Application of Orthogonal Projectors in Power System Dynamics

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***Abstract*—This paper explores the application of orthogonal projectors to simplify the analysis of complex power system dynamics. By leveraging modal information, orthogonal projections map high-dimensional dynamics onto a lower-dimensional subspace, enabling projected state variables to effectively depict the system's progression over time. A theoretical framework and its practical application are presented, illustrating how this approach enhances the analysis of high-dimensional dynamics. The method is validated using a two-machine infinite bus system and extended to the New England 39-bus system. Results demonstrate that complex trajectories in an n-dimensional space can be efficiently projected onto a 2D subspace, facilitating the observation of specific dynamics, such as those with an electromechanical nature.**

***Index Terms*—orthogonal projectors, eigenvectors, model order reduction, oscillations, power system dynamics.**

I. Introduction

The power grid is a complex dynamical system that can be studied by observing its progression over time in response to external excitations such as short circuits, changes in input setpoints, and equipment outages. Mathematically, we can use a set of differential and algebraic equations to model its behavior [1]. While this model provides a good representation when studying the entire system response, it does not allow for the decoupling of specific dynamics to observe individual contributions to system behavior.

To approach this matter, a linearization around an equilibrium point is employed. This approach enables the computation of modal information—specifically eigenvalues and eigenvectors—providing a deeper understanding of each dynamic component within the system. However, interpreting the participation of each mode in the state variables is not straightforward. Engineers and researchers typically utilize this modal information alongside phase planes of state variables. These illustrate state variables over time, indicating whether the system is within stability margins [2]. However, in systems with high dimensionality, phase planes of variables lack real interpretability of system dynamics.

Projectors are linear transformations that map vectors onto a subspace, preserving direction and magnitude. Orthogonal projectors ensure that subspaces onto which vectors are projected are orthogonal, facilitating clear separation between them [3]. This paper presents the use of orthogonal projectors to analyze complex dynamics of n-dimensional power systems through projections onto lower subspaces. These retain relevant dynamic system behavior while reducing its complexity for analytical purposes. Several engineering applications use projections, for example, in mechanical systems, they have been utilized to decompose constraining forces, allowing for a clearer analysis of the system's behavior [4]. In computational analysis, they are applied to least squared residual estimations, which measure the quality of fits in regression analysis [5]. In robotics, orthogonal projections are useful to assess the vibration accessibility in the control of flexible robotics [6].

Some applications of orthogonal projections in power systems include model order reduction, where techniques like proper orthogonal decomposition (POD) [7] provide a framework for determining efficient reduced-order dynamic models for large-scale systems [8]. Other applications include the analysis and characterization of nonlinear interarea oscillations through the POD-Galerkin characterization [9], and extraction of dynamic patterns from wide-area measurements [10]. While these methods indirectly use orthogonal projections, they do not apply them directly. Recently, the authors proposed using orthogonal projections for oscillation control applications. In this case, orthogonal projectors are used to target the damping action of one mode at a time in a multi-mode system without affecting other modes [11]–[13].

The main contribution of this work is the theoretical procedure and application of orthogonal projections to power systems models for analyzing their dynamics in lower subspaces. In the particular case of frequency oscillations, an appropriate projection can facilitate the study of dynamics that otherwise can be hard to observe and analyze. The rest of the paper is structured as follows. Section II details the theory behind projectors and how to define bases using the eigenvalue decomposition of the system matrix. Section II presents two application examples of projectors in power systems, in addition, further potential applications are discussed. Finally, section IV summarizes the conclusions of this work.

