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Classifying primitive solvable permutation groups of rank 5 and 6

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Abstract. Let G be a finite solvable permutation group acting faithfully and primitively on a finite set Ω. Let G_0 be the stabilizer of a point α in Ω. The rank of G is defined as the number of orbits of G_0 in Ω, including the trivial orbit $\{\alpha\}$. In this paper, we completely classify the cases where G has rank 5 and 6, continuing the previous works on classifying groups of rank 4 or lower.

1 Introduction

Let G be a primitive permutation group acting faithfully on a set Ω, and let G_0 be the stabilizer of an element $\alpha \in \Omega$. The *rank* of G is defined as the number of orbits of $G_0 \curvearrowright \Omega$. Some work has been done to try to characterize these groups for arbitrary ranks [4, 23]. However, there has been significantly more progress for the classification of groups of low ranks. Both solvable and insolvable groups of rank 2 have been classified [7, 12]. Various authors have studied primitive groups of rank 3 [8, 9, 14, 16, 17]. Foulser completely classified primitive solvable groups of rank 3 and gave a partial classification of primitive solvable groups of rank 4 in [5]. Revitalization of this work has been made possible due to the advent of computer algebra systems such as GAP [29], making it more feasible to study the actions of finite groups [10]. Using GAP, Dolorfino et al. completely classified primitive solvable groups of rank 4 [3]. In this work, we classify primitive solvable permutation groups of rank 5 and 6. Note that, while [3] relies on a coarse classification due to [5], our paper is self-contained and does not use such a prior classification. In recent years, there has been further interest in obtaining stronger lower bounds for the number of conjugacy classes of a finite group [6, 11, 15]. We

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believe that the results and methods in this paper may have applications in this direction.

If G is a primitive solvable permutation group acting faithfully on Ω, then $|\Omega| = p^d$ for some prime p and positive integer d (see [13]). Moreover, G has a minimal nontrivial normal subgroup V, which is an elementary abelian group such that $|V| = p^d$, so V behaves as a d-dimensional vector space over $\mathbb{F}_p$. We can now decompose G into $G = V \rtimes G_0$, where G_0 acts on V as an irreducible subgroup of $\mathrm{GL}(V)$. Conversely, if an irreducible group G_0 acts on such a vector space V, then we can construct a primitive permutation group G by taking the semidirect product $G = V \rtimes G_0$ (see [24, Section 2]). Consequently, instead of classifying the rank of G based on the number of orbits of G_0 in Ω, we can equivalently consider the number of orbits of $G_0 \curvearrowright V$, as these values are equivalent [5, Definition 2.1].

Viewing G_0 as an irreducible subgroup of $\mathrm{GL}(V)$, we classify G_0 into three classes $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, following the standard analysis of solvable linear groups in [1, 5, 12, 25]. The class $\mathfrak{A}$ consists of primitive subgroups of $\Gamma(V)$, where $\Gamma(V)$ is the semilinear group of V defined as

$$\Gamma(V) = \{x \to ax^\sigma \mid x \in \mathbb{F}_{q^n}, a \in \mathbb{F}_{q^n}^\times, \sigma \in \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)\},$$

and there are infinitely many groups in this class [10]. The class $\mathfrak{B}$ contains the remaining primitive subgroups of $\mathrm{GL}(V)$, and $\mathfrak{C}$ contains the imprimitive subgroups of $\mathrm{GL}(V)$.

Our main result is as follows.

Theorem 1.1. *Suppose $G = V \rtimes G_0$ is a finite primitive solvable permutation group of rank at most 6, where G_0 acts on V as an irreducible subgroup of $\mathrm{GL}(V)$. At least one of the following is true:*

(1) *$G_0 \in \mathfrak{A}$: G_0 is a primitive subgroup of $\Gamma(V)$;*

(2) *$G_0 \in \mathfrak{B}$: G_0 is one of the remaining primitive subgroups of $\mathrm{GL}(V)$ and appears in at least one row of Table 6; or*

(3) *$G_0 \in \mathfrak{C}$: G_0 is an imprimitive subgroup of $\mathrm{GL}(V)$, and there exists an imprimitivity decomposition $V = \bigoplus_{i=1}^r V_i$.*

   (a) *If $\mathrm{rank}(G) \le 5$, then $r = 2$, 3, or 4, and the action of G_0 on each $V_i \setminus \{0\}$ is transitive.*

   (b) *If $\mathrm{rank}(G) = 6$, then $2 \le r \le 10$ and $r \ne 5$ where the action of G_0 on each $V_i \setminus \{0\}$ is transitive, or $r = 2$ where the actions of G_0 on $V_1 \setminus \{0\}$ and $V_2 \setminus \{0\}$ are doubly transitive* [5, Proposition 2.5].

