

Dynamic Gain Adaptation in Linear Quadratic Regulators

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Abstract

In feedback control of dynamical systems, the choice of a higher loop gain is typically desirable to achieve a faster closed-loop dynamics, smaller tracking error, and more effective disturbance suppression. Yet, an increased loop gain requires a higher control effort, which can extend beyond the actuation capacity of the feedback system and intermittently cause actuator saturation. To benefit from the advantages of a high feedback gain and simultaneously avoid actuator saturation, this paper advocates a dynamic gain adaptation technique in which the loop gain is lowered whenever necessary to prevent actuator saturation, and is raised again whenever possible. This concept is optimized for linear systems based on an optimal control formulation inspired by the notion of linear quadratic regulator (LQR). The quadratic cost functional adopted in LQR is modified into a certain quasi-quadratic form in which the control cost is dynamically emphasized or deemphasized as a function of the system state. The optimal control law resulted from this quasi-quadratic cost functional is essentially nonlinear, but its structure resembles an LQR with an adaptable gain adjusted by the state of system, aimed to prevent actuator saturation. Moreover, under mild assumptions analogous to those of LQR, this optimal control law is stabilizing. As an illustrative example, application of this optimal control law in feedback design for dc servomotors is examined, and its performance is verified by numerical simulations.

Index Terms

Actuator saturation, constrained control, inverse optimality, quasi-quadratic cost, servo control.

I. INTRODUCTION

Feedback control of dynamical systems is an essential component of modern technology, which is often used to improve these systems toward a faster dynamics, more precise reference tracking, and more effective disturbance suppression. These objectives of feedback control can be all accomplished to a greater extent by increasing the gain of feedback loop. Yet, larger feedback gains generate controls of larger magnitude, which can intermittently extend beyond the physical actuation capacity of a system, causing actuator saturation. In feedback systems with a constant loop gain, the magnitude of loop gain must be taken sufficiently small to prevent actuator saturation for the worst-case scenarios, which indeed occur infrequently. This approach prevents feedback loops from exploiting the full actuation capacity of systems, which lowers their performance.

This paper investigates an alternative approach relying on an adaptable feedback gain, which is dynamically adjusted toward lower loop gains whenever necessary for preventing actuator saturation, and toward higher loop gains whenever possible to exploit the many advantages of a high loop gain. The notion of *dynamic gain adaptation* as adopted in this paper has already been introduced by other researchers as a method of avoiding actuator saturation in linear dynamical systems [1]–[7]. The main contribution of this paper is the optimization of the gain adaptation process in linear systems, within an optimal control framework inspired by the linear quadratic regulator (LQR) theory.

The control laws resulted from this framework are nonlinear in essence, even though they are developed for linear systems, are inspired by linear controllers, and are optimal in a sense similar to the LQR optimality. The core idea is to dynamically select a feedback gain from a family of LQR gains, depending on the current state of the system under control. This process leads to a nonlinear state feedback law that resembles an LQR at each instance of time, but with a variable gain dynamically adjusted as a function of the system state to confine the control effort within an acceptable range.

In LQR theory, the performance measure is a quadratic cost functional that simultaneously penalizes the deviations of both state and control from their equilibrium point at the origin. In this cost functional, the relative importance of the deviations in state and control can be adjusted by the relative size of a pair of weighting matrices parameterizing the cost functional. Thus, a control designer can decide between a low gain control regime at lower control effort, or a high gain regime with higher effort which can intermittently exceed the system actuation capacity. The decision is normally the largest possible feedback gain that insistently maintains the control effort below the actuation capacity. This gain is then held fixed during the entire course of control.

This paper modifies the quadratic cost functional adopted in LQR into a quasi-quadratic form in which the relative size of its weighting matrices is dynamically adjusted depending on the current system state. When the system is near equilibrium, it needs only a small control effort, so the feedback gain can be increased without causing actuator saturation. This is achieved by reducing the relative size of the weighting matrix penalizing the control effort. Conversely, the control effort is penalized

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more when the system state is far from equilibrium, aimed to reduce the feedback gain and consequently confine the control effort below the actuator saturation limit. The quasi-quadratic cost functional in this paper is deliberately synthesized in such a manner to inherently produce stabilizing controls and yield analytical forms for the optimal control law.

The control constraints posed by actuator saturation (or even further, those involving both state and control) can be managed properly by model predictive control (MPC) [8]–[12]. Despite its successful deployment in various advanced process control applications, MPC is computationally demanding, which limits its application to processes with slow dynamics [13] and those that can afford the cost of a powerful computer hardware for real-time implementation of the MPC algorithm. For compact, low-power, and inexpensive systems with fast dynamics, such as the servomotor studied in Section VI, the control scheme of this paper offers a low computational cost alternative to MPC. As discussed in Section V-C, this control scheme can be easily implemented by a quick search in a small lookup table, which can be executed even on inexpensive embedded systems.

The remainder of this paper is organized into six sections, including conclusion, and an appendix for the proof of results. Section II presents a detailed statement of the optimal control problem studied in this paper, and provides a short summary of its major results. The solution to this optimal control problem and the structure and properties of the optimal control law are discussed in Section III. A detailed stability analysis of the optimal control law is presented next in Section IV. Based on this optimal control law, a step-by-step procedure is offered in Section V to design practical feedback control for real-world applications. This procedure is demonstrated in Section VI by an example in design of feedback control for dc servomotors. For this example, simulation results are presented to evaluate the control performance under the designed feedback control.

II. PROBLEM STATEMENT AND SUMMARY OF RESULTS

Consider the linear state-space equation

$$\dot{x}_t = Ax_t + Bu_t \quad (1)$$

with the state vector $x_t \in \mathbb{R}^n$, control vector $u_t \in \mathbb{R}^k$, and the constant matrices A and B of the dimensions $n \times n$ and $n \times k$. The goal is to determine a feedback law $\mu(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^k$ to establish a state feedback of the form

$$u_t = \mu(x_t) \quad (2)$$

under which the state x_t of the linear system (1) decays rapidly from an initial state x_0 toward 0, while maintaining the control effort at an affordable level.

This control problem can be conveniently addressed by the LQR method, in which the feedback law $\mu(\cdot)$ is obtained by minimizing the infinite horizon quadratic cost functional

$$J_L = \int_0^\infty (x_t^T Q_L x_t + u_t^T R u_t) dt. \quad (3)$$

Here, x_t and u_t are the state and control vectors of the linear system (1), and Q_L and R are positive definite matrices of suitable dimensions. It is well known that the optimal control law minimizing (3) is a linear function of the form

$$\mu(x) = -K_L x,$$

where K_L is a constant $k \times n$ gain matrix [14]. For an idealized problem without any bound on the control vector u_t , this gain matrix can be increased indefinitely by increasing $\|Q_L\|$ in the cost functional (3).

Certainly, a larger gain matrix results in a faster closed-loop dynamics and more effective disturbance suppression; yet, this matrix cannot be increased indefinitely in practice, as a higher feedback gain requires a higher control effort which can extend beyond the physical limitations of a system. The limitations on the actuation capacity of a system are reflected mathematically by constraining $u_t \in \mathcal{U}$ for some bounded control set holding

$$\{u \in \mathbb{R}^k, \|u\| \leq a_1\} \subset \mathcal{U} \subset \{u \in \mathbb{R}^k, \|u\| \leq a_2\}, \quad (4)$$

where $0 < a_1 < a_2 < \infty$.

In a strict sense, the linear state feedback $u_t = -K_L x_t$ with any value of the gain matrix K_L eventually violates the control constraint $u_t \in \mathcal{U}$ for large enough initial states. In practice, however, the initial states are typically confined within some bounded subset $\mathcal{X}_0 \subset \mathbb{R}^n$ of the entire state space, so the gain of state feedback can be chosen small enough to only hold

$$\mu(x_t) \in \mathcal{U}, \quad t \geq 0, \quad x_0 \in \mathcal{X}_0. \quad (5)$$

This gain matrix can be determined via the LQR method by choosing Q_L in the cost functional (3) sufficiently small.¹

While the linear control $u_t = -K_L x_t$ satisfies condition (5), its performance is degraded by the low values of loop gain that are chosen reluctantly to meet this condition. If this condition was not a design constraint, the designer could ideally select a

¹If the linear system (1) is open-loop stable, always a gain matrix exists to hold (5). Otherwise, the existence of such gain matrix depends on the size of $\mathcal{X}_0$, as discussed in Section V-B.

higher feedback gain K_H for a faster closed-loop response and more effective disturbance suppression. This higher feedback gain can be also designed via LQR with a modified cost

$$J_H = \int_0^\infty (x_t^T Q_H x_t + u_t^T R u_t) dt, \quad (6)$$

where Q_H is a positive definite matrix larger than Q_L .

The key observation here is that a linear feedback law with a constant gain matrix barely achieves the most effective control performance. Therefore, this paper follows [1]–[7] in adopting a dynamic gain adaptation scheme in which the feedback gain is smoothly changed from K_L toward K_H whenever possible, and from K_H toward K_L whenever necessary to enforce the control constraint (5). Certainly, variations in the feedback gain must be scheduled in such a manner that preserves the stability of feedback loop, a concern that has been addressed in [1]–[7] via the Lyapunov stability analysis. This paper focuses on the optimality of the gain adaptation process besides its stability.

To optimize the gain adaptation process, this paper takes an optimal control approach and constructs a quasi-quadratic cost functional as a hybrid of J_L in (3) and J_H in (6), given by

$$J = \int_0^\infty (x_t^T W x_t + r(x_t) (x_t^T Q x_t + u_t^T R u_t)) dt. \quad (7)$$

Here, Q and W are, respectively, positive definite and positive semidefinite matrices, and $r(\cdot)$ is a scalar function with certain properties listed in Definition 1 below.

Definition 1: A scalar function $r(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ belongs to the $\mathcal{R}$ -class functions if it is continuous, it holds $r(x) > 1$ for $x \neq 0$ and $r(0) = 1$, $r(\xi x)$ is strictly increasing in ξ for every fixed $x \in \mathbb{R}^n - \{0\}$, and $r(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

The properties assumed for the $\mathcal{R}$ -class of functions explain the rationale for adoption of the cost functional (7). For those initial states very close to the origin, i.e., $\|x_0\| \simeq 0$, this cost functional reduces to (6) with $Q_H \simeq Q + W$, and as a result, its associated optimal control law must be similar to the high gain linear control law $\mu_H(x) = -K_H x$. For the initial states far from $x = 0$, the values of $r(x_t)$ in the cost functional (7) are initially large, therefore this cost functional behaves similar to (3) with $Q_L \simeq Q$. Correspondingly, the optimal control law for large values of x is expected to be rather close to the low gain linear control law $\mu_L(x) = -K_L x$. However, as x_t gradually decreases toward the origin, $\mu_L(\cdot)$ smoothly evolves toward $\mu_H(\cdot)$.