II. Projections

Consider a dynamical system modeled using a set of differential-algebraic equations:

$$\dot{x} = f(x, y, u) \quad (1)$$

$$0 = g(x, y, u) \quad (2)$$

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where the states variables $x \in \mathbb{R}^n$, algebraic variables $y \in \mathbb{R}^l$, and inputs $u \in \mathbb{R}^m$ are related by the multidimensional functions $f(\cdot)$ and $g(\cdot)$. For a given fixed input u_e, the linearization around an equilibrium point $(x_e \quad y_e)^T$ is

$$\begin{aligned}\Delta\dot{x} &= \left.\frac{\partial f}{\partial x}\right|_{eq.} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{eq.} \Delta y + \left.\frac{\partial f}{\partial u}\right|_{eq.} \Delta u \\ &= J_1\Delta x + J_2\Delta y + J_3\Delta u \qquad (3)\\ 0 &= \left.\frac{\partial g}{\partial x}\right|_{eq.} \Delta x + \left.\frac{\partial g}{\partial y}\right|_{eq.} \Delta y + \left.\frac{\partial g}{\partial u}\right|_{eq.} \Delta u \\ &= J_4\Delta x + J_5\Delta y + J_6\Delta u \qquad (4)\end{aligned}$$

Now, if we do not consider changes in the input Δu, the model is simplified to

$$\Delta\dot{x} = J_1\Delta x + J_2\Delta y \qquad (5)$$

$$0 = J_4\Delta x + J_5\Delta y \qquad (6)$$

Now, by eliminating algebraic variables Δy using Kron's reduction, we get

$$\Delta\dot{x} = \underbrace{\left[J_1 - J_2 J_5^{-1} J_4\right]}_{A} \Delta x \qquad (7)$$

A spectral decomposition is performed on the system matrix A using a similarity transformation: $V^{-1}AV = \Lambda$, where V is a matrix of eigenvectors with $v_i \in \mathbb{C}^{n\times 1}$ being the i-th column corresponding eigenvector, and $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is the diagonal matrix of eigenvectors with $\lambda_i \in \mathbb{C}$ being the i-th corresponding eigenvalue.

Let us say that we are interested in projecting the n-dimensional dynamics into a 2-dimensional space that captures the oscillatory behavior of a k-th targeted mode (described by eigenvalues λ_k and $\lambda_{k+1} = \lambda_k^*$ and corresponding eigenvectors v_k and $v_{k+1} = v_k^*$, with * being the complex conjugate operator). Then, we can select a projection base:

$$N = \begin{bmatrix} v_1 & v_2 & \cdots & v_{k-1} & v_{k+3} & \cdots & v_n \end{bmatrix} \qquad (8)$$

with $N \in \mathbb{C}^{n\times(n-2)}$. Note that the eigenvectors are, in general, complex-valued vectors. Then, equivalently we can consider a real basis (see Appendix A) as follows:

$$N = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \in \mathbb{R}^{n\times(n-2)} \qquad (9)$$

where

$$N_1 = \begin{bmatrix} v_1 & v_2 & \cdots & v_{k-1} \end{bmatrix} \qquad (10)$$

$$N_2 = \begin{bmatrix} \Re(v_{k+2}) & \Im(v_{k+2}) & \cdots & \Re(v_n) & \Im(v_n) \end{bmatrix} \qquad (11)$$

with N_1 the collection of all real eigenvectors and N_2 the real representation of complex eigenvectors (all but targeted mode). This matrix represents the basis of an $(n-2)$-dimensional subspace related to all but the k-th targeted mode. Consequently, an orthogonal subspace to span(N) will describe the dynamics related to only the mode of interest. The basis of such subspace is given by

$$M = \text{null}(N) \in \mathbb{R}^{n\times 2} \qquad (12)$$

In passing from Δx to its orthogonal projection $z \in$ range(M), the difference $z - \Delta x$ must be orthogonal to range(M). We can set $z = M\alpha$, with α being the components of the projection onto the basis M. Similarly, we can define the orthogonal projection and components for the basis matrix N. The components of the projections onto span(M) and span(N) are given by

$$\alpha = \underbrace{\left[(M^TM)^{-1}M^T\right]}_{\mathcal{S}} \Delta x, \ \ \alpha_n = \underbrace{\left[(N^TN)^{-1}N^T\right]}_{\mathcal{S}_n} \Delta x \qquad (13)$$

with $\alpha \in \mathbb{R}^2$ and $\alpha_n \in \mathbb{R}^{(n-2)}$. The orthogonal projectors onto range(M) and range(N) are

$$P = M(M^TM)^{-1}M^T \qquad (14)$$

$$P_n = N(N^TN)^{-1}N^T \qquad (15)$$

respectively. For details see Appendix B [3].

