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One-sided matching markets with endowments: equilibria and algorithms

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Abstract

The Arrow–Debreu extension of the classic Hylland–Zeckhauser scheme (Hylland and Zeckhauser in J Polit Econ 87(2):293–314, 1979) for a one-sided matching market—called ADHZ in this paper—has natural applications but has instances which do not admit equilibria. By introducing approximation, we define the *ϵ-approximate ADHZ model* and give the following results. 1. Existence of equilibrium under linear utility functions. We prove that the equilibrium allocation satisfies Pareto optimality, approximate envy-freeness, and approximate weak core stability. 2. A combinatorial polynomial time algorithm for an ϵ-approximate ADHZ equilibrium for the case of dichotomous, and more generally bi-valued, utilities. 3. An instance of ADHZ, with dichotomous utilities and a strongly connected demand graph, which does not admit an equilibrium. 4. A rational convex program for HZ under dichotomous utilities; a combinatorial polynomial time algorithm for this case was given in Vazirani and Yannakakis (in: Innovations in theoretical computer science, pp 59–15919, 2021). The ϵ-approximate ADHZ model fills a void in the space of general mechanisms for one-sided matching markets; see details in the paper.

Keywords Arrow–Debreu model · Hylland–Zeckhauser scheme · One-sided matching markets · Rational convex program · Dichotomous utilities

1 Introduction

A one-sided matching market is defined by a set of n indivisible goods and a set of n agents with preferences. The goal is to find a matching of each agent to a distinct good that has desirable fairness and efficiency properties. These markets can be classified along two

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directions: whether the preferences are cardinal or ordinal, and whether agents have initial endowments or not. For the cardinal setting, an allocation based on *market equilibrium* provides remarkable fairness and efficiency guarantees. The two fundamental market models are Fisher and Arrow–Debreu. An Arrow–Debreu (exchange) market is like a barter system, where each agent comes to the market with an endowment of goods and exchanges them with others to maximize her utility. On the other hand, a Fisher market is a special case of the exchange model where each agent has a fixed budget.

Hylland–Zeckhauser (HZ) is the classic mechanism [1] for one-sided matching markets without endowments, based on the Fisher model. In this paper, we define an Arrow–Debreu extension of the HZ mechanism. This fills a void in the space of general[1] mechanisms for one-sided matching markets. The other possibilities are covered as follows: (cardinal, Fisher) by the Hylland–Zeckhauser scheme [1]; (ordinal, Fisher) by Probabilistic Serial [3] and Random Priority [4]; and (ordinal, Arrow–Debreu) by Top Trading Cycles [5]. Details about these mechanisms are given in Sect. 1.1.

The two ways of expressing utilities of goods—ordinal and cardinal—have their own pros and cons and neither dominates the other. On the one hand, the former is easier to elicit from agents and on the other, the latter is far more expressive, enabling an agent to not only report if she prefers good A to good B but also by how much. Abdulkadiroğlu et al. [6] exploited this greater expressivity of cardinal utilities to give mechanisms for school choice which provides a better solution, as illustrated in the following example.

Consider an instance with three types of goods, T_1, T_2, T_3, and these goods are present in the proportion of (1%, 97%, 2%). Based on their utility functions, the agents are partitioned into two sets A_1 and A_2, where A_1 constitute 1% of the agents and A_2, 99%. The utility functions of agents in A_1 and A_2 for the three types of goods are $(1,\ \epsilon,\ 0)$ and $(1,\ 1-\epsilon,\ 0)$, respectively, for a small number $\epsilon > 0$. The main point is that whereas agents in A_2 marginally prefer T_1 to T_2, those in A_1 overwhelmingly prefer T_1 to T_2.

Clearly, the ordinal utilities of all agents in $A_1 \cup A_2$ are the same. Therefore, a mechanism based on such utilities will not be able to distinguish between the two types of agents. On the other hand, the HZ mechanism, which uses cardinal utilities, will fix the price of goods in T_3 to be zero and those in T_1 and T_2 appropriately so that by and large, the bundles of A_1 and A_2 consist of goods from T_1 and T_2, respectively.

The Arrow–Debreu setting of one-sided matching markets has several natural applications beyond the Fisher setting, e.g., allocating students to rooms in a dorm for the next academic year, assuming their current room is their initial endowment. Similarly, school choice, when a student's initial endowment is a seat in a school that they already have. The issue of obtaining such an extension of the HZ mechanism, called *ADHZ* in this paper, was studied by Hylland and Zeckhauser. However, this culminated in an example that inherently does not admit an equilibrium [1].

One recourse to this was given by Echenique, Miralles, and Zhang [7] via their notion of an *α-slack Walrasian equilibrium*: This is a hybrid between the Fisher and Arrow–Debreu settings. Agents have initial endowments of goods, and for a fixed $\alpha \in (0, 1]$, the budget of each agent, for given prices of goods, is $\alpha + (1-\alpha) \cdot m$, where m is the value for her initial endowment; the agent spends this budget to obtain an optimal bundle of goods. Via a non-trivial proof, using the Kakutani Fixed Point Theorem, they proved that an α-slack equilibrium always exists.

[1] As opposed to mechanisms for specific one-sided matching markets.

In this paper, we show that we can remain within a pure Arrow–Debreu setting provided we relax the notion of equilibrium to an *approximate equilibrium*, a notion that has become commonplace in the study of equilibria within computer science due to irrational-valued equilibria and intractability; see, e.g., [8, 9]. We call this the ϵ*-approximate ADHZ model*. For this model, we give the following results.

We prove the existence of an equilibrium for arbitrary cardinal utility functions, using the fact from the paper [7] that an α-slack equilibrium always exists for $\alpha > 0$.

We prove that the equilibrium allocation in our ϵ-approximate ADHZ model is Pareto optimal, approximately envy-free, and approximately weak core stable.[2] In contrast, the allocation found by an HZ equilibrium is Pareto optimal and envy-free [1].

For an Arrow–Debreu market under linear utilities, Gale [10] defined a *demand graph:* a directed graph on agents with an edge (i, j) if agent i likes a good that agent j has in her initial endowment. He proved that a sufficiency condition for the existence of equilibrium is that this graph be strongly connected. The following question arises naturally: Is this a sufficiency condition for equilibrium existence in ADHZ as well? We provide a negative answer to this question. We give an instance of ADHZ whose demand graph is not only strongly connected but also has dichotomous utilities (i.e., utilities in $\{0, 1\}$), and yet it does not admit an equilibrium.

For the case of dichotomous utilities, we give a combinatorial polynomial-time algorithm for computing an equilibrium for our ϵ-approximate ADHZ model. This result also extends to the case of bi-valued utilities, i.e., each agent's utility for individual goods comes from a set of cardinality two, though the sets may differ for different agents. These utilities are well-studied (see, e.g., [2, 11–14]), mainly due to their significance in practical applications. For example, it might be simpler for agents to answer whether their desire for a good is "high" or "low" with numerical values. We note that the polynomial-time algorithms for Arrow–Debreu markets under linear utilities [15, 16], as well as the recent strongly polynomial-time algorithm for the same problem [17] are quite complicated, in particular, because they resort to the use of balanced flows, which uses the l_2 norm. In contrast, we managed to avoid using the l_2 norm, and hence, we obtained a simple algorithm.

A corollary of the last result is that the equilibrium of the dichotomous utility case of the ϵ-approximate ADHZ model involves only rational numbers. In contrast, we give an instance of ADHZ, whose unique equilibrium has irrational prices and allocations. This instance is obtained by appropriately modifying an instance for the HZ model, given in [2], whose (unique) equilibrium has irrational prices and allocations. This led us to ask if there is a *rational convex program (RCP)* that captures the equilibrium in this setting.