The difficulty with the cost functional (7) is that it may not produce optimal control laws in explicit form for every $r(\cdot)$ arbitrarily taken from the $\mathcal{R}$ -class of functions. A major goal in this paper is to narrow down this class of functions into a subclass for which the cost functional (7) admits closed-form optimal control laws. The quest for such subclass of functions is an example of *inverse optimal* control [15]–[21].

The ultimate goal in this paper is to determine an optimal control law to minimize the quasi-quadratic cost functional (7) subject to the linear system (1) and the control constraint (5). This goal is achieved in two steps. In the first step, the control constraint is relaxed and Problem 1, stated below, is solved for an unconstrained control and initial state, similar to a classical LQR problem. The solution to this problem in Section III is a family of stabilizing optimal control laws parameterized by an $\mathcal{R}$ -class function $r(\cdot)$. In the second step, this function is properly constructed to maintain the control constraint (5). The procedure for construction of this function and selection of the weighting matrices W , Q , and R is presented in Section V and summarized in Algorithm 1.

Problem 1: Suppose (A, B) is stabilizable and let x_t be the state of the linear system (1) with the initial state x_0 and the control vector u_t arbitrarily chosen in $\mathbb{R}^n$ and $\mathbb{R}^k$, respectively. Subject to the state and control of this linear system, construct the infinite horizon cost functional (7) in terms of the $\mathcal{R}$ -class function $r(\cdot)$, the positive semidefinite matrix W , the positive definite matrix Q (positive semidefinite if (A, Q) is observable and W is positive definite), and the positive definite matrix R . Consider the problem of minimizing this cost functional and address the following concerns:

- i. Determine a subclass within the $\mathcal{R}$ -class of functions that only includes members $r(\cdot)$ for which this problem has an analytical solution. This subclass must include a range of functions that can be flexibly shaped by varying a set of parameters.
- ii. For each $r(\cdot)$ in this subclass, obtain an explicit feedback law $\mu(\cdot)$ that minimizes the cost functional (7) when the state feedback (2) is applied to the linear system (1).

A. Summary of Results

The solution of this paper for Problem 1 is a nonlinear state feedback law of the form

$$\mu(x) = -K(r(x))x, \quad (8)$$

where $r(\cdot)$ is an $\mathcal{R}$ -class function with a specific construction discussed later in the paper, and $K(\cdot) : [1, \infty) \rightarrow \mathbb{R}^{k \times n}$ is a matrix-valued function describing a parametric family of LQR gain matrices. Specifically, assume that the positive definite matrix $P(s)$ solves the algebraic Riccati equation

$$A^T P(s) + P(s) A - \frac{1}{s} P(s) B R^{-1} B^T P(s) + sQ + W = 0 \quad (9)$$

for each fixed $s \in [1, \infty)$. The gain matrix $K(\cdot)$ is then given in terms of $P(\cdot)$ by

$$K(s) = \frac{1}{s} R^{-1} B^T P(s), \quad s \in [1, \infty). \quad (10)$$

It is shown in the paper that under the optimal control law (8), the linear system (1) with an unconstrained u_t has a unique globally asymptotically stable equilibrium at $x = 0$, which is also globally exponentially stable under mild assumptions.

This paper presents a parametric subclass within the $\mathcal{R}$ -class of functions from which the function $r(\cdot)$ in (8) is taken. By an appropriate choice of parameters in this subclass, $r(\cdot)$ can be flexibly shaped to achieve a desirable closed-loop dynamics that confines the control u_t of (1) within a bounded control set $\mathcal{U}$. Section V presents a procedure for design of practical feedback loops by embedding the control constraint (5) within the optimal control law (8). An illustrative design example for a dc servomotor is offered in Section VI.

III. OPTIMAL CONTROL

This section presents a solution to Problem 1 which includes the detailed construction of $r(\cdot)$ in the optimal control law (8), the proof of optimality of this control law, and the study of its key properties such as stability. The solution procedure begins in Section III-A with a quick review of certain properties of the Riccati equation (9) frequently used in this paper. Next in Section III-B, a subclass of the $\mathcal{R}$ -class functions is introduced for which Problem 1 admits explicit solutions. Moreover, some properties of this subclass are discussed which are insightful for practical control design in Section V. The explicit form (8) of the optimal control law is established in Section III-C and global asymptotic stability under this control law is proven.

A. Family of Algebraic Riccati Equations

The results of this paper rely on the solutions to the family of algebraic Riccati equations (9) and the family of LQR gain matrices in (10). Certain properties of these families are crucial for the analysis of this paper, which are presented in Lemmas 1 through 3 below.

Lemma 1: In the algebraic Riccati equation (9), assume that the pair (A, B) is stabilizable, (A, Q) is observable, Q and W are positive semidefinite, and R is positive definite. Then, this equation admits a unique positive definite solution $P(s)$ for each fixed $s \in [1, \infty)$, and the family of these solutions defines a matrix-valued function $P(\cdot) : [1, \infty) \rightarrow \mathbb{R}^{n \times n}$. Furthermore, this function is twice differentiable with a positive definite first derivative $P'(\cdot)$ solving the Lyapunov equation

$$(A - BK(s))^T P'(s) + P'(s)(A - BK(s)) + \frac{1}{s^2} P(s) B R^{-1} B^T P(s) + Q = 0, \quad (11)$$

and a negative semidefinite second derivative $P''(\cdot)$ solving

$$(A - BK(s))^T P''(s) + P''(s)(A - BK(s)) - \frac{2}{s^3} P(s) B R^{-1} B^T P(s) = 0. \quad (12)$$

Here, $K(s)$ is given by (10) and $\Pi(s)$ is defined as

$$\Pi(s) = P(s) - sP'(s), \quad s \in [1, \infty). \quad (13)$$

Proof: See Appendix A. ■

The matrix-valued function (13) plays a central role in the stability analysis of Section IV. Some relevant properties of this function are summarized in the following lemma.

Lemma 2: Let the assumptions of Lemma 1 hold. Then, the matrix $\Pi(s)$ defined in (13) is the unique positive semidefinite solution to the Lyapunov equation

$$(A - BK(s))^T \Pi(s) + \Pi(s)(A - BK(s)) + W = 0 \quad (14)$$

and admits a positive semidefinite derivative $\Pi'(s)$. Moreover, if W is strictly positive definite, $\Pi(s)$ is also positive definite, and the scalar function

$$\lambda(s) = \frac{1}{2} \lambda_{\min} \left\{ \Pi^{-1/2}(s) W \Pi^{-1/2}(s) \right\}, \quad s \in [1, \infty) \quad (15)$$

is strictly positive and decreasing in s . Here, $\Pi^{-1/2}(\cdot)$ is the inverse square root of $\Pi(\cdot)$ and $\lambda_{\min} \{\cdot\}$ denotes the smallest eigenvalue of symmetric matrices.

Proof: See Appendix B. ■

The next lemma investigates the asymptotic behavior of the matrices $P'(s)$, $K(s)$, and $\Pi(s)$ as $s \rightarrow \infty$, and establishes a lower bound on the scalar function (15).

Lemma 3: Under the assumptions of Lemma 1, let $P'(s)$ be the solution to the Lyapunov equation (11), and define $K(s)$ and $\Pi(s)$ by (10) and (13), respectively. Then, the limits

$$\lim_{s \rightarrow \infty} P'(s) = P'_\infty \quad (16)$$

$$\lim_{s \rightarrow \infty} K(s) = R^{-1} B^T P'_\infty \quad (17)$$

$$\lim_{s \rightarrow \infty} \Pi(s) = \Pi_\infty$$

exist, where P'_∞ is the solution to the Riccati equation

$$A^T P'_\infty + P'_\infty A - P'_\infty B R^{-1} B^T P'_\infty + Q = 0. \quad (18)$$

Furthermore, if W is strictly positive definite, $\lambda(\cdot)$ in (15) is bounded below by

$$\lambda_L = \frac{1}{2} \lambda_{\min} \left\{ \Pi_\infty^{-1/2} W \Pi_\infty^{-1/2} \right\}. \quad (19)$$

Proof: See Appendix C. ■

B. Tractable Subclass of $\mathcal{R}$ -Class Functions

This section constructs a subclass embedded in the $\mathcal{R}$ -class of functions for which the optimal control Problem 1 admits an explicit solution of the form (8). The functions in this subclass are constructed in terms of another scalar function taken from the Γ -class functions defined below.

Definition 2: A scalar function $\gamma(\cdot) : [1, \infty) \rightarrow [0, \infty)$ is in Γ -class if it is strictly increasing, strictly convex, continuously differentiable, it holds $\gamma(1) = 0$, and its derivative $\gamma'(\cdot)$ holds the conditions $\gamma'(1) = 0$ and $\gamma'(s) \rightarrow \infty$ as $s \rightarrow \infty$. Besides, this derivative must admit a continuously differentiable inverse function $\rho(\cdot) : [0, \infty) \rightarrow [1, \infty)$ that holds

$$\rho(\gamma'(s)) = s, \quad s \in [1, \infty).$$

Remark 1: An example of a Γ -class function is given by

$$\gamma(s) = \frac{c}{1+p} (s-1)^{(1+p)} \quad (20)$$

with constant parameters $c > 0$ and $0 < p \leq 1$. The derivative of this function admits a continuously differentiable inverse

$$\rho(z) = 1 + \left(\frac{z}{c} \right)^{1/p}.$$

Remark 2: Since $\gamma(\cdot)$ is strictly convex by definition, $\gamma'(\cdot)$ is strictly increasing, and therefore, its inverse function always exists and is strictly increasing. If further, $\gamma'(\cdot)$ is continuously differentiable and $\gamma''(s) \neq 0$, $s \in [1, \infty)$, its inverse $\rho(\cdot)$ is also continuously differentiable on $[0, \infty)$. Yet, the converse is not true. In case of (20), for example, $\rho(\cdot)$ is differentiable on $[0, \infty)$ but $\gamma'(\cdot)$ is not differentiable at $s = 1$. The goal in Definition 2 is to broaden the scope of Γ -class as much as possible subject to the requirements of this paper.

Lemma 4 below explains how the $\mathcal{R}$ -class function $r(\cdot)$ in the optimal control law (8) is constructed based on a Γ -class function. Furthermore, this lemma introduces another scalar function $V(\cdot)$ later adopted in Theorem 1 as the value function for the optimal control Problem 1.

