2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Variational Observer Designs on Lie Groups, with Applications to Rigid Body Motion Estimation

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2024 IEEE 63rd Conference on Decision and Control (CDC) | 979-8-3503-1633-9/24/$31.00 ©2024 IEEE | DOI: 10.1109/CDC56724.2024.10886160

***Abstract*— Variational estimation of a mechanical system is based on the application of variational principles from mechanics to state estimation of the system evolving on its configuration manifold. If the configuration manifold is a Lie group, then the underlying group structure can be used to design nonlinearly stable observers for estimation of configuration and velocity states from measurements. Measured quantities are on a vector space on which the Lie group acts smoothly. We formulate the design of variational observers on a general finite-dimensional Lie group, followed by the design and experimental evaluation of a variational observer for rigid body motions on the Lie group $\mathrm{SE}(3)$.**

I. Introduction

State estimation schemes for mechanical systems have a wide range of applications in mobile robotics, aerospace vehicles and oceangoing vessels, among others. Mechanical systems have a second order structure, with the degrees of freedom commonly represented by generalized (local) coordinates on a configuration manifold. The velocity states are represented by generalized velocities on the tangent space to the configuration manifold for a given configuration. Based on variational principles used in geometric mechanics, one can design variational observers for mechanical systems in general. The observer design principles outlined in this article are generally applicable to mechanical systems evolving on Lie groups as configuration manifolds. Considering the geometry of the configuration manifold becomes necessary, especially when this manifold is not contractible. In particular, this is true for the Lie groups of rigid body attitude (orientation) $\mathrm{SO}(3)$ and rigid body pose $\mathrm{SE}(3)$.

State estimation schemes of mechanical systems that account for the geometry of the configuration space, are particularly advantageous if the configuration space happens to be a Lie group. This is because Lie groups are parallelizable [1], and the state space of such a mechanical system is the tangent bundle of the Lie group. This makes it possible to represent the states globally on the state space without using local coordinates, and use the symmetry properties of the Lie group to design state estimators.

However, the design of observers for systems on connected, locally compact, finite-dimensional Lie groups (or semi-direct products of such groups), is complicated by the fact that these groups are not contractible (see, e.g., [2]–[4]). This means that any smooth observer or controller design on such Lie groups have multiple equilibria, which in turn precludes global asymptotic stability of any one equilibrium. This has been pointed out, in particular, for the Lie group of rigid body rotations, $\mathrm{SO}(3)$, in prior research (e.g., [5]–[7]). Smooth estimation schemes on these Lie groups, and consequently on symmetric spaces obtained from them, can (at best) be almost globally asymptotically stable (AGAS).

Due to these considerations, an increasing number of state estimation schemes have been designed directly on the Lie groups of rigid body rotations $\mathrm{SO}(3)$ and rigid body motions $\mathrm{SE}(3)$. In particular, over the last twenty years, many estimation schemes on Lie groups have been explicitly designed using principles of geometric mechanics. These estimators do not suffer from kinematic singularities like estimators using coordinate descriptions of attitude, and they do not suffer from the unstable unwinding phenomenon encountered when using unit quaternions for attitude representation. Unwinding occurs due to the unit quaternion hypersphere $\mathbb{S}^3$ double covering $\mathrm{SO}(3)$ [5]. Early research in this direction extended traditional approaches like attitude determination using Wahba's problem, complementary filters, gradient-based observers, set-bounded filters and unscented filters; a sample of this literature can be found in [8]–[16]. Besides accounting for the geometry of the configuration space, these estimation schemes also take advantage of the symmetry properties of Lie groups in their design in continuous time (e.g., [13], [15]) or in discrete time (e.g., [7], [17]). Published research over the last decade on this topic has applied concepts like near-optimal filtering [18]–[20], invariant filter design [21]–[23], variational observer design [17], [24], stochastic filtering and Bayesian estimation [25], [26], and cascaded observer design [27].

In this work, we present a design for variational observers on finite-dimensional Lie groups, with applications to rigid body motion estimation. This variational observer design approach is based on application of the Lagrange-d'Alembert principle from variational mechanics [28], to state estimation problems for mechanical systems. In this work, we give the general approach to design of variational observers on (real) finite-dimensional Lie groups using this principle. We use a matrix representation of the Lie group and a corresponding matrix representation of its Lie algebra. This matrix representation of the Lie algebra is isomorphic to its representation as a real, finite-dimensional vector space of the same dimension as the Lie group. This general development has not appeared before in the published literature. It is followed by a brief overview of the variational pose estimator (VPE), which has appeared in our prior research [17], [29]. The VPE is a variational observer designed on the (tangent bundle of the) Lie group of rigid body motions, $\mathrm{SE}(3)$.