An RCP, defined in [18], is a nonlinear convex program whose parameters are rational numbers and always admits a rational solution in which the denominators are polynomially bounded. The quintessential such program is the Eisenberg-Gale convex program [19] for a linear Fisher market. The significance of finding such a program for a problem is that it directly implies the existence of a polynomial time algorithm for the underlying problem, since using the ellipsoid algorithm and Diophantine approximation [20, 21], an RCP can be solved exactly in polynomial time. As a result, it gives practitioners a direct way to compute a solution using general-purpose convex programming solvers. Although we could not answer this question, we did find an RCP for HZ equilibrium under dichotomous utilities. A combinatorial polynomial time algorithm for this case was given in [2].

[2] For definitions of these notions see Definitions 2 and 8

1.1 Related results

Matching markets have been intensely studied and have found many applications in various multi-agent settings. For a few recent examples, Aziz et al. [22] provided a reduction from different matching problems that preserve feasibility and stability. Beynier et al. [23] studied the fair and efficient allocation in housing markets where agents want to exchange their houses to improve their utilities. Gupta et al. [24] studied the stability in barter exchange markets. Hosseini et al. [25] studied the characteristics of one-sided matching mechanisms in which a set of items needs to be matched with a set of agents. Aziz [26] studied the strategyproofness of the exchange markets in which agents have dichotomous preferences.

We start by stating the properties of mechanisms for one-sided matching markets listed in the Introduction. Random Priority [4] is strategyproof though not efficient or envy-free; Probabilistic Serial [3] is efficient and envy-free but not strategyproof; and Top Trading Cycles [5] is efficient, strategyproof and core stable.

Recently, Vazirani and Yannakakis [2] undertook a comprehensive study of the computational complexity of the HZ scheme. They gave a combinatorial polynomial time algorithm for dichotomous utilities and an example with only irrational equilibria; consequently, this problem is not in PPAD. They showed that the problem of computing an exact HZ equilibrium is in the class FIXP, and the problem of computing an approximate equilibrium is in PPAD. Very recently, Chen et al. [27] showed that computing an approximate HZ equilibrium is PPAD-hard. To deal with the computational intractability of HZ, a Nash-bargaining-based mechanism was proposed in [28].

The study of the dichotomous case of matching markets was initiated by Bogomolnaia and Moulin [11]. They studied a two-sided matching market, and they called it an "important special case of the bilateral matching problem." Using the Gallai-Edmonds decomposition of a bipartite graph, they gave a mechanism that is Pareto optimal and group strategyproof. They also gave a number of applications of their setting, some of which are natural applications of one-sided markets, e.g., housemates distributing rooms with different features in a house. As in the HZ scheme, their mechanism also outputs a doubly stochastic matrix whose entries represent probability shares of allocations. However, they give another interesting interpretation of this matrix. They say, "Time sharing is the simplest way to deal fairly with indivisibilities of matching markets: think of a set of workers sharing their time among a set of employers." Roth, Sönmez and Ünver [29] extended these results to general graph matching under dichotomous utilities; this setting is applicable to the kidney exchange marketplace.

An interesting recent paper [30] defines the notion of a random partial improvement mechanism for a one-sided matching market. This mechanism truthfully elicits the cardinal preferences of the agents and outputs a distribution over matchings that approximates every agent's utility in the Nash bargaining solution.

Several researchers have proposed Hylland–Zeckhauser-type mechanisms for a number of applications, for instance [31–34]. The basic scheme has also been generalized in several different directions, including two-sided matching markets, adding quantitative constraints, and to the setting in which agents have initial endowments of goods instead of money, see [7, 35].

Organization of the rest of the paper Section 2 describes the HZ mechanism and its relation with linear Fisher markets. The ϵ-Approximate ADHZ model and the existence and fairness properties of its equilibria are presented in Sect. 3. Section 4 presents an algorithm for computing an equilibrium in a ϵ-Approximate ADHZ model under dichotomous utilities. An RCP for the HZ mechanism under dichotomous utilities showing that it admits a rational-valued equilibrium is presented in Sect. 5. Section 6 extends the rational-valued property to a notion of α-slack equilibrium in the ADHZ setting under dichotomous utilities. Finally, we conclude in Sect. 7.

2 The Hylland–Zeckhauser mechanism

The Hylland–Zeckhauser (HZ) mechanism can be viewed as a marriage between a fractional perfect matching and a linear Fisher market, which consists of a set A of agents and a set G of goods. Each agent i comes to the market with a budget b_i and has utilities $u_{ij} \geq 0$ for each good j. In the case of linear utilities, agent i's utility from allocation $(x_{ij})_{j \in G}$ is $\sum_j u_{ij} x_{ij}$. By fixing the units for each good, we may assume without loss of generality that there is a unit of each good in the market.

Definition 1 A *Fisher equilibrium* is a pair (x, p) consisting of an *allocation* $(x_{ij})_{i \in A, j \in G}$ and non-negative *prices* $(p_j)_{j \in G}$ with the following properties.

1. Each agent i spends at most their budget, i.e., $\sum_{j \in G} p_j x_{ij} \leq b_i$.
2. Each agent i gets an *optimal bundle*, i.e., utility maximizing bundle at prices p. Formally:

$$\sum_{j \in G} u_{ij} x_{ij} = \max \left\{ \sum_{j \in G} u_{ij} y_j \mid y \in \mathbb{R}^G_{\geq 0}, \sum_{j \in G} p_j y_j \leq b_i \right\}.$$

3. The market clears, i.e., each good with a positive price is fully allocated to the agents.

The set of equilibria of a linear Fisher market corresponds to the set of optimal solutions of the Eisenberg–Gale convex program [19], which is a *rational convex program* (RCP) and in fact it motivated the definition of this concept [18].

Fisher equilibrium allocations satisfy various nice properties, including equal-type envy-freeness and Pareto optimality.

Definition 2 (**Envy-freeness and Pareto optimality**) An allocation is *envy-free* if for any two agents $i, i' \in A$, agent i weakly prefers their allocation than those that i' gets, i.e., $\sum_{j \in G} u_{ij} x_{ij} \geq \sum_{j \in G} u_{ij} x_{i'j}$. It is *equal-type envy-free* if the above holds for any two agents with identical budgets.

An allocation x *weakly dominates* another allocation x' if no agent prefers x' to x. It *strongly* dominates x' if it weakly dominates it, and some agent prefers x to x'. An allocation x is *Pareto efficient* or *Pareto optimal* if there is no other allocation x' which strongly dominates it.

Definition 3 A *one-sided matching market* consists of a set A of *agents* and a set G of *goods*. Each agent has preferences over goods, expressed either using cardinal or ordinal utility functions. An *allocation* is a perfect matching of agents to goods. The goal of the market is to find an allocation so that the underlying mechanism has some desirable game-theoretic properties.

The HZ mechanism uses cardinal utility functions, in which each good is rendered divisible by viewing it as one unit of *probability shares*. An HZ equilibrium is defined as follows.

Definition 4 A *Hylland–Zeckhauser (HZ) equilibrium* is a pair (x, p) consisting of an *allocation* $(x_{ij})_{i\in A, j\in G}$ and non-negative *prices* $(p_j)_{j\in G}$ with the following properties.

