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Bounds and dualities of Type II Little String Theories

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Abstract: We explore the symmetry structure of Type II Little String Theories and their T-dualities. We construct these theories both from the bottom-up perspective starting with seed Superconformal Field Theories, and from the top-down using F-/M-theory. By exploiting anomaly inflow and unitarity of the LST worldsheet theory, we derive strong conditions on the possible 6D bulk theories and their flavor algebras. These constraints continue to apply if gravity is coupled to the theory. We also study the higher form symmetry structure of these theories and show how they get exchanged under T-duality. Finally, we comment on seemingly consistent bottom-up Little String Theories that cannot be constructed from the top-down approach.

Keywords: F-Theory, Global Symmetries, M-Theory, String Duality

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1 Introduction

Six-dimensional supersymmetric Quantum Field Theories (SQFTs) can have two different ultraviolet (UV) complete behaviors: one corresponds to theories that have emergent superconformal symmetry and a local energy-momentum tensor, and are characterized by a tower of tensionless strings [1–3]. The second type of theories admits strings in the spectrum that do not become tensionless in the UV. The tension of these strings sets a characteristic scale M_S, which defines the UV cutoff above which a Hagedorn growth of states emerges [4]. This is very characteristic of the closed-string sector, which contains the graviton, and although LSTs are not coupled to gravity, the UV behavior of both theories are therefore very similar. Just like SCFTs, LSTs admit a "tensor branch" description below M_s, which may not be perturbative, but allows for a description in terms of ordinary 6D quantum fields, such as vector, hyper-, and tensor multiplets.

SUGRA, LSTs and SCFTs do not exist in isolation from each other: both LSTs and SCFTs can be obtained by a careful decoupling limit of a gravity theory that sends the Planck mass to infinity while preserving a non-trivial interacting theory. In particular, one may view LSTs and SCFTs as a generic sub-sector of (almost any) gravity theory. The converse, however, is generally not true: only a finite subset of theories lying in the infinite landscape of SCFTs and LSTs can be coupled to gravity in a consistent fashion. This is also clear from the expectation that the landscape of supergravity theories is finite.[1]

LSTs and SCFTs also very closely related: in fact, there are various "tensor-decoupling limits" in which the Little String scale is taken to infinity, that give rise to SCFTs. It has been proposed that a plethora of 6D SCFTs may be obtained as a decoupling limit of a (not necessarily unique) LST [8]. Conversely, most SCFTs can be "affinized" — in a fashion described below — to at least one LST.[2] Thus the landscape of a generic UV-complete SQFT will consist mostly of LSTs and deformations thereof.

String theory — and by extension F-theory — provides an elegant and consistent way of constructing and studying SUGRAs, LSTs, and SCFTs via geometry. This systematic approach may therefore give us hints about the nature of their landscape, and the potential consistency conditions that are yet missing from a low-energy point of view. Furthermore, our current understanding of symmetries has recently undergone rapid acceleration, initiated in [10], see e.g. [11–13] and references therein for an overview of recent progress. Guided by geometry, a first step towards the study of LSTs and their non-local properties is to map out their symmetry and duality structures. In the context of six-dimensional QFTs, recent work shows that these theories generally admit a wide range of higher-form symmetries (HFS) [14–23]. In this regard, LSTs are special, since they possess a unique continuous global 1-form symmetry mixing non-trivially with other conventional 0-form symmetries to form a continuous 2-group [24, 25].[3] This improved understanding of symmetries therefore yields a finer characterization of the theories and provides novel invariants across dualities [26].

These invariants, together with their geometric engineering, give us additional leverage to chart the space of LSTs modulo T-duality, and has been used recently to study LSTs of Heterotic type [27–32]. These theories are defined by a choice of either $G_{\text{Het}} = E_8 \times E_8$ or $G_{\text{Het}} = \text{Spin}(32)/\mathbb{Z}_2$ Heterotic string theory, and by a stack of M NS5 branes probing a transverse ADE-type singularity $\mathbb{C}^2/\Gamma$, whose worldvolume is described by the 6D theory.[4] We denote these theories by

$$\mathcal{K}^{G_{\text{Het}}}(\mathfrak{g}, \mu)_M \,, \tag{1.1}$$

where $\mathfrak{g}$ is the algebra associated with the singularity, and μ is a homomorphism embedding $\Gamma \to G_{\text{Het}}$. These holonomies act on the 10D gauge groups G_{Het}, which lives on non-compact

[1] Although believed to belong to the Swampland, new classes of models seemingly consistent with all known field-theory constraints have been constructed in [5–7].

[2] In [9], the author proposes the existence of SCFTs that do not descend from LSTs.

[3] In contrast, 6D SCFTs may admit *discrete* 2-group symmetries [15].

[4] For work on twisted T-duals in theories with eight supercharges, see [33–35], and for generalization to frozen F-theory vacua see [36].

flavor 9-branes in the Hořava-Witten or Type I dual pictures. Unsurprisingly, this flavor symmetry structure can be elegantly geometrized into the Picard lattice of an elliptically polarized K3 fiber, from which other T-dual theories can be systematically be derived [28, 29].

This work aims to extend this exploration to the second type of LSTs finding their origin in Type II string theories. We will therefore refer to them as *Type II LSTs*, in contrast to the *Heterotic LSTs* described above. This type of theories is instead specified by two choices of singularities $(\mathfrak{g}_F, \mathfrak{g}_B)$, that are also of ADE type (or possibly of the special Kodaira Type II, III, IV). We denote these theories by

$$\mathcal{K}^{\text{II}}(\mathfrak{g}_F, \mathfrak{g}_B)\,. \tag{1.2}$$

In the Type IIB description, $\mathfrak{g}_F$ and $\mathfrak{g}_B$ may be viewed as 7- and 5-brane singularities, respectively.[5]

The absence of a 10D gauge group that could become generic 6D flavor branes severely restricts this class of models as compared to Heterotic LSTS. Moreover, it allows for non-trivial higher-form symmetries: first, there is a diagonal 1-form symmetry sector acting diagonally on gauge group factors [18], and secondly, there may be a non-trivial string defect group [37]. In this work we show explicitly that these higher-form symmetries $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ have a natural relation to the defining singularities via the centers Z of the associated algebras,

$$\mathcal{D}^{(1)}:\ Z(\mathfrak{g}_F)\,, \qquad \mathcal{D}^{(2)}:\ Z(\mathfrak{g}_B)\,. \tag{1.3}$$

Being yet another set of generalized symmetries, it is natural to expect that they are preserved under T-duality. Indeed, we explicitly show that T-duality acts as an exchange of the defining singularities

$$\mathfrak{g}_F \overset{\text{T-Duality}}{\longleftrightarrow} \mathfrak{g}_B \tag{1.4}$$

and hence, exchanges the higher form symmetry sectors.

Geometry gives both a beautiful and self-consistent way to construct either kind of LSTs from the top down, which can be used to prove the relation (1.4). From the bottom-up perspective, there are *a priori* substantially more theories that satisfy all known consistency conditions than those that can be realized via geometric engineering. This is true even when considering the frozen phase of F-theory, see [38, 39], and when gravity is coupled to the theory. Novel consistency conditions have recently been uncovered in the presence of extended BPS strings, whose worldsheet theories receive current algebra contributions from bulk symmetries via anomaly inflow. Unitarity considerations for these extended objects then allow to constrain bulk symmetries in SUGRA theories [40–42]. Similar BPS strings are also present in theories without gravity such as SCFTs and LSTs, and one might therefore expect analogous consistency conditions to play a similarly important role even when gravity is decoupled. Typically, many such BPS strings are present in a generic 6D theory, which makes a general analysis rather cumbersome, and the same is true for LSTs.[6] However, as their name suggests, Little String Theories have a universal characteristic string which receives contributions only from the bulk flavor symmetries for which strong constraints can be derived.

[5]In this work we denote $\mathfrak{g}_F$ as the fibral singularity while $\mathfrak{f}$ denotes the flavor algebra.

[6]Rank-one SCFTs on the other hand are simple enough to extract universal features of BPS strings [43].

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This work is structured as follows. In section 2, we give a general account of 6D LSTs from a field theory perspective: in section 2.1 we set notation and review basic concepts, and section 2.2 focuses in particular on the generalized symmetry structure of Type II and Heterotic LSTs, and we propose that they are new T-duality invariants. In section 2.3, we derive new bounds on the flavor symmetry of these theories by demanding unitarity of the Little String worldsheet description. In section 3, we give a bottom-up construction of Type II LSTs and discuss their dualities and HFS structures. In section 4 we give a top-down construction of Type II LST and present a geometric framework in which T-duality is manifest via a double fibration structure. We give an outlook and conclusion in section 5. Appendix A we summarize the possible Type II LSTs, appendix B gives more details on six- and two-dimensional SCFTs, and appendix C summarizes SUSY enhancements and possible obstructions in the 5D M-theory picture.

Note added. While writing up this work, a related article [44] appeared on the arXiv, which has some overlap with our results of matching discrete higher-form symmetries across T-duality of LSTs as described in section 2.2.

2 The symmetry structure of Little Strings

Many Little String Theories (LSTs) can be engineered by fusing 6D SCFTs together [8]. This vast landscape of theories can be ordered by their symmetries. Moreover, different LSTs can be related by T-duality, either in pairs or in some cases in larger families of T-dual theories. The notion of T-duality for LSTs is the same as for the 10D $E_8 \times E_8$ and Spin(32)/$\mathbb{Z}_2$ Heterotic strings: both theories are different in 10D, but after compactifying both theories on a circle and moving along their lower-dimensional moduli space, ones reaches a point where both theories are identical.[7] This, however, allows one to define certain 6D invariants that must match across each T-dual theory:

1. The 5D Coulomb branch dimension, $\dim(\text{CB}) = n_T + \text{rk}(G)$, where n_T is the number of tensor multiplets and $rk(G)$ is the rank of the gauge group.
2. The universal 2-group structure constants: (κ_P, κ_R) [26].
3. The rank of the 0-form flavor symmetry algebra $\mathfrak{f}$ and group $G_{\mathfrak{f}}$, $\text{rk}(\mathfrak{f})$ [26, 32, 45].
4. The 5D 1-form symmetry sector $\mathcal{D}^{(1)}_{6D} \times \mathcal{D}^{(2)}_{6D} \rightsquigarrow \mathcal{D}^{(1)}_{5D}$, which receives contribution from the 6D 1-form symmetry and the defect group as discussed in section 2.2.

These invariants — which will be reviewed in more detail below — needs to match individually across T-duality, and therefore severely constrain a candidate dual theory. Not all of the invariants listed above are independent; in fact, the most central one is the 2-group structure constant κ_P, which can only take two values [8]:

$$\begin{aligned} \kappa_P &= 2 \qquad \text{for Heterotic LST}\,, \\ \kappa_P &= 0 \qquad \text{for Type II LST}\,. \end{aligned} \tag{2.1}$$

[7] Phrased differently, T-duality means that there is a KK theory that can be UV completed by two (or more) higher dimensional theories.

In terms of F-theory, this is equivalent to the statement that LSTs can be constructed from non-compact elliptically fibered Calabi-Yau threefolds. These geometries can only have two types of birational bases,

$$B_2^{\text{Het}} = \mathbb{P}^1 \times \mathbb{C}\,, \qquad \text{and} \qquad B_2^{\text{T-II}} = (\mathbb{T}^2 \times \mathbb{C})/\Lambda\,, \tag{2.2}$$

where $\Lambda \subset \mathrm{SU}(2)$ is a discrete group associated with the algebra $\mathfrak{g}$. Physically, we may interpret the structure constants κ_P in three different ways:

1. from their 10D LST origin, which is either Type II or Heterotic 5-branes;
2. as the number of flavor 9-branes;
3. on the LST 2D worldsheet, the quantity $8\kappa_p$ counts the number of 3–7 string defects [46].

All three perspectives are consistent with each other: κ_p determines possible non-trivial bulk flavor symmetries. The three perspectives will allow us to derive new consistency conditions from the LST worldsheet in section 2.3.

2.1 Review

Before delving into the analysis of Type II LSTs and their symmetries, we shortly review the six-dimensional generalized quivers and the main properties of LSTs, which we use as an opportunity to set our notation and conventions. For a more in-depth treatment of generalized quivers and their F-theory construction, we refer the reader to the reviews [29, 47].

The tensor branch of a six-dimensional $\mathcal{N} = (1, 0)$ theory can be described in terms of weakly-coupled supermultiplets. In the presence of tensor multiplets, there exist additional BPS strings which couple naturally to the self-dual tensor fields, inducing a Dirac pairing η^{IJ} for the strings. In the F-theory description, they are obtained by wrapping branes on curves $\mathcal{C}^I$ in the base of the elliptic fibration, and the pairing is given by their intersection form[8]

$$\eta^{IJ} = -\mathcal{C}^I \cdot \mathcal{C}^J\,. \tag{2.3}$$

Since the matrix η^{IJ} has an interpretation as the Dirac pairing of strings, it must be positive semi-definite. For LSTs, we furthermore demand the presence of a *single* null eigenvector, which we refer to as the (integer-valued) LST charge ℓ^{LST},

$$\eta^{IJ}\ell_J^{\text{LST}} = 0\,, \qquad \ell_I^{\text{LST}} > 0 \;\; \forall I\,, \qquad \gcd(\ell_1^{\text{LST}}, \ell_2^{\text{LST}}, \dots) = 1\,. \tag{2.4}$$

Here and throughout, we sum over repeated indices. The LST charge is then interpreted as the coefficients of the linear combination of self-dual two-forms that couple to the Little String:

$$B_2^{\text{LST}} = \ell_I^{\text{LST}}\, B_2^I\,, \tag{2.5}$$

where B^I is the tensor associated with the curve $\mathcal{C}^I$. There then remains n_T dynamical tensors.

[8]Note that the matrix η is sometimes defined as the intersection matrix itself rather than the Dirac pairing: $\eta^{IJ} = +\mathcal{C}^I \cdot \mathcal{C}^J$. It is then negative semi-definite, since the curve have to have negative self-intersection in order for them to be shrinkable to zero size to reach the LST phase. In our convention, the diagonal elements are always positive integers.

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In addition to tensor multiplets, there can also be vector and hypermultiplets charged under gauge symmetries. To ensure gauge anomaly cancellation, each gauge algebra must be associated with a tensor multiplet, or equivalently its magnetic dual string. Indeed, in six dimensions, gauge anomalies generically do not cancel and a Green-Schwarz-West-Sagnotti mechanism, or GS mechanism for short, mediated by the tensor fields is necessary to obtain a consistent theory [48, 49]. In F-theory, the gauge sector arises naturally from non-trivial elliptic fibers over the curves $\mathcal{C}^I$, and the dictionary between the matter content and the geometry can be obtained straightforwardly [50, 51]. Gauge anomaly cancellation is then guaranteed if the theory descends from a well-defined elliptically fibered Calabi-Yau compactification.

It is common to summarize the spectrum of an $\mathcal{N} = (1,0)$ six-dimensional theory on its tensor branch in a pictorial way called a (generalized) quiver. The I^{th} tensor multiplet with $\eta^{II} = n$ associated with an algebra $\mathfrak{g}$ is denoted by

$$\overset{\mathfrak{g}}{n}\,. \tag{2.6}$$

If there is more than one tensor multiplet, they are arranged in a graph whose adjacency matrix is given by the Dirac pairing. If the algebra is trivial, the label $\mathfrak{g}$ is omitted. Furthermore, flavor symmetries of tensor multiplets are indicated with brackets, $[\mathfrak{f}]$.

For instance, the following quiver represents a Heterotic LST:

$$[\mathfrak{so}_{16}]\,\overset{\mathfrak{sp}_N}{1}\,\overset{\mathfrak{sp}_N}{1}\,[\mathfrak{so}_{16}]\,,\qquad \eta = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\,. \tag{2.7}$$

The two strings of charge 1 intersect once, and there are $2N+8$ hypermultiplets transforming in the fundamental representation of each $\mathfrak{sp}_N$, necessary to ensure anomaly cancellation. Some of these transform in the bifundamental representation $(\boldsymbol{2N}, \boldsymbol{2N})$, and there are sixteen remaining half-hypermultiplets on each side, rotated by an $\mathfrak{so}(16)$ flavor symmetry. As one can see, the quiver notation provides a convenient book-keeping device that (almost) uniquely encodes the spectrum upon demanding absence of anomalies.

In F-theory, two curves are depicted side by side if they have a normal crossing intersection. While this is enough when considering SCFTs, the requirement of a unique null eigenvector for the Dirac pairing of an LST allows for different patterns of intersections:

$$\begin{aligned}
&\text{Normal intersection:} && \overset{\mathfrak{g}_1}{m}\,\overset{\mathfrak{g}_2}{n}\,,\\
&\text{Tangential intersection:} && \overset{\mathfrak{g}_1}{m}||\overset{\mathfrak{g}_2}{n}\,,\\
&\text{Triple intersection:} && \overset{\mathfrak{g}_1}{n_1}\overset{\overset{\mathfrak{g}_2}{n_2}}{\Delta}\overset{\mathfrak{g}_3}{n_3}\,,\\
&\text{Loop configuration:} && //\overset{\mathfrak{g}_1}{n_1}\,\overset{\mathfrak{g}_2}{n_2}\,\ldots\,\overset{\mathfrak{g}_{k-1}}{n_{k-1}}\,\overset{\mathfrak{g}_k}{n_k}//
\end{aligned} \tag{2.8}$$

In the last case, the symbol // denotes the intersection between the first and last curve.

Note that the quiver notation summarizes the theory at a generic point in the tensor branch. At such a generic point, the strings, being BPS objects, have a tension T_I that is given by the vacuum expectation value of the scalar field t_I of the associated tensor multiplet, $T_I \sim \langle t_I \rangle$. For theories where η does not have a null eigenvector, moving to a point where the vacuum expectation value of all tensor fields vanish, leads to a conformal fixed point where no scales can be present. However, the existence of the null eigenvector ℓ^{LST} means that not all curves can be shrunk and hence ensures that an intrinsic LST scale remains.

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The two-form field $B_2^{\rm LST}$ acts as a background field for the $\mathrm{U}(1)^{(1)}_{\rm LST}$ one-form symmetry, and the Bianchi identity of the associated 3-form field strength $H_3^{\rm LST}$ controls the 2-group structure constant:

$$dH_3^{\rm LST} = \widehat{\kappa}_R c_2(R) - \frac{\kappa_P}{4} p_1(T) - \kappa_F^A c_2(F_A)\,, \tag{2.9}$$

where $c_2(R)$, $c_2(F_A)$ are the one-instanton-normalized second Chern classes of the R- and $\mathfrak{f} = \oplus_A \mathfrak{f}^A$ flavor symmetry bundles respectively, and $p_1(T)$ is the first Pontryagin class of the spacetime tangent bundle. The two-group structure constants can then be obtained from the quiver data by expanding $B_2^{\rm LST}$ in terms of the tensors B_2^I. At a generic point in the tensor branch, their respective Bianchi identities are given by

$$dH_3^I = h^I c_2(R) - \frac{a^I}{4} p_1(T) - B^{IA} c_2(\mathfrak{f}_A)\,, \tag{2.10}$$

where $h^I = h^\vee_{\mathfrak{g}^I}$ is the dual Coxeter number of the $I^{\rm th}$ algebra, $a^I = 2 - \eta^{II}$, and B^{IA} encodes the pairing between the $I^{\rm th}$ string and the $A^{\rm th}$ flavor algebra. Comparing with equation (2.9), we find that the 2-group structure constants are given by

$$\kappa_P = h^I \ell_I^{\rm LST}\,, \qquad \kappa_R = a^I \ell_I^{\rm LST}\,, \qquad \kappa_{\mathfrak{f}}^A = B^{IA} \ell_I^{\rm LST}\,. \tag{2.11}$$

In [26], the first two structure constants were called *universal structure constants*. They have to match across T-duality, which is not true for the third structure constant $\kappa_{\mathfrak{f}}^A$.

2.2 Higher symmetry dualities in LSTs

Beyond the 2-group symmetry structure, 6D QFTs may also admit general (discrete) higher form symmetries. In the following, we shortly review how such symmetries can arise and propose to use them as novel invariants across T-duality. Our main interest lies in the following two types of generalized symmetries

1. **Discrete center 1-form symmetries $\boldsymbol{\mathcal{D}^{(1)}_{6D}}$** associated to Wilson lines not screened by dynamical gauge or matter fields.

2. **Defect Group $\boldsymbol{\mathcal{D}^{(2)}_{6D}}$** associated to self-dual string defects that can not be screened by dynamical BPS strings [37].

We propose that a suitable combination of the above higher symmetries serves as a novel invariant across T-dual LSTs.

Since LSTs may be T-dual in 5D, we next review circle compactifications of higher form symmetries. Compactifying a 6D theory on a circle to 5D decomposes an n-form symmetry into $\mathcal{D}^{(n)}_{D-1} \times \mathcal{D}^{(n-1)}_{D-1}$ symmetries. From the perspective of a $(D-n-1)$-dimensional topological operator that generates the D-dimensional n-form symmetry, the two possibilities correspond to a case where the defect does or does not wrap the circle. Indeed, the 1-form symmetry $\mathcal{D}^{(1)}$ becomes a 1-form and a 0-form symmetry in 5D.

As discussed in [37], the 6D defect group $\mathcal{D}^{(2)}_{6D}$ is special as strings are self-dual. Upon compactification to five dimensions these objects remain either strings, or lines when they wrap the circle, and are related to one- and two-form symmetries. Due to the self-duality in six dimensions we can then make a choice of electric polarization in five dimensions which keeps only the 5D one-form symmetry [14, 52–54]. Thus we obtain two contributions to the 1-form symmetries in 5D,

$$\mathcal{D}^{(1)}_{6D} \times \mathcal{D}^{(2)}_{6D} \rightsquigarrow \mathcal{D}^{(1)}_{5D} . \tag{2.12}$$

We therefore expect T-dual LSTs to have a non-trivial 1-form and defect group structure that combines into a single 5D 1-form symmetry upon circle compactification.

6D defect group. The defect group $\mathcal{D}^{(2)}_{6D}$ of LSTs can be obtained from the Dirac pairing matrix η^{IJ}. The defect group receives contributions from string defects, modulo those that can be screened by dynamical ones, which we can represent as a lattice quotient. However, one of these combinations is exactly the little string itself appearing as a null vector in η^{IJ} and coupling to a non-dynamical tensor field. We already attribute this to the $\mathrm{U}(1)^{(1)}_{\mathrm{LST}}$ symmetry, which has to be quotiented out so that the defect group is given by

$$\mathcal{D}^{(2)}_{6D} = \frac{\mathbb{Z}^{n_T+1}}{[\eta^{IJ}]\mathbb{Z}^{n_T+1}}/\mathbb{Z} = \prod_{i=1}^{n_T} \mathbb{Z}_{m_i} . \tag{2.13}$$

A convenient way to compute the defect group is via the Smith normal form of η,

$$M = S \cdot \eta \cdot T . \tag{2.14}$$

Here, S and T are invertible matrices and M is diagonal. The integers m_i in equation (2.13) are given by the diagonal entries of M. Note that for LSTs there is always a null entry $m_0 = 0$ associated with the null eigenvector of η. The other diagonal element are then $m_i \geq 0$, $i = 1 \ldots n_T$. The defect groups of LSTs and SCFTs are therefore computed in the same way, up to the presence of m_0 in the former case.

It is easy to check that $\mathcal{D}^{(2)}_{6D}$ is invariant under blow-ups and blowdowns of smooth points [37]. It might therefore be simplest to compute the defect group for a maximal (smooth) blowdown of the quiver, a phase we will refer to as the *endpoint configuration.* For LSTs, the respective endpoint configurations and possible defect groups are

$$\begin{aligned} &\text{Heterotic LSTs:} \qquad \eta = (0)\,, \qquad \mathcal{D}^{(2)}_{6D} = \emptyset\,, \\ &\text{Type II LSTs:} \qquad \eta = \widehat{A}(G)\,,\; \mathcal{D}^{(2)}_{6D} = Z(G)\,. \end{aligned} \tag{2.15}$$

Here, $\widehat{A}(G)$ is the affine extended Dynkin diagram of the gauge algebra G and $Z(G)$ is its center. For all ADE groups, we have listed the respective center symmetries in table 1. As already remarked in [26], Heterotic LSTs cannot have a non-trivial defect group. In contrast, Type II LSTs generally have a non-trivial defect group which, however, is bounded by the ADE center symmetries. Note that this strongly constrains the higher symmetry structure of Heterotic LSTs: since the defect group is always trivial, two Heterotic LSTs can only be T-dual if they have the same global center 1-form symmetry. However, in all examples discussed in the literature [26–29, 45], only gauged center 1-form symmetries were found.

1-form symmetries. Center 1-form symmetries are the simplest higher form symmetries and are hence well-explored and understood. A center 1-form symmetry is associated to line defects, such as Wilson lines of a gauge group G with non-trivial center $Z(G)$. The 1-form symmetry $\mathcal{D}^{(1)}$ acts on Wilson lines that are in the weight lattice of G modulo the screening by dynamical W-bosons living in the root lattice. In $\mathcal{N} = (1, 0)$ theories, there can be additional sources that can break the symmetry, such as hypermultiplets charged under $Z(G)$. Depending on their representations, Wilson lines may then end on those hypermultiplets and get screened, such that the 1-form symmetry is also broken. This can be generalized to non-simple gauge groups, $G = \prod_I G_I$ with center $Z(G) = \prod_I Z(G_I)$. This combined center symmetry cannot be fully realized, since various matter representations charged under G (typically bifundamentals of $G_I \times G_{I\pm 1}$) have to be present in the theory to ensure gauge anomaly cancellation. Nevertheless, there is generically a diagonal combination of center symmetries $\mathcal{Z} \in Z(G)$ that acts trivially on all matter representations, which can then be a center 1-form symmetry.

The putative center 1-form symmetry identified above can be rendered inconsistent by BPS instanton strings that are charged under it. The consistency of the 1-form symmetry can be studied as follows [18, 22, 55]: for a consistent 1-form symmetry $\mathcal{D}^{(1)}_{6D} = \mathcal{Z} \in Z(G)$, one should be able to treat it as a non-trivial background field. The background is parameterized by a twist vector $\vec{k}$ for $\mathcal{Z}$ whose entries are integral values in the centers

$$k_I \in Z(G_I)\,. \tag{2.16}$$

Upon switching on the above background twist, the respective gauge instanton fractionalizes [55]. By performing large gauge transformations of the 2-form fields, we are then left with a fractional shift that leads to a phase in the path integral. This phase is trivial if

$$\sum_j \eta^{IJ} k_J^2 \alpha_{G_J} = 0 \text{ mod } 1 \quad \forall I\,, \tag{2.17}$$

which signals consistency of the chosen 1-form symmetry background. Note that the k_i have to be trivial for the flavor symmetry factors (which are mostly absent in Type II LSTs anyways), and hence do not appear in the sum (2.17).

