{N}(0, \frac{1}{d}I_d), \text{ then } \|Z\|_2 \approx 1 \text{ with high probability.} \quad (\text{II})$$

This correspondence has been used to argue about either model – in some arguments more helpful is independence of Gaussian coordinates [14, 16] while in others orthonormality of the Gegenbauer basis over the sphere [24]. We exploit this correspondence in both directions.

We also show that the two models are qualitatively different in the sparse regime (see Fig. 1). The cause of this difference is the perhaps benign looking fact that Eq. (II) is only an approximate

statement. This creates dependence between edges in the Gaussian case for edges which are independent in the spherical case: For example, under $G \sim \text{RGG}(n, \mathbb{S}^{d-1}, p)$, the edges G_{21} and G_{31} are independent. In contrast, under $H \sim \text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$, H_{21}, H_{31} are positively correlated as both are monotone in $\|Z_1\|_2$. The dependence turns out to be quite strong for small values of p to the point where (signed) wedges are better than signed triangles for testing against Erdős-Rényi.

THEOREM 4.6. *Consider testing between $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$ and $G(n, p)$ under Eq. (A).*

- (1) *When $d \geq \max\{(n^3p^3)^{1+c}, (n^{3/2}p)^{1+c}\}$ for any constant $c > 0$, no degree $(\log n)^{1.1}$ algorithm can distinguish the two graph models with probability $\Omega(1)$.*
- (2) *When $d \leq n^{3/2}p(\log n)^{-5}$ and $p \leq 0.49$, the signed wedge count succeeds w.h.p.*
- (3) *When $d \leq n^3p^3(\log n)^{-5}$, the signed triangle count succeeds w.h.p.*

In the non-adaptive query complexity model, the difference turns out to be even more dramatic. One can exploit the fact that wedges are highly informative by querying a star-graph, as star graphs maximize the number of wedges for a fixed number of edges.

THEOREM 4.7. *Consider some M where $\tilde{n} = \sqrt{M}$, d, p satisfy the assumptions in Eq. (A). Let M be any graph on M edges with no isolated vertices. Let N be the number of vertices in M . Consider testing between $G_0 \odot M$ and $G_1 \odot M$ for $G_0 \sim G(N, p)$, $G_1 \sim \text{RGG}(N, \mathcal{N}(0, \frac{1}{d}I_d), p)$.*

- (1) *When $d \geq \max\{(M^{3/2}p^3)^{1+c}, (Mp)^{1+c}\} = \max\{(\tilde{n}^3p^3)^{1+c}, (\tilde{n}^2p)^{1+c}\}$ for any constant $c > 0$, no degree $(\log M)^{1.1}$ algorithm can distinguish the two graph models with probability $\Omega(1)$ for any mask M .*
- (2) *When $d \leq Mp(\log M)^{-5} = \tilde{\Theta}(\tilde{n}^2p)$ and $p \leq 0.49$, the signed wedge count succeeds w.h.p. when M is a union of $A = \lceil (\log M)^{17} \rceil$ disjoint stars with $\lfloor M/A \rfloor$ edges each.*
- (3) *When $d \leq (M^{3/2}p^3)(\log M)^{-5} = \tilde{\Theta}(\tilde{n}^3p^3)$, the signed triangle count succeeds w.h.p. when M is $K_{\tilde{n}}$.*

The main message of Section 4.3 is that even though the Gaussian and spherical models are closely related and each useful for reasoning about the other, they are also fundamentally different.

The proofs are similar to the ones in Section 4.2, except that we need to take extra care of graphs with leaves (as their Fourier coefficients are non-zero, unlike in the spherical case).

Remark 2. The work [10] studies the convergence of masked Wishart matrices to GOE (that is $A_{ji} = \langle Z_i, Z_j \rangle$ instead of $G_{ji} = \mathbb{1}[\langle Z_i, Z_j \rangle \geq \rho_d^p]$). Of course, for $p = o(1)$ the RGG testing problem becomes very different from the Wishart versus GOE problem [12, 26].

4.4 Application III: Testing Between Spherical RGG and Planted Coloring

In the regime $d \leq (np)^{3-c}$, RGG is very different from Erdős-Rényi. But is it, perhaps, closely approximated by some other simple model? We show that, with respect to low-degree polynomial tests, $\text{RGG}(n, \mathbb{S}^{d-1}, 1/2)$ is indistinguishable from a slight variation of

the planted coloring distribution in [23]. We focus on the density 1/2 case, but our arguments can be easily extended (we only use Theorem 4.1, not Proposition 4.3).

Definition 8. $\text{PCol}(n, q)$ is the following distribution over n vertex graphs. First, each node $i \in [n]$ independently receives a uniform label $x_i \in [q]$. Then, if $x_i = x_j$, nodes i and j are adjacent with probability 1. If $x_i \neq x_j$, nodes i and j are adjacent with probability $\frac{1}{2} - \frac{1}{2(q-1)}$.

In comparison, [23] have i and j adjacent with probability 1/2 when $x_i \neq x_j$. Choosing a value of q so that the signed triangle counts of $\text{RGG}(n, \mathbb{S}^{d-1}, \frac{1}{2})$ and $\text{PCol}(n, q)$ (nearly) match, we prove the following fact.

THEOREM 4.8. *Suppose that $d \geq n^{8/3+c}$ for any constant $c > 0$. Then, there exists some $q \in [d^{1/4}/(\log d), d^{1/4}(\log d)]$ such that no $(\log n)^{1.1}$ -degree polynomial test distinguishes $\text{PCol}(n, q)$ and $\text{RGG}(n, \mathbb{S}^{d-1}, 1/2)$ with probability $\Omega(1)$.*

Remark 3. The condition $q = \tilde{\Theta}(d^{1/4})$ establishes a statistical-computational gap when $d \leq n^{4-\kappa}$ for any constant $\kappa > 0$. An instance of $\text{PCol}(n, q)$ has a clique of size $n/q = \tilde{\Omega}(nd^{-1/4})$ with probability 1. However, $\text{RGG}(n, \mathbb{S}^{d-1}, \frac{1}{2})$ does not contain a clique of size more than $3 \log_2 n$ with high probability under Eq. (A) by [16]. Perhaps surprisingly, our result holds in the exact same regime as the results of [23] for refuting q -colorability. Namely, $q = \tilde{\Theta}(d^{1/4}), d \geq n^{8/3}$ is equivalent to $q = \tilde{\Omega}(n^{2/3})$. Our contribution here is not the analysis, but the realisation that RGG is indistinguishable from PCol. We prove hardness for detecting q -colorability against the natural PCol model and do not need to construct a more sophisticated “quiet distribution” as in [23].