1. x is a fractional perfect matching, i.e., $\sum_{j\in G} x_{ij} = 1$ for all i and $\sum_{i\in A} x_{ij} = 1$ for all j.
2. Each agent i spends at most their budget, i.e., $\sum_{j\in G} p_j x_{ij} \le b_i$ (usually $b_i = 1$).
3. Each agent i gets an *optimal bundle*, which is defined to be a cheapest utility maximizing bundle, i.e.,

$$\sum_{j\in G} u_{ij}x_{ij} = \max\left\{ \sum_j u_{ij}y_j \mid \sum_j y_j = 1;\ \sum_j p_j y_j \le b_i \right\} \quad \text{and}$$

$$\sum_{j\in G} p_j x_{ij} = \min\left\{ \sum_j p_j y_j \mid \sum_j y_j = 1; \sum_j u_{ij}y_j \ge \sum_j u_{ij}x_{ij} \right\}.$$

Unlike in a linear Fisher market, an agent's utility in the HZ setting can be *satiated*, i.e., they may be able to reach their optimum utility without spending their entire budget. This is why the additional condition that each agent receives their *cheapest* optimal bundle is added. HZ equilibrium allocations are known to be Pareto optimal and envy-free (assuming unit budgets).[3]

The allocation x found by the HZ mechanism is a fractional perfect matching or a doubly stochastic matrix. To get an integral perfect matching from x, a lottery can be carried out using the Theorem of Birkhoff [36] and von Neumann [37]. It states that any doubly-stochastic matrix can be written as a convex combination of integral perfect matchings; moreover, this decomposition can be found efficiently. Picking a perfect matching according to the discrete probability distribution determined by this convex combination yields the resulting allocation in the HZ mechanism.

3 The ϵ-approximate ADHZ model

In this paper, we are interested in an exchange version of the HZ mechanism. Before defining it, we introduce the Arrow–Debreu (exchange) market under linear utility functions, which consists of a set A of *agents* and a set G of *goods*. Each agent i comes to the market with an *endowment* $e_{ij} \ge 0$ of each good j and also has a utility $u_{ij} \ge 0$. Each good j must be fully owned by the agents, i.e., $\sum_{i\in A} e_{ij} = 1$ for all $j \in G$.

[3] Pareto optimality for HZ requires that each agent receives a *cheapest* utility maximizing bundle. Consider an instance with two agents a_1 and a_2, and two goods g_1 and g_2 with $u_{11} = u_{21} = u_{22} = 1; u_{12} = 0$. The agents are assumed to have unit budgets. The prices (2, 0) together with the allocation $x_{11} = x_{12} = x_{21} = x_{22} = 0.5$ are optimal bundles, though not cheapest. The utilities in this equilibrium are 0.5 for agent a_1 and 1 for agent a_2. However, there is another HZ equilibrium with prices $(1, p)$, for any $p \in [0, 1]$ with utility 1 for both agents.

Definition 5 An *Arrow–Debreu (AD) equilibrium* for a given AD market is a pair (x, p) consisting of an *allocation* $(x_{ij})_{i \in A, j \in G}$ and non-negative *prices* $(p_j)_{j \in G}$ with the following properties.

1. Each agent spends at most the budget earned from the endowment, i.e., $\sum_j p_j x_{ij} \leq b_i := \sum_j p_j e_{ij}$.
2. Each agent i gets an *optimal bundle*, i.e.,

$$\sum_{j \in G} u_{ij} x_{ij} = \max \left\{ \sum_{j \in G} u_{ij} y_j \mid y \in \mathbb{R}^G_{\geq 0}, \sum_{j \in G} p_j y_j \leq b_i \right\}.$$

3. The market clears, i.e., each good with a positive price is fully allocated to the agents.

The AD model generalizes the Fisher model in the sense that any Fisher market can be easily transformed into an AD market by giving each agent a fixed proportion of every good. Clearly, AD equilibria satisfy the condition of individual rationality, defined below, since every agent could always buy back their endowment.

Definition 6 An allocation in an AD market is *individually rational* if for every agent i we have $\sum_j u_{ij} x_{ij} \geq \sum_j u_{ij} e_{ij}$, i.e., no agent loses utility by participating in the market.

However, individual rationality fundamentally clashes with envy-freeness. Consider a market consisting of two agents, each owning a distinct good. Assume that both agents prefer the good of agent 2 over the good of agent 1, then in any allocation, either agent 1 envies agent 2 or agent 2's individual rationality is violated. For this reason, we primarily consider a version of equal-type envy-freeness in exchange markets, which demands envy-freeness only for agents with the same initial endowment.

AD equilibria do not always exist. However, there is a simple necessary and sufficient condition for their existence based on *strong connectivity of demand graph*, due to Gale [10]. An RCP for this problem was given by Devanur, Garg, and Végh [38].

We now turn to the extension of the HZ mechanism to exchange markets. In the *ADHZ market*, we have a set A of *agents* and a set G of *goods* with $|A| = |G| = n$. Each agent i comes with an *endowment* $e_{ij} \geq 0$ of each good j and utilities $u_{ij} \geq 0$. The endowment vector e is a fractional perfect matching.

Definition 7 An *ADHZ equilibrium* for a given ADHZ market is a pair (x, p) consisting of an *allocation* $(x_{ij})_{i \in A, j \in G}$ and non-negative *prices* $(p_j)_{j \in G}$ with the following properties.

1. x is a fractional perfect matching, i.e., $\sum_{j \in G} x_{ij} = 1$ for all i and $\sum_{i \in A} x_{ij} = 1$ for all j.
2. Each agent spends at most the budget earned from the endowment, i.e., $\sum_j p_j x_{ij} \leq b_i := \sum_j p_j e_{ij}$.
3. Each agent i gets an *optimal bundle*, which is defined to be a cheapest utility maximizing bundle, i.e.,

$$\sum_{j\in G} u_{ij}x_{ij} = \max\left\{\sum_j u_{ij}y_j \mid \sum_j y_j = 1;\ \sum_j p_j y_j \le b_i\right\} \text{ and}$$

$$\sum_{j\in G} p_j x_{ij} = \min\left\{\sum_{j\in G} p_j y_j \mid \sum_j y_j = 1;\ \sum_{j\in G} u_{ij}y_j \ge \sum_{j\in G} u_{ij}x_{ij}\right\}.$$

Theorem 1 *ADHZ equilibrium allocations are Pareto optimal, individually rational, and equal-type envy-free.*

Proof Pareto optimality follows from the fact that any ADHZ equilibrium is an HZ equilibrium with certain budgets b. Since any HZ equilibrium allocation is Pareto optimal, we get the same for ADHZ.

Note that the budget of any agent is always enough to buy back their initial endowment. Since they get an optimal bundle, they must get something that they value at least as high as their initial endowment. Thus, individual rationality is guaranteed.

If two agents, say 1 and 2, have the same endowment, then their budget will be the same, and so agent 1 will never value the 2's bundle higher than their own. Thus, ADHZ equilibrium allocations are equal-type envy-free. □

In addition, ADHZ equilibrium allocations also satisfy the following notion of core stability.

Definition 8 An allocation x in an ADHZ market is *weakly core stable* if for any subsets $A' \subseteq A$ and $G' \subseteq G$, there does not exist an allocation $x' \in \mathbb{R}_{\ge 0}^{A'\times G'}$ such that

- x' allocates at most one unit of goods to every agent in A',
- Every good $j \in G'$ is allocated at most to the extent of the endowments of the agents in A', i.e., $\sum_{i\in A'} x'_{ij} \le \sum_{i\in A'} e_{ij}$, and
- Every agent in A' receives strictly better utility in x' than in x.

Theorem 2 *ADHZ equilibrium allocations are weakly core stable.*

Proof Let (x, p) be some ADHZ equilibrium. For the sake of a contradiction, assume that there are $A' \subseteq A$, $G' \subseteq G$, and $x' \in \mathbb{R}_{\ge 0}^{A'\times G'}$ as excluded by the definition of weak core stability. Now consider the total money spent "along allocation x'", i.e., the quantity $\sum_{i\in A'} \sum_{j\in G'} p_j x'_{ij}$.