To get a clear grasp, consider the following system:

$$\dot{x_1} = x_2 + k_1x_3 \qquad (16)$$

$$\dot{x_2} = -x_1 + k_2x_3 \qquad (17)$$

$$\dot{x_3} = k_3x_3 \qquad (18)$$

with $k_1, k_2 > 0$ and $k_3 \leq 0$ to ensure stability. It is simple to notice that this system is, in essence, a harmonic oscillator perturbed by x_3 dynamics. Given an initial condition $x_0 = [x_{10} \ \ x_{20} \ \ x_{30}]^T$ and the constants k_1, k_2, and k_3, the explicit solution is given by

$$x_1(t) = C_2\sin(t) - C_1\cos(t) + (k_2x_{30} - C_3k_3)\, e^{k_3t} \qquad (19)$$

$$x_2(t) = C_1\sin(t) - C_2\cos(t) + C_3e^{k_3t} \qquad (20)$$

$$x_3(t) = x_{30}e^{k_3t} \qquad (21)$$

where $C_1 = k_2x_{30} - x_{10} - k_3C_3$, $C_2 = x_{20} - C_3$, $C_3 = K/(k_3^2+1)$, and $K = k_2k_3x_{30} - k_1x_{30}$.

For $x_0 = [1.24 \ 0.65 \ 0.45]^T$, $k_1 = 0.1$, $k_2 = 1.5$, and $k_3 = -0.1$, the system solution is shown in Fig. 1a. Note that (x_1, x_2, x_3) dynamics are projected onto two trivial 2D subspaces: $x_1 - x_2$, and $x_1 - x_3$, shown in blue. The subspace $x_1 - x_2$ (Fig. 1b) resembles an oscillatory behavior in steady-state, however, the other (Fig. 1c and d) do not capture any particularly comprehensible behavior.

Next, based on the eigenvalue/eigenvector decomposition, projections can be defined. The system matrix is given by

$$A = \begin{bmatrix} 0 & 1 & k_1 \\ -1 & 0 & k_2 \\ 0 & 0 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0.1 \\ -1 & 0 & 1.5 \\ 0 & 0 & -0.1 \end{bmatrix}$$

whose eigenvectors and eigenvalues are

$$V = \begin{bmatrix} 0.707 & 0.707 & 0.770 \\ 0.707i & -0.7071i & -0.345 \\ 0 & 0 & 0.534 \end{bmatrix}, \ \Lambda = \begin{bmatrix} j & 0 & 0 \\ 0 & -j & 0 \\ 0 & 0 & -0.1 \end{bmatrix}$$

In Fig. 1d the projection using the basis $M = [\Re(v_1), \Im(v_1)]$ is shown. At first, this may sound logical since it comprises the eigenvalue related to the targeted oscillatory

