Prediction-Driven Surge Planning with Application to Emergency Department Nurse Staffing

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Determining emergency department (ED) nurse staffing decisions to balance quality of service and staffing costs can be extremely challenging, especially when there is a high level of uncertainty in patient demand. Increasing data availability and continuing advancements in predictive analytics provide an opportunity to mitigate demand uncertainty by utilizing demand forecasts. In this work, we study a two-stage prediction-driven staffing framework where the prediction models are integrated with the base (made weeks in advance) and surge (made nearly real-time) nurse staffing decisions in the ED. We quantify the benefit of having the ability to use the more expensive surge staffing and identify the importance of balancing demand uncertainty versus system stochasticity. We also propose a near-optimal two-stage staffing policy that is straightforward to interpret and implement. Lastly, we develop a unified framework that combines parameter estimation, real-time demand forecasts, and nurse staffing in the ED. High-fidelity simulation experiments for the ED demonstrate that the proposed framework has the potential to reduce annual staffing costs by 10%–16% ($2 M–$3 M) while guaranteeing timely access to care.

Key words: Nurse Staffing, Demand Forecast, Surge Capacity, Emergency Department, Healthcare

1. Introduction

Emergency department (ED) crowding is a significant problem in many countries around the world, leading to adverse effects on patient outcomes, patient satisfaction, and staff morale (Bernstein et al. 2009). Nurses provide a substantial portion of patient care and are often a bottleneck resource in the ED (Green 2010). Inadequate nurse staffing is found as a major contributor to the significant increase in the waiting time experienced by patients and the percentage of patients who leave without being seen (LWBS) (Ramsey et al. 2018). In addition, nursing costs comprise a substantial fraction of hospital operating budgets. Therefore, developing effective nurse staffing policies to ensure timely access to care is of great importance.

Optimally balancing the ED nurse staffing levels to ensure good quality of service versus increasing staffing costs can be extremely challenging. One of the major complications comes from the high level of uncertainty in patient demand and the relatively static nature of nurse staffing decisions. Poisson processes have been standard assumptions in modeling the arrival processes in service systems due to their analytical tractability. Their validity has also been statistically verified in

some healthcare settings (Kim and Whitt 2014). However, several recent empirical studies suggest the presence of a higher level of uncertainty (dispersion) relative to standard Poisson processes in real ED arrival data (Maman 2009, Armony et al. 2015), and in other service systems such as call centers (Brown et al. 2005, Steckley et al. 2009, Zhang et al. 2014). Random events such as weather conditions or severity of the flu strain in circulation can cause a high level of fluctuation in ED demand. On the other hand, ED staffing decisions are often made well ahead of time and the staffing level is difficult (or very expensive) to change in real time (Chan et al. 2021). In particular, it is common for EDs to divide a day into multiple nursing shifts, e.g., two 12-hour nursing shifts, with the day shift lasting from 7am to 7pm, and the night shift from 7pm to 7am the next day. As a typical practice, a "base" staffing level, which consists of the majority of the staff, is determined several weeks in advance, when the actual demand is largely unknown. This allows the nurses to plan their working schedule ahead of time. As the time approaches to several hours before the shift, if the ED manager senses a surge in patient volume, he/she can add an extra level of "surge" staffing by calling in overtime or agency nurses at a higher compensation (e.g., overtime salary). The nurse staffing level is then held more or less at a constant level throughout the shift. The surge staffing provides some flexibility to cope with the demand surge, but there currently lack any systematic guidelines on how to optimally utilize this partial flexibility.

Meanwhile, in recent years, increasing data availability and continuing development in statistical learning tools provide an emerging opportunity to mitigate demand uncertainty by building advanced demand forecast models. There have been considerable efforts devoted to developing prediction models for ED patient volume and flow (see, e.g., Marcilio et al. (2013), Calegari et al. (2016), Chang et al. (2018), Whitt and Zhang (2019), Bertsimas et al. (2021)). However, despite the vast amount of literature on demand forecasts, how to effectively incorporate the predictive information to improve ED nurse staffing decisions is less studied. In particular, while advanced prediction models that utilize real-time information generate more accurate short-term forecast of the ED demand in comparison to using traditional historical averages (Schweigler et al. 2009), it remains unclear how the increased prediction accuracy can be translated to improved system performance (e.g., reduction in patient waiting time and LWBS rate) and/or reduced staffing costs. In this paper, we study prediction-driven surge planning. The key tradeoff in this two-stage staffing problem is the long-term staffing commitments which have a lower costs but face a higher level of demand uncertainty (larger prediction error) versus the short-term staffing commitments which have a higher cost but face a lower level of demand uncertainty (smaller prediction error).

To capture the highly uncertain demand faced by the ED, we assume that patients arrive to the ED according to a doubly stochastic Poisson process as in Maman (2009), Bassamboo et al. (2010), Koçağa et al. (2015). The arrival rate for a particular type of shift is a random variable that takes the form of

$$\Lambda = \lambda + \lambda^{\alpha} X, \tag{1}$$

where λ is the mean arrival rate, $\alpha \in (0,1)$ captures the order of arrival-rate uncertainty, and X is a random variable with zero mean. At the base-stage, our prediction model is only able to capture the long-run average pattern that defines the type of the shift, e.g., day of the week effect and day versus night effect. Thus, we assume the base-stage prediction model predicts $\mathbb{E}[\Lambda] = \lambda$ accurately. At the surge stage, as we gather more real-time information, we can build more sophisticated prediction models. Motivated by value of real-time information identified in Hu et al. (2021), we assume in our main model that the surge-stage prediction model is able to predict the realized arrival rate $\ell = \lambda + \lambda^{\alpha} x$ where x is a particular realization of X for the specific shift. Conditional on ℓ, the ED operates as a Markovian multi-server queue with Poisson arrival process, exponentially distributed service times, and exponentially distributed patience times. Note that even with the surge-stage predictive information, we still incur a certain level of uncertainty due to the randomness in the interarrival times between patients, patients' service requirements, and their patience times (time before abandoning).

The ED manager makes two staffing decisions for each shift: a base staffing level and a surge staffing level. The base staffing decision is based on the base prediction, i.e., λ, and knowledge of the arrival rate distribution, i.e., the distribution of $\lambda^{\alpha} X$. The surge staffing decision is based on the surge prediction, i.e., ℓ. The surge staff are assumed to be more costly than the base staff. Our objective is to minimize the sum of the staffing cost and the performance cost which consists of the costs incurred by patients' waiting and patients' LWBS. Our main contributions can be summarized as:

The benefit of surge staffing. To quantify the benefit of having the more expensive surge staff, we compare the two-stage stochastic optimization problem to a single-stage benchmark where only base staffing is allowed. We quantify the cost saving of the optimal two-stage staffing rule over the optimal single-stage policy. Our result shows that the magnitude of cost-saving depends on the order of arrival-rate uncertainty captured by α in (1). In particular, the cost saving is $o(\sqrt{\lambda})$ if $\alpha < 1/2$, $O(\sqrt{\lambda})$ if $\alpha = 1/2$, and $\Theta(\lambda^{\alpha})$ if $\alpha > 1/2$. As we will explain in more details, the three regimes of cost saving are divided by the interplay between the order of arrival-rate uncertainty, which is $O(\lambda^{\alpha})$, and stochastic variability in patient arrival and services, which is

$O(\sqrt{\lambda})$. The cost-saving quantification suggests that surge staffing is most beneficial when the arrival-rate uncertainty dominates the system stochasticity, i.e., $\alpha > 1/2$. In this regime, the larger the arrival-rate uncertainty, the more cost savings we gain from the flexibility of surge staffing.

Near-optimal two-stage staffing rule. Focusing on the regime where the arrival-rate uncertainty dominates the system stochasticity, i.e., $\alpha > 1/2$, we propose a near-optimal two-stage staffing rule that is easy to interpret and implement. In particular, at the base stage, the base staffing level is set to meet the mean demand, together with a hedging that is of the same order as the arrival-rate uncertainty. After the random arrival rate is realized at the surge stage, the surge staffing level is brought up to meet the realized offered load, together with a hedging against the stochastic variability catered to the realized arrival rate. The parameters of the staffing rule, which dictate the amount of hedging, are the optimal solutions to a two-stage newsvendor problem, which can be viewed as a stochastic-fluid approximation to the optimal staffing problem, and the optimal solutions to a square-root staffing problem based on a diffusion approximation of the queue length process. We prove that our proposed policy has an optimality gap of $o(\sqrt{\lambda})$ compared to the exact two-stage optimum. We also extend the two-stage staffing rule to allow more general prediction errors at the surge stage. In particular, we consider the case where we are not able to predict the realized arrival rate ℓ exactly. Instead, we may incur different levels of prediction error. We quantify how prediction error affect the staffing rule and its corresponding performance.

Practical insights and ED implementation. To facilitate real-world implementation, we propose an integrated framework to which includes 1) parameter estimation, 2) a two-stage prediction model, and 3) a two-stage prediction-driven staffing rule. Using data from the ED in New York Presbyterian Columbia University Medical Center (NYP CUMC), we estimate its arrival-rate uncertainty to be $\alpha = 0.769$. We then build a two-stage prediction model to inform the staffing policy. At the base stage, a simple linear regression model that incorporates the day of the week and day v.s. night effect works well. For the surge stage, we implement a recently developed prediction model in Hu et al. (2021), which utilizes concurrent information such as weather, patient comorbidity profile, ED congestion level, etc. Lastly, we extend the two-stage staffing rule developed based on the parsimonious queueing model to accommodate realistic patient-flow characteristics in our collaborating ED. We extend our two-stage staffing rule to allow adjustment for the transient-shift effect. This includes a base-stage adjustment which accounts for the difference in average queue length between day and night shifts, and a surge-stage adjustment which takes the concurrent queue length information into account. With these adjustments, our policy achieves significant cost savings for the simulated ED. For example, compared to the newsvendor solution (Bassamboo

et al. 2010), our policy achieves a reduction of 16% ($3 M) in the annual staffing cost while the average waiting time is kept below 30 minutes.

Remark 1 *In this paper, we focus on the nurse staffing problem under the assumption that nurses are the bottleneck resource. This is because nurses are the primary staff who execute care plans during patients' length of stay in the ED. In addition, burnout and high turnover rates among ED nurses have been widely reported by healthcare systems (Phillips et al. 2022, Susila and Laksmi 2022). These problems further worsened during and after the COVID-19 pandemic. That said, our staffing framework can be applied to plan other resources, such as physicians and technicians, as long as the two-stage planning with the corresponding information and cost tradeoff is relevant.*

1.1. Related Literature

Classic square-root staffing rule. The standard stream of capacity planning problems for service operations focuses on systems where model parameters are exactly known. In this setting, the square-root staffing principle dates back to Erlang (1917) in the study of automatic telephone exchanges. The principle is more recently explained based on an infinite-server queue heuristic in Kolesar and Green (1998). In particular, it is shown that the stochastic fluctuation of the system is of square root order of the offered load. Thus, the square-root staffing can be viewed as an uncertainty hedging against system stochasticity. Halfin and Whitt (1981) establish a formal diffusion limit for $M/M/N$ queues under the square-root staffing as the arrival rate goes to infinity. Borst et al. (2004) further establishes that the square-root staffing rule optimally balances the staffing cost and the service quality. For this reason, the many-server asymptotic scaling under the square root staffing is often referred to as the quality-and-efficiency driven (QED) regime. A few extensions have been considered to incorporate features not captured by the $M/M/N$ model. Garnett et al. (2002) generalize the diffusion limit under the square-root staffing to the $M/M/N+M$ queue where customers can abandon the system if waiting for too long under the exponentially distributed patience time; more general patience time distributions are considered in Mandelbaum and Zeltyn (2009). Jennings et al. (1996) and Liu and Whitt (2012) extend the square-root staffing rule to systems with time-varying arrival rates. Our work extends this stream of literature by allowing the arrival rate to be random and considering a two-stage staffing problem in two time scales. Relevantly, after the random arrival rate is realized at the surge stage, our proposed two-stage QED staffing rule brings the total staffing level up to the square-root staffing prescription if the base-stage capacity is inadequate. In addition, similar to the literature, our theoretical analysis

takes an asymptotic approach, where we send the mean arrival rate λ to infinity and study how the optimal staffing level scales with λ.

Managing queues with parameter uncertainty. Motivated by the high level of demand uncertainty in many service systems, more sophisticated models for arrival processes that account for characteristics not captured by standard Poisson processes have been proposed in the literature. Whitt (1999) is one of the first to study a random arrival rate for call centers and its implications on staffing decisions. Chen and Henderson (2001), Avramidis et al. (2004), Brown et al. (2005) and Steckley et al. (2009) provide empirical evidence of arrival-rate uncertainty and explore its modeling implications. Maman (2009) finds empirical evidence of high arrival-rate uncertainty in an Israeli ED. Our work is closely related to works that study staffing decisions in the presence of arrival-rate uncertainty. Whitt (2006) investigates a fluid-based staffing prescription catered to arrival-rate uncertainty and absenteeism of servers. Harrison and Zeevi (2005) and Bassamboo et al. (2010) propose a newsvendor-based solution method whose effectiveness is pronounced when the order of arrival-rate uncertainty is larger than stochastic variability. Their proposed staffing rule is set to meet the mean demand plus a hedging against the arrival-rate uncertainty. More recently, moving from single-stage to two-stage decisions, Koçağa et al. (2015) formulate a joint staffing and co-sourcing problem, where the staffing decision is made before the random arrival-rate is realized, and the co-sourcing decision is made in real time after the arrival-rate uncertainty is resolved. Our two-stage optimization problem has similar decision epochs to those in Koçağa et al. (2015), i.e., before and after the random demand is realized. However, different from Koçağa et al. (2015), we consider a two-stage staffing problem and allow the arrival-rate uncertainty to be of a larger magnitude than stochastic variability. The solution method we use to solve the two-stage stochastic optimization problem leverages the stochastic fluid approximation introduced in Harrison and Zeevi (2005), but we considered a more refined version of this approximation, which takes the system stochasticity into account at the surge stage.

Predictive analytics and data-driven methods in capacity sizing. Several works take a data-driven approach for capacity sizing with demand uncertainty. Zheng et al. (2018) and Sun and Liu (2021) propose statistical methods to estimate the arrival-rate distribution. See also Ibrahim et al. (2016) for a comprehensive review of literature on modeling and forecasting for call center arrivals. Bassamboo and Zeevi (2009) develop a data-driven approach that yields staffing prescriptions that are asymptotically optimal, as both the system scale and data size increase to infinity. There is a large literature on studying demand uncertainty in inventory systems without queueing dynamics (see for example (Chen et al. 2007, Perakis and Roels 2008, Levi et al. 2015, Ban and Rudin 2019,

Boada-Collado et al. 2020)). Motivated by the operations of EDs, our work takes into account the arrival-rate distribution at the base stage, the demand visibility at the surge stage, and the stochasticity of queueing dynamics.

Two-stage stochastic optimization problem. Our work is related to the mathematical programming literature on two-stage stochastic optimization problems for staffing and resource planning; see representative works from Kim and Mehrotra (2015), Bodur and Luedtke (2017), Rath and Rajaram (2022). However, our work has important differences from the existing literature and adds new insights by taking an analytical approach that allows us to 1) develop simple, explicit, and interpretable staffing policies, 2) provide more managerial insights by quantifying cost savings from the surge staffing in different demand uncertainty regimes and the effects of prediction errors on system performance, and 3) capturing detailed queueing dynamics.

ED capacity planning Our work relates to the growing literature on using queueing theory to address capacity planning problems in the ED. Green et al. (2006) model the ED as an $M_t/M/s$ queue and use a Lag SIPP (stationary independent period by period) approach to gain insights into the staffing prescriptions. Yankovic and Green (2011) develop a finite source queueing model with two types of resources —nurses and beds—to study the interplay between bed occupancy level and demand for nursing. Véricourt and Jennings (2011) study nurse staffing using a closed queueing model, where patients alternate between being needy of service and stable without service need. Similar patient reentrant behavior is studied by Yom-Tov and Mandelbaum (2014) using an Erlang-R model in time-varying environments. Chan et al. (2021) use a multiclass queue to study the dynamic assignment of nurses to different areas of the ED at the beginning of each shift. Batt et al. (2019) empirically investigate the impact of discrete work shifts on service rates and patient handoffs (i.e., passing patients in treatment to the next care provider at the end of a shift). Compared to the literature, we focus on studying the effect of demand uncertainty on ED staffing, where we investigate how demand prediction can be utilized to make better staffing decisions.