The structure of this paper is as follows. First, we describe the structure of primitive solvable permutation groups G acting faithfully on a vector space V in Section 2. Specifically, we do this for the case when $G_0 \in \mathfrak{B}$. Then, in Section 3, we leverage this structure in order to enumerate the groups in $\mathfrak{B}$ we are interested in using the computer algebra system GAP [29]. These groups are fully detailed in Section 4. In particular, we provide examples illustrating the choice of wording in Theorem 1.1 (2) where we state $G_0 \in \mathfrak{B}$ appears in at least one row rather than exactly one row of Table 6 (see Examples 4.1, 4.2 and 4.3). Finally, we pose possible directions for future work in Section 5.

2 Preliminary results

Let G be a primitive solvable permutation group acting faithfully on a set Ω. As described before, this can be decomposed into a solvable subgroup $G_0 \leq \mathrm{GL}(V)$ acting irreducibly on a d-dimensional vector space V over $\mathbb{F}_p$. We now proceed to describe the structure when $G_0 \in \mathfrak{B}$. Note that, in this case, G_0 acts primitively on V. It follows that G_0 also acts quasi-primitively, meaning that all non-trivial normal subgroups of G_0 act homogeneously on V.

Theorem 2.1 ([26, Theorem 2.2], [27, Theorem 2.2], and [28, Theorem 2.1]). *Suppose a finite solvable group G_0 acts faithfully, irreducibly, and quasi-primitively on a d-dimensional vector space V over a finite field $\mathbb{F}$ of characteristic p. Then every normal abelian subgroup of G_0 is cyclic, and G_0 has normal subgroups $Z(E) \leq U \leq F \leq A \leq G_0$ and a characteristic subgroup $E \leq F$ such that the following statements hold.*

(1) *$F = EU$ is a central product, where $Z(E) = E \cap U$.*

(2) *$F/U \cong E/Z(E)$ is a direct sum of completely reducible (or semi-simple) G_0/F-modules.*

(3) *There is a decomposition $E = E_1 \times \cdots \times E_s$, where each E_i is an extraspecial q_i-group. We have that $|E_i| = q_i^{2m_i+1}$ for some distinct primes $q_1, \ldots, q_s$ and some integer $m_i \geq 1$. Denote*

$$e_i := \sqrt{|E_i/Z(E_i)|} = q_i^{m_i} \quad \textit{and} \quad e := e_1 \cdots e_s.$$

We have that $e \mid d$ and $\gcd(p, e) = 1$.

(4) *$A = C_{G_0}(U)$, and A/F acts faithfully on $E/Z(E)$.*

(5) *U is cyclic and acts fixed point freely on W, where W is an irreducible submodule of V_U.*

(6) $|U|$ *divides* $p^k - 1$ *for some* $k \geq 1$, *and* W *can be identified with the span of* U *which is isomorphic to* $\mathrm{GF}(p^k)$.

(7) $|V| := n = |W|^{eb}$ *for some integer* b.

Note that, for $G_0 \in \mathfrak{B}$, we must have that $e > 1$, as it is the product of prime powers.

We now state a series of lemmas and propositions with the goal of restricting the various parameters stated in Theorem 2.1 for the case that $\mathrm{rank}(G) \leq 6$. The value e as defined in Theorem 2.1 (3) is of great importance to our work. Our first step is to set a bound on the value e in the case of $\mathrm{rank}(G) \leq 6$. One immediate lower bound on the rank of G is

$$\mathrm{rank}(G) \geq \left\lceil \frac{|V| - 1}{|G_0|} \right\rceil + 1. \tag{2.1}$$

This follows from the facts that at least one trivial orbit exists, that all orbits of $G_0 \curvearrowright V$ must partition V, and that the largest possible orbit size is $|G_0|$. We now build on equation (2.1) further.

Lemma 2.2 ([27, Lemma 2.3]). $|G_0|$ *divides* $\dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)$.