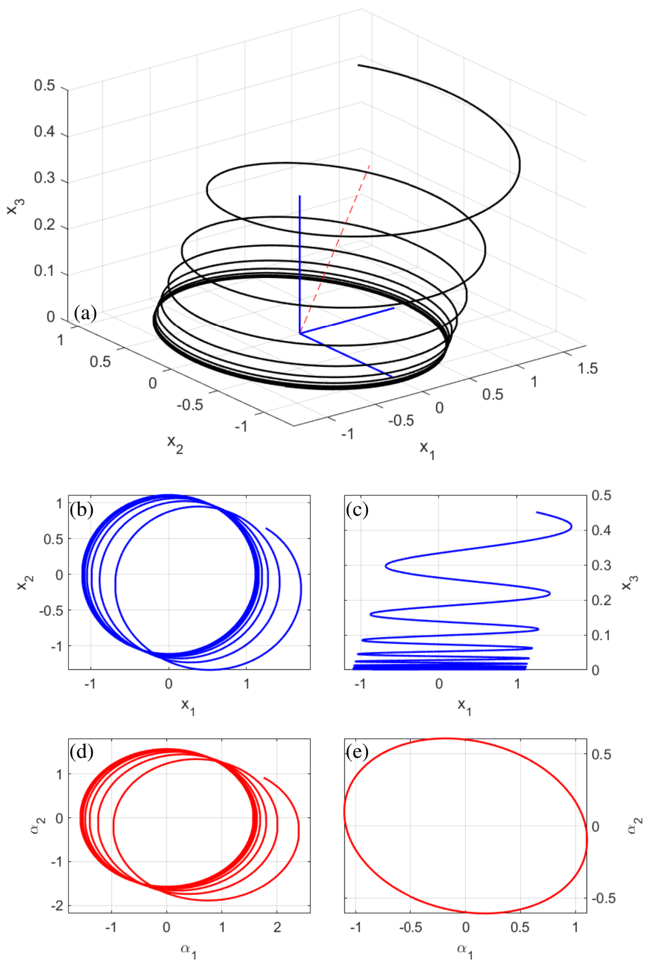

Fig. 1. Dynamical system a) 3D trajectory, and projections onto: b) x_1-x_2 subspace, c) x_1-x_3 subspace, d) x_2-x_3 subspace, and e) $\perp v_3$ subspace.

mode. Nevertheless, it only depicts the same information as the x_1-x_2 plane. Now, take the basis of the desired projected subspace as $M = \text{null}(v_3) \in \mathbb{R}^{3\times2}$, i.e., two vectors on the plane orthogonal to the dashed red line on Fig. 1a. Then, the components of the projections onto span(M) are given by $\alpha = \mathcal{S}x = (M^TM)^{-1}M^Tx$ and depicted on the red projection plot in Fig. 1e. The projection onto the α-plane shows only the behavior related to the complex mode: $\lambda_{1,2} = \pm j$, thus describing a harmonic oscillator.

In summary, the concept of projections becomes useful when the objective is to depict, in a lower-order system, some eigenvalue-driven dynamics of interest. In a linearized system, these dynamics are fundamentally related to the eigen-decomposition of the system matrix. Consequently, system eigenvectors, and proper basis definition, define hyper-planes where relevant dynamics are projected.

III. Applications

Projections of power system dynamics may become useful when trying to unfold complex interactions among components of the system. In this sense, a linear model of the system representation or a linearization around an equilibrium point is considered to ultimately get the proper projection matrices. In

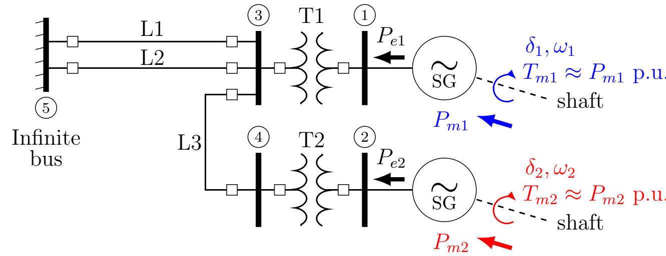

Fig. 2. TMIB system diagram.

the following subsections, a linear model of a two-machine system and a nonlinear model of the IEEE 39-bus system are examined. Orthogonal projections are applied to reveal the oscillatory behavior of system modes.

A. Two-machine infinite bus (TMIB) system

Consider the system of Fig. 2 modeled using the SG classical model. The mathematical representation with state variables: angle δ [rad] and speed ω [p.u.] of each generator, is as follows

$$d\delta_1/dt = \omega_s(\omega_1 - 1) \quad (22)$$
$$2H_1 d\omega_1/dt = P_{m1} - P_{e1} - K_{d1}(\omega_1 - 1) \quad (23)$$
$$d\delta_2/dt = \omega_s(\omega_2 - 1) \quad (24)$$
$$2H_2 d\omega_2/dt = P_{m2} - P_{e2} - K_{d2}(\omega_2 - 1) \quad (25)$$

with $\omega_s = 120\pi$ rad/s. Given the two generators' operation, the electric powers P_{e1} and P_{e2} depend on both: the individual generator angle and angle difference.