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This is followed by experimental results from a hardware implementation of the VPE using a depth camera sensor.

The remainder of this paper is organized as follows. Section II gives the general principles behind design of variational state estimation schemes on finite-dimensional Lie groups and obtains the structure of the variational observer. Section III shows how this general design gives the variational pose estimator (VPE) for state estimation of rigid body motions, which has appeared in our prior research, when applied to $\mathrm{TSE}(3)$. Both continuous- and discrete-time versions of this variational observer are provided in this section. This is followed by the implementation and evaluation of the VPE in experiments using a depth camera (vision) sensor, in section IV. Finally, section V concludes this article by summarizing its contributions and giving planned future directions of related research.

II. Variational Observer Design on Finite-dimensional Lie Groups

Consider a (real) finite-dimensional Lie group G that has a regular action on a finite-dimensional vector space $\mathrm{W} \subseteq \mathbb{R}^m$. The group configuration is usually not measured directly in applications. Instead, measurements on the vector space W are used to estimate states on the Lie group. For an n-dimensional Lie group G, its Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R}^n$, i.e., $\mathfrak{g} \cong \mathbb{R}^n$. A matrix representation of the group G on $\mathbb{R}^{m\times m}$ is used for state estimation, which expresses it as a subgroup of $\mathrm{GL}(m,\mathbb{R})$. This enables representation of its action on W by the matrix product. Let $g(t)$ denote a trajectory (integral curve) on the Lie group for a given $t \in \mathbb{R}$. The tangent vector to the Lie group at $g(t)$ is given by the kinematic expression:

$$\dot{g} = g\xi^\vee, \tag{1}$$

where $g \in \mathrm{G}$, $\xi^\vee \in \mathfrak{g}$ where $\xi \in \mathbb{R}^n$. Here $(\cdot)^\vee : \mathbb{R}^n \to \mathfrak{g}$ denotes the vector space isomorphism from $\mathbb{R}^n$ to $\mathfrak{g}$. Note that the RHS of eq. (1) gives a left-invariant vector field on G with the integral curve $g(t)$ for a given $g(0)$.

A. Formulation of Variational Observer Design

The problem of state estimation on the Lie group G is based on instantaneous vector measurements on W. An instantaneous set of l vectors measured on W are represented by the column vectors of matrix $L \in \mathbb{R}^{m\times l}$. It is assumed that the action of G on these vectors give a set of known vectors and enough of these vectors are measured at each instant to give full observability of the states $(g,\xi) \in \mathrm{G}\times\mathfrak{g}$. In addition, if measurements of some components of the velocity state ξ on the Lie algebra $\mathfrak{g}$ are available, they can also be used to estimate velocity states. This sets us up for the observer design for full state estimation of $(g,\xi) \in \mathrm{G}\times\mathfrak{g}$ from these measurements.

As mentioned in §I, the design framework of variational estimation schemes follows applying the Lagrange-d'Alembert principle from variational mechanics to an energy-like quantity in the state estimation errors with dissipative terms that dissipate this energy. Denoting the state estimates at an instant t by $(\hat{g}(t), \hat{\xi}(t)) \in \mathrm{TG} \cong \mathrm{G}\times\mathfrak{g}$, we have these satisfy the kinematic relation:

$$\dot{\hat{g}} = \hat{g}\hat{\xi}^\vee, \tag{2}$$

which gives an integral curve $\hat{g}(t)$ for a given $\hat{g}(0)$. Define the (right-invariant) estimation error in the configuration on the group by:

$$h = g\hat{g}^{-1} \in \mathrm{G}, \tag{3}$$

which also defines an integral curve given the initial estimation error $h(0) = g(0)\hat{g}(0)^{-1} \in \mathrm{G}$. From (1) and (2), this estimation error satisfies:

$$\dot{h} = h\eta^\vee \text{ where } \eta^\vee \in \mathfrak{g} \text{ and } \eta = \mathrm{Ad}_{\hat{g}}\big(\xi - \hat{\xi}\big). \tag{4}$$

Here η represents the estimation error in the velocity state, and Ad_g denotes the adjoint representation of $g \in \mathrm{G}$ on $\mathfrak{g}$. A quadratic positive definite function of this velocity estimation error is given by the kinetic energy-like function:

$$\mathcal{T}(\eta) = \frac{1}{2}\eta^{\mathrm{T}} J\eta \text{ where } J = J^{\mathrm{T}} \succ 0. \tag{5}$$