Dual sourcing problem in supply chain management. Though our work is motivated by the staffing problem for service systems, a similar core tradeoff between cost and responsiveness arises in dual sourcing inventory systems, in which one supplier is cheaper but slower, while the other is more costly but faster. In this setting, a tailored base-surge (TBS) sourcing policy is found to be effective in both continuous and periodic review models (Allon and Van Mieghem 2010, Janakiraman et al. 2015). Xin and Goldberg (2018) formally prove that the TBS policy is asymptotically optimal as the lead time of the cheaper supplier grows without bound. Different from the dual sourcing problem, our theoretical framework further incorporates queueing dynamics into the optimization problem.

We quantify how the cost savings of our proposed policy increase with the order of arrival-rate uncertainty.

1.2. Organization

The rest of the paper is organized as follows. In Section 2 we introduce the model and formulate the two-stage staffing problem. In Section 3 we quantify the cost savings from surge staffing. In Section 4 we propose near-optimal two-stage staffing rules that are easy to interpret and implement. The optimality gap between the proposed policy and the exact two-stage optimum is also derived. The performance of the two-stage staffing rule is further illustrated through numerical experiments in Section 5, where we compare the performance of our proposed staffing rule to several benchmark policies. In Section 6, we extend the two-stage staffing rule to accommodate more general prediction errors at the surge stage. Lastly, in Section 7, we develop a holistic framework to implement the prediction-driven staffing policy in the actual ED, which includes parameter estimation, demand forecast, and capacity sizing that takes the transient shift effect into account. We conclude in Section 8. All the proofs appear in the appendix.

1.3. Notation

As we take an asymptotic approach to performance analysis, we define some notations following the convention in the literature (see, e.g., Chapter 3 in Cormen et al. (2022)) For a sequence of positive real numbers $\{a^n : n \in \mathbb{R}_+\}$ and a sequence of real numbers $\{b^n : n \in \mathbb{R}_+\}$, we write (i) $b^n = o(a^n)$ if $|b^n/a^n| \to 0$ as $n \to \infty$, (ii) $b^n = O(a^n)$ if $|b^n/a^n|$ is bounded from above, and (iii) $b^n = \Theta(a^n)$ if $|b^n/a^n|$ is bounded from above and from below by a strictly positive real number, i.e., if $m \le |b^n/a^n| \le M$ for some $0 < m < M < \infty$ for all $n > 0$. For a sequence of random variables $\{X^n : n \in \mathbb{R}_+\}$ and a sequence of positive real numbers $\{a^n : n \in \mathbb{R}_+\}$, we write (i) $X^n = o(a^n)$ if $|X^n/a^n| \to 0$ as $n \to \infty$ with probability 1, and (ii) $X^n = o_{UI}(a^n)$ if $X^n = o(a^n)$ and there exists some random variable Y with $\mathbb{E}[Y] < \infty$ such that $|X^n/a^n| < Y$ for all $n > 0$.

2. The Model

To gain insights into the potential benefits of two-stage staffing, we start with a stylized model of the ED using a parsimonious multi-server queueing system where patients arrive according to a doubly stochastic Poisson process. The arrival rate for a shift Λ is a random variable with cumulative distribution function F_Λ and mean $\mathbb{E}[\Lambda] = \lambda$. Conditional on Λ, the arrival process is a homogeneous Poisson process with that rate. Patients are served on a first-come first-served (FCFS) basis, and wait in an infinite capacity buffer when all servers (nurses) are busy. While waiting for service, a delayed patient abandons the system (LWBS) after an exponentially distributed amount

of time with mean $1/\gamma$. Patients have service requirements that are independently and identically distributed (i.i.d.) exponential random variables with mean $1/\mu$. Hence, conditioned on Λ, the ED operates as an $M/M/N+M$ queue (also known as the Erlang-A queue; see, e.g., Zeltyn and Mandelbaum (2005)), where the staffing level N is the decision variable.

The ED manager makes two decisions: an upfront base staffing level and a surge staffing level, both of which are non-negative integers. At the base stage, which is often a few weeks/months before the start of the actual shift, the prediction model can only predict the average arrival rate level, λ. We assume the arrival rate distribution is known. Thus, the base staffing level $N_1 := N_1(F_\Lambda) \in \mathbb{N}$ is made before the arrival rate is realized, based on knowledge of the arrival rate distribution, F_Λ, only. At the surge stage, as we gather more real-time information, the prediction model can predict the realized arrival rate ℓ quite accurately. Thus, the surge staffing level $N_2(N_1, \ell) \in \mathbb{N}$ is made based on the base staffing level, N_1, and the realized arrival rate, ℓ. We do not allow $N_2(N_1, \ell)$ to take negative values, because in most EDs, the manager cannot make a last-minute decision to reduce the staffing level, e.g., by canceling shifts for the nurses who are staffed at the base stage. We denote the joint staffing decision as $\pi := (N_1, N_2(N_1, \ell))$, and use Π to denote the set of all feasible staffing rules. Note that in this parsimonious model, the prediction at the base stage is captured by the expected arrival rate, $\lambda := \mathbb{E}[\Lambda]$, and the prediction errors are captured by the distribution of $\Lambda - \lambda$. To start, we assume perfect prediction at the surge stage. We will relax this assumption in Section 6 to explicitly incorporate prediction errors at the surge stage.

There are costs associated with patients' waiting, patients' LWBS (abandonments), and staffing. In particular, a holding cost is incurred at a rate of h per patient per unit time spent waiting. Each abandoning patient incurs a fixed cost of a. The staffing cost is c_1 per base server per unit time, and c_2 per surge server per unit time. Let $Q(n, \ell)$ denote the steady-state queue length of an $M/M/n+M$ queue with arrival rate ℓ. Then, we consider the following two-stage cost minimization problem.

$$\min_{\pi \in \Pi} \mathcal{C}_\pi = \min_{N_1} \left\{ c_1 N_1 + \mathbb{E}\left[\min_{N_2(N_1,\Lambda)} \{ c_2 N_2(N_1, \Lambda) + (h + a\gamma)\, \mathbb{E}\left[Q(N_1 + N_2(N_1, \Lambda), \Lambda) | \Lambda \right] \} \right] \right\}. \quad (2)$$

For an $M/M/n+M$ queue with arrival rate ℓ, $\gamma \mathbb{E}[Q(n, \ell)]$ is the steady-state abandonment rate. Thus, $a\gamma \mathbb{E}[Q(n, \ell)]$ captures the abandonment cost while $h\mathbb{E}[Q(n, \ell)]$ captures the holding cost in steady state. Note that there are two expectations in (2). The inner expectation is taken with respect to the stochasticity in the steady-state queue length, i.e., randomness in $Q(n, \Lambda)$ conditional on $\Lambda = \ell$. The outer expectation is taken with respect to the arrival-rate uncertainty, i.e., randomness in Λ.

2.1. Parameter Regime

It makes intuitive sense that if the waiting and abandonment costs are excessively lower than the staffing costs, there is no motivation to staff any server. In addition, if the base staffing cost is higher than the surge staffing cost, i.e., $c_1 > c_2$, it is cost-effective to staff all servers at the surge stage when the arrival-rate uncertainty is resolved. This intuition is formalized in Proposition 1.

Proposition 1 *For the optimal solution* $(N_1^*, N_2^*(N_1^*, \Lambda))$ *to problem (2):*

(I) If $\min\{c_1, c_2\} \geq h\mu/\gamma + a\mu$, *then* $N_1^* = 0$ *and* $N_2^*(N_1^*, \Lambda) = 0$.

(II) If $\min\{c_1, h\mu/\gamma + a\mu\} \geq c_2$, *then* $N_1^* = 0$.

(III) If $c_2 \geq h\mu/\gamma + a\mu \geq c_1$, *then* $N_2^*(N_1, \Lambda) = 0$ *for any base staffing level* N_1.

Based on Proposition 1, the cost parameters can be divided into four regimes as summarized in Table 1.

Table 1 Optimal staffing combination for different cost parameters

Cost parameters	Staffing decisions
$\min\{c_1, c_2\} \geq h\mu/\gamma + a\mu$	No staffing
$\min\{c_1, h\mu/\gamma + a\mu\} \geq c_2$	Complete surge staffing
$c_2 \geq h\mu/\gamma + a\mu \geq c_1$	Complete base staffing
$h\mu/\gamma + a\mu > c_2 > c_1$	Base + surge staffing

In this paper, we are interested in the non-trivial regime that provides motivation to staff both base and surge servers.

Assumption 1 *The cost rates satisfy* $h\mu/\gamma + a\mu > c_2 > c_1$.

2.2. Arrival-Rate Uncertainty

Solving (2) explicitly is challenging due to the two sources of randomness. In addition, $\mathbb{E}[Q(N_1 + N_2(N_1, \ell), \ell)]$ has no closed-form expression. To gain analytical insights, we take an asymptotic approach by sending the mean arrival rate λ to infinity and study how the optimal staffing rule scales with λ.

To facilitate the theoretical development, we assume that the random arrival rate takes the form

$$\Lambda = \lambda + X\lambda^{\alpha}\mu^{1-\alpha}, \tag{3}$$

for some constant $\alpha \in (0, 1)$ and X is a random variable with $\mathbb{E}[|X|] < \infty$ [1]. Note that because $\mathbb{E}[\Lambda] = \lambda$, $\mathbb{E}[X] = 0$. Let F_X denote the cumulative distribution function (cdf) of X. We assume that

[1] This form of arrival-rate uncertainty, i.e., (3) is equivalent to the one introduced in (1); we factor out $\mu^{1-\alpha}$ to facilitate technical derivations.

X has a proper probability density function (pdf). The second term in (3) captures the fluctuation of the arrival rate around its mean. It is further decomposed into two parts: X and $\lambda^{\alpha}\mu^{1-\alpha}$, where the second part captures the order of fluctuation in relation to λ. We refer to the exponent α as the *order of arrival-rate uncertainty*. A random arrival rate of the form (3) is proposed in Maman (2009). Similar arrival rate formula has been used in Bassamboo et al. (2010), Koçağa et al. (2015).

In what follows, we use the superscript λ to denote quantities that scale with λ. To simplify notations, we sometimes suppress the superscript when it is clear from the context.

3. When is Surge Staffing Beneficial?

As mentioned in Section 1, implementing the two-stage staffing requires knowing the realized arrival rate with high precision. In practice, this often involves investing in sophisticated prediction models, which can be costly to develop and maintain. In addition, even though surge staffing is paid at a higher rate, it may not be a desirable working mode for nurses. Therefore, it is important to know how much cost saving we can gain by having the flexibility of surge staffing.

Analogous to the two-stage optimization problem (2), we define the single-stage optimal staffing problem as

$$\min_{\pi\in\Pi} \mathcal{C}_{\pi} = \min_{N_1} \left\{ c_1 N_1 + \mathbb{E}\left[(h+a\gamma)\, Q(N_1,\Lambda) \right] \right\}. \tag{4}$$

Note that the single-stage problem is equivalent to the two-stage staffing problem (2) by imposing the surge staffing level to be $N_2(N_1,\Lambda)=0$ for any base staffing level N_1.

For the sequence of systems indexed by λ, we use $\mathcal{C}^{\lambda}_{1,*}$ to denote the optimal total cost for the single-stage optimization problem (4). Correspondingly, we use $\mathcal{C}^{\lambda}_{2,*}$ to denote the optimal total cost for the two-stage optimization problem (2).

Theorem 1 (benefit of surge staffing) *Given the order of uncertainty α, the difference in optimal costs for the single-stage versus two-stage optimization problem can be summarized as:*

(I) If $\alpha<1/2$, then $\mathcal{C}^{\lambda}_{1,*}-\mathcal{C}^{\lambda}_{2,*}=o(\sqrt{\lambda})$.

(II) If $\alpha=1/2$, then $\mathcal{C}^{\lambda}_{1,*}-\mathcal{C}^{\lambda}_{2,*}=O(\sqrt{\lambda})$.

(III) If $\alpha>1/2$, then $\mathcal{C}^{\lambda}_{1,*}-\mathcal{C}^{\lambda}_{2,*}=\Theta(\lambda^{\alpha})$.

We next provide some intuition behind Theorem 1. We first note that when $\gamma=\mu$, for a given realization of the arrival rate, i.e., $\Lambda=\ell$, the steady-state number of patients in the system follows a Poisson distribution with mean ℓ/μ. Its standard deviation is equal to $\sqrt{\ell/\mu}=O(\sqrt{\lambda})$, which is known as the system stochasticity and cannot be resolved by the prediction model. On the other

hand, the arrival-rate uncertainty characterized by (3) is of order λ^α. This parameter uncertainty can be resolved by the prediction model at the surge stage. When $\alpha < 1/2$, the system stochasticity dominates the parameter uncertainty. The gain by conducting two-stage staffing is restricted to $o(\sqrt{\lambda})$. The cost savings are $O(\sqrt{\lambda})$ if the parameter uncertainty and system stochasticity are of the same order, i.e., $\alpha = 1/2$. When $\alpha > 1/2$, the parameter uncertainty dominates the system stochasticity. This is when we gain the most cost savings from the flexibility offered by surge staffing. In this regime, the larger the order of arrival-rate uncertainty is, the larger magnitude of cost savings we gain from surge staffing.

4. Near-Optimal Surge Staffing Policy

As derived in Section 3, when the order of arrival-rate uncertainty is strictly larger than that of system stochasticity, the cost saving of implementing the two-stage staffing optimally is significant, i.e., $\Theta(\lambda^\alpha)$. We thus consider this regime as the most meaningful scenario to execute the two-stage staffing, and assume throughout this section that $\alpha > 1/2$. We next derive solutions to the two-stage staffing problem.

Due to the convoluted system dynamics, solving the two-stage stochastic optimization problem (2) explicitly is hard. Part of the difficulty lies in characterizing the expected steady-state queue length which depends intricately on the staffing decisions. While the problem can be solved numerically, e.g., via simulation optimization, limited insights about the optimal policy can be generated. Hence, we take the approach of solving more tractable approximations of the two-stage optimization problem (2). These approximations can be viewed as asymptotic limits of (2) under appropriate scalings as the system scale λ grows to infinity. Thus, policies derived based on them work really well for relatively large systems and provide insights into how the optimal policy scales with λ. We also discuss small system adaptions in Section 4.3.

4.1. Stochastic-Fluid Based Solution

Since the parameter uncertainty is of a larger order than system stochasticity, we start by approximating the objective function in (2) via suppressing the system stochasticity and focusing solely on the uncertainty in the arrival rate. This relaxation is known as the *stochastic-fluid approximation* (Harrison and Zeevi 2005, Bassamboo et al. 2010). In particular, conditional on the arrival rate Λ, we approximate the steady-state queue length of the $M/M/n+M$ queue via $(\Lambda - n\mu)/\gamma$, which is the equilibrium queue length of a deterministic fluid model with the same arrival rate, service rate, and abandonment rate.

Before introducing the stochastic-fluid approximation for the two-stage optimization problem (2), we illustrate the idea by reviewing the single-stage newsvendor policy (denoted by $u_{1,NV}$)

proposed by Bassamboo et al. (2010). Given the staffing level N_1, the steady-state abandonment rate is approximately $(\Lambda - \mu N_1)$ and the steady state queue length is approximately $(\Lambda - N_1\mu)/\gamma$. Then, the single-stage optimization problem (4) can be approximated by

$$\min_{N_1} \left\{ c_1 N_1 + (h\mu/\gamma + a\mu)\, \mathbb{E}\left[(\Lambda/\mu - N_1)^+ \right] \right\}. \tag{5}$$

Note that (5) is a typical newsvendor problem, with unit capacity cost c_1, unit sales price $h\mu/\gamma + a\mu$, random demand Λ/μ, and capacity decision N_1. The optimal solution is given by

$$N_1 = \bar{F}^{-1}_{\Lambda/\mu}\left(\frac{c_1}{h\mu/\gamma + a\mu} \right),$$

where $\bar{F}_{\Lambda/\mu} := 1 - F_{\Lambda/\mu}$ is the complementary cumulative distribution function (ccdf) of Λ/μ, and $\bar{F}^{-1}_{\Lambda/\mu}$ is its inverse. Equivalently, we can write

$$N_1 = \frac{\lambda}{\mu} + \bar{F}^{-1}_{X}\left(\frac{c_1}{h\mu/\gamma + a\mu} \right)\left(\frac{\lambda}{\mu} \right)^{\alpha}, \tag{6}$$

where $\bar{F}_X$ is the ccdf of X. We remark that for all staffing rules discussed in the paper, we do not explicitly restrict N_1 and N_2 to satisfy the integer constraints. Since rounding becomes immaterial when we consider the asymptotic performance of the policy as $\lambda \to \infty$, we assume without loss of generality that each staffing prescription is rounded up to its nearest integer.

