MATHEMATICS OF COMPUTATION
Volume 93 Number 350 November 2024 Pages 2679–2710
https://doi org/10 1090/mcom/3940
Article electronically published on February 13 2024

ON MEDIAN FILTERS FOR MOTION BY MEAN CURVATURE

SELIM ESEDOḠLU, JIAJIA GUO, AND DAVID LI

Abstract. The median filter scheme is an elegant, monotone discretization of the level set formulation of motion by mean curvature. It turns out to evolve every level set of the initial condition precisely by another class of methods known as threshold dynamics. Median filters are, in other words, the natural level set versions of threshold dynamics algorithms. Exploiting this connection, we revisit median filters in light of recent progress on the threshold dynamics method. In particular, we give a variational formulation of, and exhibit a Lyapunov function for, median filters, resulting in energy based unconditional stability properties. The connection also yields analogues of median filters in the multiphase setting of mean curvature flow of networks. These new multiphase level set methods do not require frequent redistancing, and can accommodate a wide range of surface tensions.

1. Introduction

Motion by mean curvature of an interface, or networks of interfaces (curves in ^{2}, or surfaces in ^{3}) arises in a great variety of applications. Formally, it can be seen as gradient flow (steepest descent) dynamics associated with surface tension, i.e. perimeter (length in ^{2}, or surface area in ^{3}). At any point on an interface, away from free boundaries known as triple junctions in the network case, the interface moves in the normal direction with speed proportional to its mean curvature at that point. If the interface is parametrized, the evolution is described by a system of parabolic partial differential equations satisfied by the time dependent parametrization. Prominent among the applications this dynamics plays a central role in are materials science (evolution of microstructure in polycrystalline materials, see e.g. [19–21, 29]) and computer vision (variational models for image segmentation, see e.g. [6, 30, 39]). In both, as well as in many others, the more challenging network (also known as the multiphase) case of the problem is relevant. Here, there is the additional complication of enforcing natural boundary conditions at triple junctions along which three distinct interfaces intersect.

Both in the scalar two-phase and especially in the vectorial multi-phase case, the evolution can entail singularities and topological changes. All these features make efficient and accurate simulation of the dynamics challenging. Numerical methods that rely on explicit, parametrized representation of the interfaces face the daunting task of detecting, classifying, and “manually carrying out topological changes that are all but inevitable in the long run. Implicit methods, such as phase field, level set [32], and threshold dynamics [27], on the other hand, have to contend with

Received by the editor December 9, 2022, and, in revised form, November 12, 2023, and December 17, 2023.

2020 *Mathematics Subject Classification.* Primary 65M12; Secondary 35K93.

The first author was supported by NSF DMS-2012015. The second author was supported by NSF DMS-2012015. The third author was supported by REU components of NSF DMS-2012015.

nonlinear, degenerate, or singular PDE descriptions, and have to ensure correct conditions at free boundaries are indirectly enforced.

In this paper, we focus on two related approaches that represent the interfaces implicitly: The level set method [32], and threshold dynamics [27, 28]. The former has a complete theory in the scalar (two-phase) case for which elegant, convergent (with proof) numerical methods have also been developed that are, nevertheless, typically low order accurate in time and come with oppressive time step restrictions. A major advantage is spatial accuracy on uniform grids: The interface can be located at sub-grid precision via interpolation. Its main weakness is in extension to networks (the multi-phase case) at the generality demanded by applications, where verifying that correct junction conditions hold at free boundaries remains far from obvious. The latter is unconditionally stable, with very low per time step cost, and has a fairly complete theory even in the network (multi-phase) case, including the verification of the correct junction conditions along free boundaries. Its main drawback is low accuracy when implemented naively on uniform spatial grids: Because it represents interfaces by characteristic functions, the interface cannot be located at subgrid precision via interpolation. As a result, even a smooth interface the curvature of which is small enough compared to the choice of time step size can get “pinned (stuck).