Let $P \in \mathbb{R}^{m\times l}$ be the known constant matrix whose column vectors correspond to the set of vector measurements that are the column vectors in L, i.e., $P = gL$ (where the group action on W is represented by the matrix multiplication). Since the configuration on the group G is not directly measured, we express the estimation error in the configuration in terms of the measurement residual. The measurement residual $P - \hat{g}L$ can be expressed in terms of the estimation error in configuration on the group h, defined by eq. (3), as follows:

$$E := P - \hat{g}L = P - \hat{g}g^{-1}P = (I - h^{-1})P, \tag{6}$$

where I denotes the identity element (matrix) on G. A class $\mathcal{K}$ function of this estimation error in the configuration, $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$, is used as a measure of this error, using this matrix representation of G. This leads to a potential energy-like function:

$$\mathcal{U}(\hat{g}, L, P) = \Psi\big(\|P - \hat{g}L\|\big), \tag{7}$$

where any matrix norm $\|\cdot\| : \mathbb{R}^{m\times l} \to \mathbb{R}^+$ may be used on the right side of eq. (7). This potential function can be expressed in terms of h (in an abuse of notation) as:

$$\mathcal{U}(h, P) = \Psi\big(\|(I - h^{-1})P)\|\big). \tag{8}$$

Assuming measurements of P and η, the Lagrangian used to design the variational estimation scheme is defined as:

$$\mathcal{L}(\eta, h, P) = \mathcal{T}(\eta) - \mathcal{U}(h, P). \tag{9}$$

An action integral is obtained from this Lagrangian for $(h(t), \eta(t))$ over a time interval $t \in [0, T]$ as follows:

$$\mathcal{S}(\mathcal{L}(\eta, h, P)) = \int_0^T \mathcal{L}(\eta, h, P)\mathrm{d}t. \tag{10}$$

After the Lagrangian and action functional are expressed in terms of the state estimation errors $(h, \eta) \in \mathrm{G}\times\mathfrak{g} \cong \mathrm{TG}$, the first variation of this action functional, which involves first variations of these estimation errors, is obtained. We apply *reduced variations* [28], [30] on the Lie group G as follows:

$$\delta h = h\upsilon^\vee,\ \delta\eta = \dot{\upsilon} + \mathrm{ad}_\eta \upsilon, \tag{11}$$

and ad_η denotes the adjoint representation of the Lie algebra $\mathfrak{g}$. In addition, a dissipation term linear in η that is obtained from a Rayleigh dissipation function, is introduced to dissipate the energy and guarantee asymptotic stability of the resulting observer.

B. Variational Observer on the Lie group G

Using the Rayleigh dissipation function defined by:

$$\mathcal{R}(\eta) = \frac{1}{2}\eta^{\mathrm{T}} D\eta, \tag{12}$$

we obtain the dissipation term:

$$\tau_D = -\nabla_\eta \mathcal{R}(\eta) = -D\eta. \tag{13}$$

Application of the Lagrange-d'Alembert principle to the Lagrangian (9) with the dissipation term (13) for $t \in [0, T]$, leads to the following expression:

$$\delta_{h,\eta}\mathcal{S}\big(\mathcal{L}(\eta, h, P)\big) + \int_0^T \tau_D^{\mathrm{T}} \upsilon \mathrm{d}t = 0. \tag{14}$$

The first term in the left side of (14) is the first variation of the action functional with respect to the observer error states (h, η), with the reduced variations on $\mathrm{G}\times\mathfrak{g}$ satisfying (11).

The first variation of the potential energy-like term is obtained as:

$$\delta\mathcal{U}(h, P) = \Psi'(\cdot)\upsilon^{\mathrm{T}} T_P(h), \tag{15}$$

where $\Psi'(\cdot)$ is the derivative of the class-$\mathcal{K}$ function $\Psi(\cdot)$ and $T_P : \mathrm{G} \to \mathbb{R}^n$ (where $bR^n \cong \mathfrak{g}^\star$) is obtained from the gradient of $\|P - \hat{g}L\|$ along the variation vector field $h\upsilon^\vee$. With measurements of P and η, the variational observer obtained is given by the following result.

Theorem 1: The variational estimation scheme on the tangent bundle of the Lie group G is given by:

$$\begin{cases} J\dot{\eta} &= \mathrm{ad}^*_\eta J\eta - \Psi'(\|P - \hat{g}L\|)T_P(h) - D\eta, \\ \hat{\xi} &= \xi - \mathrm{Ad}_{g^{-1}}\eta, \\ \dot{\hat{g}} &= \hat{g}\hat{\xi}^\vee, \end{cases} \tag{16}$$

where $\mathrm{ad}^*_\zeta = (\mathrm{ad}_\zeta)^{\mathrm{T}}$ and (P, η) are measured.