Consider the system with the following parameters:

$$P_{m1} = 1 \text{ p.u.}, \; E_1 = 1.05 \text{ p.u.}, \; H_1 = 5 \text{ s.}, \; K_{d1} = 6 \text{ p.u.}$$
$$P_{m2} = 1 \text{ p.u.}, \; E_2 = 1.05 \text{ p.u.}, \; H_2 = 5 \text{ s.}, \; K_{d2} = 6 \text{ p.u.}$$
$$X_{SG1} = X_{SG2} = 0.2 \text{ p.u.}, \; X_{T1} = X_{T2} = 0.1 \text{ p.u.}$$
$$X_{L1} = X_{L2} = 0.4 \text{ p.u.}, \; X_{L3} = 0.2 \text{ p.u.}$$

The system is in steady-state and a three-phase short circuit occurs at bus 3, which is self-cleared after 100 ms. The phase plane system dynamics is complicated since it is composed of four states. We can arbitrarily select three or two variables to generate a 3D or 2D description, respectively. For example, in Fig. 3a shows the phase plane of variables δ_1, δ_2, and ω_1, Fig. 3b and c show the phase plane of SG1 variables and SG1-SG2 difference, respectively. Noticeably, it is hard to draw any conclusion from these graphical descriptions of the dynamics.

From the system eigen-decomposition, two oscillatory modes are present. Mode 1 (local): $\lambda_{1,2} = -0.3 \pm j9.44$, $f = 1.49$ Hz, mode shapes in counter-phase $91.8°(\omega_1)$, $-88.2°(\omega_2)$, describes the power exchange between SG1 and SG2. Mode 2 (interarea): $\lambda_{3,4} = -0.3 \pm j5.57$, $f = 0.89$ Hz, mode shapes in phase $86.9°(\omega_1)$, $86.9°(\omega_2)$, represents the energy trade of the two SGs together with the infinite bus.

Now, when the state vector is projected over the subspace of the modes of interest, we can get interesting representations of the system dynamics. Fig. 3d and e show these projections

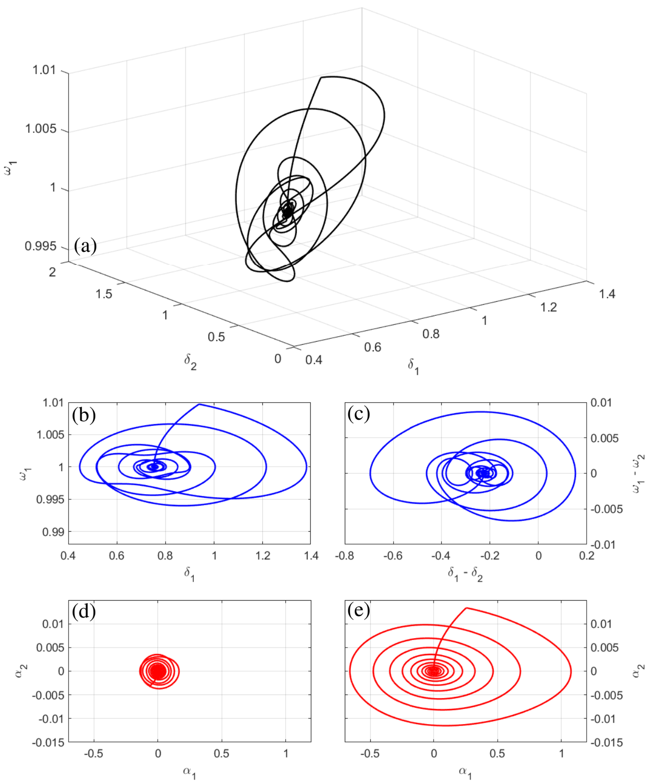

Fig. 3. TMIB system a) 3D trajectory, b) SG1 phase plane, c) SG2 phase plane, d) mode 1 (SG1–SG2) subspace, and e) mode (SG1+SG2–Infinite bus) 2 subspace.

onto the subspaces related to the local and interarea modes, respectively. From these projected dynamics it is clear that mode 1 is not significantly excited (smaller amplitude), and mode 2 exhibits a much pronounced excitation and lower oscillation frequency. Particularly from this last one, the harmonic oscillator plus damping behavior is explicit, which confirms the capturing of the mode dynamics from the initially complex behavior. Overall, the system evidences a good stability margin, i.e., both projected dynamics are far from any saddle-node point.