Remark 4. [15] studies a similar question for Wishart matrices in the regime $d = o(n^3)$ when Wishart and GOE are distinguishable. The authors obtain a sequence of phase transitions for the Wishart density. The approximating densities are defined in terms of an inverse Fourier transform and are not easily interpretable, in contrast to the simple PCol distribution.

4.5 Application IV: The Second Eigenvalue of Spherical RGG

THEOREM 4.9. *Suppose that $G \sim \text{RGG}(n, \mathbb{S}^{d-1}, 1/2)$ (equivalently $G \sim \text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), \frac{1}{2})$).*

- If $d \leq n(\log n)^8$, then $|\lambda_2(G)| \leq n(\log n)^{10}/\sqrt{d}$ w.h.p.
- If $d \geq n(\log n)^8$, then $|\lambda_2(G)| \leq (\log n)^{10}\sqrt{n}$ w.h.p.

Here, we need the strong bounds in Proposition 4.4 for sparse graphs. As these bounds provably do not hold for OEI, more work is needed to extend to $p \neq 1/2$.

5 PROVING THE BOUNDS ON FOURIER COEFFICIENTS

Here, we prove our bounds on Fourier coefficients of random geometric graphs by formalizing the argument in Section 5. Specifically, in Section 5.1, we prove Theorem 4.1. In Section 5.3, we modify the argument slightly to prove the stronger Proposition 4.3 in the density 1/2 case. In Section 5.2 we prove the bounds on the edge independence numbers stated in Propositions 4.2 and 4.4.

5.1 The Main Argument in Theorem 4.1

Fix a connected graph H on $k = |V(H)|$ vertices and $m = |E(H)|$ edges such that $Ckm(\log d)^{3/2} \leq \sqrt{d}$. Let π be any bijective labeling of its vertices by $[k]$. We will identify vertices by their labelling in π and optimize over π at the end. We prove Theorem 4.1 in the Gaussian setting and state the necessary modifications for the spherical setting at the end.

Step 1: High-Probability Bound on $\sum_{\ell < i} Z_{i\ell}Z_{j\ell} + Z_{ji}(Z_{ii} - 1)$. Recall Eq. (6). As discussed, dependence between edges is due to the term

$$Q_{ji}^{N(0, \frac{1}{d}I_d)}(\{Z_{j\ell}\}_{1 \leq \ell \leq j}, \{Z_{i\ell}\}_{1 \leq \ell \leq i}) := \sum_{\ell < i} Z_{i\ell}Z_{j\ell} + Z_{ji}(Z_{ii} - 1). \quad (7)$$

We bound the size of this term, along the way introducing notation that will be used later. Let

$$[R_L, R_U] = \left[-C\sqrt{\frac{m \log d}{d}}, C\sqrt{\frac{m \log d}{d}} \right]$$

be the “reasonable interval” for each summand in (7): By Gaussian and χ^2 -concentration, for any desired constant C' , there exists some absolute constant C such that under Eq. (A) (which implies $\log 1/p = O(\log d)$ and $\log n = O(\log d)$) we have $\mathbb{P}[Z_{i\ell} \in [R_L, R_U]] \geq 1 - e^{-C'm \log d}$ and the same for $(Z_{ii} - 1)$. Denote by $\mathcal{RSC}$ the “reasonable set of configurations”:

$$\{Z_{uv} \in [R_L, R_U] \text{ and } (Z_{uu} - 1) \in [R_L, R_U] \text{ for all } 1 \leq u \leq v \leq k\}.$$

By the union bound, its complement has probability

$$\mathbb{P}[\mathcal{RSC}^c] \leq k^2 e^{-C'm \log d} \leq p^m d^{-m}. \quad (8)$$

As $Q_{ji}^{N(0, \frac{1}{d}I_d)}$ is a sum of at most k terms of order $(C\sqrt{m \log d/d})^2$ under the high probability event $\mathcal{RSC}$, we conclude that with probability at least $1 - p^m d^{-m}$, simultaneously for all $(ji) \in E(H)$,

$$\left| Q_{ji}^{N(0, \frac{1}{d}I_d)}(\{Z_{j\ell}\}_{1 \leq \ell \leq j}, \{Z_{i\ell}\}_{1 \leq \ell \leq i}) \right| \leq C^2 \frac{km \log d}{d} = \Delta \quad (9)$$

where we defined $\Delta := C^2 \frac{km \log d}{d}$. We condition on $\mathcal{RSC}$. Since $|\prod_{(ji) \in E(H)} (G_{ji} - p)| < 1$ a.s.,

$$\begin{aligned} & \left| \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \right] \right| \\ & \leq \left| \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \middle| \mathcal{RSC} \right] \right| + \mathbb{P}[\mathcal{RSC}^c] \\ & \leq \left| \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \middle| \mathcal{RSC} \right] \right| + p^m d^{-m}. \end{aligned} \quad (10)$$

As $p^m d^{-m} = p^{|E(H)|} d^{-|E(H)|} \leq p^{|E(H)|} \sqrt{d}^{-\text{OEI}(H)}$, it remains only to bound the first term.

Step 2: Fragile Edges. Observe that under the high probability event in Eq. (9), as long as $Z_{ji} \notin [\rho_d^p - \Delta, \rho_d^p + \Delta]$, it is the case that

$$\mathbb{1}[Z_{ji} \geq \rho_d^p] = \mathbb{1}[Z_{ji} + Q_{ji}^{N(0, I_d)} \geq \rho_d^p] = \mathbb{1}[Z_{ji} \geq \rho_d^p]$$

and variables Z_{ji} are independent (even conditioned on $\mathcal{RSC}$). Thus, all edges besides the ones for which Z_{ji} is close to ρ_d^p are independent. We localize dependence to the following fragile edges.

Definition 9 (Fragile Interval and Fragile Edges). Denote $F_L = \rho_d^p - \Delta$ and $F_U = \rho_d^p + \Delta$. The *fragile interval* is $[F_L, F_U]$ and an edge ji is called *fragile* if $Z_{ji} \in [F_L, F_U]$.