Proof: The first variation of the action functional is obtained as:

$$\begin{aligned} \delta_{h,\eta}\mathcal{S}\big(\mathcal{L}(\eta, h, P)\big) &= \int_0^T \big[\delta\eta^{\mathrm{T}} J\eta - \delta\mathcal{U}(h, P)\big]\mathrm{d}t \\ &= \int_0^T \Big[\big(\dot{\upsilon}^{\mathrm{T}} + \upsilon^{\mathrm{T}}\mathrm{ad}^*_\eta\big)J\eta - \Psi'(\cdot)\upsilon^{\mathrm{T}} T_P(h)\Big]\mathrm{d}t. \end{aligned} \tag{17}$$

Applying integration by parts to the first term on the right side of the second line of expression (17) with fixed endpoint variations ($\upsilon(0) = \upsilon(T) = 0 \in \mathfrak{g}$), we can reduce this expression to the following:

$$\begin{aligned} \delta_{h,\eta}\mathcal{S}\big(\mathcal{L}(\eta, h, P)\big) = \int_0^T \upsilon^{\mathrm{T}} \big[&\mathrm{ad}^*_\eta J\eta - J\dot{\eta} \\ &- \Psi'(\|P - \hat{g}L\|)T_P(h)\big]\mathrm{d}t. \end{aligned} \tag{18}$$

Now applying the Lagrange-d'Alembert principle as given by (14), we obtain the following expression:

$$\begin{aligned} \int_0^T \upsilon^{\mathrm{T}} \big[&\mathrm{ad}^*_\eta J\eta - J\dot{\eta} - D\eta \\ &- \Psi'(\|P - \hat{g}L\|)T_P(h)\big]\mathrm{d}t = 0. \end{aligned} \tag{19}$$

As eq. (19) holds for all variations with fixed endpoints, the expression within the square brackets in the integral, must vanish identically. This gives the first of eqs. (16), while the second of eqs. (16) arises from eq. (4), and the third of eqs. (16) is identical to eq. (2). ■

Note that the statement of Theorem 1 is applicable when measurements of vectors in W and all velocities η are available. However, it is possible to adapt the theory to cases when the velocities η can be obtained (perhaps taking time derivatives) from vectors in P. This was done for the variational pose estimator described next.

III. The Variational Pose Estimator for State Estimation of Rigid Body Motion

The variational pose estimator (VPE) was first obtained in continuous and discrete time in [17], using the same design principles outlined in §II for finite-dimensional Lie groups. We provide a brief recap of this VPE design, now in the context of this general theory.

A. Pose and Velocities Measurement Model

Consider the rotational and translational motion of a rigid body. Let $\mathcal{I}$ denote the inertial frame and $\mathcal{B}$ denote the body coordinate frame. Let $R \in \mathrm{SO}(3)$ denote the rotation matrix from frame $\mathcal{B}$ to frame $\mathcal{I}$ and $b \in \mathbb{R}^3$ denote the position of origin of $\mathcal{B}$ expressed in frame $\mathcal{I}$. Then the pose of the rigid body is given by

$$\mathrm{g} = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3). \tag{20}$$

Consider optical measurements of j points at time t with known and fixed positions in frame $\mathcal{I}$, denoted q_j. These points generate $\binom{j}{2}$ unique relative position vectors from pairwise differences. Let a_i be the position of the i-th stationary point measured in frame $\mathcal{B}$. The measured vectors in the presence of additive noise are expressed as:

$$\bar{a}^m = R^{\mathrm{T}}(\bar{q} - b) + \bar{\wp},$$

where $\bar{q}$ and $\bar{a}^m$ are defined as follows:

$$\bar{q} = \frac{1}{j}\sum_{i=1}^{j} q_i,\ \bar{a}^m = \frac{1}{j}\sum_{i=1}^{j} a_i^m, \tag{21}$$

and $\bar{\wp}$ is the additive measurement noise obtained by averaging the measurement noise vectors for each a_i.

