2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

A Variational Observer on Spheres and its Application to Pointing Direction Motion Estimation

Neon Srinivasu, Zijian He, and Amit K. Sanyal[1]

2024 IEEE 63rd Conference on Decision and Control (CDC) | 979-8-3503-1633-9/24/$31.00 ©2024 IEEE | DOI: 10.1109/CDC56724.2024.10886583

Abstract— **We consider estimation of motion on spheres as a variational problem. The concept of variational estimation for mechanical systems is based on application of variational principles from mechanics, to state estimation of mechanical systems evolving on configuration manifolds. If the configuration manifold is a symmetric space, then the overlying connected Lie group of which it is a quotient space, can be used to design nonlinearly stable observers for estimation of configuration and velocity states from measurements. If the configuration manifold is a sphere, then it can be globally represented by an unit vector. We illustrate the design of variational observers for mechanical systems evolving on spheres, through its application to estimation of pointing directions (reduced attitude) on the regular sphere $\mathbb{S}^2$.**

I. Introduction

We present the design of a state estimation scheme based on variational principles used in geometric mechanics, for *reduced attitude* state estimation. Specifically, this estimation scheme can be applied to pointing direction estimation of a body-fixed sensor on a rigid body in spatial motion. More generally, the observer design principles outlined here are applicable to mechanical systems evolving on Lie groups or symmetric spaces as configuration manifolds. Considering the geometry of the configuration manifold becomes necessary, especially when this manifold is not contractible. In particular, this is true for the Lie group of rigid body attitudes $\mathrm{SO}(3)$, and the space of reduced attitudes (pointing directions) $\mathbb{S}^2$.

Geometric mechanics-based state estimation schemes that account for the geometry of the configuration space, have appeared in research publications over the last twenty years. In particular, some of these estimation schemes have explicitly been designed using principles of geometric mechanics. The design of observers for systems on connected, locally compact, non-trivial, finite-dimensional Lie groups, symmetric spaces, and semi-direct products of such groups, is further complicated by the fact that these groups are not contractible (see, e.g., [1]–[3]). This means that any smooth observer or controller design on such Lie groups have multiple equilibria, which in turn precludes global asymptotic stability of any one equilibrium. This has been pointed out for the Lie group of rigid body rotations, $\mathrm{SO}(3)$, in prior research (e.g., [4]–[6]), as well as for the regular sphere $\mathbb{S}^2$ in [5]. Smooth estimation schemes on these Lie groups, and consequently on symmetric spaces obtained from them, can (at best) be almost globally asymptotically stable (AGAS).

Due to these considerations, an increasing number of estimation schemes have been designed directly on the Lie groups of rigid body rotations $\mathrm{SO}(3)$ and rigid body motions $\mathrm{SE}(3)$ over the last twenty years or so. Early research in this direction extended traditional approaches like attitude determination using Wahba's problem, complementary filters, gradient-based observers, set-bounded filters and unscented filters; a sample of this literature can be found in [7]–[15]. Besides accounting for the geometry of the configuration space, these estimation schemes also take advantage of the symmetry properties of Lie groups in their design in continuous time (e.g., [12], [14]) or in discrete time (e.g., [6], [16]). Published research over the last decade on this topic has applied concepts like near-optimal filtering [17]–[19], invariant filter design [20]–[22], variational observer design [16], [23], stochastic filtering and Bayesian estimation [24], [25], and cascaded observer design [26].

These techniques for estimation on Lie groups can easily be adapted for estimation of systems evolving on symmetric spaces as configuration manifolds. Although the current published literature on estimation on Lie groups (and their tangent bundles) is quite voluminous, the literature on observer designs on symmetric spaces is scant. Existing literature on control design on symmetric spaces is less scant, although not quite as voluminous as either controller or estimator design problems on Lie groups. Pointing direction control, also termed *reduced attitude* control, has been treated in prior literature using the tools of geometric mechanics; see, e.g., [5], [27]. Variational and optimal control problems on Stiefel manifolds, which are also symmetric spaces, were treated in [28], [29].

The design of variational observers is based on the Lagrange-d'Alembert principle from variational mechanics, applied to state estimation problems for mechanical systems [30]. In this work, we start with a general treatment of variational problems on spheres $\mathbb{S}^n$ for integer $n > 1$. This is followed by a treatment of variational observer design on the regular sphere $\mathbb{S}^2$. Among symmetric spaces, the sphere $\mathbb{S}^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$ is of great interest due to its application to pointing direction estimation and control. We focus our attention on variational observer design on the tangent bundle of the sphere. Besides its application to pointing direction estimation, the sphere is an interesting case study as the simplest non-trivial example of a symmetric space.

The remainder of this paper is organized as follows. In section II, we provide a background of a variational problem set up on the sphere $\mathbb{S}^{n-1} \equiv \mathrm{SO}(n)/\mathrm{SO}(n-1)$. Section III obtains a continuous-time variational observer design on the

[1] Department of Mechanical and Aerospace Engineering, Syracuse University, Syracuse, NY 13244, USA. neonsrin, zhe135, aksanyal@syr.edu

sphere $\mathbb{S}^2$ applying the Lagrange-d'Alembert principle. This is followed by section III-B, which designs the variational observer for reduced attitude and angular velocity in discrete time as a Lie group variational integrator [31]. Section IV provides numerical simulation results for the discrete-time variational observer. These simulation results provide numerical validation of the proposed estimation scheme. Section V concludes this paper with a summary of the results obtained here, and possible future research directions related to this work.