On the one hand, we know that only the endowment of the agents in A' is allocated by x'. Thus

$$\sum_{i\in A'}\sum_{j\in G'} p_j x'_{ij} \le \sum_{i\in A'}\sum_{j\in G'} p_j e_{ij}.$$

On the other hand, every agent i receives strictly better utility from x' than from x. But since agents buy optimal bundles in (x, p), this implies that the bundles in x' must be worth more than their budget, i.e.,

$$\sum_{j\in G'} p_j x'_{ij} > \sum_{j\in G} p_j e_{ij} \ge \sum_{j\in G'} p_j e_{ij}.$$

Summing this inequality over all $i \in A'$ contradicts the previous inequality. □

Like in the case of HZ, equilibrium prices in ADHZ are invariant under the operation of *scaling* the difference of prices from 1, as shown in the following lemma.

Lemma 3 *Let $p \geq 0$ be an equilibrium price vector. For any $r > 0$, let $p' \geq 0$ be such that $p'_j - 1 = r(p_j - 1)$ for all $j \in G$. Then p' is also an equilibrium price vector.*

Proof Let $y \in \mathbb{R}^G_{\geq 0}$ be some vector with $\sum_{j \in G} y_j = 1$. Then we observe that

$$\sum_{j \in G} y_j p'_j = \sum_{j \in G} y_j (r(p_j - 1) + 1) = r \sum_{j \in G} y_j p_j + (1 - r).$$

In other words, the total price under p' is a strictly increasing function of the total price under p. Let x be an equilibrium allocation at prices p. Using this monotonicity observation, we show that the pair (x, p') is also an equilibrium.

First, note that, by definition, for any agent i, we have $\sum_{j \in G} x_{ij} p_j \leq \sum_{j \in G} e_{ij} p_j$. But then we also have that $\sum_{j \in G} x_{ij} p'_j \leq \sum_{j \in G} e_{ij} p'_j$ by monotonicity. In other words, x consists of feasible bundles under p'.

On the other hand, note that for any agent i, there cannot be any better bundle than x_i since such a bundle would be feasible under p as well (by monotonicity), contradicting the fact that (x, p) is an equilibrium. The same argument also shows that there cannot be any cheaper optimal bundle under prices p'. □

Unlike HZ, which always admits an equilibrium, ADHZ has instances that do not admit an equilibrium, as observed by Hylland and Zeckhauser [1]. Below, we give a counterexample in which the demand graph is strongly connected and utilities are dichotomous.

Proposition 4 *The ADHZ market with dichotomous utilities in Fig. 1 does not admit an equilibrium.*

Proof Assume this market has an equilibrium (x, p). Further, using Lemma 3, we can assume that the minimum price is zero at p. This implies that no agent will buy a zero utility good at a positive price since otherwise they are clearly not getting a cheapest optimal bundle.

Each agent buys a total of one unit of goods, and s is the only agent with positive utility for goods a and b. Therefore, at least one of these goods is not fully sold to s and must be sold to an agent deriving zero utility from it. Therefore, this good must have zero price. Without loss of generality, assume $p_a = 0$ (in fact, one could show that $p_b = 0$ as well by the cheapest bundle condition, but we do not need this). Since a has no budget and c and d are desired only by a, $p_c = p_d = 0$, otherwise c and d cannot be sold. For the same reason, $p_e = 0$. Now observe that both agents c and d have a utility 1 edge to a good of price zero, namely e. Therefore, the optimal bundle of both c and d is e. But then e would have to be matched twice, which is a contradiction. □

Even if ADHZ equilibria *do* exist, computing them is at least as hard as computing HZ equilibria. This follows from the following reduction.

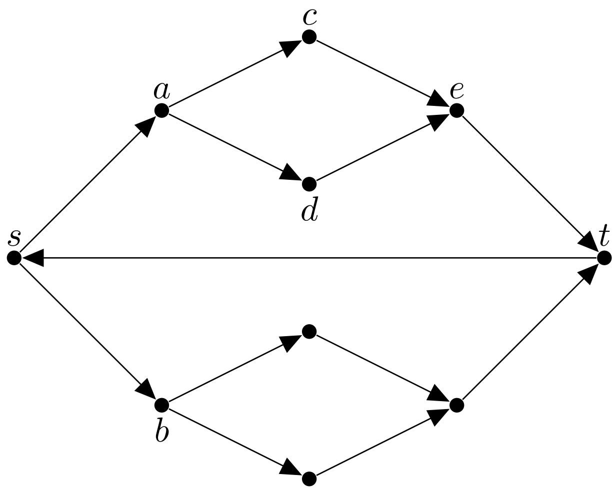

Fig. 1 The demand graph of an ADHZ market with dichotomous utilities and no equilibrium. Each agent in this market has an endowment consisting of one unit of a unique good. Each node in the graph represents an agent together with their good. An arrow from i to j means that agent i has utility 1 for good j (i.e., the good which is in agent j's endowment). All other utilities are 0

Proposition 5 *Consider an HZ market with unit budgets. Define an ADHZ market by giving every agent as an endowment an equal amount of every good. Then, every HZ equilibrium in which the prices sum up to n is an ADHZ equilibrium, and every ADHZ equilibrium yields an HZ equilibrium by rescaling all prices by* $n/\sum_{j\in G} p_j$.

Vazirani and Yannakakis [2] gave an instance of HZ with four agents and four goods which has one equilibrium in which all agents fully spend their budgets, and allocations and prices are irrational. Since this example satisfies the conditions of Proposition 5, we get that the modification of the example of [2], as stated in the Proposition, is an instance for ADHZ having only irrational equilibria.

3.1 Existence and properties of ϵ-approximate ADHZ equilibria

Since ADHZ equilibria do not always exist, we study the following approximate equilibrium notion instead.

Definition 9 An *ϵ-approximate ADHZ equilibrium* is an HZ equilibrium (x, p) for a budget vector b with

$$(1-\epsilon)\sum_{j\in G} p_j e_{ij} \le b_i \le \epsilon + \sum_{j\in G} p_j e_{ij} \quad \text{for all } i \in A \quad .$$

We also require that if two agents have the same endowment, their budget should be the same.

The additive error term in the upper bound is necessary since otherwise the counterexample from Proposition 4 still works. On the other hand, the multiplicative lower bound is useful to get approximate individual rationality. However, one can always find approximate equilibria in which the sum of prices is bounded by n using Lemma 3, so we also get

$$\sum_{j\in G} p_j e_{ij} - \epsilon' \leq b_i \leq \sum_{j\in G} p_j e_{ij} + \epsilon' \quad \text{for } \epsilon' := n\epsilon.$$

This implies that we can equivalently define the above notion with additive error terms on both upper and lower bounds.

In our notion of approximate equilibrium, we do not relax the fractional perfect matching constraints or the optimum bundle condition. We only allow agents' budgets to differ slightly from the money they would normally obtain in an ADHZ market. Hence, the step of randomly rounding the equilibrium allocation to an integral perfect matching is the same as in the HZ scheme.

Theorem 6 *Any ϵ-approximate ADHZ equilibrium allocation is Pareto optimal, ϵ-approximately individually rational, equal-type envy-free.*

Proof Pareto optimality follows just as for the non-approximate ADHZ setting from the fact that an ϵ-approximate ADHZ equilibrium is, first and foremost, an HZ equilibrium. For approximate individual rationality, note that every agent gets a budget of at least $(1-\epsilon)$ times the cost of their endowment. Hence, their utility can decrease by at most a factor of $(1-\epsilon)$. Equal-type envy-freeness follows immediately from the condition that agents with the same endowment have the same budget. □

One can also define a suitably ϵ-approximate notion of weak core stability, where instead of demanding that every agent strictly improves in the seceding coalition, we instead require that every agent improves by a factor of more than $\frac{1}{1-\epsilon}$.