Now, denote the angular and translational velocity of the rigid body expressed in frame $\mathcal{B}$ by Ω and ν, respectively. Therefore, the kinematics of the rigid body is given by

$$\dot{R} = R\Omega^\times, \quad \dot{b} = R\nu \ \Rightarrow \dot{\mathrm{g}} = \mathrm{g}\xi^\vee, \tag{22}$$

where $\xi = \begin{bmatrix} \Omega \\ \nu \end{bmatrix} \in \mathbb{R}^6$, $\xi^\vee = \begin{bmatrix} \Omega^\times & \nu \\ 0 & 0 \end{bmatrix}$. Consider the $\binom{j}{2}$ relative position vectors for j measured points from optical sensor measurements, denoted as $d_j = q_\lambda - q_\ell$ in frame $\mathcal{I}$ and the corresponding vectors in frame $\mathcal{B}$ as $e_j = a_\lambda - a_\ell$, where λ, ℓ are any two measured points such that, $\lambda \neq \ell$. Therefore,

$$d_j = Re_j \Rightarrow D = RE, \tag{23}$$

where $D = [d_1 \ \dots \ d_n]$, $E = [e_1 \ \dots \ e_n] \in \mathbb{R}^{3\times n}$ with $n = 3$ if $\binom{j}{2} = 2$ and $n = \binom{j}{2}$ if $\binom{j}{2} > 2$. In the presence of measurement noise, the measured value of matrix E is given by,

$$E^m = R^{\mathrm{T}} D + \mathfrak{L},$$

where the columns of matrix $\mathfrak{L} \in \mathbb{R}^{3\times n}$ are additive noise vectors in the vector measurements made in frame $\mathcal{B}$.

B. Pose Estimation in SE(3)

The estimated pose and its kinematics are represented as

$$\widehat{\mathrm{g}} = \begin{bmatrix} \widehat{R} & \widehat{b} \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3), \ \ \dot{\widehat{\mathrm{g}}} = \widehat{\mathrm{g}}\widehat{\xi}^\vee, \tag{24}$$

where $\widehat{b}$ is the position estimate, $\widehat{R}$ is the attitude estimate, and $\widehat{\xi}$ is the rigid body velocities estimate, with $\widehat{\mathrm{g}}_0$ as the initial pose estimate. The pose estimation error is defined as

$$h = \mathrm{g}\widehat{\mathrm{g}}^{-1} = \begin{bmatrix} Q & b - Q\widehat{b} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Q & \chi \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3), \tag{25}$$

where $Q = R\widehat{R}^{\mathrm{T}}$ is the attitude estimation error, and $\chi = b - Q\widehat{b}$. Velocity kinematics for the estimation error is:

$$\dot{h} = h\eta^\vee, \text{ where } \eta(\widehat{\mathrm{g}}, \xi^m, \widehat{\xi}) = \begin{bmatrix} \omega \\ v \end{bmatrix} = \mathrm{Ad}_{\widehat{\mathrm{g}}}(\xi - \widehat{\xi}), \tag{26}$$

where $\xi^m = \xi \in \mathbb{R}^6$ is the measured rigid body velocities, v and ω are translational and angular velocity estimation errors respectively, and $\mathrm{Ad}_{\mathfrak{g}} = \begin{bmatrix} \mathfrak{R} & 0 \\ \mathfrak{b}^\times \mathfrak{R} & \mathfrak{R} \end{bmatrix}$ for $\mathfrak{g} = \begin{bmatrix} \mathfrak{R} & \mathfrak{b} \\ 0 & 1 \end{bmatrix}$.

The potential energy-like quantity in the pose estimation error can be expressed as the sum of rotational and translational measurement residuals:

$$\begin{aligned} \mathcal{U}(\hat{\mathrm{g}}, E^m, D, \bar{a}^m, \bar{q}) &= \mathcal{U}_r(\mathrm{g}, E^m, D) + \mathcal{U}_t(g, \bar{a}^m, \bar{q}) \\ &= \Phi\big(\mathcal{U}_r^0(\mathrm{g}, E^m, D)\big) + \frac{1}{2}\alpha y^{\mathrm{T}} y, \end{aligned} \tag{27}$$

where

$$y = \bar{q} - \widehat{R}\bar{a}^m - \widehat{b} \tag{28}$$

$$\mathcal{U}_r^0(\mathrm{g}, E^m, D) = \frac{1}{2}\left\langle D - \widehat{R}E^m, (D - \widehat{R}E^m)W \right\rangle, \tag{29}$$

W is positive definite, $\Phi : [0, \infty) \mapsto [0, \infty)$ is a $\mathcal{C}^2$ function that satisfies $\Phi(0) = 0$ and $\Phi'(z) > 0$ for all $z \in [0, \infty)$. Additionally, $\Phi'(\cdot) \leq \alpha(\cdot)$ where $\alpha(\cdot)$ is a class $\mathcal{K}$ function [31] and $\Phi'(\cdot)$ is the derivative of $\Phi(\cdot)$ with respect to its argument. This ensures that $\mathcal{U}_r^0(\hat{\mathrm{g}}, R^m)$ and $\mathcal{U}_r(\hat{\mathrm{g}}, R^m)$ have the same indices and minimizer $\hat{R} \in \mathrm{SO}(3)$. Now, define the kinetic energy-like function as