Note that each edge is fragile independently as $\{Z_{uv}\}_{1 \leq u \leq v \leq k}$ are independent. Let $\mathcal{F}$ be the set of fragile edges. Now, $Z_{ji} \sim \mathcal{N}(0, 1/d)$ and $\Delta = C^2 \frac{km \log d}{d} = o(1/\sqrt{d}) = o(\rho_d^p / \log d)$ imply

$$\mathbb{P}[Z_{ji} \in [F_L, F_U]] \leq \Delta p \sqrt{C'' d \log d} \quad (11)$$

for some absolute constant C'' , because $[F_L, F_U]$ has length Δ and the Gaussian density around ρ_d^p is $\tilde{O}(p\sqrt{d})$ as $p = \mathbb{P}[\langle Z_1, Z_2 \rangle \geq \rho_d^p] = \mathbb{P}[Z_{11}Z_{21} \geq \rho_d^p] \approx \mathbb{P}[Z_{21} \geq \rho_d^p]$ and $Z_{21} \sim \mathcal{N}(0, 1/d)$. Conditioning on the fragile set yields

$$\begin{aligned} & \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \middle| \mathcal{RSC} \right] \\ &= \sum_{F \subseteq E(H)} \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \middle| \mathcal{RSC}, \mathcal{F} = F \right] \times \\ & \quad \times \mathbb{P}[\mathcal{F} = F | \mathcal{RSC}] \\ &\leq 2 \sum_{F \subseteq E(H)} \left| \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \middle| \mathcal{RSC}, \mathcal{F} = F \right] \right| \times \\ & \quad \times (\Delta p \sqrt{C'' d \log d})^{|F|}. \end{aligned} \quad (12)$$

We used the fact that $\mathbb{P}[\mathcal{RSC}] \geq 1/2$ so $\mathbb{P}[\mathcal{F} = F | \mathcal{RSC}] \leq 2\mathbb{P}[\mathcal{F} = F]$. This last conditioning is useful, because our noise operator depends on the set of fragile edges.

Step 3: The noise operator. Conditioned on the reasonable event $\mathcal{RSC}$ and on the set of fragile edges $\mathcal{F} = F$, we define the following noise operator. It rerandomizes all variables Z_{uv} such that the value of Z_{uv} does not appear in the expression $Q_{ji}^{\mathcal{N}(0, I_d)}$ for any fragile $(ji) \in F$. In particular, as $Q_{ji}^{\mathcal{N}(0, I_d)}$ is only a function of $\{Z_{j\ell}\}_{1 \leq \ell \leq j}, \{Z_{i\ell}\}_{1 \leq \ell \leq i}$, we rerandomize all $(uv) \in E(H) \setminus (F \cup \partial_H^\pi(F))$. For $(uv) \in E(H) \setminus (F \cup \partial_H^\pi(F))$, conditioned on $\mathcal{RSC}$ and $\mathcal{F} = F$, the variable Z_{uv} uniquely determines G_{uv} (as (uv) is not fragile). Furthermore, $Z_{(uv)}$ is independent of $(G_{ji})_{(ji) \in E(H) \setminus \{(uv)\}}$ (as there is no fragile edge of the form (ji) for which $(uv) \in \{j\ell\}_{1 \leq \ell \leq j} \cup \{i\ell\}_{1 \leq \ell \leq i}$). For clarity and uniformity with [Observation 3.1](#), we spell this out separately.

OBSERVATION 5.1 (NOISE OPERATOR VIEW ON RGG). *The noise operator T_p^π on the distribution $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$ is parametrized by an ordering of the vertices π and marginal edge probability p . To sample from RGG, one first samples a fragile set $\mathcal{F}$ by including each edge independently with probability $\mathbb{P}[Z_{ji} \in [F_L, F_U]]$ (recall that edges are fragile independently). $\mathcal{F}$ together with π determines $\partial_H^\pi \mathcal{F}$. Then, one samples $H_{\mathcal{F} \cup \partial_H^\pi \mathcal{F}}$ from the marginal distribution on edges $\mathcal{F} \cup \partial_H^\pi \mathcal{F}$ from the distribution $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$ conditioned on $\mathcal{F}$ being the fragile set with respect to π . Conditioned on $\mathcal{F}$, $H_{\mathcal{F} \cup \partial_H^\pi \mathcal{F}}$, T_p^π acts independently on edges with the following noise rates:*

$$T_p^\pi(G)_{uv} = \begin{cases} H_{uv} & \text{for } (uv) \in \mathcal{F} \cup \partial_H^\pi \mathcal{F} \text{ (case of no noise),} \\ \text{Bern}(\tilde{p}) & \text{otherwise (case of full noise).} \end{cases}$$

Above, $\tilde{p} := \mathbb{P}[Z_{ji} \geq \rho_d^p | Z_{ji} \in [R_L, R_U] \setminus [F_L, F_U]] \approx p$. That is, there is no noise on the edges in $\mathcal{F} \cup \partial \mathcal{F}$ and the rest of the

edges are fully rerandomized. Hence, $T_p^\pi(\delta_{K_n})$ is a sample from $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d}I_d), p)$.

Phrased differently, the above noise operator represents RGG restricted to the edges of H as a small planted subgraph in an ambient Erdős-Rényi $G(n, \tilde{p})$.

One key difference with [Observation 3.1](#) is that the distribution of the edge set $H_{\mathcal{F} \cup \partial_H^\pi \mathcal{F}}$ is much more complicated than the distribution on the planted clique A . The latter is simply a clique, while $H_{\mathcal{F} \cup \partial_H^\pi \mathcal{F}}$ is a subgraph of a random geometric graph which is further conditioned on its fragile set.

Nevertheless, just as in [Observation 3.1](#), the independent rerandomization over the rest of the edges $(\mathcal{F} \cup \partial_H^\pi \mathcal{F})^c$ yields an (exponentially fast) decay of Fourier coefficients, which we discuss next.