II. A Variational Problem on the Sphere $\mathbb{S}^{n-1}$

Consider the sphere $\mathbb{S}^{n-1}$ that can be expressed as the space of unit vectors in the vector space $\mathbb{R}^n$. Therefore, if $\Gamma \in \mathbb{R}^n$ is used to represent the configuration on the sphere, then $\mathbb{S}^{n-1}$ is defined by:

$$\mathbb{S}^{n-1} := \left\{ \Gamma \in \mathbb{R}^n \,|\, \Gamma^{\mathrm{T}}\Gamma = 1 \right\}. \tag{1}$$

The sphere $\mathbb{S}^{n-1}$ is obtained as the symmetric space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, where the Lie group $\mathrm{SO}(n)$ is defined as the set of proper orthogonal $n \times n$ matrices:

$$\mathrm{SO}(n) := \left\{ Q \in \mathbb{R}^{n\times n} \,|\, Q^{\mathrm{T}}Q = I_n \right\},$$

where I_n denotes the $n \times n$ identity matrix, which is also the identity element on $\mathrm{SO}(n)$. The Lie algebra of $\mathrm{SO}(n)$, denoted $\mathfrak{so}(n)$, is defined by:

$$\mathfrak{so}(n) := \left\{ U \in \mathbb{R}^{n\times n} \,|\, U^{\mathrm{T}} + U = 0_n \right\},$$

where 0_n is the $n \times n$ zero matrix consisting of all zero entries. Clearly, this makes $\mathfrak{so}(n)$ identical to the vector space of $n \times n$ skew-symmetric matrices.

Now consider a trajectory (an integral curve) on $\mathrm{SO}(n)$, given by $Q : \mathbb{R} \to \mathrm{SO}(n)$, with $Q(t) \in \mathrm{SO}(n)$ corresponding to a value of $t \in \mathbb{R}$. This leads to a trajectory on $\mathrm{TSO}(n) \equiv \mathrm{SO}(n) \times \mathfrak{so}(n)$ given by:

$$\dot{Q} := \frac{\mathrm{d}Q}{\mathrm{d}t} = QU, \text{ where } U \in \mathfrak{so}(n), \tag{2}$$

where we omit the variable parameter t in the trajectories for Q and $U \in \mathfrak{so}(n)$ in eq. (2) for notational convenience. This also induces a trajectory on $\mathbb{S}^{n-1}$ and $\mathrm{T}\mathbb{S}^{n-1}$ as follows. If $p \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ is a known fixed (constant) unit vector, then a trajectory on the sphere $\mathbb{S}^{n-1}$ corresponding to this trajectory on $\mathrm{SO}(n)$, is given by:

$$\Gamma(t) = Q(t)^{\mathrm{T}} p. \tag{3}$$

Therefore, the tangent vector to this trajectory at $\Gamma(t)$, where $t \in \mathbb{R}$, is given by:

$$\dot{\Gamma} := \frac{\mathrm{d}\Gamma}{\mathrm{d}t} = -U\Gamma, \text{ where } U \in \mathfrak{so}(n). \tag{4}$$

Eq. (4) can be used to generate the integral curve given the value of Γ at an instant, say $\Gamma(0) = \Gamma_0$. It is easy to verify that the trajectory given by eq. (3) is on $\mathbb{S}^{n-1}$ and the tangent vector given by eq. (4) is indeed on $\mathrm{T}_{\Gamma(t)}\mathbb{S}^{n-1}$.

A trajectory on $\mathbb{S}^{n-1}$ can be obtained in an optimal manner by setting up a Lagrangian for a corresponding variational problem. Consider the kinetic energy-like term given by:

$$\mathcal{T}(U) = \langle U, U\Lambda \rangle \text{ where } \Lambda = \Lambda^{\mathrm{T}} \succ 0. \tag{5}$$

Here $\langle \cdot, \cdot \rangle : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a suitable inner product and the positive definite matrix $\Lambda \in \mathbb{R}^{n\times n}$ ensures that this function is positive definite. This, along with eq. (4), also makes $\mathcal{T}(U)$ a metric on $\mathbb{S}^{n-1}$. In addition, a potential energy-like function on $\mathbb{S}^{n-1}$ is defined by:

$$\mathcal{U}(\chi, \Gamma) = \kappa(1 - \chi^{\mathrm{T}}\Gamma) \text{ where } \kappa > 0, \tag{6}$$

and $\chi \in \mathbb{S}^{n-1}$ is a known reference point or trajectory on $\mathbb{S}^{n-1}$. This potential function has a maximum of 2κ when $\Gamma = -\chi$ and a minimum of 0 when $\Gamma = \chi$, which makes it a positive definite function of the "error" between Γ and χ on $\mathbb{S}^{n-1}$. In fact, $\mathcal{U}(\chi, \Gamma)$ defines a measure of the error between two configurations (or elements) χ and Γ on $\mathbb{S}^{n-1}$. Define a Lagrangian from the energy-like functions as:

$$\mathcal{L}(U, \chi, \Gamma) = \mathcal{T}(U) - \mathcal{U}(\chi, \Gamma) = \langle U, U\Lambda \rangle - \kappa(1 - \chi^{\mathrm{T}}\Gamma). \tag{7}$$

This Lagrangian is used to form the action functional:

$$\mathcal{S}(\mathcal{L}(U, \chi, \Gamma)) = \int_0^T \mathcal{L}(U, \chi, \Gamma)\mathrm{d}t. \tag{8}$$

Extremal values of this action functional generate smooth trajectories on $\mathbb{S}^{n-1}$ that satisfy eq. (4). Extremization of this action involves taking *reduced variations* on $\mathbb{S}^{n-1}$ for the states $(\Gamma, \dot{\Gamma}) \in \mathrm{T}\mathbb{S}^{n-1}$, or alternately, $(\Gamma, U) \in \mathbb{S}^{n-1} \times \mathfrak{so}(n)$. Similar approaches, without the potential energy-like term given by eq. (6), have been used in the past to obtain geodesics on spheres and other symmetric spaces [28], [29], [32]. Although this Lagrangian is not directly related to the variational estimation problem on $\mathbb{S}^2$ that is analyzed and solved in the remainder of this paper, it sets up the basic framework that we use.

III. Variational Observer Design on the Sphere $\mathbb{S}^2$

As mentioned in the introduction §I, state estimation on the regular sphere $\mathbb{S}^2$ has applications to pointing direction estimation and control. Here we explore the state estimation problem, exploiting the variational framework set up in the previous section, §II.