The above consistency check is the same for center 1-form *quotient symmetries* that appears in the total symmetry group $\widetilde{G} = ([G_{\mathfrak{f}}] \times G)/\mathcal{Z}$, which acts simultaneously on gauge and flavor group factors. This is precisely what happens in Heterotic LSTs: the global gauge group structure is fixed by the flavor branes. The $\mathbb{Z}_2$ quotient of the $\text{Spin}(32)/\mathbb{Z}_2$ flavor group acts diagonally on all gauge group factors.[9]

Examples. We illustrate the above considerations with a few concrete examples. First, let us consider a theory with a $G = \text{SU}(n)$ singularity, whose quiver has the necklace shape of the corresponding affine Dynkin diagram with Dirac pairing matrix

$$\eta = \begin{pmatrix} 2 & -1 & 0 & .. & -1 \\ -1 & 2 & -1 & .. & 0 \\ 0 & -1 & 2 & .. & 0 \\ .. & .. & .. & .. & .. \\ -1 & 0 & 0 & .. & 2 \end{pmatrix} \tag{2.18}$$

[9]The global quotient structure can be engineered in F-theory, for both compact and non-compact cases via a non-trivial Mordell-Weil torsion group [16, 18, 19, 56, 57].

G	$Z(G)$	α_G
$\mathrm{SU}(N)$	$\mathbb{Z}_N$	$\frac{N-1}{2N}$
$\mathrm{Sp}(N)$	$\mathbb{Z}_2$	$\frac{N}{4}$
$Spin(2N+1)$	$\mathbb{Z}_2$	$\frac{1}{2}$
$Spin(4N+2)$	$\mathbb{Z}_4$	$\frac{2N+1}{4}$
$Spin(4N)$	$\mathbb{Z}_2^s \times \mathbb{Z}_2^c$	$(\frac{N}{4}, \frac{1}{2})$
G_2	$\mathbb{Z}_1$	–
F_4	$\mathbb{Z}_1$	–
E_6	$\mathbb{Z}_3$	$\frac{2}{3}$
E_7	$\mathbb{Z}_2$	$\frac{3}{4}$
E_8	$\mathbb{Z}_1$	–

Table 1. Summary of center symmetries and group theory coefficients for all simple groups, reproduced from [36]. For $Spin(4N)$, the two $\mathbb{Z}_2$ factors act on the spinor and co-spinor, respectively; the vector representation is charged under both.

and LST vector $\vec{\ell}^{\mathrm{LST}} = (1,1,1,\ldots 1)$. The defect group can be computed using the Smith normal form, which is given by

$$M = \mathrm{diag}(0,1,1,\ldots,1,n)\,. \tag{2.19}$$

Hence, the defect group of the theory is

$$\mathcal{D}^{(2)} = \mathbb{Z}_n\,. \tag{2.20}$$

We may now decorate the theory with an $\mathrm{SU}(m)$ gauge symmetry, which gives the quiver

$$//\underbrace{\overset{\mathfrak{su}_m}{2}\ \overset{\mathfrak{su}_m}{2}\ \overset{\mathfrak{su}_m}{2}\ \ldots\ \overset{\mathfrak{su}_m}{2}}_{n\times}//\,. \tag{2.21}$$

The theory has a non-trivial diagonal 1-form symmetry: all matter transforms in bifundamental representations $(\boldsymbol{m},\overline{\boldsymbol{m}})$, which leaves the diagonal $\mathbb{Z}_m$ 1-form symmetry unbroken upon choosing all twists $k_i = 1$. The 2-group structure constants and Coulomb branch dimension of this theory are

$$\kappa_P = 0\,, \qquad \kappa_R = m\cdot n\,, \qquad \dim(\mathrm{CB}) = m\cdot n - 1\,, \tag{2.22}$$

using the formulas given in equation (2.11) Together with the higher form symmetries

$$\mathcal{D}^{(2)}:\ \mathbb{Z}_n^{(2)}\,, \qquad \mathcal{D}^{(1)}:\ \mathbb{Z}_m^{(1)}\,, \tag{2.23}$$

this gives the T-duality invariant data. An obvious T-dual of the above theory is given by exchanging fiber and base singularities, which results just in a flip of the respective 1-form and 2-form symmetries, leaving CB and 2-group structure constant invariant.

The situation at hand is in fact even more general: recall that the total space of the fibration can be represented by the orbifold of $X_3 = (\mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{C}^1)/\Lambda$ with Λ given by the two actions

$$\begin{aligned} \mathbb{Z}_n : \quad & g_n : (e^{2\pi \mathrm{i}/n}, \mathbb{1}, e^{-2\pi \mathrm{i}/n}) \\ \mathbb{Z}_m : \quad & g_m : (\mathbb{1}, e^{2\pi \mathrm{i}/m}, e^{-2\pi \mathrm{i}/m}). \end{aligned}$$

If m and n are co-prime, the Chinese Remainder Theorem tells us that we can write

$$\Lambda = \mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_{m\cdot n}\,, \tag{2.24}$$

leading to a $\mathbb{Z}^{(1)}_{n\cdot m}$ 1-form symmetry in 5D. This hints at the possibility that there might be more T-dual Type II LSTs of type $\mathcal{K}^{\text{II}}(\mathfrak{su}_k, \mathfrak{su}_l)$ with

$$k \cdot l = m \cdot n\,, \tag{2.25}$$

as those have the same $\mathbb{Z}^{(1)}_{nm}$ 1-form symmetry, Coulomb branch dimension and 2-group structure constants. Indeed, it is has been proposed in [58, 59] that there exist flop transitions in the extended Kähler cone of the threefold which lead to multiple T-duals for each k, l partition of n, m. Since the 1-form symmetry group in 5D is insensitive to flop transitions, this perspective would be consistent with conditions imposed by the matching of higher form symmetries. Similarly, for $\gcd(n, m) = k$ we can then write $\Gamma = \mathbb{Z}_{n\cdot m/k} \times \mathbb{Z}_k$, which gives the maximal 1-form symmetry in 5D. In fact, any other partition $nm = \tilde{n}\tilde{m}$ that leads to the above symmetry is also consistent, which implies the additional condition $\gcd(\tilde{n}, \tilde{m}) = k$. In section 4, we give an explicit geometric construction for the case of two inequivalent elliptic fibrations.

The computation of the defect group is generally straightforward by either moving to the endpoint configuration, where the base has an ADE singularity and taking its center, or by directly computing the Smith Normal Form on the full tensor branch geometry. Determining the center 1-form symmetry can be more involved, so we discuss two more examples in the following to illustrate the computation. Generically, we might need to identify a center symmetry transformation on the gauge group factors that acts faithfully on all representations.

Our next example is a slightly more complicated LST, which is the theory $\mathcal{K}^{\text{II}}(\mathfrak{su}_n, \mathfrak{so}_{12})$ with tensor branch quiver

$$\mathcal{K}^{\text{II}}(\mathfrak{su}_n, \mathfrak{so}_{12}) : \qquad \begin{array}{ccccc} & \overset{\mathfrak{su}_n}{2} & & \overset{\mathfrak{su}_n}{2} & \\ \overset{\mathfrak{su}_n}{2} & \overset{\mathfrak{su}_{2n}}{2} & \overset{\mathfrak{su}_{2n}}{2} & \overset{\mathfrak{su}_{2n}}{2} & \overset{\mathfrak{su}_n}{2} \end{array}\,. \tag{2.26}$$

The total gauge group has a $\mathbb{Z}_n^4 \times \mathbb{Z}_{2n}^3$ center, which is broken to a diagonal subgroup by the various bifundamental matter representations of the quiver. For example, the fundamental representation $\boldsymbol{n}$ of $\mathfrak{su}_n$ has center charge 1 and transforms with phase

$$\mathbb{Z}_n : \phi(\mathbf{n}) = e^{\frac{2\pi i}{n}}\,. \tag{2.27}$$

Bifundamental representations such as $(\boldsymbol{n}, \overline{\boldsymbol{2n}})$ may leave a diagonal center symmetry $\mathcal{Z} = \mathbb{Z}_n \subset \mathbb{Z}_n \times \mathbb{Z}_{2n}$ invariant, since

$$(\phi_1 \otimes \phi_2^2)(\boldsymbol{n}, \overline{\boldsymbol{2n}}) = e^{\frac{2\pi i}{n}} \cdot e^{\frac{-2\cdot 2\pi i}{2n}} = \mathbb{1}\,. \tag{2.28}$$

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The above $\mathbb{Z}_n^{(1)}$ generator must be further extended to encompass all gauge symmetry factors in order to account for all bifundamental hypermultiplets. Furthermore, we need to check the BPS string obstruction in (2.17). We find a consistent $\mathbb{Z}_n$ background flux for the following twist choice for k_i in (2.17): $k_i = 1$ for $G_i = \mathrm{SU}(n)$ and $k_j = 2$ for $G_j = \mathrm{SU}(2n)$. In summary, the HFS structure of the above LST is

$$\mathcal{D}^{(2)} : \mathbb{Z}_2^{s\,(2)} \times \mathbb{Z}_2^{c\,(2)} , \qquad \mathcal{D}^{(1)} : \mathbb{Z}_n^{(1)} \tag{2.29}$$

We may proceed similarly for the $\mathcal{K}^{\mathrm{II}}(\mathfrak{so}_{12}, \mathfrak{su}_n)$ theory, whose quiver has the necklace shape

$$\mathcal{K}^{\mathrm{II}}(\mathfrak{so}_{12}, \mathfrak{su}_n) : \qquad // \overset{\mathfrak{so}_{12}}{4} \overset{\mathfrak{sp}_2}{1} \overset{\mathfrak{so}_{12}}{4} \overset{\mathfrak{sp}_2}{1} \ldots \overset{\mathfrak{so}_{12}}{4} \overset{\mathfrak{sp}_2}{1} // \tag{2.30}$$

with n pairs of $\overset{\mathfrak{so}_{12}}{4} \overset{\mathfrak{sp}_2}{1}$. Each $\mathfrak{so}_{12}$ gauge factor has a $\mathbb{Z}_2^s \times \mathbb{Z}_2^c$ center symmetry and each $\mathfrak{sp}_2$ factor has a $\mathbb{Z}_2$ symmetry. The fundamentals of $\mathfrak{so}_{4N}$ are charged under both $\mathbb{Z}_2^{s/c}$ factors, and $\mathfrak{sp}_M$ fundamentals are charged under the $\mathbb{Z}_2$ center symmetry of $\mathrm{Sp}(2)$. Hence, we obtain two invariant generators. One is the diagonal combination of the $\mathbb{Z}_2^s \times \mathbb{Z}_2$ generator, and the other of the $\mathbb{Z}_2^c \times \mathbb{Z}_2$ generator, $\phi_{\mathfrak{so}_{12}}^{s/c} \otimes \phi_{\mathfrak{sp}_2}$, which acts trivially on the bifundamental representations. With this, the full quiver has a

$$Z = \prod_{i=1}^{n} (\mathbb{Z}_{2,i}^s \times \mathbb{Z}_{2,i}^c \times \mathbb{Z}_{2,i}) \tag{2.31}$$

center symmetry. The bifundamental matter will be invariant under a diagonal combination of the invariant generators of each block,

$$\mathbb{Z}_2^{s\,(1)} \times \mathbb{Z}_2^{c\,(1)} : \qquad \bigotimes_{i=1}^{n} \left(\phi_{\mathfrak{so}_{12},i}^{s/c} \otimes \phi_{\mathfrak{sp}_2,i} \right) . \tag{2.32}$$

Note that there are no spinor or co-spinor representations that would break the respective $\mathbb{Z}_2^{s/c}$ center symmetries. Further we also have to check that the BPS strings are consistent with the center symmetries.

Checking the BPS string obstruction works out just as in the previous case: there are two non-trivial center twists (k^s, k^c) for an $\mathrm{SO}(4N)$, each of which contributes a term [22]

$$\eta^{ij} \left(\frac{N_j}{4} (k_j^s + k_j^c)^2 + \frac{1}{2} (k_j^s \cdot k_j^c) \right) \tag{2.33}$$

to the condition (2.17). For each of the n blocks $\overset{\mathfrak{so}_{12}}{4} \overset{\mathfrak{sp}_2}{1}$ in the quiver (2.30), we need to choose three $\mathbb{Z}_2$-valued entries for k_i, which encode the $\mathbb{Z}_{2,i}^s \times \mathbb{Z}_{2,i}^c \times \mathbb{Z}_{2,i}$ for the $\mathrm{SO}(12)_i \times \mathrm{Sp}(2)_i$ centers. There are two consistent choices: we can choose $k_i = (1, 0, 1)$ or $k_i = (0, 1, 1)$. Both choices correspond to a 1-form symmetry background generator of order two, and it is straightforward to check that the path integral phase (2.17) is trivial for either of them.

To summarize, the higher form symmetries of this example are

$$\mathcal{D}^{(2)} : \mathbb{Z}_n^{(2)} , \qquad \mathcal{D}^{(1)} : \left(\mathbb{Z}_2^{(1)} \right)^2 . \tag{2.34}$$

The two Type II LSTs in the example are T-dual, which is consistent with the exchange of the higher form symmetry sectors we observe.

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2.3 Flavor bounds from the LST worldsheet theory

We now turn to the 0-form flavor symmetries of LSTs. The presence of BPS string in the spectrum is known to impose constraints on the flavor symmetries demanding unitarity of the worldsheet theory [40–42].

From the perspective of the six-dimensional effective description at a generic point of the tensor branch, the D3-branes wrapping two-cycles in the geometry arise as strings with finite tension. Their worldsheet description is that of a two-dimensional $\mathcal{N} = (0, 4)$ gauge theory, which flows in the deep infrared (IR) to a 2D SCFT. By studying the consequences of unitarity for such SCFTs, one can obtain constraints on the allowed spectrum and the properties of extended objects, since the two-dimensional central charges of the theory are inherited from protected quantities in six dimensions. We will follow the methods developed in [40], which were used to obtain bounds for the rank of the possible gauge groups appearing in supergravity theories containing BPS strings.

The relevant quantities are encoded in the anomaly polynomial of the 2D worldsheet theory

$$I_4 = \sum_a k^a c_2(F_a) - \frac{k_G}{24} p_1(T_2)\,, \tag{2.35}$$

where $c_2(F_a) = \frac{1}{4}\operatorname{Tr} F_a^2$ are the one-instanton-normalized second Chern classes of the worldsheet flavor symmetries, and $p_1(T_2)$ is the first Pontryagin class of the worldsheet tangent bundle. Note that both the gauge and flavor symmetries of the 6D bulk are seen as flavor symmetries from the worldsheet point of view. The anomaly polynomial I_4 can be written in term of the string charges and their intersection pairing, as well as other quantities of the 6D theory [60, 61]. In the deep IR, the emergent superconformal invariance fixes the left- and right-handed central charges in terms of the level of the $\mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r$ R-symmetry, and their difference is furthermore fixed by the gravitational anomaly k_G.[10] In the normalization defined in equation (2.35), we have

$$c_l = 6k_l\,, \qquad c_r = 6k_r\,, \qquad c_r - c_l = k_G\,. \tag{2.36}$$

On the worldsheet, the 6D bulk flavor and gauge symmetries $\mathfrak{f}_a$ manifest themselves through the presence of holomorphic currents associated with a Kac-Moody algebra $\widehat{\mathfrak{f}}^a$ at level k^a in the spectrum, whose contribution to the left-handed central charge is given by

$$c_F = \sum_a \frac{k^a \dim(\mathfrak{f})}{k^a + h^\vee_{\mathfrak{f}}}\,, \tag{2.37}$$

where $h^\vee_{\mathfrak{f}^a}$ is the dual Coxeter number of the algebra $\mathfrak{f}^a$. We are therefore led to a lower bound on the central charge, which constrains the possibles flavor symmetries:

$$c_F \leq c_l\,. \tag{2.38}$$

While these constraints have been derived for supergravity theories in the context of the Swampland Program [40, 41], they are very generic, and we will now apply them to

[10] We use lowercase subscripts to indicate the worldsheet left and right R-symmetry, reserving the letter R for the bulk R-symmetry as the focus of this work is on six-dimensional theories. In the literature, $c_2(I)$ is sometimes used for the Chern class of the bulk R-symmetry bundle.

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LSTs.[11] All the necessary quantities can be obtained from the anomaly polynomial of the six-dimensional effective theory — which can be computed directly from the tensor-branch description — and are reviewed in appendix B.1 for convenience and to set our conventions.

This gives us the spectrum on the D3-brane, and the anomaly polynomial of the CFT is obtained by removing the contributions of the center-of-mass universal hypermultiplet:

$$I_4^{\text{CFT}} = I_4 - I_4^{\text{CoM}} . \tag{2.39}$$

Furthermore, the left and right R-symmetries of the UV description might mix with the other worldsheet flavor factor, changing that of the IR CFT. In our case, the right-handed part of the R-symmetry $\mathfrak{su}(2)_r$ remains unchanged. The left-handed central charges can then be inferred from the gravitational anomaly, see equation (2.36), which is easily read off from the anomaly polynomial. We obtain

$$c_l = 6k_l = 6k_r + k_G = 3(3Q_I a^I - Q_I \eta^{IJ} Q_J) + 2 , \tag{2.40}$$

where Q_I denotes the charge of the string, and $a^I = 2 - \eta^{II}$. A short summary and additional details on the derivation of the central charges can be found in appendix B.1. A bound on the possible symmetries then depends on the values of the levels k^a. Those arise from the Green-Schwarz coupling in the 6D bulk theory, and are given by

$$k^a = B^{aI} Q_I , \tag{2.41}$$

where B^{aI} is the pairing between the a^{th} flavor and the I^{th} string charge.

Let us now specialize to the little string itself. In that case, the charge of the string is by definition

$$Q_I = \ell_I^{\text{LST}} , \qquad \eta^{IJ} \ell_I^{\text{LST}} = 0 , \tag{2.42}$$

see equation (2.4). Moreover, the string charge also fixes the two-group structure constant,

$$\kappa_P = a^I \ell_I^{\text{LST}} , \qquad a^I = 2 - \eta^{II} . \tag{2.43}$$

On the worldsheet we can therefore simply rewrite the left-handed central charge as

$$c_l = 9\,\kappa_P + 2 . \tag{2.44}$$

The bound then follows from the value of the possible worldsheet flavor symmetry. As already mentioned above, from the two-dimensional point of view, both bulk gauge $\mathfrak{g}^I$ and flavor $\mathfrak{f}^A$ symmetries appear as flavor symmetries. We must therefore distinguish between the two types of possibilities, namely when $a = I$ where the flavor pairing is simply $B^{IJ} = \eta^{IJ}$, and those of the bulk flavor symmetries

$$k^I = \eta^{IJ} \ell_J^{\text{LST}} = 0 , \qquad k^A = B^{AI} \ell_I^{\text{LST}} \tag{2.45}$$

We then conclude that only bulk flavor symmetries can contribute to the bound given in equation (2.38). Moreover, in the geometric picture, the flavor pairing matrix is given by

[11] Via generalized blow-up equations, the elliptic genus of LSTs has been computed recently in [62].

the intersection of D7-branes wrapping non-compact divisors $\overline{D}^A$ with quiver curves w^I, $B^{AI} = \overline{D}^A \cdot w^I$, so that in the overwhelming majority of cases $B^{AI} \in \{0,1\}$, and the level of the flavor symmetry is simply the charge of the tensor the flavor is attached to in the quiver description.[12] Putting everything together, we therefore obtain the bound

$$\sum_A \frac{k^A \dim(\mathfrak{f}^A)}{k^A + h^\vee_A} \leq 9\,\kappa_P + 2\,, \tag{2.46}$$

where we remind the reader that the sum is taken only over the flavor symmetry factors $\mathfrak{f} = \oplus_A \mathfrak{f}^A$ of the 6D theory. Equation (2.46) can be used to derive a bound on the total rank of the flavor symmetry. Indeed, a consequence of the strange formula of Freudenthal and de Vries is that for any simple algebra

$$\mathrm{rk}(\mathfrak{g}) \leq \frac{\dim(\mathfrak{g})}{(1 + h^\vee_{\mathfrak{g}})}\,. \tag{2.47}$$

This relation is saturated when $\mathfrak{g}$ is simply laced, and is easily found to be correct by inspection, see table 10.

We conclude that given an LST with two-group structure constant κ_P, the rank of its total flavor symmetry $\mathfrak{f} = \oplus_A \mathfrak{f}^A$ must satisfy

$$\mathrm{rk}(\mathfrak{f}) \;\leq\; 9\,\kappa_P + 2\,. \tag{2.48}$$

Note that this bound is weaker than the one given in equation (2.46), as the levels k^A related to the little string charges ℓ_I^{LST} can be larger than one. At the end of this section, we give an example of a would-be LST that satisfies the bound given in equation (2.48), but violates the stronger constraints of equation (2.46).

Using the two possible values of κ_P in the Heterotic and Type II cases, we obtain the bounds

$$\begin{aligned} \text{Heterotic LSTs:} \qquad & \mathrm{rk}(\mathfrak{f}) \leq 20\,, \\ \text{Type II LSTs:} \qquad & \mathrm{rk}(\mathfrak{f}) \leq 2\,. \end{aligned} \tag{2.49}$$

For Type II LSTs, we see that there is very little room for a potential flavor symmetry; the only possibilities are $\mathfrak{f} \in \{\mathfrak{u}(1), \mathfrak{u}(1)^2, \mathfrak{u}(1) \oplus \mathfrak{su}(2), \mathfrak{so}(4), \mathfrak{su}(3)\}$ flavor algebras. On the other hand, it is well known that a wealth of other algebras are permitted for Heterotic LSTs. For instance, a rank-16 flavor arises from the 3–7 strings stretching between the LST curve and the two (unbroken) M9-branes. When fixing $\mathfrak{e}_8^2$ as the flavor symmetry, one has $k_G = 16$ as expected, and there are very few possible additional contributions, such as $\mathfrak{u}(1)^4, \mathfrak{su}(2)^4$ or a combination thereof. Notably, a concrete case of an Heterotic LST with rank-18 flavor has been discussed in [28]. We are however not aware of any theory saturating the bounds above.

To obtain the bound in equation (2.48), we have used the anomaly polynomial for strings arising in six-dimensional $\mathcal{N} = (1,0)$ theories derived in [61]. The numerical values can be confirmed from the F-theory point of view by looking at the matter content arising from

[12] As pointed out in [63], there are a few cases where the flavor pairing entries can be larger than one, occurring for instance in the presence of undecorated (-1)-curves in the quiver description, where the $\mathfrak{e}_8$ flavor is decomposed into a non-maximal subalgebra.

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(0,4) Multiplet	Multiplicity	(c_l, c_r)	Het LST $(g = 0)$	Type II LST $(g = 1)$
Hyper	$\mathcal{C}^2 + 1 - g$	(4,6)	1	0
Twisted Hyper	1	(4,6)	1	1
Fermi	g	(2,0)	0	1
Half-Fermi	$8\mathcal{C}^2 + 16(1 - g)$	(1,0)	16	0

Table 2. Summary of the $(0, 4)$ 2D SCFT field content and their contributions to left- and right-handed central charges of a D3-brane wrapping a curve $\mathcal{C}$ of genus g. The multiplicities are evaluated for Heterotic $(g = 0)$ and Type II $(g = 1)$ LSTs using $\mathcal{C}^2 = 0$.

D3-branes wrapping a curve $\mathcal{C} \subset B_2$. For an LST, the curve has by definition self-intersection zero, $\mathcal{C} \cdot \mathcal{C} = 0$, and its genus g is either $g = 1$ for the Heterotic cases, or $g = 0$ for Type II cases. The multiplicity of the $\mathcal{N} = (0, 4)$ multiplets was derived in [46] for curves of arbitrary genera, which we have collected in table 2 for convenience.[13] The anomaly polynomial is then obtained by summing the contributions of each supermultiplets. It is then easy to see that, taking the center-of-mass modes into account, the left- and right-handed central charges are those discussed above.

3 LSTs from minimal affinization of SCFTs

We have seen that the possible flavor symmetries of general LSTs are quite constrained, in particular those of Type II LSTs. Conversely, Superconformal Field Theories (SCFTs) can host a vast zoo of flavor symmetries, which can be used to construct LSTs. Indeed, through an operation called *fusion* [64], certain SCFTs can be glued together to obtain an LST.

From the field-theory point of view, this procedure gauges a common flavor symmetry (or a subalgebra thereof) of two decoupled SCFTs to obtain a new theory. The presence of a new vector multiplet in the spectrum may lead to gauge anomalies that need to be cancelled through a Green-Schwarz-West-Sagnotti mechanism [48, 49], requiring the introduction of an additional tensor multiplet to mediate it. The fusion procedure can, however, substantially alter the UV behavior of the new theory. Depending on the (anti-)self duality of the new tensor, the SCFT can be converted to an LST, or even to a SUGRA theory [36].

The reverse procedure, where a tensor multiplet and the associated gauge symmetry are decoupled, is called *fission*. Since the pairing matrix of an LST has by definition a single null eigenvector, see equation (2.3), fission leads a positive definite pairing matrix, i.e., that of an SCFT. This is known as the *tensor-decoupling criterion* [8].

Geometrically, fusion can be understood as Kähler deformations rendering a curve in the base compact. Schematically, given two SCFTs $\mathcal{T}_1$ and $\mathcal{T}_2$ with a common flavor symmetry $\mathfrak{f}$, they can be fused together to obtain a new theory $\mathcal{K}$:

$$\begin{array}{ccccc} \mathcal{T}_1 & \oplus & \mathcal{T}_2 & = & \mathcal{K}\,, \\ \cdots \overset{\mathfrak{g}_1}{n_1}\, \overset{\mathfrak{g}_2}{n_2}\, [\mathfrak{f}] & \oplus & [\mathfrak{f}]\, \overset{\mathfrak{h}_1}{k_1}\, \overset{\mathfrak{h}_2}{k_2} \cdots & = & \cdots \overset{\mathfrak{g}_1}{n_1}\, \overset{\mathfrak{g}_2}{n_2}\, \overset{\mathfrak{f}}{m}\, \overset{\mathfrak{h}_1}{k_1}\, \overset{\mathfrak{h}_2}{k_2}\,, \end{array} \tag{3.1}$$

[13] We have used that by adjunction, $2(g - 1) = \mathcal{C}^2 - \mathcal{C} \cdot c_1(B)$ to write the multiplicities purely in terms of the self-intersection of the curve and its genus.