Lemma 4: Assume that (A, B) is a stabilizable pair, (A, Q) is observable, Q and W are positive semidefinite matrices, and matrix R is positive definite. Let $P(\cdot) : [1, \infty) \rightarrow \mathbb{R}^{n \times n}$ be a matrix-valued function defined as the positive definite solution to the algebraic Riccati equation (9) for each fixed $s \in [1, \infty)$. Denote the derivative of this function by $P'(\cdot)$ and its second derivative by $P''(\cdot)$. Using $P(\cdot)$ and the Γ -class function $\gamma(\cdot)$, construct $V(\cdot) : \mathbb{R}^n \rightarrow [0, \infty)$ and $r(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ as the scalar functions

$$V(x) = \max_{s \geq 1} \{x^T P(s)x - \gamma(s)\} \quad (21)$$

$$r(x) = \arg \max_{s \geq 1} \{x^T P(s)x - \gamma(s)\}. \quad (22)$$

Then, these functions are well-defined with unique values and hold the following properties:

- i. The scalar function $r(\cdot)$ is in the $\mathcal{R}$ -class of functions and is differentiable with a gradient explicitly given by

$$\nabla r(x) = \frac{2\rho'(\gamma'(r(x))) P'(r(x)) x}{1 - \rho'(\gamma'(r(x))) x^T P''(r(x)) x}, \quad (23)$$

where $\rho(\cdot)$ is the inverse function of $\gamma'(\cdot)$ and $\rho'(\cdot)$ is its derivative.

- ii. The scalar function $V(\cdot)$ is positive definite in the sense that it holds $V(0) = 0$ and $V(x) > 0$ for every $x \neq 0$. In addition, this function is continuously differentiable with a gradient of the explicit form

$$\nabla V(x) = 2P(r(x))x. \quad (24)$$

Proof: See Appendix D. ■

Remark 3: The scalar function $r(\cdot)$ in (22) can be obtained either by solving the optimization problem (21) or, as shown in the proof of Lemma 4, by solving the algebraic equation

$$\gamma'(s) = x^T P'(s)x \quad (25)$$

with respect to s for each fixed x . Since $s = r(x)$ solves this equation, the level sets of $r(\cdot)$ can be easily obtained by noting that the set of points holding $r(x) = \bar{r}$ for some constant $\bar{r}$ is the same as the set of points on the hyperellipsoid

$$x^T \left(\frac{P'(\bar{r})}{\gamma'(\bar{r})} \right) x = 1, \quad \bar{r} \in [1, \infty).$$

C. Optimal Control Law

The core results of this paper are presented in Theorem 1 below. Using the results of Lemma 4, this theorem proves that for any $\mathcal{R}$ -class function $r(\cdot)$ defined by (22), the optimal control Problem 1 has a value function $V(\cdot)$ explicitly given by (21). Moreover, this theorem presents the explicit form (8) for the optimal control law and ensures that under this control law, the closed-loop system is globally asymptotically stable.

Theorem 1: Suppose that W is positive semidefinite, Q is positive definite (positive semidefinite if (A, Q) is observable and W is positive definite), and R is positive definite. Define the scalar functions $V(\cdot)$ and $r(\cdot)$ via (21) and (22), and in terms of $r(\cdot)$ construct the infinite horizon cost functional (7) along the state and control trajectories of the linear system (1) with a stabilizable (A, B) . Using the gain matrix $K(\cdot)$ in (10) establish the nonlinear state feedback

$$u_t = -K(r(x_t))x_t \quad (26)$$

and apply it to the linear system (1) to generate the closed-loop nonlinear dynamics

$$\dot{x}_t = (A - BK(r(x_t)))x_t. \quad (27)$$

Then, the following statements hold:

- i. For any initial state $x_0 \in \mathbb{R}^n$, the ordinary differential equation (27) admits a unique solution on $t \in [0, \infty)$.
- ii. The closed-loop system (27) has a unique equilibrium at the origin $x = 0$, which is globally asymptotically stable.
- iii. Along the closed-loop state trajectory, $r(x_t)$ is decreasing in t and tends to 1 as $t \rightarrow \infty$.
- iv. The state feedback (26) minimizes the cost functional (7) among all control laws, and the minimum value of this cost functional for an initial state x_0 is given in terms of the value function $V(\cdot)$ by

$$J^* = V(x_0). \quad (28)$$

Proof: For the sake of readability, the proof is outlined here, while its technical details are presented in Appendix E.

Statement i: It is shown in Appendix E that the right-hand side of (27) is globally Lipschitz continuous. Then, according to [22, Thm. 3.2], the state-space equation (27) admits a unique solution on $t \in [0, \infty)$.

Statement ii: As $K(s)$ is an LQR gain matrix, $A - BK(s)$ is a stable matrix [14, Thm. 21.2], and as a result, nonsingular. Consequently, the algebraic equation $(A - BK(r(x)))x = 0$ cannot admit any solution except for $x = 0$, i.e., $x = 0$ is the unique equilibrium of the state-space equation (27).

To analyze the stability of this equilibrium, $V(\cdot)$ in (21) is taken as a Lyapunov function. By Lemma 4, this function is positive definite, and as shown in the proof of this lemma, it holds the inequality $V(x) \geq x^T P(1)x$ for $x \in \mathbb{R}^n$. As $P(1)$ is positive definite, the right hand side of this inequality tends to ∞ as $\|x\| \rightarrow \infty$, which ensures $V(\cdot)$ is radially unbounded, i.e., $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$. The gradient of $V(\cdot)$ in (24) and the algebraic Riccati equation (9) imply that

$$(\nabla V(x))^T (A - BK(r(x)))x = -x^T \left(r(x) (Q + K^T(r(x))RK(r(x)) + W) \right) x \quad (29)$$

is strictly negative for all $x \neq 0$, since $r(x)Q + W$ is positive definite. Then, the unique equilibrium at $x = 0$ is globally asymptotically stable by [22, Thm. 4.2].

Statement iii: Using the gradient (23) of $r(\cdot)$ and the results of Lemma 1, it is shown in Appendix E that

$$\frac{d}{dt} r(x_t) = (\nabla r(x_t))^T \dot{x}_t \leq 0. \quad (30)$$

This inequality implies that $r(x_t)$ is decreasing in t . Moreover, since $x = 0$ is the unique stable equilibrium of (27), it holds that $\lim_{t \rightarrow \infty} x_t = 0$. Then, the continuity of $r(\cdot)$ and $r(0) = 1$ imply that

$$\lim_{t \rightarrow \infty} r(x_t) = 1.$$

Statement iv: Since (A, B) is stabilizable, there exist linear feedback laws to asymptotically stabilize the linear state-space equation (1), and under such stabilizing feedback control, the cost functional (7) stays bounded. Hence, the optimal control law is necessarily stabilizing, otherwise, its cost value would be unbounded (Q or W is positive definite), which exceeds the cost of any stabilizing linear control. As a result, the search for an optimal control law is restricted to the set of asymptotically stabilizing control laws. For controls in this set, Appendix E rewrites the cost functional (7) as

$$J = V(x_0) + \int_0^\infty r(x_t) \|u_t + K(r(x_t))x_t\|_R^2 dt, \quad (31)$$

where $\|\cdot\|_R$ denotes the norm $\|z\|_R = \sqrt{z^T R z}$. Therefore, to minimize J , the control u_t must be taken as (26), which leads to the minimum cost (28). ■

Remark 4: The minimum cost (28) indicates that $V(\cdot)$ must be the value function for the cost functional (7) subject to the linear system (1). This result is consistent with the fact implied by (24) and (9) that $V(\cdot)$ solves the Hamilton-Jacobi-Bellman (HJB) equation

$$\min_{u \in \mathbb{R}^k} \left\{ (\nabla V(x))^T (Ax + Bu) + x^T W x + r(x) (x^T Q x + u^T R u) \right\} = 0.$$

As a result of the infinite horizon of the cost functional (7), this HJB equation is stationary with $(\partial/\partial t)V(\cdot) = 0$.

The classical setting of LQR problem allows the weighting matrix Q in the quadratic cost functional

$$J = \int_0^\infty (x_t^T Q x_t + u_t^T R u_t) dt$$

to be only positive semidefinite, provided that the pair (A, Q) is observable. This case is not easily extendable to the results of Theorem 1 that requires Q (or alternatively W) to be strictly positive definite. The difficulty originates in the proof of global asymptotic stability in statement ii of the theorem, which in turn, undermines the argument of statement iv on optimality of the control law (26). Indeed, the Lyapunov function $V(\cdot)$ can only prove the marginal stability of $x = 0$ if both Q and W are positive semidefinite, since the right-hand side of (29) can be only proven to be nonnegative. If by any possible means, the global stability of $x = 0$ can be verified, all other results of Theorem 1 hold identically, as stated in the following corollary.

Corollary 1: Suppose (A, Q) is an observable pair while Q and W are positive semidefinite matrices. Then, statement i of Theorem 1 holds and the closed-loop system (27) has a unique equilibrium at $x = 0$. Moreover, if this equilibrium is globally asymptotically stable, statements iii and iv of Theorem 1 hold identically.

Proof: The proof parallels that of Theorem 1 with minor modification. The proof of statement iv uses the fact that the cost functional (7) is unbounded under non-stabilizing controls as a result of positive definiteness of Q or W . This statement is proven via the following argument under the assumptions of this corollary. Assume that the linear system (1) is under some non-stabilizing control. If for this control, $\|u_t\|$ does not tend to 0 as $t \rightarrow \infty$, the cost functional (7) will be unbounded due to its strictly positive integrand $r(x_t) u_t^T R u_t$. In the opposite case of $\lim_{t \rightarrow \infty} \|u_t\| = 0$, the state x_t of the system tends to the unbounded solution of $\dot{x}_t = A x_t$, which again results in an unbounded cost, due to the term $x_t^T Q x_t$ and the observability assumption on (A, Q) . ■

IV. EXPONENTIAL STABILITY

Theorem 1 guarantees that the closed-loop system (27) has a globally asymptotically stable equilibrium at the origin $x = 0$. This section investigates the exponential stability of $x = 0$ and estimates the convergence rate toward this equilibrium. First, Proposition 1 proves the local exponential stability of $x = 0$ for the most general case in which both Q and W are positive semidefinite matrices. Moreover, it is shown that for the initial states close enough to the equilibrium, the convergence rate is close to that attained under the high feedback gain $K(1)$. Next, the global exponential stability is studied in Theorem 2 and a lower bound on the convergence rate is established. This lower bound is comparable to the convergence rate achieved under the low feedback gain $K(\infty)$. The weakness of Theorem 2 is an assumption requiring W to be strictly positive definite. To relax this assumption, Theorem 3 establishes the exponential stability of the closed-loop equilibrium in a quasi-global sense, which is weaker but still comparable to the global sense.

Proposition 1: Let the same assumptions of Lemma 4 hold, highlighting that Q and W are positive semidefinite matrices and (A, Q) is observable. Then, the unique equilibrium $x = 0$ of the closed-loop system (27) is locally exponentially stable.