Proposition 2.3. *Using the notation from Theorem 2.1,*

$$\mathrm{rank}(G) \geq \left\lceil \frac{|W|^e - 1}{\log_2(|W|) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)} \right\rceil + 1. \tag{2.2}$$

Proof. Combining equation (2.1) with Theorem 2.1 (7) and Lemma 2.2, we get

$$\mathrm{rank}(G) \geq \left\lceil \frac{|V| - 1}{|G_0|} \right\rceil + 1 \geq \left\lceil \frac{|W|^{eb} - 1}{\dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)} \right\rceil + 1. \tag{2.3}$$

Then note that $|W| = p^k$ for some prime p and integer k, so $\dim(W) \mid k$. Also, $k = \log_p(|W|)$, which yields $\dim(W) \mid \log_p(|W|)$. Thus, $\dim(W) \leq \log_p(|W|)$. Since p is a prime, we have that $\dim(W) \leq \log_p(|W|) \leq \log_2(|W|)$. Using this and equation (2.3), we get that

$$\begin{aligned}\mathrm{rank}(G) &\geq \left\lceil \frac{|W|^{eb} - 1}{\dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)} \right\rceil + 1 \\ &\geq \left\lceil \frac{|W|^{e} - 1}{\dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)} \right\rceil + 1 \\ &\geq \left\lceil \frac{|W|^{e} - 1}{\log_2(|W|) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)} \right\rceil + 1.\end{aligned}$$

□

Lemma 2.4. *A/F acts completely reducibly on $E/Z(E)$.*

Proof. By Theorem 2.1 (2), we have that $E/Z(E)$ is a direct sum of completely reducible G_0/F-modules. Also, by Theorem 2.1, $A \trianglelefteq G_0$, so $E/Z(E)$ is a direct sum of completely reducible A/F-modules. Therefore, A/F acts completely reducibly on $E/Z(E)$. □

Lemma 2.5. *Let $\alpha = \log_9(96 \cdot \sqrt[3]{3})$ and $\lambda = 2 \cdot \sqrt[3]{3}$. Then $|A/F| \leq \frac{e^{2\alpha}}{\lambda}$.*

Proof. This is an immediate application of [18, Theorem 3.5]. Note that $E/Z(E)$ is a direct sum of completely reducible A/F-modules by Lemma 2.4 and that A/F is solvable since G is solvable. Therefore, by [18, Theorem 3.5],

$$|A/F| \leq \frac{|E/Z(E)|^{\alpha}}{\lambda} = \frac{e^{2\alpha}}{\lambda}.$$ □

Lemma 2.6. *Let $\mathrm{rad}(e)$ be the product of distinct prime factors of e. Then*

$$\mathrm{rad}(e) \mid (|W| - 1).$$

Proof. By Theorem 2.1 (5), U acts fixed point freely on W. This implies that

$$|U| \mid (|W| - 1).$$

By Theorem 2.1 (1) we have that $E \cap U = Z(E)$, so $Z(E) \leq U$. Then, by Theorem 2.1 (3),

$$\mathrm{rad}(e) := q_1 \cdots q_s \mid |Z(E)|.$$

Therefore,

$$\mathrm{rad}(e) \mid |Z(E)| \mid |U| \mid (|W| - 1).$$ □

Proposition 2.7. *Suppose $\mathrm{rank}(G) \leq 6$ and $G_0 \in \mathfrak{B}$. Then the only possible values for e are 2, 3, 4, 5, 6, 7, 8, 9, and 16.*

Proof. Combining equation (2.2) with Lemma 2.5 yields

$$\begin{aligned}\mathrm{rank}(G) &\geq \left\lceil \frac{|W|^e - 1}{\log_2(|W|) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)} \right\rceil + 1 \\ &\geq \left\lceil \frac{\lambda \cdot (|W|^e - 1)}{\log_2(|W|) \cdot e^{2\alpha+2} \cdot (|W| - 1)} \right\rceil + 1. \end{aligned} \tag{2.4}$$

By Lemma 2.6 and the fact that the smallest possible value of e is 2, $|W| \geq 3$. One can see that the bound in equation (2.4) is increasing with respect to $|W|$ for $|W| \geq 3$. By finding e that satisfies the inequality

$$\left\lceil \frac{\lambda \cdot (3^e - 1)}{\log_2(3) \cdot e^{2\alpha+2} \cdot (3-1)} \right\rceil + 1 > 6,$$

we see that if $\text{rank}(G) \leq 6$, then $e \leq 18$. We can now improve this bound on e by considering individual cases.

e	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\text{rad}(e)$	2	3	2	5	6	7	2	3	10	11	6	13	14	15	2	17	6
$\|W\| \geq$	3	4	3	11	7	8	3	4	11	23	7	27	29	16	3	103	7

Table 1. Minimum values of $|W|$ for each value of e.