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where we have highlighted the new compact curve in blue, whose precise self-intersection $(-m)$ is dictated by anomaly cancellation. We again stress that the theory obtained after fusion must be free of any gauge anomaly, and not all fusions lead to consistent theories.

The SCFT $\mathcal{T}_2$ can be taken to be trivial, in which case we are simply gauging the flavor symmetry:

$$\cdots \overset{\mathfrak{g}_1}{n_1}\,\overset{\mathfrak{g}_2}{n_2}\,[\mathfrak{f}] \qquad \longrightarrow \qquad \cdots \overset{\mathfrak{g}_1}{n_1}\,\overset{\mathfrak{g}_2}{n_2}\,\overset{\mathfrak{f}}{m}\,. \tag{3.2}$$

To study LSTs, we can therefore start with SCFTs, and through fusion introduce a null eigenvalue. However, this can drastically change the (higher) symmetry structure of the new theory compared to the original SCFT. First, we have removed the flavor symmetry $\mathfrak{f}$, which may have an impact on the one-form (gauge) symmetry. Second, the introduction of a new compact curve, or equivalently new dynamical BPS strings, may also modify the defect group $\mathcal{D}^{(2)}$. This in turn gives additional constraints on the set of consistent fusions. For example, for Heterotic LSTs, the defect group must be trivialized and the flavor group reduced to at most rank 20. For Type II LSTs, the flavor rank must be almost trivialized to at most rank two, and $\mathcal{D}^{(2)}$ reduced to the center of a simple ADE algebra. The landscape of those theories is therefore much more sparse than their Heterotic cousins, and the possible bases giving rise to the defect group have been classified in [8]. Through the tensor-decoupling criterion, one might then ask which SCFT is closest to being an LST in the sense that their defect group agrees, and the only operation needed is to gauge the flavor symmetry. We will define such an operation as follows:

Definition 1. Given a 6D SCFT $\mathcal{T}$ with a flavor symmetry $\mathfrak{f}$, a Little String Theory $\mathcal{K}$ is reached through *minimal affinization* if:

- At a generic point of the tensor branch, gauging the flavor symmetry $\mathfrak{f}$ introduces a single curve, and a null eigenvalue to the Dirac pairing, turning the resulting theory into an LST.
- Only vector multiplets are added to the spectrum of the tensor branch theory.
- The defect group is preserved, $\mathcal{D}^{(2)}(\mathcal{T}) = \mathcal{D}^{(2)}(\mathcal{K})$.

In particular, this means that this procedure does not require additional matter to cancel possible new gauge anomalies. Furthermore, the symmetry $\mathfrak{f}$ could be part of a larger flavor symmetry, as long as the resulting LST $\mathcal{K}$ satisfies the bound given in equation (2.46).

To give an example, let us consider the minimal affinization of the so-called $A_2^{\mathfrak{su}_k}$ SCFT, discussed in more detail below. It has a flavor symmetry $\mathfrak{su}_k \oplus \mathfrak{su}_k$, which can be trivialized by gauging the common diagonal subalgebra:

$$[\mathfrak{su}_k]\,\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_k}{2}\,[\mathfrak{su}_k] \qquad \longrightarrow \qquad \overset{\mathfrak{su}_k}{2}\overset{\overset{\mathfrak{su}_k}{2}}{}\overset{\mathfrak{su}_k}{2} \quad = \quad //\,\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_k}{2}\,//\,. \tag{3.3}$$

Fusion has introduced a new curve, depicted in blue, and we remind the reader that the symbols // indicate identification of the ends of the quiver, see equation (2.8). Anomaly cancellation demands that the new curve has self-intersection (-2), and it is easy to check

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that the new pairing matrix now has a null eigenvalue. The resulting quiver is therefore an LST, and the defect group of both theories is $\mathcal{D}^{(2)} = \mathbb{Z}_3$. Generalization of this example to more complicated quivers will be discussed in section 3.1.

While there is a unique way to obtain an LST in the previous example, in general multiple SCFTs can lead to the same LST through fusion. Minimal affinization therefore defines a canonical way to introduce a null eigenvalue to the pairing matrix without modifying the matter spectrum of the theory beyond the changes coming from the trivialisation of the flavor symmetry.

Another by-product of minimal affinization is that it enables us to obtain the two-group structure constants of an LST from the anomaly polynomial of the 6D SCFT. The anomaly polynomial of an $\mathcal{N} = (1,0)$ theory is generically[14] of the form [65]

$$\begin{aligned} I_8 = &\frac{\alpha}{24} c_2(R)^2 + \frac{\beta}{24} c_2(R) p_1(T) + \frac{\gamma}{24} p_1(T)^2 + \frac{\delta}{24} p_2(T) \\ &+ \sum_a \operatorname{Tr} F_a^2 \left(k^a c_2(R) + \mu^a p_1(T) + \sum_b \rho^{ab} \operatorname{Tr} F_b^2 \right) + \sum_a \nu^a \operatorname{Tr} F_a^4 \,, \end{aligned} \tag{3.4}$$

where the traces are one-instanton normalized, and we use the same notation as around equation (2.9). Given a six-dimensional quiver describing the tensor branch of the theory, there is a simple procedure to extract the anomaly polynomial of the theory, see e.g. [63] for a concise review of the algorithm. Via 't Hooft anomaly matching, the anomaly coefficients are preserved as we move in the tensor branch to reach the singular point.

When performing a minimal affinization, we only introduce a vector multiplet mediating the gauge interactions, and since the new curve cannot participate in the Green-Schwarz-West-Sagnotti mechanism [48, 49], the anomaly polynomial of an LST $\mathcal{K}$ obtained from an SCFT $\mathcal{T}$ is simply given by

$$I_8(\mathcal{K}) = I_8(\mathcal{T}) + I_8^{\text{vec}}(\mathfrak{f}) \,, \tag{3.5}$$

where I_8^{vec} is the contribution of the vector multiplet, defined in appendix B.

This gives us an alternative way to find the two-group structure constants defined in equation (2.11). Indeed, for an LST, they can also be read off directly from the anomaly polynomial [25],

$$I_8(\mathcal{K}) = \left(-\kappa_R \, c_2(R) - \frac{\kappa_P}{4} p_1(T) + \kappa_F^A \, c_2(F_A) \right) c_2(F) + \dots \,, \tag{3.6}$$

where $c_2(F) = \frac{1}{4} \operatorname{Tr} F^2$ is the second Chern class of the background field strength of the background symmetry of the $\mathrm{U}(1)^{(1)}$ 1-form symmetry. In the case of minimal affinization, this is simply that of the (now gauged) symmetry $\mathfrak{f}$.

Note that the above procedure is very much a feature of minimal affinization, and not true in general: generic fusions might require the introduction of additional matter, change the Green-Schwarz term, or $c_2(F)$ might not be associated directly with that of the SCFT flavor symmetry. As a result, the anomaly polynomial will not take the form given in equations (3.5) and (3.6) in those cases.

[14] We ignore possible Abelian flavor symmetries for simplicity.

Using the expression for the contribution of the vector multiplet, in the convention defined in equation (3.4), we find

$$\kappa_G(\mathcal{K}) = -16\mu^{\mathfrak{f}}(\mathcal{T}) + \frac{1}{3}h^\vee_{\mathfrak{f}}\,, \qquad \kappa_R(\mathcal{K}) = -4k^{\mathfrak{f}}(\mathcal{T}) + h^\vee_{\mathfrak{f}}\,, \qquad \kappa^A_F(\mathcal{K}) = 16\rho^{A,\mathfrak{f}}(\mathcal{T})\,. \tag{3.7}$$

Beyond their usefulness as a cross-check for the two-group structure constants computed from equation (2.11), these expressions will also enable us to explain some of the coincidences we observe for numerical factors for certain T-dual dual pairs of LSTs. Indeed, for SCFTs with a base that is part of an infinite series — so-called long quivers — closed-form expressions have been found in [66], relying only on the rank N of the base and group-theoretic data of $\mathfrak{g}$, possibly supplemented with nilpotent orbit data if the theory is reached through a Higgs-branch Renormalization Group (RG) flow.

In the sequel, we exemplify minimal affinization by considering SCFTs with a base associated to an ADE orbifold of $\mathbb{C}^2$ in F-theory, and then move on to other more general types of theories and their duals.

3.1 LSTs from ADE orbifolds

Our goal is to study LSTs using SCFTs. In the F-theory picture, $\mathcal{N} = (1,0)$ SCFTs are obtained from compactification on an elliptically fibered Calabi-Yau with a non-compact base B_2. The possible bases have been classified, and are known to correspond to discrete orbifolds of $\mathbb{C}^2$ [50, 51]:

$$B_2 = \mathbb{C}^2/\Lambda\,, \qquad \Lambda \subset \mathrm{U}(2)\,. \tag{3.8}$$

The simplest cases, which we will focus on first, are those where we further restrict Λ to be a discrete subgroup of SU(2) rather than U(2). These are well known to follow an ADE classification:

$$\Lambda \subset \mathrm{SU}(2): \qquad A_{N-1}\,, \quad D_N\,, \quad E_6\,, \quad E_7\,, \quad E_8\,. \tag{3.9}$$

The first two infinite series are given by the cyclic ($\mathbb{Z}_N$) and binary dihedral groups, while the exceptional series correspond to the tetrahedral, octahedral and icosahedral finite groups, in ascending order of rank. When the fibers are trivial, it is well known that 16 supercharges are preserved and we find a collection of undecorated (-2)-curves intersecting in the same pattern as the Dynkin diagram of the associated simple algebra: these are the celebrated $\mathcal{N} = (2,0)$ theories [1–3], see appendix C.

Generically, to preserve eight supercharges in the six-dimensional effective description, the fiber over the curves can be singular. However, for a given choice of orbifold action Λ, the possible fibers are severely constrained by demanding a well-defined elliptic fibration. For instance, when $\Lambda = A_{N-1}$, only fibers associated with a simply-laced algebra $\mathfrak{g} = \mathfrak{su}_k, \mathfrak{so}_{2k}, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ are consistent. Except for $\mathfrak{g} = \mathfrak{su}_k$, the resulting geometry contains non-minimal singularities, and a series of blow-ups is necessary, leading to a repeating sequence of curves called rank-N $(\mathfrak{g},\mathfrak{g})$ conformal matter, which we denote as $A^{\mathfrak{g}}_{N-1}$:

$$\begin{aligned}
A^{\mathfrak{su}_k}_{N-1}:&\quad [\mathfrak{su}_k]\,\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_k}{2}\,\dots\,\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_k}{2}\,[\mathfrak{su}_k]\,,\\
A^{\mathfrak{so}_{2k}}_{N-1}:&\quad [\mathfrak{so}_{2k}]\,\overset{\mathfrak{sp}_{k-4}}{1}\,\overset{\mathfrak{so}_{2k}}{4}\,\overset{\mathfrak{sp}_{k-4}}{1}\,\dots\,\overset{\mathfrak{sp}_{k-4}}{1}\,\overset{\mathfrak{so}_{2k}}{4}\,\overset{\mathfrak{sp}_{k-4}}{1}\,[\mathfrak{so}_{2k}]\,,\\
A^{\mathfrak{e}_6}_{N-1}:&\quad [\mathfrak{e}_6]1\overset{\mathfrak{su}_3}{3}1\overset{\mathfrak{e}_6}{6}1\overset{\mathfrak{su}_3}{3}1\overset{\mathfrak{e}_6}{6}\dots1\overset{\mathfrak{su}_3}{3}1\overset{\mathfrak{e}_6}{6}1\overset{\mathfrak{su}_3}{3}1[\mathfrak{e}_6]\,,\\
A^{\mathfrak{e}_7}_{N-1}:&\quad [\mathfrak{e}_7]1\overset{\mathfrak{su}_2}{2}\overset{\mathfrak{so}_7}{3}\overset{\mathfrak{su}_2}{2}1\overset{\mathfrak{e}_7}{8}\dots\overset{\mathfrak{e}_7}{8}1\overset{\mathfrak{su}_2}{2}\overset{\mathfrak{so}_7}{3}\overset{\mathfrak{su}_2}{2}1[\mathfrak{e}_7]\,,\\
A^{\mathfrak{e}_8}_{N-1}:&\quad [\mathfrak{e}_8]12\overset{\mathfrak{su}_2}{2}\overset{\mathfrak{g}_2}{3}1\overset{\mathfrak{f}_4}{5}1\overset{\mathfrak{g}_2}{3}\overset{\mathfrak{su}_2}{2}21\overset{\mathfrak{e}_8}{(12)}\dots\overset{\mathfrak{e}_8}{(12)}12\overset{\mathfrak{su}_2}{2}\overset{\mathfrak{g}_2}{3}1\overset{\mathfrak{f}_4}{5}1\overset{\mathfrak{g}_2}{3}\overset{\mathfrak{su}_2}{2}21[\mathfrak{e}_8]\,.
\end{aligned}\tag{3.10}$$

In each case, the curve with the lowest self-intersection — that is $-2\,,-4\,,-6\,,-8\,,-12$ for $\mathfrak{g}=\mathfrak{su}_k\,,\mathfrak{so}_{2k}\,,\mathfrak{e}_6\,,\mathfrak{e}_7\,,\mathfrak{e}_8$, respectively — appears $(N-1)$ times and is often referred to as a node of the quiver. With the links of minimal conformal matter, the theory $A^{\mathfrak{g}}_0$, this lends the name to the notion of generalized quivers, where bifundamental hypermultiplets are replaced by a generalization of matter with $\mathfrak{g}\oplus\mathfrak{g}$ flavor, glued together via fusion [67].

Up to a few outliers, the overwhelming majority of 6D SCFTs can be obtained from deformations of a few so-called "parent" theories with a base associated with the discrete ADE group Λ and a flavor symmetry dictated by an algebra $\mathfrak{g}$. We will follow the notation used extensively in [66] and denote the parent theories by $\Lambda^{\mathfrak{g}}$. The rank N conformal matter theories $A^{\mathfrak{g}}_{N-1}$ described above are an example of parent theories. If an SCFT is obtained by moving onto the Higgs branch of the parent theory via a deformation encoded in e.g. a nilpotent orbit $\mathcal{O}$, we label it as $\Lambda^{\mathfrak{g}}(\mathcal{O})$, see below for an example. The LST reached through minimal affinization is then denoted as $\widehat{\Lambda}^{\mathfrak{g}}(\mathcal{O})$, in the obvious notation.

For conformal matter, minimal affinization is a generalization of the example given in equation (3.3). In every case, we gauge the diagonal subalgebra of the $\mathfrak{g}\oplus\mathfrak{g}$ flavor, and one needs to introduce a new node with the appropriate self-intersection, as given below equation (3.10).

This can also be seen on the partial tensor branch, also called the endpoint of the theory, which is the point where all (-1)-curves have been successively blown down. There, we obtain a collection of (-2)-curves intersecting like the Dynkin diagram of the affine simple algebra $\widehat{A}_{N-1}$:

$$A^{\mathfrak{g}}_{N-1}:\quad [\mathfrak{g}]\,\underbrace{\overset{\mathfrak{g}}{2}\cdots\overset{\mathfrak{g}}{2}}_{N-1}\,[\mathfrak{g}]\quad\longrightarrow\quad \widehat{A}^{\mathfrak{g}}_{N-1}:\quad //\underbrace{\overset{\mathfrak{g}}{2}\cdots\overset{\mathfrak{g}}{2}}_{N-1}\overset{\mathfrak{g}}{2}//\,.\tag{3.11}$$

Due to their loop configuration, these theories are sometimes referred to as necklace LSTs, and we will denoted them as $\widehat{A}^{\mathfrak{g}}_{N-1}$, in the notation summarized above. At a generic point in the tensor branch, or in geometric terms when enough blow-ups have been performed to obtain a configuration with only non-minimal singularities, we have for each algebra

$$\begin{aligned}
\widehat{A}^{\mathfrak{su}_k}_{N-1}:&\quad //\,\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_k}{2}\,\dots\,\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_k}{2}\,//\,,\\
\widehat{A}^{\mathfrak{so}_{2k}}_{N-1}:&\quad //\,\overset{\mathfrak{sp}_{k-4}}{1}\,\overset{\mathfrak{so}_{2k}}{4}\,\overset{\mathfrak{sp}_{k-4}}{1}\,\dots\,\overset{\mathfrak{sp}_{k-4}}{1}\,\overset{\mathfrak{so}_{2k}}{4}\,\overset{\mathfrak{sp}_{k-4}}{1}\,\overset{\mathfrak{so}_{2k}}{4}\,//\,,\\
\widehat{A}^{\mathfrak{e}_6}_{N-1}:&\quad //1\overset{\mathfrak{su}_3}{3}1\overset{\mathfrak{e}_6}{6}1\overset{\mathfrak{su}_3}{3}1\overset{\mathfrak{e}_6}{6}\dots1\overset{\mathfrak{su}_3}{3}1\overset{\mathfrak{e}_6}{6}1\overset{\mathfrak{su}_3}{3}1\overset{\mathfrak{e}_6}{6}//\,,\\
\widehat{A}^{\mathfrak{e}_7}_{N-1}:&\quad //1\overset{\mathfrak{su}_2}{2}\overset{\mathfrak{so}_7}{3}\overset{\mathfrak{su}_2}{2}1\overset{\mathfrak{e}_7}{8}\dots\overset{\mathfrak{e}_7}{8}1\overset{\mathfrak{su}_2}{2}\overset{\mathfrak{so}_7}{3}\overset{\mathfrak{su}_2}{2}1\overset{\mathfrak{e}_7}{8}//\,,\\
\widehat{A}^{\mathfrak{e}_8}_{N-1}:&\quad //12\overset{\mathfrak{su}_2}{2}\overset{\mathfrak{g}_2}{3}1\overset{\mathfrak{f}_4}{5}1\overset{\mathfrak{g}_2}{3}\overset{\mathfrak{su}_2}{2}21\overset{\mathfrak{e}_8}{(12)}\dots\overset{\mathfrak{e}_8}{(12)}12\overset{\mathfrak{su}_2}{2}\overset{\mathfrak{g}_2}{3}1\overset{\mathfrak{f}_4}{5}1\overset{\mathfrak{g}_2}{3}\overset{\mathfrak{su}_2}{2}21\overset{\mathfrak{e}_8}{(12)}//\,.
\end{aligned}\tag{3.12}$$

A similar behavior occurs for the other types of SCFTs with ADE bases. For those of exceptional types $\Lambda = E_r$, each (-2)-curve can be decorated with an $\mathfrak{su}(d^i k)$ algebra, where d^i is the Kac label corresponding node of the base algebra, see table 3. The resulting theory has a flavor symmetry $\mathfrak{su}_k$ which can be gauged in order to obtain a well-defined LST, where the (-2)-curves now intersect like the Dynkin diagram of the corresponding affine algebra. For instance, for an E_6 base, we have:

$$E_6^{\mathfrak{su}_k}: \quad \begin{array}{ccccc} & & [\mathfrak{su}_k] & & \\ & & \overset{\mathfrak{su}_{2k}}{2} & & \\ \overset{\mathfrak{su}_k}{2} & \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_{3k}}{2} & \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_k}{2} \end{array} \quad \longrightarrow \quad \widehat{E}_6^{\mathfrak{su}_k}: \quad \begin{array}{ccccc} & & \overset{\mathfrak{su}_k}{2} & & \\ & & \overset{\mathfrak{su}_{2k}}{2} & & \\ \overset{\mathfrak{su}_k}{2} & \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_{3k}}{2} & \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_k}{2} \end{array} . \tag{3.13}$$

The same can be done for the other two cases, giving rise to the following LST quivers:

$$\widehat{E}_7^{\mathfrak{su}_k}: \quad \begin{array}{ccccccc} & & & \overset{\mathfrak{su}_{2k}}{2} & & & \\ \overset{\mathfrak{su}_k}{2} & \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_{3k}}{2} & \overset{\mathfrak{su}_{4k}}{2} & \overset{\mathfrak{su}_{3k}}{2} & \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_k}{2} \end{array} \qquad \widehat{E}_8^{\mathfrak{su}_k}: \quad \begin{array}{cccccccc} & & \overset{\mathfrak{su}_{3k}}{2} & & & & & \\ \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_{4k}}{2} & \overset{\mathfrak{su}_{6k}}{2} & \overset{\mathfrak{su}_{5k}}{2} & \overset{\mathfrak{su}_{4k}}{2} & \overset{\mathfrak{su}_{3k}}{2} & \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_k}{2} \end{array} \tag{3.14}$$

Finally, the case of D-type bases is slightly more subtle. With fibers of type $\mathfrak{su}_L$, gauge-anomaly-cancellation conditions demand that on the arbitrarily-long spine, L must be even:

$$D_N^{\mathfrak{su}_{2k}}: \quad \begin{array}{ccc} & \overset{\mathfrak{su}_k}{2} & \\ \overset{\mathfrak{su}_k}{2} & \overset{\mathfrak{su}_{2k}}{2} & \underbrace{\overset{\mathfrak{su}_{2k}}{2} \cdots \overset{\mathfrak{su}_{2k}}{2}}_{N-3} [\mathfrak{su}_{2k}] \end{array} . \tag{3.15}$$

Simply gauging the $\mathfrak{su}_{2k}$ flavor of the $D_N^{\mathfrak{su}_{2k}}$ theory does not give rise to an LST but rather another SCFT, $D_{N+1}^{\mathfrak{su}_{2k}}$. We can, however, break the $\mathfrak{su}_{2k}$ flavor algebra to $\mathfrak{su}_k$ by moving onto the Higgs branch of the SCFT. These types of deformations are classified by nilpotent orbits [68], and for $\mathfrak{su}_L$ algebras, they can be labelled by partitions of L. We denote a partition $L = \sum_{i=1}^{L} i m_i$ as $[1^{m_1}, 2^{m_2}, \ldots, L^{m_L}]$, and we omit entries where $m_i = 0$ for ease of reading.

Given a nilpotent orbit, the quiver configuration can be read off directly, and some of the data of the resulting SCFT follows straightforwardly from the associated group theory [66, 68, 69]. For the nilpotent orbit $[2^k]$ of $\mathfrak{su}_{2k}$, the quiver is given by

$$D_N^{\mathfrak{su}_{2k}}([2^k]): \quad \begin{array}{ccccc} & \overset{\mathfrak{su}_k}{2} & & [\mathfrak{su}_k] & \\ \overset{\mathfrak{su}_k}{2} & \overset{\mathfrak{su}_{2k}}{2} & \underbrace{\overset{\mathfrak{su}_{2k}}{2} \cdots \overset{\mathfrak{su}_{2k}}{2}}_{N-5} & \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_k}{2} \end{array} . \tag{3.16}$$

The LST is then obtained via minimal affinization by gauging the flavor symmetry, which is achieved without changing the defect group:

$$\widehat{D}_N^{\mathfrak{su}_{2k}}([2^k]): \quad \begin{array}{ccccc} & \overset{\mathfrak{su}_k}{2} & & \overset{\mathfrak{su}_k}{2} & \\ \overset{\mathfrak{su}_k}{2} & \overset{\mathfrak{su}_{2k}}{2} & \underbrace{\overset{\mathfrak{su}_{2k}}{2} \cdots \overset{\mathfrak{su}_{2k}}{2}}_{N-5} & \overset{\mathfrak{su}_{2k}}{2} & \overset{\mathfrak{su}_k}{2} \end{array} . \tag{3.17}$$

Note that as for the exceptional bases — and trivially for $A_{N-1}^{\mathfrak{su}_k}$ — all fibers are now of the form $\mathfrak{su}(d^i \cdot k)$ with d^i the Kac labels of $\mathfrak{so}_{2N}$, see table 3.

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To summarize, in all ADE cases, we start with an SCFT $\Lambda^{\mathfrak{g}}$ where Λ is a discrete ADE group associated to a simple algebra $\mathfrak{g}_B$ that describes the singularity structure of the base of the F-theory geometry (and thus, after resolution, the intersection properties of the base divisors), and $\mathfrak{g}$ describe both the generic fibers at the origin of the tensor branch and the flavor of the parent theory. Up to additional possible deformations, the SCFT has flavor symmetry $\mathfrak{f}$, and after the minimal affinization procedure we obtain an LST labelled uniquely by the two algebras $\mathfrak{g}_F$ and $\mathfrak{g}_B$. Pictorially, we have

$$\text{SCFT:} \quad \Lambda^{\mathfrak{g}}(\mathcal{O}) \qquad \xrightarrow{\text{min. affinization}} \qquad \text{LST:} \quad \mathcal{K}^{\text{II}}(\mathfrak{g}_F, \mathfrak{g}_B) = \widehat{\Lambda}^{\mathfrak{g}}(\mathcal{O})\,. \tag{3.18}$$

The three LSTs with ADE base are given by

$$\mathcal{K}^{\text{II}}(\mathfrak{g}_F, \mathfrak{su}_N) = \widehat{A}_N^{\mathfrak{g}_F}\,, \qquad \mathcal{K}^{\text{II}}(\mathfrak{su}_k, \mathfrak{so}_{2N}) = \widehat{D}_N^{\mathfrak{su}_{2k}}([2^k]) \qquad \mathcal{K}^{\text{II}}(\mathfrak{su}_k, \mathfrak{e}_N) = \widehat{E}_N^{\mathfrak{su}_k}\,. \tag{3.19}$$

For all these theories, one can show by direct computation using equation (2.11) that the two-group invariants are arrange neatly in terms of quantities related to $\mathfrak{g}_F$ and $\mathfrak{g}_B$:

$$\mathcal{K}^{\text{II}}(\mathfrak{g}_F, \mathfrak{g}_B): \qquad \kappa_R = \Gamma_{\mathfrak{g}_F}\Gamma_{\mathfrak{g}_B}\,, \qquad \kappa_P = 0\,, \qquad \dim(\text{CB}) = h^\vee_{\mathfrak{g}_F} h^\vee_{\mathfrak{g}_B} - 1\,, \tag{3.20}$$

where $\Gamma_{\mathfrak{g}}$ is the order of the discrete ADE group associated with the algebra $\mathfrak{g}$. Their values are given in table 10. The factorization of these quantities in terms of group-theoretic data of the fiber and base makes the T-duality between the two classes of LSTs manifest:

$$\mathcal{K}^{\text{II}}(\mathfrak{su}_N, \mathfrak{g}) \qquad \overset{\text{T-duality}}{\longleftrightarrow} \qquad \mathcal{K}^{\text{II}}(\mathfrak{g}, \mathfrak{su}_N)\,. \tag{3.21}$$

It is quite instructive to see how these structure constants arise from the point of view of minimal affinization and the anomaly polynomial of the associated SCFTs. One of the corollaries of fusion is that the anomaly polynomial of the vast majority of SCFTs can be obtained from that of minimal conformal matter, $A_0^{\mathfrak{g}}$ or a deformation thereof [65, 66, 70]. This can be used to write the anomaly polynomial of conformal matter in a compact form, depending only on the rank N of the base and group-theoretic quantities related to the algebra $\mathfrak{g}$ [65]:

$$\begin{aligned} I_8(A_{N-1}^{\mathfrak{g}}) = {} & \frac{N^3}{24}(c_2(R)\Gamma_{\mathfrak{g}})^2 - \frac{1}{2}(c_2(R)\Gamma)\left(J(F_L) + J(F_R)\right) - \frac{1}{2N}\left[J(F_L) - J(F_R)\right]^2 \\ & + I_{\text{sing}} - I_8^{\text{tensor}} - \frac{1}{2}\left[I_8^{\text{vec}}(F_L) - I_8^{\text{vec}}(F_R)\right]\,, \end{aligned} \tag{3.22}$$

where F_L, F_R refer to the background fields for the two flavor symmetries $\mathfrak{g}$. The contributions I_8^{tensor} and I_8^{vec} of the tensor and vector multiplets, together with I_{sing}, are given in appendix B. Furthermore, the four-form $J(F)$ is

$$J(F) = \frac{\chi}{48}\left(4c_2(R) + p_1(T)\right) + c_2(F)\,, \qquad \chi = \text{rk}(\mathfrak{g}) + 1 - \frac{1}{\Gamma_{\mathfrak{g}}}\,. \tag{3.23}$$

Since the Bianchi identity of the associated two-form is $dH_3^{\text{CM}} \sim J(F)$, it can be understood as the charge of the BPS string associated with conformal matter at the conformal fixed point. This means that the first line of equation (3.22) is generated entirely by the Green-Schwarz

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term. Furthermore, this equation makes it clear that minimal affinization for conformal matter sets $F_L = F_R$, and adds the contribution of the vector multiplet introduced through the gauging of the flavor, as explained around equation (3.5). Doing so automatically removes all quartic contributions $(\mathrm{Tr}\, F^2)^2$ and $\mathrm{Tr}\, F^4$, which would otherwise lead to gauge anomalies, and the two-group constants are readily found to be given by equation (3.20).