Proof: The Jacobian matrix of the right-hand side of (27) is obtained in Appendix E, which takes a value $(A - BK(1))$ at $x = 0$. Since $K(1)$ is an LQR gain matrix, $(A - BK(1))$ is a stable matrix [14, Thm. 21.2], which implies that $x = 0$ is locally exponentially stable [23, Cor. 5.5.26]. ■

Remark 5: The core idea in Proposition 1 is that the closed-loop system (27) behaves similar to its linear approximation

$$\dot{x}_t = (A - BK(1))x_t \quad (32)$$

in a neighborhood of its equilibrium $x = 0$. Therefore, after the closed-loop state reaches this neighborhood, it continues to decay toward $x = 0$ with an exponential rate comparable to that of the linear system (32) with a high gain linear feedback.

To study the global exponential stability, the scope of results is narrowed down to a strictly positive definite W for which the stability analysis is facilitated by the Lyapunov function

$$U(x) = x^T \Pi(r(x))x, \quad (33)$$

where the positive definite matrix $\Pi(\cdot)$ is given by (13). In the next lemma, an exponentially decaying upper bound with a time-varying decay rate is established on $U(x_t)$ along the state trajectories of the closed-loop system (27). This upper bound is adopted by Theorem 2 to prove the global exponential stability of the closed-loop equilibrium point at $x = 0$.

Lemma 5: Suppose that the assumptions of Theorem 1 hold and that W is positive definite. Let $x_t, t \geq 0$ be the state of the closed-loop system (27) with an initial state x_0 . Define the scalar functions $\lambda(\cdot)$ as (15) and $U(\cdot)$ as (33). Then, $U(x_t)$ is upper bounded by

$$U(x_t) \leq U(x_0) \exp\left(-2 \int_0^t \lambda(r(x_\tau)) d\tau\right), \quad t \geq 0. \quad (34)$$

Proof: See Appendix F. ■

The next theorem derives an upper bound on $\|x_t\|$ from (34) to prove the global exponential stability of $x = 0$ for a strictly positive definite W .

Theorem 2: Under the assumptions of Theorem 1 narrowed down to a strictly positive definite W , let $x_t, t \geq 0$ be the state of the closed-loop system (27) with the initial state x_0 . Then, a positive constant $c_1 > 0$ exists such that

$$\|x_t\| \leq c_1 \|x_0\| \exp\left(-\int_0^t \lambda(r(x_\tau)) d\tau\right), \quad t \geq 0, \quad (35)$$

where $\lambda(\cdot)$ is a scalar function given by (15). Moreover, the equilibrium point of the closed-loop system (27) at $x = 0$ is globally exponentially stable with a rate of convergence not smaller than λ_L defined in (19).

Proof: Lemmas 2 and 3 indicate that $\Pi(\cdot)$ is increasing in the matrix ordering sense and $\Pi_\infty = \lim_{s \rightarrow \infty} \Pi(s)$ exists. It is then concluded for every $x \in \mathbb{R}^n$ that

$$\pi_1 \|x\|^2 \leq x^T \Pi(1)x \leq U(x) \leq x^T \Pi_\infty x \leq \pi_\infty \|x\|^2,$$

where π_1 and π_∞ denote the smallest eigenvalue of $\Pi(1)$ and the largest eigenvalue of Π_∞ , respectively. Also, π_1 and π_∞ are strictly positive since $\Pi(\cdot)$ is positive definite by Lemma 2. Then, by taking $c_1 = \sqrt{\pi_\infty/\pi_1}$, the upper bound (35) is easily derived from (34).

By Lemma 3, $\lambda(\cdot)$ is lower bounded by λ_L , which implies the right-hand side of (35) is bounded above by $e^{-\lambda_L t}$. Then,

$$\|x_t\| \leq c_1 \|x_0\| e^{-\lambda_L t}, \quad t \geq 0 \quad (36)$$

indicates that the unique equilibrium $x = 0$ of (27) is globally exponentially stable with the convergence rate $\lambda_L > 0$. ■

Remark 6: In the upper bound (35), $\lambda(r(x_t))$ is interpreted as the instantaneous rate at which $\|x_t\|$ decays exponentially toward 0. This instantaneous rate monotonically increases with time, since $r(x_t)$ is decreasing in t by Theorem 1 and $\lambda(s)$ is decreasing in s by Lemma 2.

Remark 7: The positive constant λ_L in (19) is an estimate for the rate of convergence of the low gain linear system

$$\dot{x}_t = (A - BK(\infty))x_t. \quad (37)$$

Therefore, Theorem 2 suggests that the state of the nonlinear closed-loop system (27) converges to the equilibrium at $x = 0$ at least as fast as the linear system (37).

For the case in which Q in the cost functional (7) is positive definite but W is only positive semidefinite, the stability of the closed-loop system (27) can be analyzed using

$$\tilde{U}(x) = x^T P(r(x))x \quad (38)$$

as a Lyapunov function. The difficulty with this function when compared to its counterpart (33) is that $P(\infty)$ is not bounded, as is the case for $\Pi(\infty)$. Despite this difficulty, an upper bound analogous to (34) can be established on $\tilde{U}(x_t)$; however, due to unboundedness of $P(\infty)$, an upper bound on $\|x_t\|$ similar to (35) cannot be derived, unless c_1 is taken a function of x_0 . Yet, from such x_0 -dependent upper bound, conclusions can be drawn which are slightly weaker but still comparable to global exponential stability, as stated in the next theorem.

Theorem 3: Under the same assumptions of Theorem 1, the unique equilibrium of the closed-loop system (27) at $x = 0$ is quasi-globally exponentially stable in the following sense: for any arbitrarily large $\varrho > 0$, a positive constant $c_2(\varrho)$ exists to hold the inequality

$$\|x_t\| \leq c_2(\varrho) \|x_0\| e^{-\tilde{\lambda}_L t}, \quad \|x_0\| \leq \varrho, \quad t \geq 0 \quad (39)$$

for some $\tilde{\lambda}_L > 0$ not depending on ϱ . Here, x_t is the state of the closed-loop system (27) with the initial state x_0 .

Proof: See Appendix G. ■

V. ENFORCEMENT OF CONTROL CONSTRAINT

This section presents a feedback design procedure to exploit the optimal control law (8) in practical applications that require the enforcement of the control constraint (5). First, the required technical background is provided in Section V-A, and next, the design procedure is detailed in Section V-B. A computationally efficient numerical implementation of the resulting controller is presented in Section V-C. Later in Section VI, the overall design procedure is demonstrated for a practical application in feedback control of dc servomotors.

A. Technical Background

The control design procedure presented in this paper relies on Theorem 4 of this section, which in turn, is concluded from the following lemma.

Lemma 6: Suppose that $\mathcal{U}$ is a compact set holding (4), B is full rank, and $K(\cdot)$ is given by (10). Under the assumptions of Lemma 1 define the scalar function

$$g(s) = \inf_{v \notin \mathcal{U}} v^T \left(K(s) (P'(s))^{-1} K^T(s) \right)^{-1} v. \quad (40)$$

Let $r(\cdot)$ be an $\mathcal{R}$ -class function expressed as (22) in terms of a Γ -class function $\gamma(\cdot)$ that for some $r_m > 1$ satisfies

$$\gamma'(s) \leq g(s), \quad s \in [1, r_m]. \quad (41)$$

Then, any $x \in \mathbb{R}^n$ that holds $r(x) \leq r_m$ also holds

$$-K(r(x))x \in \mathcal{U}. \quad (42)$$

Proof: See Appendix H. ■

The following theorem is the keystone of the control design procedure in Section V-B.

Theorem 4: Under the assumptions of Lemma 6, let $\gamma(\cdot)$ be a Γ -class function holding (41) and $\gamma'(r_m) = g(r_m)$ for some constant $r_m > 1$. Associated with $\gamma(\cdot)$, construct the $\mathcal{R}$ -class function $r(\cdot)$ via (22). Let x_t be the state of the closed-loop system (27) for $t \geq 0$, starting from an initial state x_0 in the bounded set $\mathcal{X}_0$ that holds

$$\mathcal{X}_0 \subset \{x \in \mathbb{R}^n \mid x^T P'(r_m) x \leq g(r_m)\}. \quad (43)$$

Then, under the same assumptions of Theorem 1, the feedback control (26) stays in the compact control set $\mathcal{U}$ for every $t \geq 0$, that is

$$-K(r(x_t))x_t \in \mathcal{U}, \quad t \geq 0. \quad (44)$$

Proof: It is noted that $r(x_0) \leq r_m$ holds for all $x_0 \in \mathcal{X}_0$, since otherwise, $r(x_0) > r_m$ leads to the inequality

$$x_0^T P'(r(x_0)) x_0 \leq x_0^T P'(r_m) x_0 \leq g(r_m) = \gamma'(r_m) < \gamma'(r(x_0))$$

which contradicts the fact that $r(\cdot)$ is the solution to (25). This inequality is concluded from the facts that $P'(\cdot)$ is decreasing in the matrix ordering sense (by Lemma 1), and $\gamma'(\cdot)$ is strictly increasing (as $\gamma(\cdot)$ is strictly convex by definition). As $r(x_t)$ is decreasing in t by Theorem 1, $r(x_t) \leq r(x_0) \leq r_m$ holds for $t \geq 0$. Then, Lemma 6 implies (44). ■

B. Control Design Procedure

The control design procedure offered in this section is aimed at determining proper parameter values for the optimal control law (8) under which the closed-loop system (27) achieves two major objectives. First, $r(x_t)$ must stay as close as possible to its minimum value 1 in order to exploit the advantages of the largest possible feedback gain $K(r(x_t))$. Second, for any initial state in the bounded set $\mathcal{X}_0$, the feedback control (26) must stay in the compact control set $\mathcal{U}$ over the entire course of control, as expressed mathematically by (5). The parameters to be chosen are the weighting matrices W , Q , and R in the cost functional (7), the Γ -class function $\gamma(\cdot)$ generating $r(\cdot)$ in this cost functional, and the constant $r_m > 1$ in Theorem 4. These parameters are chosen to meet the assumptions of Theorem 4, and therefore, satisfy the control constraint (44). The design procedure is summarized in Algorithm 1 and is discussed in detail as follows.

In the first step, the positive definite matrices $Q_H = Q + W$ and R are chosen such that minimizing the cost functional (6) results in a high gain control $u_t = -K_H x_t$ which perfectly meets the control design objectives when applied to the linear system (1) in the absence of any control constraints. This is a classic LQR problem and is not discussed in detail here.