A. Reduced attitude state observer in continuous time

A pointing direction fixed to a rotating rigid body in Euclidean three-dimensional space, is often referred to as its *reduced attitude* [5]. Reduced attitude can be defined in one of two ways: an inertially fixed unit vector expressed in a body-fixed coordinate frame, or a body-fixed unit vector expressed in an inertial frame. We follow the first definition, which is relevant for applications like a space telescope required to point at distant stars or star clusters. Let $p \in \mathbb{S}^2 \subset \mathbb{R}^3$ denote a unit vector along an inertially-fixed direction.

Then the reduced attitude representing this direction in the body-fixed frame is defined as:

$$\Gamma = R^{\mathrm{T}} p \in \mathbb{S}^2, \tag{9}$$

where $R \in \mathrm{SO}(3)$ denotes the body's (full) attitude, given by the rotation matrix from the body-fixed frame to the inertial frame. If $R(t)$ denotes a trajectory on $\mathrm{SO}(3)$, the tangent vector at $R(t)$ is given by:

$$\dot{R} = R\Omega^{\times}, \text{ where } \Omega^{\times} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \in \mathfrak{so}(3), \tag{10}$$

for $\Omega = [\Omega_1\ \Omega_2\ \Omega_3]^{\mathrm{T}} \in \mathbb{R}^3$. Here $(\cdot)^{\times} : \mathbb{R}^3 \to \mathfrak{so}(3)$ is the vector space isomorphism from 3D vectors to 3×3 skew-symmetric matrices, and the cross superscript is because this is equivalent to the cross product operator matrix, i.e., $a^{\times} b = a \times b$ for vectors $a, b \in \mathbb{R}^3$. The time rate of change of reduced attitude given in (9) is therefore given by the kinematic relation:

$$\dot{\Gamma} = \dot{R}^{\mathrm{T}} p = (R\Omega^{\times})^{\mathrm{T}} p = -\Omega^{\times} R^{\mathrm{T}} p = \Gamma^{\times}\Omega, \tag{11}$$

where $\Omega \in \mathbb{R}^3 \equiv \mathfrak{so}(3)$ denotes the angular velocity of the body as measured in the body-fixed frame. Note that eqs. (10) and (11) correspond to eqs. (2) and (4) in §II, respectively.

We assume that measurements of Γ and Ω are available, and a variational observer (deterministic) is designed for these states. Denote the estimate of the reduced attitude by

$$\hat{\Gamma} \in \mathbb{S}^2, \text{where } \hat{\Gamma} = \hat{R}^{\mathrm{T}} p \tag{12}$$

and $\hat{R} \in \mathrm{SO}(3)$ is the (full) attitude estimate, i.e., the estimated rotation matrix from body to inertial frame. *Note that we are not estimating the full attitude here, but just using it as a convenient tool for certain steps in our derivation of the observer for reduced attitude and angular velocity.* Therefore, the velocity kinematics of the estimated reduced attitude is:

$$\dot{\hat{\Gamma}} = -\hat{\Omega}^{\times} \hat{R}^{\mathrm{T}} p = \hat{\Gamma}^{\times}\hat{\Omega} \tag{13}$$

where, $\hat{\Omega} \in \mathbb{R}^3 \equiv \mathfrak{so}(3)$ is the estimated angular velocity vector. The error in estimating the reduced attitude can be represented by the following potential energy-like function:

$$\mathcal{U}(\Gamma, \hat{\Gamma}) = k(1 - \Gamma^{\mathrm{T}}\hat{\Gamma}), \text{where } k > 0. \tag{14}$$

The function $\mathcal{U}$ in eq. (14) has two critical points: a minimum of zero when $\hat{\Gamma} = \Gamma$ and a maximum of $2k$ when $\hat{\Gamma} = -\Gamma$. The time derivative of eq. (14) is given by:

$$\frac{d}{dt}\mathcal{U}(\Gamma, \hat{\Gamma}) = -k(\dot{\Gamma}^{\mathrm{T}}\hat{\Gamma} + \Gamma^{\mathrm{T}}\dot{\hat{\Gamma}}). \tag{15}$$

Using eqs. (11) and (13) in eq. (15) and simplifying gives,

$$\dot{\mathcal{U}}(\Gamma, \hat{\Gamma}) = -k\Gamma^{\mathrm{T}}\hat{\Gamma}^{\times}\tilde{\Omega}, \tag{16}$$

where $\tilde{\Omega} := \hat{\Omega} - \Omega$.

Proposition 1: First variations of $\Gamma, \hat{\Gamma}$ in $\mathbb{S}^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$, a symmetric space, are of the form:

$$\delta\Gamma = \Gamma^{\times}\Sigma \text{ and } \delta\hat{\Gamma} = \hat{\Gamma}^{\times}\hat{\Sigma}, \tag{17}$$

where $\Sigma,\ \hat{\Sigma} \in \mathfrak{so}(3)$, are variation vector fields on the Lie algebra $\mathfrak{so}(3)$.

Note that the variations given by eq. (17) are on the respective tangent spaces at $\Gamma, \hat{\Gamma} \in \mathbb{S}^2$, i.e., $\Gamma^{\mathrm{T}}\delta\Gamma = 0$ and $\hat{\Gamma}^{\mathrm{T}}\delta\hat{\Gamma} = 0$. Using eqs. (14) and (17), the first variation in $\mathcal{U}(\Gamma, \hat{\Gamma})$ is obtained as,

$$\delta\mathcal{U}(\Gamma, \hat{\Gamma}) = -k\Gamma^{\mathrm{T}}\hat{\Gamma}^{\times}\tilde{\Sigma} \text{ where } \tilde{\Sigma} = \hat{\Sigma} - \Sigma \tag{18}$$