As a consequence of 't Hooft anomaly matching, the anomaly polynomial does not change as we move in the tensor branch by successively blowing down all (-1)-curves. We can therefore work directly on the partial tensor branch: even though the quiver consists of multiple tensor, hyper- and vector multiplets, we can treat each of the fused N minimal conformal matter links $A_0^{\mathfrak{g}}$ as a single entity. By the argument above, there are no quartic gauge anomalies, and by carefully tracking the changes in the different quantities, as explained in appendix B, one finds that the Green-Schwarz contribution of the SCFT *on the partial tensor branch* takes the form

$$I_8^{\mathrm{GS}} = \frac{1}{2} A_{ij} I^i I^j \,, \qquad I^i = -A^{ij} c_2(F_i) + B^{ia} J(F_a) + d^i\, \Gamma_{\mathfrak{g}} c_2(R)\,. \tag{3.24}$$

Here, A^{ij} is the pairing matrix at the conformal fixed point and A_{ij} is its inverse. This pairing matrix is simply given by the Cartan matrix of A_{N-1}, and should be distinguished from the pairing matrix at a generic point, which we have denoted by η. Moreover, we have $d^i = 1$ for all i, and the flavor pairing matrix can be written as

$$B^{ia} = (A^{ij} d_j)\, \delta^{ia} \,, \qquad A \cdot d = (1, 0,\ \ldots\ , 0, 1)\,. \tag{3.25}$$

While writing the pairing matrix in this form might seem *ad hoc*, if we interpret d^i as the Kac labels of the base A_{N-1}, the factorization of the structure constants shown in equation (3.20) becomes apparent. Indeed, using that for a simply-laced algebra $\mathfrak{g}$, we have the relation

$$\Gamma_{\mathfrak{g}} = 1 + \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} (d^i)^2 \,, \tag{3.26}$$

and the Green-Schwarz term of the SCFT can be found to be

$$I_8^{\mathrm{GS}} = -\frac{1}{2} \Gamma_{\mathfrak{su}_k} \Gamma_{\mathfrak{g}} c_2(R) \big(J(F_L) + J(F_R)\big) + \ldots \tag{3.27}$$

This generalizes to all three families of SCFTs with an ADE base discussed above. The Green-Schwarz term in equation (3.24) is given in terms of the Kac labels d^i of the algebra of base, in the convention of table 3.[15] The flavor pairing matrix is again given in terms of $(A \cdot d)$, and a similar reasoning explains the numerical coincidences observed in the structure constants.[16] The Kac labels will similarly play an important role in the geometric engineering of the Type II LSTs in section 4.

[15] Note that here, the coefficients d^i are Kac labels of the simple algebra $\mathfrak{g}$, and not its affine version. For simply-laced algebras, they also correspond to the highest root of $\mathfrak{g}$ in the Serre-Chevalley basis. Moreover, they are sometimes also referred to as Dynkin multiplicities in the literature.

[16] In the case of $D_N^{\mathfrak{su}_{2k}}([2^k])$, $\Gamma_{\mathfrak{su}_{2k}}$ must be substituted for that of the flavor symmetry $\mathfrak{su}_k$.

We find that not only the structure constants, but the complete anomaly polynomial, can be described purely in terms of group-theoretic data of the base and fiber algebras,

$$I_8\left(\mathcal{K}(\mathfrak{g}_F,\mathfrak{g}_B)\right) = \frac{\mathrm{rk}(\mathfrak{g}_B)+1}{24}(\Gamma_{\mathfrak{g}_B}\Gamma_{\mathfrak{g}_F}\, c_2(R))^2 - \Gamma_{\mathfrak{g}_B}\Gamma_{\mathfrak{g}_F}\, c_2(R)c_2(F) + (\mathrm{rk}(\mathfrak{g}_B)+1)I_8^{\mathrm{sing}} \\ - \frac{1}{48}\left[\Gamma_{\mathfrak{g}_B}\Gamma_{\mathfrak{g}_F}^2 - (\mathrm{rk}(\mathfrak{g}_B)+1)\right] c_2(R)(4c_2(R)+p_1(T)) - I_8^{\mathrm{tensor}}\,. \tag{3.28}$$

This generalizes the minimal affinization of the anomaly polynomial of necklace quivers given in equation (3.22) to the dual LSTs, and can be checked explicitly for $\widehat{E}_N^{\mathfrak{su}_k}$ and $\widehat{D}_N^{\mathfrak{su}_{2k}}([2^k])$ by using the algorithm to determine the anomaly polynomial on the tensor branch.

A similar analysis can be performed for the dimension of the Coulomb branch. It is easy to check that for minimal conformal matter, the dimension can be written similarly compactly in terms of group-theory data of the fiber algebra:

$$\dim(\mathrm{CB}_{A_0^{\mathfrak{g}}}) = h^\vee_{\mathfrak{g}} - \mathrm{rk}(\mathfrak{g}) - 1\,. \tag{3.29}$$

On the partial tensor branch, the i^{th} curve of the base is decorated with an algebra $\mathfrak{g}^i$ of rank $\mathrm{rk}(\mathfrak{g}^i) = d^i\ \mathrm{rk}(\mathfrak{g}_F)$. Taking into account the vector multiplets associated with each node, we then recover the expected result given in equation (3.20).

We can therefore explain the numerical coincidences behind T-duality at the level of the anomaly polynomial from group theory, and in particular by looking on the partial tensor branch directly. In equation (3.28), the duality is realized easily by simply exchanging the two algebras $\mathfrak{g}_F \leftrightarrow \mathfrak{g}_B$.

Note that while we mainly used the language of the F-theory construction throughout this section, every step has an interpretation in the field-theory description. Fusion is a well-defined six-dimensional operation for the gauge theories even at the non-perturbative level, and blowing down curves corresponds to moving to a specific point of the tensor branch. The expressions we have used are then guaranteed to be correct by 't Hooft anomaly matching. This remains true for other types of LSTs as well: we can always start by computing the anomaly polynomial at a generic, weakly-coupled point of the tensor branch, and track how quantities change at the conformal fixed point. Minimal affinization then simply adds a contribution from a single vector multiplet to the anomaly polynomial from which the two-group structure constants can be extracted.

Before moving on to LSTs beyond necklace quivers and their T-dual theories, let us comment on the possibility of minimal affinization for theories of type $\widehat{\Lambda}^{\mathfrak{su}_k}$, but where the rank of every $\mathfrak{su}$-type algebra over any curve is not fixed by the value of the Kac labels of the base algebra. In other words, can we pick any choice of fibers for ADE bases? The possibilities are of course severely constrained by anomaly-cancellation conditions: allowing for different gauge algebras $\mathfrak{su}_{k_I}$, quartic traces $\mathrm{Tr}\, F_I^4$ must be cancelled. This imposes that there must be $2k_I$ hypermultiplets in the fundamental representation of $\mathfrak{su}_{k_I}$. Taking into account bifundamental hypermultiplets $(\boldsymbol{k}_I, \bar{\boldsymbol{k}}_J)$ arising at the intersection of two curves, there generically remains f^I hypermultiplets, which are rotated by an $\mathfrak{su}_{f^I}$ flavor symmetry. It is easy to check that the constraints on the possible values for f^I is given in terms of the Cartan matrix of $\widehat{A}^{IJ}$ associated with the base $\widehat{\Lambda}$:

$$f^I = \widehat{A}^{IJ} k_J\,. \tag{3.30}$$

While there are non-trivial integer solutions to this equation for SCFTs, the only consistent choice of k_I for LSTs are null vectors of $\widehat{A}^{IJ}$, so that $f^I = 0$. This forces the k^I to be a multiple of the Kac labels of the affine algebra of the base, and the consistent choices reduce to the cases discussed above. For $\widehat{A}_{N-1}$ bases, the generalization to fiber algebras of type DE is straightforward, and we conclude that there cannot be any necklace LST with non-trivial flavor symmetry, and similarly for their T-duals. Here, we have considered only non-Abelian algebras associated with a particular curve I as we have relied on cancellation of quartic traces, and there is therefore still a possibility for Abelian symmetries, or flavor factors "delocalized" along the quiver rotating composite gauge-invariant operators. The possibilities are, however, severely constrained by the worldsheet bound.

3.2 Beyond necklace quivers

We have seen that for ADE orbifolds, minimal affinization simply corresponds to considering the affine extension of the Dynkin diagram of the discrete group $\Lambda \subset \mathrm{SU}(2)$, which is obtained by gauging the flavor symmetry. All the relevant quantities can then be directly computed on the partial tensor branch.

SCFTs whose bases are associated with discrete groups $\Lambda \subset \mathrm{U}(2)$ that are not of ADE type are usually referred to as generalized orbifolds. In the M-theory picture, these SCFTs are — up to a few outliers — associated with fractional M5-branes probing frozen singularities [38, 70–74]. However, these non-ADE bases cannot be affinized minimally.

Let us give an example. The affinization of the base

$$\Lambda : 3\underbrace{22\ldots22}_{N-2}3 \qquad \longrightarrow \qquad \widehat{\Lambda} : \qquad //3\underbrace{22\ldots22}_{N-2}31// \tag{3.31}$$

does not preserve the defect group. Indeed, when blowing down the new (-1)-curve, we get $\widehat{\Lambda} \simeq \widehat{A}_{N-2}$, and the resulting LST has a defect group $\mathcal{D}^{(2)}(\widehat{\Lambda}) = \mathbb{Z}_{N-1}$. However, the SCFT has $\mathcal{D}^{(2)}(\Lambda) = \mathbb{Z}_{4N}$, and this violates one of the defining conditions of minimal affinization.

It is a straightforward exercise to show that only ADE bases can be affinized minimally. One of course recovers the original geometric classification [8]: on the partial tensor branch, the pairing matrix of all LSTs corresponds the Dynkin diagram of an affine ADE algebra, see table 3. There are two exceptions, associated with the affinization of an (endpoint) (-1)-curve:

$$11 \to 0\,, \qquad 1||4 \to 0\cdot\,. \tag{3.32}$$

The former corresponds to the endpoint of Heterotic LSTs I_0, while the latter gives rise to a Type II LST with a base associated with a Kodaira singularity of Type II. We differentiate these two bases by including an additional dot in the latter case, and write $0\cdot$, as in equation (3.32). In two further cases corresponding to Type III and IV singularities in the Kodaira classification, the bases of the LST have the same pairing matrix as those of A_1 and A_2, respectively, but the geometry of the base is different. They correspond to a tangential intersection of two curves or a triple intersection point,

$$\mathrm{III}: \quad 2||2\,, \qquad \qquad \mathrm{IV}: \quad 2\overset{2}{\Delta}2\,. \tag{3.33}$$

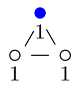
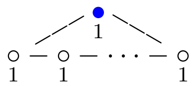
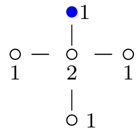
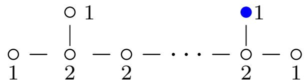
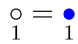
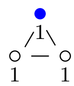
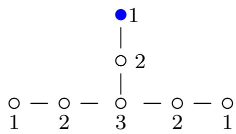
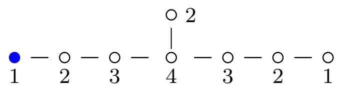
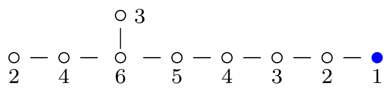
Kodaira	G	Endpoint	$\mathcal{D}^{(2)}$	Affine Dynkin diagram
$\mathbb{P}^1$	$\varnothing$	$11 \to 0$	$\varnothing$	
I_0	$\varnothing$	0	$\varnothing$	
I_3	A_2	$//222// = 2\overset{2}{}2$	$\mathbb{Z}_3$	
I_N	A_{N-1}	$//\underbrace{22\cdots22}_{N-1}2//$	$\mathbb{Z}_n$	
I_0^*	D_4	$\underset{2}{\overset{2}{222}}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	
I_{N-4}^*	D_N	$\overset{2}{2}2\,\underbrace{22\ldots22}_{N-5}\,\overset{2}{2}2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$	
II	$H_1 \simeq A_0$	$1\|\|4 \to 0\cdot$	$\varnothing$	
III	$H_2 \simeq A_1$	$2\|\|2$	$\mathbb{Z}_2$	
IV	$H_3 \simeq A_2$	$\overset{2}{2\Delta2}$	$\mathbb{Z}_3$	
IV^*	E_6	$22\overset{\overset{2}{2}}{2}22$	$\mathbb{Z}_3$	
III^*	E_7	$222\overset{2}{2}222$	$\mathbb{Z}_2$	
II^*	E_8	$22\overset{2}{2}22222$	$\varnothing$	

Table 3. Possible (endpoint) bases associated with LSTs obeying the tensor-decoupling condition and preserving the defect groups of the associated SCFTs. We show those of the Heterotic LSTS in the first line for completeness. The algebras for the Kodaira singularities of type II, III, IV are sometimes denoted H_i to differentiate them from those of the I_n series. Each node of the Dynkin diagram is denoted by its Kac label d^i. The curve defined by minimal affinization and the corresponding affine node in the Dynkin diagram are colored blue.

Given a Type II LST base, there are very few choices of compatible algebras. From the minimal affinization point of view, this is partly explained from the discussion above. A_N-type bases only give rise to conformal matter, and we have seen that their nilpotent deformations cannot lead to a gauge-anomaly free LST, and similarly for the three exceptional classes $\widehat{E}_N^{\mathfrak{su}_k}$. Leaving out the special bases given in equations (3.32) and (3.33) for a moment, we are then left with D_N bases. The trivalent pattern severely constrains the possible SCFT [75, 76], which in addition to the $\mathfrak{su}_{2k}$ treated above, can only have $\mathfrak{g} \in \{\mathfrak{su}_3, \mathfrak{so}_8, \mathfrak{e}_6\}$. As with $\mathcal{K}(\mathfrak{su}_k, \mathfrak{so}_{2L})$, the corresponding flavor must first be broken via a nilpotent deformation before minimal affinization can be performed. This once again fixes the choice of nilpotent orbit, and there is a unique possibility for each SCFT, leading to the following three infinite series of LSTs:

$$\widehat{D}_N^{\mathfrak{su}_3}([3^1]): \quad 2\,\overset{\mathfrak{su}_2}{\underset{}{\overset{2}{2}}}\,\underbrace{\overset{\mathfrak{su}_3}{2}\,\overset{\mathfrak{su}_3}{2}\cdots\overset{\mathfrak{su}_3}{2}}_{N-5}\,\overset{\mathfrak{su}_2}{\overset{\color{blue}2}{2}}\,2\,, \qquad \widehat{D}_N^{\mathfrak{so}_8}([4^2]): \quad \overset{\mathfrak{su}_2}{2}\,\overset{\overset{\mathfrak{su}_2}{2}}{\overset{\mathfrak{so}_7}{3}}\,\underbrace{1\,\overset{\mathfrak{so}_8}{4}\,1\cdots1\,\overset{\mathfrak{so}_8}{4}\,1}_{N-5}\,\overset{\color{blue}\overset{\mathfrak{su}_2}{2}}{\overset{\mathfrak{so}_7}{3}}\,\overset{\mathfrak{su}_2}{2}\,,$$
$$\widehat{D}_N^{\mathfrak{e}_6}(D_4): \quad \overset{\mathfrak{su}_3}{3}\,1\,\overset{\overset{\overset{\mathfrak{su}_3}{3}}{1}}{\overset{\mathfrak{e}_6}{6}}\,\underbrace{1\,\overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6}\cdots\overset{\mathfrak{e}_6}{6}\,1\,\overset{\mathfrak{su}_3}{3}\,1}_{N-5}\,\overset{\color{blue}\overset{\overset{\mathfrak{su}_3}{3}}{1}}{\overset{\mathfrak{e}_6}{6}}\,1\,\overset{\mathfrak{su}_3}{3}\,. \tag{3.34}$$

Note that in the case of $D_N^{\mathfrak{e}_6}$, the breaking is done with the so-called D_4 nilpotent orbit in the Bala-Carter notation [77, 78]. More information about these three families of SCFTs and their nilpotent breaking can be found in [66].

For the case of $\widehat{D}_N^{\mathfrak{su}_3}([3^1])$, we see that the minimal affinization does not involve any flavor. Indeed, the nilpotent orbit associated with the partition $[3^1]$ breaks the original $\mathfrak{su}_3$ flavor completely. The new curve, however, has a fiber with a singularity of Type II, which we will explore in more detail in section 4. The other two series of LSTs also have special singularities of Type III and IV rather than their ADE counterparts. Their 2-group invariants are straightforwardly found to be

$$\begin{aligned}
\mathcal{K}(\mathrm{II}, \mathfrak{so}_{2N}) &= \widehat{D}_N^{\mathfrak{su}_3}([3^1]): \quad & \kappa_R &= 6(N-3)\,, \quad & \dim(\mathrm{CB}) &= 3(N-2)-2\,, \\
\mathcal{K}(\mathrm{III}, \mathfrak{so}_{2N}) &= \widehat{D}_N^{\mathfrak{so}_8}([4^2]): \quad & \kappa_R &= 16(N-3)\,, \quad & \dim(\mathrm{CB}) &= 6(N-2)-2\,, \\
\mathcal{K}(\mathrm{IV}, \mathfrak{so}_{2N}) &= \widehat{D}_N^{\mathfrak{e}_6}(D_4): \quad & \kappa_R &= 48(N-3)\,, \quad & \dim(\mathrm{CB}) &= 12(N-2)-2\,.
\end{aligned} \tag{3.35}$$

We see that minimal affinization gives us precious information about the dual theory. Since the SCFTs have a flavor algebra $\mathfrak{su}_3$, $\mathfrak{so}_8$, or $\mathfrak{e}_6$, as can be seen from the spine, one might naively expect that we should look for LSTs with such bases. However, the correct bases are of Kodaira type II, III, IV. This will become clear when we consider the geometric engineering of these theory in section 4. Moreover, knowing that the affine curve should have an algebra $\mathfrak{g} = \mathfrak{so}_{2N}$ is helpful to find the correct T-dual theories,

$$\mathcal{K}(\mathfrak{so}_{2N}, \mathrm{II}): \quad {\color{blue}\overset{\mathfrak{so}_{2N}}{0}\cdot} \quad \longrightarrow \quad \overset{\mathfrak{su}_{2(N-4)}}{1} \,||\, {\color{blue}\overset{\mathfrak{so}_{2N}}{4}}$$

$$\mathcal{K}(\mathfrak{so}_{2N}, \mathrm{III}): \quad \overset{\mathfrak{so}_{2N}}{2} \,{\color{blue}||}\, {\color{blue}\overset{\mathfrak{so}_{2N}}{2}} \quad \longrightarrow \quad \overset{\mathfrak{so}_{2N}}{4}\,\overset{\overset{\mathfrak{su}_{2(N-4)}}{2}}{\overset{\mathfrak{sp}_{2(N-4)}}{1}}\,{\color{blue}\overset{\mathfrak{so}_{2N}}{4}}$$

$$\mathcal{K}(\mathfrak{so}_{2N},\mathrm{IV}):\quad \overset{\mathfrak{so}_{2N}}{2}\;\overset{\overset{\mathfrak{so}_{2N}}{2}}{\Delta}\;\overset{\mathfrak{so}_{2N}}{2} \qquad\longrightarrow\qquad \overset{\mathfrak{so}_{2N}}{4}\;\overset{\mathfrak{sp}_{2(N-4)}}{1}\;\overset{\overset{\overset{\mathfrak{so}_{2N}}{4}}{\overset{\mathfrak{sp}_{2(N-4)}}{1}}}{\overset{\mathfrak{so}_{6N-16}}{4}}\;\overset{\mathfrak{sp}_{2(N-4)}}{1}\;\overset{\mathfrak{so}_{2N+8}}{4} \tag{3.36}$$

We have chosen $N > 4$ as there are certain special enhancements for $\mathfrak{so}_8 \simeq D_4$ bases which do not fit the patterns above. We will treat those cases in more detail in the next section.

The LST $\mathcal{K}^{\mathrm{II}}(III, \mathfrak{so}_{2N})$ and its dual exhibit an additional peculiar feature. The LST is reached from the $D_N^{\mathfrak{so}_8}$ SCFT via the nilpotent orbit associated with the very-even partition $[4^2]$. However, there are *two* nilpotent orbits related to this partition, usually denoted $[4^2]_I$ and $[4^2]_{II}$. While the resulting quiver is the same, the Nambu-Goldstone modes that decouple along the RG flow are different, and this leads to two inequivalent SCFTs. The difference stems from the fact that the SCFT $\overset{\mathfrak{sp}_{m-4}}{1}[\mathfrak{so}_{4m}]$ has a Higgs-branch operator in its spectrum transforming in one of the two spin representations of the $\mathfrak{so}_{4m}$ flavor symmetry [79, 80]. This choice of spin representation selects one of the two theories. When that curve configuration is part of a larger quiver there can also be physically different theories, which is revealed after a careful analysis of certain gauge-invariant operators. This was shown to be the case for nilpotent deformations of D-type conformal matter associated with very-even partitions [80]. Similarly, since the SCFT associated with the LST $\mathcal{K}^{\mathrm{II}}(\mathfrak{so}_{2N}, \mathrm{III})$ also has a single (-1)-curve decorated with an $\mathfrak{sp}_m$ algebra, we again have two different theories. Performing minimal affinization, it is natural to expect two different LSTs as well, giving a nice additional cross-check of T-duality.

Let us comment on the possible flavor symmetries arising in Type II LSTs, which we know to be constrained from the bound given equation (2.49). We find that there is single pair of T-dual LSTs with a flavor symmetry. They have an $\mathfrak{su}_2$ symmetry, and are given by the two quivers

$$\mathcal{K}(\mathfrak{e}_6,\mathrm{II}):\quad [\mathfrak{su}_2]\,2\,\overset{\overset{\mathfrak{e}_6}{6}}{1}\,\overset{\mathfrak{su}_3}{3}\,, \qquad\qquad \mathcal{K}(\mathrm{II},E_6):\quad 2\,\overset{\mathfrak{su}_2}{2}\,\underset{[\mathfrak{su}_2]}{\overset{\overset{\overset{2}{\overset{\mathfrak{su}_2}{2}}}{\mathfrak{g}_2}}{2}}\,\overset{\mathfrak{su}_2}{2}\,2\,. \tag{3.37}$$

The first LST $\mathcal{K}(\mathfrak{e}_6, \mathrm{II})$ has a trival defect group and a one-form symmetry $\mathcal{D}^{(1)} = \mathbb{Z}_3^{(1)}$, converted into the defect group $\mathcal{D}^{(2)} = \mathbb{Z}_3^{(2)}$ associated with the center of the affine $\mathfrak{e}_6$ base under the duality. Moreover, the pair has the following duality invariants:

$$\kappa_G = 0\,, \qquad \kappa_R = 27\,, \qquad \mathrm{rk}(\mathfrak{f}) = \mathrm{rk}(\mathfrak{su}_2) = 1\,, \qquad \dim(\mathrm{CB}) = 11\,. \tag{3.38}$$

In addition, we find that in both cases, $\kappa_{\mathfrak{f}}^{\mathfrak{su}_2} = 3$.

Minimal affinization therefore gives us the correct information to find the T-dual theory. Indeed, we can systematically scan over all possible SCFT bases and affinize them to LSTs. Demanding minimal affinization then greatly simplifies the search for dual pairs. All Type II LSTs and their two-group invariants are collected in appendix A, which we will now construct geometrically.

4 The geometry of Type II LSTs

In this section, we first review the geometric construction of Type II LSTs, and illustrate the procedure using F-theory on elliptic Calabi-Yau threefolds with multiple fibrations. For a givem choice of algebras associated with fiber and base singularities, this will allow us to construct the corresponding LST and to identify T-dual pairs.