In the next step, the positive definite matrix $Q_L \leq Q_H$ and the constant $1 < r_m < \infty$ are chosen to ensure that (43) holds under $Q + W/r_m = Q_L$. The selection procedure for this pair of parameters is explained shortly. Upon choosing Q_H , Q_L , and r_m , the matrices W and Q are simply obtained from

$$W = \frac{r_m}{r_m - 1} (Q_H - Q_L) \quad (45a)$$

$$Q = \frac{r_m}{r_m - 1} \left(Q_L - \frac{1}{r_m} Q_H \right). \quad (45b)$$

Let P_L be the solution to the algebraic Riccati equation

$$A^T P_L + P_L A - P_L B R^{-1} B^T P_L + Q_L = 0 \quad (46)$$

and define the low gain matrix

$$K_L = R^{-1} B^T P_L. \quad (47)$$

Also, assume that Π_L is the solution to the Lyapunov equation

$$(A - B K_L)^T \Pi_L + \Pi_L (A - B K_L) + Q_H - Q_L = 0.$$

Then, it is concluded from (9), (10), (13), (14), and (45) that

$$K(r_m) = K_L \quad (48a)$$

$$P'(r_m) = P_L - (r_m - 1)^{-1} \Pi_L. \quad (48b)$$

Fix a constant $0 < \varepsilon \ll 1$ and choose Q_L sufficiently small to hold the inequality

$$\sup_{x \in \mathcal{X}_0} x^T P_L x \leq (1 - \varepsilon) \inf_{v \notin \mathcal{U}} v^T R (B^T P_L B)^{-1} R v. \quad (49)$$

For an open-loop stable system (i.e., the eigenvalues of A have negative real parts), this objective is always achievable for any bounded $\mathcal{X}_0$ regardless of its size. The reason is that as Q_L in the Riccati equation (46) decreases toward 0, the solution P_L to this equation decreases toward 0 as well, and as a result, the left-hand side of (49) decreases toward 0, while its right-hand side increases toward ∞ . Hence, small enough values of Q_L must exist to satisfy (49). For open-loop unstable systems, on the other hand, P_L tends to a nonzero matrix as Q_L tends to 0. As a consequence, suitable values of Q_L to hold (49) may or may not exist, depending on the size of $\mathcal{X}_0$. In case such values of Q_L do not exist, the control design specifications must be realistically modified for a smaller $\mathcal{X}_0$.

After selecting Q_L , the parameter $1 < r_m < \infty$ is chosen as the smallest value that simultaneously holds

$$Q_L - \frac{1}{r_m} Q_H > 0 \quad (50a)$$

$$\varepsilon P_L - \frac{1}{r_m - 1} \Pi_L \geq 0. \quad (50b)$$

The first inequality here guarantees that Q is positive definite, while the second one enforces (43) by the following argument.

For K_L defined as (47), it is straightforward to show that

$$(1 - \varepsilon) R (B^T P_L B)^{-1} R = \left(K_L ((1 - \varepsilon) P_L)^{-1} K_L^T \right)^{-1}.$$

Moreover, (50b) can be rewritten as

$$(1 - \varepsilon) P_L \leq P_L - (r_m - 1)^{-1} \Pi_L$$

which together with (48) leads to

$$(1 - \varepsilon) R (B^T P_L B)^{-1} R \leq \left(K(r_m) (P'(r_m))^{-1} K^T(r_m) \right)^{-1}.$$

Then, applying this inequality to (49) using the definition (40) of $g(\cdot)$, and noting from (48b) that $P'(r_m) < P_L$ result in

$$\sup_{x \in \mathcal{X}_0} x^T P'(r_m) x \leq g(r_m)$$

which implies (43).

The control design procedure is completed by constructing a Γ -class function $\gamma(\cdot)$ to satisfy the conditions of Theorem 4, namely, $\gamma'(s)$ must lower bound $g(s)$ over $s \in [1, r_m]$ and hold $\gamma'(r_m) = g(r_m)$. For the best control performance, $\gamma'(\cdot)$

Algorithm 1 Control Design Procedure.

Input: $A, B, \mathcal{U}, \mathcal{X}_0$
Output: $W, Q, R, \gamma(\cdot), r_m$

- 1: Select Q_H and R such that minimizing (6) subject to (1) yields an LQR gain K_H that perfectly meets the control design objectives in the absence of control constraints.
 - 2: Fix $0 < \varepsilon \ll 1$ and choose Q_L sufficiently small such that the solution P_L to the Riccati equation (46) holds (49).
 - 3: Determine $r_m > 1$ as the smallest value that holds (50).
 - 4: Compute Q and W from (45).
 - 5: Construct $g(\cdot)$ on $[1, r_m]$ from (40).
 - 6: Construct a Γ -class $\gamma(\cdot)$ such that $\gamma'(\cdot)$ be the tightest lower bound of $g(\cdot)$ on $[1, r_m]$ that holds $\gamma'(r_m) = g(r_m)$.
-

must be ideally the tightest possible lower bound of $g(\cdot)$ which satisfies the structural constraints Definition 2 imposes on the Γ -class functions. Such tightest lower bound yields an instance of $r(\cdot)$ with lowest possible values, which in turn, leads to the largest possible gain $K(r(x_t))$ over the course of control.

As a practical example, in the design problem of Section VI, the parametric form (20) is adopted as $\gamma(\cdot)$ and its c parameter is chosen as $c = g(r_m)(r_m - 1)^{-p}$ to ensure that

$$\gamma'(s) = g(r_m) \left(\frac{s-1}{r_m-1} \right)^p \quad (51)$$

holds $\gamma'(r_m) = g(r_m)$. The parameter p is optimized then to render (51) the tightest lower bound of $g(s)$ over $s \in [1, r_m]$. Since (51) is decreasing in $p \in (0, 1]$ for each fixed $s \in [1, r_m]$, the smallest p that holds (41) is the optimal value.

C. Numerical Implementation

Upon completing the design procedure of Section V-B, the resulting control law of the form (8) must be implemented on a digital computer for practical use. Since digital computers can only process discrete-time signals, the implemented controller will be a discrete-time approximation of the continuous-time state feedback (26), and is mathematically represented as

$$u_t = -K(r(x_{iT}))x_{iT}, \quad iT + d \leq t < (i+1)T + d, \quad i = 0, 1, 2, \dots$$

Here, x_{iT} , $i = 0, 1, 2, \dots$ are the samples of the state vector x_t taken at the multiples of the sampling time T , and $0 < d \leq T$ is an inherent delay introduced by the time needed to compute the control for each new sample x_{iT} . A shorter computation time d clearly allows for a higher sampling rate, which in turn, results in a lower approximation error introduced by sampling. This can be achieved either by employing faster computers or more efficient computational techniques.

At each sampling time iT , a fresh sample x_{iT} of the state vector is taken and during the next d seconds, the numerical value of $-K(r(x_{iT}))x_{iT}$ is computed via a procedure that includes solving the optimization problem (22) for $r(x_{iT})$, and computation of $K(r(x_{iT}))$ from (10). Both these steps rely on numerical solution of the Riccati equation (9), which typically, is computationally expensive and can be overwhelming for the optimization problem (22) that requires many times solving this equation. Yet, real-time solution of the Riccati equation (9) can be circumvented using a precomputed lookup table [24].

To construct this lookup table, the search space $s \in [1, r_m]$ of the optimization problem (22) is approximated by a discrete set $\{s_1, s_2, \dots, s_N\}$, in which, $1 = s_1 < s_2 < \dots < s_N = r_m$ are chosen sufficiently close to keep the approximation error at an acceptable level. The lookup table is then constructed in N rows, with row $j = 1, 2, \dots, N$ containing s_j , $\gamma(s_j)$, and the elements of the matrix $P(s_j)$. Based on this table, real-time computation reduces to a quick search for the maximum value of $x_{iT}^T P(s_j) x_{iT} - \gamma(s_j)$ on $j = 1, 2, \dots, N$, and then, using the maximizer $s^* \in \{s_1, s_2, \dots, s_N\}$ to compute the numeric value of the control $-R^{-1}B^T P(s^*) x_{iT}/s^*$.

Since $P(s_j)$ is $n \times n$ and symmetric, each row of the lookup table includes $n(n+1)/2 + 2$ elements, and as a result, a table of N rows includes a total of $N(n(n+1)/2 + 2)$ elements. For the servomotor example of Section VI with $n = 3$ state variables, a lookup table with $N = 500$ rows consists of 4000 elements, which needs only 64K bytes of computer memory for storage in a double-precision floating-point format. Such a small block of memory is often affordable even for inexpensive embedded systems.

VI. FEEDBACK DESIGN FOR DC SERVOMOTORS

This section demonstrates the feedback design procedure of Section V-B for a practical application in position control of dc servomotors. First in Section VI-A, the open-loop dynamics of dc motors is represented by a linear state-space equation with a constrained control input. In Section VI-B, this dynamics is stabilized by a nonlinear regulator of the form (8) satisfying the control constraint. This nonlinear regulator is then modified in Section VI-C for the purpose of trajectory tracking. In both Sections VI-B and VI-C, numerical results for evaluation of the control performance are presented.

A. Dynamical Model of DC Motors

A dc motor is regarded in this paper as a dynamical system with a single control input and three state variables. The motor is controlled via its armature voltage that ranges in $[-V_m, V_m]$, and the state variables are its armature current, shaft speed, and shaft position. For the sake of simplicity, these variables are normalized in amplitude, and then, represented respectively by the functions v_t , i_t , ω_t , and θ_t of the normalized time t . The procedure for normalization is explained below.

The armature voltage is simply normalized by its maximum value V_m such that $|v_t| \leq 1$. Suppose under this maximum voltage, the motor runs in steady state at an angular velocity Ω under the armature current I_a . The actual armature current and its shaft speed are normalized then by I_a and Ω to construct i_t and ω_t , respectively. The angular position θ_t is dimensionless in essence and is measured in radians.

The time axis is normalized by the mechanical time constant of the dc motor, i.e., this time constant is chosen as the unit of time. The mechanical time constant τ_m is given by the ratio of the moment of inertia to friction coefficient of the armature and its mechanical load. Heuristically, $1/\tau_m$ represents the rate of exponential decay in the shaft speed when the armature is open circuited. The dynamics of a dc motor further includes an electrical time constant τ_e defined as the RL time constant of its armature winding when the armature is stalled. This time constant is typically smaller than the mechanical time constant by an order of magnitude.

The dynamics of dc motors has been extensively studied in the literature (see for example [25, p. 188]). For the specific normalization scheme adopted in this paper, this dynamics is represented by the linear state-space equation

$$\frac{d}{dt} \begin{bmatrix} i_t \\ \omega_t \\ \theta_t \end{bmatrix} = \begin{bmatrix} -\alpha & -\frac{\eta\alpha}{1-\eta} & 0 \\ 1 & -1 & 0 \\ 0 & \beta & 0 \end{bmatrix} \begin{bmatrix} i_t \\ \omega_t \\ \theta_t \end{bmatrix} + \begin{bmatrix} \frac{\alpha}{1-\eta} \\ 0 \\ 0 \end{bmatrix} v_t \quad (52)$$

with the control v_t in the compact control set

$$\mathcal{U} = \{u \in \mathbb{R} \mid -1 \leq u \leq 1\}. \quad (53)$$

This equation is parameterized by $\alpha = \tau_m/\tau_e$, the efficiency η of the motor, and $\beta = \tau_m\Omega$. Here, the efficiency is defined as the ratio of the output mechanical power to the input electrical power when the motor runs at a constant speed. The numerical values of these parameters are chosen as $\alpha = 25$, $\eta = 0.8$, and $\beta = 100$.