Theorem 1: The observer equations for the variational estimator for reduced attitude and angular velocity in continuous time are given by

$$\begin{aligned} J\dot{\tilde{\Omega}} &= -\Omega^{\times} J\tilde{\Omega} - D\tilde{\Omega} - k\hat{\Gamma}^{\times}\Gamma \\ \hat{\Omega} &= \tilde{\Omega} + \Omega,\ \hat{\Omega}(0) \text{ is known,} \\ \dot{\hat{\Gamma}} &= \hat{\Gamma}^{\times}\hat{\Omega},\ \hat{\Gamma}(0) \text{ is known.} \end{aligned} \tag{19}$$

Proof: Let the estimation error in angular velocity be represented by the kinetic energy-like term:

$$\mathcal{T}(\tilde{\Omega}) = \frac{1}{2}\tilde{\Omega}^{\mathrm{T}} J\tilde{\Omega} \text{ where } J = J^{\mathrm{T}} \succ 0 \tag{20}$$

We now relate the first variation in $\tilde{\Omega}$ with the first variation $\tilde{\Sigma}$ of the reduced attitude estimation error. Note that if the full attitude estimation error were to be defined as $Q = R\hat{R}^{\mathrm{T}} \in \mathrm{SO}(3)$, so that $\Gamma^{\mathrm{T}}\hat{\Gamma} = p^{\mathrm{T}}Qp$, then the first variation in $\Gamma^{\mathrm{T}}\hat{\Gamma}$ (alternately $\mathcal{U}(\Gamma, \hat{\Gamma})$) relates to the first variation in Q (because $p \in \mathbb{S}^2$ is constant). This observation leads to:

$$\delta\tilde{\Omega} = \dot{\tilde{\Sigma}} + \tilde{\Omega}^{\times}\tilde{\Sigma},\ \tilde{\Sigma} = \hat{\Sigma} - \Sigma, \tag{21}$$

and therefore $\delta\mathcal{T}(\tilde{\Omega}) = \tilde{\Omega}^{\mathrm{T}} J\delta\tilde{\Omega} = \tilde{\Omega}^{\mathrm{T}} J(\dot{\tilde{\Sigma}} + \tilde{\Omega}^{\times}\tilde{\Sigma})$. An action functional is created from the energy-like terms in eqs. (14) and (20), as follows:

$$\mathcal{S}(\mathcal{L}(\tilde{\Omega}, \Gamma, \hat{\Gamma})) = \int_0^T \left[\frac{1}{2}\tilde{\Omega}^{\mathrm{T}} J\tilde{\Omega} - k(1 - \Gamma^{\mathrm{T}}\hat{\Gamma})\right] dt \tag{22}$$

where $[0, T]$ is the time interval over which this action integral is defined. Thereafter, we apply the Lagrange-d'Alembert principle to this action integral, along with a dissipation term that is linear in the angular velocity estimation error. This term can be obtained from a Rayleigh dissipation function [30], [33], and is used to dissipate the energy in the reduced attitude and angular velocity estimation errors. The variational observer is obtained by applying the Lagrange-d'Alembert principle, as follows:

$$\delta\mathcal{S}(\mathcal{L}(\tilde{\Omega}, \Gamma, \hat{\Gamma})) = \int_0^T \tilde{\Sigma}^{\mathrm{T}} D\tilde{\Omega}\, dt \tag{23}$$

where $D = D^{\mathrm{T}} \succ 0$ and $D\tilde{\Omega}$ is the dissipation term. Applying reduced variations for $\Gamma^{\mathrm{T}}\hat{\Gamma}$ and $\tilde{\Omega}$ as outlined earlier with fixed endpoints, we get:

$$\begin{aligned} &\int_0^T \left[\tilde{\Omega}^{\mathrm{T}} J(\dot{\tilde{\Sigma}} + \tilde{\Omega}^{\times}\tilde{\Sigma}) + k\Gamma^{\mathrm{T}}\hat{\Gamma}^{\times}\tilde{\Sigma} - \tilde{\Omega}^{\mathrm{T}} D\tilde{\Sigma}\right] dt = 0 \\ &\implies \tilde{\Omega}^{\mathrm{T}} J\tilde{\Sigma}\Big|_0^T - \int_0^T \dot{\tilde{\Omega}}^{\mathrm{T}} J\tilde{\Sigma} dt \\ &+ \int_0^T \left[\tilde{\Omega}^{\mathrm{T}}\Big\{J(\tilde{\Omega}^{\times}\tilde{\Sigma}) - D\tilde{\Sigma}\Big\} + k\Gamma^{\mathrm{T}}\hat{\Gamma}^{\times}\tilde{\Sigma}\right] dt \end{aligned} \tag{24}$$

Due to fixed endpoints, $\tilde{\Sigma}(0) = \tilde{\Sigma}(T) = 0$. Substituting this into eq. (24) gives,

$$\int_0^T \tilde{\Sigma}^{\mathrm{T}} \Big[- J\dot{\tilde{\Omega}} - \tilde{\Omega}^{\times} J\tilde{\Omega} - D\tilde{\Omega} - k\hat{\Gamma}^{\times}\Gamma \Big] dt = 0 \quad (25)$$

As $\tilde{\Sigma}(t)$ is variable for $t \in (0, T)$, eq. (25) leads to the result. ■

Eq. (19) describes the continuous-time variational observer for reduced attitude and angular velocity states on $\mathrm{T}\mathbb{S}^2$. The observer in eqs. (19) is initialized by selecting appropriate initial state estimates $(\hat{\Gamma}(0), \hat{\Omega}(0)) \in \mathbb{S}^2 \times \mathbb{R}^3$. Moreover using the total energy function

$$\mathcal{E}(\tilde{\Omega}, \Gamma, \hat{\Gamma}) = \mathcal{T}(\tilde{\Omega}) + \mathcal{U}(\Gamma, \hat{\Gamma}) \quad (26)$$

as a Lyapunov function and applying a generalization of the invariance principle (given by Theorem 8.4 of Khalil), it can be readily shown that the observer is almost globally asymptotically stable (AGAS) at $(\hat{\Gamma}, \hat{\Omega}) = (\Gamma, \Omega)$ and unstable at $(\hat{\Gamma}, \hat{\Omega}) = (-\Gamma, \Omega)$, that is, it converges to the true states in an AGAS manner.