For each case, we find the minimum value of $|W|$ by Lemma 2.6 and the fact that $|W|$ is a prime power by Theorem 2.1 (6). These minimum values of $|W|$ are given in Table 1. If we take these values for e and the corresponding lower bounds on $|W|$ and apply them to equation (2.4), then we get that

$$\text{rank}(G) > 6 \quad \text{for } e \in \{10, 11, 12, 13, 14, 15, 17, 18\}.$$

Therefore, if $\text{rank}(G) \leq 6$, then $e \in \{2, 3, 4, 5, 6, 7, 8, 9, 16\}$ as claimed. □

Now that we have reduced the possible values of e to a finite list, we aim to obtain upper bounds on $|A/F|$ for each value of e. By equation (2.2), larger values of $|A/F|$ decrease the rank bound. Thus, the largest values of $|A/F|$ yield the worst case bound on $\text{rank}(G)$.

Lemma 2.8. *$E/Z(E)$ is a symplectic vector space.*

Proof. By [18, Corollary 1.10 (iii)],

$$E/Z(E) = E_1/Z(E) \times \cdots \times E_n/Z(E),$$

where $E_i \leq C_{G_0}(E_j)$ for $i \neq j$. In the proof of [18, Corollary 1.10], it is stated that $E_i/Z(E)$ has a non-degenerate symplectic form over $\mathbb{F}_{q_i}$. Therefore, $E/Z(E)$ is a symplectic vector space. □

Proposition 2.9. *For each value of e in Proposition 2.7, the following bounds for* $|A/F|$ *hold.*

$e := q^m$	$\|A/F\|$ divides one of the following values:	$\|A/F\| \leq$
2	$\{6\}$	6
3	$\{24\}$	24
4	$\{60, 6^2 \cdot 2\}$	$6^2 \cdot 2$
5	$\{24\}$	24
6	$\{24 \cdot 6\}$	$24 \cdot 6$
7	$\{48\}$	48
8	$\{42, 54, 6^3 \cdot 2, 6^4\}$	6^4
9	$\{40, 24^2 \cdot 2\}$	$24^2 \cdot 2$
16	$\{136, 6^4 \cdot 24\}$	$6^4 \cdot 24$

Table 2. Upper bounds on $|A/F|$.

Proof. For $e \in \{2, 3, 4, 5, 7, 8, 9, 16\}$, we let $e := q^m$ for some prime q. We have that $E/Z(E)$ is a symplectic vector space of dimension $2m$ over $\mathbb{F}_q$ by Lemma 2.8, and A/F acts completely reducibly on $E/Z(E)$ by Lemma 2.4. Thus, we can apply [26, Lemma 2.17] with $G = A/F$ and $V = E/Z(E)$ to obtain the desired upper bounds for $|A/F|$ when $e \in \{2, 3, 4, 5, 7, 8, 9, 16\}$. Furthermore, the proof of [26, Lemma 2.17] describes many possible values that $|A/F|$ divides for each value of e.

For $e = 6$, we let $q = 6$ and $m = 1$. In this case, we see from the proof of [27, Theorem 3.1] that we have $A/F \leq \mathrm{SL}(2,3) \times \mathrm{SL}(2,2)$, so $|A/F| \mid 24 \cdot 6$. □

Using these values of e and the corresponding bounds on $|A/F|$, we can obtain upper bounds on $|W|$.

Proposition 2.10. *For each value of e in Proposition 2.7, the following upper bounds on* $|W|$ *hold when* $\mathrm{rank}(G) \leq 6$. *In particular,* $e = 5$ *and* $e = 7$ *are ruled out when* $\mathrm{rank}(G) \leq 6$.

e	2	3	4	6	8	9	16
$\max\|W\|$	1511	79	31	7	7	4	3

Table 3. Upper bounds for $|W| = p^k$ for all possible values of e.

Proof. By Proposition 2.3, $\mathrm{rank}(G) > 6$ if

$$\frac{|W|^e - 1}{\log_2(|W|) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)} + 1 > 6. \tag{2.5}$$

For each value of e, by using the largest corresponding bound of $|A/F|$ from Proposition 2.9, we can solve equation (2.5) for $|W|$ to get an upper bound on $|W|$ for which it is possible that $\mathrm{rank}(G) \leq 6$. By Theorem 2.1 (6) and Lemma 2.6, we can further improve the bound by restricting these values of $|W|$ to be prime powers such that $\mathrm{rad}(e) \mid (|W| - 1)$. This improvement completely eliminates the possibility of $e = 5$ and $e = 7$. Therefore, we obtain upper bounds on $|W|$ as listed in Table 3. □

We now give an algorithm which optimizes the lower bound on $\mathrm{rank}(G)$ in the case that e is a prime power. The first lower bound given in Proposition 2.3 was attained by using equation (2.1). This bound is due to the fact that the largest possible size of an orbit of $G_0 \curvearrowright V$ is $|G_0|$. In this algorithm, we extend this technique to account for the existence of smaller orbit sizes.