B. 39-bus system

A nonlinear model of the IEEE 39-bus system is used to validate the application of projections. Simulations are carried out in MATLAB using a two-axis model for SGs, with IEEE Type I exciter, standard governor model, and classical load flow formulation for the grid. This system accounts for 100 states and 108 algebraic variables. Data is taken from [14].

Generators' speed dynamics after a 5-cycle self-cleared fault at bus 20 are shown in Fig. 4. The response is very complex and composed, to a significant degree, of the interaction of the nine electromechanical modes present in this system. However, it is unclear to what extent the dynamics are dominated by some modes more than others. Under this scenario, projections can become handy by separating dynamics related to each mode. Fig. 5 shows the projections of the entire state vector (all 100 variables) onto a 2D space related to each of the nine electromechanical mode subspaces. The frequency and damping ratio of each mode are indicated on each subplot. From the projected dynamics, it is clear that the main excited mode (dominant) is the 0.56 Hz mode, which corresponds to the interarea mode between SG1 and all the rest of the generators combined. However, the response has a not negligible contribution from the 0.91, 0.97, 1.08, and 1.13 Hz modes, giving the response its complex characteristic. Certainly, local modes of 1.19, 1.38, 1.39, and 1.40 Hz are not significantly excited.

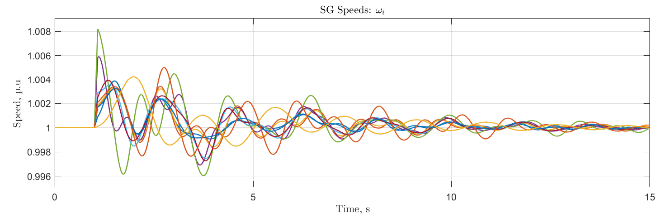

Fig. 4. 39-bus system SG speed dynamics.

These results highlight the ability of projections to depict uncoupled dynamics in the projected subspaces, considering the system's dimension, complexity, and nonlinearity. Note that, the accuracy of the results depends on the linearity of the dynamics around the equilibrium point, i.e., the validity of the linearization.

C. Discussion

The concept of projected dynamics described in this paper has been shown to successfully unfold complex dynamics in lower-dimension planes. Such capability finds practical applications in areas such as:

Oscillation control: projected dynamics that describe excited modes after an event on the system are useful to display control systems that can utilize these planes to attenuate dominant modes. In fact, in [11]–[13], the authors have exploited these ideas by employing a discrete change in the equilibrium point in projected spaces. By incorporating inverter-based resources (IBRs) that can provide a step-wise change in their power output, the oscillations of dominant excited modes are significantly reduced. Orthogonal projectors are of use to project onto dominant modes-related subspaces and define the required power injection from controllable components.

Dynamics classification: the description of excited modes onto projected subspaces allows the understanding of fundamental or critical dynamics behind an event in the power system. This permits the classification of the underlying nature of an event: voltage-driven, electromechanical oscillations, frequency-driven, controller-related (governor, AVR, PSS, etc), and IBR-related. This classification can also be directly related to relevant locations, elements, controllers, and/or variables through participation factors. Particularly, for IBRs where their dynamic interactions with the power grid are not well understood and new unknown connections can unravel. These event-specific dynamics for situational awareness can provide useful information to operators when the power system is subjected to a disturbance.