Theorem 7 *Any ϵ-approximate ADHZ equilibrium is ϵ- approximately weak-core stable.*

Proof Let (x, p) be an ϵ-approximate ADHZ equilibrium for some budget vector b. Assume for the sake of a contradiction that there are $A' \subseteq A$, $G' \subseteq G$ and $x' \in \mathbb{R}_{\geq 0}^{A'\times G'}$ such that the allocation x' redistributes the endowments of agents in A' in such a way that every agent in A' improves their utility by a factor of more than $\frac{1}{1-\epsilon}$. As in the proof of Theorem 2, we note that

$$\sum_{i\in A'}\sum_{j\in G'} p_j x'_{ij} \leq \sum_{i\in A'}\sum_{j\in G'} p_j e_{ij} \tag{1}$$

since for every good j, only the part that is endowed to agents in A' can be redistributed by x'.

On the other hand, in order for x' to improve i's utility by a factor of more than $\frac{1}{1-\epsilon}$, i must spend more than $\frac{b_i}{1-\epsilon}$, i.e.

$$\sum_{j\in G'} p_j x'_{ij} > \frac{b_i}{1-\epsilon} \geq \sum_{j\in G} p_j e_{ij}$$

where the second inequality follows from the fact that (x, p) is an ϵ-approximate ADHZ equilibrium. Summing this inequality over i yields a contradiction to (1). □

While approximate equilibrium notions are more amenable to computation, they generally do not lend themselves well to existence proofs. However, our notion of ϵ-approximate ADHZ equilibrium is a slight relaxation of the notion of an α-slack equilibrium introduced in [7].

Definition 10 An *α-slack ADHZ equilibrium* for $\alpha \in (0, 1]$ is an HZ equilibrium (x, p) for a budget vector b in which $b_i = \alpha + (1 - \alpha) \sum_{j \in G} p_j e_{ij}$ for all $i \in A$.

Theorem 8 (Theorem 2 in [7]) *In any ADHZ market, α-slack equilibria always exist if $\alpha > 0$.*

Note that any α-slack equilibrium is automatically also an α-approximate equilibrium. Thus we get:

Theorem 9 *In any ADHZ market, ϵ-approximate equilibria always exist if $\epsilon > 0$.*

4 Algorithm for ϵ-approximate ADHZ under dichotomous utilities

In this section we will focus on *dichotomous utilities*, i.e. we assume that each $u_{ij} \in \{0, 1\}$ for all i, j. Before we can tackle the ADHZ setting, let us first give an algorithm that can compute HZ equilibria with non-uniform budgets. This extends the algorithm presented in [2]. In the following, fix some HZ market consisting of n agents and goods with $u_{ij} \in \{0, 1\}$ for all $i \in A$ and $j \in G$. If $u_{ij} = 1$, we will say that i *likes* j (and *dislikes* otherwise). We assume that every agent likes at least one good.

Remark 1 Any HZ equilibrium (x, p) for the utilities u_{ij} is also an equilibrium for $\tilde{u}_{ij}$ where $\tilde{u}_{ij} = a_i$ if $u_{ij} = 0$ and b_i if $u_{ij} = 1$ for for all agents i, goods j, and arbitrary $0 \le a_i < b_i$ for every agent. This is because $\sum_{j \in G} \tilde{u}_{ij} x_{ij} = a_i + (b_i - a_i) \sum_{j \in G} u_{ij} x_{ij}$ since x is a fractional perfect matching. Hence utility function $\tilde{u}$ is an affine transformation of utility function u; the former is called a *bi-valued utility function*.

Definition 11 Let $(p_j)_{j \in G}$ be non-negative prices. For any $\rho \ge 0$, let $G(\rho)$ be the goods that have price ρ and let $A(\rho)$ be those agents for which the cheapest price of any liked good is ρ. We call the graph consisting of the utility 1 edges restricted to $A(\rho) \cup G(\rho)$ the *price class ρ*.

Lemma 10 *Let $(p_j)_{j \in G}$ be non-negative prices. Assume that*

- There is a matching in price class 0 which covers all agents in $A(0)$ and
- If $\rho > 0$ is equal to the price of some good, then the flow network shown in Fig. 2 has a maximum flow of size $\rho |G(\rho)| = \sum_{i \in A(\rho)} \beta_i$.

Then, we can find a fractional perfect matching x, which makes (x, p) an HZ equilibrium in polynomial time.

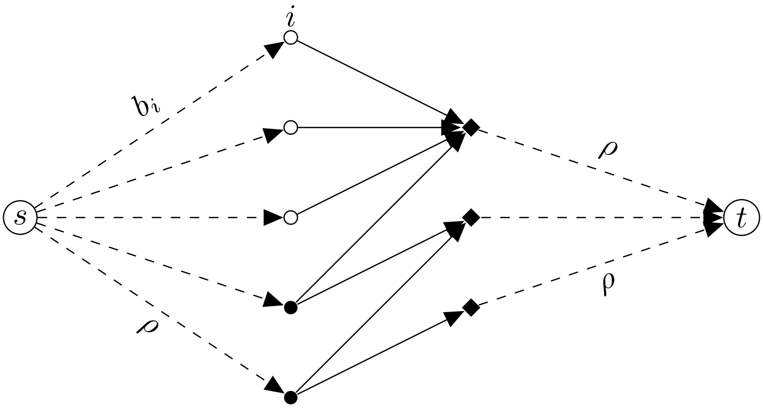

Fig. 2 Shown is the flow network corresponding to finding an equilibrium allocation in price class ρ. Filled circles represent agents in $A(\rho)$ with $b_i \geq \rho$, empty circles are agents in $A(\rho)$ with $b_i < \rho$, and diamond vertices are goods in $G(\rho)$. The contiguous edges represent all utility 1 edges and have infinity capacity (utility 0 edges are not part of the network). Dashed edges to empty circle vertices i have capacity b_i, whereas the other dashed edges have capacity ρ

Proof Allocate every agent in $A(0)$ to some good in $G(0)$ according to the matching that exists by assumption. Let $\rho > 0$ be the price of some good. Then we compute the maximum flow $f^{(\rho)}$ in the flow network from Fig. 2 and allocate $x_{ij} = f^{(\rho)}_{i,j}/\rho$ for all $i \in A(\rho)$ and $j \in G(\rho)$. Lastly, extend x to a fractional perfect matching by matching the remaining capacity of the agents to the remaining capacity of goods in $G(0)$.

Clearly, no agent exceeds their budget. To see that this yields an HZ equilibrium, note that every agent only spends money on cheapest liked goods and if they do not get allocated entirely to liked goods, then they additionally spend all of their budget. This ensures that every agent gets an optimum bundle. □

Definition 12 Given some prices $(p_j)_{j \in G}$ and some agent $i \in A$, their *effective budget* β_i is the price of their cheapest liked goods. Moreover, for any $S \subseteq G$, let $\Gamma(S)$ be the set of agents which have a cheapest liked good in S.

Let H be the bipartite graph on $A \cup G$ which has an edge $\{i, j\}$ whenever $u_{i,j} = 1$.
Compute a minimum vertex cover $A_2 \cup G_1$ in H where $G_1 \subseteq G$, $A_2 \subseteq A$ and G_1 is minimal.
Let $A_1 := A \setminus A_2$ and $G_2 := G \setminus G_1$.
Set prices $p_j := 0$ for all $j \in G_2$ and $p_j := \min_{i \in A_1} b_i$.
while $A_1 \neq \emptyset$ **do**
    Raise all prices of goods in G_1 such that there is a non-empty set $S \subseteq G_1$ with $\sum_{i \in \Gamma(S) \cap A_1} \beta_i = \sum_{j \in S} p_j$ but still $\sum_{i \in \Gamma(S') \cap A_1} \beta_i \leq \sum_{j \in S'} p_j$ for all $S' \subseteq G_1$.
    Set $A_1 \setminus \Gamma(S)$ and $G_1 \setminus S$.
The final prices p are equilibrium prices. Use max-flow (Lemma 10) to obtain the allocation.