B. Discrete time variational observer design for reduced attitude and angular velocity

A discrete-time version of the observer in (19) is obtained in the form of a Lie group variational integrator (LGVI). In addition to maintaining the energy-momentum properties of the continuous-time observer, this LGVI scheme also preserves the structure of the configuration space of the system. Readers are directed to, e.g., [6], [31], for use of LGVI to discretize systems obtained from variational principles. We discretize the Lagrangian assuming a fixed time step size Δt between measurements:

$$\mathcal{L}(\Gamma_i, \hat{\Gamma}_i, \tilde{\Omega}_i) = \frac{1}{2}\tilde{\Omega}_i^{\mathrm{T}} J\tilde{\Omega}_i - k(1 - \Gamma_i^{\mathrm{T}}\hat{\Gamma}_i), \quad (27)$$

for $i = 0, 1, 2, ..., N-1$, over the time interval $[0, T]$. The discrete action sum is defined as:

$$\begin{aligned} \mathcal{S}_d(\mathcal{L}(...)) &= \Delta t \sum_{i=0}^{N-1} \mathcal{L}(\Gamma_i, \hat{\Gamma}_i, \tilde{\Omega}_i) \\ &= \Delta t \sum_{i=0}^{N-1} \Big[\frac{1}{2}\tilde{\Omega}_i^{\mathrm{T}} J\tilde{\Omega}_i - k(1 - \Gamma_i^{\mathrm{T}}\hat{\Gamma}_i) \Big] \end{aligned} \quad (28)$$

Discretize the reduced attitude kinematics on $\mathbb{S}^2$ as:

$$\hat{\Gamma}_{i+1} = \hat{F}_i^{\mathrm{T}}\hat{\Gamma}_i,\ \Gamma_{i+1} = F_i^{\mathrm{T}}\Gamma_i,\ \text{where } F_i, \hat{F}_i \in \mathrm{SO}(3). \quad (29)$$

Considering again $\Gamma^{\mathrm{T}}\hat{\Gamma} = p^{\mathrm{T}}Qp$ where Q is the full attitude estimation error, the kinematic relation in (29) corresponds to $Q_{i+1} = Q_i\tilde{F}_i$ where $\tilde{F}_i = F_i\hat{F}_i^{\mathrm{T}}$. With first variations of $\Gamma_i, \hat{\Gamma}_i$ (and therefore, Q_i) defined as in the continuous-time system, this leads to the first variation in $\tilde{F}_i$ as follows:

$$\begin{aligned} &\delta\tilde{F}_i = -\tilde{\Sigma}_i^{\times}\tilde{F}_i + \tilde{F}_i\tilde{\Sigma}_{i+1}^{\times}, \text{ and} \\ &\delta(\Gamma_i^{\mathrm{T}}\hat{\Gamma}_i) = (\Gamma_i^{\times}\Sigma_i^{\times})^{\mathrm{T}}\hat{\Gamma}_i + \Gamma_i^{\mathrm{T}}(\hat{\Gamma}_i^{\times}\hat{\Sigma}_i) = \Gamma_i^{\mathrm{T}}\hat{\Gamma}_i^{\times}\tilde{\Sigma}_i \end{aligned} \quad (30)$$

where $\tilde{\Sigma}_i = \hat{\Sigma}_i - \Sigma_i$. In addition, we discretize the angular momentum as follows:

$$\begin{aligned} (J\tilde{\Omega}_i)^{\times} &= \tilde{\Omega}_i^{\times}\mathcal{J} + \mathcal{J}\tilde{\Omega}_i^{\times} \\ &\approx \frac{1}{\Delta t}[(\tilde{F}_i - I)\mathcal{J} - \mathcal{J}(\tilde{F}_i^{\mathrm{T}} - I)] \\ &= \frac{1}{\Delta t}\big(\tilde{F}_i\mathcal{J} - \mathcal{J}\tilde{F}_i^{\mathrm{T}}\big) \end{aligned} \quad (31)$$

where I is the identity matrix and $\mathcal{J} = (1/2)\mathrm{tr}(J)I - J$. This leads to the first variation in the discrete angular velocity:

$$(J\delta\tilde{\Omega}_i)^{\times} = \frac{1}{\Delta t}(\delta\tilde{F}_i\mathcal{J} - \mathcal{J}\delta\tilde{F}_i^{\mathrm{T}}) \quad (32)$$

with $\delta\tilde{F}_i$ given in eq. (30). Therefore, the first variation of the energy-like terms are: $\delta\mathcal{U}(\Gamma_i, \hat{\Gamma}_i) = -k\Gamma_i^{\mathrm{T}}\hat{\Gamma}^{\times}\tilde{\Sigma}_i$ and

$$\begin{aligned} &\delta\mathcal{T}(\tilde{\Omega}_i) = \delta\mathcal{T}(\tilde{F}_i) \\ &= \frac{1}{\Delta t^2}\langle \mathcal{J}(\tilde{F}_i - I), \delta\tilde{F}_i \rangle \\ &= \frac{1}{\Delta t^2}\langle \mathcal{J}(\tilde{F}_i - I), \tilde{F}_i\tilde{\Sigma}_{i+1}^{\times} - \tilde{\Sigma}_i^{\times}\tilde{F}_i \rangle \\ &= \frac{1}{2\Delta t^2}\Big[\langle \tilde{\Sigma}_i^{\times}, \mathcal{J}\tilde{F}_i^{\mathrm{T}} \rangle - \langle \tilde{\Sigma}_{i+1}^{\times}, \tilde{F}_i^{\mathrm{T}}\mathcal{J} \rangle \Big] \\ &= \frac{1}{2\Delta t^2}\Big[\langle \tilde{\Sigma}_i^{\times}, \mathcal{J}\tilde{F}_i^{\mathrm{T}} - \tilde{F}_i^{\mathrm{T}}\mathcal{J} \rangle - \langle \tilde{\Sigma}_{i+1}^{\times}, \tilde{F}_i^{\mathrm{T}}\mathcal{J} - \mathcal{J}\tilde{F}_i \rangle \Big] \\ &= \frac{1}{2\Delta t}\Big[\langle \tilde{\Sigma}_{i+1}^{\times}, \tilde{F}_i^{\mathrm{T}}(J\tilde{\Omega}_i)^{\times}\tilde{F}_i \rangle - \langle \tilde{\Sigma}_i^{\times}, (J\tilde{\Omega}_i)^{\times} \rangle \Big] \end{aligned} \quad (33)$$