B. Controller Design

The control design procedure in Algorithm 1 is applied to the dynamical system (52) with the state vector $x = (i, \omega, \theta)$ and the control variable $u = v$ in the compact control set (53). This linear system has $n = 3$ open-loop real poles at 0, -6.37 , and -19.63 . The weighting matrix $Q_H = \text{diag}(1, 10, 1000)$ is chosen together with $R = 1$ in order to place the closed-loop poles at $-37.35 \pm j40.90$ and -128.87 in a high gain regime. Since the control goal is to regulate the angular position θ_t , the weighting matrix Q_H penalizes this variable with a large coefficient 1000, versus smaller coefficients 1 and 10 for the other two state variables.

The controller in this paper is designed for the initial states taken from the bounded set

$$\mathcal{X}_0 = \{(0, 0, \theta_0) \mid -\pi \leq \theta_0 \leq \pi\}.$$

Then, it is ensured that the control variable stays in (53) if the dc motor is initially at rest while its shaft position deviates up to 180° from its reference.

The weighting matrix Q_L is chosen proportional to Q_H and the proportionality constant is determined as 5×10^{-5} in order to satisfy (49) with $\varepsilon = 0.05$. Next, $r_m = 4 \times 10^5$ is taken as the smallest value holding (50). In terms of Q_H , Q_L , and r_m , the weighting matrices W and Q are computed from (45) as

$$\begin{aligned} W &= \text{diag}(0.99995, 9.9995, 999.95) \\ Q &= \text{diag}(4.75 \times 10^{-5}, 4.75 \times 10^{-4}, 4.75 \times 10^{-2}). \end{aligned}$$

Based on the numerical values of W , Q , and r_m , the scalar function $g(\cdot)$ is numerically computed from (40) over $[0, r_m]$. The parametric form (20) is next adopted for $\gamma(\cdot)$ and via the procedure explained in Section V-B [see (51)], the parameters of this family are determined as $c = 0.0022$ and $p = 0.288$. Under these parameter values, the resulting $\gamma'(\cdot)$ is the tightest lower bound of $g(\cdot)$ over $[1, r_m]$ that holds $\gamma'(r_m) = g(r_m)$, as shown in Fig. 1.

For the purpose of computer simulations, the control law (8) was numerically implemented on MATLAB via the procedure of Section V-C. The lookup table prescribed in this procedure was generated with $N = 500$ logarithmically distanced points distributed on the interval $[1, 4 \times 10^5]$. Under the developed feedback controller, the dynamical system (52) was simulated numerically. As baselines for comparison, this system was also simulated under two alternative controls: a low gain linear state feedback $u_t = -K_L x_t$ and a high gain $u_t = -\text{sat}(K_H x_t)$ with the saturation nonlinearity

$$\text{sat}(z) = \min\{|z|, 1\} \text{sign}(z).$$

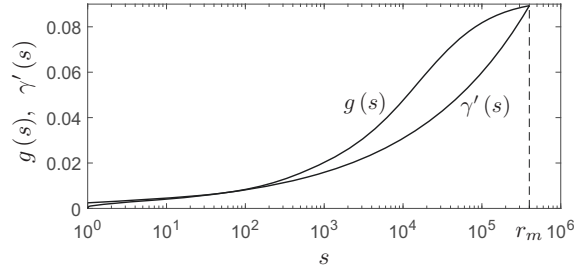


Fig. 1. Scalar functions $\gamma'(\cdot)$ and $g(\cdot)$ in (40) plotted over $[1, r_m]$. The parametric form $c(s-1)^p$ is assumed for $\gamma'(s)$ and the parameters c and p are determined in such a manner that $\gamma'(\cdot)$ be the tightest lower bound of $g(\cdot)$ that holds $\gamma'(r_m) = g(r_m)$ for $r_m = 4 \times 10^5$.

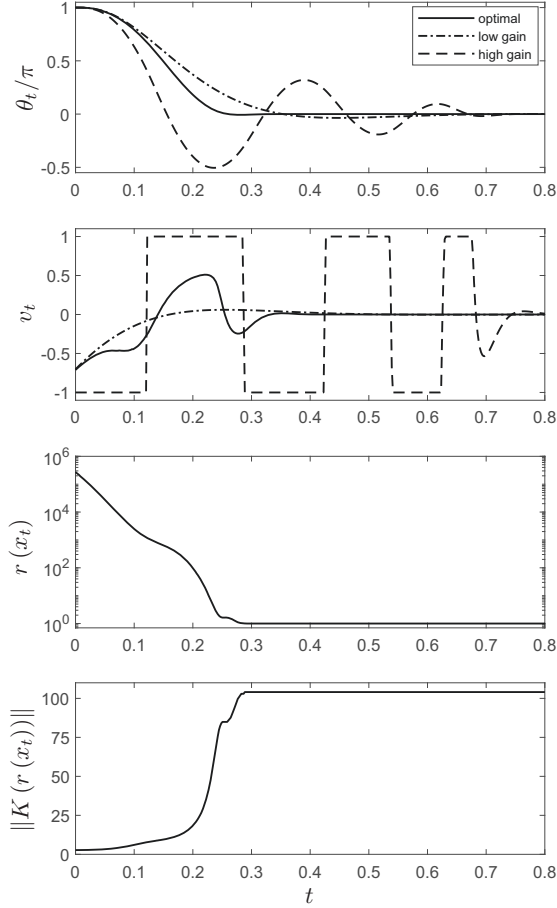


Fig. 2. Regulation performance for an initial state $x_0 = (0, 0, \pi)$ under the optimal control (26), the low gain linear control $u_t = -K_L x_t$, and the high gain nonlinear control $u_t = -\text{sat}(K_H x_t)$. The graphs from top to bottom represent the angular position θ_t (output), the armature voltage v_t (control), the scalar function $r(x_t)$, and the Euclidean norm of the gain matrix (for the optimal control) versus time.

The simulation results are illustrated in Fig. 2, which verifies the effectiveness of the optimal control law developed in this paper, particularly compared to its constant gain alternatives. As observed from this figure, a low gain linear state feedback necessarily yields a slow dynamics in order to keep the control effort below the saturation level. On the other hand, a high gain linear control requires a large control effort causing frequent actuator saturation, which in turn, results in a rough oscillatory response with a long settling time. Fig. 2 further explains how temporal variations of $r(x_t)$ dynamically adjust the feedback gain to prevent actuator saturation whenever necessary.

C. Trajectory Tracking

The optimal regulator designed in Section VI-B can be used for trajectory tracking by slight modifications in its structure. The goal in this control scenario is to constrain the position θ_t of the motor shaft to closely track a reference input θ_t^r over the course of control. This goal is achieved by replacing θ_t in the optimal regulator (8) with $\theta_t - \theta_t^r$. Then, this regulator drives

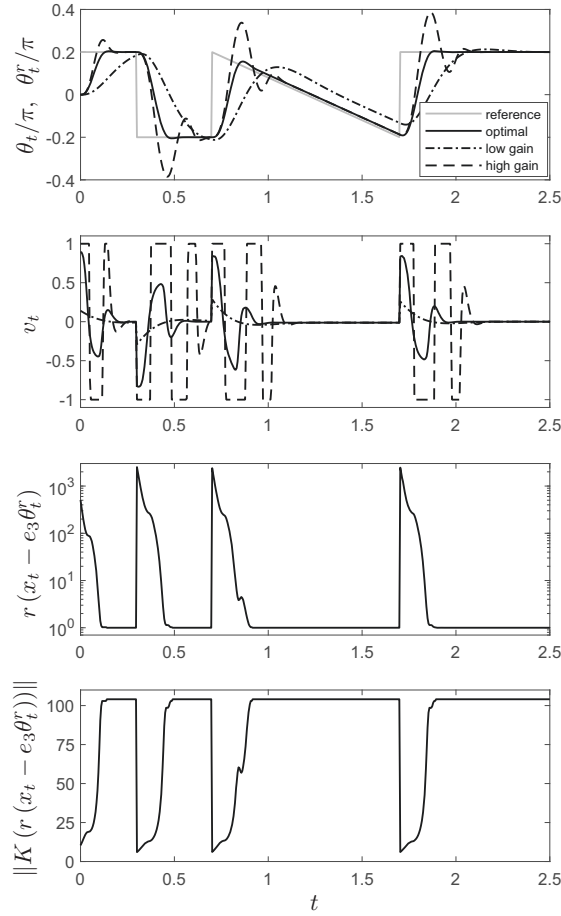


Fig. 3. Tracking performance for a typical reference trajectory (light solid line) under the nonlinear control (54) and the baseline controls (55). The graphs from top to bottom represent the angular position θ_t (output), the armature voltage v_t (control), the scalar function $r(x_t - e_3 \theta_t^r)$, and the Euclidean norm of the gain matrix [for the nonlinear control (54)] versus time.

the tracking error $\theta_t - \theta_t^r$ toward 0 to fulfil the requirement of trajectory tracking. The resulting feedback law is expressed as

$$u_t = -K(r(x_t - e_3 \theta_t^r))(x_t - e_3 \theta_t^r), \quad (54)$$

where e_3 denotes the unit vector $e_3 = (0, 0, 1)$.

The dynamical system (52) was simulated under this control law for a typical reference trajectory. The simulation results are illustrated in Fig. 3 together with their baseline counterparts generated under the low gain and high gain controllers

$$u_t = -K_L(x_t - e_3 \theta_t^r) \quad (55a)$$

$$u_t = -\text{sat}(K_H(x_t - e_3 \theta_t^r)). \quad (55b)$$

According to this figure, the nonlinear controller (54) performs effectively in tracking the reference trajectory (light solid line), and substantially outperforms both its counterparts (55). Fig. 3 also demonstrates the gain adaptation mechanism embedded in the control law (54) to effectively adjust the feedback gain during the course of control.

VII. CONCLUSION

Within an optimal control framework inspired by the notion of LQR, a class of nonlinear control laws was developed aimed at feedback control of linear systems with control constraints. The control laws in this class resemble a linear state feedback with a gain matrix chosen dynamically from a family of LQR gains. An adaption law was introduced to continuously adjust this gain to prevent any violations of the control constraints by reducing the gain whenever necessary and increasing it again whenever possible. Relying on this adaptation law, a feedback system can benefit from the advantages of a high feedback gain without violating the control constraints typically promoted by large loop gains. Moreover, it was shown that the developed control law is stabilizing, and is optimal in a sense analogous to the LQR optimality. Practical application of this control law was demonstrated for feedback control of dc servomotors, and the performance of control for this application was verified by computer simulations.