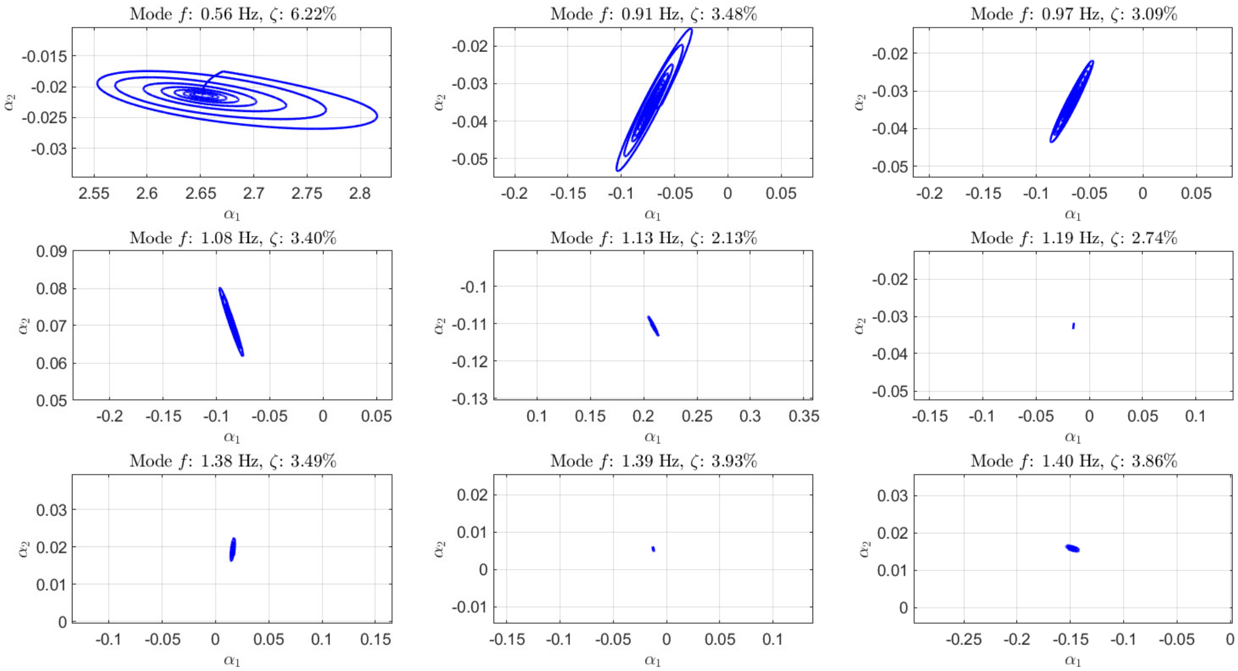

Fig. 5. 39-bus system projected dynamics onto electromechanical modes.

IV. Conclusion

This paper presents the application of orthogonal projectors in power systems. This technique allows the description of the system behavior in a lower-dimension space, that captures dynamics of interest related to its modes. To do so, the system linearization and eigen-decomposition define the proper basis of projected subspaces and the corresponding projection transformations. A two-machine infinite bus system and the New England 39-bus system are used to validate the use of orthogonal projectors. Results show the ability of projections to depict uncoupled dynamics in the projected subspaces, considering the system's complexity and dimension. This is particularly useful when the objective is to capture specific eigenvalue-driven dynamics, such as electromechanical oscillatory modes in power systems.

Appendix A
Complex to Real Eigenvalues/Eigenvectors

Consider a linear transformation $A \in \mathbb{R}^{n\times n}$ that acts onto real eigenvectors $v_i \in \mathbb{R}^n, \forall i$. From the definition of the eigenvalue-eigenvector pair:

$$Av_i = \lambda_i v_i, \quad \lambda_i = a + jb \in \mathbb{C}, \tag{26}$$

$$v_i = \Re(v_i) + j\Im(v_i) \in \mathbb{C}^n, \forall i \tag{27}$$

then, separating into real and imaginary parts for Av_i, we have

$$A\Re(v_i) = \Re(\lambda_i v_i) = a\Re(v_i) - b\Im(v_i) = V\begin{pmatrix} a \\ -b \end{pmatrix} \tag{28}$$

$$A\Im(v_i) = \Im(\lambda_i v_i) = b\Re(v_i) + a\Im(v_i) = V\begin{pmatrix} b \\ a \end{pmatrix} \tag{29}$$

with $V = [\Re(v_i) \quad \Im(v_i)] \in \mathbb{R}^{n\times 2}$. Then, rearranging the equivalency

$$AV = \left[V\begin{pmatrix} a \\ -b \end{pmatrix} \quad V\begin{pmatrix} b \\ a \end{pmatrix}\right] = V\underbrace{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}}_{\Lambda} \tag{30}$$

which shows that the complex eigenvalue/eigenvector equation $Av_i = \lambda_i v_i$ with $\lambda_i \in \mathbb{C}, v_i \in \mathbb{C}^n$ is equivalent to $AV = V\Lambda$ with $V \in \mathbb{R}^{n\times 2}, \Lambda \in \mathbb{R}^{2\times 2}$.