$$\mathcal{T}(\eta) = \frac{1}{2}\eta^{\mathrm{T}}\mathbb{J}\eta, \tag{30}$$

where $\eta \equiv \eta(\hat{\mathrm{g}}, \xi^m, \hat{\xi})$ and $\mathbb{J} \in \mathbb{R}^{6\times 6} > 0$ is an artificial inertia-like kernel matrix. The Lagrangian is defined as

$$\mathcal{L}(\hat{\mathrm{g}}, E^m, D, \bar{a}^m, \bar{q}, \eta) = \mathcal{T}(\eta) - \mathcal{U}(\hat{\mathrm{g}}, E^m, D, \bar{a}^m, \bar{q}), \tag{31}$$

and the corresponding action functional over an arbitrary time interval $[t_0, T]$ where $T > 0$ is given by,

$$\mathcal{S}\Big(\mathcal{L}(\hat{\mathrm{g}}, E^m, D, \bar{a}^m, \bar{q}, \eta)\Big) = \int_{t_0}^{T} \mathcal{L}(\hat{\mathrm{g}}, E^m, D, \bar{a}^m, \bar{q}, \eta)\mathrm{d}t, \tag{32}$$

such that $\dot{\hat{\mathrm{g}}} = \hat{\mathrm{g}}\xi^\vee$. The following lemma gives the Lagrangian expression in case of perfect measurements.

Lemma 1: The Lagrangian in the absence of measurement noise is given by

$$\mathcal{L}(\hat{\mathrm{g}}, b, R, \varphi) = \frac{1}{2}\varphi^{\mathrm{T}}\mathbb{J}\varphi - \Phi\big(\langle I - Q, K\rangle\big) - \frac{1}{2}\alpha y^{\mathrm{T}} y, \tag{33}$$

where $K = DWD^{\mathrm{T}}$ is a positive definite matrix and $y \equiv y(h) = (I - Q^{\mathrm{T}})b + Q^{\mathrm{T}}\chi$.

C. VPE for Rigid Body Motion in Continuous Time

Proposition 1: The nonlinear variational pose estimator of a rigid body in continuous time is given by

$$\begin{cases} \mathbb{J}\dot{\eta} &= \mathrm{ad}_\eta^*\mathbb{J}\eta - Z(\hat{\mathrm{g}}, E^m, D, \bar{a}^m, \bar{q}) - \mathbb{D}\eta, \\ \hat{\xi} &= \xi^m - \mathrm{Ad}_{\hat{\mathrm{g}}^{-1}}\eta, \\ \dot{\hat{\mathrm{g}}} &= \hat{\mathrm{g}}\hat{\xi}^\vee, \end{cases} \tag{34}$$

where $\mathrm{ad}_\zeta^* = (\mathrm{ad}_\zeta)^{\mathrm{T}}$ and ad_ζ is defined as

$$\mathrm{ad}_\zeta = \begin{bmatrix} w^\times & 0 \\ v^\times & w^\times \end{bmatrix} \text{ for } \zeta = \begin{bmatrix} w \\ v \end{bmatrix} \in \mathbb{R}^6, \tag{35}$$

and

$$Z(\hat{\mathrm{g}}, E^m, D, \bar{a}^m, \bar{q}) = \begin{bmatrix} \Phi'\Big(\mathcal{U}_r^0(\cdots)\Big)S_K(\hat{R}) + \alpha(\bar{p})^\times y \\ \alpha y \end{bmatrix}, \tag{36}$$

where $\mathcal{U}_r^0(\cdots) = \mathcal{U}_r^0(\hat{\mathrm{g}}, E^m, D)$ is defined in (29), $y \equiv y(\hat{\mathrm{g}}, \bar{a}^m, \bar{q})$ is defined in (28), and

$$S_K(\hat{R}) = \mathrm{vex}\big(DW(E^m)^{\mathrm{T}}\hat{R}^{\mathrm{T}} - \hat{R}(L^m)WD^{\mathrm{T}}\big), \tag{37}$$

where $\mathrm{vex}(\cdot) : \mathfrak{so}(3) \to \mathbb{R}^3$ is the inverse of the $(\cdot)^\times$ map and $\mathbb{D} \in \mathbb{R}^{6\times 6} > 0$.

In our prior work [17], this estimator was shown to be almost globally asymptotically stable, with convergence to zero errors from almost all initial estimates except for those in a set of (volume) measure zero in the state space $\mathrm{TSE}(3)$. In practice, the presence of measurement noise leads to ultimate boundedness of estimation errors, where the bounds are determined by observer gains.