Exploiting a precise connection between a particularly elegant discretization of the level set method known as the median filter scheme [16, 31] and threshold dynamics, the present study offers the following contributions:

(1) A new, variational formulation of median filter schemes for the level set method, in any dimension. In particular, a Lyapunov functional is given in Section 4 that implies unconditional energy stability. A minimizing movements interpretation is also offered. In earlier work, the comparison principle had been the main tool for investigating stability. A potential application is indicated.
(2) A new, fast and accurate algorithm for approximating median filters in two dimensions (three dimensions viable, left to future work), in Section 5. In the scalar (two-phase) case, convergence of this approximation to the unique viscosity solution is verified.
(3) A new, *monotone* discretization of the level set equation for mean curvature motion that is *second order accurate in time* in dimension two. Convergence to the unique viscosity solution is again ensured; Section 6.
(4) Some barriers to finding second order accurate in time versions of median filter schemes in dimensions three and higher; Section 7.
(5) New, multiphase analogues of median filter schemes in any dimension, based entirely on exploiting the connection to threshold dynamics and our rather complete understanding of the latter in the multiphase context of networks [11, 22, 23]; Section 8. This results in a new level set method for multiphase mean curvature motion that allows locating the interface via interpolation and enforces the correct junction condition at the free boundaries, at the generality demanded by applications (e.g. the unequal, non-additive surface tension case [11]).

In addition to presenting new analysis and extensions of median filters (and hence the level set method) as summarized above, the present work can also be seen as a contribution to threshold dynamics, in addressing its difficulties on uniform grids by finding natural level set versions of it.

2. Background

The level set formulation of two-phase motion by mean curvature is described by the partial differential equation

$$\phi_t = |\nabla\phi|\nabla\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right). \tag{2.1}$$

A complete well-posedness theory is developed in [7, 13] in the framework of viscosity solutions. Discretization of the equation, which is degenerate parabolic, and singular wherever $\nabla\phi = 0$, has been an interesting problem of numerical analysis. Preserving qualitative features of the viscosity solution, such as the comparison principle that forms its backbone, is an important challenge in the design of numerical schemes.

Among the many interesting contributions to the numerical treatment of (2.1), the *local median filter* based algorithm proposed in [31] is one of the most elegant (see also [16] and references therein for earlier related work). If we denote by $\mathbf{M}_r\phi^n(x)$ the median of the level set values at time step n around the periphery $\partial B_r(x)$ of the ball $B_r(x)$ of radius r centered at x, the scheme is simply

$$\phi^{n+1}(x) = \mathbf{M}_r\phi^n(x). \tag{2.2}$$

It was derived from the level set formulation (2.1) based on the observation that the right hand side can be written, at least when ϕ is sufficiently differentiable and $\nabla\phi(x) \neq 0$, as the Laplacian of ϕ in the tangent plane of its level set passing through x:

$$|\nabla\phi|\nabla\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right) = \Delta\phi(x) - \left\langle D^2\phi(x)\frac{\nabla\phi}{|\nabla\phi|}, \frac{\nabla\phi}{|\nabla\phi|}\right\rangle. \tag{2.3}$$

Based on this, scheme (2.2) can immediately be seen to be at least plausible in dimension $d = 2$ by observing that

$$\phi\left(x \pm \frac{\nabla^\perp\phi(x)}{|\nabla^\perp\phi(x)|}r\right) = \mathbf{M}_r\phi(x) + O(r^3) \text{ as } r \qquad 0 \tag{2.4}$$

so that

$$\left\langle D^2\phi(x)\frac{\nabla^\perp\phi(x)}{|\nabla^\perp\phi(x)|}, \frac{\nabla^\perp\phi(x)}{|\nabla^\perp\phi(x)|}\right\rangle \approx \frac{\mathbf{M}_r\phi(x) - 2\phi(x) + \mathbf{M}_r\phi(x)}{r^2}. \tag{2.5}$$

Equation (2.2) then implies

$$\phi^{n+1}(x) = \phi^n(x) + k|\nabla\phi^n(x)|\nabla\cdot\left(\frac{\nabla\phi^n(x)}{|\nabla\phi^n(x)|}\right) + O(k^2) \tag{2.6}$$

provided we choose the time step size k as $k = \frac{1}{2}r^2$.