Let $\mathcal{C}^{\lambda}_{1,NV}$ denote the expected total cost defined in (2) under the one-stage newsvendor solution. Recall that $\mathcal{C}^{\lambda}_{1,*}$ is the optimal total cost for the single-stage optimization problem (4). Theorem 1 in Bassamboo et al. (2010) establishes that

$$\mathcal{C}^{\lambda}_{1,NV} - \mathcal{C}^{\lambda}_{1,*} = O(\lambda^{1-\alpha}). \tag{7}$$

Note that when $\alpha > 1/2$, $O(\lambda^{1-\alpha}) = o(\sqrt{\lambda})$. Thus, the single-stage newsvendor solution works remarkably well in the single-stage optimal staffing problem.

We next extend the single-stage newsvendor solution to the two-stage newsvendor solution where surge staffing is allowed after we observed the realized arrival rate. The stochastic-fluid approximation of the two-stage optimization problem (2) takes the form

$$\min_{N_1} \left\{ c_1 N_1 + \mathbb{E}\left[\min_{N_2(N_1,\Lambda)} \left\{ c_2 N_2(N_1,\Lambda) + (h/\gamma + a)\left(\Lambda - \mu(N_1 + N_2(N_1,\Lambda))\right)^+ \right\} \right] \right\}. \tag{8}$$

Given N_1, Assumption 1 implies that the optimal surge-stage staffing level in (8) is given by

$$N_2(N_1,\Lambda) = (\Lambda/\mu - N_1)^+.$$

Hence, the optimal base-stage staffing level is the optimal solution to

$$\min_{N_1} \left\{ c_1 N_1 + c_2 \mathbb{E}\left[(\Lambda/\mu - N_1)^+ \right] \right\}. \tag{9}$$

Similar to (5), (9) is a newsvendor problem, with unit capacity cost c_1, unit sales price c_2, random demand Λ/μ, and capacity decision N_1. The optimal solution is given by

$$N_1 = \bar{F}^{-1}_{\Lambda/\mu}(c_1/c_2) = \lambda/\mu + \bar{F}^{-1}_{X}(c_1/c_2)(\lambda/\mu)^{\alpha}.$$

Let $\beta^* := \bar{F}^{-1}_{X}(c_1/c_2)$. We propose the following two-stage newsvendor solution denoted by $u_{2,NV}$.

Definition 1 (two-stage newsvendor solution) *For $\alpha \in (1/2, 1)$, the parameters of the two-stage newsvendor solution $u_{2,NV}$ are set as follows:*

1. *At the base stage, the base-stage staffing level is*

$$N_1 := \lambda/\mu + \beta^*(\lambda/\mu)^{\alpha} + o((\lambda/\mu)^{\alpha}).$$

2. *At the surge stage, the surge-stage staffing level is*

$$N_2(N_1, \Lambda) := (X - \beta^*)^+ (\lambda/\mu)^{\alpha} + o_{UI}((\lambda/\mu)^{\alpha}).$$

In the two-stage newsvendor solution, the base-stage capacity is equal to the average offered load, λ/μ, together with a hedging term that is in the same order as the arrival-rate uncertainty. The additional $o((\lambda/\mu)^{\alpha})$ term can be set as zero or some number that is of a smaller order than $(\lambda/\mu)^{\alpha}$. As we will show in Theorem 2, this term will not affect the asymptotic performance of the staffing rule. After the arrival rate is realized at the surge stage, the capacity is brought up to the realized offered load if $X > \beta^*$, with some flexibility of order $o_{UI}((\lambda/\mu)^{\alpha})$; see Section 1.3 for a formal definition of $o_{UI}(\cdot)$. Note that the surge staffing is of a smaller order than the base staffing. Since X is a continuous random variable, by the definition of β^*, the probability of assigning nonzero surge staffing is equal to c_1/c_2. Moreover, it follows from Assumption 1 that $c_1/(h\mu/\gamma + a\mu) < c_1/c_2$. Thus, in comparison to the single-stage newsvendor solution described in (6), the two-stage newsvendor solution prescribes less capacity at the base stage. This is intuitive, because with the flexibility to respond to surges in demand by raising the staffing level at the surge stage, the two-stage newsvendor solution can be less aggressive in assigning base-stage uncertainty hedging.

Note that Definition 1 defines a family of two-stage solutions, where some flexibility of order $o((\lambda/\mu)^{\alpha})$ in the base stage staffing and flexibility of order $o_{UI}((\lambda/\mu)^{\alpha})$ in the surge stage staffing are

allowed. For ease of exposition, we refer to this family of staffing rules as the two-stage newsvendor solution, and let $\mathcal{C}^{\lambda}_{2,NV}$ denote the expected total cost defined in (2) under the two-stage newsvendor solution. Recall that $\mathcal{C}^{\lambda}_{2,*}$ is the optimal total cost for the two-stage optimization problem (2).

Theorem 2 (optimality gap of $u_{2,NV}$) *For $\alpha \in (1/2, 1)$, the two-stage newsvendor solution in Definition 1 has $\mathcal{C}^{\lambda}_{2,NV} - \mathcal{C}^{\lambda}_{2,*} = o(\lambda^{\alpha})$.*

Since $\alpha > 1/2$, Theorem 1 implies that $\mathcal{C}^{\lambda}_{1,NV} - \mathcal{C}^{\lambda}_{2,*} = \Theta(\lambda^{\alpha})$. This, together with Theorem 2 and the gap in (7), suggests that $\mathcal{C}^{\lambda}_{1,NV} - \mathcal{C}^{\lambda}_{2,NV} = \Theta(\lambda^{\alpha})$.

We provide numerical demonstrations of Theorem 2 in Section 5.

Remark 2 *Our development so far has assumed a single pool of nurses, the ability to recruit surge nurses as needed, and all nurses show up (i.e. no nurse no-shows). The model can be generalized to relax these assumptions in a relatively straightforward manner. First, it is possible to distinguish between base and surge nurses by assuming different service rates. Second, we can incorporate a capacity cap on surge nurses and create an on-call pool with a small amount of monetary compensation. That is, a compensation $c_2^0 \in \mathbb{R}_+$ per nurse per shift is paid at the base stage to staff a total of $N_2^0 \in \mathbb{N}$ nurses in the on-call pool. Then at the surge stage, the ED manager calls N_2 ($N_2 \leq N_2^0$) nurses from the on-call pool to serve as surge staff in the upcoming shift. If called, these nurses will be paid at the surge rate. In this setting, N_1 and N_2^0 are determined at the base stage, while N_2 is determined at the surge stage. Third, we can consider nurse no-shows by modeling the number of nurses who show up to work as a Bernoulli random variable. In all cases mentioned above, similar lines of analysis can be followed to develop a "generalized" two-stage newsvendor solution. More detailed discussions are relegated to Appendix H.*

4.2. Refinement for The Two-Stage Newsvendor Solution

We have established in Theorem 2 that the two-stage newsvendor solution achieves an optimality gap of $o(\lambda^{\alpha})$ compared to the exact two-stage optimum. In this section, we propose a refinement for the two-stage newsvendor solution which further reduces the optimality gap to $o(\sqrt{\lambda})$. The improvement is achieved by characterizing the $o_{UI}(\lambda^{\alpha})$ term in the two-stage newsvendor solution more carefully.

To provide intuition for the refinement, we shall ignore the $o(\lambda^{\alpha})$ and $o_{UI}(\lambda^{\alpha})$ terms for now, i.e., setting them to zero, in the two-stage newsvendor solution. The key observation is that depending on the realized arrival rate, the two-stage newsvendor solution will result in the system being either

underloaded (capacity exceeding offered load), or critically loaded (capacity equal to offered load). In particular, for any realized arrival rate $\ell = \lambda + x\lambda^{\alpha}\mu^{1-\alpha}$, if $x < \beta^*$, then

$$N_1 + N_2(N_1, \ell) - \ell/\mu = (\beta^* - x)\,(\lambda/\mu)^{\alpha} = \Theta(\lambda^{\alpha}).$$

In this case, the steady-state average queue length is "negligibly" small, i.e., $o(\sqrt{\lambda})$ (more details is provide in Appendix C.1, see (32) in the proof of Lemma 3) In the case where $x \geq \beta^*$, the total staffing level is equal to ℓ/μ, under which the system operates in the QED regime (Mandelbaum and Zeltyn 2009). We can then add a square-root hedging against the stochastic fluctuation of the queue process. In particular, consider

$$N_1 + N_2(N_1, \ell) = \ell/\mu + \eta\sqrt{\ell/\mu} + o(\sqrt{\ell/\mu}), \quad \text{for some } \eta \in \mathbb{R}. \tag{10}$$

Under the capacity prescription in (10), the expected steady-state queue length is $\Theta(\sqrt{\lambda})$. This fact is well-known and will be made rigorous for our system in the proof of Theorem 3 in Appendix E. Thus, to "optimize" queue length of this magnitude, we refine the two-stage newsvendor solution by restricting the $o_{UI}(\lambda^{\alpha})$ term to $O(\sqrt{\lambda}) + o_{UI}(\sqrt{\lambda})$, so that it serves as a safety capacity against system stochasticity.

A few more definitions are needed to formally introduce the refined staffing rule. Let ϕ and Φ be the pdf and cdf of the standard normal distribution, respectively. The hazard rate of the standard normal distribution is given by

$$H(t) = \phi(t)/\Phi(-t), \quad t \in \mathbb{R}.$$

Define

$$\eta^* := \arg\min_{\eta \in \mathbb{R}} \; c_2\eta + \left(\frac{h\mu}{\gamma} + a\mu\right) \underbrace{\frac{\sqrt{\frac{\gamma}{\mu}}\left[H\left(\eta\sqrt{\frac{\mu}{\gamma}}\right) - \eta\sqrt{\frac{\mu}{\gamma}}\right]}{1 + \sqrt{\frac{\gamma}{\mu}}\frac{H\left(\eta\sqrt{\frac{\mu}{\gamma}}\right)}{H(-\eta)}}}_{(a)}. \tag{11}$$

η^* is the optimal solution of the square-root staffing problem in (Mandelbaum and Zeltyn 2009). In particular, the term (a) on the right-hand side of (11) is the diffusion approximation (and a bona-fide limit in the QED regime) of the expected steady-state queue length of an $M/M/n+M$ queue with service rate μ, abandonment rate γ, staffing cost c_2, abandonment cost a, and staffing level prescribed in (10) (i.e., with square root staffing parameter η).

We are now ready to introduce the following refinement to the two-stage newsvendor solution. Since the system operates in the QED regime when $X \geq \beta^*$, we refer to this policy as the *two-stage QED staffing rule* and denote it by $u_{2,QED}$.

Definition 2 (two-stage QED staffing rule) *For $\alpha \in (1/2, 1)$, the two-stage QED staffing rule prescribes staffing levels as follows:*

1. *At the base stage, the base-stage staffing level is*

$$N_1 := \lambda/\mu + \beta^*(\lambda/\mu)^\alpha + O(\sqrt{\lambda/\mu}).$$

2. *At the surge stage, the surge-stage staffing level is*

$$N_2(N_1, \Lambda) := (\Lambda/\mu + \eta^* \sqrt{\Lambda/\mu} - N_1)^+ + o_{UI}(\sqrt{\lambda/\mu}).$$

Similar to Definition 1, Definition 2 characterizes a family of two-stage QED staffing rules, where some flexibility of order $O(\sqrt{\lambda/\mu})$ is allowed at the base stage staffing and flexibility of order $o_{UI}(\sqrt{\lambda/\mu})$ is allowed at the surge stage staffing; see Section 1.3 for the definition of $o_{UI}(\cdot)$. To simplify the exposition, we use the two-stage QED staffing rule to refer to any staffing specification in this family. In the two-stage QED staffing rule, the base-stage staffing level is of a similar form as in the two-stage newsvendor solution. After the arrival rate is realized at the surge stage, we first compute the optimal staffing level in the QED regime, and then bring up the staffing level to meet that target. Let $\mathcal{C}^\lambda_{2,QED}$ denote the expected total cost in (2) under the two-stage QED staffing rule. The two-stage QED staffing rule guarantees a smaller optimality gap than the two-stage newsvendor solution as quantified in the following theorem.

Theorem 3 (optimality gap of $\boldsymbol{u_{2,QED}}$) *For $\alpha \in (1/2, 1)$, the two-stage QED staffing rule in Definition 2 has $\mathcal{C}^\lambda_{2,QED} - \mathcal{C}^\lambda_{2,*} = o(\sqrt{\lambda})$.*

Theorem 3 establishes that any two-stage QED staffing rule achieves the same $o(\sqrt{\lambda})$ optimality gap. While it is intuitive that the $o_{UI}(\sqrt{\lambda/\mu})$ flexibility term in N_2 does not influence the optimality gap, it is less straightforward to see the effect of the $O(\sqrt{\lambda/\mu})$ flexibility term in N_1. We next provide a brief explanation for this (a more detailed explanation can be found in the proof of Theorem 3 in Appendix E). Let D_1 denote the $O(\sqrt{\lambda/\mu})$ term we add in N_1. This generates a staffing cost of $c_1 D_1$ at the base stage. At the surge stage, for λ sufficiently large, When $X \geq \beta^*$, adding D_1 to N_1 leads to a reduction of D_1 in N_2, which decreases the surge staffing cost by $c_2 D_1$. When $X < \beta^*$, adding D_1 to N_1 does not change N_2. By the construction of β^*, we have $\mathbb{P}(X \geq \beta^*) = c_1/c_2$. Then, the total staffing cost change is $c_1 D_1 - c_2 D_2 \mathbb{P}(X \geq \beta^*) = 0$. Furthermore, in both scenarios, the holding cost (expected steady-state queue length) does not change significantly (i.e., the change is of order $o(\sqrt{\lambda/\mu})$). Therefore, having a flexible term of order $O(\sqrt{\lambda/\mu})$ in N_1 does not impact the optimality gap.

We provide numerical demonstrations of Theorem 3 in Sections 4.3 and 5. Importantly, in Section 4.3, we numerically examine which specification of the flexibility terms, i.e., the $O(\sqrt{\lambda/\mu})$ term in N_1 and $o_{UI}(\sqrt{\lambda/\mu})$ term in N_2, achieves better performance for pre-limit finite stochastic systems.

4.3. Effective Translation of The Two-Stage QED Staffing Rule to Small Systems

Theorem 3 shows that any policy that belongs to the family of the two-stage QED staffing rules in Definition 2 achieves an optimality gap of $o(\sqrt{\lambda})$. The specification of the $O(\sqrt{\lambda/\mu})$ term in N_1 and the $o_{UI}(\sqrt{\lambda/\mu})$ term in N_2, though asymptotically indistinguishable in the context of Theorem 3, may have non-negligible impact on system performance for a finite system, especially when λ is small. We next numerically investigate system performance under different specifications of the two-stage QED staffing rule.

To this end, we consider staffing prescriptions of the form

$$N_1 = \lambda/\mu + \beta^*(\lambda/\mu)^\alpha + k\sqrt{\lambda/\mu} \quad \text{and} \quad N_2(N_1,\Lambda) = (\Lambda/\mu + \eta^*\sqrt{\Lambda/\mu} - N_1)^+, \quad \text{for } k \in \mathbb{R}. \tag{12}$$

We consider systems with small arrival rates, namely, setting $\lambda = 25, 50, 75, 100$. We vary the value of k in (12) from -3 to 3 in increments of 1. In each experiment, we estimate the steady-state cost by averaging over 1000 realizations of the random variable X. For each mean arrival rate λ, we compare the costs under different values of k, and report the percentage gap between each cost under the examined policy and the exact optimal cost (obtained by exhaustive search) in Tables 2 and 3. For example, in Table 2, when $\lambda = 25$, the exact optimal cost is 39.47. In this case, the policy specification with $k = 1$ achieves a cost of 39.48 and thus has an optimality gap of $(39.48 - 39.47)/39.48 = 0.03\%$, which is the smallest among different values of k. The system with $k = -3$ achieves a cost of 49.75 which corresponds to a percentage gap of 20.66%. In all experiments, the random variable X is assumed to follow a standard normal distribution. The other system parameters and the resulting value of (β^*, η^*) are listed in the caption of the tables.