D. VPE for Rigid Body Motion in Discrete Time

The variational pose estimation scheme presented in Proposition 1 is discretized for implementation on onboard computers. It is discretized using the framework of discrete geometric mechanics, and the resulting discrete-time estimator is obtained in the form of a Lie group variational integrator (LGVI) [32]. The advantages of LGVIs are twofold: they discretize the motion on the Lie group without the need for local maps or projection, and they are variational in nature which implies preservation of energy-momentum properties of the continuous dynamics [33], [34].

Since the estimation scheme in Proposition 1 is obtained from a variational principle of mechanics, it can be discretized by applying the discrete Lagrange–d'Alembert principle [35]. Consider an interval of time $[t_0, T] \in \mathbb{R}^+$ separated into N equal-length subintervals $[t_i, t_{i+1}]$ for $i = 1, 2, \ldots, N$ with $t_N = T$ and $\Delta t = t_{i+1} - t_i$ is the time step size. Then, an LGVI is applied to the following discrete-time form of the Lagrangian (33) to obtain the discrete-time estimator. Therefore, the discrete-time variational pose and velocities estimator corresponding to the continuous-time estimator given by Proposition 1, is obtained as

$$
\begin{aligned}
&(J\omega_i)^\times = \frac{1}{\Delta t}\big(F_i\mathcal{J} - \mathcal{J}F_i^{\mathrm{T}}\big),\\
&(M + \Delta t\mathbb{D}_t)\upsilon_{i+1} = F_i^{\mathrm{T}} M\upsilon_i\\
&\quad + \Delta t\alpha(\hat{b}_{i+1} + \hat{R}_{i+1}\bar{a}^m_{i+1} - \bar{p}_{i+1}),\\
&(J + \Delta t\mathbb{D}_r)\omega_{i+1} = F_i^{\mathrm{T}} J\omega_i + \Delta t M\upsilon_{i+1} \times \upsilon_{i+1}\\
&\quad + \Delta t\alpha\bar{p}^\times_{i+1}(\hat{b}_{i+1} + \hat{R}_{i+1}\bar{a}^m_{i+1})\\
&\quad - \Delta t\Phi'\Big(\mathcal{U}^0_r(\hat{\mathrm{g}}_{i+1}, E^m_{i+1}, D_{i+1})\Big)S_{K_{i+1}}(\hat{R}_{i+1}),\\
&\hat{\xi}_i = \xi^m_i - \mathrm{Ad}_{\hat{\mathrm{g}}_i^{-1}}\eta_i,\\
&\hat{\mathrm{g}}_{i+1} = \hat{\mathrm{g}}_i \exp(\Delta t\hat{\xi}_i^\vee),
\end{aligned}
\tag{38}
$$

where $F_i \in \mathrm{SO}(3)$, $\mathcal{J}$ is defined in terms of the matrix J as $\mathcal{J} = \frac{1}{2}\mathrm{trace}[J]I - J$, M is a positive definite matrix, $\varphi_i = [\omega_i^{\mathrm{T}}\ \upsilon_i^{\mathrm{T}}]^{\mathrm{T}}$, $\big(\hat{\mathrm{g}}(t_0), \hat{\xi}(t_0)\big) = (\hat{\mathrm{g}}_0, \hat{\xi}_0)$ and $S_{K_i}(\hat{R}_i)$ is the value of $S_K(\hat{R})$ at time t_i, where $S_K(\hat{R})$ is defined by (37). $\mathbb{D}_t$ and $\mathbb{D}_r$ are positive definite matrices used in two Rayleigh dissipation terms linear in the translational velocity and angular velocity estimation errors, respectively. This discrete-time version of the VPE was first obtained in [17]. It is implemented using a vision-inertial sensor, as reported in the following section.

IV. Experimental Results for VPE using Vision and Inertial Sensing

This section outlines results from an experimental evaluation of the variational pose estimator (VPE) in discrete-time, given by (38). For our experiment, we used the ZED 2i stereo camera sensor for onboard implementation, and the Vicon motion capture system for external verification. The instantaneous vector measurements for VPE in the form of point clouds are obtained from the ZED stereo camera. These vector measurements are the feature points in 3D space that are observed by the stereo camera. The Scale-Invariant Feature Transform (SIFT) is the feature matching algorithm in OpenCV that is used for identifying and matching features points between two successive camera frames. Ground truth in the form of pose measurements are obtained from the Vicon motion capture system. The VPE-generated estimates are then compared with the Vicon measurements and are shown in time plots here. The experiment is conducted for 43 s with a time step size of $\Delta t = 1/15$ s, while the ZED camera is operating at a frequency of 15 Hz.