Theorem 11 *For any rational budget vector b, Algorithm 1 computes an HZ equilibrium in polynomial time.*

Proof First, note that the algorithm terminates. The main observation here is that if prices keep rising, eventually, agents' effective budgets become their actual budgets, and so there is always, in fact, a set S as demanded by line 6 of the algorithm. In each while loop iteration, both A_1 and G_1 strictly decrease, so the algorithm terminates.

Next, we need to show correctness, i.e., that the prices established by the algorithm satisfy the constraints of Lemma 10. First, observe that in the very first iteration of the loop, we can indeed raise the price to some positive amount. If this were not the case, then this would imply that there is some set $S \subseteq G_1$ such that $|\Gamma(S) \cap A_1| = |S|$. But then $(A_2 \cup \Gamma(S)) \cup (G_1 \setminus S)$ would be a vertex cover which contradicts the minimality of G_1. This implies that $A(0) = A_2$ and so the first condition of Lemma 10 is satisfied since $A_2 \cup G_1$ is a vertex cover.

The second condition of Lemma 10 is satisfied because the condition on line 6 is chosen exactly so that the flow network as shown in Fig. 2 has a minimum cut of weight $\rho|G(\rho)|$ for each price ρ. This is because a set S corresponds to the s-t-cut $\{s\} \cup (A_1 \setminus \Gamma(S)) \cup (G_1 \setminus S)$ and vice versa.

Lastly, we want to see that the algorithm can be implemented to run in polynomial time, for which the main challenge lies in line 6: how do we compute the raised prices and the set S efficiently? We need to raise the price of all goods in G_1 to some ρ such that $\sum_{i \in \Gamma(S') \cap A_1} \beta_i \leq |S'|\rho$ for all $S' \subseteq G_1$ and there exists some non-empty $S \subseteq G_1$ with $\sum_{i \in \Gamma(S) \cap A_1} \beta_i = \rho|S|$. Observe that the set of all ρ for which a suitable S exists is some interval $[\rho^-, \rho^+]$. This makes it possible to find ρ using binary search or parametric search as long as we can decide for some ρ whether $\rho < \rho^-$, $\rho \in [\rho^-, \rho^+]$, or $\rho > \rho^+$.

As mentioned above, the sets S correspond to s-t-cuts in the flow network in Fig. 2. If $\rho > \rho^+$, there is some overtight S, which gives a cut of weight less than $\rho|G_1|$, which can be detected with a max-flow algorithm. On the other hand, if $\rho < \rho^-$, this means that the minimum s-t-cut in the flow network is unique and given by $\{s\} \cup A_1 \cup G_1$ (this essentially corresponds to $S = \emptyset$). Whereas for $\rho \in [\rho^-, \rho^+]$, there will be at least one more min-cut corresponding to a non-empty S. This can be decided – and the set S obtained – from the residual graph after a max-flow computation in polynomial time. Thus, line 6 can be implemented in polynomial time using binary search or even in strongly polynomial time using parametric search. □

Lemma 12 *Let b and b′ be two budget vectors with* $0 \leq b \leq b'$. *Assume we are given an HZ equilibrium* (x, p) *for the budgets b. Then we can compute a new HZ equilibrium* (x', p') *with* $p \leq p'$ *for the budgets b′ in polynomial time.*

Proof We will run the same algorithm as in the proof of Theorem 11, except that this time we start with the prices p. More precisely, we increase the lowest non-zero price until a set goes tight or it becomes equal to the next higher price, then repeat this process until we once again get $\sum_{i \in \Gamma(S)} \beta_i \geq \sum_{j \in A} p_j$ and $\sum_{i \in \Gamma(G_1)} \beta_i = \sum_{j \in G_1} p_j$ where G_1 is now defined as the set of goods with positive prices in (x, p). As in the proof of Theorem 11, this will freeze all prices in polynomial time, at which point we can use a max-flow min-cut argument to construct the new equilibrium allocation x' in polynomial time. □

Let us now return to the approximate ADHZ setting. Instead of budgets, fix now some fractional perfect matching of endowments $(e_{ij})_{i\in A, j\in G}$.

Theorem 13 *An ϵ-approximate ADHZ equilibrium for rational $\epsilon \in (0,1)$, can be computed in time polynomial in $\frac{1}{\epsilon}$ and n, i.e. by a fully polynomial time approximation scheme.*

Proof We will iteratively apply Lemma 12. Start by setting $b_i^{(1)} := \frac{\epsilon}{2}$ for all $i \in A$ and computing an HZ equilibrium $(x^{(1)}, p^{(1)})$ according to Theorem 11. Beginning with $k := 1$, we run the following algorithm.

1. Let $b_i^{(k+1)} := \frac{\epsilon}{2} + (1-\frac{\epsilon}{2}) \sum_{j\in G} p_j^{(k)} e_{ij}$ for all $i \in A$.
2. Compute a new HZ equilibrium $(x^{(k+1)}, p^{(k+1)})$ for budgets $b^{(k+1)}$ according to Lemma 12 using the old equilibrium $(x^{(k)}, p^{(k)})$ as the starting point. Note that since $p^{(k)} \geq p^{(k-1)}$ we always have $b^{(k+1)} \geq b^{(k)}$ and so this is well-defined.
3. Set $k := k+1$ and go back to step 1.

Note that

$$\sum_{i\in A} b_i^{(k+1)} = \frac{\epsilon}{2}n + \left(1-\frac{\epsilon}{2}\right)\sum_{j\in G} p_j^{(k)} \quad \leq \frac{\epsilon}{2}n + \left(1-\frac{\epsilon}{2}\right)\sum_{i\in A} b_i^{(k)}$$

and thus

$$\sum_{j\in G} p_j^{(k)} \leq \sum_{i\in A} b_i^{(k)} \leq n$$

as otherwise we would get $\sum_{i\in A} b_i^{(k+1)} < \sum_{i\in A} b_i^{(k)}$.

Let K be the first iteration such that $p^{(K)} \leq \frac{1-\epsilon/2}{1-\epsilon} p^{(K-1)}$. Note that

$$K \leq n \log_{\frac{1-\epsilon/2}{1-\epsilon}} \left(\frac{n}{\epsilon}\right) = O\left(\frac{n}{\epsilon}\log\left(\frac{n}{\epsilon}\right)\right)$$

since all non-zero prices are initialized to at least ϵ but are bounded by n. Then $(x^{(K)}, p^{(K)})$ is an ϵ-approximate ADHZ equilibrium with budget vector $b^{(K)}$ because for all $i \in A$ we have

$$\begin{aligned} b_i^{(K)} &= \frac{\epsilon}{2} + \left(1-\frac{\epsilon}{2}\right)\sum_{j\in G} p_j^{(K-1)} e_{ij} \\ &\in \left[(1-\epsilon)\sum_{j\in G} p_j^{(K)} e_{ij}, \epsilon + \sum_{j\in G} p_j^{(K)} e_{ij}\right]. \end{aligned}$$

Lastly, we note that since the number of iterations is bounded by $O(\frac{n}{\epsilon}\log(\frac{n}{\epsilon}))$ and each iteration runs in polynomial time, the total runtime is polynomial in $\frac{1}{\epsilon}$ and n as claimed. □

5 An RCP for the HZ scheme under dichotomous utilities

Without loss of generality, we will assume that each agent $i \in A$ likes some good $j \in G$, i.e., $u_{ij} = 1$. We will show that the program (2) given below is the required RCP.

$$\begin{array}{rcc} & \max & \sum_{i\in A} \log \sum_{j\in G} u_{ij}x_{ij} \\ \text{subject to} & \forall j \in G: & \sum_{i\in A} x_{ij} \leq 1 \\ & \forall i \in A: & \sum_{j\in G} x_{ij} \leq 1 \\ & \forall i \in A, j \in G: & x_{ij} \geq 0 \end{array} . \tag{2}$$

Let p_j's and α_i's denote the non-negative dual variables for the first and second constraints, respectively.