Table 2 System performance (optimality gap) under different specifications of the two-stage QED staffing rule with $\beta^* = 0, \eta^* = 0.610$

($\mu = 1, \gamma = 0.1, \alpha = 0.75, h = 1.5, a = 3, c_1 = 1, c_2 = 2$)

λ \ k	-3	-2	-1	0	1	2	3
25	20.66%	13.73%	6.91%	2.08%	0.03%	2.01%	7.37%
50	15.04%	9.61%	4.77%	1.47%	0.00%	1.28%	4.98%
75	12.56%	7.69%	3.99%	1.18%	0.00%	0.98%	4.09%
100	10.44%	6.35%	3.02%	0.87%	0.00%	1.04%	3.76%

We first observe from the tables that even though all the staffing prescriptions, i.e., k ranging from -3 to 3, are asymptotically optimal, there are substantial differences in the pre-limit performances.

Table 3 System performance (optimality gap) under different specifications of the two-stage QED staffing rule with $\beta^* = 1.282, \eta^* = -0.140$ ($\mu = 1, \gamma = 0.1, \alpha = 0.75, h = 1.5, a = 3, c_1 = 1, c_2 = 10$)

λ \ k	-3	-2	-1	0	1	2	3
25	43.49%	25.83%	10.35%	1.28%	2.64%	9.64%	17.46%
50	31.77%	17.38%	6.51%	0.42%	1.39%	6.57%	12.90%
75	25.86%	14.42%	5.18%	0.20%	1.10%	5.60%	11.35%
100	20.84%	10.35%	3.40%	0.04%	1.67%	5.71%	10.66%

In Table 2, $k = 1$ leads to the best performance across all system scales tested. In Table 3, $k = 0$ leads to the best performance. Second, k has a highly nonlinear effect on the cost. Staffing too few servers tends to result in a larger optimality gap than staffing too many servers at the base stage. In particular, in both tables, $k = -3$ leads to the worst performance. In Table 3, when $\lambda = 25$ and $k = -3$, the percentage gap can be as large as 43.49%. Third, we note that as the system scale grows, the performance gap among different policies shrinks. This is consistent with our optimality gap quantification in Theorem 3. Lastly, we note that when k is properly tuned, $u_{2,QED}$ can achieve a very small optimality gap even for very small systems. For example, when $\lambda = 25$, the gap is 0.03% for $k = 1$ in Table 2 and 1.28% for $k = 0$ in Table 3.

Besides the experiments reported in Tables 2 and 3, we also summarize a few more sets of simulation results with different surge staffing costs in Appendix I.1. Among all the numerical experiments, we find the following specification of the two-stage QED staffing rule to be effective and robust for small-scale systems:

$$N_1 = \lambda/\mu + \beta^*(\lambda/\mu)^\alpha + \eta^*\sqrt{\lambda/\mu}, \quad \text{and} \quad N_2(N_1, \Lambda) = (\Lambda/\mu + \eta^*\sqrt{\Lambda/\mu} - N_1)^+. \tag{13}$$

The capacity prescription in (13) lends itself to an intuitive explanation. At the base stage, the staffing level consists of the offered load, a hedging against arrival-rate uncertainty, and a hedging against system stochasticity catered to the mean arrival rate λ. At the surge stage, the staffing level is raised to reach the optimal value in the QED regime catered to the realized arrival rate.

5. Numerical Experiments

In this section, we perform numerical experiments to demonstrate the cost savings of our proposed two-stage staffing rules over single-stage benchmark policies for different levels of arrival rate uncertainty and cost rates. We also examine the optimality gaps between the two-stage staffing rules and the numerically-solved exact optimal staffing levels. Moreover, we check the robustness of our proposed staffing rules when the service time distribution is lognormal. For comparison, we consider the following five staffing rules:

(I). Our proposed two-stage QED staffing rule $u_{2,QED}$ prescribes staffing levels

$$N_1 = \lambda/\mu + \beta^*(\lambda/\mu)^\alpha + \eta^*\sqrt{\lambda/\mu}, \quad \text{and} \quad N_2(N_1,\Lambda) = (\Lambda/\mu + \eta^*\sqrt{\Lambda/\mu} - N_1)^+,$$

for $\beta^* = \bar{F}_X^{-1}(c_1/c_2)$, and η^* defined in (11).

(II). Our derived two-stage newsvendor solution $u_{2,NV}$ assigns staffing levels

$$N_1 = \lambda/\mu + \beta^*(\lambda/\mu)^\alpha, \quad \text{and} \quad N_2(N_1,\Lambda) = (X - \beta^*)^+(\lambda/\mu)^\alpha.$$

(III). The single-stage newsvendor solution $u_{1,NV}$ prescribes staffing levels

$$N_1 = \lambda/\mu + \bar{F}_X^{-1}\left(\frac{c_1}{h\mu/\gamma + a\mu}\right)(\lambda/\mu)^\alpha, \quad \text{and} \quad N_2(N_1,\Lambda) = 0.$$

This policy accounts for arrival-rate uncertainty, but does not allow surge staffing.

(IV). The conventional single-stage square-root staffing rule, denoted by $u_{1,QED}$, makes a one-time staffing decision at the base stage, assuming a staffing cost of c_1 and a deterministic arrival rate of λ. In particular, the staffing levels are given by

$$N_1 = \lambda/\mu + \eta^*_{1,QED}\sqrt{\lambda/\mu}, \quad \text{and} \quad N_2(N_1,\Lambda) = 0,$$

where $\eta^*_{1,QED}$ is defined as

$$\eta^*_{1,QED} := \underset{\eta\in\mathbb{R}}{\arg\min}\ c_1\eta + \left(\frac{h\mu}{\gamma} + a\mu\right)\frac{\sqrt{\frac{\gamma}{\mu}}\left[H\left(\eta\sqrt{\frac{\mu}{\gamma}}\right) - \eta\sqrt{\frac{\mu}{\gamma}}\right]}{1 + \sqrt{\frac{\gamma}{\mu}}\frac{H\left(\eta\sqrt{\frac{\mu}{\gamma}}\right)}{H(-\eta)}}. \tag{14}$$

This policy ignores arrival-rate uncertainty. It is important to distinguish $\eta^*_{1,QED}$ in (14) (used in the single-stage square-root staffing rule) from η^* in (11) (used in the two-stage QED staffing rule). While both serve as coefficients in front of the hedging against system stochasticity, $\eta^*_{1,QED}$ is calculated assuming a staffing cost of c_1 (base-stage cost) and η^* is calculated assuming a staffing cost of c_2 (surge-stage cost).

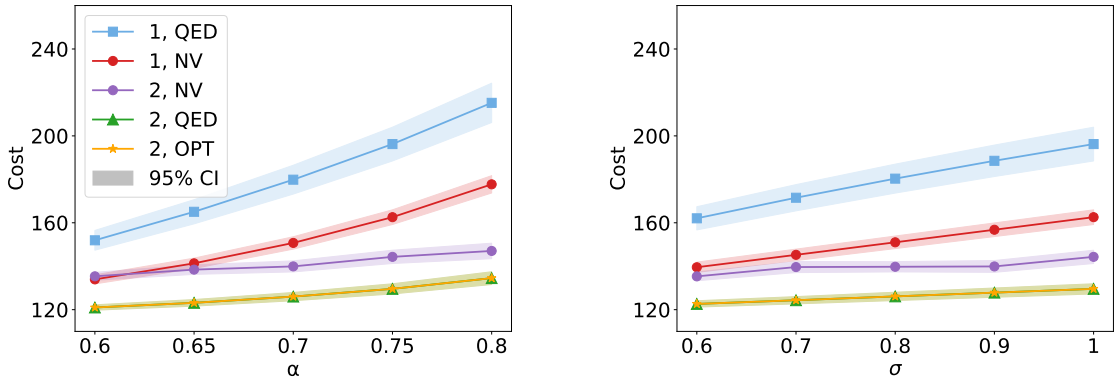
(V). The optimal two-stage staffing rule, denoted as $u_{2,OPT}$. We numerically solve for the optimal staffing levels via simulation optimization. Calculating the exact optimal staffing levels enables us to examine the optimality gaps characterized asymptotically in Theorems 2 and 3 for finite stochastic systems.

5.1. Level of Arrival-Rate Uncertainty

In the first set of experiments, we examine the cost savings of the proposed two-stage QED rule as we vary the magnitude of arrival-rate uncertainty. In particular, we assume that the random variable X is normally distributed with mean 0 and standard deviation σ. We vary the order of arrival-rate uncertainty, α, and the standard deviation of X, σ, respectively, with everything else held constant. We simulate 1000 realizations of X and calculate the expected steady-state cost (where the expectation is taken over the stochastic fluctuations) for each realization. The expected total cost (where the expectation is taken over the random variable X) is then averaged over the expected steady-state costs for all realizations of X.

Figure 1 illustrates the expected total costs under the five policies, with α increasing from 0.6 to 0.8 in Figure 1(a) and σ increasing from 0.6 to 1 in Figure 1(b). We observe that $u_{1,QED}$ performs the worst as it does not take the arrival rate uncertainty into account. As the level of arrival-rate uncertainty increases, the performance gap between the one-stage policies ($u_{1,QED}$ or $u_{1,NV}$) and two-stage policies ($u_{2,NV}$ or $u_{2,QED}$) increases as suggested by Theorem 1. Lastly, compared to $\mu_{2,OPT}$, $u_{2,QED}$ achieves almost the exact optimal performance (i.e., the lines for $\mu_{2,OPT}$ and $u_{2,QED}$ are almost identical). This suggests that the $o(\sqrt{\lambda})$ bound for the optimality gap developed in Theorem 3 is likely to be conservative. Meanwhile, $u_{2,NV}$ still has a considerable optimality gap.

Figure 1 **Sensitivity analysis with respect to the order of arrival-rate uncertainty**
((a): $\lambda = 100, \mu = 1, \gamma = 0.1, h = 1.5, a = 3, c_1 = 1, c_2 = 1.5, \sigma = 1$
(b) : $\lambda = 100, \mu = 1, \gamma = 0.1, h = 1.5, a = 3, c_1 = 1, c_2 = 1.5, \alpha = 0.75$)



(a) Sensitivity with respect to α

(b) Sensitivity with respect to σ

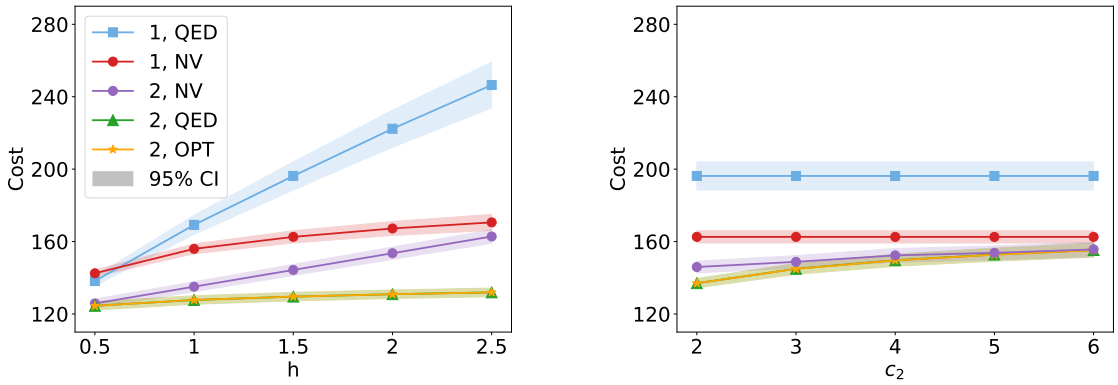
5.2. Cost Rates

We next investigate the performance of our proposed two-stage policy with respect to the cost parameters. We first compare the costs of the three policies under different holding costs, h, in

Figure 2(a). Note $u_{2,QED}$ outperforms $u_{1,QED}$ and $u_{1,NV}$ by a larger magnitude as the holding cost becomes larger. However, the cost saving under $u_{2,NV}$ relative to $u_{1,NV}$ increases as h increases. This is likely because as h increases we put more emphasis on the queueing cost, but $u_{1,NV}$ and $u_{2,NV}$ do not take the detailed queueing dynamics into account. Figure 3 demonstrates the distribution of the average steady-state queue length for a given value of h over 1000 realizations of X under $u_{1,NV}$ and $u_{2,QED}$. We observe that the average steady-stage queue length under $u_{1,NV}$ is either very high or very low, while $u_{2,QED}$ leads to more stable performance. This is not surprising. On one hand, the two-stage QED staffing rule is able to circumvent understaffing when the realized arrival rate is excessively large by adding staff at the surge stage. In contrast, due to the inability to adjust staffing levels at the surge stage, the single-stage policies can result in a relatively larger queue when the realized arrival rate is large. On the other hand, $u_{1,NV}$ tends to assign more base staff to hedge against arrival-rate uncertainty, which can result in overstaffing when the realized arrival rate is low.

Besides the holding cost, we also vary the surge-stage staffing cost, c_2. Recall from Assumption 1 that the surge staffing cost is larger than the base staffing cost c_1, but smaller than the performance cost $h\mu/\gamma + a\mu$. In the numerical experiment depicted in Figure 2(b), we set $c_1 = 1, h\mu/\gamma + a\mu = 18$, and vary c_2 from 2 to 6. We see that the cost saving of the proposed two-stage policies decreases as c_2 increases. For example, the performance of $u_{2,QED}$ becomes nearly indistinguishable from that of $u_{1,NV}$ when c_2 reaches 6. This is because when the surge staffing costs are very large, the two-stage policy will hardly ever surge, even though it has the option to. Lastly, we again observe that $u_{2,QED}$ achieves almost the optimal performance in various scenarios tested.

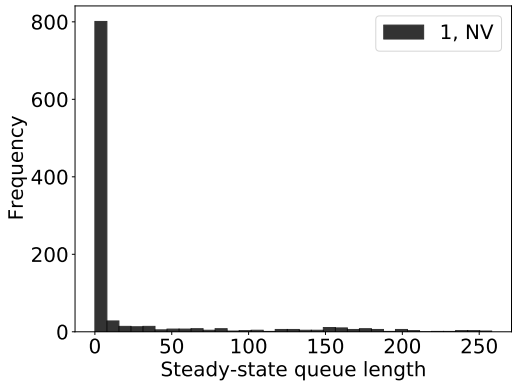
Figure 2 **Sensitivity analysis with respect to the cost rates**
((a): $\lambda = 100, \mu = 1, \gamma = 0.1, a = 2h, c_1 = 1, c_2 = 1.5, \alpha = 0.75, \sigma = 1$
(b): $\lambda = 100, \mu = 1, \gamma = 0.1, h = 1.5, a = 3, c_1 = 1, \alpha = 0.75, \sigma = 1$)



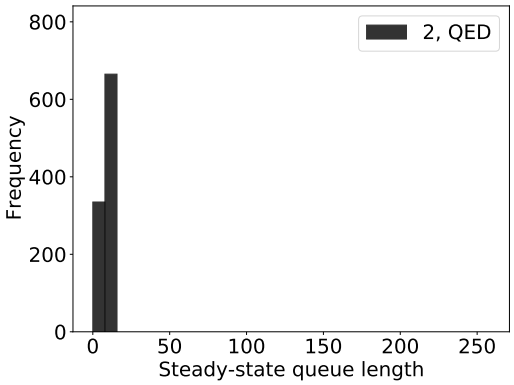
(a) Sensitivity with respect to h

(b) Sensitivity with respect to c_2

Figure 3 **Distribution of the average steady-state queue length**
$(\lambda = 100, \mu = 1, \gamma = 0.1, h = 1.5, a = 3, c_1 = 1, c_2 = 1.5, \alpha = 0.75, \sigma = 1)$



(a) Single-stage newsvendor solution (mean = 19.131, std = 49.070)



(b) Two-stage QED staffing rule (mean = 8.029, std = 5.305)

Remark 3 *The numerical experiments in Sections 5.1 and 5.2 suggest that the cost difference between the two-stage QED staffing rule and the exact two-stage optimum is very small. This suggests that it may be possible to refine the optimality gap* $o(\sqrt{\lambda})$ *in Theorem 3. This requires substantial methodological developments, such as those in Gurvich et al. (2014) and Randhawa (2016), which we reserve as an interesting future research direction.*

5.3. Lognormal Service Time Distribution

To achieve analytical tractability, we assume an exponential service time distribution. However, it is quite common to have service times with heavier tails in real healthcare systems; see, e.g., Armony et al. (2015) for inpatient wards and Section 7 for our partner ED. As such, a lognormal service time distribution can often be a better fit than an exponential service time distribution. In this section, we conduct numerical experiments to examine the performance of the proposed staffing rules under lognormal service time distributions. We consider lognormal service times with a fixed mean at 1 and vary the variance with values from 0.25 to 2.25 in increments of 0.25. For each value of the variance, we numerically solve for the optimal staffing levels. We then compare $u_{2,OPT}$, $u_{2,QED}$, and $u_{1,NV}$ Note that the latter two policies are based on the assumption of exponential service time distribution.