We separate the conditional signed expectation into the portion rerandomized by the noise operator, i.e. $E(H) \setminus (F \cup \partial_H^\pi(F))$, and a portion that is not rerandomized, i.e. $F \cup \partial_H^\pi(F)$:

$$\begin{aligned} & \left| \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \middle| \mathcal{RSC}, \mathcal{F} = F \right] \right| \\ &= \left| \prod_{(ji) \in E(H) \setminus (F \cup \partial_H^\pi(F))} \mathbb{E}[(G_{ji} - p) | \mathcal{RSC}, \mathcal{F} = F] \times \right. \\ & \quad \left. \mathbb{E} \left[\prod_{(ji) \in F \cup \partial_H^\pi(F)} (G_{ji} - p) \middle| \mathcal{RSC}, \mathcal{F} = F \right] \right| \\ &\leq \prod_{(ji) \in E(H) \setminus (F \cup \partial_H^\pi(F))} \left| \mathbb{E}[\mathbb{1}[Z_{ji} \geq \rho_d^p] - p] \right| \\ & \quad \left| Z_{ji} \in [R_L, R_U] \setminus [F_L, F_U] \right| \\ & \quad \times \mathbb{E} \left[\prod_{(ji) \in F \cup \partial_H^\pi(F)} |G_{ji} - p| \middle| \mathcal{RSC}, \mathcal{F} = F \right]. \end{aligned} \quad (13)$$

Next, we bound the factors for $E(H) \setminus (F \cup \partial_H^\pi(F))$ and then the factors for $F \cup \partial_H^\pi(F)$.

Step 4: Factors $E(H) \setminus (F \cup \partial_H^\pi(F))$ Rerandomized By The Noise Operator. A simple calculation with 1-dimensional Gaussian variables using [Eqs. \(8\)](#) and [\(11\)](#) and the fact that $\mathbb{P}[Z_{ji} \geq \rho_d^p] = p + O(p(\log d)/\sqrt{d})$ gives

$$\left| \mathbb{E}[\mathbb{1}[Z_{ji} \geq \rho_d^p] - p | Z_{ji} \in [R_L, R_U] \setminus [F_L, F_U]] \right| \leq \Delta p \sqrt{C'' d \log d}. \quad (14)$$

Step 5: Factors $F \cup \partial_H^\pi(F)$. Convexity of $|\cdot|$, the fact $G_{ji} \leq 1$ a.s., and triangle inequality give

$$\begin{aligned} & \mathbb{E} \left[\prod_{(ji) \in F \cup \partial_H^\pi(F)} |G_{ji} - p| \middle| \mathcal{RSC}, \mathcal{F} = F \right] \\ &\leq \sum_{B \subseteq F \cup \partial_H^\pi(F)} p^{|F \cup \partial_H^\pi(F)| - |B|} \mathbb{E} \left[\prod_{(ji) \in B} G_{ji} \middle| \mathcal{RSC}, \mathcal{F} = F \right] \\ &\leq \sum_{B \subseteq F \cup \partial_H^\pi(F)} p^{|F| + |\partial_H^\pi(F)| - |B|} \mathbb{E} \left[\prod_{(ji) \in B \setminus F} G_{ji} \middle| \mathcal{RSC}, \mathcal{F} = F \right]. \end{aligned} \quad (15)$$

For each $(ji) \in B \setminus F$, the edge G_{ji} is determined by Z_{ji} , conditioned on $\mathcal{RSC}, \mathcal{F} = F$. Thus,

$$\begin{aligned} & \mathbb{E} \left[\prod_{(ji) \in B \setminus F} G_{ji} \mid \mathcal{RSC}, \mathcal{F} = F \right] \\ &= \mathbb{E} \left[\mathbb{1}[Z_{ji} \geq \rho_d^p] \mid \mathcal{RSC}, \mathcal{F} = F \right]^{|B| - |F|}. \end{aligned}$$

By Eq. (14), the last expression is at most $(2p)^{|B| - |F|}$. Plugging this into Eq. (15),

$$\begin{aligned} & \mathbb{E} \left[\prod_{(ji) \in F \cup \partial_H^\pi(F)} |G_{ji} - p| \mid \mathcal{RSC}, \mathcal{F} = F \right] \\ & \leq \sum_{B \subseteq F \cup \partial_H^\pi(F)} p^{|F| + |\partial_H^\pi(F)| - |B|} \times (2p)^{|B| - |F|} \leq 4^{|E(H)|} (2p)^{|\partial_H^\pi(F)|}. \end{aligned} \quad (16)$$

Step 6: Putting It All Together. Plugging (14) and (16) into (13), the conditional Fourier coefficients are bounded as

$$\begin{aligned} & \left| \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \mid \mathcal{RSC}, \mathcal{F} = F \right] \right| \\ & \leq (\Delta p \sqrt{C''' d \log d})^{|E(H)| - |F| - |\partial_H^\pi(F)|} \times 4^{|E(H)|} \times (2p)^{|\partial_H^\pi(F)|}. \end{aligned}$$

We combine this with Eq. (12) to obtain

$$\begin{aligned} & \left| \mathbb{E} \left[\prod_{(ji) \in E(H)} (G_{ji} - p) \mid \mathcal{RSC} \right] \right| \\ & \leq 2 \sum_{F \subseteq E(H)} (\Delta p \sqrt{C''' d \log d})^{|E(H)| - |F| - |\partial_H^\pi(F)|} \\ & \quad \times 4^{|E(H)|} \times (2p)^{|\partial_H^\pi(F)|} \times (\Delta p \sqrt{C''' d \log d})^{|F|} \\ & \leq 2(8p)^{|E(H)|} (\Delta \sqrt{C d \log d})^{|E(H)| - |\partial_H^\pi(F)|} \\ & \leq 2(8p)^{|E(H)|} \left(\frac{Ckm(\log d)^{3/2}}{\sqrt{d}} \right)^{|E(H)| - |\partial_H^\pi(F)|} \end{aligned}$$

All that is left to show is $|E(H)| - |\partial_H^\pi(F)| \geq \text{OEI}_\pi(H)$. This follows immediately because for any $F \subseteq E(H)$, the set $E(H) \setminus \partial_H^\pi(F)$ satisfies the covering properties Definition 4 (note that $F \subseteq E(H) \setminus \partial_H^\pi(F)$). Finally, we can choose π as the maximizer of $\text{OEI}_\pi(H)$ and conclude Theorem 4.1 in the Gaussian case.

Remark 5 (Energy-Entropy Trade-off). $|E(H)| - |\partial_H^\pi(F)| \geq \text{OEI}_\pi(H)$ highlights the following energy-entropy trade-off phenomenon in RGG. Rewrite it as $|F| + |E(H) \setminus (F \cup \partial_H^\pi(F))| \geq \text{OEI}_\pi(H)$. The term $|E(H) \setminus (F \cup \partial_H^\pi(F))|$ corresponds to entropy in the distribution as it is the size of the subset of edges which the noise operator rerandomizes (and are independent with all edges in $E(H)$). The term $|F|$ measures energy as $\mathcal{F} = F$ is the subset of edges with non-trivial interactions (dependence) with other edges in $E(H)$. The inequality shows that energy and entropy cannot both be small, and either one being large results in small Fourier coefficient: entropy due to randomness and energy due to low probabilities.