Theorem 14 *Any HZ equilibrium is an optimal solution to* (2), *and every optimal solution of* (2) *can be trivially extended to an HZ equilibrium. Furthermore, the latter can be expressed via rational numbers whose denominators have polynomial, in n, number of bits, thereby showing that* (2) *is a rational convex program.*

Proof Let $u_i := \sum_{j\in G} u_{ij}x_{ij}$. Clearly, in any HZ equilibrium, since each agent i is allocated an optimal bundle of goods, she will be allocated a non-zero amount of a unit-utility good and hence will satisfy $u_i > 0$. Furthermore, in an optimal solution x of (2), every agent must have positive utility because otherwise the objective function value will be $-\infty$. Therefore, $\forall i \in A: \ u_i > 0$.

The KKT conditions of this program are:

1. $\forall i \in A: \alpha_i \geq 0$.
2. $\forall j \in G: p_j \geq 0$.
3. $\forall i \in A:$ If $\alpha_i > 0$ then $\sum_j x_{ij} = 1$.
4. $\forall j \in G:$ If $p_j > 0$ then $\sum_i x_{ij} = 1$.
5. $\forall i \in A, j \in G: \ u_{ij} \leq u_i(p_j + \alpha_i)$.
6. $\forall i \in A, j \in G: \ \ x_{ij} > 0 \Rightarrow u_{ij} = u_i(p_j + \alpha_i)$.

To prove the forward direction of the first statement, let (x, p) be an HZ equilibrium. Since x is a fractional perfect matching on agents and goods, it satisfies the constraints of (2) and is hence a feasible solution for it. We are left with proving optimality.

The KKT conditions 2, 3 and 4 are clearly satisfied by (x, p). Next, consider agent i. If there is a good j such that $p_j \leq 1$ and $u_{ij} = 1$, then i will be allocated one unit of the cheapest such goods. Assume the price of the latter is p. Define $\alpha_i = 1 - p$. Clearly $u_i = 1$. It is easy to check that Conditions 1, 5, and 6 are also holding.

Next, assume that every good j such that $u_{ij} = 1$ has $p_j > 1$ and let p be the cheapest such price. Clearly, i's optimal bundle will contain $1/p$ amount of these goods, giving her total utility $1/p$. Since the equilibrium always has a zero-priced good, that good, say j, must have $u_{ij} = 0$. Now, i must buy such zero-utility, zero-priced goods to get to one unit of goods. We will define $\alpha_i = 0$. Again, it is easy to check that Conditions 1, 5, and 6 are holding. Hence, we get that (x, p) is an optimal solution to (2).

Next, we prove the reverse direction of the first statement. Let (x, p) be an optimal solution to (2). Assume that agent i is allocated good j, i.e. $x_{ij} > 0$. We consider the following two cases:

(a) $u_{ij} = 0$. Using Condition 6 and $u_i > 0$, we get that $p_j = \alpha_i = 0$.
(b) $u_{ij} = 1$. Using Conditions 5 and 6 and $u_i > 0$, we get that the price of good j is the cheapest among all goods for which i's utility is 1.

For each agent i, multiply the equality in Condition 6 by x_{ij} and sum over all j to get:

$$\sum_j x_{ij} u_{ij} \;=\; u_i \sum_j x_{ij}(p_j + \alpha_i)$$

After canceling u_i from both sides, we obtain

$$\sum_j x_{ij}(p_j + \alpha_i) = 1 = \sum_j x_{ij} p_j + \alpha_i \sum_j x_{ij}.$$

Now, if $\alpha_i > 0$, then $\sum_j x_{ij} = 1$ and if $\alpha_i = 0$, then $\alpha_i \sum_j x_{ij} = 0 = \alpha_i$. Therefore, in both cases $\alpha_i \sum_j x_{ij} = \alpha_i$. Hence,

$$\sum_j x_{ij} p_j = 1 - \alpha_i. \tag{3}$$

We will view the dual variables p of the optimal solution (x, p) as prices of goods. The above statement then implies that agent i's bundle costs $1 - \alpha_i$.

Let S denote the set of agents who get less than one unit of goods, i.e., $S := \{i \in A \mid \sum_j x_{ij} < 1\}$, and let T denote the set of partially allocated goods, i.e. $T := \{j \in G \mid \sum_i x_{ij} < 1\}$. By Condition 4, $p_j = 0$ for each $j \in T$. Observe that if for $i \in S$ and $j \in T$, $u_{ij} = 1$, then by allocating a positive amount of good j to i, the objective function value of program (2) strictly increases, giving a contradiction. Therefore, $u_{ij} = 0$.

Since the number of agents equals the number of goods, the total deficiency of agents in solution x equals the total amount of unallocated goods. Therefore, we can arbitrarily allocate unallocated goods in T to deficient agents in S to obtain a fractional perfect matching, say x'. Clearly, (x', p) is still an optimal solution to (2) and is also an HZ equilibrium.

For the second statement, we will start with this solution (x', p). Let $G' \subseteq G$ denote the set of goods with prices bigger than 1, i.e., $G' = \{j \in G \mid p_j > 1\}$ and let $A' \subseteq A$ be the set of agents who have allocations from G'. By Cases (a) and (b), for each $i \in A'$, there is a $j \in G'$ such that $u_{ij} = 1$; moreover this is the cheapest good for which i has utility 1. We first show that each agent $i \in A'$ satisfies $\alpha_i = 0$. If $\sum_{j \in G} x_{ij} < 1$, this follows from KKT Condition 3. Otherwise, there exists $j \in G$ such that $x_{ij} > 0$ and $u_{ij} = 0$. The last statement follows from the fact that $\sum_j x_{ij} p_j \le 1$, which follows from (3). Again, by Case (a), $\alpha_i = 0$. Now, by (3), the money spent by each agent in A' is exactly 1 dollar on goods in G'.

Consider the connected components of bipartite graph (A', G', E), where the set $E = \{(i,j) \in (A', G') \mid x_{ij} > 0\}$. Cases (a) and (b) imply that all goods in a connected component C must have the same price, say p_C. Clearly, the sum of prices of all goods in C equals the total money of agents in C; the latter is simply the number of agents in C. This implies that p_C is rational. Clearly, there is a rational allocation of $1/p_C$ amount of goods to every agent in C.

Let $i \in A$ such that the cheapest good for which i has utility 1 has price 1. If $\alpha_i = 0$, by (3), i buys 1 dollar, and hence 1 unit, of such goods. If $\alpha > 0$, by KKT Condition 3, $\sum_{j \in G} x_{ij} = 1$ and therefore again i has bought 1 unit of such goods. Now, without loss of generality, we will assign to i an entire unit of one such good.

Finally, let $G'' \subseteq G$ denote the set of goods with prices in the interval (0, 1), i.e. $G'' = \{j \in G \mid 0 < p_j < 1\}$ and let $A'' \subseteq A$ be the set of agents who have allocations from G''. Let $i \in A''$. Since $\sum_j x_{ij}p_j < 1$, by (3) $\alpha > 0$. Therefore, each agent in A'' buys one unit of goods from G''. Hence, the allocation of goods from G'' to A'' forms a fractional perfect matching on (G'', A''). Therefore, we can pick any perfect matching consistent with this fractional perfect matching and allocate goods from G'' integrally to A''.

Hence in all cases, the allocation consists of rational numbers, completing the proof.

□

Remark 2 The proof of Theorem 14 shows that for the dichotomous case, the dual of (2) yields equilibrium prices. In contrast, for arbitrary utilities, there is no known mathematical construct, no matter how inefficient its computation, that yields equilibrium prices. In a sense, this should not be surprising since there is a polynomial time algorithm for computing an equilibrium for the dichotomous case [2].