Table 4 summarizes the base staffing levels and average surge staffing levels under each policy for different variances of the service time distribution. Because the two-stage QED rule and single-stage newsvendor solution only depend on the service distribution through its mean, they are identical for all variances. We also report the optimality gaps of $u_{2,QED}$ (relative to $u_{2,OPT}$) in the second-to-last column, as well as the cost savings of $u_{2,QED}$ compared to $u_{1,NV}$ in the last column.

We observe that $u_{2,QED}$ performs similarly to the optimal staffing policy. The optimality gaps are less than 6%. $u_{2,QED}$ achieves significant cost savings over $u_{1,NV}$ in all cases. Moreover, there do not exist any directional trends in the optimal base and surge staffing levels as the variance of the lognormal service time distribution increases. Therefore, we do not recommend any adjustment to the two-stage QED rule in situations where service times follow a lognormal distribution.

Table 4 Optimality gaps and cost savings under lognormal service time distributions
($\lambda = 100, \gamma = 0.1, h = 1.5, a = 3, c_1 = 1, c_2 = 1.5, \alpha = 0.75, \sigma = 1$)

Variance	Two-stage OPT		Two-stage QED		Single-stage NV	Opt gap	Cost savings over $u_{1,NV}$
	Base	Avg surge	Base	Avg surge	Base		
0.25	90	20.86	94	19.34	150	0.92%	18.17%
0.5	90	20.78	94	19.34	150	1.57%	17.47%
0.75	96	17.26	94	19.34	150	1.13%	18.52%
1	96	17.26	94	19.34	150	1.06%	18.36%
1.25	90	21.02	94	19.34	150	0.99%	18.36%
1.5	90	20.58	94	19.34	150	2.21%	17.75%
1.75	94	18.44	94	19.34	150	1.85%	17.49%
2	86	19.32	94	19.34	150	5.86%	18.88%
2.25	86	19.48	94	19.34	150	5.29%	19.70%

6. Model Extension: Incorporation of Surge-Stage Prediction Error

In the two-stage optimization problem (2), we assume that the realization of the random arrival rate Λ is known exactly at the surge stage. That is, the surge-stage prediction model provides perfect arrival rate information. However, in practice, the surge-stage predictive models may incur some prediction errors. In this section, we investigate a model extension where we allow prediction errors in the surge stage.

To incorporate prediction error, we further decompose the random arrival rate into two terms: predictable and unpredictable arrival rate uncertainty. In particular, we consider a random arrival rate of the form

$$\Lambda = \lambda + Y\lambda^{\alpha}\mu^{1-\alpha} + Z\lambda^{\nu}\mu^{1-\nu}, \tag{15}$$

where $\alpha \in (1/2, 1)$, $\nu \in (0, \alpha]$, and Y and Z are continuous random variables independent of each other. We assume that $\mathbb{E}[Y] = \mathbb{E}[Z] = 0$, $\mathbb{E}[|Y|] < \infty$, and $\mathbb{E}[|Z|] < \infty$. In (15), Y and Z can be understood as the *predictable* and *unpredictable* arrival-rate uncertainty, respectively. If there is a prediction model to forecast demand at the surge stage, then $Y\lambda^{\alpha}\mu^{1-\alpha}$ is the predicted arrival rate and $Z\lambda^{\nu}\mu^{1-\nu}$ is the error (residual) of the prediction model. α captures the scale of the arrival-rate uncertainty and ν captures the scale of the prediction error. It is reasonable to assume that

the distributions of Y and Z are known at the base stage. The two-stage staffing problem with prediction error is then formulated as

$$\min_{N_1}\left\{c_1N_1+\mathbb{E}\left[\min_{N_2(N_1,Y)}\{c_2N_2(N_1,Y)+(h+a\gamma)\,\mathbb{E}\left[Q(N_1+N_2(N_1,Y),\Lambda)|Y\right]\}\right]\right\}. \tag{16}$$

To differentiate notation from that of problem (2), we denote the optimal objective value of (16) as $\mathcal{C}^{e,\lambda}_{2,*}$ when there is prediction error at the surge stage.

Similar to problem (2), we compare to the single-stage optimization problem (4) for Λ in form of (15), and use $\mathcal{C}^{e,\lambda}_{1,*}$ to denote its optimal objective value. To facilitate the connection between the arrival rates in (3) and (15), we can let X be such that

$$X\lambda^{\alpha}\mu^{1-\alpha}=Y\lambda^{\alpha}\mu^{1-\alpha}+Z\lambda^{\nu}\mu^{1-\nu}. \tag{17}$$

In this context, problem (2) can be seen as an "oracle" problem that knows the exact realized arrival rate at the surge stage. We use $\mathcal{C}^{o,\lambda}_{2,*}$ to denote the optimal objective value of the oracle problem (2) for Λ in form of (15). In particular, the oracle problem does not incur any unpredictable arrival-rate uncertainty (prediction error). Intuitively, the impact of the prediction error should depend on how substantial it is. We formalize this for "small" and "moderate/large" prediction errors in the next subsections. The error regime depends on the relationship between the scale of the arrival-rate uncertainty and that of the prediction error.

6.1. Small Prediction Error: $0<\nu<1/2$

When $\nu\in(0,1/2)$, the prediction error is sufficiently small to be "ignored." Doing so does not impact performance. For problem (16), we propose the *two-stage error policy* and denote it by $u_{2,ERR}$.

Definition 3 (two-stage error policy for $\nu<1/2$) *For $\alpha\in(1/2,1)$ and $\nu\in(0,1/2)$, the two-stage error policy prescribes staffing levels as follows:*

1. *At the base stage, the base-stage staffing level is*

$$N_1:=\lambda/\mu+\bar{F}_Y^{-1}\left(c_1/c_2\right)(\lambda/\mu)^{\alpha}+O(\sqrt{\lambda/\mu}).$$

2. *At the surge stage, the surge-stage staffing level is*

$$N_2(N_1,Y):=((\lambda+Y\lambda^{\alpha}\mu^{1-\alpha})/\mu+\eta^*\sqrt{(\lambda+Y\lambda^{\alpha}\mu^{1-\alpha})/\mu}-N_1)^+ +o_{UI}(\sqrt{\lambda/\mu}),$$

for η^* defined in (11).

When $\nu \in (0, 1/2)$, $u_{2,ERR}$ is similar to $u_{2,QED}$, the latter of which is defined for the case without prediction error. In particular, $u_{2,ERR}$ completely ignores the existence of prediction error Z and makes staffing decisions based on Y only. Let $\mathcal{C}^{e,\lambda}_{2,ERR}$ denote the expected total cost under $u_{2,ERR}$ when the mean arrival rate is λ. Analogous results to Theorems 1 and 3 hold for $u_{2,ERR}$.

Proposition 2 *For $\alpha \in (1/2, 1)$ and $\nu \in (0, 1/2)$, we have*

(I) Cost saving: $\mathcal{C}^{e,\lambda}_{1,*} - \mathcal{C}^{e,\lambda}_{2,*} = \Theta(\lambda^\alpha)$.

(II) Optimality gap: $\mathcal{C}^{e,\lambda}_{2,ERR} - \mathcal{C}^{e,\lambda}_{2,*} = o(\sqrt{\lambda})$.

(III) Cost of prediction error: $\mathcal{C}^{e,\lambda}_{2,*} - \mathcal{C}^{o,\lambda}_{2,*} = o(\sqrt{\lambda})$.

Item (III) in Proposition 2 quantifies the gap between the two-stage optimal costs with and without prediction error. We observe that when the prediction error is small, i.e., $\nu < 1/2$, there is not much value, from the cost-saving perspective, to further improve the prediction accuracy.

6.2. Moderate to Large Prediction Error: $1/2 \le \nu \le \alpha$

When $\nu \in [1/2, \alpha]$, the prediction error is of a larger order than the system stochasticity and thus can no longer be ignored for staffing. To derive a near-optimal solution to problem (16), we consider the following stochastic-fluid optimization problem

$$\min_{N_1} \left\{ c_1 N_1 + \mathbb{E}\left[\min_{N_2(N_1,Y)} \left\{ c_2 N_2(N_1, Y) + (h\mu/\gamma + a\mu)\mathbb{E}\left[(\Lambda/\mu - N_1 - N_2(N_1, Y))^+ \,|Y \right] \right\} \right] \right\}. \quad (18)$$

Let $(\bar{N}_1, \bar{N}_2(\bar{N}_1, Y))$ denote an optimal solution to (18), whose existence is rigorously established in the proof of Proposition 3. When $\nu \in [1/2, \alpha]$, we define the two-stage error policy, $u_{2,ERR}$, to prescribe staffing levels $(\bar{N}_1, \bar{N}_2(\bar{N}_1, Y))$.

When $1/2 < \nu < \alpha$ (moderate prediction error), the prediction error is of a smaller order than the predictable arrival-rate uncertainty. In this case, we still expect that resolving some of the arrival-rate uncertainty at the surge stage can bring a cost saving as large as $O(\lambda^\alpha)$ compared to the optimal single-stage staffing rule. When $\nu = \alpha$ (large prediction error), the prediction error is of the same order as the predictable arrival-rate uncertainty. The following assumption requires that the predictable arrival-rate uncertainty is sufficiently large compared to the unpredictable arrival-rate uncertainty when $\nu = \alpha$. If Assumption 2 holds, resolving the predictable arrival rate uncertainty could still lead to $\Theta(\lambda^\alpha)$ cost savings when compared to the optimal single-stage staffing rule. In contrast, if Assumption 2 does not hold, the predictable uncertainty is so small compared to the unpredictable uncertainty that resolving Y only leads to limited cost savings.

Assumption 2 *There exists $p \in (0, 1]$ such that*

$$Y + \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right) - \bar{F}_{Y+Z}^{-1}\left(\frac{c_1}{h\mu/\gamma + a\mu}\right) > 0 \quad \textit{with probability } p.$$

Note that Assumption 2 can be violated when Y has a bounded support, c_2 is large, and/or Z has a large standard deviation. For a concrete example, consider $Y \sim \text{Uniform}[-1,1]$, $Z \sim \text{Normal}(0,10^2)$, $h\mu/\gamma + a\mu = 1$, $c_1 = 0.1$, and $c_2 = 0.9$. In this case, $Y + \bar{F}_Z^{-1}\left(c_2/(h\mu/\gamma + a\mu)\right) < 0$ and $\bar{F}_{Y+Z}^{-1}\left(c_1/(h\mu/\gamma + a\mu)\right) > 0$ with probability 1.

Proposition 3 *For $\alpha \in (1/2, 1)$ and $\nu \in [1/2, \alpha]$, we have*

(I) Cost saving: *If $\nu < \alpha$, then $\mathcal{C}_{1,*}^{e,\lambda} - \mathcal{C}_{2,*}^{e,\lambda} = \Theta(\lambda^\alpha)$. If $\nu = \alpha$ and Assumption 2 holds, then $\mathcal{C}_{1,*}^{e,\lambda} - \mathcal{C}_{2,*}^{e,\lambda} = \Theta(\lambda^\alpha)$. If $\nu = \alpha$ and Assumption 2 does not hold, then $\mathcal{C}_{1,*}^{e,\lambda} - \mathcal{C}_{2,*}^{e,\lambda} = o(\lambda^\alpha)$.*

(II) Optimality gap: $\mathcal{C}_{2,ERR}^{e,\lambda} - \mathcal{C}_{2,*}^{e,\lambda} = O(\sqrt{\lambda})$.

(III) Cost of prediction error: $\mathcal{C}_{2,*}^{e,\lambda} - \mathcal{C}_{2,*}^{o,\lambda} = \Theta(\lambda^\nu)$.

Comparing item (III) in Proposition 3 to item (III) in Proposition 2, we note that when having a large prediction error, there is potentially more cost savings we can gain by improving the prediction accuracy. In particular, when $\nu \geq 1/2$, the cost saving brought by a more accurate prediction model can be as large as $\Theta(\lambda^\nu)$.

6.3. Numerical Experiments for Models with Prediction Error

We conduct numerical experiments in the presence of prediction errors, and focus on the case where the magnitude of prediction error is the most salient, namely, $\nu = \alpha$.

We compare the following five staffing rules:

(I) The two-stage error policy $u_{2,ERR}$ introduced in Section 6.2. It has near-optimal performance as established in Proposition 3.

(II) The two-stage QED rule $u_{2,QED}$, which is a straightforward extension of the two-stage QED rule defined in Definition 2 by ignoring the prediction error: For X defined in (17) (namely, $X := Y + Z$), it assigns

$$N_1 = \lambda/\mu + \bar{F}_X^{-1}\left(c_1/c_2\right)(\lambda/\mu)^\alpha + \eta^*\sqrt{\lambda/\mu}$$
$$N_2(N_1, Y) = \left((\lambda + Y\lambda^\alpha\mu^{1-\alpha})/\mu + \eta^*\sqrt{(\lambda + Y\lambda^\alpha\mu^{1-\alpha})/\mu} - N_1\right)^+.$$

The staffing prescription takes into account the distribution of X at the base stage, but uses the realization of Y as a proxy for the realization of X at the surge stage. To simplify notation, we still refer to this policy as $u_{2,QED}$ in the following experiments.

(III) The single-stage newsvendor solution $u_{1,NV}$ as defined in Section 5, assuming we know the distribution of X. Note that for a fixed distribution of X, the single-stage staffing rule and its performance will not be affected by the surge-stage prediction errors.

(IV) The single-stage square-root staffing rule $u_{1,QED}$ as defined in Section 5.

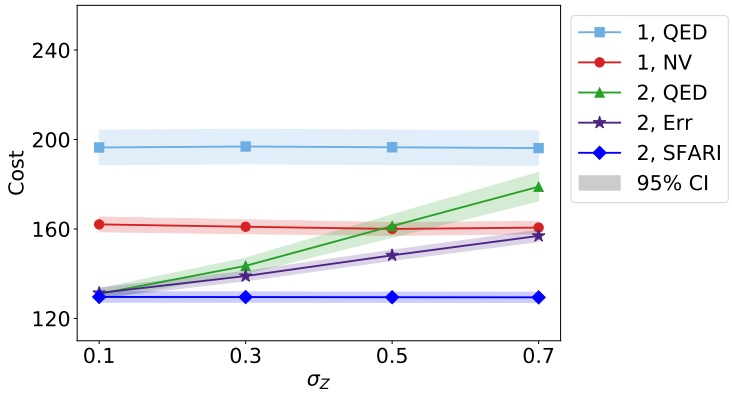
(V) To demonstrate the cost of prediction error, we also consider an oracle policy termed *second-stage full arrival rate information* (SFARI), and denote it by $u_{2,SFARI}$. It prescribes staffing levels

$$N_1 = \lambda/\mu + \bar{F}_X^{-1}(c_1/c_2)(\lambda/\mu)^{\alpha} + \eta^*\sqrt{\lambda/\mu} \quad \text{and} \quad N_2(N_1,\Lambda) = (\Lambda/\mu + \eta^*\sqrt{\Lambda/\mu} - N_1)^+,$$

for η^* defined in (11). Note that $u_{2,SFARI}$ is identical to $u_{2,QED}$ when there is full arrival rate information at the surge stage. It provides a performance lower bound.