4.1 LSTs and double fibrations

We first take a more general perspective on the F-theory geometry X_3 and discuss which type of LSTs can be obtained on general grounds. Our starting point will be compact threefolds X_3 and their possible elliptic fibration structures. By taking a decompactification limit of X_3 which keeps a curve of self-intersection 0 in the base compact, we can decouple gravity and obtain an LST. Note that there are infinite classes of LSTs that cannot be coupled consistently to gravity, and therefore cannot be obtained via a decompactification limit of a supergravity theory.

Indeed, we want to argue that only Heterotic and Type II LSTs can be obtained by decoupling gravity. We start by imposing two conditions on X_3 that are necessary to give rise to an LST via F-theory:

1. **F-theory condition:** X_3 needs to have a torus fibration with base B_2.
2. **Little String condition:** B_2 has to have a curve D_b of self-intersection $D_b^2 = 0$.

Both conditions have appeared in works by Kollár, Oguiso, and Wilson [81–83] in their general study of the possible fibration structures of Calabi-Yau threefolds. For each linear independent nef divisor D_F that satisfies $D_F^3 = 0$ but $D_F^2 \neq 0$, there exists an inequivalent torus fibration.[17] Intuitively, one may view D_F as a vertical divisor $D_B \in B_2$. Such divisors identify a fibration structure and have vanishing triple self-intersection in the full threefold.

In his proof of Kollár's conjecture for threefolds [82], Oguiso classified the general fibration structures X_3 can have, employing only the two following intersection properties of an ample nef divisor D_B:

$$D_B^{\nu+1} = 0\,, \quad D_B \cdot c_2(X_3) = \delta\,, \tag{4.1}$$

with $c_2(X_2)$ the second Chern class of the tangent bundle of X_3. Depending on the values of ν and δ, X_3 admits a fibration structure with $\pi(X_3) = B_2$ and with general fibers $\pi^{-1}(B_2) = F$, as summarized in table 4.

Oguiso Type II corresponds to the F-theory condition, i.e., X_3 has a torus fibration with $B = \Sigma_2$ a rational surface. The LST condition requires a curve $\mathcal{C} \in B_2$ with $\mathcal{C}^2 = 0$, which, when pulled back to a surface $S \in X_3$, satisfies $S^2 \cdot D = 0$ for any divisor $D \in X_3$. This tells us that we have a divisor of Oguiso Type I. Thus, when combining these two conditions, we require a divisor D_F with $D_F^3 = 0$ and a divisor S with $S^2 = 0$, compatible with the double fibration structure. The two subtypes of Oguiso Type I divisors correspond to different fibration structures and hence two different LSTs: Type I_+ has a K3 fibration

[17] A divisor D_F is nef if $D_F \cdot \mathcal{C} \geq 0$ for every algebraic curve $\mathcal{C} \in X_3$.

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Type	ν	δ	Structure of B and F
I_+	1	$\delta > 0$	$B_2 = \mathbb{P}^1$ and $F = K3$
I_0	1	$\delta = 0$	$B_2 = \mathbb{P}^1$ and $F = \mathbb{T}^4$
II_+	2	$\delta > 0$	$B_2 = \Sigma_2$ and $F = \mathbb{T}^2$

Table 4. Fibration structures of a threefold X_3 with a divisor D_B satisfying (4.1). There are also fibrations of Oguiso Type II_0 and III, which we omitted since they are not relevant in this work.

and together with a compatible Oguiso Type II structure, it must be an elliptic K3, while for the Type I_0 case the fiber is $\mathbb{T}^2 \times \mathbb{T}^2$. We conclude that there can be only two types of LSTs that can be obtained by decoupling gravity in F-theory compactification, where the possible bases B_2 of the elliptic fibration are

Oguiso Type	LST Type	Base Topology
I_+	Heterotic	$B_2 \sim \mathbb{P}^1 \times \mathbb{C}$
I_0	Type II	$B_2 \sim (\mathbb{T}^2 \times \mathbb{C})/\Lambda$

(4.2)

Note that we have added the possibility of a quotient Λ, as the fibration may have Kodaira fibers. Typical examples of Oguiso Type I_0 bases are rational elliptic surfaces, such as dP_9. These may have reducible Kodaira fibers at the origin of $\mathbb{C}$.

4.2 Type II LSTs from gravity decoupling

To describe an elliptically fibered threefold with a base that is itself and elliptic fibration, we can start with a compact geometry and subsequently decompactify it. A good starting point for Type II LSTs with a double elliptic fibration structure is the Schoen manifold,[18] which features prominently in string constructions. The simplest description for the Schoen manifold is given by the split bi-cubic

$$\left[\begin{array}{c|cc} \mathbb{P}^2 & 3 & 0 \\ \mathbb{P}^2 & 0 & 3 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right] \tag{4.3}$$

We can then define two hypersurfaces P_F and P_B which are cubics in either of the two ambient $\mathbb{P}^2$, and linear in their common $\mathbb{P}^1$. Via a change of coordinates, this can be mapped into a Tate model given by the GSLM matrix

X	Y	Z	$\hat{X}$	$\hat{Y}$	$\hat{Z}$	t_0	t_1	P_B	P_F
2	3	1	0	0	0	0	0	0	6
0	0	0	2	3	1	0	0	6	0
0	0	−1	0	0	−1	1	1	0	0

(4.4)

[18] In fact, in [84] it was shown that there exists an infinite family of elliptic fibration structures for the Schoen manifold.

The two hypersurface equations are simply the following two polynomials in Tate form:

$$\begin{aligned} P_F &= Y^2 + X^3 + XYZa_1 + X^2Z^2a_2 + YZ^3a_3 + XZ^4a_4 + Z^6a_6\,, \\ P_B &= \hat{Y}^2 + \hat{X}^3 + \hat{X}\hat{Y}\hat{Z}b_1 + \hat{X}^2\hat{Z}^2b_2 + \hat{Y}\hat{Z}^3b_3 + \hat{X}\hat{Z}^4b_4 + \hat{Z}^6b_6\,, \end{aligned} \tag{4.5}$$

with a_i and b_i generic polynomials in the $\mathbb{P}^1$ coordinates $[t_0 : t_1]$ of degree i. The split bi-cubic and the double Tate model differ slightly in their singularity structures. However, this difference will disappear upon decompactification. The description makes the double fibration over $\mathbb{P}^1$ manifest: we can treat either $\{p_F = 0\}$ or $\{p_B = 0\}$ as the F-theory torus. In either case, the discriminant of the base is of degree $\deg(\Delta_b) = 12$ and hence a rational elliptic surface.

For concreteness, we take $\{p_B = 0\}$ to be the base of the fibration. There are two divisors in the base B_2 that are of main interest: first, $\{\hat{Z} = 0\}$ is the section of the rational elliptic surface and hence a copy of the base $\mathbb{P}^1_{t_0,t_1}$. Secondly, the two linear equivalent divisors $\{t_i = 0\}$ are points in the base $\mathbb{P}^1$ and thus a copy of the torus fiber, which in the full threefold is $\mathbb{T}^2 \times \mathbb{T}^2$. Before engineering various singularities in the fiber and the base, we need to decompactify the theory to a Type II LST. We can perform this decompactification at the level of the ambient space by taking the volume of the $\mathbb{P}^1_t$ factor to infinity, replacing

$$\mathbb{P}^1_{[t_0,t_1]} \to \mathbb{C}_{t_0}\,. \tag{4.6}$$

We can therefore simply focus on the patch $t_1 = 1$, where we rename $t_0 = t$. We can then factorize a_i and b_i in t to obtain singularities in either F-theory fiber or base.

4.3 Fiber and base singularity structure

A Type II LST $\mathcal{K}^{\mathrm{II}}(\mathfrak{g}_F, \mathfrak{g}_B)$ is specified by a fiber and base with a singularity in the Kodaira classification. To reduce notational burden and increase readability, we will use the name of Kodaira singularities in the text, but otherwise refer to them in terms of their ADE algebra $\mathfrak{g}$, as in table 3, unless they are of special type II, III and IV.

We can engineer those singularities using the Tate classification (see e.g. [85]), which amounts to specifying the vanishing orders of the Tate coefficients a_i of the fiber and b_i of the base. We collect this data in Tate vectors $\vec{n}$ and $\vec{\hat{n}}$ for the fiber and the base,

$$a_i \to t^{n_i}\hat{a}_i\,, \quad \text{and} \quad b_i \to t^{\hat{n}_i}\hat{b}_i\,. \tag{4.7}$$

This allows for a compact notation, and to easily read off the resolution of the fiber and base singularities in terms of toric tops [86] and possible gauge enhancements.

Let us assume we engineered a fiber a singularity $\mathfrak{g}_F$ and base singularity $\mathfrak{g}_B$ with Tate vectors given in equation (4.7). We start by discussing the singularity structure in the base at $t = 0$. The elliptic fiber $\mathbb{T}^2_B$ needs to be resolved first by replacing it with a set of $\mathfrak{g}_B \in ADE$ resolution divisors D_{f_i} with $i = 0, \ldots, r$, with $r = \mathrm{rk}(\mathfrak{g}_b)$ such that

$$[\mathbb{T}^2_B] \sim \sum_{i=0}^{r} d_i [D_{f_i}] \tag{4.8}$$

with $D_{f_i} \cdot D_{f_j} = \widehat{A}_G^{i,j}$ the affine (negative) Cartan matrix and d_i the Kac labels.[19] The latter are collected in table 3 for the reader's convenience. At the same time, we need to replace t by

$$t \to \hat{t} = \prod_{i=0}^{r} f_i^{d_i} \,. \tag{4.9}$$

This is required to ensure $\hat{t}$ stays invariant under $\mathbb{C}^*$ scalings of the respective resolution divisors D_{f_i}. Since the fiber $\{p_F = 0\}$ is a section in the new variable $\hat{t}$, it is invariant under those scalings as well.

Note that $\{\hat{t} = 0\}$ is a reducible divisor with components f_i of vanishing order d_i. This is relevant when considering additional fiber singularities $\mathfrak{g}_F$ in $\{p_F = 0\} \cap \{\hat{t} = 0\}$, engineered with the Tate vector $\vec{n}$. First, note that the singularity $\mathfrak{g}_F$ splits into r components over $\{\hat{t} = 0\}$. Since $\hat{t}$ vanishes to order d_i over $\{f_i = 0\}$, the original Tate vector is modified to $\vec{n} \to d_i \cdot \vec{n}$ over each resolution divisor D_{f_i}, which enhances the singularity type. This construction is very useful in the study of the more exotic singularities of Kodaira Type II, III, IV.

We are furthermore required to choose two non-trivial singularities for fiber and base to obtain a threefold with full SU(3) holonomy. If one of the singularities is chosen to be trivial, the geometry becomes a direct product $X_3 = \mathbb{T}^2 \times (\mathbb{T}^2 \times \mathbb{C})/\Lambda$ and supersymmetry enhances to $\mathcal{N} = 2$ in 5D. The two different F-theory lifts yield $\mathcal{N} = (2,0)$ or $\mathcal{N} = (1,1)$ LSTs, which we review in more detail in appendix C. Note that there exist non-simply laced $\mathcal{N} = (1,1)$ LSTs that were argued to be T-dual to $\mathcal{N} = (2,0)$ LSTs with an (affine) outer automorphism twist [26]. The geometry and T-duality structure of such theories has been discussed in [33].

The above structure readily explains our notation of the two singularities defining a Type II LST: the singularity $\mathfrak{g}_B$ denotes the (affine) Dynkin diagram characterising the shape of the quiver at the endpoint configuration. The fibral singularity $\mathfrak{g}_F$ then encodes the gauge algebra factor over Dynkin multiplicity 1 nodes over the affine quiver base, which is also the flavor symmetry gauged during minimal affinization. This is the geometric avatar of the partial tensor branch description discussed in section 3.1.

4.4 ADE singularities

In the following, we show how to construct LSTs of type $\mathcal{K}^{\text{II}}(\mathfrak{g}_F, \mathfrak{g}_B)$. The strategy is to first tune a singularity $\mathfrak{g}_B$ in the base, resolve it, and then discuss the possible choices for $\mathfrak{g}_F$. We engineer all models discussed in section 3 and lay the grounds for the more exotic Kodaira singularities of Type II, III, IV.

As a warm-up, let us start with a simple I_3 singularity, i.e., $\mathfrak{g}_B = \mathfrak{su}_3$ in the base, with Tate vector

$$\vec{n} = \{0,1,1,2,3\} \,. \tag{4.10}$$

The resolution requires three divisors, which replace the central fiber with

$$\hat{t} \to f_0 f_1 f_2 , \tag{4.11}$$

[19]As the base itself is complex one-dimensional, there is no monodromy that could act on the $\mathbb{T}_B^2$ fibers, which means they are all split and of ADE type.

and the resolved hypersurface for the base takes the form

$$p_B = Y^2 + f_1^2 f_2 X^3 + X^2 Z^2 c_2 + f_0^2 f_2 X Z^4 c_4 + XYZc_1 + f_0^4 f_2^2 Z^6 c_6 + f_0^2 f_2 Y Z^3 a_3 \,, \tag{4.12}$$

where the c_i are generic polynomials in $\hat{t}$. We can easily compute the various reducible components of the fiber $\mathbb{P}^1$'s at $\{p_B = 0\}$ upon replacing t by $\hat{t} = f_0 f_1 f_2$. Next we engineer an ADE singularity $\mathfrak{g}_F$ in the F-theory fiber over $t = 0$. We choose to take a Type IV* Kodaira singularity — that is $\mathfrak{g}_F = \mathfrak{e}_6$ — with Tate vector

$$\vec{n} = (1, 2, 2, 3, 5)\,. \tag{4.13}$$

Recall that upon resolving the base, we have to replace $t \to \hat{t} = f_0 f_1 f_2$, leading to three copies of $\mathfrak{e}_6$ in the F-theory fiber in which each singularity intersects the other two. This leads to non-minimal singularities that can be resolved as chains of minimal conformal matter $A_0^{\mathfrak{e}_6}$ [67], and we obtain the necklace quiver $\widehat{A}_2^{\mathfrak{e}_6}$ discussed around equation (3.12):

$$\mathcal{K}^{\mathrm{II}}(\mathfrak{e}_6, \mathfrak{su}_3): \quad //1\,\overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6}\,1\,\overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6}\,1\,\overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6}//\,. \tag{4.14}$$

In section 3, we have seen that the curve denoted in blue is the one involved in minimal affinization. In this section, the fiber singularity of such curves define $\mathfrak{g}_F$. As mentioned above, in the geometric construction this is the algebra associated with the affine Kac label of multiplicity one.

Next, we exchange the role of the fiber and base $\mathbb{T}^2$ prior to resolution. We then resolve the $\mathfrak{e}_6$ singularity in the base by replacing

$$t \to \hat{t} = f_0 f_1 f_2 g_1^2 g_2^2 g_3^2 h_1^3 \,, \tag{4.15}$$

which leads to the base hypersurface equation

$$\begin{aligned} p_B = {} & f_1 f_2^2 g_3 X^3 + f_0^2 f_1 f_2^2 g_1 g_2^2 g_3^2 h_1^2 X^2 Z^2 a_2 + f_0^3 f_2 g_2^2 g_3 h_1 X Z^4 a_4 \\ & + f_0 f_1 f_2 g_1 g_2 g_3 h_1 XYZa_1 + f_0^5 f_2 g_1 g_2^4 g_3^2 h_1^3 Z^6 a_6 + f_0^2 g_2 Y Z^3 a_3 + f_1 g_1 Y^2 \,. \end{aligned} \tag{4.16}$$

Having resolved the central Type IV* fiber in the base, we turn to the F-theory fiber. For simplicity, we only discuss the Weierstrass model corresponding to the Tate model of the fiber. Since we have an I_3 singularity, f and g do not vanish but the discriminant vanishes to order 3 over $\{t = 0\}$. Replacing t by $\hat{t}$, we thus obtain the reducible discriminant locus

$$\Delta = (\hat{t})^3 \hat{\Delta} = (f_0 f_1 f_2 g_1^2 g_2^2 g_3^2 h_1^3)^3 \hat{\Delta}\,. \tag{4.17}$$

The I_3 singularity over $\{t = 0\}$ is hence enhanced to $(\mathrm{I}_3)^3 \times (\mathrm{I}_6)^3 \times \mathrm{I}_9$, resulting in the quiver

$$\mathcal{K}^{\mathrm{II}}(\mathfrak{su}_3, \mathfrak{e}_6): \quad \begin{array}{ccccc} & & \overset{\mathfrak{su}_3}{2} & & \\ & & \overset{\mathfrak{su}_6}{2} & & \\ \overset{\mathfrak{su}_3}{2} & \overset{\mathfrak{su}_6}{2} & \overset{\mathfrak{su}_9}{2} & \overset{\mathfrak{su}_6}{2} & \overset{\mathfrak{su}_3}{2} \end{array}\,. \tag{4.18}$$

It is straightforward to generalize this procedure to an I_k singularity in the fiber, which is then enhanced to an $\mathrm{I}_{d_i k}$ singularity where d_i are the Dynkin multiplicities of the base singularity. For the $\mathfrak{e}_6$ case, those are the $\widehat{E}_6^{\mathfrak{su}_k}$ theories given in (3.13).

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The above construction proves T-duality explicitly for the LSTs $\mathcal{K}^{II}(\mathfrak{su}_k, \mathfrak{e}_6)$ and $\mathcal{K}^{II}(\mathfrak{e}_6, \mathfrak{su}_k)$ as they are engineered from the very same singular threefold. After full resolution of fibers and base, both would be birational phases of the same (extended) Kähler moduli space. It is straightforward to engineer all $\mathcal{K}^{\rm II}(\mathfrak{g}_F, \mathfrak{g}_B)$ type of LSTs and their T-duals $\mathcal{K}^{\rm II}(\mathfrak{g}_F, \mathfrak{g}_B)$ when choosing $\mathfrak{g}_B = \mathfrak{su}_K$ and $\mathfrak{g}_F \in$ ADE. This reproduces the list given in section 3.1 and table 6.[20]

From this geometric perspective, it becomes clear why there are no theories $\mathcal{K}^{\rm II}(\mathfrak{g}_F, \mathfrak{g}_B)$ where $\mathfrak{g}_F, \mathfrak{g}_B$ are both singularities of type I_0^* or higher: consider an I_n^* singularity, which engineers a $\mathfrak{g}_F = \mathfrak{so}_{2n+8}$ gauge algebra with Weierstrass coefficients

$$f = t^2 \hat{f}\,, \qquad g = t^3 \hat{g}\,. \tag{4.19}$$

When combining this with another singularity of type I_0^* or higher, the reducible locus $\{\hat{t} = 0\}$ is given by a set of irreducible divisors, at least one of which has $d_i > 1$. This leads to a non-minimal singularity of vanishing orders $(4, 6)$ or higher in the fiber. Such a singularity has no crepant resolution and does not lead to a 6D supersymmetric theory.

4.5 Type II, III, IV fibers

The above procedure is very useful to discuss the exotic Kodaira singularities of Type II, III, IV. These singularities correspond to strong coupling versions of the ordinary $\mathrm{I}_1, \mathrm{I}_2$ and I_3 singularities. For a given fiber singularity of Type II, III, IV, we can enumerate all compatible base singularities by identifying the highest integer d that still leads to a crepantly resolvable threefold, see table 5. Hence, we can get the following combinations of fiber and base singularities:

Fiber	Base
II	A_{M-1}, D_N, E_6, E_7
III	A_{M-1}, D_N, E_6
IV	A_{M-1}, D_N

(4.20)

We have left out the possible Kodaira Type II, III and IV base singularities, which we will discuss separately in the next section. We will also omit a separate discussion of the exotic singularities over A_{M-1} bases, as those are indistinguishable from their I_k counterparts for $k = 1, 2, 3$. Our main interest here lies in the D and E type singularities, and their gauge enhancements upon resolution. We can read off the final (enhanced) gauge algebra from table 5.

Note that a Kodaira Type II singularity over a base singularity I_N^* with a $D_{N+4} \simeq \mathfrak{so}_{2N+8}$ algebra enhances the fiber singularity to Type IV on multiplicity 2 nodes. The two endpoints of the multiplicity-two chain (i.e., those with multiplicity-one neighbors), have a non-split fiber of singularity Type IV^{ns} and the gauge algebra is reduced to $\mathfrak{su}_2$. In the other cases, the fiber singularities are split and hence host an $\mathfrak{su}_3$ algebra:

$$\mathcal{K}^{\rm II}(\mathrm{II}, \mathfrak{so}_{2N+10}):\quad 2\ \overset{\displaystyle 2}{\overset{\mathfrak{su}_2}{2}}\ \underbrace{\overset{\mathfrak{su}_3}{2}\ \overset{\mathfrak{su}_3}{2}\ \ldots\ \overset{\mathfrak{su}_3}{2}}_{\times N}\ \overset{\displaystyle 2}{\overset{\mathfrak{su}_2}{2}}\ 2\,, \qquad (\mathrm{CB}, \kappa_R) = (3N+7, 6N+12)\,. \tag{4.21}$$

[20] For a similar recent LST construction via brane webs, see [87].

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$\vec{n}$	d	Fiber	Algebra
$(0,0,1,1,1)$	d	I_d	$\mathfrak{su}_d$
$(1,1,1,1,1)$	1	II	–
	2	$\mathrm{IV}^{s/ns}$	$\mathfrak{sp}_1/\mathfrak{su}_3$
	3	$\mathrm{I}_0^{*,s/ss/ns}$	$\mathfrak{so}_8/\mathfrak{so}_7/\mathfrak{g}_2$
	4	$\mathrm{IV}^{*,s/ns}$	$\mathfrak{e}_6/\mathfrak{f}_4$
	5	II^*	$\mathfrak{e}_8$
	6	non-minimal	–
$(1,1,1,1,2)$	1	III	$\mathfrak{sp}_1$
	2	$\mathrm{I}_0^{*,s/ss/ns}$	$\mathfrak{so}_8/\mathfrak{so}_7/\mathfrak{g}_2$
	3	III^*	$\mathfrak{e}_7$
	≥ 4	non-minimal	–
$(1,1,1,2,3)$	1	IV^s	$\mathfrak{su}_3$
	2	IV^*	$\mathfrak{e}_6$
	≥ 3	non-minimal	–

Table 5. Singularity and gauge algebra structure for Type I_1, II, III, IV singularities and possible enhancements by a factor d.

The matter above originates from intersections of Type II$-$IV^{ns} and IV$^{s/ns}-$IV fiber singularities that have been partially computed in [88, 89]. The Type IV$-$IV gives a bifundamental hypermultiplet of $\mathfrak{su}_3^2$, while the II$-$IV^{ns} collision gives a fundamental of $\mathfrak{su}_2$.

For generic N, the above theory has a U(1) flavor symmetry [45, 90], necessitated by the small "ramp" under which the first and last $\mathfrak{su}_3$ fundamental multiplets are charged. The D_4 and D_5 case are special since they have only $\mathfrak{su}_2$ gauge algebra factors over the middle curves.

Moving on to $\mathfrak{e}_6$ singularities in the base, we get the quiver

$$\mathcal{K}^{\mathrm{II}}(\mathrm{II},\mathfrak{e}_6):\quad \begin{array}{ccccc} & & \color{blue}{2} & & \\ & & \overset{\mathfrak{su}_2}{2} & & \\ 2 & \overset{\mathfrak{su}_2}{2} & \underset{[\mathfrak{su}_2]}{\overset{\mathfrak{g}_2}{2}} & \overset{\mathfrak{su}_2}{2} & 2 \end{array} \qquad \text{with } (\mathrm{CB},\kappa_R)=(11,27) \tag{4.22}$$

The $\mathfrak{g}_2$ gauge algebra requires an $\mathfrak{sp}_4$ flavor symmetry which upon gauging of an $(\mathfrak{su}_2)^3$ subalgebra leads to a residual $\mathfrak{su}_2$ flavor algebra. Since the LST charge ℓ_I^{LST} of the curve with a $\mathfrak{g}_2$ algebra is 3, the induced $\mathfrak{su}_2$ flavor current on the LST curve lies within the unitarity bound,

$$c_{3,\mathfrak{sp}_1} = \frac{9}{5} < 2\,. \tag{4.23}$$

Geometrically, this is the only type of model we can obtain, as the I_0^* singularity in the middle must always be non-split.

For an $\mathfrak{e}_7$ base we find

$$\mathcal{K}^{\mathrm{II}}(\mathrm{II},\mathfrak{e}_7):\quad \begin{array}{ccccccccc} & & & & \overset{\mathfrak{su}_3}{3} & & & & \\ & & & & 1 & & & & \\ \color{blue}{2} & \overset{\mathfrak{su}_2}{2} & \overset{\mathfrak{g}_2}{3} & 1 & \overset{\mathfrak{f}_4}{5} & 1 & \overset{\mathfrak{g}_2}{3} & \overset{\mathfrak{su}_2}{2} & 2\,, \end{array} \qquad \text{with } (\mathrm{CB},\kappa_R)=(22,96) \tag{4.24}$$

The resolved geometry above has fixed the singularity on the upper (-3)-curve to be a IV^s singularity,[21] i.e., an $\mathfrak{su}_3$ algebra as can be seen in table 5. This is also consistent with the considerations made in [76] for SCFTs.

We move on to Type III singular fibers with I_{n+1}^* base singularities. The cases I_0^* and I_1^* are special and we will discuss them momentarily. The generic quiver is

$$\mathcal{K}^{\text{II}}(\text{III},\mathfrak{so}_{2N+10}): \overset{\mathfrak{su}_2}{2}\,\overset{\overset{\mathfrak{su}_2}{2}}{\overset{\mathfrak{so}_7}{3}}\,1\,\overset{\mathfrak{so}_8}{4}\,1\ldots 1\,\overset{\mathfrak{so}_8}{4}\,1\,\overset{\overset{\mathfrak{su}_2}{2}}{\overset{\mathfrak{so}_7}{3}}\,\overset{\mathfrak{su}_2}{2}\,, \qquad (\text{CB},\kappa_R)=(6N+16,16N+32)\,, \tag{4.25}$$

where N is the number of $\mathfrak{so}_8$ factors. Again, the geometry requires the first and last I_0^* singularity to be semi-split, i.e., an $\mathfrak{so}_7$. Field theoretically, one can again wonder whether they could be enhanced to $\mathfrak{so}_8$. However, this would violate the worldsheet unitarity bound, since $\mathfrak{so}_8$ would come with an extra $\mathfrak{su}_2$ flavor symmetry at level two, which induces an $c_l = 3$ left-moving current on the LST worldsheet, and is thus inconsistent with the bound derived in section 2.