Simulation parameters for this experiment are given here. The initial attitude and position estimates are

$$
\hat{R}_0 = \exp_{\mathrm{SO}(3)}\left(\frac{\pi}{4} \times \begin{bmatrix} \frac{3}{7} & \frac{6}{7} & \frac{2}{7} \end{bmatrix}^{\mathrm{T}}\right)^\times \text{ and } \hat{b}_0 = \begin{bmatrix} 2.5 \\ 0.5 \\ -1.8 \end{bmatrix},
\tag{39}
$$

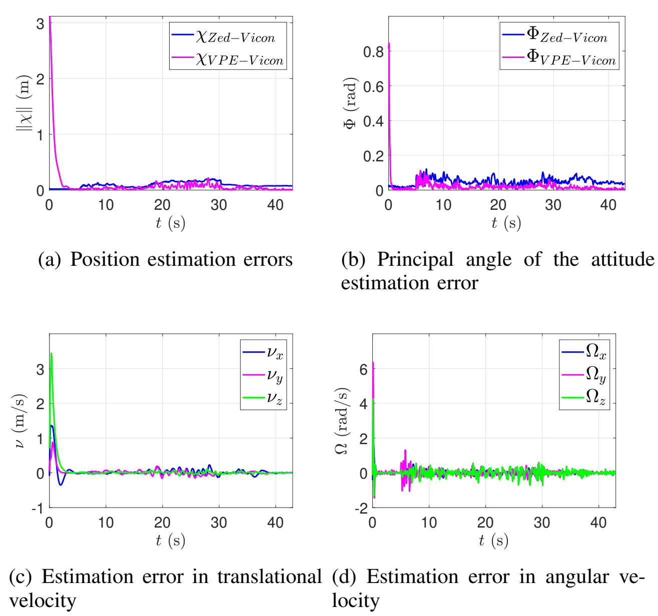
respectively. The initial angular and translational velocities, respectively, are

$$
\hat{\Omega}_0 = [-0.41976\ \ -0.1305\ \ -0.1305]^{\mathrm{T}} \text{ rad/s}, \tag{40}
$$

and $\hat{\nu}_0 = [-0.09\ 0.05\ 0.7]^{\mathrm{T}}$ m/s. The positive definite estimator gain matrices are:

$$
\begin{aligned}
J &= \mathrm{diag}([2.1\ 2.0\ 1.6]),\\
M &= \mathrm{diag}([0.095\ 0.037\ 0.019]),\\
\mathbb{D}_r &= \mathrm{diag}([59.01\ 42.63\ 25.86]),\\
\mathbb{D}_t &= \mathrm{diag}([0.184\ 0.294\ 0.357]).
\end{aligned}
\tag{41}
$$

Fig. 1(a) shows a comparison between position estimation errors from VPE and unfiltered data from the ZED camera, by taking the respective differences from the ground truth given by the Vicon mo-cap system. The position estimation error from VPE converges from non-zero initial conditions to a bounded neighborhood of the true states. The attitude estimation error, parameterized by the principal rotation angle $\Phi = \cos^{-1}(\frac{1}{2}(\mathrm{tr}(Q) - 1))$ of the attitude estimation error matrix $Q \in \mathrm{SO}(3)$, is shown in Fig. 1(b). The principal angle from VPE converges from the given initial condition to a bounded neighborhood of zero. Figs. 1(c) and 1(d) show the estimation errors in translational and angular velocity (as obtained by the VPE from the point cloud measurements) with time, respectively. These errors are also seen to converge to small neighborhoods of zero errors.



(a) Position estimation errors

(b) Principal angle of the attitude estimation error

(c) Estimation error in translational velocity

(d) Estimation error in angular velocity

Fig. 1: Experimental results

V. Conclusion

This paper describes variational estimators for state estimation of systems evolving on finite-dimensional Lie groups as their configuration manifolds. These variational estimators are designed by applying the Lagrange-d'Alembert principle to a Lagrangian consisting of energy-like quantities in the state estimation errors and dissipative terms that dissipate this total energy. The resulting observer estimates the configuration and velocity states from vector measurements by using the underlying structure of the Lie group. A particular application is given for the Lie group of rigid body motions, from our prior published research. The variational pose estimator (VPE) is formulated in both continuous time and discrete time. The VPE in discrete time is then experimentally validated using a stereo camera and a Vicon motion capture system. It is seen that state estimates converge to a bounded neighborhood of the true states. These experimental results corroborate the theory behind design of this variational estimation scheme. Future work will show the nonlinear stability of this general variational estimator design for finite-dimensional Lie groups.

Acknowledgement

The authors acknowledge support from U.S. NSF grants 2132799 and 2343062.

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