We assume that Y and Z are normally distributed with standard deviation σ_Y and σ_Z, respectively. We then fix the standard deviation of X to be equal to 1, i.e., $\sigma_Y^2 + \sigma_Z^2 = 1$, and vary σ_Z from 0.1 to 0.7 in increments of 0.2. For each policy and each value of σ_Z, we simulate 1000 independent and identically distributed realizations of the random arrival rate, and use the average to approximate the expected total cost. Figure 4 compares the costs under the five policies with different values of σ_Z. Note that, as expected, the single-stage benchmark policies ($u_{1,NV}$ and $u_{1,QED}$) and the oracle policy ($u_{2,SFARI}$) are unaffected by prediction accuracy. In contrast, the performance of our proposed two-stage policies ($u_{2,ERR}$ and $u_{2,QED}$) degrades as the prediction error increases. When σ_Z is larger than or equal to 0.5, $u_{2,QED}$ yields higher expected total cost than $u_{1,NV}$. On the other hand, $u_{2,ERR}$, which properly accounts for prediction errors, outperforms the benchmark single-stage policies for all σ_Z. As σ_Z increases from 0.1 to 0.7, the expected total cost under $u_{2,ERR}$ increases from 131.356 to 156.897. This further demonstrates the cost savings we can gain by improving the prediction accuracy. In practice, this can often be achieved by employing more sophisticated machine learning models or including more relevant real-time features.

Figure 4 **Sensitivity analysis with respect to prediction error**
($\lambda = 100, \mu = 1, \gamma = 0.1, h = 1.5, a = 3, c_1 = 1, c_2 = 1.5, \alpha = \nu = 0.75$)



7. Application to the Emergency Department

In this section, we develop a unified framework to guide the implementation of the proposed two-stage staffing policy in ED nurse staffing. Our framework consists of three key elements:

1) Estimating the arrival rate distribution, especially the order of arrival-rate uncertainty. This helps us determine whether the ED operates in an environment where surge staffing could be beneficial. In our partner hospital, α is estimated to be 0.769. According to Theorem 1, we can gain substantial cost savings by utilizing the surge staffing in this case.

2) Building an integrated two-stage prediction model that is synchronized with the staffing decision epochs. At the base stage we can only capture the day-of-the-week and day-versus-night effects, while at the surge stage, we can utilize more real-time information such as the severity profile of patients currently in the ED, the weather condition, etc.

3) Evaluating the prediction-driven staffing rule in a set of more complex simulation experiments that incorporates more realistic ED operational features. For our partner hospital, we incur a non-negligible prediction error at the surge stage. Thus, we employ $u_{2,ERR}$. We also modify $u_{2,ERR}$ to adjust for the transient-shift effects.

7.1. Background and Data

Our partner hospital, NYP CUMC, is an urban academic medical center in New York City. We focus on the Milstein ED at NYP CUMC, which is the main adult ED of the hospital and treats more than 90,000 patients annually. Nurses are typically scheduled for 12-hour shifts that begin at 7am (day shift) or 7pm (night shift) each day. The nursing schedules are set 4–8 weeks in advance and the staffing level is difficult to change in real time. If the ED manager anticipates a high patient volume close to the start of a shift, he/she can call in extra nurses. Currently, there lacks a data-driven approach to determine the appropriate surge staffing levels.

We provide the following remarks on the timing of the base and surge staffing epochs. For the theoretical model, the exact timing of the base and surge epochs can be flexible, as long as there are significant differences between the arrival-rate prediction accuracy and staffing costs at these two stages. Our theoretical results suggest that the two-stage staffing framework is able to achieve significant cost savings if 1) the order of arrival-rate uncertainty dominates the order of system stochasticity, and 2) the surge-stage prediction model is able to resolve much of the arrival-rate uncertainty. For the real-world application, the timing of the base and surge epochs depends on the feasible practice of the hospital, which determines what information is available at these stages (especially the surge stage) for prediction and staffing. Throughout Section 7, we assume an idealistic setting where the surge-stage planning can happen right before the focal shift, so we

"maximize" the amount of real-time information available. That said, having a reasonable amount of lead time (e.g., several hours before the shift start time) for surge staffing should not affect the results significantly. In the subsequent sections, we will discuss further how the lead time may affect surge-stage prediction and transient-shift adjustment.

We collect 12 months of data from February 1, 2018 to January 31, 2019. The data contain patient-level records that include 1) patient-flow time stamps (i.e., time stamps for arrival, first evaluation, lab and imaging orders, admission decision, and departure), 2) patient's demographics and severity (i.e., age, gender, arriving source, emergency severity index, chief complaint, comorbidities, and deposition decision), and 3) patient's lab and imaging requests (i.e., different tests and imaging that are ordered for the patient). We also collect data from two other sources: weather information (i.e., temperature, precipitation, snow, wind, etc.) and Google trend data (i.e., search volume for keywords such as "flu," "heart attack," "abuse," etc.). These data allow us to a) estimate arrival-rate uncertainty, b) build a two-stage prediction model where the surge-stage prediction model can utilize rich real-time information, and c) calibrate a simulation model that incorporates more real-world ED features to evaluate different staffing policies.

We first group the shifts into 14 different types based on the day of the week and day versus night. Table 5 provides some summary statistics for different shifts. We observe that the day shifts see more arrivals than the night shifts, and weekday day shifts see more arrivals than the weekend day shifts. We also note that the coefficient of variation can be as high as 14% for some shifts (e.g., Sunday night shift and Thursday night shift). This suggests that even after we control for day-of-the-week and day-versus-night effects, there can still be significant uncertainty in demand.

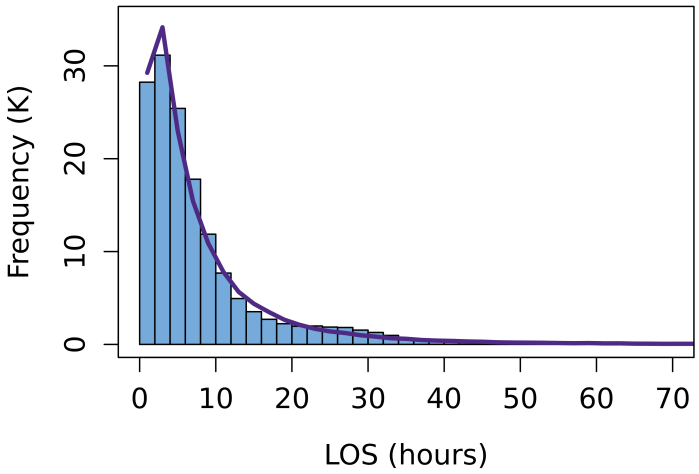
Table 5 Mean and standard deviation (std) of the shift-level arrival counts

Day shift							
	Sun	Mon	Tue	Wed	Thur	Fri	Sat
Mean	141.019	207.385	188.769	186.942	185.208	175.173	147.058
Std	15.788	21.503	20.701	23.657	21.004	16.124	12.095
Night shift							
	Sun	Mon	Tue	Wed	Thur	Fri	Sat
Mean	89.462	97.058	97.769	93.711	95.189	96.692	94.115
Std	12.698	12.064	10.547	12.508	13.602	12.199	11.514

The length of stay (LOS) for each patient is defined as the time interval between the first evaluation time and departure time. The average LOS in our ED is 8.156 hours due to a long boarding time for patients who need to be admitted into the hospital; see Figure 5 for the empirical LOS distribution. The average waiting time (calculated as the time between arrival and first evaluation)

is close to an hour, i.e., 0.975 hours, and the proportion of patients who left without being seen is 4.35%. Properly managing congestion is a key challenge faced by the ED. In what follows, we look into how our data-driven surge planning can help reduce congestion and staffing costs.

Figure 5 **Patient LOS distribution (The solid line illustrates the fitted lognormal distribution whose logarithm has a mean equal to 1.597 and a standard deviation equal to 1.050.)**



7.2. Estimating Arrival-Rate Uncertainty

In this section, we introduce statistical procedures to estimate the arrival-rate uncertainty. Because there are significant day-of-the-week and day-versus-night effects, the shifts are classified into 14 different types as demonstrated in Table 5. Let λ_i denote the mean arrival rate for type $i \in \mathcal{I} := \{1, ..., 14\}$ shift. Since we have one year of data, each shift type i has $n_i = 52$ observations. For each type of shift, we assume that the random arrival rate takes the form

$$\Lambda_i = \lambda_i + \lambda_i^{\alpha} \mu^{1-\alpha} X, \quad i \in \mathcal{I},$$

for μ equal to the inverse of the average LOS. In particular, the uncertainty scaling parameter α and the distribution of X is the same across different types of shifts. We also make the parametric assumption that $X \sim N(0, \sigma^2)$ for some $\sigma \in \mathbb{R}_+$; see Appendix G.4 for the normal probability plot to validate this assumption. Then $\Lambda_i \sim N(\lambda_i, \lambda_i^{2\alpha} \mu^{2(1-\alpha)} \sigma^2)$, $i \in \mathcal{I}$.

Let $L_i^{(k)}$ denote the observed arrival count for the kth shift of type i, $1 \le k \le n_i$. We also define $\bar{L}_i := \frac{1}{n_i} \sum_{k=1}^{n_i} L_i^{(k)}$ and $\Sigma_i^2 := \frac{1}{n_i} \sum_{k=1}^{n_i} (L_i^{(k)} - \bar{L}_i)^2$, i.e., the corresponding sample mean and sample variance. Based on the method of moments, we have the following system of equations for the estimators

$$\bar{L}_i = \hat{\lambda}_i, \quad \Sigma_i^2 = \hat{\lambda}_i^{2\hat{\alpha}} \mu^{2(1-\hat{\alpha})} \hat{\sigma}^2, \quad i \in \mathcal{I}. \tag{19}$$

It follows from (19) that

$$\log \Sigma_i = \hat{\alpha} \log \bar{L}_i + \log(\mu^{1-\hat{\alpha}} \hat{\sigma}), \quad i \in \mathcal{I}. \tag{20}$$

Then, we can fit $\hat{\alpha}$ and $\hat{\sigma}$ by solving the following least squares problem

$$\min_{\alpha \in (0,1), \gamma \in \mathbb{R}} \sum_{i=1}^{14} \left(\log \Sigma_i - \gamma - \alpha \log \bar{L}_i \right)^2. \tag{21}$$

In particular, let γ^* and α^* denote the optimal solution to the least squares problem (21). Then, $\hat{\alpha} = \alpha^*$ and $\mu^{1-\hat{\alpha}} \hat{\sigma} = \exp(\gamma^*)$.

In Table 6, the first row below the header (with $|\mathcal{I}| = 14$) summarizes the point estimates for α and $\mu^{1-\alpha}\sigma$; see Appendix G.1 for the full estimation results. We also report their corresponding 95% confidence intervals. Based on our estimation, for the Milstein ED, $\alpha = 0.769$ and $\mu^{1-\alpha} X \sim N(0, 0.348^2)$.

To check the robustness of our estimation, we also run a similar analysis by dividing the shifts into 56 different types. In particular, in addition to the day-of-the-week and day-versus-night effects, we also incorporate the season-of-the-year effect. The second row below the header (with $|\mathcal{I}| = 56$) summarizes estimation results (see also Appendix G.1), which are very close to our original estimation. Lastly, we also consider a non-parametric estimation proposed in Maman (2009), which works for $\alpha > 1/2$ only (see Appendix G.2 for more details). It gives the same results as our original estimation. Since it is a priori unclear for a real-world system whether $\alpha > 1/2$, our parametric estimation method, which allows $\alpha \in (0, 1)$, is preferred.

Table 6 Estimated α and standard deviation of X

Number of shift types	$\hat{\alpha}$	95% CI for $\hat{\alpha}$	$\mu^{1-\alpha}\hat{\sigma}$	95% CI for $\mu^{1-\alpha}\hat{\sigma}$
Day-of-week and day/night: $\lvert\mathcal{I}\rvert = 14$	0.769	(0.543, 0.994)	0.344	(0.114, 1.034)
Day-of-week, day/night and seasons: $\lvert\mathcal{I}\rvert = 56$	0.746	(0.558, 0.933)	0.362	(0.135, 0.902)

7.3. Two-Stage Prediction Model

To facilitate base and surge staffing decisions, we need to develop a two-stage prediction model that is synchronized with these decision epochs.

At the base stage, which is several weeks before the start of the shift, there is very limited information we can utilize for demand forecasting. The key features based on our analysis are the day-of-the-week effect and the day-versus-night effect. The stratified historical averages based on these features are able to capture 88.26% of the variability in shift-level arrival counts.

At the surge stage, which we assume is right before the start of the shift, we have access to more real-time information. We employ a recently developed linear regression model in Hu et al. (2023)

to forecast the realized arrival rate. The model utilizes the following five categories of features: (i) *Time-of-the-shift information*, including season of the year, day of the week, day versus night, and whether the shift takes place on, before, or after a national holiday; (ii) *Previous-shift arrival counts*, including the shift-level arrival count 1 day before the shift, the shift-level arrival count 7 days before the shift (a week ago), and a moving average of shift-level arrival counts over the past 30 days; (iii) *Patient severity*, which is the average of the weighted sum of a total of 17 Charlson comorbidity indices in ICD-10-CM coding for each patient over the past 3 days; (iv) *Google trends*, including the Google search volume for the keywords "depression" and "flu" in New York State for the week before the shift; (v) *Weather forecast*, including the minimum temperature, precipitation, snow, wind, and whether the maximum temperature exceeds 86°F on the day of the shift. The estimated coefficients for the covariates in the model are provided in Table 13 in Appendix I.2. This linear regression model is able to capture 90.84% of the variability in shift-level arrival counts. (Since we are fitting simple linear regression models, we use the entire one-year of data as the training set.)

The root mean-square error (RMSE) of the prediction model is 15.860 at the base stage, and 14.009 at the surge stage. The real-time information is able to reduce the RMSE by 11.67%. That said, what we are more interested in is the value of this gained accuracy in making staffing decisions. We shall investigate this in the next subsection.

We use the random arrival-rate model with prediction error, i.e., (15), and estimate ν and the distribution of Z next. We assume Z follows a normal distribution with zero mean; see Appendix G.4 for the normal probability plot to validate this assumption. Then, we can estimate ν and the standard deviation of Z following a similar procedure as that developed in Section 7.2 for α and the standard deviation of X (the detailed estimation procedure is provided in Appendix G.3). This gives us $\hat{v} = 0.508$ and $Z \sim N(0, 1.067)$.

We conclude this subsection with two remarks. First, in situations where the surge-stage decision epoch has a lead time (e.g., several hours before the start time of the focal shift), the surge-stage prediction model needs to be modified by only using the available information at the decision epoch. This is likely to reduce the prediction accuracy. Second, the base-stage prediction model can be improved by including more features (e.g., holiday information) or using more advanced prediction techniques. We view our results here as a simple proof of concept and refer to Hu et al. (2023) for more informatics-oriented development on ED demand prediction.

7.4. ED-Adapted Two-Stage Staffing Rule

To examine the performance of the proposed two-stage staffing rule, we build a queueing model to simulate the patient flow process in Milstein ED over 52 weeks from February 1, 2018 to January 31, 2019.