Turning to the special case I_1^* in the base, we do not get an $\mathfrak{so}_8$ algebra but only

$$\mathcal{K}^{\text{II}}(\text{III},\mathfrak{so}_{10}): \quad \overset{\mathfrak{su}_2}{2}\,\overset{\overset{\mathfrak{su}_2}{2}}{\overset{\mathfrak{so}_7}{3}}\,1\,\overset{\overset{\mathfrak{su}_2}{2}}{\overset{\mathfrak{so}_7}{3}}\,\overset{\mathfrak{su}_2}{2}\,, \quad \text{with } (\text{CB},\kappa_R)=(16,32)\,. \tag{4.26}$$

Similarly, for a base singularity of type I_0^*, we get

$$\mathcal{K}^{\text{II}}(\text{III},\mathfrak{so}_8): \quad \overset{\mathfrak{su}_2}{2}\,\underset{\overset{\mathfrak{su}_2}{2}}{\overset{\overset{\mathfrak{su}_2}{2}}{\overset{\mathfrak{so}_7}{2}}}\,\overset{\mathfrak{su}_2}{2} \quad \text{with } (\text{CB},\kappa_R)=(11,18)\,. \tag{4.27}$$

The $\mathfrak{so}_7$ algebra requires an additional hypermultiplet in the vector representation $\mathbf{7}$, and therefore has an $\mathfrak{su}_2$ flavor symmetry. This flavor symmetry has $k^{\mathfrak{su}_2}=2$, and is consistent with unitarity of the LST worldsheet. Field-theoretically, we could also further enhance the middle algebra to $\mathfrak{so}_8$, but this would-be quiver has an $\mathfrak{su}_2$ flavor symmetry that violate the unitarity bound. We could also try to replace $\mathfrak{so}_7$ algebra by $\mathfrak{g}_2$. This quiver looks promising, since its invariants would be the $N=-1$ case of $\mathcal{K}^{\text{II}}(\text{III},\mathfrak{so}_{2N+10})$ in equation (4.25). There is, however, no candidate T-dual theory; the invariants of the $\mathcal{K}^{\text{II}}(\mathfrak{so}_8,\text{III})$ LST also exhibit a jump as compared to the rest of the infinite series, as we will see below. In either cases, these algebras cannot be constructed with our methods, where consistency of the geometry forces us to have an $\mathfrak{so}_7$ algebra on that curve. A possibility that is allowed for the quiver (4.27) is to break its flavor symmetry $\mathfrak{so}_7 \to \mathfrak{so}_6 \simeq \mathfrak{su}_4$ via a Higgs mechanism. Geometrically, this corresponds to a deformation of the Type III Kodaira singularity to I_2.

[21]Field theory may have suggested an $[\mathfrak{f}_4,\mathfrak{e}_8]=\mathfrak{g}_2$ subalgebra, due to the attached E-string. The enhanced $\mathfrak{su}_2$ flavor symmetry of this $\mathfrak{g}_2$ algebra would in principle be consistent with the LST unitarity bound. However, there is no T-dual model of type $\mathcal{K}^{\text{II}}(\mathfrak{e}_7,\text{II})$ with consistent flavor rank and 2-group structure data.

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The final base consistent with a Type III fiber singularity is of Type IV*. After blowing up the base, we first end up with the seemingly consistent configuration:

$$\begin{array}{c} \overset{\mathfrak{su}_2}{2} \\ \overset{\mathfrak{g}_2}{3} \\ 1 \\ \overset{\mathfrak{su}_2}{2}\,\overset{\mathfrak{g}_2}{3}\,1\,\overset{\mathfrak{f}_4}{5}\,1\,\overset{\mathfrak{g}_2}{3}\,\overset{\mathfrak{su}_2}{2} \end{array} \qquad ? \tag{4.28}$$

However, it was shown in [76] that three (-3)-curves with $\mathfrak{g}_2$ algebras cannot be linked to a (-5)-curve in a trivalent pattern. We are therefore forced to perform additional blow-ups, which leads to the quiver

$$\mathcal{K}^{\mathrm{II}}(\mathrm{III},\mathfrak{e}_6):\ \begin{array}{c} \color{blue}{\overset{\mathfrak{su}_2}{2}} \\ \overset{\mathfrak{so}_7}{3} \\ \overset{\mathfrak{su}_2}{2} \\ 1 \\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{so}_7}{3}\ \overset{\mathfrak{su}_2}{2}\ 1\ \overset{\mathfrak{e}_7}{8}\ 1\ \overset{\mathfrak{su}_2}{2}\ \overset{\mathfrak{so}_7}{3}\ \overset{\mathfrak{su}_2}{2} \end{array} \quad \text{with } (\mathrm{CB},\kappa_R)=(34,144)\,. \tag{4.29}$$

Finally, for Type IV fiber singularities, we can only have I_N^* base singularities, which results in the quiver

$$\mathcal{K}^{\mathrm{II}}(\mathrm{IV},\mathfrak{so}_{2N+8}):\ \begin{array}{c} \overset{\mathfrak{su}_3}{3}\qquad\qquad\qquad\qquad \color{blue}{\overset{\mathfrak{su}_3}{3}}\quad \\ 1\qquad\qquad\qquad\qquad\quad 1\quad \\ \overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6}\,1\,\overset{\mathfrak{su}_3}{3}\,1\overset{\mathfrak{e}_6}{6}\cdots\overset{\mathfrak{e}_6}{6}1\,\overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6}\,1\,\overset{\mathfrak{su}_3}{3} \end{array} \ \text{with } (\mathrm{CB},\kappa_R)=(12N+22,48N+48), \tag{4.30}$$

with $N+1$ corresponding to the number of $\mathfrak{e}_6$ gauge algebra factors.

4.6 Type II, III, IV bases

While the typical ADE bases are straightforward to discuss and to resolve, the bases of Kodaira Type II, III and IV need special treatment since we need to blow up double- or triple-point singularities. The Tate model for Type II singularities is given in table 5, but we repeat it here for convenience:

$$p_B = Y^2 + X^3 + c_1 tXYZ + c_2 tX^2Z^2 + c_3 tYZ^3 + c_4 tXZ^4 + c_6 tZ^6\,. \tag{4.31}$$

At $t = 0$, the above model has a double-point singularity at $X = Y = 0$. In order to distinguish the quiver from the regular the regular torus, we recall that we write it as $0\cdot$. We can then perform a blow-up, which can be done locally by replacing $Y \to Xe_1$. The curve over $\{t = 0\}$ is reducible and splits into

$$X^2(e_1^2 + X) = 0 \tag{4.32}$$

with a double intersection at $e_1 = X = 0$. In terms of curves, we write the above configuration over $\{\hat{t} = 0\}$ as

$$4||1 \qquad \text{with} \qquad \vec{\ell}^{\,\mathrm{LST}} = (1,2)\,, \tag{4.33}$$

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where || denotes the double intersection. Setting the stage for more general cases, we can also perform another blow-up of the double intersection by taking $X \to Xe_2$ and $e_1 \to e_1 e_2$. Upon taking the proper transform, we end up with the equation

$$X^2(e_1^2 e_2 + X) = 0\,. \tag{4.34}$$

All three components meet at a single point, which we write as the quiver

$$5 \overset{1}{\Delta} 2\,. \tag{4.35}$$

These blow-ups are necessary in all cases except for fiber singularities of type I_n. We expect them to work out similar to the I_n base case. An interesting case appears in $\mathcal{K}^{\mathrm{II}}(\mathfrak{su}_n, \mathrm{II})$ theories, i.e., for I_n fibers over Type II bases: it was proposed in [8] that at the cusp, the adjoint is split into a symmetric and an antisymmetric representation of the $\mathfrak{su}_n$ gauge algebra. This matter breaks the 1-form symmetry group $\mathbb{Z}_n$ to either $\mathbb{Z}_2$ if n is even or to nothing if n is odd. This is, however, at odds with the $\mathbb{Z}_n$ defect group in the T-dual theory $\mathcal{K}^{\mathrm{II}}(\mathrm{II}, \mathfrak{su}_n)$. It would be interesting to return to this puzzle in the future.

Next, we want to give a simple example of an I_0^* fiber singularity for quivers with exotic bases. For simplicity, we only discuss the Weierstrass model here, which has vanishing orders $(f, g, \Delta) = (2, 3, 6)$ in the F-theory fiber over $\{t = 0\}$. Since the cusp in the base is a self-intersection of a curve, the singularity is enhanced to $(4, 6, 12)$, which requires one blow-up in the base, resulting in the quiver

$$\mathcal{K}^{\mathrm{II}}(\mathfrak{so}_8, \mathrm{II}): \quad \overset{\mathfrak{so}_8}{4}||1 \quad \text{with} \quad (\mathrm{CB}, \kappa_R) = (5, 8)\,. \tag{4.36}$$

Notably, this theory has no flavor symmetry, since the E-string sees two $\mathfrak{so}_8$ factors due to the double intersection.

For an I_N^* singularity, the F-theory Weierstrass model has vanishing orders $(2, 3, N + 6)$. Blowing up the self-intersection point and taking the proper transform, the singularity on the exceptional divisor becomes $(0, 0, 2N)$. The gauge algebra is fixed by the local monodromy, which is only affected by the I_N^* singularity. In other cases, we have the monodromy cover equation

$$\psi^2 + \frac{g}{f} = 0 \tag{4.37}$$

which does not split if f and g are (locally) quadratic and cubic monomials, respectively. However, due to the double intersections with the exceptional divisor, the order of f and g is (locally) twice as large and hence g/f is a perfect square. The I_{2n} singularity is therefore split, and the quiver is

$$\mathcal{K}^{\mathrm{II}}(\mathfrak{so}_{2N+8}, \mathrm{II}): \quad \overset{\mathfrak{so}_{2N+8}}{4}||\underset{[\Lambda^2=1]}{\overset{\mathfrak{su}_{2N}}{1}} \quad \text{with} \quad (\mathrm{CB}, \kappa_R) = (3N + 4, 6N + 6)\,. \tag{4.38}$$

Note that there is an extra anti-symmetric representation of $\mathfrak{su}_{2N}$ for $N > 2$ leading to an $\mathfrak{su}_2$ flavor symmetry consistent with the worldsheet unitarity bound.

For a IV^* singularity, we have to perform additional blow-ups: the first blow-up yields a Type IV fiber over the exceptional divisor, resulting in yet another $(4, 6, 12)$ singularity at the double intersection with the $\mathfrak{e}_6$. This requires two blow-ups to be fully removed, resulting in the quiver

$$\mathcal{K}^{\mathrm{II}}(\mathfrak{e}_6,\mathrm{II}):\ \overset{\mathfrak{e}_6}{4}||\overset{\mathfrak{su}_3}{1} \to \overset{\mathfrak{e}_6}{6}\ \ \overset{[\mathfrak{su}_2]}{\overset{2}{1}}\ \ \overset{\mathfrak{su}_3}{3} \quad \text{with} \quad (\mathrm{CB},\kappa_R)=(11,27)\,. \tag{4.39}$$

Note that the empty (-2)-curve is endowed with an enhanced $\mathfrak{su}_2$ flavor symmetry, as that curve is part of a rank two E-string. As with the T-dual theory, this flavor has level 3 and is consistent with unitarity.

For singularities of Type II*, we also find a non-minimal singularity at the self-intersection point. Upon blow-up, this yields a Type I_0^* singularity where the double intersection removes the monodromy and hence results in an $\mathfrak{so}_8$ gauge algebra, as in the $\mathcal{K}^{\mathrm{II}}(\mathfrak{so}_{2N+8},\mathrm{II})$ case. Performing the next blow-up of the double intersection results in the quiver (4.35), with a Type III fiber on the (-1)-curve. At the triple intersection point, we obtain another I_0^* singularity, which is now semi-split, i.e., an $\mathfrak{so}_7$. Performing the residual blow-ups then results in the quiver

$$\mathcal{K}^{\mathrm{II}}(\mathfrak{e}_7,\mathrm{II}):\ \overset{\mathfrak{e}_7}{4}||\overset{\mathfrak{so}_8}{1} \to \overset{\mathfrak{e}_7}{8}1\overset{\mathfrak{su}_2}{2}\overset{\overset{\mathfrak{su}_2}{2}}{\overset{\mathfrak{so}_7}{3}}1\overset{\mathfrak{so}_8}{4} \quad \text{with} \quad (\mathrm{CB},\kappa_R)=(22,96)\,, \tag{4.40}$$

which is T-dual to the $\mathcal{K}^{\mathrm{II}}(\mathrm{II},\mathfrak{e}_7)$ theory discussed in the previous section.

Finally, one might consider the case of an II*, that is an $\mathfrak{e}_8$ singularity, which after the first blow-up gives the quiver $\overset{\mathfrak{e}_8}{4}||\overset{\mathfrak{e}_6}{1}$. However, upon further blowing up this quiver, we end up with a (-13)-curve, and hence an inconsistent geometry. This was to be expected, since we also encountered a non-crepant singularity when interchanging fiber and base in the putative $\mathcal{K}^{\mathrm{II}}(\mathrm{II},\mathfrak{e}_8)$ theory discussed in the previous section. All other types of quivers such as $\mathcal{K}^{\mathrm{II}}(\mathrm{III},\mathrm{II})$ and $\mathcal{K}^{\mathrm{II}}(\mathrm{IV},\mathrm{II})$ have an $\mathfrak{su}_2$ and $\mathfrak{su}_3$ gauge algebra with a symmetric representation.

Moving on to the Type III base, we are required to perform a first resolution over $t=0$ and replace it by $\hat{t}=f_0f_1$. The resulting Tate Model is

$$p_B = Y^2+f_1X^3+c_1f_0f_1XYZ+c_2f_0^2f_1^2X^2Z^2+c_3f_0^3f_1^2YZ^3+c_4f_0XZ^4+c_6f_0^3f_1Z^6\,, \tag{4.41}$$

with the two resolution divisors f_0, f_1 having components

$$\begin{aligned} p_B = f_0 = 0: &\qquad Y^2+f_1X^3\,.\\ p_B = f_1 = 0: &\qquad Y^2+c_4f_0XZ^4\,. \end{aligned} \tag{4.42}$$

Both components have a double intersection at $Y=0$ which we write as

$$2||2\,. \tag{4.43}$$

In the following we want to resolve the tangential intersection at $f_0=f_1=Y=0$ by taking

$$\{f_0\to f_0e_1, \quad f_1\to f_1e_1\,, \quad Y\to Ye_1\}\,. \tag{4.44}$$

Upon resolving p_B and taking the proper transform, we obtain

$$\begin{aligned} p_B = {}& f_1X^3+e_1Y^2+c_1e_1^2f_0f_1XYZ+c_2e_1^3f_0^2f_1^2X^2Z^2+c_3e_1^5f_0^3f_1^2YZ^3+\\ & c_4f_0XZ^4+c_6e_1^3f_0^3f_1Z^6\,. \end{aligned} \tag{4.45}$$

The divisors D_{f_0}, D_{f_1} and D_{e_1} all intersect in a single point. Hence, we write the resulting base as

$$\overset{1}{3\,\Delta\,3}\,. \tag{4.46}$$

We may perform a second resolution at $f_0 = f_1 = e_1 = 0$ and take the proper transform, which yields

$$\begin{aligned} p_B =& f_1X^3 + e_1Y^2 + c_1e_1^2e_2^3f_0f_1XYZ + c_2e_1^3e_2^6f_0^2f_1^2X^2Z^2 \\ &+ c_3e_1^5e_2^9f_0^3f_1^2YZ^3 + c_4f_0XZ^4 + c_6e_1^3e_2^6f_0^3f_1Z^6\,. \end{aligned} \tag{4.47}$$

The resulting quiver is

$$4\,\overset{2}{1}\,4\,. \tag{4.48}$$

It can be checked that the above quivers are LSTs with base singularities of Type III, which have a defect group $\mathbb{Z}_2$.

We start by considering an I_k^* fiber singularity, which yields the following quivers

$$\mathcal{K}^{\rm II}(\mathfrak{so}_{2k+8},{\rm III}):\ \overset{\mathfrak{so}_{2k+8}}{2}\,||\,\overset{\mathfrak{so}_{2k+8}}{2}\ \to\ \overset{\mathfrak{so}_{2k+8}}{4}\,\overset{\overset{\mathfrak{su}_{2k}}{2}}{\overset{\mathfrak{sp}_{2k}}{1}}\,\overset{\mathfrak{so}_{2k+8}}{4}\ \text{ with } (\text{CB},\kappa_R) = (6k+10, 16k+16)\,, \tag{4.49}$$

whose Coulomb branch dimension and 2-group structure data matches those of equation (4.25). Note that the $N = 0$ case is again special,

$$\mathcal{K}^{\rm II}(\mathfrak{so}_8,{\rm III}):\ \overset{\mathfrak{so}_8}{4}\,\overset{\overset{[\mathfrak{su}_2]}{2}}{1}\,\overset{\mathfrak{so}_8}{4}\ \text{ with } (\text{CB},\kappa_R) = (11, 18)\,. \tag{4.50}$$

The data as well as the $\mathfrak{su}_2$ flavor symmetry on the (-2)-curve matches that of the $\mathcal{K}^{\rm II}({\rm III},\mathfrak{so}_8)$ theory. Note that the empty (-2)-curve has LST charge 2 and hence the $\mathfrak{su}_2$ worldsheet current contribution is given by $c_{2,\mathfrak{su}_2} = 3/2$, which is within the unitarity bounds.

Moving on to a IV* singularity, we find a vanishing order $(f, g, \Delta) = (6, 8, 16)$ for the associated Weierstrass model. After the first blow-up, this gives a Type IV fiber over the exceptional divisor. At the triple intersection point, there is a singularity with vanishing order $(8, 10, 20)$, see equation (4.46), which results after another blow-up and proper transform in a $(4, 4, 8)$ singularity, i.e another $\mathfrak{e}_6$. Performing all other residual blow-ups, we find the base quiver as

$$\mathcal{K}^{\rm II}(\mathfrak{e}_6,{\rm III}):\ \overset{\mathfrak{e}_6}{2}||\overset{\mathfrak{e}_6}{2}\ \to\ \overset{\mathfrak{e}_6}{6}\,1\,\overset{\mathfrak{su}_3}{3}\,1\,\overset{\overset{\overset{\mathfrak{su}_3}{3}}{1}}{\overset{\mathfrak{e}_6}{6}}\,1\,\overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6}\qquad (\text{CB},\kappa_R) = (34, 144)\,. \tag{4.51}$$

One may proceed in a similar fashion for type III* and II* singularities. However, in both cases we find non-crepantly resolvable singularities at codimension two with vanishing orders $(f, g, \Delta) \geq (8, 12, 24)$ after the first blow-up. This is consistent with the non-crepant singularities we encountered when exchanging the fiber and base.

Finally, there are two exotic cases, given by (III,III) and (IV,III), neither of which require additional blow-ups of the base. The first case is self-T-dual and is given by

$$({\rm III},{\rm III}):\ \overset{\mathfrak{su}_2}{2}\,||\,\overset{\mathfrak{su}_2}{2}\,,\qquad (\text{CB},\kappa_R) = (3, 4)\,, \tag{4.52}$$

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with two bifundamental matters, respecting the $\mathbb{Z}_2$ 1-form symmetry. The second case is

$$\mathcal{K}^{\rm II}(\text{IV},\text{III}):\ \overset{\mathfrak{su}_3}{2}||\overset{\mathfrak{su}_3}{2}\,,\qquad (\text{CB},\kappa_R)=(5,6)\,, \tag{4.53}$$

which also admits $(\mathbf{3},\overline{\mathbf{3}})$ bifundamental hypermultiplets respecting the diagonal $\mathbb{Z}_3$ 1-form symmetry. The fact that we have two bifundamental hypermultiplets suggests that we an $\mathfrak{su}_2$ flavor symmetry. This is consistent with the expected $\mathfrak{su}_2$ flavor symmetry in the T-dual $\mathcal{K}^{\rm II}(\text{III},\text{IV})$ theory.

Finally, we discuss the structure of a IV base. The Tate vector is given in table 5 and the model requires two more resolution divisors, which replace $\hat{t}=f_0f_1f_2$ and result in the Tate-Model

$$p_b=f_2X^3+f_1Y^2+c_1f_0f_1f_2XYZ+c_2f_0^2f_1f_2^2X^2Z^2+c_3f_0YZ^3+c_4f_0^2f_2XZ^4+c_6f_0^3f_2Z^6\,. \tag{4.54}$$

The three curves $\{f_i=0\}$ all meet in a single point, which we write as the quiver

$$\overset{2}{2\Delta 2}\,. \tag{4.55}$$

In the following we want to blow up this point by letting $e_i\to e_is$ with s the resolution divisor. After taking the proper transform, the hypersurface equation becomes

$$p=f_2X^3+f_1Y^2+f_0f_1f_2e_1^2XYZ+f_0^2f_1f_2^2e_1^4X^2Z^2+f_0YZ^3+f_0^2f_2e_1^2XZ^4+f_0^3f_2e_1^3Z^6\,. \tag{4.56}$$

The three curves are given by

$$\begin{aligned} f_0=0:&\qquad f_2X^3+f_1Y^2\\ f_1=0:&\qquad f_2X^3+f_0YZ^3+f_0^2f_2e_1^2XZ^4+f_0^3f_2e_1^3Z^6\\ e_2=0:&\qquad f_1+f_0Z^3\\ e_1=0:&\qquad f_2X^3+f_1Y^2+f_0YZ^3\,. \end{aligned} \tag{4.57}$$

All curves intersect the exceptional divisor, but not each other. The resulting quiver is given by

$$\overset{3}{313}\,. \tag{4.58}$$

Having discussed the minimal bases and their blow-ups, we engineer a I_N^* singularity in the F-theory fiber p_F. For simplicity, we take the minimal Weierstrass model with coefficients

$$f=\hat{t}^2\hat{f}\qquad g=\hat{t}^3\hat{g}\qquad \Delta=\hat{t}^{N+6}\hat{\Delta} \tag{4.59}$$

Since $\hat{t}=f_0f_1f_2$, and all three f_i collide in a single point, the singularity is enhanced to vanishing order $(6,9,3k+18)$. Blowing up the singular point in the base and taking the proper transform lowers the vanishing order over the exceptional locus to $(2,3,3k+6)$, which corresponds to an $\mathfrak{so}_{6k+8}$ gauge algebra. Performing three more resolutions at the three collisions, we obtain the quiver

$$\mathcal{K}^{\rm II}(\mathfrak{so}_{2k+8},\text{IV}):\ \begin{array}{ccc} & \overset{\mathfrak{so}_{2k+8}}{2} & \\ \overset{\mathfrak{so}_{2k+8}}{2} & \Delta & \overset{\mathfrak{so}_{2k+8}}{2} \end{array} \ \to\ \begin{array}{ccccc} & & \overset{\mathfrak{so}_{2k+8}}{4} & & \\ & & \overset{\mathfrak{sp}_{2k}}{1} & & \\ \overset{\mathfrak{so}_{2k+8}}{4} & \overset{\mathfrak{sp}_{2k}}{1} & \overset{\mathfrak{so}_{6k+8}}{4} & \overset{\mathfrak{sp}_{2k}}{1} & \overset{\mathfrak{so}_{2k+8}}{4} \end{array} \tag{4.60}$$

with the Coulomb branch dimension and structure constant given by

$$\dim(\text{CB})=12k+22\,,\qquad \kappa_R=48k+48\,. \tag{4.61}$$

The above data matches that of the T-dual $\mathcal{K}^{\rm II}(\text{IV},\mathfrak{so}_{2N+8})$ theory shown in equation (4.30). From this perspective, it is also clear that we can have no exceptional gauge algebra for this base. For example, with an $\mathfrak{e}_6$ algebra we have a vanishing order enhancement $(3,4,8)\to(9,12,24)$ in the fiber, which does not admit a crepant resolution. This is consistent with the fact that we could not construct a crepant $\mathcal{K}^{\rm II}(\text{IV},\mathfrak{e}_n)$ model.

Two more exotic cases come from the theories $\mathcal{K}^{\rm II}(\text{III},\text{IV})$ and $\mathcal{K}^{\rm II}(\text{IV},\text{IV})$. The quiver of the former is given by

$$\mathcal{K}^{\rm II}(\text{III},\text{IV}):\ \overset{\mathfrak{su}_2}{2}\ \overset{\overset{\mathfrak{su}_2}{2}}{\Delta}\ \overset{\mathfrak{su}_2}{2}\qquad\text{with}\qquad(\text{CB},\kappa_R)=(5,6)\,. \tag{4.62}$$

Due to the triple intersection, we expect that the matter content of the theory are two half-hypermultiplets in the tri-fundamental representation $(\mathbf{2},\mathbf{2},\mathbf{2})$. We therefore expect an $\mathfrak{su}_2$ flavor symmetry, which is compatible with the T-dual theory. The above model is T-dual to the $\mathcal{K}^{\rm II}(\text{IV},\text{III})$ theory shown in equation (4.53).

Finally, The $\mathcal{K}^{\rm II}(\text{IV},\text{IV})$ theory has a $(6,6,12)$ singularity at the triple intersections, which requires a single blow-up. The resulting self-T-dual theory is

$$\mathcal{K}^{\rm II}(\text{IV},\text{IV}):\ \overset{\mathfrak{su}_3}{3}\ \overset{\overset{\mathfrak{su}_3}{3}}{1}\ \overset{\mathfrak{su}_3}{3}\qquad\text{with}\qquad(\text{CB},\kappa_R)=(9,12)\,. \tag{4.63}$$

A remark on the flavor symmetry of the above theory is in order: due to the undecorated (-1)-curve, we expect to find an additional $\mathfrak{su}_3$ flavor factor attached to that curve due to the maximal breaking $\mathfrak{e}_8\to\mathfrak{su}_3^4$. As the LST charge of the curve is $\ell_I^{\rm LST}=3$, the contribution of the induced worldsheet current to the central charge is

$$c_F=\frac{3\cdot 8}{3+3}=4\,. \tag{4.64}$$

This does not satisfy the unitarity bound $c_F\leq 2$. A possible explanation could be that the flavor symmetry is in fact broken to a smaller sub-algebra. Furthermore, the associated SCFT has an $\mathfrak{su}_3^2$ flavor symmetry. To perform minimal affinization, we need to gauge a subalgebra of the full flavor symmetry, but since the two simple factors are the same, it is not clear *which* subalgebra must be gauged. At the level of the anomaly polynomial, adding a vector multiplet transforming in the adjoint representation of either $\mathfrak{su}_3$ factors leads to consistent results. But so does the choice of having the vector multiplet transform in a combination of the two. Since the worldsheet bounds seem to indicate a smaller flavor symmetry, it is tempting to conjecture that the correct gauging is a subalgebra of $\mathfrak{su}_3^2$ with an $\mathfrak{su}_2$ commutant, so that the $\mathcal{K}^{\rm II}(\text{IV},\text{IV})$ LST is a consistent theory with an $\mathfrak{su}_2$ flavor symmetry.