The first variation of the discrete action sum is obtained as:

$$\begin{aligned} \delta\mathcal{S}_d(...) &= \sum_{i=0}^{N-1} \Big[-\frac{1}{2}\langle \tilde{\Sigma}_i^{\times}, (J\tilde{\Omega}_i)^{\times} \rangle + \frac{1}{2}\langle \tilde{\Sigma}_{i+1}^{\times}, (\tilde{F}_i^{\mathrm{T}}\mathcal{J}\tilde{\Omega}_i)^{\times} \rangle \\ &\quad + k\tilde{\Sigma}_i^{\mathrm{T}}\hat{\Gamma}_i^{\times}\Gamma_i\Delta t \Big] \\ &= \sum_{i=0}^{N-1} \Big[\tilde{\Sigma}_i^{\mathrm{T}}\tilde{F}_i^{\mathrm{T}}\mathcal{J}\tilde{\Omega}_i^{\times} - \tilde{\Sigma}_i^{\mathrm{T}}\mathcal{J}\tilde{\Omega}_i + k\tilde{\Sigma}_i^{\mathrm{T}}\hat{\Gamma}_i^{\times}\Gamma_i\Delta t \Big] \end{aligned} \quad (34)$$

Adding the dissipation term $-D\tilde{\Omega}_i$ and applying the discrete Lagrange-d'Alembert principle, leads to:

$$\delta\mathcal{S}_d(...) - \sum_{i=0}^{N-1} \tilde{\Sigma}_i^{\mathrm{T}} D\tilde{\Omega}_i\Delta t = 0 \quad (35)$$

Substituting eq. (34) in eq. (35) with fixed endpoint variations ($\tilde{\Sigma}_0 = \tilde{\Sigma}_N = 0$), readily leads to the following observer equations for the discrete variational reduced attitude and angular velocity estimation scheme:

$$\begin{aligned} &\hat{\Omega}_i = \tilde{\Omega}_i + \Omega_i,\ \hat{\Omega}_0, \hat{\Gamma}_0 \text{ known}, \\ &\hat{\Gamma}_{i+1} = \exp(-\Delta t\hat{\Omega}_i^{\times})\hat{\Gamma}_i,\ i = 0, 1, \ldots, N-1, \\ &(J\tilde{\Omega}_i)^{\times} = \frac{1}{\Delta t}(\tilde{F}_i\mathcal{J} - \mathcal{J}\tilde{F}_i^{\mathrm{T}}), \\ &\text{where } \mathcal{J} = \frac{1}{2}\mathrm{Tr}[J]\mathcal{I} - J, \text{ and} \\ &(J + D\Delta t)\tilde{\Omega}_{i+1} = \tilde{F}_i^{\mathrm{T}} J\tilde{\Omega}_i + k\Gamma_{i+1}^{\times}\hat{\Gamma}_{i+1}\Delta t. \end{aligned} \quad (36)$$

IV. Simulation Results

This section presents numerical simulation results for the discrete time variational observer design for reduced attitude and angular velocity obtained in Section III-B. These simulation results are provided for a time period of $T = 60$ s, and with a time step size of $\Delta t = 0.01$ s. The vehicle's initial reduced attitude and angular velocity estimates, respectively, are

$$\begin{aligned} \hat{\Gamma}_0 &= [0.1952 \quad -0.9759 \quad 0.0976]^{\mathrm{T}}\mathrm{m}. \\ \text{and } \hat{\Omega}_0 &= [0.2000 \quad -0.0500 \quad 0.1000]^{\mathrm{T}}\mathrm{rad/s}. \end{aligned} \tag{37}$$

Terminal conditions for measured reduced attitude unit vectors, respectively, are

$$\begin{aligned} \Gamma_0 &= [0.5774 \quad -0.5774 \quad 0.5774]^{\mathrm{T}}\mathrm{m}. \\ \text{and } \Gamma_{\mathrm{f}} &= [-0.4082 \quad 0.4082 \quad -0.8165]^{\mathrm{T}}\mathrm{m}. \end{aligned} \tag{38}$$

The control gain for the potential energy-like function is set to $k = 3$. The observer gain matrices $J = \mathrm{diag}([0.0512 \;\; 0.0602 \;\; 0.0596])$ and, $D = \mathrm{diag}([0.2940 \; 0.2625 \; 0.3150])$. We now consider a reduced-attitude maneuver that transfers an initial reduced attitude $\Gamma_0 \in \mathbb{S}^2$ and an initial angular velocity $\Omega_0 = 0 \in \mathbb{R}^3$ to a terminal reduced attitude $\Gamma_f \in \mathbb{S}^2$ and a terminal angular velocity $\Omega_f = 0 \in \mathbb{R}^3$ in the fixed maneuver time $T > 0$. According to Euler's principal rotation theorem, there exists an axis $a \in \mathbb{S}^2$ and an angle $\alpha \in [0, 2\pi)$ that satisfy $e^{\alpha a^\times} = \Gamma_f \Gamma_0^{-1}$. We treat only a rest-to-rest reduced-attitude maneuver that meets the specified boundary conditions given by:

$$\begin{aligned} \Gamma(t) &= e^{\theta(t) a^\times}\Gamma_0 \\ &= [I + a^\times \sin\theta(t) + (a^\times)^2(1 - \cos\theta(t))]\Gamma_0, \\ \Omega(t) &= \dot{\theta}(t) a, \end{aligned} \tag{39}$$