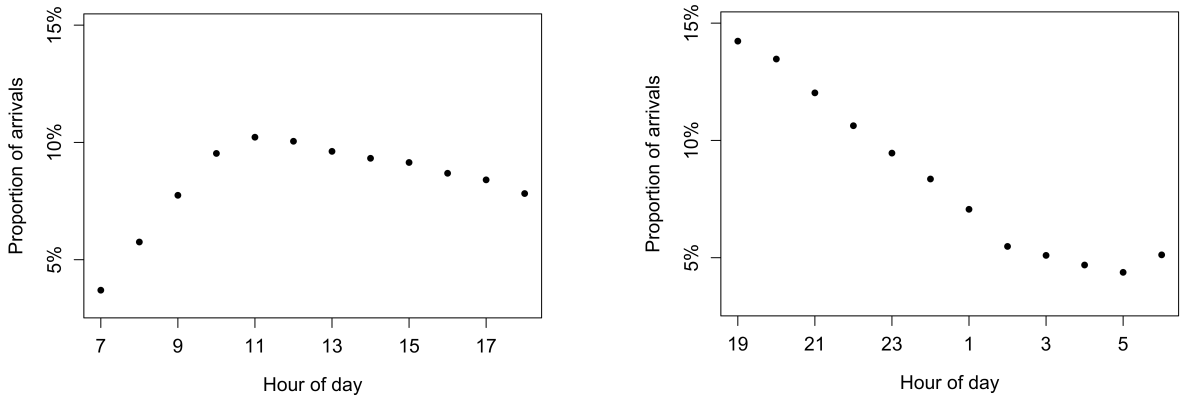
7.4.1. Model Calibration The hospital system is modeled as an $M_t/G/N_t + M$ queue, a multi-server queue with time-varying arrival rate at the hourly level, log-normal service time distribution, and time-varying staffing at the shift level, where the servers are the nurses. For a shift of type i in the kth week, we assume that the realized arrival rate for that shift is equal to the observed arrival count in data, $L_i^{(k)}$, $1 \le i \le 14$, $1 \le k \le 52$. The hourly arrival rate for each of the 12 hours in a shift is obtained by scaling $L_i^{(k)}$ according to the empirical hourly proportion of arrivals as illustrated in Figure 6. In what follows, we shall refer to the $L_i^{(k)}$'s as the realized arrival rates. As shown in Figure 5, the LOS can be well fitted by a lognormal distribution whose logarithm has a mean equal to 1.597 and a standard deviation equal to 1.050. While waiting in the queue, patients can leave the system without being seen after an exponentially distributed amount of patience time with a mean equal to 27.5 hours (fitted using the maximum likelihood estimation; see Appendix G.5 for details). Patients are served in a FCFS manner and once a patient begins service, he/she will not abandon the system. Note that in practice while patients within a severity class (e.g., within the same ESI) are often served FCFS, this is not necessarily the case across different classes. As we are interested in assessing the impact of the new staffing approach on system-level performance (e.g. average waiting time across all patients), rather than on specific individual patients, FCFS is a reasonable simplification. Furthermore, we consider a nurse-to-patient ratio of 1-to-3, which is the number of patients that an ED nurse can treat simultaneously. We scale down the staffing levels suggested by the staffing policies by the nurse-to-patient ratio to get the actual number of nurses needed for the shift. We also assume the boarding patients require the same level of nursing care as other ED patients (i.e., the LOS includes the boarding time). Note that in some EDs, boarding patients may be taken care of by inpatient nurses rather than ED nurses. Thus, our assumption gives a conservative estimation of ED nursing requirements.

At the end of each shift, patients who have not finished service are queued up in a FCFS manner (according to their arrival times) for the nurses who are staffed for the upcoming shift to continue treatment, and do not abandon the system while waiting to resume service. When calculating the performance metrics, the waiting time includes the time he/she waits to be first evaluated by a nurse upon arrival, as well as the period during which his/her treatment is in suspension due to the

change of shifts. We remark that while there are different ways to handle patient hand-off at shift transitions (such as having nurses work overtime or allowing multi-tasking), our assumption on having the patients wait to resume service has practically insignificant impact on the system-level performance.

In terms of the costs, we assume that the salary is \$45 per hour for nurses who are staffed at the base stage, and \$67.5 ($\$45 \times 1.5$) per hour for nurses who are staffed at the surge stage (Weiss et al. 2011). We vary the holding and abandonment costs in our numerical experiments.

Figure 6 Proportion of patient arrivals by hour within day/night shift



(a) Day shift (b) Night shift

7.4.2. Adjustment to the Staffing Rules The queueing dynamics during each shift in the ED are different from the stylized model considered in Section 2. In particular, based on our model calibration in Section 7.4.1, i) the arrival rate is time-varying, ii) the service-time distribution is lognormal (not exponential), and iii) each shift is only 12 hours, which may not be long enough for the system to reach stationarity. We single out these deviations and run simulation experiments to check the performance of our two-stage error policy (Appendix I.3). It turns out that our two-stage error policy achieves robust performance to non-exponential service time distributions and time-varying arrival rates. However, the fact that each shift only lasts for 12 hours and the arrival rate for the day shift can be twice as large as that for the night shift degrades the performance of our proposed policy. Specifically, since the night shift has a much lower arrival rate than the day shift, the day shift usually starts with a lower patient volume than an otherwise stationary system. Similarly, the night shift usually starts with a higher patient volume than an otherwise stationary system. Our proposed policy based on the stylized model is not able to capture these transient shift effects well. We next propose an adjustment to our two-stage error policy that takes these transient shift effects into account. At the base stage, we increase the base staffing level for night

shifts and decrease the base staffing level for day shifts based on the steady-state mean arrival rates. Then at the surge stage, we further adjust the surge staffing level based on the current state of the system, i.e., the number of patients in the system towards the end of the current shift. Formally, the two-stage error policy is adjusted as follows:

Base Stage: For $1 \leq i \leq 14$, let $N_{1,i}$ denote the base staffing level for shift of type i under $u_{2,ERR}$, which is calculated using the base-stage prediction $\hat{\lambda}_i$. For a shift of type i, calculate the expected steady-state queue length for an $M/M/n+M$ queue with arrival rate $\hat{\lambda}_i$ and number of servers equal to $N_{1,i}$, and denote it by $\bar{Q}_i$. Let Δ_i denote the difference in the expected queue length between two consecutive shifts, i.e., $\Delta_i := \bar{Q}_{i-1} - \bar{Q}_i$, where $\bar{Q}_0 \equiv \bar{Q}_{14}$. The adjusted base-stage staffing level is given by $N_{1,i}^{Adj} := N_{1,i} + \xi_1 \Delta_i$, where $\xi_1 \in \mathbb{R}$ is some base adjustment parameter to be determined. Intuitively, the base-stage adjustment accounts for the difference in the expected steady-state queue length for two adjacent shift types. For example, if the expected steady-state queue length of shift $(i-1)$ is higher than that of shift i, e.g., when transitioning from a day shift to a night shift, then the base staffing level for the ith shift is adjusted up to account for the high number of patient handoffs from the previous shift.

Surge Stage: For $1 \leq i \leq 14$, $1 \leq k \leq 52$, let $N_{2,i}^{(k)}$ denote the surge staffing level for shift of type i in the kth week under $u_{2,ERR}$, which is calculated using the surge-stage prediction $\hat{\ell}_i^{(k)}$. For each shift, calculate the expected steady-state queue length for an $M/M/n+M$ queue with arrival rate $\hat{\ell}_i^{(k)}$ and $N_{1,i}^{Adj} + N_{2,i}^{(k)}$ servers, and denote it by $\bar{Q}_i^{(k)}$. Let $Q_i^{(k)}$ be the number of patients in the ED at the end of the previous shift, and let $D_i^{(k)} := Q_i^{(k)} - \bar{Q}_i^{(k)}$. The adjusted surge-stage staffing level is given by $N_{2,i}^{(k),Adj} := N_{2,i}^{(k)} + \xi_2 D_i^{(k)}$, where $\xi_2 \in \mathbb{R}$ is some surge adjustment parameter to be determined. Intuitively, the surge-stage adjustment accounts for the concurrent difference in the actual and expected steady-state queue length for the focal shift. For example, if the observed queue length at the beginning of the focal shift is much higher than the expected value, then the surge-stage staffing level is adjusted up to account for the high initial value.

When determining the base and surge adjustment parameters, we see from extensive numerical experiments that setting $\xi_1 \in [4, 8]$ and $\xi_2 \in [1, 2]$ gives consistently good performance. Thus, we set $\xi_1 = 5$ and $\xi_2 = 1$ in the subsequent numerical experiments and suggest using these values in practice.

In what follows, we compare the ED-adapted two-stage error policy to the single-stage newsvendor solution using simulation. To make the comparison fair, a similar base adjustment is applied to the single-stage newsvendor solution, i.e., $N_{1,i}^{Adj} = N_{1,i} + 5\Delta_i$. For ease of reference, we keep the same names and acronyms for these ED-adapted policies.

We remark that the transient adjustment parameters, ξ_1 and ξ_2, can be optimized for different systems, for example, by enumerating of all possible combinations. In Appendix I.4, we show through numerical experiments that setting $\xi_1 = 5$ and $\xi_2 = 1$ in general achieves good and robust performance. We also remark that in situations where the surge-stage decision epoch has a lead time, the surge-stage staffing adjustment can be modified by using the observed queue length at the surge-stage decision epoch.

7.4.3. Performance Evaluation In practice, it can be challenging to calibrate the holding and abandonment costs. To circumvent this difficulty, we fix the abandonment cost to be 1.5 times the holding cost, and calculate the staffing levels for a wide range of holding costs under each policy. In particular, for each holding cost, we calculate the staffing levels under $u_{2,ERR}$ and $u_{1,NV}$, and simulate the ED over 52 weeks to estimate various system performance measures, such as the average waiting time, average queue length, percentage of patients who left without been seen, and percentage of patients whose waiting time exceeds 60 minutes. The same experiment is repeated 5 times using different random seeds to construct the 95% confidence intervals for the performance measures. This allows us to construct a tradeoff curve between the staffing costs and the system performances under different staffing rules; see Figure 7. We observe that the tradeoff curve of $u_{2,ERR}$ is strictly below those of $u_{1,NV}$. This suggests that for a fixed system performance target, we are able to achieve it with a much lower staffing cost under the two-stage staffing policy than the single-stage staffing policy.

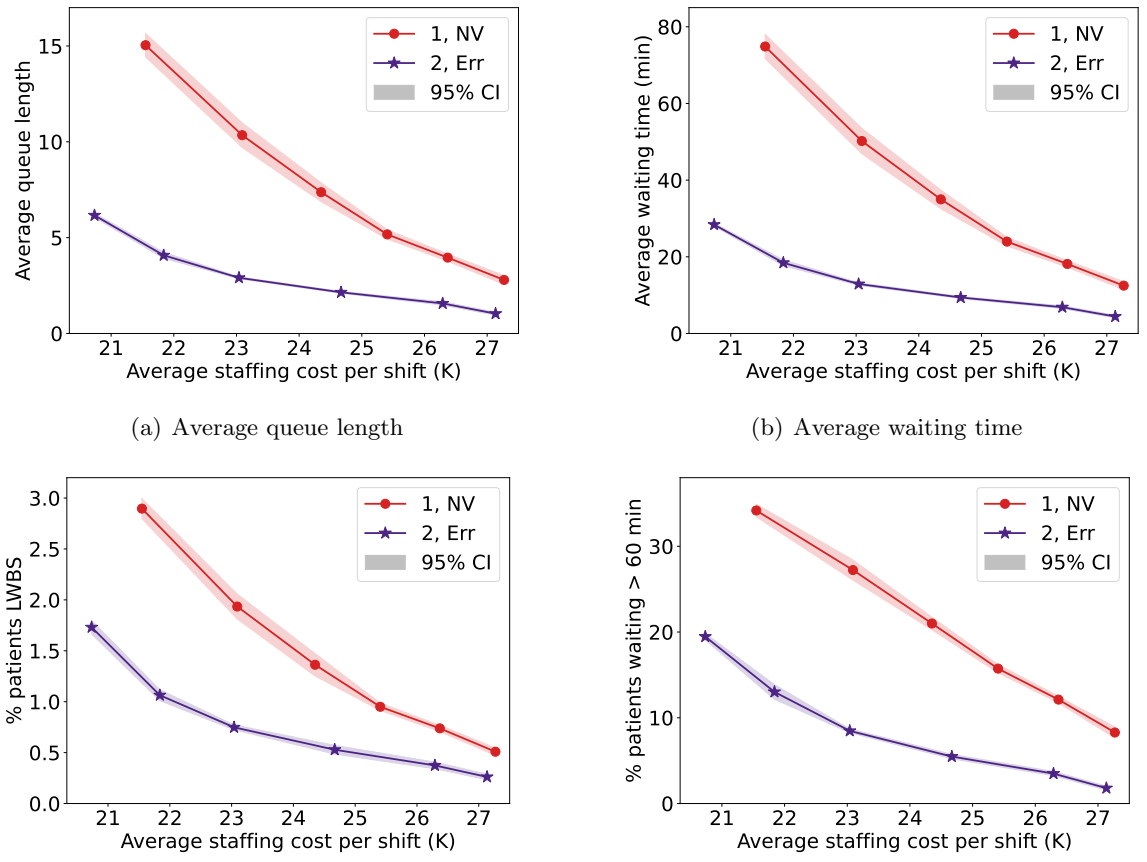
Given some specific performance targets, we calculate the staffing cost needed to achieve the desired service quality under each policy. Table 7 lists the savings in the annual staffing cost of $u_{2,ERR}$ in comparison to $u_{1,NV}$ in order to guarantee that (i) the average queue length is below 5, or (ii) the average waiting time is below 30 minutes, or (iii) the percentage of patients who left without been seen is less than 2%, or (iv) less than 20% of patients wait for more than 60 minutes. We observe that we are able to achieve 9.799% to 16.492% ($1.644 M to $3.059 M) in annual cost savings for different performance requirements. In a setting where many hospitals are operating on thin margins, such savings can have a significant impact on the bottom line. Lastly, recall from Section 7.3 that the surge-stage linear regression model is able to improve the prediction accuracy in terms of RMSE at the base stage by 11.16%. Our numerical results suggest that even with this modest gain in prediction accuracy, this information, together with the real-time queue length information, can lead to significant cost savings while ensuring timely access to care.

In addition to examining the tradeoff curves between various performance targets and the staffing costs under $u_{2,ERR}$ and $u_{1,NV}$, we also compare the expected total costs under these two policies for

some fixed cost parameters. Specifically, we vary the holding cost so that its ratio to the base-stage staffing cost ranges from 0.7 to 1.7 in increments of 0.2. The other parameters and experiment setups are the same as those in Figure 7. Figure 8 below demonstrates the expected total costs over 52 weeks under $u_{2,ERR}$ and $u_{1,NV}$ for a variety of holding costs. As expected, we observe that $u_{2,ERR}$ outperforms $u_{1,NV}$ in all scenarios.

We conclude this section by acknowledging that despite our efforts to comprehensively incorporate a number of ED patient-flow characteristics, the simulation experiments are not able to capture many important nuances in reality. Practitioners need to take this limitation into account when interpreting our reported cost savings.

Figure 7 Tradeoff between staffing cost and quality of service



(a) Average queue length

(b) Average waiting time

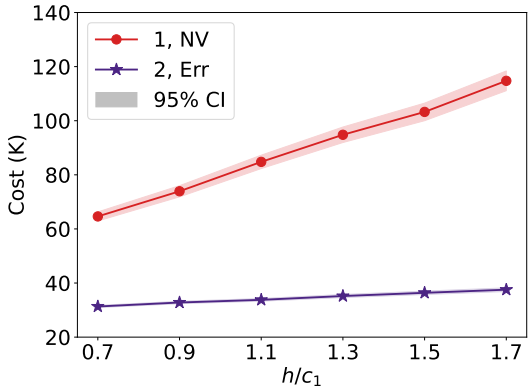
(c) % patients LWBS

(d) % patients waiting > 60 min

Table 7 Annual saving in staffing cost to achieve target performance

Policy	Avg queue length < 5	Avg waiting time < 30 min	% patients LWBS $< 2\%$	$< 20\%$ patients wait > 60 min
V.s. $u_{1,NV}$	\$3.059 M (16.407 %)	\$2.989 M (16.492 %)	\$1.644 M (9.799%)	\$2.786 M (15.547%)

Figure 8 **Expected total costs per shift for fixed cost parameters**



8. Conclusion

In this paper, we study the prediction-driven surge staffing problem motivated by ED nurse staffing. A key tradeoff in this problem is between base-stage staffing, which is cheaper but faces a higher level of uncertainty versus surge-stage staffing, which is more expensive but faces a lower level of uncertainty. Our analysis quantifies when surge staffing is beneficial and provides prescriptive staffing rules that are highly interpretable, easy to implement, and achieve near-optimal performance. Our analysis demonstrates that the benefits of surge staffing are substantial when the arrival-rate uncertainty dominates the system stochasticity. To capture this benefit, at the base stage, our proposed policy solves a two-stage newsvendor problem to serve the expected offered load plus an uncertainty hedging term. At the surge stage, we increase the staffing level to meet the realized demand plus a square-root hedging against the system stochasticity. We then extend the analysis to study the effect of prediction errors at the surge stage. Lastly, to facilitate implementation in the actual ED setting, we develop a unified framework that includes parameter estimation, building a two-scale prediction model that is synchronized with the staffing decision epochs, and modifying the prediction-driven staffing rule to account for the transient-shift effects. Using data from the Milstein ED in NYP CUMC, we demonstrate via high-fidelity simulation that our proposed staffing rule can achieve significant cost savings.

We conclude by discussing several limitations of our work and identifying a few interesting future research directions.