Fix some value $e = q^m$, where q is prime, from Proposition 2.7. Since G_0 has a trivial orbit in V, the other orbits contain a total of $|V| - 1$ elements. For each orbit O, $|O|$ divides $|G_0|$. Let $B := \dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)$. From Lemma 2.2, $|G_0|$ divides B. Thus, $|O|$ divides B, and the divisors of B are possible sizes for the orbits of G_0.

We now obtain specific parameters q, m, p, and k. For each $e = q^m$, we get upper bounds on $|W|$ and $\dim(W)$ over $\mathbb{F}_p$ from Proposition 2.10. Let p^k be some prime power less than or equal to this upper bound on $|W|$, and the possible values for $\dim(W)$ are the different values of possible k. We also get possible values of $|A/F|$ to consider from the second column of Table 2 since we know that $|A/F|$ divides one of those given values. Note that we do not need to consider any proper divisors of the values in the second column of Table 2 since, by equation (2.2), smaller values of $|A/F|$ can only make the rank bound greater.

The other values of consideration are b and d, where d is $\dim(V)$ over $\mathbb{F}_p$ such that $d = b \cdot k \cdot q^m$. For each set of parameters p, k, q, and m, we check both of the cases where $b = 1$ and where $b > 1$. For a given set of parameters p, k, q, and m, we first let $b = 1$. Later, we describe how to consider $b > 1$.

Algorithm 2.11. Suppose we have parameters p, k, q, m, and $b = 1$ as described as well as the corresponding values of e, $|W|$, $\dim(W)$, and $|A/F|$. We know that, for each orbit O, $|O|$ divides

$$B := \dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1).$$

Let $d_1, d_2, \dots, d_t$ be the divisors of B, the possible sizes of these orbits. In order to obtain a lower bound on the number of orbits of $G_0 \curvearrowright V$, we want to find the worst-case "packing" of elements of V into orbits. We know that there is at least one trivial orbit. Thus, our goal is to pick n_i orbits of size d_i for each d_i such that

$$\sum_{i=1}^{t} n_i d_i = |V| - 1 = p^{b \cdot k \cdot q^m} - 1$$

while minimizing

$$N = \sum_{i=1}^{t} n_i.$$

This is exactly the change-making problem, the problem of representing some chosen value using as few coins as possible from some fixed set of denominations [20]. Let N be the optimal solution to the change-making problem with the chosen value $|V| - 1$ and the denominations being the set of divisors $\{d_1, \dots, d_t\}$. Finding N is quite simple and can be done via a standard dynamic programming algorithm. Then the value $N + 1$ sets a lower bound on the number of orbits of $G_0 \curvearrowright V$, i.e. a lower bound on the rank of G.

Algorithm 2.11 above helps restrict possible sets of parameters $e = q^m$ and $|W| = p^k$ when e is a prime power. When $e = 6$, we let $q = 6$ and $m = 1$, and we have $p = 7$ and $k = 1$ from Proposition 2.10. We now describe how we consider $b > 1$. For each of these sets of parameters, we use equation (2.3) to bound the value of b. For a given set of parameters $e = q^m$, $|W| = p^k$, and $b \geq 1$, if equation (2.3) gives a bound less than or equal to 6 for any of the values of $|A/F|$ from Proposition 2.9, we keep that set of parameters. Otherwise, we eliminate that b value and higher b values for the corresponding set of q, m, p, and k, as the rank bound given by equation (2.3) increases as b increases.

Hence, Algorithm 2.11 and equation (2.3) yield possible sets of parameters q, m, p, k, and b for families of groups $G_0 \in \mathfrak{B}$ which have 6 or fewer orbits when acting on V. We further classify $G_0 \in \mathfrak{B}$ into two cases based on if $b = 1$ or $b > 1$, denoted as $\mathfrak{B}_1$ and $\mathfrak{B}_>$, respectively. It is due to the algorithmic process in Algorithm 3.1 that we split $\mathfrak{B}$ into these two cases rather than based on the action $E \curvearrowright V$ as in [3] or the action of a minimal normal abelian subgroup of G_0 acting on V as in [5].