APPENDIX
PROOF OF THEOREMS AND LEMMAS

A. Proof of Lemma 1

By [26, Lem. 4.1], the pair $(A, sQ + W)$ is observable, and as a result, the Riccati equation (9) admits a unique positive definite solution [26, Thm. 2.1]. Differentiating both sides of this equation with respect to s and then rearranging terms yield the Lyapunov equation (11) for $P'(s)$. Similar procedure applied to (11) results in (12) for $P''(s)$. As $K(s)$ is an LQR gain matrix, $A - BK(s)$ is stable [26, Thm. 2.1], and thereby, the Lyapunov equations (11) and (12) admit unique solutions which are positive and negative semidefinite, respectively [27]. The existence and uniqueness of these solutions imply that the functions $P'(\cdot)$ and $P''(\cdot)$ are well-defined. Furthermore, since (A, Q) in the Lyapunov equation (11) is an observable pair, an argument paralleling [14, Note 12, p. 202] confirms that $P'(s)$ is indeed strictly positive definite.

B. Proof of Lemma 2

By multiplying both sides of (11) by s and then subtracting the resulting equation from (9), the Lyapunov equation (14) is determined for $\Pi(s)$. As $A - BK(s)$ is a stable matrix, this equation admits a unique positive semidefinite solution for a positive semidefinite W , and strictly positive definite if W is positive definite [27]. The derivative $\Pi'(s) = -sP''(s)$ taken from (13) is positive semidefinite by Lemma 1. The scalar function (15) can be equivalently expressed as

$$\lambda(s) = \frac{1}{2} \min_{z \neq 0} \frac{z^T W z}{z^T \Pi(s) z}$$

which is well-defined and strictly positive when W is positive definite. Moreover, since $z^T \Pi(s) z$ is increasing in s , $\lambda(\cdot)$ is a decreasing function.

C. Proof of Lemma 3

For any fixed $z \in \mathbb{R}^n - \{0\}$ define $f(s) = z^T (P(s)/s) z$ as a scalar function of s . The derivative of this function is given by $f'(s) = -z^T (\Pi(s)/s^2) z$ in terms of $\Pi(s)$ in (13), which is positive semidefinite by Lemma 2. This implies $f'(s) \leq 0$, which verifies that $f(\cdot)$ is a decreasing function. Since $f(\cdot)$ is also nonnegative, its limit as $s \rightarrow \infty$ exists, which implies that $\lim_{s \rightarrow \infty} P(s)/s$ exists. Application of this limit to (10) verifies that $K(\infty) = \lim_{s \rightarrow \infty} K(s)$ exists, which in turn, proves the existence and boundedness of $\Pi_\infty = \lim_{s \rightarrow \infty} \Pi(s)$ as the unique solution to the Lyapunov equation

$$(A - BK(\infty))^T \Pi_\infty + \Pi_\infty (A - BK(\infty)) + W = 0.$$

The existence of $\Pi_\infty = \lim_{s \rightarrow \infty} \Pi(s)$ further indicates that the scalar function $\lambda(\cdot)$ in (15) has a limit at ∞ and that limit is the constant λ_L in (19). Moreover, since $\lambda(\cdot)$ is a decreasing function by Lemma 2, it is lower bounded by its limit λ_L .

Dividing both sides of (9) by s and taking limit as $s \rightarrow \infty$ confirm that $P'_\infty = \lim_{s \rightarrow \infty} P(s)/s$ must solve (18), which in turn, verifies (17). Moreover, the existence and boundedness of $\lim_{s \rightarrow \infty} \Pi(s)$ result in

$$\lim_{s \rightarrow \infty} \left(\frac{1}{s} P(s) - P'(s) \right) = \lim_{s \rightarrow \infty} \frac{1}{s} \Pi(s) = 0$$

which proves (16).

D. Proof of Lemma 4

For $x = 0$, the optimization problem (21) admits a unique maximizer $r(0) = 1$ since $-\gamma(s)$ is strictly negative for $s > 1$ and it holds $-\gamma(1) = 0$. For $x \neq 0$, this optimization problem admits a unique maximizer $r(x) > 1$ since $x^T P(\cdot) x - \gamma(\cdot)$ is a strictly concave function with one and only one stationary point on $(1, \infty)$. To maintain $\frac{d}{ds} (x^T P(s) x - \gamma(s)) = 0$, this point must solve the algebraic equation $x^T P'(s) x = \gamma'(s)$, which indeed has a unique solution by the following argument. At $s = 1$, the left-hand side $x^T P'(1) x$ is strictly positive by Lemma 1 while its right-hand side $\gamma'(1)$ is 0 by definition. In addition, the left-hand side of the equation is decreasing by Lemma 1, while its right-hand side $\gamma'(s)$ is strictly increasing, as $\gamma(\cdot)$ is strictly convex by definition. Moreover, $\gamma'(s)$ tends to ∞ as $s \rightarrow \infty$, by definition. Hence, the continuous graphs of the left- and right-hand sides necessarily intersect at one and only one point on the interval $(1, \infty)$. Since the optimization problem (21) admits a unique maximizer, both functions (21) and (22) are well-defined.

Statement i: By construction, $r(x)$ holds the identity

$$x^T P'(r(x)) x - \gamma'(r(x)) = 0, \quad x \in \mathbb{R}^n \quad (56)$$

which is rewritten in terms of the inverse $\rho(\cdot)$ of $\gamma'(\cdot)$ as

$$r(x) = \rho(x^T P'(r(x)) x), \quad x \in \mathbb{R}^n. \quad (57)$$

If $r(\cdot)$ admits a gradient, that gradient $\nabla r(\cdot)$ must necessarily solve the algebraic equation

$$\begin{aligned}\nabla r(x) &= \nabla \rho(x^T P'(r(x))x) \\ &= \rho'(x^T P'(r(x))x) \nabla(x^T P'(r(x))x) \\ &= \rho'(\gamma'(r(x))) \left(2P'(r(x))x + x^T P''(r(x))x \nabla r(x) \right)\end{aligned}$$

for every $x \in \mathbb{R}^n$. Furthermore, if this equation has a unique solution $\nabla r(x)$ for every $x \in \mathbb{R}^n$, that solution is the gradient of $r(\cdot)$. This indeed is the case, and as a linear equation, it has a unique solution explicitly given by (23). The denominator of the solution (23) is strictly positive, implying that the solution exists for every $x \in \mathbb{R}^n$, and therefore, is well-defined. This is concluded from the fact that $\rho(\cdot)$ is increasing, so $\rho'(\cdot) \geq 0$, and from Lemma 1 which states $P''(\cdot)$ is negative semidefinite and thereby $x_t^T P''(r(x_t))x_t \leq 0$.

Since $r(\cdot)$ is differentiable, it is also continuous. Moreover, it is already shown that $r(x) > 1$ for $x \neq 0$ and $r(0) = 1$. To prove that (22) is an $\mathcal{R}$ -class function, its other two properties are verified next. First, to show that $r(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, the smallest eigenvalue of $P'_\infty = \lim_{s \rightarrow \infty} P'(s)$ (which exists by Lemma 3) is denoted by $\eta_1 > 0$ and the matrix inequality

$$r(x) \geq \rho(x^T P'_\infty x) \geq \rho(\eta_1 \|x\|^2) \quad (58)$$

is concluded from (57), the fact that $\rho(\cdot)$ is increasing, and the matrix inequality $P'_\infty \leq P'(r(x))$ implied by $P''(\cdot) \leq 0$ in Lemma 1. It is next observed that the right-hand side of (58) tends to ∞ as $\|x\| \rightarrow \infty$, noting that $\rho(z) \rightarrow \infty$ as $z \rightarrow \infty$, since by definition, $\gamma'(s) \rightarrow \infty$ as $s \rightarrow \infty$.

Finally, it is shown that $r(\xi x)$ is strictly increasing in ξ for each fixed $x \neq 0$. To that end, $\frac{d}{d\xi} r(\xi x) = \xi^T \nabla r(\xi x)$ is considered and by replacing $\nabla r(\cdot)$ from (23), it is confirmed that $\xi^T \nabla r(\xi x) \geq 0$. This ensures that $r(\xi x)$ is increasing in ξ . To show $r(\xi x)$ is strictly increasing, by contradiction assume that $\xi_1 \neq \xi_2$ exist to hold $r(\xi_1 x) = r(\xi_2 x) = \bar{s}$. Then, $\bar{s}$ must solve the algebraic equation $\xi_1 x^T P'(\bar{s})x = \xi_2 x^T P'(\bar{s})x$, but this equation does not admit any solution unless $\xi_1 = \xi_2$.

Statement ii: It is already shown that $r(0) = 1$, which leads to $V(0) = 0$. To verify $V(x) > 0$ for $x \neq 0$, the maximum in (21) is first lower bounded by $x^T P(1)x - \gamma(1)$, and then, the property $\gamma(1) = 0$ is applied to obtain $V(x) \geq x^T P(1)x$. However, $P(1)$ is positive definite by Lemma 1, which implies that $V(x) > 0$ for $x \neq 0$.

To show that (24) is the gradient of the scalar function $V(\cdot)$, this function is expressed equivalently as

$$V(x) = x^T P(r(x))x - \gamma(r(x)),$$

and its gradient is obtained then as

$$\nabla V(x) = 2P(r(x))x + (x^T P'(r(x))x - \gamma'(r(x))) \nabla r(x).$$

This gradient reduces to (24), noting that $r(x)$ satisfies (56). Clearly, (24) is a continuous function as both $P(\cdot)$ and $r(\cdot)$ are continuous.