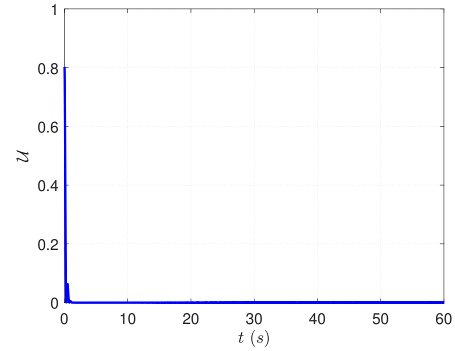
where the rotation angle $\theta : [0, T] \to \mathbb{S}^1$ satisfies

$$\begin{aligned} &\theta(0) = 0, \;\; \dot{\theta}(0) = 0, \\ &\theta(T) = \begin{cases} \alpha, & 0 \le \alpha < \pi \\ -2\pi + \alpha, & \pi \le \alpha < 2\pi \end{cases}, \dot{\theta}(T) = 0. \end{aligned} \tag{40}$$

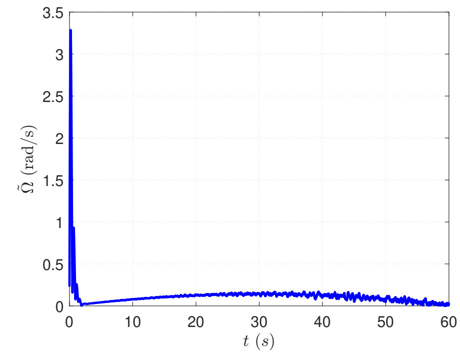
The rotation angle $\theta(t)$ is chosen as a polynomial in time, with coefficients determined to satisfy the initial and terminal boundary conditions in (40). We follow the strategy of a rotation about an inertially-fixed axis, given in [5]. This axis, normalized to lie in $\mathbb{S}^2$, is given by $a = \Gamma_0 \times \Gamma_f / ||\Gamma_0 \times \Gamma_f|| \in \mathbb{S}^2$, assuming Γ_0 and Γ_f are not collinear. The angle $\alpha \in [0, 2\pi)$ satisfies:

$$\cos(\alpha) = \Gamma_0^{\mathrm{T}} \Gamma_f. \tag{41}$$

If Γ_0 and Γ_f are collinear, then these vectors are either equal or differ by a sign; if they are equal, then no attitude maneuver is required, whereas if they differ by a sign, then a is chosen to be an arbitrary direction perpendicular to Γ_0 with $\alpha = \pi$. Simulated measurements in attitude are obtained by adding uniformly distributed random noise in $[0, \pi/240]$ rad to $\theta(t)$ in eq. (39). Angular velocity measurements are corrupted by adding uniformly distributed random noise in $[0, (\pi/240)^2]$ rad/s to $\Omega(t)$ in eq. (39). The results of the simulation are summarized in Fig. 1a and Fig. 1b. The simulated rest-to-rest maneuver occurs over a time period of $T = 60$ s. The plot in Fig. 1a shows the reduced attitude estimation error in terms of the potential energy-like function given in eq. (14) converging to small values close to zero within the simulated period, implying that the rigid body pointing direction converges to the desired pointing direction, Γ_f. Fig. 1b shows the norm of the error between the estimated angular velocity and measured angular velocity vectors of the rigid body converging to zero. These simulation studies support the stability of the proposed estimation scheme as well as its robustness to sensor measurement noise, as estimates converge to small neighborhoods of true states.



(a) Reduced attitude estimation error given by the potential energy-like function with time



(b) Norm of angular velocity estimation error with time

Fig. 1: Simulation Results

V. Conclusion

This article presents a variational observer for reduced attitude (pointing direction) and angular velocity of a rigid body in rotational motion. It starts with an overview of a variational problem on the configuration space of the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$ embedded in $\mathbb{R}^n$. This is then used to formulate the variational observer designed to estimate states on the tangent bundle of the sphere $\mathbb{S}^2$, using a Lagrangian constructed from energy-like terms that quantify the state estimation errors, and applying the Lagrange-d'Alembert principle with a dissipation term to dissipate these energies. This is known to give a nonlinearly stable observer. This is followed by a discrete-time version obtained using the discrete Lagrange-d'Alembert principle. The discrete variational observer equations are in the form of a variational integrator. Numerical simulation results obtained with the discrete observer and simulated measurement noise align with the theoretical stability properties, demonstrating robustness to noise. Future work will provide complete proofs of nonlinear stability and robustness for continuous and discrete time versions of this variational observer.

ACKNOWLEDGEMENT

The authors acknowledge support from U.S. NSF grants 2132799 and 2343062.