Theorem 2.12. *Let $G = V \rtimes G_0$, where $G_0 \in \mathfrak{B}$, and V is a vector space over a field of characteristic p. Then, by Theorem 2.1, G_0 has an extraspecial subgroup E of order q^{2m+1} for some prime q, and $q^m = e$. Let $d = \dim(V)$ and let k be the integer guaranteed by Theorem 2.1 (6) such that $d = b \cdot k \cdot q^m$. If G_0 is*

No.	q	m	p	k	d	Rank $\geq$
1	2	1	3	1	2	2
2	2	1	3	2	4	3
3	2	1	3	3	6	3
4	2	1	3	4	8	4
5	2	1	3	5	10	4
6	2	1	5	1	2	2
7	2	1	5	2	4	3
8	2	1	5	3	6	4
9	2	1	7	1	2	2
10	2	1	7	2	4	3
11	2	1	11	1	2	2
12	2	1	11	2	4	5
13	2	1	13	1	2	3
14	2	1	13	2	4	6
15	2	1	17	1	2	3
16	2	1	19	1	2	3
17	2	1	23	1	2	2
18	2	1	29	1	2	3
19	2	1	31	1	2	3
20	2	1	37	1	2	4
21	2	1	41	1	2	4
22	2	1	43	1	2	4
23	2	1	47	1	2	3
24	2	1	53	1	2	4
25	2	1	59	1	2	4
26	2	1	61	1	2	5
27	2	1	67	1	2	5
28	2	1	71	1	2	4
29	2	1	73	1	2	5
30	2	1	79	1	2	5
31	2	1	83	1	2	5
32	2	1	89	1	2	6
33	2	1	97	1	2	6
34	2	1	101	1	2	6
35	2	1	103	1	2	6
36	2	1	107	1	2	6
37	2	2	3	1	4	2
38	2	2	3	2	8	4
39	2	2	5	1	4	3
40	2	2	7	1	4	3
41	2	2	11	1	4	4
42	2	2	13	1	4	5
43	2	3	3	1	8	4
44	2	3	5	1	8	5
45	3	1	2	2	6	3
46	3	1	2	4	12	4
47	3	1	5	2	6	4
48	3	1	7	1	3	3
49	3	1	13	1	3	4
50	3	1	19	1	3	5
51	3	2	2	2	18	6
52	6	1	7	1	6	2

Table 4. Parameters for $G_0 \in \mathfrak{B}_1$ which possibly have 6 or fewer orbits on V.

No.	q	m	p	k	d	Rank $\geq$
53	2	1	3	1	4	4
54	2	2	3	1	8	6

Table 5. Parameters for $G_0 \in \mathfrak{B}_>$ which possibly have 6 or fewer orbits on V.

not described by the sets of parameters in Tables 4 and 5, then $\mathrm{rank}(G) > 6$. *In addition, the case where* $e = 6$ *is also included.*

Proof. Table 4 and Table 5 result from computing the minimum rank of all possible sets of parameters allowed by Proposition 2.10 using Algorithm 2.11 and keeping only those whose minimum ranks are below 6. □

3 Computation

Using the parameters in Tables 4–5, we can enumerate possible groups $G_0 \curvearrowright V$ with 6 or fewer orbits using GAP [29]. The procedure used mirrors that of [3, 10], and our computational process is split into two cases.

Algorithm 3.1. When $G_0 \in \mathfrak{B}_1$, we take the following four steps.

1: Construct the extraspecial group E as guaranteed by Theorem 2.1 (3) as well as its normalizer N_E as subgroups of $\mathrm{GL}(e, p^k)$.

2: Embed N_E into $\mathrm{GL}(k \cdot e, p)$ using the embedding $\mathrm{GL}(e, p^k) \hookrightarrow \mathrm{GL}(k \cdot e, p)$.

3: Construct the normalizer N of N_E in $\mathrm{GL}(k \cdot e, p)$.

4: Since G_0 is a subgroup of N, enumerate all subgroups G_0 of N up to conjugacy in $\mathrm{GL}(k \cdot e, p)$ and check for primitive solvable groups of 6 or fewer orbits which contain E as an isomorphic subgroup of G_0.

For $G_0 \in \mathfrak{B}_>$, we introduce an intermediate step after step 1.

1.5: Embed N_E from $\mathrm{GL}(e, p^k)$ into $\mathrm{GL}(b \cdot e, p^k)$ by taking the tensor product of N_E with $\mathrm{GL}(b, p^k)$. Call this new product N_E and replace $k \cdot e$ with $b \cdot k \cdot e$ as needed for the rest of the algorithm.

Notice that $e = 6$ is the only value of e which is not a prime power. Since 6 is the product of two distinct primes, we can take advantage of a decomposition lemma.

Lemma 3.2 ([2, Lemma 2.1 (6)]). *Let* $G_0 \curvearrowright V$ *be a primitive solvable subgroup of* $\mathrm{GL}(6, 7)$. *Then* G_0 *is conjugate in* $\mathrm{GL}(6, 7)$ *to the Kronecker product* $G_2 \dot{\times} G_3$, *where* G_2 *is a primitive solvable subgroup of* $\mathrm{GL}(2, 7)$, *and* G_3 *is a primitive solvable subgroup of* $\mathrm{GL}(3, 7)$.