Step 5 In The Spherical Case. The analysis of the spherical case is nearly the same. We generate $V_1, V_2, \dots, V_k \sim_{iid} \text{Unif}(\mathbb{S}^{d-1})$ as $V_i = Z_i / \|Z_i\|_2$, where $Z_1, Z_2, \dots, Z_k \sim_{iid} \mathcal{N}(0, \frac{1}{d} I_d)$. Then, we apply the Gram-Schmidt process on $Z_1, Z_2, \dots, Z_k$. With respect to

the Bartlett decomposition,

$$\langle V_i, V_j \rangle = \frac{\langle Z_i, Z_j \rangle}{\|Z_i\|_2 \times \|Z_j\|_2} = Z_{ji} + Q_{(ji)}^{\mathbb{S}^{d-1}}(\{Z_{j\ell}\}_{1 \leq \ell \leq j}, \{Z_{i\ell}\}_{1 \leq \ell \leq i}),$$

where

$$\begin{aligned} & Q_{(ji)}^{\mathbb{S}^{d-1}}(\{Z_{j\ell}\}_{1 \leq \ell \leq j}, \{Z_{i\ell}\}_{1 \leq \ell \leq i}) := \\ & \left(\frac{\sum_{\ell < i} Z_{i\ell} Z_{j\ell} + Z_{ji}(Z_{ii} - 1)}{\|Z_i\|_2 \times \|Z_j\|_2} + Z_{ji} \times \left(\frac{1}{\|Z_i\|_2 \times \|Z_j\|_2} - 1 \right) \right). \end{aligned}$$

The function $Q_{(ji)}^{\mathbb{S}^{d-1}}$ depends on the exact same set of variables as $Q_{ji}^{N(0, \frac{1}{d} I_d)}$ and takes value in $[-\frac{C''' pmk \log d}{\sqrt{d}}, \frac{C''' pmk \log d}{\sqrt{d}}]$ under $\mathcal{RSC}$. The rest of the analysis is identical.

5.2 Bounds on the Ordered Edge Independence Numbers

PROOF OF FIRST PART OF PROPOSITION 4.2. We first bound $\text{OEI}_\pi(H)$ by another quantity. Denote by $\mathcal{SN}_\pi(H)$ the set of vertices j of H for which there exists some $i < j$ such that $(ji) \in E(H)$, i.e. with a smaller neighbour. Then, $\text{OEI}_\pi(H) \geq \lceil |\mathcal{SN}_\pi(H)|/2 \rceil$. To prove this, suppose, for the sake of contradiction, that there exists a set of $r < |\mathcal{SN}_\pi(H)|/2$ edges $(j_1, i_1), (j_2, i_2), \dots, (j_r, i_r)$ that satisfy the covering properties from Definition 4. Since $r < |\mathcal{SN}_\pi(H)|/2$, $\{j_1, i_1, j_2, i_2, \dots, j_r, i_r\} < |\mathcal{SN}_\pi(H)|$. Thus, there exists some vertex $j^* \in \mathcal{SN}_\pi(H)$ such that $j^* \notin \{j_1, i_1, j_2, i_2, \dots, j_r, i_r\}$. Let $i^* < j^*$ be a vertex such that $(j^* i^*)$ is an edge. Such an i^* exists by the definition of $\mathcal{SN}_\pi(H)$. But then, clearly, $(j^* i^*) \notin \partial_H^\pi F$. This contradicts the fact that the edges $(j_1, i_1), (j_2, i_2), \dots, (j_r, i_r)$ satisfy the covering properties.

Now, we need to show that $\max_\pi |\mathcal{SN}_\pi(H)| \geq |V(H)| - 1$ for connected H . Let T be a rooted spanning tree of H . Define π to be any labelling of H , such that for all i , all vertices on level $i + 1$ have a larger label than the vertices on level i . Clearly, the root is the only vertex without a neighbour with a smaller label. $\square$

PROOF OF SECOND PART OF PROPOSITION 4.2. Recall that $\delta(H) := \max_{S \subseteq V(H)} \{|S| - |N_H(S)|\}$, where $N_H(S) := \{j \in V(H) : \exists i \in S \text{ s.t. } (ji) \in E(H)\}$. The inequality $\text{OEI}(H) \geq \delta(H) + 1$ is clear if $\delta(H) = 0$, so we assume throughout that $\delta(H) \geq 1$. We need the following fact.

PROPOSITION 5.1 ([1]). *There exists an independent set $S' \in \arg \max_S \{|S| - |N_H(S)|\}$.*

PROOF. Take any $S'' \in \arg \max_S \{|S| - |N_H(S)|\}$. Then, $S' = S'' \setminus N(S'')$ is an independent set and satisfies $|S'| - |N_H(S')| \geq |S''| - |N_H(S'')|$. $\square$

Let $S' \in \arg \max_S |S| - |N_H(S)|$ be independent and $N_H(S') = \{u_1, u_2, \dots, u_\ell\}$. Define the sets

- (1) $S'_1 = N_H(u_1) \cap S'$ and
- (2) $S'_i = (N_H(u_i) \setminus (N_H(u_1) \cup N_H(u_2) \cup \dots \cup N_H(u_{i-1}))) \cap S'$.