An alternative way to see this equivalency is by using the polar form of the eigenvalue $\lambda_i = r\angle\theta = |\lambda_i|(\cos\theta + j\sin\theta)$. Then, the matrix Λ can be expressed as

$$\Lambda = |\lambda_i|\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = |\lambda_i| \cdot R_{(-\theta)} \tag{31}$$

where $|\lambda_i|$ can be seen as an scale factor and $R_{(-\theta)}$ as a clockwise rotation. In this definition, Λ is defined as the real standard form of the 2×2 matrix A with complex eigenvalues $\lambda_{1,2} = a \pm jb$. Now, consider the following change of basis $V^{-1}AV = \Lambda$, with

$$\mathcal{B} := \{v_1 = \Re(V), v_2 = \Im(V)\} \tag{32}$$

a basis in $\mathbb{R}^2$. The matrix A acts on $\mathcal{B}$ as

$$Av_1 = av_1 - bv_2 \tag{33}$$

$$Av_2 = bv_1 + av_2 \tag{34}$$

which means that a linear transformation $T \in \mathbb{R}^2$ with operator A in the canonical basis $\mathcal{B}_0$ is given by Λ in the basis $\mathcal{B}$, i.e.

$$[T]_{\mathcal{B}_0} = A \quad ; \quad [T]_{\mathcal{B}} = \Lambda \tag{35}$$

with $V = [v_1 \quad v_2]$ as the change of basis matrix.

Appendix B
Orthogonal Projectors

Given a vector x in the space $\mathbb{R}^n$, we can formulate a projection into a lower dimensional space by using orthogonal projectors. In general, an orthogonal projector onto a r-dimensional subspace with $r < n$ can be constructed from an arbitrary basis, not necessarily orthogonal.

Suppose that an r-dimensional subspace is spanned by r linearly independent vectors $\{m_1, \dots, m_r\}$, and let $M \in \mathbb{R}^{n \times r}$ be the matrix whose j-th column is m_j. In passing from x to its orthogonal projection $z \in \text{range}(M)$, the difference $z - x$ must be orthogonal to range(M). This is equivalent to stating that z must satisfy

$$m_j^T(z - x) = 0, \forall j. \tag{36}$$

Since $z \in \text{range}(M)$ we can set $z = M\alpha$, $\alpha \in \mathbb{R}^r$ and write the condition as

$$m_j^T(M\alpha - x) = 0, \forall j \tag{37}$$

or equivalently

$$M^T(M\alpha - x) = 0 \Leftrightarrow M^T M\alpha = M^T x. \tag{38}$$

Now, given that M is full column rank (rank$(M) = r$), $M^T M$ is nonsingular. Therefore

$$\alpha = \left[(M^T M)^{-1} M^T\right] x. \tag{39}$$

Finally, the projection of x, $z = M\alpha$ is

$$z = \underbrace{\left[M(M^T M)^{-1} M^T\right]}_{P} x \tag{40}$$

where $P \in \mathbb{R}^{n \times n}$ is the orthogonal projector onto range(M).

Note that this same analysis can be carried out for any subspace of $\mathbb{R}^n$. Particularly for the subspace orthogonal to span(M), let us define $N = \text{null}(M) \in \mathbb{R}^{n \times (n-r)}$. This subspace, together with span(M), completes the description of the n-dimensional space.

Then, we can define the orthogonal projection of x onto span(N) as $z_n = N\alpha_n \in \text{range}(N)$, and similarly to the case with M, we get

$$\alpha_n = \left[(N^T N)^{-1} N^T\right] x \tag{41}$$

$$z_n = \underbrace{\left[N(N^T N)^{-1} N^T\right]}_{P_n} x \tag{42}$$

with $P_n \in \mathbb{R}^{n \times n}$ being the orthogonal projector onto range(N). It is important to highlight that the vector x can be expressed as a sum of its projections onto the orthogonal subspaces range(M) and range(N) =null(M) as

$$x = z + z_n = Px + P_n x = M\alpha + N\alpha_n. \tag{43}$$

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