Our problem for $e = 6$ is then reduced to enumerating primitive solvable subgroups of $\mathrm{GL}(2, 7)$ and $\mathrm{GL}(3, 7)$ and taking their Kronecker product by Lemma 3.2. This is a less computationally intensive process than Algorithm 3.1 in the case of $e = 6$. However, this cannot be extended to the cases of e being a prime power since it relies on the fact that 6 is the product of distinct primes.

4 Results

Table 6 lists all lines from Tables 4 and 5 that correspond to families of groups $G = V \rtimes G_0$ such that $\mathrm{rank}(G) \leq 6$. Recall that there are exactly two types of extraspecial groups of order q^{2m+1} for a prime q and integer $m \geq 1$ (see [22]). One of them is of exponent q, and the other is of exponent q^2. We denote these as extraspecial types, or et, + and −, respectively. For each set of parameters q, m, p, k, d, b, et we describe the number of groups G_0 of rank 2 through 6 with these parameters. However, we note that there are some caveats to this list.

Example 4.1. For a given q and m, we may have that a group G_0 contains both extraspecial groups of type + and of type −. One such example of this is line 3 of Table 4. When taking these parameters and the extraspecial group of type +, one of the generated groups G_0 is $\mathbb{Z}_{13} \times \mathrm{QD}_{16}$. This group is also generated when taking the same parameters and the extraspecial group of type −. We count the group in both sets of parameters in this case.

Example 4.2. We can have that a group is of multiple ranks. What this means is that two isomorphic copies of a group G_0 may be in separate conjugacy classes of the larger group N constructed in Algorithm 3.1 and act on V differently, resulting in different ranks. One case where this occurs is line 48 of Table 4. When considering the extraspecial group of type +, one of the rank 4 groups and one of the rank 5 groups are isomorphic to $((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes Q_8$. We also count the group in both ranks in this case.

Example 4.3. We may have a group in $\mathfrak{B}_1$ that also appears in $\mathfrak{B}_>$. Furthermore, in each of these cases, the rank may even be the same. One such pair of parameters that demonstrates this is line 2 of Table 4 with an extraspecial group of type − and line 53 of Table 5 with an extraspecial group of type −. Both of these sets of parameters generate a group isomorphic to $((\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)) \rtimes \mathbb{Z}_3$. Furthermore, both of these groups are of rank 4. Such examples are also considered in [3]. Again, we count such groups in both sets of parameters.

It is because of these examples that Theorem 1.1 states that a given $G_0 \in \mathfrak{B}$ appears in at least one row of Table 6 rather than exactly one row.

We provide all families of G_0 for the parameters described in Table 6 as separate GAP files. These files, along with the details for the implementation of the algorithms in Section 3 in GAP, can be found on GitHub[1]. This repository also contains notes on naive optimizations used to make Algorithm 3.1 faster as well as use less

[1] https://github.com/Spamakin/Solvable-Primitive-Permutation-Groups-of-Rank-5-and-6

No.	q	m	p	k	d	b	et	# Rank 2	# Rank 3	# Rank 4	# Rank 5	# Rank 6
1	2	1	3	1	2	1	−	4	0	0	0	0
1	2	1	3	1	2	1	+	1	0	0	0	0
2	2	1	3	2	4	1	−	0	8	3	0	0
3	2	1	3	3	6	1	−	0	3	3	1	1
3	2	1	3	3	6	1	+	0	1	0	1	0
4	2	1	3	4	8	1	−	0	0	0	3	5
5	2	1	3	5	10	1	−	0	0	1	0	1
6	2	1	5	1	2	1	−	3	0	0	0	0
7	2	1	5	2	4	1	−	0	0	4	5	6
8	2	1	5	3	6	1	−	0	0	0	1	0
9	2	1	7	1	2	1	−	3	3	1	1	0
9	2	1	7	1	2	1	+	1	1	1	0	0
10	2	1	7	2	4	1	−	0	0	0	3	3
11	2	1	11	1	2	1	−	2	1	1	0	2
11	2	1	11	1	2	1	+	0	1	0	0	0
13	2	1	13	1	2	1	−	0	1	1	1	2
15	2	1	17	1	2	1	−	0	3	0	2	0
16	2	1	19	1	2	1	−	0	1	2	0	2
16	2	1	19	1	2	1	+	0	0	1	0	0
17	2	1	23	1	2	1	−	1	1	1	0	0
17	2	1	23	1	2	1	+	0	1	0	1	0
18	2	1	29	1	2	1	−	0	1	1	0	1
19	2	1	31	1	2	1	−	0	1	0	3	0
19	2	1	31	1	2	1	+	0	0	1	0	1
20	2	1	37	1	2	1	−	0	0	1	0	1
21	2	1	41	1	2	1	−	0	0	1	1	2
22	2	1	43	1	2	1	−	0	0	1	0	1
23	2	1	47	1	2	1	−	0	1	0	1	0
23	2	1	47	1	2	1	+	0	0	0	1	0
24	2	1	53	1	2	1	−	0	0	1	0	1
25	2	1	59	1	2	1	−	0	0	1	0	1
26	2	1	61	1	2	1	−	0	0	0	1	0
27	2	1	67	1	2	1	−	0	0	0	1	0