4.7 Higgs branches

Although most Type II LSTs do not have a flavor algebra, some do admit non-trivial Higgs branches. In particular, the cases involving exotic Kodaira Type II, III and IV singularities discussed in the previous sections have flavor symmetry factors. From a geometric perspective, these singularities arise from deformations of classical I_n Kodaira fibers,

$$\text{I}_1\to\text{II}\,,\qquad \text{I}_2\to\text{III}\,,\qquad \text{I}_3\to\text{IV}\,, \tag{4.65}$$

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where the complex structure modulus τ of the F-theory torus is tuned to a special point. Having discussed the LSTs of type $\mathcal{K}^{\mathrm{II}}(\mathfrak{g},\mathfrak{su}_N)$, as well as their duals and enhancements, we now turn to their (partial) Higgs branches in cases where a flavor symmetry exists.

For instance, the theory $\mathcal{K}^{\mathrm{II}}(\mathrm{II},\mathfrak{so}_{2k+10})$, whose quiver is given in equation (4.21), has a U(1) flavor symmetry [45, 90] under which the $(\mathbf{3},\mathbf{3})$ hypermultiplets are non-trivially charged. The Higgsing breaks the $\mathfrak{su}_3$ factors to $\mathfrak{su}_2$, resulting in the LST $\mathcal{K}^{\mathrm{II}}(\mathrm{I}_1,\mathfrak{so}_{2k+10})$. Its T-dual, on the other hand, is more interesting: here we have a U(1) flavor symmetry under which the bifundamentals and anti-symmetric representations are charged. "Higgsing" the singularity of the base from II $\rightarrow$ I_1 gives an expectation value (vev) to these fields, splitting the double intersection:

$$\overset{\mathfrak{so}_{2k+10}}{4}\;||\;\underset{[\Lambda^2=1]}{\overset{\mathfrak{su}_{2n+2}}{1}} \qquad \overset{\text{Higgs}}{\rightsquigarrow} \qquad //\;\overset{\mathfrak{so}_{2k+10}}{4}\;\overset{\mathfrak{sp}_{n+1}}{1}\;//\,. \tag{4.66}$$

The $\mathcal{K}^{\mathrm{II}}(\mathrm{II},E_6)$ theory and its duals work similarly:

$$\begin{array}{ccccc} & & 2 & & \\ & & \overset{\mathfrak{su}_2}{2} & & \\ 2 & \overset{\mathfrak{su}_2}{2} & \underset{[\mathfrak{su}_2]}{\overset{\mathfrak{g}_2}{2}} & \overset{\mathfrak{su}_2}{2} & 2 \end{array} \qquad \overset{\text{Higgs}}{\rightsquigarrow} \qquad \begin{array}{ccccc} & & 2 & & \\ & & \overset{\mathfrak{su}_2}{2} & & \\ 2 & \overset{\mathfrak{su}_2}{2} & \overset{\mathfrak{su}_3}{2} & \overset{\mathfrak{su}_2}{2} & 2 \end{array} \tag{4.67}$$

The theory has an $\mathfrak{sp}_1 \simeq \mathfrak{su}_2$ flavor symmetry under which the $\mathfrak{g}_2$ fundamental representations are charged. Giving them a vev breaks $\mathfrak{g}_2 \rightarrow \mathfrak{su}_3$, preserving the Coulomb branch dimension of the LST while reducing the 2-group structure constant κ_R. The same transition is again more exotic in the T-dual. There we have an $\mathfrak{su}_2$ flavor symmetry on the unpaired (-2)-curve which, under the same transition as above, gives rise to the quiver

$$\overset{\mathfrak{e}_6}{6}\;\overset{\overset{[\mathfrak{su}_2]}{2}}{1}\;\overset{\mathfrak{su}_3}{3} \qquad \overset{\text{Higgs}}{\rightsquigarrow} \qquad //\overset{\mathfrak{e}_6}{6}\;1\;\overset{\mathfrak{su}_3}{3}\;1//\,. \tag{4.68}$$

We close by recalling an important point concerning the Higgs branches of LSTs. An LST is also defined by a point in its tensor-branch moduli space. In particular, a tensor-branch quiver of an LST comes with the choice of a "contraction map" [56] determining a maximal set of curves that can be collapsed. While this choice is unique for SCFTs, it is not for LSTs due to the presence of the curve of self-intersection zero, and there are n_T such choices. Each choice leads to a different theory although the tensor-branch geometry is the same. Furthermore, the invariants we have discussed in this work are the same, as they are computed on the tensor branch, but the structure of their Higgs branch can change from one choice to the other. Being associated with the volumes of the curves, different choices for the curve that remains of finite size at the contracted point will have an impact on the Higgs branch of the theory. This feature was explored in [32] for a variety of Heterotic LSTs. From the fusion perspective, this correspond to the different ways of fusing SCFTs together to obtain an LST, and minimal affinization defines a particular choice of such a contraction map.

The structure of Higgs branches can be probed using the technology of magnetic quivers. While the precise map between 3D magnetic quivers and 6D generalized quivers is not fully understood, in particular in the presence of exceptional algebras, they offer a very potent

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apparatus to extract information about the Higgs branch of 6D $\mathcal{N}=(1,0)$ theories. In the context of Heterotic LSTs, they have been used recently in [30–32, 91], and for Type II LSTs in the upcoming work [92].

4.8 Outlier theories

So far, we have geometrically engineered LSTs from a given base and fiber algebra by using the double fibration structure described in section 4.2. There are a few quivers that look consistent, but have fibers that cannot be reproduced geometrically. In addition, there are a few more quivers that cannot be thought of as deformations of other LSTs, some of which have already appeared in the literature [8, 93]. These outliers all elude the geometric construction and do not admit a candidate T-dual theory with the same two-group invariants. They satisfy all known field-theoretic constraints, such as anomaly cancellation and the worldsheet bound given in equation (2.46).

For instance, consider the following quiver, with $(\mathfrak{g}_F,\mathfrak{g}_B)=(\mathfrak{so}_{2N+8},\mathrm{III})$:

$$\overset{\mathfrak{so}_{2N+8}}{2}\,||\,\overset{\mathfrak{so}_{2N+8}}{2}\ \rightarrow\ \overset{\mathfrak{so}_{2N+8}}{4}\ \underset{\displaystyle\overset{\mathfrak{sp}_{2N}}{1}}{\overset{\overset{\mathfrak{so}_{2N+8}}{4}}{}}\ \overset{\mathfrak{so}_{4N}}{4}\ \overset{\mathfrak{sp}_{2N-8}}{1}\ \overset{\mathfrak{so}_{4N-16}}{4}\ \ldots\ \overset{\mathfrak{sp}_{2N-8k}}{1}\ \underset{\displaystyle\overset{\mathfrak{so}_{4N-16k}}{4}}{\overset{\overset{\mathfrak{sp}_{N-4k-4}}{1}}{}}\ \overset{\mathfrak{sp}_{N-4k-4}}{1}\ . \tag{4.69}$$

From the minimal affinization point of view, this quiver can be reached from an $\mathfrak{so}_{2N+8}$ orbi-instanton with an $\mathfrak{so}_{2N+8}\oplus\mathfrak{e}_8$ symmetry, albeit through a few modifications. The $\mathfrak{so}_{2N+8}$ flavor can first be gauged, leading to another SCFT, the so-called $\frac{1}{2}$-fractional D-type orbi-instanton [66, 69, 72]. The $\mathfrak{e}_8$ symmetry is then broken by moving to the Higgs branch. The effect of the breaking propagates through the spine to the other side of the quiver, and the resulting remnant flavor symmetry can then be gauged to obtain the LST. Both the rank of $\mathfrak{so}_{2N+8}$ and the breaking pattern set the length of the quiver. We are not aware of a T-dual theory for this LST, but it exhibits peculiar features. For instance, from the M-theory perspective, one would expect it to be constructed via M5-branes probing a $\mathbb{C}^2/D_{N+4}$ singularity, with a single M9-brane giving rise the $\mathfrak{e}_8$ symmetry. However, the partial tensor branch description of this quiver is that of a Type II LST with $\kappa_P=0$, where no M9-brane is expected.

Furthermore, while the quiver above has a Type III base, there are two further outliers theories with D-type fibers:

$$\overset{\mathfrak{so}_{2N+8}}{0\cdot}\ \longrightarrow\ \overset{\mathfrak{so}_{2N+8}}{4}\,||\,\overset{\mathfrak{sp}_{N}}{1}\ ,\qquad \overset{\mathfrak{so}_{4(p+4)}}{2}\ \underset{\displaystyle\Delta}{\overset{\overset{\mathfrak{so}_{4(p+4)}}{2}}{}}\ \overset{\mathfrak{so}_{4(p+4)}}{2}\ \longrightarrow\ \overset{\mathfrak{sp}_p}{1}\ \overset{\mathfrak{so}_{4(p+4)}}{4}\ \underset{\displaystyle\overset{\mathfrak{sp}_{3p+8}}{1}}{\overset{\overset{\mathfrak{sp}_p}{1}}{\overset{\mathfrak{so}_{4(p+4)}}{4}}}\ \overset{\mathfrak{so}_{4(p+4)}}{4}\ \overset{\mathfrak{sp}_p}{1}\ . \tag{4.70}$$

We have not found T-duals for these LSTs. It is, however, quite intriguing that these three quivers all have D-type fibers with a base of Kodaira Type II, III, or IV. While we constructed the T-dual pairs for these singularity types, the quivers shown in equations (4.69) and (4.70) are qualitatively different: their duality invariants do not match any of the $\mathcal{K}^{\mathrm{II}}(D_N,\mathfrak{g}_B)$ LSTs.

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Similarly, quivers with a Type II base also fall in this category, e.g.

$$[\mathfrak{su}_2]\,1\,\overset{\mathfrak{e}_8}{8}\,1\,\underset{\displaystyle \overset{\mathfrak{su}_2}{2}}{\overset{\displaystyle \overset{\mathfrak{g}_2}{3}}{}}\,2 \quad \overset{\text{Higgs}}{\rightsquigarrow} \quad \overset{\mathfrak{e}_8}{7}\,1\,\underset{\displaystyle \overset{\mathfrak{su}_2}{2}}{\overset{\displaystyle \overset{\mathfrak{g}_2}{3}}{}}\,2\,. \tag{4.71}$$

These two theories seem to be on the same Higgs branch, as the second quiver is obtained by shrinking the left-most (-1)-curve. One would expect such a transition to be described by the closure of the minimal nilpotent orbit of the $\mathfrak{su}_2$ flavor symmetry. Interestingly, only the second quiver can be reached through minimal affinization. For the first one, sending any curve to infinite volume will give rise to an SCFT with a different defect group.

Another LST for which we do not have a dual is a deformation of the $\mathcal{K}^{\mathrm{II}}(\mathfrak{e}_6, \mathrm{III})$ theory that we have realized geometrically in this section:

$$\overset{\mathfrak{e}_6}{6}\,1\,\overset{\mathfrak{su}_3}{3}\,1\,\underset{\displaystyle \overset{\mathfrak{e}_6}{6}}{\overset{\displaystyle \overset{\mathfrak{su}_3}{3}}{\overset{}{1}}}\,1\,\overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6} \qquad \overset{?}{\rightsquigarrow} \qquad \overset{\mathfrak{f}_4}{5}\,1\,\underset{\displaystyle \overset{\mathfrak{g}_2}{3}}{\overset{\displaystyle \overset{\mathfrak{su}_2}{2}}{}}\,1\,\overset{\mathfrak{f}_4}{5}\,. \tag{4.72}$$

While this configuration does not have apparent gauge anomalies there are no simple deformation of the T-dual parent theory $\mathcal{K}^{\mathrm{II}}(\mathfrak{e}_6, \mathrm{III})$ that can be obtained by further blowing up or down curves, and we have not been able to engineer it geometrically. The quiver furthermore contains exceptional algebras, and we are not aware of any magnetic quiver realization that could shed light on the structure of the Higgs branch of these two theories. We therefore do not know whether such a deformation is even allowed, or whether it is obstructed either geometrically or in field theory.

We have collected the outliers presented here in table 9 along with their T-dual invariant quantities for the reader's convenience. We do not know whether these quivers are consistent. For instance, certain SCFTs can be shown to be gauge anomalous due the presence of extra matter arising at trivalent intersections [75, 76], and similar arguments might apply here. Another possibility is that they are simply not captured by the construction above. Indeed, we have assumed that the two hypersurface of the double fibrations can be written in Tate form. It is then tempting to conjecture that, if the outliers exist at all and are consistent quivers, their T-dual belong to the frozen phase of F-theory or arise from twisted T-dualities, and are therefore not part of the analysis performed in this work.

5 Conclusion and outlook

Novel invariants, such as higher group symmetries, have initiated a recent exploration of Heterotic LSTs via geometry and uncovered a very rich landscape connected via T-dualities [27–29, 45]. In this work, we have considered another class of LSTs, so called Type II LSTs, which are disconnected from their Heterotic cousins. From a higher-dimensional perspective, the most direct consequence is the absence of flavor 9-branes that intersect the spacetime boundary. While the absence of such branes severely restricts possible flavor symmetries, it allows for non-trivial one-form symmetries and defect group structure, characterized by the centers of two algebras $(\mathfrak{g}_F, \mathfrak{g}_B)$ that define the theory. These higher symmetries are exchanged under T-duality, and thus give rise to novel invariants of Type II LSTs that do not exist in Heterotic models.

We have furthermore studied the possible flavor symmetries of these theories from the perspective of the fundamental 2D little string worldsheet theory through anomaly inflow. The contributions of the induced worldsheet currents to the central charge are strongly constrained by unitarity, which allowed us to derive universal bounds on the flavor symmetry algebra. This results in a new set of consistency conditions for 6D LSTs, analogous to those derived for SUGRA models [40, 41], to which they apply as well. The presence of these flavor symmetries, together with the duality constraints of the higher form symmetries imply that Type II LSTs are much more constrained than their Heterotic counterparts.

To further discuss these conditions, we contrast bottom-up and top-down constructions. Field-theoretically, certain seed SCFTs can be turned into Type II LSTs by gauging (part of) their flavor symmetry while preserving the defect group, and without changing its matter spectrum at a generic point of the tensor branch. We have referred to this operation as *minimal affinization*, and this enabled us to explain some of the factorization of the 2-group invariants in terms of group-theoretical quantities for a class LSTs. In a top-down approach, we have used F-theory to engineer Type II LSTs systematically, where we have provided a simple geometric framework using a non-compact toric complete intersection Calabi-Yau threefold with a double elliptic fibration structure. An LST is then fixed by specifying two types of Kodaira singularities for the two elliptic fibers, from which the resulting 6D quiver can be easily read off. Our results are consistent with the field-theory considerations described above. By exploiting F/M-theory duality, the double elliptic fibration structures also makes T-duality of theories manifest.

These two approaches therefore provide a framework to compare geometrically-realizable LSTs with those that appear consistent in field theory. We have also computed the T-duality invariants for all the theories we have encountered, such as the dimension of their Coulomb branch, higher group symmetry structure constants, and higher symmetries. The are collated in appendix A for convenience.

Note that our exploration of Type II LSTs only considers the unfrozen phase of F-theory, as well as untwisted T-dualities. The frozen phase has recently been investigated for Heterotic LSTs from a top-down perspective in [36, 39]. Similarly, twisted LSTs can be engineered via genus-one fibrations [94] that have recently been investigated using toric geometry methods [33–35], for upcoming work on twisted Heterotic LSTs. Both discussions can be extended to Type II LSTs, which we will return to in the future.

Moreover, we have shown that there is a very small and exotic class Type II LSTs with non-trivial flavor symmetries. More generally Type II LSTs are endowed with non-trivial Higgs branches, albeit very restricted due to the constrains described above. Some of these theories admit a description in terms of magnetic quivers [31, 92], and it would be interesting to study the Higgs branch of Type II LSTs in more detail, both field-theoretically and via geometry.

Finally, the toolbox to compute all types of higher form symmetries directly in M-theory have been fully developed in [16, 17, 95–97]. It would be an important cross-check to compute all 1-form symmetries directly in M-theory and match them to their 6D origins. This approach might also allows us to access all p-form symmetries (which might also be present in Heterotic LSTs), and could result in yet additional duality constraints.

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A Tables of Type II Little String Theories

In this appendix, we collate an exhaustive list of the Type II LSTs discussed in the main text, along with their 2-group structure constants κ_P and κ_R, their Coulomb branch dimension dim(CB), and higher-form data $\mathcal{D}^{(1)} \times \mathcal{D}^{(2)}$. We moreover group them in T-dual pairs. In table 6, we give the necklace quivers $\mathcal{K}^{\text{II}}(\mathfrak{g}_F, \mathfrak{su}_N)$ and their duals; in table 7 those involving D-type algebras; in table 8 those of exceptional types — along with Kodaira type II, III, and IV. In addition, table 9 shows all outlier theories we could not construct, and who do not have a known T-dual. Note that in all cases, the theories always satisfy the worldsheet bound given in equation (2.46).

$\mathcal{K}(\mathfrak{g}_F, \mathfrak{b}_B) = \widehat{\Lambda}^{\mathfrak{g}}$	Quiver	$\mathcal{D}^{(1)} \times \mathcal{D}^{(2)}$	κ_P	κ_R	dim(CB)
$\widehat{A}_N^{\mathfrak{g}}$	$//\underbrace{\overset{\mathfrak{g}}{2}\overset{\mathfrak{g}}{2}\ldots\overset{\mathfrak{g}}{2}}_{N-1}\overset{\mathfrak{g}}{2}//$	$Z(\mathfrak{g})^{(1)} \times \mathbb{Z}_N^{(2)}$	0	$N\Gamma_{\mathfrak{g}}$	$h^\vee_{\mathfrak{g}} N - 1$
$\widehat{D}_N^{\mathfrak{su}_{2k}}([2^k])$	$\overset{\mathfrak{su}_k}{2}\,\overset{\overset{\mathfrak{su}_k}{2}}{\overset{\mathfrak{su}_{2k}}{2}}\,\underbrace{\overset{\mathfrak{su}_{2k}}{2}\cdots\overset{\mathfrak{su}_{2k}}{2}}_{N-5}\,\overset{\overset{\mathfrak{su}_k}{2}}{\overset{\mathfrak{su}_{2k}}{2}}\,\overset{\mathfrak{su}_k}{2}$	$\mathbb{Z}_k^{(1)} \times Z(D_N)^{(2)}$	0	$(4N-8)k$	$(2N-2)k-1$
$\widehat{E}_6^{\mathfrak{su}_k}$	$\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_{2k}}{2}\,\overset{\overset{\overset{\mathfrak{su}_k}{2}}{\overset{\mathfrak{su}_{2k}}{2}}}{\overset{\mathfrak{su}_{3k}}{2}}\,\overset{\mathfrak{su}_{2k}}{2}\,\overset{\mathfrak{su}_k}{2}$	$\mathbb{Z}_k^{(1)} \times \mathbb{Z}_3^{(2)}$	0	$24k$	$12k-1$
$\widehat{E}_7^{\mathfrak{su}_k}$	$\overset{\mathfrak{su}_k}{2}\,\overset{\mathfrak{su}_{2k}}{2}\,\overset{\mathfrak{su}_{3k}}{2}\,\overset{\overset{\mathfrak{su}_{2k}}{2}}{\overset{\mathfrak{su}_{4k}}{2}}\,\overset{\mathfrak{su}_{3k}}{2}\,\overset{\mathfrak{su}_{2k}}{2}\,\overset{\mathfrak{su}_k}{2}$	$\mathbb{Z}_k^{(1)} \times \mathbb{Z}_2^{(2)}$	0	$48k$	$18k-1$
$\widehat{E}_8^{\mathfrak{su}_k}$	$\overset{\mathfrak{su}_{2k}}{2}\,\overset{\mathfrak{su}_{4k}}{2}\,\overset{\overset{\mathfrak{su}_{3k}}{2}}{\overset{\mathfrak{su}_{6k}}{2}}\,\overset{\mathfrak{su}_{5k}}{2}\,\overset{\mathfrak{su}_{4k}}{2}\,\overset{\mathfrak{su}_{3k}}{2}\,\overset{\mathfrak{su}_{2k}}{2}\,\overset{\mathfrak{su}_k}{2}$	$\mathbb{Z}_k^{(1)}$	0	$120k$	$30k-1$

Table 6. Little String Theories associated with ADE orbifolds. The blue curve refers to the node obtained through minimal affinization of the corresponding SCFT. For the necklace LSTs $\mathcal{K}(\mathfrak{g}, \mathfrak{su}_N) = \widehat{A}^{\mathfrak{g}}_{N-1}$ obtained from conformal matter, only the partial tensor branch description is shown for brevity, the general quivers can be found in equation (3.12). Note that κ_R and the dimension of the Coulomb branch can be written in a closed form, see equation (3.12).

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$(\mathfrak{g}_F, \mathfrak{g}_B)$	Quiver	$\mathcal{D}^{(1)} \times \mathcal{D}^{(2)}$	$\widehat{\kappa}_P$	$\widehat{\kappa}_R$	dim(CB)
$(\text{II}, \mathfrak{so}_8)$	$\begin{array}{c} 2 \\ 2\,\overset{\mathfrak{su}_2}{2}\,2 \\ 2 \end{array}$	$Z(D_4)^{(1)}$	0	8	5
$(\mathfrak{so}_8, \text{II})$	$\overset{\mathfrak{so}_8}{4}\,\|\|1$	$Z(D_4)^{(2)}$	0	8	5
$(\text{II}, \mathfrak{so}_{2k})$	$\begin{array}{c} \;\;2 \qquad\qquad\qquad\qquad 2 \\ 2\,\overset{\mathfrak{su}_2}{2}\,\underbrace{\overset{\mathfrak{su}_3}{2}\;\overset{\mathfrak{su}_3}{2}\ldots\overset{\mathfrak{su}_3}{2}}_{\times(k-5)}\,\overset{\mathfrak{su}_2}{2}\,2 \end{array}$	$Z(D_k)^{(2)}$	0	$6(k-3)$	$3k-8$
$(\mathfrak{so}_{2k}, \text{II})$	$\overset{\mathfrak{so}_{2k}}{4}\,\|\|\;\overset{\mathfrak{su}_{2k-8}}{1}$	$Z(D_k)^{(1)}$	0	$6(k-3)$	$3k-8$
$(\text{III}, \mathfrak{so}_8)$	$\begin{array}{c} \overset{\mathfrak{su}_2}{2} \\ \overset{\mathfrak{su}_2}{2}\;\overset{\mathfrak{so}_7}{2}\;\overset{\mathfrak{su}_2}{2} \\ \overset{\mathfrak{su}_2}{2} \end{array}$	$\mathbb{Z}_2^{(1)} \times Z(D_4)^{(2)}$	0	18	11
$(\mathfrak{so}_8, \text{III})$	$\begin{array}{c} \overset{\mathfrak{so}_8}{4} \\ 2\;1\;\overset{\mathfrak{so}_8}{4} \end{array}$	$Z(D_4)^{(1)} \times \mathbb{Z}_2^{(2)}$	0	18	11
$(\text{III}, \mathfrak{so}_{2k})$	$\begin{array}{c} \overset{\mathfrak{su}_2}{2} \qquad\qquad\qquad\qquad \overset{\mathfrak{su}_2}{2} \\ \overset{\mathfrak{su}_2}{2}\,\overset{\mathfrak{so}_7}{3}\,\underbrace{1\,\overset{\mathfrak{so}_8}{4}\,1\ldots1\,\overset{\mathfrak{so}_8}{4}\,1}_{(k-5)\times\overset{\mathfrak{so}_8}{4}}\,\overset{\mathfrak{so}_7}{3}\,\overset{\mathfrak{su}_2}{2} \end{array}$	$\mathbb{Z}_2^{(1)} \times Z(D_k)^{(2)}$	0	$16(k-3)$	$6k-14$
$(\mathfrak{so}_{2k}, \text{III})$	$\begin{array}{c} \overset{\mathfrak{so}_{2k}}{4} \\ \overset{\mathfrak{su}_{2k-8}}{2}\;\overset{\mathfrak{sp}_{2k-8}}{1}\;\overset{\mathfrak{so}_{2k}}{4} \end{array}$	$Z(D_k)^{(1)} \times \mathbb{Z}_2^{(2)}$	0	$16(k-3)$	$6k-14$
$(\text{IV}, \mathfrak{so}_{2k})$	$\begin{array}{c} \overset{\mathfrak{su}_3}{3} \qquad\qquad\qquad\qquad \overset{\mathfrak{su}_3}{3} \\ 1 \qquad\qquad\qquad\qquad 1 \\ \overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6}\,\underbrace{1\,\overset{\mathfrak{su}_3}{3}\,1\,\overset{\mathfrak{e}_6}{6}\cdots\overset{\mathfrak{e}_6}{6}\,1\,\overset{\mathfrak{su}_3}{3}\,1}_{(k-5)\times\overset{\mathfrak{e}_6}{6}}\,\overset{\mathfrak{e}_6}{6}\,1\,\overset{\mathfrak{su}_3}{3} \end{array}$	$\mathbb{Z}_3^{(1)} \times Z(D_k)^{(2)}$	0	$48(k-3)$	$12k-26$
$(\mathfrak{so}_{2k}, \text{IV})$	$\begin{array}{c} \overset{\mathfrak{so}_{2N+8}}{4} \\ \overset{\mathfrak{sp}_{2N}}{1} \\ \overset{\mathfrak{so}_{2N+8}}{4}\;\overset{\mathfrak{sp}_{2N}}{1}\;\overset{\mathfrak{so}_{6N+8}}{4}\;\overset{\mathfrak{sp}_{2N}}{1}\;\overset{\mathfrak{so}_{2N+8}}{4} \end{array}$	$Z(D_k)^{(1)} \times \mathbb{Z}_3^{(1)}$	0	$48(k-3)$	$12k-26$

Table 7. Type II Little String Theories involving D-type algebras. The blue curve refers to the node obtain through minimal affinization of the corresponding SCFT.