E. Technical Details of the Proof of Theorem 1

Statement i: It is shown that the vector-valued function

$$f(x) = (A - BK(r(x)))x \quad (59)$$

is globally Lipschitz. By [22, Lem. 3.1], it is equivalently shown that the Jacobian matrix $F(\cdot)$ of this function holds

$$\|F(x)\| \leq L < \infty, \quad x \in \mathbb{R}^n \quad (60)$$

for some bounded constant L . By substituting $K(\cdot)$ from (10) into (59), the Jacobian matrix is determined as

$$F(x) = A - \frac{1}{r(x)} BR^{-1}B^T P(r(x)) + BR^{-1}B^T \Pi(r(x)) \frac{x(\nabla r(x))^T}{r^2(x)},$$

where the matrix $\Pi(\cdot)$ is given by (13). Based on the triangle inequality and the submultiplicative property of induced matrix norms, it is concluded that

$$\|F(x)\| \leq \|A\| + \|BR^{-1}B^T\| \cdot \left\| \frac{P(r(x))}{r(x)} \right\| + \|BR^{-1}B^T\| \cdot \|\Pi(r(x))\| \frac{\|x\| \cdot \|\nabla r(x)\|}{r^2(x)}. \quad (61)$$

The derivative of $P(s)/s$ is determined as $-\Pi(s)/s^2$, and by Lemma 2, $\Pi(s)$ is positive semidefinite under a positive semidefinite W . Hence, $P(s)/s$ is a decreasing function of s in the matrix ordering sense, and therefore, it takes its largest value at $s = 1$, i.e., $\|P(r(x))/r(x)\| \leq \|P(1)\|$. Lemmas 2 and 3 further imply that $\|\Pi(r(x))\| \leq \|\Pi_\infty\|$. By defining

$$L' = \sup_{x \in \mathbb{R}^n} \frac{\|x\| \cdot \|\nabla r(x)\|}{r^2(x)} \quad (62)$$

and applying these results to (61), it is concluded that

$$\|F(x)\| \leq \|A\| + \|BR^{-1}B^T\| (\|P(1)\| + L' \|\Pi_\infty\|).$$

By showing next that L' is bounded and the fact that all other terms on the right-hand side are bounded, (60) is established.

To show that L' is bounded, $\nabla r(x)$ is substituted from (23) into (62) and the upper bound

$$L' \leq 2 \|P'(1)\| \sup_{x \in \mathbb{R}^n} \frac{\|x\|^2 \rho'(\gamma'(r(x)))}{r^2(x)} \quad (63)$$

is established. To derive this inequality, the submultiplicative property $\|P'(r(x))x\| \leq \|P'(r(x))\| \cdot \|x\|$ of the induced matrix norms is applied to $\|\nabla r(x)\|$ and two additional facts are used. First, $\|P'(r(x))\| \leq \|P'(1)\|$ implied by Lemma 1, and second, the denominator of (23) is not less than 1 as $\rho'(\cdot)$ is positive and $P''(\cdot)$ is negative semidefinite by Lemma 1.

Since $\gamma(\cdot)$ is strictly convex by definition, $\gamma'(\cdot)$ is strictly increasing. Using this fact and noting that $\gamma'(\cdot)$ is the inverse of $\rho(\cdot)$, it is concluded from (58) that $\|x\|^2 \leq \gamma'(r(x)) / \eta_1$. Applying this result to (63) leads to

$$\begin{aligned} L' &\leq \frac{2 \|P'(1)\|}{\eta_1} \sup_{x \in \mathbb{R}^n} \frac{\gamma'(r(x)) \rho'(\gamma'(r(x)))}{r^2(x)} \\ &= \frac{2 \|P'(1)\|}{\eta_1} \sup_{z \geq 0} \frac{z \rho'(z)}{\rho^2(z)}. \end{aligned}$$

However, the function $z \rho'(z) / \rho^2(z)$ is continuous, and as a result, bounded on any closed interval. The proof is completed by confirming that its limit is also bounded as $z \rightarrow \infty$. To that end, consider the integral

$$\begin{aligned} \int_0^\infty \frac{1}{z} \cdot \frac{z \rho'(z)}{\rho^2(z)} dz &= \int_0^\infty \left(-\frac{d}{dz} \frac{1}{\rho(z)} \right) dz \\ &= 1, \end{aligned}$$

determined based on the facts that $\rho(0) = 1$ and $\rho(z) \rightarrow \infty$ as $z \rightarrow \infty$. Since $z \rho'(z) / \rho^2(z)$ is nonnegative, to maintain the integral on the left-hand side bounded, it is necessary that

$$\lim_{z \rightarrow \infty} \frac{z \rho'(z)}{\rho^2(z)} = 0.$$

Statement iii: By substituting the gradient $\nabla r(\cdot)$ from (23) into (30), it is shown that $\frac{d}{dt} r(x_t) \leq 0$ holds. This substitution results in

$$\frac{d}{dt} r(x_t) = \frac{2 \rho'(\gamma'(r(x_t))) x_t^T P'(r(x_t)) \dot{x}_t}{1 - \rho'(\gamma'(r(x_t))) x_t^T P''(r(x_t)) x_t}. \quad (64)$$

Noting that $P'(\cdot)$ is the solution to the Lyapunov equation (11) and that $\dot{x}_t$ is given by (27), it is concluded that

$$x_t^T P'(r(x_t)) \dot{x}_t = -x_t^T \left(\frac{P(r(x_t)) B R^{-1} B^T P(r(x_t))}{r^2(x_t)} + Q \right) x_t \leq 0.$$

In addition, it is shown in the proof of Lemma 4 that $\rho'(\cdot) \geq 0$ and that the denominator of (64) is strictly positive. These facts confirm that the right-hand side of (64) is nonpositive.

Statement iv: For the gradient $\nabla V(\cdot)$ given by (24) and the gain matrix $K(\cdot)$ given by (10) in terms of the solution $P(\cdot)$ to the Riccati equation (9), it can be shown that the equality

$$x^T W x + r(x) (x^T Q x + u^T R u) = \|u + K(r(x))x\|_R^2 - (\nabla V(x))^T (Ax + Bu)$$

holds for every x and u (see for example [14, p. 194]). In this equality, x and u are respectively replaced by the state x_t and the control u_t of the linear state-space equation (1) to obtain

$$x_t^T W x_t + r(x_t) (x_t^T Q x_t + u_t^T R u_t) = \|u_t + K(r(x_t))x_t\|_R^2 - \frac{d}{dt} V(x_t).$$

Integrating both sides of this equation on $[0, \infty)$ results in

$$J = V(x_0) + \int_0^\infty r(x_t) \|u_t + K(r(x_t))x_t\|_R^2 dt - \lim_{t \rightarrow \infty} V(x_t).$$

However, for the class of globally asymptotically stabilizing controls, $\lim_{t \rightarrow \infty} V(x_t) = 0$ holds, which proves (31).

F. Proof of Lemma 5

The upper bound (34) is established by solving a differential inequality for $U(x_t)$, obtained via the following steps:

$$\begin{aligned} \frac{d}{dt} U(x_t) &= 2x_t^T \Pi(r(x_t)) \dot{x}_t + x_t^T \Pi'(r(x_t)) x_t \frac{d}{dt} r(x_t) \\ &\leq 2x_t^T \Pi(r(x_t)) (A - BK(r(x_t))) x_t \\ &= -x_t^T W x_t \\ &= -\frac{x_t^T W x_t}{x_t^T \Pi(r(x_t)) x_t} U(x_t) \\ &\leq -2\lambda(r(x_t)) U(x_t). \end{aligned}$$

Here, the first inequality is derived by replacing $\dot{x}_t$ in the first equality with (27) and using the fact that the second term on the right-hand side of this equality is nonpositive, as $\Pi'(\cdot)$ is positive semidefinite by Lemma 2 and $\frac{d}{dt} r(x_t)$ is nonpositive by Theorem 1. The second equality is concluded from the fact that $\Pi(\cdot)$ solves the Lyapunov equation (14). The last equality is a trivial result of $U(x_t) = x_t^T \Pi(r(x_t)) x_t \neq 0$. Finally, the last inequality is determined by lower bounding the fraction in the third equality with $2\lambda(r(x_t))$.

G. Proof of Theorem 3

Theorem 1 assumes that either Q or W is positive definite. For the case that W is positive definite, the theorem statement is immediately concluded from (36) in the proof of Theorem 2 by taking $c_2(\varrho) = c_1$ and $\tilde{\lambda}_L = \lambda_L$. For the case of a positive definite Q , the statement of theorem is proven by adopting (38) as a Lyapunov function. Following a procedure paralleling the proof of Lemma 5, it is first shown that

$$\tilde{U}(x_t) \leq \tilde{U}(x_0) e^{-2\tilde{\lambda}_L t}, \quad t \geq 0, \quad (65)$$

where $\tilde{\lambda}_L$ is the strictly positive constant

$$\tilde{\lambda}_L = \frac{1}{2} \lambda_{\min} \left\{ P^{-1/2}(1) Q P^{-1/2}(1) \right\}.$$

Similar to the proof of Lemma 5, this result is derived from a differential inequality of the form $\frac{d}{dt} \tilde{U}(x_t) \leq -2\tilde{\lambda}_L \tilde{U}(x_t)$, obtained by first differentiating $\tilde{U}(x_t) = x_t^T P(r(x_t)) x_t$ and then using the facts that $P(\cdot)$ solves the Riccati equation (9) and $P'(\cdot)$ is positive definite (by Lemma 1).

Since $P(s)$ is increasing in s (in the matrix ordering sense), it is concluded that

$$\tilde{U}(x_t) \geq x_t^T P(1) x_t \geq \lambda_{\min} \{P(1)\} \|x_t\|^2.$$

Moreover, for any x_0 in the ball $\|x_0\| \leq \varrho$, it holds that

$$\begin{aligned} \tilde{U}(x_0) &\leq \sup_{\|z\| \leq \varrho} x_0^T P(r(z)) x_0 \\ &\leq \|x_0\|^2 \sup_{\|z\| \leq \varrho} \lambda_{\max} \{P(r(z))\}, \end{aligned}$$

where $\lambda_{\max} \{\cdot\}$ denotes the largest eigenvalue of a symmetric matrix. Applying these inequalities to (65) leads to (39) with

$$c_2(\varrho) = \left(\frac{\sup_{\|z\| \leq \varrho} \lambda_{\max} \{P(r(z))\}}{\lambda_{\min} \{P(1)\}} \right)^{1/2}.$$

H. Proof of Lemma 6

Using the definition of $K(s)$ in (10), it is first shown that

$$\begin{aligned} \inf_{-K(s)z \notin \mathcal{U}} z^T P'(s) z &= \inf_{v \notin \mathcal{U}} \min_{-K(s)z=v} z^T P'(s) z \\ &= \inf_{v \notin \mathcal{U}} v^T \left(K(s) (P'(s))^{-1} K^T(s) \right)^{-1} v \\ &= g(s). \end{aligned}$$

Let x be a vector such that $r(x) = s \in [1, r_m]$. Then, the fact that $s = r(x)$ solves the algebraic equation (25) implies that

$$x^T P'(s) x = \gamma'(s) \leq g(s) = \inf_{-K(s)z \notin \mathcal{U}} z^T P'(s) z.$$

Since the convex function $z^T P'(s) z$ has only one stationary point at $z = 0$ not belonging to $-K(s)z \notin \mathcal{U}$, the infimum on the right-hand side is attained on the boundary of the compact set $\mathcal{U}$. Therefore, $x^T P'(s) x$ is strictly larger than this infimum for any x holding $-K(s)x \notin \mathcal{U}$. This implies $-K(s)x \in \mathcal{U}$, which together with $s = r(x)$ results in (42).

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