REFERENCES

[1] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*. New York: Academic Press New York, 1978.
[2] B. Hoffmann, "A compact contractible topological group is trivial," *Archiv der Mathematik*, vol. 32, no. 1, pp. 585–587, Dec 1979. [Online]. Available: https://doi.org/10.1007/BF01238544
[3] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry (PMS-38)*. Princeton University Press, 1989. [Online]. Available: http://www.jstor.org/stable/j.ctt1bpmb28
[4] S. Bhat and D. Bernstein, "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon," *Systems & Control Letters*, vol. 39, no. 1, pp. 63–70, 2000.
[5] N. Chaturvedi, A. Sanyal, and N. McClamroch, "Rigid body attitude control—Using rotation matrices for continuous, singularity-free control laws," *IEEE Control Systems Magazine*, vol. 31, no. 3, pp. 30–51, 2011.
[6] M. Izadi and A. Sanyal, "Rigid body attitude estimation based on the Lagrange-d'Alembert principle," *Automatica*, vol. 50, no. 10, pp. 2570 – 2577, 2014. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0005109814003112
[7] H. Rehbinder and X. Hu, "Drift-free attitude estimation for accelerated rigid bodies," *Automatica*, vol. 40, no. 4, pp. 653–659, 2004.
[8] F. Markley, "Attitude filtering on SO(3)," *The Journal of the Astronautical Sciences*, vol. 54, no. 4, pp. 391–413, 2006.
[9] J. F. Vasconcelos, C. Silvestre, and P. Oliveira, "A nonlinear gps/imu based observer for rigid body attitude and position estimation," in *IEEE Conf. on Decision and Control*, Cancun, Mexico, Dec. 2008, pp. 1255–1260.
[10] C. Lageman, J. Trumpf, and R. Mahony, "Gradient-like observers for invariant dynamics on a Lie group," *IEEE Transaction on Automatic Control*, vol. 55, pp. 367 – 377, 2010.
[11] A. Sanyal, T. Lee, M. Leok, and N. McClamroch, "Global optimal attitude estimation using uncertainty ellipsoids," *Systems & Control Letters*, vol. 57, no. 3, pp. 236–245, 2008.
[12] R. Mahony, T. Hamel, and J.-M. Pfimlin, "Complementary filters on the special orthogonal group," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1217, 2008.
[13] A. Sanyal, "Optimal attitude estimation and filtering without using local coordinates, Part 1: Uncontrolled and deterministic attitude dynamics," in *American Control Conference, 2006*. Minneapolis, MN: IEEE, 2006, pp. 5734–5739.
[14] S. Bonnabel, P. Martin, and P. Rouchon, "Nonlinear symmetry-preserving observers on Lie groups," *IEEE Transactions on Automatic Control*, vol. 54, no. 7, pp. 1709–1713, 2009.
[15] J. Bohn and A. Sanyal, "Unscented state estimation for rigid body attitude estimation," in *IEEE Conf. on Decision and Control*, Maui, HI, Dec. 2012, pp. 7498–7503.
[16] M. Izadi and A. K. Sanyal, "Rigid body pose estimation based on the Lagrange-d'Alembert principle," *Automatica*, vol. 50, pp. 2570–2577, 2016.
[17] M. Zamani, J. Trumpf, and R. Mahony, "Near-optimal deterministic filtering on the rotation group," *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 1411–1414, 2011.
[18] ——, "Minimum-energy filtering for attitude estimation," *Automatic Control, IEEE Transactions on*, vol. 58, no. 11, pp. 2917–2921, 2013.
[19] J. Berger, A. Neufeld, F. Becker, F. Lenzen, and C. Schnörr, "Second order minimum energy filtering on SE(3) with nonlinear measurement equations," in *Scale Space and Variational Methods in Computer Vision*, J.-F. Aujol, M. Nikolova, and N. Papadakis, Eds. Cham: Springer International Publishing, 2015, pp. 397–409.
[20] S. Goffin, S. Bonnabel, O. Brüls, and P. Sacré, "Invariant Kalman filtering with noise-free pseudo-measurements," in *2023 62nd IEEE Conference on Decision and Control (CDC)*, 2023, pp. 8665–8671.
[21] K. S. Phogat and D. E. Chang, "Invariant extended Kalman filter on matrix Lie groups," *Automatica*, vol. 114, p. 108812, 2020. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0005109820300108
[22] T. Bouazza, K. Ashton, P. van Goor, R. Mahony, and T. Hamel, "Equivariant filter for feature-based homography estimation for general camera motion," in *2023 62nd IEEE Conference on Decision and Control (CDC)*, 2023, pp. 8463–8470.
[23] P. Cruz, P. Batista, and A. Sanyal, "Design and analysis of attitude observers based on the Lagrange-d'Alembert principle applied to constrained three-vehicle formations," *Advances in Space Research*, vol. 69, no. 11, pp. 4001–4012, 2022. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0273117722001892
[24] A. S. Jayaraman, J. Ye, and G. S. Chirikjian, "A lie-theoretic approach to propagating uncertainty jointly in attitude and angular momentum," in *2023 62nd IEEE Conference on Decision and Control (CDC)*, 2023, pp. 3212–3219.
[25] S. Labsir, A. Giremus, B. Yver, and T. Benoudiba–Campanini, "An intrinsic bayesian bound for estimators on the Lie groups SO(3) and SE(3)," *Signal Processing*, vol. 214, p. 109232, 2024. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S0165168423003067
[26] P. Gintrand, M.-D. Hua, T. Hamel, and G. Varra, "A novel observer design for monocular visual SLAM," *IFAC-PapersOnLine*, vol. 56, no. 2, pp. 1661–1666, 2023, 22nd IFAC World Congress. [Online]. Available: https://www.sciencedirect.com/science/article/pii/S2405896323022796
[27] A. Dongare, R. Hamrah, and A. K. Sanyal, "Attitude pointing control using artificial potentials with control input constraints," in *2021 American Control Conference (ACC)*, 2021, pp. 1–6.
[28] A. M. Bloch, P. E. Crouch, and A. K. Sanyal, "A variational problem on Stiefel manifolds," *Nonlinearity*, vol. 19, no. 10, pp. 22–47, aug 2006. [Online]. Available: https://dx.doi.org/10.1088/0951-7715/19/10/002
[29] K. Hüper, I. Markina, and F. S. Leite, "A Lagrangian approach to extremal curves on Stiefel manifolds," *Journal of Geometric Mechanics*, vol. 13, no. 1, pp. 55–72, mar 2021. [Online]. Available: https://www.aimsciences.org/article/doi/10.3934/jgm.2020031
[30] A. Bloch, J. Ballieul, P. Crouch, and J. Marsden, "Nonholonomic mechanics and control, volume 24 of interdisciplinary applied mathematics," 2003.
[31] T. Lee, M. Leok, and N. McClamroch, "A Lie group variational integrator for the attitude dynamics of a rigid body with applications to the 3D pendulum," in *Proceedings of the IEEE Conference on Control Applications*, 2005, pp. 962–967.
[32] J. Moser and A. P. Veselov, "Discrete versions of some classical integrable systems and factorization of matrix polynomials," *Communications in Mathematical Physics*, vol. 139, no. 2, pp. 217–243, Aug 1991. [Online]. Available: https://doi.org/10.1007/BF02352494
[33] J. Marsden and T. Ratiu, *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, 2nd ed. Springer Verlag, 1999, vol. 17.