A desirable and most helpful property of scheme (2.2) is its *monotonicity*:

$$\phi_1^0(x) \geq \phi_2^0(x) \text{ for all } x \Longrightarrow \phi_1^n(x) \geq \phi_2^n(x) \text{ for all } x \text{ and } n, \tag{2.7}$$

which of course also holds for the viscosity solutions of the PDE (2.1). Another useful property, especially for numerical implementation, is that any global Lipschitz bound is preserved:

$$\sup_{x\neq y}\frac{|\phi^n(x)-\phi^n(y)|}{|x-y|}\leq\sup_{x\neq y}\frac{|\phi^0(x)-\phi^0(y)|}{|x-y|}. \tag{2.8}$$

This implies, in particular, that no *steepening* of the level-set function will take place, which is to be expected since *every* level set approximates motion by mean curvature, which is well known to enjoy a related property.

In practice, to implement scheme (2.2), we can choose m points $y_1, y_2, \ldots, y_m$ that sample approximately uniformly the sphere $\partial B_r(0)$, with $r=\sqrt{2k}$. The value of the level set function ϕ, which is typically presented on a uniform grid, can be evaluated at any $x+y_j$ via bilinear interpolation [38], which preserves monotonicity. The algorithm is then as follows:

Algorithm 1 Median Filter for Motion by Mean Curvature [31]

1: **sort** the level set values $\{\phi^n(x+y_1), \phi^n(x+y_2), \ldots, \phi^n(x+y_m)\}$ so that the permutation $p : \{1,2,\ldots,m\} \to \{1,2,\ldots,m\}$ satisfies
$$\phi^n\big(x+y_{p(1)}\big)\leq\phi^n\big(x+y_{p(2)}\big)\leq\cdots\leq\phi^n\big(x+y_{p(m)}\big).$$

2: $\phi^{n+1}(x)=\frac{1}{2}\Big(\phi^n(x+y_{p(\lfloor\frac{m}{2}\rfloor)})+\phi^n(x+y_{p(\lceil\frac{m}{2}\rceil)})\Big)$.

The main task in Algorithm 1 is *sorting* the values $\{\phi^n(x+y_j)\}_{j=1}^m$. In what follows, it will be useful to regard this reliance on the sort operation (applied to level set values) as the distinctive feature of median based scemes; doing so will facilitate some of the extensions that will be introduced.

Algorithm 1 is discrete in time but continuous in space, save for the discrete set of points y_j sampled on $\partial B_r(0)$. It is convenient to consider a discrete in time, but fully continuous in space version that relies on the *continuum* median. In doing so, it's also worth discussing a slight generalization, namely the *weighted local median.* To that end, let K be a positive, radially symmetric kernel with rapid decay; let $\|K\|$ denote its mass. We will informally allow K to be concentrated (a δ function) on finitely many circles ($d=2$) or spherical shells ($d=3$) to cover algorithms such as Algorithm 1. For $r>0$, define

$$K_r(x)=\frac{1}{r^d}K\left(\frac{x}{r}\right). \tag{2.9}$$

For a bounded, continuous function ϕ and $\lambda\in\mathbb{R}$, let

$$\mathbf{T}_\lambda\phi=\big\{x\,:\,\phi(x)\geq\lambda\big\}. \tag{2.10}$$

Assume $\|K\|=1$. The function

$$\psi_K\phi(x,\lambda)=\int_{T_\lambda\phi}K(x-y)\,dy \tag{2.11}$$

is decreasing (since $K\geq 0$) and left continuous in λ, satisfying

$$\lim_{\lambda\to\infty}\psi_K\phi(x,\lambda)=0 \text{ and } \lim_{\lambda\to-\infty}\psi_K\phi(x,\lambda)=1.$$

Define the weighted median of ϕ at x with respect to the weight K as

$$\mathbf{M}_K\phi(x)=\sup\left\{\lambda\,:\,\psi_K\phi(x,\lambda)\geq\frac{1}{2}\right\}. \tag{2.12}$$