Take any ordering π such that the following vertices appear in the following decreasing order:

$$S'_1, u_1, S'_2, u_2, \dots, S'_\ell, u_\ell, V(H) \setminus (S' \cup N_H(S')),$$

i.e. $\pi(x) > \pi(u_1)$ for all $x \in S'_1$, $\pi(u_1) > \pi(y)$ for all $y \in S'_2$, and so on. Let F be any set satisfying the covering properties in [Definition 4](#) with respect to this ordering. Observe that for each $k \in [\ell]$, $v \in S'_k$, the vertex u_k is the only neighbour of v among $S'_1 \cup S'_2 \cdots \cup S'_k \cup \{u_1, u_2, \dots, u_k\}$. The set F must include the edge (vu_k) as otherwise $(vu_k) \notin \partial_H^\pi F$. As this holds for each $k \in [\ell]$, $v \in S'_k$, there are at least $\sum_{k=1}^\ell |S'_k| = |S'| = \delta(H) + |N(S')| \geq \delta(H) + 1$ edges in F . $\square$

PROOF OF PROPOSITION 4.4. Suppose that there exists a set F satisfying the strong covering property for which $|F| < 2|V(H)| - |E(H)| - 2$. In particular, the graph defined by vertices $V(H)$ and edges F has $t \geq |V(H)| - (2|V(H)| - |E(H)| - 2) = |E(H)| + 2 - |V(H)|$ connected components. Let $C_1, C_2, \dots, C_t$ be these connected components. Since H is connected, there exist at least $t - 1$ pairs of different connected components C_i, C_j with an edge between them. However, if (uv) is such an edge between different connected components, where $u \in C_i, v \in C_j$ and $u > v$, v must have another neighbour in C_i as F satisfies the strong covering property in [Definition 6](#). Thus, whenever there is an edge between C_i and C_j , there are at least two such edges. Hence, the total number of edges in H is at least

$$2(t-1) + \sum_{i=1}^t (|V(C_i)| - 1) = |V(H)| + t - 2 > |E(H)|,$$

which is a contradiction. $\square$

Remark 6. This proof holds for any ordering π . We expect that choosing an optimal π will yield an improved bound.

5.3 Improving The Bound in The Density 1/2 Case in Proposition 4.3

The density 1/2 case is special as there is a measure preserving map between $[R_L, F_L]$ and $[F_U, R_U]$ which also preserves norms – namely, $\Xi(Z)$, where $\Xi(Z) := -Z$. In particular, the analogue of [Eq. \(14\)](#) is

$$\mathbb{E} \left[\mathbb{1}[Z_{ji} \geq \rho_d^{1/2}] - 1/2 \mid Z_{ji} \in [R_L, R_U] \setminus [F_L, F_U] \right] = 0.$$

Thus, unless $(F \cup \partial_H^\pi F) = \emptyset$, the expression in [Eq. \(13\)](#) is equal to 0. This immediately yields an improved bound on the Fourier coefficient of the form $2 \times p^{|E(H)|} \times (C''' p k m (\log d)^{3/2} / \sqrt{d})^{\text{SOE}(\pi(H))}$.

A more powerful noise operator. The map $\Xi(Z)$, however, allows us to do more. One can apply a noise operator on certain edges adjacent to fragile edges. The reason is that

$$\begin{aligned} Q_{ji}^{N(0, \frac{1}{d} I_d)}(\{Z_{j\ell}\}_{1 \leq \ell \leq j}, \{Z_{i\ell}\}_{1 \leq \ell \leq i}) \\ = Q_{ji}^{N(0, \frac{1}{d} I_d)}(\{\Xi(Z_{j\ell})\}_{1 \leq \ell \leq j, i \neq j}, \{\Xi(Z_{i\ell})\}_{1 \leq \ell < i}, Z_{ii}, Z_{ji}) \end{aligned}$$

and similarly for the spherical analogue $Q_{ji}^{\mathbb{S}^{d-1}}$.

Namely, condition on $\mathcal{F} = F$. Take any $(ji) \in E(H) \setminus F$ such that, furthermore, j is the unique neighbour of i according to H in $V(F_{\geq i, \ni j}^\pi)$ (equivalently, $(ji) \in E(H) \setminus (F \cup \partial_H^\pi F)$ by [Definition 6](#)). This means that the operation $(Z_{ai})_{a \in V(F_{\geq i, \ni j}^\pi)} \rightarrow (\Xi(Z_{ai}))_{a \in V(F_{\geq i, \ni j}^\pi)}$ changes G_{ji} to $1 - G_{ji}$ but leaves all other edges G_{uv} unchanged. Indeed, consider the cases for (uv) :

- (1) $(uv) = (ja)$ for some $a \in V(F_{\geq i, \ni j}^\pi)$. However, (ji) is the unique edge with this property.
- (2) (uv) is not fragile and not of the form (ja) for some $a \in V(F_{\geq i, \ni j}^\pi)$. Then, Z_{uv} remains unchanged and so does G_{uv} as uv is not fragile.
- (3) (uv) is fragile and at least one of u and v is not in $V(F_{\geq i, \ni j}^\pi)$. Then, either $\min(u, v) \leq i$, in which case G_{uv} can only depend on Z_{ai} via its norm, Z_{ai}^2 (this follows from the definitions of $Q_{uv}^{N(0, \frac{1}{d} I_d)}$ and $Q_{uv}^{\mathbb{S}^{d-1}}$). However, $Z_{ai}^2 = \Xi(Z_{ai})^2$.
- (4) (uv) is fragile and $u, v \in V(F_{\geq i, \ni j}^\pi)$, in which case G_{uv} depends on terms of the form Z_{ai} via the product $Z_{ui}Z_{vi}$, but $Z_{ui}Z_{vi} = \Xi(Z_{ui})\Xi(Z_{vi})$ (again, this follows from the definitions of $Q_{uv}^{N(0, \frac{1}{d} I_d)}$ and $Q_{uv}^{\mathbb{S}^{d-1}}$).

With this property in mind, we define the following noise operator: for each edge $(ji) \in E(H) \setminus (F \cup \partial_H^\pi F)$, one independently applies with probability 1/2 the operation $(Z_{ai})_{a \in V(F_{\geq i, \ni j}^\pi)} \rightarrow (\Xi(Z_{ai}))_{a \in V(F_{\geq i, \ni j}^\pi)}$. As long as there is at least one such edge, the corresponding expected signed weight $\text{SW}_H^{1/2}$ becomes zero.

By [Definition 6](#), this is always the case if $|F| < \text{SOE}(\pi(H))$. An analogous argument to the one in [Section 5.1](#) allows us to bound

$$\begin{aligned} \left| \mathbb{E} \left[\prod_{ji \in E(H)} (G_{ji} - 1/2) \right] \right| \\ \leq 2 \times p^{|E(H)|} \times (C''' p k m (\log d)^{3/2} / \sqrt{d})^{\text{SOE}(\pi(H))}. \end{aligned}$$

Optimizing over the labelling π yields [Proposition 4.3](#).