$(\mathfrak{g}_F, \mathfrak{g}_B)$	Quiver	$\mathcal{D}^{(1)} \times \mathcal{D}^{(2)}$	$\widehat{\kappa}_P$	$\widehat{\kappa}_R$	dim(CB)
(II, II)	$0\cdot$	$\varnothing$	0	0	0
(II, III)	$2\|\|2$	$\mathbb{Z}_2^{(2)}$	0	2	1
(III, II)	$\overset{\mathfrak{su}_2}{0}\cdot$	$\mathbb{Z}_2^{(1)}$	0	2	1
(II, IV)	$\begin{array}{c} 2 \\ 2\Delta2 \end{array}$	$\mathbb{Z}_2^{(2)}$	0	3	2
(IV, II)	$\overset{\mathfrak{su}_3}{0}\cdot$	$\mathbb{Z}_2^{(1)}$	0	3	2

Table 8. Type II Little String Theories involving only exceptional algebras, and the special Kodaira fibers of type II, III, and IV. The blue curve refers to the node obtained through minimal affinization of the corresponding SCFT *(continues. . .)*.

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$(\mathfrak{g}_F, \mathfrak{g}_B)$	Quiver	$\mathcal{D}^{(1)} \times \mathcal{D}^{(2)}$	$\widehat{\kappa}_P$	$\widehat{\kappa}_R$	dim(CB)
$(\mathfrak{e}_6, \text{II})$	$\overset{\mathfrak{e}_6}{6}$ $[\mathfrak{su}_2]\; 2\; 1\; \overset{\mathfrak{su}_3}{3}$	$\mathbb{Z}_3^{(1)}$	0	27	11
$(\text{II}, \mathfrak{e}_6)$	2 $\overset{\mathfrak{su}_2}{2}$ $2\; \overset{\mathfrak{su}_2}{2}\; \overset{\mathfrak{g}_2}{2}\; \overset{\mathfrak{su}_2}{2}\; 2$ $[\mathfrak{su}_2]$	$\mathbb{Z}_3^{(2)}$	0	27	11
$(\text{II}, \mathfrak{e}_7)$	$\overset{\mathfrak{su}_3}{3}$ 1 $2\; \overset{\mathfrak{su}_2}{2}\; \overset{\mathfrak{g}_2}{3}\; 1\; \overset{\mathfrak{f}_4}{5}\; 1\; \overset{\mathfrak{g}_2}{3}\; \overset{\mathfrak{su}_2}{2}\; 2$	$\mathbb{Z}_2^{(1)}$	0	96	22
$(\mathfrak{e}_7, \text{II})$	$\overset{\mathfrak{su}_2}{2}$ $\overset{\mathfrak{e}_7}{8}\; 1\; \overset{\mathfrak{su}_2}{2}\; \overset{\mathfrak{so}_7}{3}\; 1\; \overset{\mathfrak{so}_8}{4}$	$\mathbb{Z}_2^{(2)}$	0	96	22
$(\text{III}, \mathfrak{e}_6)$	$\overset{\mathfrak{su}_2}{2}$ $\overset{\mathfrak{so}_7}{3}$ $\overset{\mathfrak{su}_2}{2}$ 1 $\overset{\mathfrak{su}_2}{2}\; \overset{\mathfrak{so}_7}{3}\; \overset{\mathfrak{su}_2}{2}\; 1\; \overset{\mathfrak{e}_7}{8}\; 1\; \overset{\mathfrak{su}_2}{2}\; \overset{\mathfrak{so}_7}{3}\; \overset{\mathfrak{su}_2}{2}$	$\mathbb{Z}_2^{(1)} \times \mathbb{Z}_3^{(2)}$	0	144	34
$(\mathfrak{e}_6, \text{III})$	$\overset{\mathfrak{su}_3}{3}$ 1 $\overset{\mathfrak{e}_6}{6}\; 1\; \overset{\mathfrak{su}_3}{3}\; 1\; \overset{\mathfrak{e}_6}{6}\; 1\; \overset{\mathfrak{su}_3}{3}\; 1\; \overset{\mathfrak{e}_6}{6}$	$\mathbb{Z}_3^{(1)} \times \mathbb{Z}_2^{(2)}$	0	144	34
(III, III)	$\overset{\mathfrak{su}_2}{2} \,\vert\vert\, \overset{\mathfrak{su}_2}{2}$	$\mathbb{Z}_2^{(1)} \times \mathbb{Z}_3^{(2)}$	0	4	3
(IV, III)	$\overset{\mathfrak{su}_3}{2} \,\vert\vert\, \overset{\mathfrak{su}_3}{2}$	$\mathbb{Z}_2^{(2)} \times \mathbb{Z}_3^{(1)}$	0	6	5
(III, IV)	$\overset{\mathfrak{su}_2}{2}$ $\overset{\mathfrak{su}_2}{2}\; \Delta\; \overset{\mathfrak{su}_2}{2}$	$\mathbb{Z}_2^{(1)} \times \mathbb{Z}_3^{(2)}$	0	6	5
(IV, IV)	$\overset{\mathfrak{su}_3}{3}$ $\overset{\mathfrak{su}_3}{3}\; 1\; \overset{\mathfrak{su}_3}{3}$	$\mathbb{Z}_3^{(1)} \times \mathbb{Z}_3^{(2)}$	0	12	9

Table 8. Type II Little String Theories involving only exceptional algebras, and the special Kodaira fibers of type II, III, and IV. The blue curve refers to the node obtained through minimal affinization of the corresponding SCFT.

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Quiver	$\widehat{\kappa}_R$	dim(CB)
$\begin{array}{ccccccccc} & \overset{\mathfrak{so}_{2k}}{4} & & & & & & \overset{\mathfrak{sp}_{k-4m}}{1} & \\ \overset{\mathfrak{so}_{2k}}{4} & \overset{\mathfrak{sp}_{2(k-4)}}{1} & \overset{\mathfrak{so}_{4(k-4)}}{4} & \overset{\mathfrak{sp}_{2(k-8)}}{1} & \overset{\mathfrak{so}_{4(k-8)}}{4} & \cdots & \overset{\mathfrak{sp}_{2(k+4-4m)}}{1} & \overset{\mathfrak{so}_{4(k+4-4m)}}{4} & \overset{\mathfrak{sp}_{k-4m}}{1} \end{array}$	$8(k-2m)(2m-1)$	$1+(2+4k)m-8m^2$
$\overset{\mathfrak{so}_{2k}}{4} \mid\mid \overset{\mathfrak{sp}_{k-4}}{1}$	$4(k-2)$	$2k-3$
$\begin{array}{ccccc} & & \overset{\mathfrak{sp}_{p}}{1} & & \\ & & \overset{\mathfrak{so}_{4(p+4)}}{4} & & \\ \overset{\mathfrak{sp}_{p}}{1} & \overset{\mathfrak{so}_{4(p+4)}}{4} & \overset{\mathfrak{sp}_{3p+8}}{1} & \overset{\mathfrak{so}_{4(p+4)}}{4} & \overset{\mathfrak{sp}_{p}}{1} \end{array}$	$24(p+3)$	$12p+38$
$\begin{array}{ccccc} & & \overset{\mathfrak{su}_2}{2} & & \\ \overset{\mathfrak{f}_4}{5} & 1 & \overset{\mathfrak{g}_2}{3} & 1 & \overset{\mathfrak{f}_4}{5} \end{array}$	48	16
$\begin{array}{cccccc} & & & & \overset{\mathfrak{g}_2}{3} & \\ [\mathfrak{su}_2] & 1 & \overset{\mathfrak{e}_8}{8} & 1 & \overset{\mathfrak{su}_2}{2} & 2 \end{array}$	49	15
$\begin{array}{cccc} & & \overset{\mathfrak{g}_2}{3} & \\ \overset{\mathfrak{e}_8}{7} & 1 & \overset{\mathfrak{su}_2}{2} & 2 \end{array}$	48	14
$\begin{array}{ccc} & \overset{\mathfrak{su}_2}{2} & \\ \overset{\mathfrak{su}_2}{2} & \overset{\mathfrak{g}_2}{2} & \overset{\mathfrak{su}_2}{2} \\ & \overset{\mathfrak{su}_2}{2} & \end{array}$	16	10

Table 9. Type II LSTs with no known dual six-dimensional quivers. All of them have $\kappa_P = 0$, and satisfy the worldsheet bound given in equation (2.46).

B Anomalies in six dimensions

The anomaly of a D-dimensional theory is encoded in a formal $(D+2)$-form, the anomaly polynomial I_{D+2}. In six dimensions with $\mathcal{N}=(1,0)$ supersymmetry, it takes the form[22]

$$\begin{aligned} I_8 = & \frac{\alpha}{24} c_2(R)^2 + \frac{\beta}{24} c_2(R) p_1(T) + \frac{\gamma}{24} p_1(T)^2 + \frac{\delta}{24} p_2(T) \\ & + \sum_a \operatorname{Tr} F_a^2 \left(\mu^a p_1(T) + k^a c_2(R) + \sum_b \rho^{ab} \operatorname{Tr} F_b^2 \right) + \sum_a \nu^a \operatorname{Tr} F_a^4 \,, \end{aligned} \tag{B.1}$$

where $c_2(F) = \frac{1}{4} \operatorname{Tr} F^2$ are one-instanton normalized traces, $c_2(R)$ is the second Chern class associated with the $\mathfrak{su}(2)_R$ R-symmetry bundle, and $p_1(T)$, $p_2(T)$ are the first and second Pontryagin classes of the spacetime tangent bundle, respectively. Given a generalized quiver, the anomaly polynomial of an SCFT can be directly computed from its tensor branch description [65], and one distinguishing two different types of contributions,

$$I_8 = I_8^{\text{1-loop}} + I_8^{\text{GS}} \,. \tag{B.2}$$

The first is obtained by summing the contributions of the various supermultiplets in the spectrum. It is well known that in D spacetime dimensions, fermions appearing in the various supermultiplets, possibly transforming in a representation $\mathcal{R}$ of the gauge and flavor symmetries, lead to a contribution [98]:

$$I_{D+2}^{\text{fermion}} = \widehat{A}(T) \operatorname{ch}_{\mathcal{R}}(F) \Big|_{(d+2)\text{-form}} \,, \qquad \operatorname{ch}_{\mathcal{R}}(F) = \operatorname{Tr}_{\mathcal{R}} e^{iF} \,, \tag{B.3}$$

where $\widehat{A}(T)$ is the A-roof genus and $\operatorname{ch}_{\mathcal{R}}(F)$ is the Chern character of the field strength of the associated symmetry, potentially also including the R-symmetry depending on the supermultiplet. Similar anomalies are induced by self-dual two-forms, and one must also consider contributions from more complicated objects such as E-strings. The anomaly polynomial contributions for the supermultiplets that are relevant in this work are

$$\begin{aligned} I_8^{\text{tensor}} = & \frac{1}{24} c_2(R)^2 + \frac{1}{48} c_2(R) p_1(T) + \frac{1}{5760} \left(23 p_1(T)^2 - 116 p_2(T) \right) , \\ I_8^{\text{vec}}(F) = & -\frac{1}{24} \left(\operatorname{tr}_{\mathbf{adj}} F^4 + 6 c_2(R) \operatorname{tr}_{\mathbf{adj}} F^2 + \dim(\mathfrak{g}) c_2(R)^2 \right) \\ & - \frac{1}{48} p_1(T) \left(\operatorname{tr}_{\mathbf{adj}} F^2 + \dim(\mathfrak{g}) c_2(R) \right) - \frac{\dim(\mathfrak{g})}{5760} \left(7 p_1(T)^2 - 4 p_2(T) \right) . \end{aligned} \tag{B.4}$$

Note that the traces $\operatorname{tr}_{\mathcal{R}} F^n$ must be converted to one-instanton-normalized traces $\operatorname{Tr}_{\mathcal{R}} F^n$. We follow the conventions of [66] which outlines general procedure. The conversion coefficients for the most common representations appearing in the F-theory construction can be found in appendix F of [47]. The other group-theoretic quantities appearing explicitly in the main text are summarized in table 10. Another one-loop contribution that arises is in this work is

$$I_{\text{sing}} = \frac{1}{24} \left(\frac{1}{2} p_1(T) c_2(R) + \frac{1}{8} p_1(T)^2 - \frac{1}{2} p_2(T) \right) , \tag{B.5}$$

which finds its origin in the M-theory construction of conformal matter, where it corresponds to the modes localized on the orbifold singularity probed by M5-branes [65].

[22] We ignore possible Abelian symmetries in this work.

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$\mathfrak{g}$	$\text{rk}(\mathfrak{g})$	$\dim(\mathfrak{g})$	$h^\vee_{\mathfrak{g}}$	Γ
$\mathfrak{su}_k$	$k-1$	k^2-1	k	k
$\mathfrak{so}_8$	4	28	6	8
$\mathfrak{so}_{p\neq 8}$	$\lfloor \frac{p}{2} \rfloor$	$\frac{1}{2}p(p-1)$	$p-2$	$2p-8$
$\mathfrak{sp}_k$	k	$k(2k+1)$	$k+1$	—
$\mathfrak{g}_2$	2	14	4	—
$\mathfrak{f}_4$	4	52	9	—
$\mathfrak{e}_6$	6	78	12	24
$\mathfrak{e}_7$	7	133	18	48
$\mathfrak{e}_8$	8	248	30	120

Table 10. Relevant quantities of simple Lie algebras. Note that the order Γ of the discrete subgroups of SU(2) are only defined for ADE algebra $\mathfrak{su}_n$, $\mathfrak{so}_{2k}$, $\mathfrak{e}_{6,7,8}$.

The one-loop term will generically be gauge anomalous. Due to the presence of tensors in the spectrum, there is a Green-Schwarz-West-Sagnotti mechanism [48, 49] that harnesses the non-trivial Bianchi identities to cure anomalies via anomaly inflow. This contribution, often referred to as the Green-Schwarz or GS term, takes the generic form

$$I_8^{\text{GS}} = \frac{1}{2} A_{ij} I^i I^j \,, \qquad I^i = A^{ik} c_2(F_k) + B^{ia} c_2(F_a) - a^i p_1(T) + h^i c_2(R) \,. \tag{B.6}$$

At a generic point of the tensor branch, $A^{ij} = \eta^{ij}$ (we denote the inverse by A_{ij}), B^{ia} is the flavor pairing matrix, $a^i = 2 - A^{ii}$, and $h^i = h^\vee_{\mathfrak{g}^i}$. When the curve is undecorated, we set $h^i = 1$. However, the GS term is usually computed away from the tensor branch by blowing down (-1)-curves. Since such a move does not break any of the symmetries of the theory, the complete anomaly polynomial remains invariant by 't Hooft anomaly matching, which we can use to track the changes in the various quantities appearing in equation (B.6). One finds that after blowing down a (-1)-curve

$$\begin{array}{rcccc} \text{Quiver:} & \cdots \overset{\mathfrak{g}_1}{m_1} \overset{\mathfrak{g}_2}{1} \overset{\mathfrak{g}_3}{m_3} \cdots & & \cdots \overset{\mathfrak{g}_1}{(m_1-1)} \; \overset{\mathfrak{g}_3}{(m_3-1)} \cdots \\ \ell_I^{\text{LST}} : & \cdots \; \ell_1 \; \ell_2 \; \ell_3 \; \cdots & \longrightarrow & \cdots \quad \ell_1 \quad\quad \ell_3 \quad \cdots \\ y^I : & \cdots \; y^1 \; y^2 \; y^3 \; \cdots & & \cdots (y_1+y_2) \; (y_2+y_3) \cdots \end{array} \tag{B.7}$$

Additional details can be found in e.g. [26, 37, 63, 65].

B.1 Anomalies of strings in 6D $\mathcal{N} = (1,0)$ theories

In six dimensions, self-dual tensor fields couple naturally to strings, giving rise to an $\mathcal{N} = (0,4)$ theory on their worldsheet, whose central charges and flavor levels are related to those of the six-dimensional bulk theory. We review here how one can derive bounds using unitarity and the central charges of the worldsheet theory, following [40, 41, 61].

From the worldsheet point of view, the strings have the flavor symmetry

$$\mathfrak{so}(4)_{\mathcal{N}} \oplus \mathfrak{su}(2)_R \oplus_I \mathfrak{g}^I \oplus_A \mathfrak{f}^A \,, \tag{B.8}$$

corresponding to the directions normal to the string, as well as the R-symmetry, flavor and gauge symmetries of the bulk theory. The latter appear as flavor symmetries on the worldsheet. The indices I and A runs over the 6D gauge and flavor symmetries, respectively. Furthermore, we decompose the normal directions as $\mathfrak{so}(4)_{\mathcal{N}} = \mathfrak{su}(2)_{\mathcal{A}} \oplus \mathfrak{su}(2)_{\mathcal{B}}$.[23] Ignoring the flavor- and gauge-symmetry factors for a moment, we decompose the supercharges into two-dimensional quantities,

$$\begin{aligned} \mathfrak{so}(6) \oplus \mathfrak{su}(2)_R &\longrightarrow \mathfrak{u}(1) \oplus \mathfrak{su}(2)_{\mathcal{A}} \oplus \mathfrak{su}(2)_{\mathcal{B}} \oplus \mathfrak{su}(2)_R\,, \\ (\mathbf{4},\mathbf{2}) &\longrightarrow (\mathbf{2},\mathbf{1},\mathbf{2})_+ \oplus (\mathbf{1},\mathbf{2},\mathbf{2})_-\,, \end{aligned} \tag{B.9}$$

where the subscript refers to the chirality of the corresponding worldsheet fermions; the R-symmetry of the $\mathcal{N} = (0, 4)$ theory is then identified with $\mathfrak{so}(4)_{R,\text{2D}} = \mathfrak{su}(2)_{\mathcal{B}} \oplus \mathfrak{su}(2)_R$.

As for their six-dimensional counterparts, the levels of the flavor symmetries, as well as the gravitational anomalies, are encoded in the anomaly polynomial of the theory. Given a collection of strings with intersection pairing η^{IJ} associated with two-forms B_I, it takes the generic form [61]

$$\begin{aligned} I_4 &= \frac{1}{2} Q_I \eta^{IJ} Q_J \chi(N) + Q_I I^I\,, \\ I^I &= B^{Ia} c_2(F_a) - a^I p_1(T_6) + h^I c_2(R)\,, \end{aligned} \tag{B.10}$$

where we defined $a^I = (2 - \eta^{II})$, and B^{Ia} is the 6D flavor pairing matrix. The coefficients Q_I are the charges of the string and $h^I = h^\vee_{\mathfrak{g}^I}$ the dual Coxeter number of the associated algebra. Since the Euler density $\chi(N)$ of the $\mathfrak{so}(4)_{\mathcal{N}}$ normal bundle, and the Pontryagin class of the bulk tangent bundle of the bulk $p_1(T_6)$ can be decomposed as

$$\chi(\mathcal{N}) = c_2(\mathcal{A}) - c_2(\mathcal{B})\,, \qquad p_1(T_6) = p_1(T_2) + p_1(\mathcal{N}) = p_1(T_2) - 2c_2(\mathcal{A}) - 2c_2(\mathcal{B})\,, \tag{B.11}$$

we can rewrite (B.10) in terms of worldsheet quantities as

$$\begin{aligned} I_4 = &-\frac{6Q_I a^I}{24} p_1(T_2) + Q_I h^I\, c_2(R) + Q_I B^{Ia}\, c_2(F_a) \\ &+ \frac{1}{2}(Q_I a^I + Q_I \eta^{IJ} Q_J) c_2(\mathcal{A}) + \frac{1}{2}(Q_I a^I - Q_I \eta^{IJ} Q_J) c_2(\mathcal{B})\,, \end{aligned} \tag{B.12}$$

The levels can then be directly inferred from this expression. In particular, the level of the $\mathfrak{su}(2)_r = \mathfrak{su}(2)_{\mathcal{B}}$ is easily found to be

$$k_r = k_{\mathcal{B}} = \frac{1}{2}\left(Q_I a^I - Q_I \eta^{IJ} Q_J\right)\,. \tag{B.13}$$

In the deep IR, the worldsheet theory flows to a two-dimensional CFT. The coefficients of the UV anomaly polynomial can be related to the central charges. These are the quantities associated with the relevant poles of the OPE of the energy-momentum tensor $(\mathcal{T}, \overline{\mathcal{T}})$ and

[23] The normal directions are often denoted as $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$, while the six-dimensional R-symmetry is denoted $\mathfrak{su}(2)_I$. Our nomenclature differs slightly from that of [61], since we are mainly interested in six-dimensional quantities in the main text. We therefore reserve uppercase letters for the 6D R-symmetry, and use lowercase letters for the left-/right-handed modes on the worldsheet.

the holomorphic currents $\mathcal{J}^A$ of the non-Abelian symmetries, see e.g. appendix A of [99] for a concise review:

$$\mathcal{T}(z)\mathcal{T}(0) \sim \frac{c_l}{2z} + \dots, \quad \overline{\mathcal{T}}(\overline{z})\overline{\mathcal{T}}(0) \sim \frac{c_r}{2\overline{z}} + \dots, \quad \mathcal{J}^A(z)\mathcal{J}^B(0) \sim \frac{k^A \delta^{AB}}{z^2} + \dots, \tag{B.14}$$

where $(z, \overline{z})$ are the coordinates on the worldsheet. The central charges c_l, c_r are positive by unitarity, and their difference is given by the gravitational anomaly,

$$c_r - c_l = k_G, \qquad I_4^{\text{CFT}} \supset -\frac{k_G}{24} p_1(T_2). \tag{B.15}$$

Note that along the flow, some of the modes will decouple as they become massive, and those must be taken into account when computing the anomaly polynomial of the worldsheet CFT,

$$I_4 = I_4^{\text{CFT}} + I_4^{\text{mas}}, \tag{B.16}$$

where I_4 is given in equation (B.12). In the cases relevant in this work, the massive modes are those associated with the center-of-mass of the string, and correspond to the contribution of a universal hypermultiplet

$$I_4^{\text{mas}} = -\frac{2}{24} p_1(T_2) - c_2(\mathcal{A}). \tag{B.17}$$

The gravitational anomaly of the IR SCFT is therefore given by

$$k_G = 6\, Q_I a^I + 2. \tag{B.18}$$

Along the flow, the $\mathfrak{su}(2)_l \oplus \mathfrak{su}(2)_r$ R-symmetry might mix with the other $\mathfrak{su}(2)$ factors. However, in the cases relevant here, it was shown through direct computations as well as holographic considerations that the $\mathfrak{su}(2)_r = \mathfrak{su}(2)_{\mathcal{B}}$ R-symmetry component survives unchanged at the conformal fixed point [46, 100, 101]. We can therefore infer that the right-handed central charges, related to the level of the corresponding factor, are given by

$$c_r = 6\, k_r^{\text{IR}} = 6\, k_{\mathcal{B}}^{\text{UV}}. \tag{B.19}$$

Finally, the presence of the holomorphic current implies that the IR description is at least that of a Wess-Zumino-Witten (WZW) model with a Kac-Moody algebra with flavor algebra $\mathfrak{g} = \oplus_a \mathfrak{g}^a$, each factor being at level k^a. Here, the index a runs *a priori* over both bulk flavor and gauge symmetries. The central charge of such a theory is fixed by the symmetry data, and must be smaller than the actual value of c_l, giving a constraint on the possible flavor data from the left-handed central charge [40]:

$$c_{\text{WZW}} = \sum_a \frac{k^a \dim(\mathfrak{g}^a)}{k^a + h^\vee_{\mathfrak{g}^a}} \leq c_l. \tag{B.20}$$

C LSTs with enhanced supersymmetry

When one of the singularities $\mathfrak{g}$ that defines the LST is trivial, the number of unbroken SUSY generators is doubled. There are two kinds of supersymmetry enhancement in six dimensions, depending on which singularity is trivial. In terms of $\mathcal{N} = (1, 0)$ multiplets, we have

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1. $\mathcal{N} = (1,0) \to \mathcal{N} = (1,1)$: an adjoint-valued hypermultiplet and a vector multiplet recombine into a $\mathcal{N} = (1,1)$ vector multiplet.

2. $\mathcal{N} = (1,0) \to \mathcal{N} = (2,0)$: singlet hypermultiplets and tensor multiplets recombine into a $\mathcal{N} = (2,0)$ tensor multiplet.

The two theories are related by T-duality [8, 26], which becomes evident when viewed from the M-theory geometry perspective. There, the total space is given as

$$X_3 = (\mathbb{T}^2_A \times \mathbb{C})/\Lambda) \times \mathbb{T}^2_B \tag{C.1}$$

which is endowed with $\mathcal{N} = 2$ supersymmetry in five dimensions since we do not have the full SU(3) holonomy. The theory has two torus fibrations and thus two 6D lifts: when lifting the theory by sending $\text{vol}(\mathbb{T}^2_B) \to \infty$, we obtain the $\mathcal{N} = (2,0)$ IIB limit, which is just the constant F-theory torus over the LST base $B_2 = (\mathbb{T}^2_A \times \mathbb{C})/\Lambda_{\text{ADE}}$. When lifting to 6D by sending $\text{vol}(\mathbb{T}^2_A) \to \infty$, the resulting base is simply $B_2 = \mathbb{C} \times \mathbb{T}^2_B$. From the F-theory perspective, there is a 7-brane stack at the origin of $\mathbb{C}$ that wraps $\mathbb{T}^2_B$, which yields an extra hypermultiplet in the adjoint representation. Combined with the vector multiplet of the seven-brane, this enhances SUSY to $\mathcal{N} = (1,1)$ in 6D.

The above structure can be deformed by deforming the base slightly when moving to a Type I_1 or Type II base [8]. In this picture, when the IIA base has a I_1 degeneration, the adjoint representation decomposes as

$$\mathbf{Adj} \to \mathbf{S}^2 + \mathbf{A}^2\,, \tag{C.2}$$

modulo a singlet. Indeed, the degeneration locus of the F-theory elliptic fiber I_m signals the precence of a defect that breaks half of the supersymmetries. Similarly, when switching to the IIB dual, we have an A_{N-1} LST, but with I_1 fibers. From the IIB perspective, there is a single $D7$ brane that breaks half of the supersymmetry. In field theory, this corresponds again to an obstruction to enhance $\mathcal{N} = (1,0) \to \mathcal{N} = (2,0)$. Indeed, while there are m copies of the I_1 D7 brane over each tensor multiplet which do not carry a gauge group, there are nevertheless some matter states. Due to the I_2 enhanced singularity at each intersection locus, a massless hypermultiplet charged under a massive U(1) trapped between each tensor arises. These $\mathcal{N} = (1,0)$ hypermultiplets cannot recombine with the $\mathcal{N} = (1,0)$ tensors and therefore obstructs any possibility of enhancement to $\mathcal{N} = (2,0)$.

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