Since the objective function in (2) is strictly concave, the utility derived by each agent i must be the same in all solutions of (2). Hence, we get the following corollary which could be seen as an analogue of the *Rural Hospital Theorem* from the theory of stable matchings; see [39].

Corollary 15 *Each agent gets the same utility under all HZ equilibria with dichotomous utilities.*

6 Rationality of α-slack equilibria under dichotomous utilities

If one wishes to compute exact equilibria (if they exist) instead of approximate ones, clearly, a necessary condition is that equilibria are always rational. As noted in Sect. 3.1, with general utilities, both HZ and ADHZ may have only irrational equilibria. On the other hand, with $\{0, 1\}$-utilities, there are always rational HZ equilibria. In this section, we extend this result to α-slack equilibria in the ADHZ setting.

Fix some ADHZ market with $\{0, 1\}$-utilities, rational endowment vectors e, and some rational $\alpha > 0$. Our rationality proof will work in two steps: First, we show that as a consequence of Theorem 11, there always exists a *special* α-slack equilibrium in which prices are minimal in some sense. Then, we will show that the price vector of such a special equilibrium is the unique solution to a system of linear equations with rational coefficients, proving the rationality of the prices (and hence, there also exists a rational allocation).

Definition 13 An HZ equilibrium (x, p) is called *special* if

1. There is a good $j \in G$ with $p_j = 0$, and
2. For every price $\rho > 0$ in p, there is an agent i whose cheapest liked goods have price ρ and whose budget is at most ρ, i.e., $\rho = \min\{p_j \mid u_{ij} = 1,\ j \in G\}$ and $b_i \leq \rho$.

Lemma 16 *The algorithm described in Theorem 11 always computes a special equilibrium.*

Proof We will show that at any point in the algorithm, if there is some good of price $\rho > 0$, then there is some $i \in A_1$ such that i's cheapest desirable goods have price ρ and $\beta_i = b_i$. Note that this property holds at the beginning of the algorithm since the prices are set to the minimum budget of an agent in A_1. Furthermore, as prices increase, the number of agents $i \in A_1$ with $\beta_i = b_i$ can only increase.

So the only way in which this property could be lost is at the points where prices are frozen and the remaining prices are increased, thus decreasing the number of cheapest desirable goods for some agents. Let $S \subseteq G_1$ be the set of goods that have been frozen at some point in the algorithm and assume that we have raised prices so that the price ρ of items in $G_1 \setminus S$ is strictly larger than the prices in S. Furthermore, assume for the sake of a contradiction that for all $i \in \Gamma(G_1 \setminus S)$, we have that $\beta_i = \rho < b_i$. But then

$$\sum_{i \in \Gamma(G_1 \setminus S)} \beta_i = \rho \cdot |\Gamma(G_1 \setminus S)| \geq \rho \cdot |G_1 \setminus S| = \sum_{j \in G_1 \setminus S} p_j.$$

This means that $G_1 \setminus S$ would have already been frozen in the algorithm, contradicting the fact that ρ is strictly greater than the prices in S. □

Lemma 17 *The prices of the HZ equilibrium as computed in Theorem 11 depend continuously on the budgets, assuming the initial vertex cover is chosen consistently.*

Proof Let b and b' be two distinct positive budget vectors with $\|b - b'\|_\infty \leq \epsilon$ for some $\epsilon > 0$. Consider running the algorithm on b and b' simultaneously; note that initially, prices differ by at most ϵ everywhere. Whenever a set S is frozen for the budget b, all prices in that set must also be frozen for b' soon afterward since there is at most $n\epsilon$ more budget available (otherwise S would go overtight).

Let p and p' be the prices computed for budgets b and b', respectively. Then we have just observed that $p' \leq p + n\epsilon$ and symmetrically $p \leq p' + n\epsilon$. Thus, p depends continuously on b. □

Theorem 18 *There exists a special α-slack equilibrium.*

Proof Let $P := \{p \in \mathbb{R}^G_{\geq 0} \mid \sum_{j \in G} p_j \leq n\}$ be the set of feasible price vectors. Given some $p \in P$, define $f(p)$ to be the prices output by the algorithm from Theorem 11 when applied to the budgets

$$b_i := \alpha + (1 - \alpha) \sum_{j \in G} p_j e_{ij}$$

for all $i \in A$.

Clearly, f maps P into P and by Lemma 17, f is continuous. So, by Brouwer's fixed point theorem, it has a fixed point $p^* \in P$. But by definition of f, this fixed point yields an α-slack equilibrium, and by Lemma 16, this equilibrium is special. □

Lemma 19 *Special α-slack equilibria have rational prices.*

Proof Let (x, p) be a special α-slack equilibrium. Let $0 = \rho_1 \leq \cdots \leq \rho_k$ be the distinct prices in p. For each $\rho_l > 0$, we let $A_m(\rho_l) \subseteq A(\rho_l)$ be those agents whose budget is at most ρ_l and we let $A_s(\rho_l) \subseteq A(\rho_l)$ be the remaining agents whose budget is more than ρ_l.

Since (x, p) is an α-slack equilibrium, we then have that

$$\sum_{i \in A_m(\rho_l)} \left(\alpha + (1-\alpha) \sum_{l'=2}^{k} \rho_{l'} \sum_{j \in G(\rho_{l'})} e_{ij} \right) + \rho_l |A_s(\rho_l)| = \rho_l |G(\rho_l)|. \tag{4}$$

where $A_m(\rho_l) \neq \emptyset$ since (x, p) is a special equilibrium. Together with $\rho_1 = 0$, this gives us a system of linear equations with rational coefficients that $(\rho_1, \dots, \rho_k)$ is a solution to.

Finally, let us show that this system has unique solutions. To see this, let there be some other solution vector $(\rho'_1, \dots, \rho'_k)$. Assume without loss of generality that there is some l with $\rho_l > \rho'_l$; otherwise, we can swap ρ and ρ'. Let l^* be the index maximizing $\frac{\rho_l}{\rho'_l}$ and consider constraint (4) for this l^*. But now assuming that ρ' satisfies this constraint, ρ_l cannot satisfy it since, compared to ρ', the right-hand side increases by a factor of $\frac{\rho_l}{\rho'_l}$ whereas the left-hand side increases by strictly less due to the presence of $\sum_{i \in A_m(\rho_{l^*})} \alpha > 0$. □

Theorem 20 *There exists a rational α-slack equilibrium.*

Proof By Theorem 18, there always exists a special α-slack equilibrium and by Lemma 19, this equilibrium must have rational prices. To get a rational allocation, one can obtain an allocation via a flow network in each price class as shown in Lemma 10. The theorem follows since max-flows in networks with rational weights can always be chosen to be rational. □

7 Discussion

In this paper, we defined an ϵ-approximate ADHZ model for one-sided matching markets with endowments. We showed that ϵ-approximate ADHZ equilibrium always exists for every $\epsilon > 0$. We strengthened the non-existence of ADHZ equilibrium for the case when the demand graph is not strongly connected and agents have dichotomous utilities. We derived a novel combinatorial polynomial-time algorithm for computing an ϵ-ADHZ equilibrium under dichotomous utilities. Finally, we presented a rational convex program (RCP) for the HZ model under dichotomous utilities, which also implies that the problem is polynomial-time solvable.

Since finding an HZ equilibrium is PPAD-complete [2, 27], it will be interesting to obtain a similar result for the ϵ-approximate ADHZ model. In Sect. 1.1, we stated a number of results that build on the HZ scheme and others that are generalizations of the HZ scheme. It will also be interesting to explore similar extensions of the ϵ-approximate ADHZ model.

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Declarations

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