Remark 7 (Why not other densities?). In principle, we could have carried out the same argument for other densities by fixing some measurable bijection $\Xi_p : [R_L, F_L] \rightarrow [F_U, R_U], \Xi_p : [R_L, F_L] \rightarrow [F_L, R_L]$. Then, we apply it independently to variables $(Z_{ai})_{a \in V(F_{\geq i, \ni j}^\pi)}$ for edges $(ji) \in E(H) \setminus (F \cup \partial_H^\pi F)$ with some probability $\tilde{p}$ so that marginal distributions remain Gaussian. The difficulty with this approach is that for essentially any other density besides 1/2, the equality $Z_{ui}Z_{vi} = \Xi_p(Z_{ui})\Xi_p(Z_{vi})$ will not hold. Thus, the values of $Q^{N(0, \frac{1}{d} I_d)}$ (resp. $Q^{\mathbb{S}^{d-1}}$) will change and so it is not clear how fragile edges are affected by the respective noise operator.

6 DISCUSSION

We introduced a novel strategy for bounding the Fourier coefficients of graph distributions with high-dimensional latent geometry. It is based on localizing dependence to few edges and applying a noise operator to (some of) the remaining edges. Not only is this method useful for our concrete goal of bounding Fourier coefficients, but it also explains how and where dependence among edges is created. In the setting of $\text{RGG}(n, \mathcal{N}(0, \frac{1}{d} I_d), p)$, $\text{RGG}(n, \mathbb{S}^{d-1}, p)$, dependence is localized to fragile edges. We anticipate future applications of the fragile edges approach.

One future direction is tightening our bounds. Extending [Proposition 4.3](#) to all densities is appealing as it will also extend [Theorem 4.9](#). Can one improve further or is [Proposition 4.3](#) tight (up to lower-order terms)?

A related question to the tightness of [Proposition 4.3](#) is finding lower bounds and precise estimates of the Fourier coefficients. Those are useful for the design of low-degree algorithms. Our upper bounds on Fourier coefficients are mostly suited to showing hardness.

Finally, the information-theoretic counterparts of many of the questions addressed in this paper remain open. Is it possible to prove such information theoretic convergence using χ^2 -like arguments based on squares of Fourier coefficients? A simple calculation shows that bounds scaling as $p^{|E(H)|}d^{-\theta|V(H)|}$ for constant θ (of which form [Theorem 4.1](#) is) are insufficient to show $\chi^2(\text{RGG}(n, \mathbb{S}^{d-1}, p) \| G(n, p)) = o_n(1)$. One could hope to surpass this barrier by using a tensorization argument [\[26, 29\]](#) and/or conditional χ^2 -divergence [\[17, 31\]](#).

ACKNOWLEDGMENTS

We thank Chenghao Guo for insightful discussions at the initial stages of this project and Dheeraj Nagaraj for many conversations on random geometric graphs over the years. We are also grateful to three anonymous reviewers for the feedback and suggestions on the exposition. KB was supported by a Siebel Scholarship. GB is supported by NSF Career Award CCF-1940205.