Table 6. All parameters of $G_0 \in \mathfrak{B}$ which describe families of groups of rank 6 or lower.

No.	q	m	p	k	d	b	et	# Rank 2	# Rank 3	# Rank 4	# Rank 5	# Rank 6
28	2	1	71	1	2	1	−	0	0	1	0	0
28	2	1	71	1	2	1	+	0	0	0	0	1
29	2	1	73	1	2	1	−	0	0	0	0	1
30	2	1	79	1	2	1	−	0	0	0	1	0
31	2	1	83	1	2	1	−	0	0	0	1	0
32	2	1	89	1	2	1	−	0	0	0	0	1
34	2	1	101	1	2	1	−	0	0	0	0	1
35	2	1	103	1	2	1	−	0	0	0	0	1
36	2	1	107	1	2	1	−	0	0	0	0	1
37	2	2	3	1	4	1	−	3	4	0	0	0
37	2	2	3	1	4	1	+	0	7	6	0	0
39	2	2	5	1	4	1	−	0	0	0	1	4
39	2	2	5	1	4	1	+	0	0	5	0	8
40	2	2	7	1	4	1	−	0	1	0	1	1
40	2	2	7	1	4	1	+	0	0	0	0	3
43	2	3	3	1	8	1	−	0	0	0	5	5
43	2	3	3	1	8	1	+	0	0	0	2	3
45	3	1	2	2	6	1	+	0	5	2	0	0
46	3	1	2	4	12	1	+	0	0	0	1	3
48	3	1	7	1	3	1	+	0	0	3	2	2
49	3	1	13	1	3	1	+	0	0	0	0	1
53	2	1	3	1	4	2	−	0	10	5	0	0
53	2	1	3	1	4	2	+	0	8	3	0	0
54	2	2	3	1	8	2	−	0	0	0	2	0

Table 6 (continued)

memory. Furthermore, it also contains an implementation of Algorithm 2.11 and files explicitly describing Examples 4.1, 4.2, and 4.3.

We also note some minor corrections to the classification of primitive solvable permutation groups of rank 4 due to [3]. This prior work found only 3 distinct groups for line 37 in Table 4 in the case of groups of rank at most 4. However, we find 4 distinct groups. Furthermore, the case found in [3] with $b > 1$ and $k = 1$ as stated has parameters which define a family of groups of rank significantly larger than 4. Instead, that case must have $b = 1$ and $k = 5$, which then corresponds to line 5 in Table 6.

5 Future work

The methodology of this paper can be extended in a straightforward manner to classify groups of higher rank. The analysis in Proposition 2.7 can be extended by considering the relevant inequalities with a value higher than 6. One could apply the analysis of Section 2, replacing 6 as desired with some larger rank, and then list possible parameters for higher ranks in a similar manner to Table 4 and Table 5. Starting with Proposition 2.7, we may consider setting equation (2.4) to be greater than or equal to some higher value than 6 and proceed with similar analysis from there obtaining bounds on the parameters p, k, q, and m. The same computational method described in Section 3 can then be applied.

The main source of difficulty of extending our method to higher ranks such as 7 and 8 comes from the computational intensity of constructing groups with larger parameters for higher ranks. Steps 1 and 4 of Algorithm 3.1 require us to solve the subgroup isomorphism problem, and step 4 requires us to enumerate subgroups. Both of these problems are some of the hardest problems in computational group theory [19, 21]. As one considers larger and larger ranks, the sizes of our vector spaces V, and thus their general linear groups, grow at a considerable rate. As a concrete example, Line 51 in Table 4 took 8 days of computation to verify that no groups of rank 6 or below exist for that particular set of parameters.

Further analysis on the structure of primitive solvable groups is required to overcome this difficulty. While this issue of extending to higher ranks could be solved with further computational power, this avenue becomes prohibitively expensive. Rather, future work should focus on theoretical improvements to obtaining bounds on the parameters q, m, p, k, and d as well as optimizations to Algorithm 3.1.

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