REFERENCES

- [1] Noga Alon. 1981. On the number of subgraphs of prescribed type of graphs with a given number of edges. *Israel Journal of Mathematics* 38 (1981), 116–130. <https://doi.org/10.1007/BF02761855>
- [2] Noga Alon, Oded Goldreich, Johan Hästad, and René Peralta. 1992. Simple Constructions of Almost k -wise Independent Random Variables. *Random Structures & Algorithms* 3, 3 (1992), 289–304. <https://doi.org/10.1002/rsa.3240030308>
- [3] Afonso S. Bandeira, Dmitriy Kunisky, and Alexander S. Wein. 2020. Computational Hardness of Certifying Bounds on Constrained PCA Problems. In *11th Innovations in Theoretical Computer Science Conference (ITCS 2020) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 151)*, Thomas Vidick (Ed.), Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 78:1–78:29. <https://doi.org/10.4230/LIPIcs.ITCS.2020.78>
- [4] Kiril Bangachev and Guy Bresler. 2023. Detection of L_∞ Geometry in Random Geometric Graphs: Suboptimality of Triangles and Cluster Expansion. arXiv:2310.14501 [math.ST]
- [5] Kiril Bangachev and Guy Bresler. 2023. Random Algebraic Graphs and Their Convergence to Erdos-Renyi. arXiv:2305.04802 [math.PR]
- [6] Kiril Bangachev and Guy Bresler. 2024. On The Fourier Coefficients of High-Dimensional Random Geometric Graphs. arXiv:2402.12589 [math.ST]
- [7] Boaz Barak, Samuel B Hopkins, Jonathan Kelner, Pravesh Kothari, Ankur Moitra, and Aaron Potechin. 2016. A nearly tight sum-of-squares lower bound for the planted clique problem. In *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*. IEEE, 428–437. <https://doi.org/10.1137/17M1138236>
- [8] M. S. Bartlett. 1934. On the Theory of Statistical Regression. *Proceedings of the Royal Society of Edinburgh* 53 (1934), 260–283. <https://doi.org/10.1017/S0370164600015637>
- [9] Matthew Brennan, Guy Bresler, Samuel B. Hopkins, Jerry Li, and Tselil Schramm. 2021. Statistical Query Algorithms and Low-Degree Tests Are Almost Equivalent. *Conference on Learning Theory* (2021).
- [10] Matthew Brennan, Guy Bresler, and Brice Huang. 2021. De Finetti-Style Results for Wishart Matrices: Combinatorial Structure and Phase Transitions. arXiv:2103.14011
- [11] Matthew Brennan, Guy Bresler, and Brice Huang. 2022. Threshold for Detecting High Dimensional Geometry in Anisotropic Random Geometric Graphs. [10.48550/arXiv.2206.14896](https://arxiv.org/abs/2206.14896) To appear in Random Structures and Algorithms.
- [12] Matthew Brennan, Guy Bresler, and Dheeraj M. Nagaraj. 2019. Phase transitions for detecting latent geometry in random graphs. *Probability Theory and Related Fields* 178 (2019), 1215 – 1289. <https://doi.org/10.1007/s00440-020-00998-3>
- [13] Nader Bshouty and Christino Tamon. 1996. On the Fourier Spectrum of Monotone Functions. *J. ACM* 43 (04 1996). <https://doi.org/10.1145/234533.234564>
- [14] Sébastien Bubeck, Jian Ding, Ronen Eldan, and Miklós Rácz. 2014. Testing for high-dimensional geometry in random graphs. *Random Structures & Algorithms* 49 (11 2014). <https://doi.org/10.1002/rsa.20633>
- [15] Didier Chételat, Martin T Wells, et al. 2019. The middle-scale asymptotics of Wishart matrices. *The Annals of Statistics* 47, 5 (2019), 2639–2670. <https://doi.org/10.1214/18-AOS1760>
- [16] Luc Devroye, András Györfi, Gábor Lugosi, and Frederic Udina. 2011. High-Dimensional Random Geometric Graphs and their Clique Number. *Electronic Journal of Probability* 16 (2011), 2481 – 2508. <https://doi.org/10.1214/EJP.v16-967>
- [17] Abhishek Dhawan, Cheng Mao, and Alexander S. Wein. 2023. Detection of Dense Subhypergraphs by Low-Degree Polynomials. arXiv:2304.08135 [cs.DS]
- [18] Ronen Eldan and Dan Mikulincer. 2020. *Information and Dimensionality of Anisotropic Random Geometric Graphs*. Springer International Publishing, Cham, 273–324. https://doi.org/10.1007/978-3-030-36020-7_13
- [19] Alexandros Eskenazis and Paata Ivanisvili. 2022. Learning low-degree functions from a logarithmic number of random queries. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (Rome, Italy) (STOC 2022)*. Association for Computing Machinery, New York, NY, USA, 203–207. <https://doi.org/10.1145/3519935.3519981>
- [20] Tobias Friedrich, Andreas Göbel, Maximilian Katzmann, and Leon Schiller. 2023. A simple statistic for determining the dimensionality of complex networks. arXiv:2302.06357 [cs.SI]
- [21] Samuel Hopkins. 2018. STATISTICAL INFERENCE AND THE SUM OF SQUARES METHOD.
- [22] S. B. Hopkins and D. Steurer. 2017. Efficient Bayesian Estimation from Few Samples: Community Detection and Related Problems. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE Computer Society, Los Alamitos, CA, USA, 379–390. <https://doi.org/10.1109/FOCS.2017.42>
- [23] Pravesh Kothari, Santosh S. Vempala, Alexander S. Wein, and Jeff Xu. 2023. Is Planted Coloring Easier than Planted Clique?. In *Annual Conference Computational Learning Theory*. <https://api.semanticscholar.org/CorpusID:257255498>
- [24] Shuangping Li and Tselil Schramm. 2023. Spectral clustering in the Gaussian mixture block model. arXiv:2305.00979 [stat.ML]
- [25] Suqi Liu and Morgane Austern. 2024. Random Geometric Graph Alignment with Graph Neural Networks. arXiv:2402.07340 [cs.LG]
- [26] Siqi Liu, Sidhanth Mohanty, Tselil Schramm, and Elizabeth Yang. 2022. Testing thresholds for high-dimensional sparse random geometric graphs. In *Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing (Rome, Italy) (STOC 2022)*. Association for Computing Machinery, New York, NY, USA, 672–677. <https://doi.org/10.1145/3519935.3519989>
- [27] Siqi Liu, Sidhanth Mohanty, Tselil Schramm, and Elizabeth Yang. 2023. Local and Global Expansion in Random Geometric Graphs. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing (Orlando, FL, USA) (STOC 2023)*. Association for Computing Machinery, New York, NY, USA, 817–825. <https://doi.org/10.1145/3564246.3585106>
- [28] Suqi Liu and Miklos Racz. 2021. Phase transition in noisy high-dimensional random geometric graphs. arXiv:2103.15249
- [29] Suqi Liu and Miklós Z. Rácz. 2023. A probabilistic view of latent space graphs and phase transitions. *Bernoulli* 29, 3 (2023), 2417 – 2441. <https://doi.org/10.3150/22-BE1547>
- [30] Zhuang Ma, Zongming Ma, and Hongsong Yuan. 2020. Universal Latent Space Model Fitting for Large Networks with Edge Covariates. *J. Mach. Learn. Res.* 21 (2020), 4:1–4:67. <https://api.semanticscholar.org/CorpusID:211529745>
- [31] Cheng Mao, Alexander S. Wein, and Shenduo Zhang. 2024. Information-Theoretic Thresholds for Planted Dense Cycles. arXiv:2402.00305 [math.ST]
- [32] Jay Mardia, Kabir Aladin Verchand, and Alexander S. Wein. 2024. Low-degree phase transitions for detecting a planted clique in sublinear time. arXiv:2402.05451 [cs.DS]
- [33] G. A. Margulis. 1974. Probabilistic Characteristics of Graphs with Large Connectivity. *Probl. Peredachi Inf.* 10 (1974), 101–108. Issue 2.
- [34] Andrea Montanari and Alexander S. Wein. 2022. Equivalence of Approximate Message Passing and Low-Degree Polynomials in Rank-One Matrix Estimation. arXiv:2212.06996 [math.ST]
- [35] Luke O’Connor, Muriel Médard, and Soheil Feizi. 2020. Maximum Likelihood Embedding of Logistic Random Dot Product Graphs. *Proceedings of the AAAI Conference on Artificial Intelligence* 34 (04 2020), 5289–5297. <https://doi.org/10.1609/aaai.v34i04.5975>
- [36] Ryan O’Donnell. 2014. *Analysis of Boolean Functions*. Cambridge University Press, Cambridge. <https://doi.org/10.1017/CBO9781139814782>
- [37] Cynthia Rush, Fiona Skerman, Alexander S. Wein, and Dana Yang. 2023. Is It Easier to Count Communities Than Find Them?. In *14th Innovations in Theoretical Computer Science Conference, ITCS 2023, January 10–13, 2023, MIT, Cambridge, Massachusetts, USA (LIPIcs, Vol. 251)*, Yael Tauman Kalai (Ed.), 94:1–94:23. <https://doi.org/10.4230/LIPIcs.ITCS.2023.94>
- [38] Lucio Russo. 1981. On the critical percolation probabilities. *Probability Theory and Related Fields* 56 (06 1981), 229–237. <https://doi.org/10.1007/BF00535742>
- [39] Haoyu Wang, Yihong Wu, Jiaming Xu, and Israel Yolou. 2022. Random Graph Matching in Geometric Models: the Case of Complete Graphs. In *Proceedings of Thirty Fifth Conference on Learning Theory (Proceedings of Machine Learning Research, Vol. 178)*. PMLR, 3441–3488. <https://proceedings.mlr.press/v178/wang22a.html>