First, we assume a linear waiting/holding cost for analytical tractability. This assumption is reasonable when the waiting time is relatively short, i.e., where a linear interpolation is accurate, which is the case in the QD (quality-driven) and QED regimes. These are also the regimes where the system operates under our proposed two-stage QED rule. When non-linear holding costs are

concerned, we can heuristically modify the two-stage newsvendor solution by applying the holding cost function to the approximating queue length in (8); see Appendix I.5 for details on the heuristic and some numerical experiments. That said, it would be interesting to extend the model and analysis to non-linear, especially convexly increasing, waiting costs.

Second, we do not explicitly model multiple patient classes. Heuristically, we can incorporate multiple patient classes by first predicting the demand and making the corresponding staffing decisions for each class individually. Then, we can pool the staffing decisions for each class together. The heuristic development and numerical experiments are provided in Appendix I.5. A more refined extension to a multi-class queue is an interesting future research direction. To do so, we need to jointly optimize the patient scheduling decision, e.g., which patient class to prioritize, and the staffing decision.

Third, in our work, we focus on the staffing problem for EDs and do not consider the use of floating nurses. This is because many hospitals only have a single ED and ED nurses require specific training and qualification. Floating nurse pools are commonly employed for inpatient wards, where similar nursing skill sets are required. There, floating nurses are scheduled in advance (i.e., at the base stage) but can be assigned to a specific unit in near real-time (i.e., at the surge stage). The use of the floating nurse pools to handle demand uncertainty for various inpatient wards is an interesting future research direction.

Fourth, while our theoretical model is unable to capture all features of the real ED (e.g., time-varying arrivals, lognormal service times, etc.), we find that it is able to capture core tradeoffs to provide insights into the management of ED staffing. That said, we also find that transient-shift effects can have a measurable impact on system performance. As such, it would be interesting as future research to explore a transient (rather than steady-state) analysis of our system. Since closed-form expressions for transient queuing dynamics are limited, new approximation techniques may need to be developed.

Fifth, our model considers two discrete staffing epochs with different levels of demand information. Our view of the two-stage decision is informed by the current nurse staffing practice in hospitals. An interesting extension is to examine more granular decision epochs or even a continuous-time model, where both demand information and staffing cost increase as the time approaches the start of the shift. This requires a more granular model of arrival-rate uncertainty, such as those developed in Zhang et al. (2014), Daw and Pender (2018). However, increasing the granularity of decision epochs may also come with certain implementation challenges from the practical perspective.

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Appendix A: Roadmap for The Main Proofs

In this section, we introduce the notations used throughout the appendices, present a useful lemma, and give a roadmap for the organization of the main proofs.

Let $\alpha \in (0,1)$. Consider an admissible staffing policy $\pi \in \Pi$ with base staffing level N_1 and surge staffing level $N_2(N_1,\Lambda)$. For any realized arrival rate ℓ, the total cost under π is denoted by

$$\mathcal{C}_\pi(\ell) := c_1 N_1 + c_2 N_2(N_1,\ell) + (h+a\gamma)\,\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right]. \tag{22}$$

We also write

$$\mathcal{C}_\pi(\Lambda) := c_1 N_1 + c_2 N_2(N_1,\Lambda) + (h+a\gamma)\,\mathbb{E}\left[Q(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda\right], \quad \text{and} \quad \mathcal{C}_\pi := \mathbb{E}\left[\mathcal{C}(\Lambda)\right].$$

We use the following notations, in addition to the notations introduced in the main paper:

1. For an $M/M/m+M$ queue with m servers and arrival rate λ, we let $\mathbb{P}(AB,m,\lambda)$ denote the steady-state abandonment probability, $W(m,\lambda)$ denote the steady-state waiting time, and $V(m,\lambda)$ denote the steady-state virtual waiting time. $V(m,\lambda)$ is the time that a patient with infinite patience would wait and $W(m,\lambda)$ is the minimum of $V(m,\lambda)$ and the patient's patience time. Let $\mathbb{1}_{(AB,m,\lambda)}$ be the indicator of whether or not a customer arriving to a system in steady-state will abandon, i.e., $\mathbb{P}(AB,m,\lambda) = \mathbb{E}\left[\mathbb{1}_{(AB,m,\lambda)}\right]$. In what follows, we use $\mathbb{P}(AB,m,\Lambda)$ to denote the steady-state abandonment probability conditional on the random arrival rate, i.e., $\mathbb{P}(AB,m,\Lambda) := \mathbb{E}\left[\mathbb{1}_{(AB,m,\Lambda)}|\Lambda\right]$. In particular, $\mathbb{P}(AB,m,\Lambda)$ is a random variable. Similar convention for notation has been used in the literature; see, e.g., Koçağa et al. (2015).
2. For an $M/M/m/m$ queue with m servers and arrival rate λ, we let $\mathbb{P}(BL,m,\lambda)$ denote the steady-state blocking probability, $L(m,\lambda)$ denote the steady-state loss rate, and $\mathbb{1}_{(BL,m,\lambda)}$ be the indicator of whether or not a customer will be blocked in steady state. Note that $L(m,\lambda) = \lambda\mathbb{P}(BL,m,\lambda)$, and $\mathbb{P}(BL,m,\lambda) = \mathbb{E}\left[\mathbb{1}_{(BL,m,\lambda)}\right]$. In what follows, we let $\mathbb{P}(BL,m,\Lambda)$ denote the steady-state blocking probability conditional on the random arrival rate, i.e., $\mathbb{P}(BL,m,\Lambda) := \mathbb{E}\left[\mathbb{1}_{(BL,m,\Lambda)}|\Lambda\right]$. Similar to $\mathbb{P}(AB,m,\Lambda)$, $\mathbb{P}(BL,m,\Lambda)$ is a random variable.
3. For functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$, we use the relation $f\sim k$ to denote that $\lim_{\lambda\to\infty} f(\lambda)/k(\lambda) = 1$.

The following lemma will be used in the subsequent development.

Lemma 1 *For the multi-server queue with abandonment,*

$$\mathbb{E}\left[Q(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda=\ell\right] \le \max\{\mu/\gamma,1\}\left((\ell/\mu - N_1 - N_2(N_1,\ell))^+ + \sqrt{4\pi/\mu}\sqrt{\ell} + 1/\log 2\right). \tag{23}$$

Proof: We conduct the proof in three cases: $\mu=\gamma$, $\mu<\gamma$, and $\mu>\gamma$.

Case 1: $\mu=\gamma$. In this case, Lemma 3 in Bassamboo et al. (2010) directly implies that

$$\mathbb{E}\left[Q(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda=\ell\right] \le (\ell/\mu - N_1 - N_2(N_1,\ell))^+ + \sqrt{4\pi/\mu}\sqrt{\ell} + 1/\log 2,$$

from which (23) follows.

Case 2: $\mu < \gamma$. In this case, we consider a sequence of auxiliary systems with abandonment rate μ (as opposed to γ), and every other parameter is held the same as in the original system. Comparing the underlying Markov chains of these two sequences of systems, we see that the steady-state queue length in the auxiliary system is stochastically larger than that in the original system. In particular, let $\mathbb{E}\left[\tilde{Q}(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda\right]$ denote the conditional expectation of the steady-state queue in the auxiliary system. It holds that

$$\mathbb{E}\left[Q(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda=\ell\right] \le \mathbb{E}\left[\tilde{Q}(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda=\ell\right].$$

We can apply the same arguments as in Case 1 to the auxiliary system, and infer (23).

Case 3: $\mu > \gamma$. In this case, we consider a sequence of auxiliary systems with abandonment rate μ (as opposed to γ), and every other parameter is held the same as in the original system. Following similar arguments as in the proof of Theorem 3 in Bassamboo et al. (2010), we get that the steady-state abandonment rate in the auxiliary system is larger than that in the original system. In particular, let $\mathbb{P}\left(\tilde{AB}, N_1+N_2(N_1,\Lambda),\Lambda\right)$ denote the steady-state abandonment rate in the auxiliary system. It holds that

$$\mathbb{P}\left(AB, N_1+N_2(N_1,\ell),\ell\right) \le \mathbb{P}\left(\tilde{AB}, N_1+N_2(N_1,\ell),\ell\right).$$

Since the steady-state abandonment rate must be equal to the steady-state arrival rate of abandoning patients, we have

$$\mu\mathbb{E}\left[\tilde{Q}(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda=\ell\right] = \ell\mathbb{P}\left(\tilde{AB}, N_1+N_2(N_1,\ell),\ell\right),$$

and

$$\gamma\mathbb{E}\left[Q(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda=\ell\right] = \ell\mathbb{P}\left(AB, N_1+N_2(N_1,\ell),\ell\right).$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[Q(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda=\ell\right] &= (\ell/\gamma)\mathbb{P}\left(AB, N_1+N_2(N_1,\ell),\ell\right)\\ &\le (\ell/\gamma)\mathbb{P}\left(\tilde{AB}, N_1+N_2(N_1,\ell),\ell\right)\\ &= (\mu/\gamma)\mathbb{E}\left[\tilde{Q}(N_1+N_2(N_1,\Lambda),\Lambda)|\Lambda=\ell\right].\end{aligned}$$

We can apply the same arguments as in Case 1 to the auxiliary system, and (23) follows. Q.E.D.

Appendices B–F contain the proofs of the main results. In Appendix B, we prove Proposition 1 which specifies the nontrivial cost parameter regime for the staffing problem. In Appendix C, we introduce a general family of two-stage staffing policies for all $\alpha \in (0,1)$. We refer to this policy as the two-stage uncertainty hedging rule, and derive its asymptotic performance in Appendices C.1 (for $\alpha > 1/2$) and C.2 (for $\alpha \le 1/2$). In Appendix C.3, we prove that the two-stage uncertainty hedging rule with properly selected parameters achieves an optimality gap of $o(\lambda^{\max\{1/2,\alpha\}})$ compared to the exact two-stage optimum. As the two-stage newsvendor solution is a special case of the two-stage two-stage uncertainty hedging rule when $\alpha > 1/2$, the optimality gap of the two-stage newsvendor solution (Theorem 2) follows (see Appendix C.4). In Appendix D, we prove Theorem 1 which characterizes the cost saving of the optimal two-stage staffing rule compared to the optimal single-stage policy. This is done by combining the cost quantification under different near-optimal staffing rules and the corresponding optimality gap results. For example, when $\alpha > 1/2$, we first compare the cost under the two-stage newsvendor rule and the single-stage newsvendor rule. We then use

the optimality gap of the single-stage newsvendor solution (compared to the single-stage optimal) and the optimality gap of the two-stage newsvendor solution (compared to the two-stage optimal) to quantify the cost saving. In Appendix E, we prove Theorem 3, where we show that the two-stage square-root staffing rule refines the two-stage newsvendor solution and further reduces the optimality gap. Lastly in Appendix F, we analyze the two-stage staffing problem with surge-stage prediction errors. The results for small prediction errors (Proposition 2) are proved in Appendix F.1 and the results for moderate to large prediction errors (Proposition 3) are proved in Appendix F.2

Appendix B: Proof of Proposition 1

PROOF: Consider an admissible staffing policy $\pi \in \Pi$ with base staffing level N_1 and surge staffing level $N_2(N_1, \Lambda)$. For any realized arrival rate ℓ, we let $B_1(N_1, N_2(N_1,\ell),\ell)$ denote the steady-state number of busy servers among those that are staffed at the base stage, and let $B_2(N_1, N_2(N_1,\ell),\ell)$ denote the steady-state number of busy servers among those that are staffed at the surge stage. It holds that

$$B_1(N_1, N_2(N_1,\ell),\ell) \le N_1 \quad \text{and} \quad B_2(N_1, N_2(N_1,\ell),\ell) \le N_2(N_1,\ell). \tag{24}$$

Note that for $B_1(N_1, N_2(N_1,\ell),\ell)$ and $B_2(N_1, N_2(N_1,\ell),\ell)$ to be well-defined, we need to specify the assignment policy of patients to the base and surge servers. Since the model does not distinguish base and surge servers (i.e., they provide the same quality of service), we assume that patients are randomly assigned to the available servers with equal probability. That said, (24) holds regardless of the assignment policy.

Proof of (I). Following (22), the total cost satisfies

$$\begin{aligned}
&\mathcal{C}_\pi(\ell) = c_1 N_1 + c_2 N_2(N_1,\ell) + (h+a\gamma)\,\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right] \\
&\ge c_1 \mathbb{E}\left[B_1(N_1,N_2(N_1,\ell),\ell)\right] + c_2 \mathbb{E}\left[B_2(N_1,N_2(N_1,\ell),\ell)\right] + \left(\frac{h\mu}{\gamma}+a\mu\right)\frac{\gamma}{\mu}\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right] \\
&\ge \min\left\{c_1, c_2, \frac{h\mu}{\gamma}+a\mu\right\}\left(\mathbb{E}\left[B_1(N_1,N_2(N_1,\ell),\ell)\right] + \mathbb{E}\left[B_2(N_1,N_2(N_1,\ell),\ell)\right] + \frac{\gamma}{\mu}\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right]\right) \\
&= \left(\frac{h\mu}{\gamma}+a\mu\right)\frac{\ell}{\mu} \\
&= \left(\frac{h}{\gamma}+a\right)\ell,
\end{aligned} \tag{25}$$

where the second to last equality in (25) follows from the steady-state balance equation:

$$\begin{aligned}
\ell &= \mu\mathbb{E}\left[B_1(N_1,N_2(N_1,\ell),\ell)\right] + \mu\mathbb{E}\left[B_2(N_1,N_2(N_1,\ell),\ell)\right] + \gamma\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right] \\
\frac{\ell}{\mu} &= \mathbb{E}\left[B_1(N_1,N_2(N_1,\ell),\ell)\right] + \mathbb{E}\left[B_2(N_1,N_2(N_1,\ell),\ell)\right] + \frac{\gamma}{\mu}\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right].
\end{aligned} \tag{26}$$

Moreover, the cost lower bound in (25) can be achieved by staffing zero base and zero surge servers. To see this, let π_0 denote the "zero-staff" policy under which all customers abandon. The long-run average cost for the realized arrival rate ℓ under π_0 is

$$\mathcal{C}_{\pi_0}(\ell) = c_1 0 + c_2 0 + (h+a\gamma)\,\mathbb{E}\left[Q(0,\ell)\right] = (h+a\gamma)\,\mathbb{E}\left[Q(0,\ell)\right].$$

By flow balance, the steady-state rate at which abandoning customers arrive must be equal to the abandonment rate, namely,

$$\ell = \gamma\mathbb{E}\left[Q(0,\ell)\right],$$

which gives that $\mathcal{C}_{\pi_0}(\ell) = (h+a\gamma)\,\ell/\gamma$. Hence, π_0 achieves the cost lower bound, and is optimal to the optimization problem (2).

Proof of (II). Based on π, we construct another admissible policy π' where $\pi' := (0, N_2(N_1,\Lambda)+N_1)$. Namely, if π assigns N_1 base servers and $N_2(N_1,\Lambda)$ surge servers, then π' assigns zero base servers and $N_2(N_1,\Lambda)+N_1$ surge servers. By assumption, either $h\mu/\gamma + a\mu \geq c_1 \geq c_2$ or $c_1 \geq h\mu/\gamma + a\mu \geq c_2$. It follows from (22) that $\mathcal{C}_{\pi'}(\Lambda) \leq \mathcal{C}_{\pi}(\Lambda)$. Thus, it is optimal to set $N_1^* = 0$.

Proof of (III). Based on π, we construct another admissible policy π' where $\pi' := (N_1, 0)$. Namely, π' assigns the same number of base servers as π but zero surge servers for any realized arrival rate. Following (22), the total cost satisfies

$$\begin{aligned}
\mathcal{C}_{\pi}(\ell) &= c_1 N_1 + c_2 N_2(N_1,\ell) + (h+a\gamma)\,\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right] \\
&\geq c_1 N_1 + c_2 \mathbb{E}\left[B_2(N_1,N_2(N_1,\ell),\ell),\ell)\right] + \left(\frac{h\mu}{\gamma}+a\mu\right)\frac{\gamma}{\mu}\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right] \\
&\geq c_1 N_1 + \left(\frac{h\mu}{\gamma}+a\mu\right)\left(\mathbb{E}\left[B_2(N_1,N_2(N_1,\ell),\ell),\ell)\right] + \frac{\gamma}{\mu}\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right]\right) \\
&\geq c_1 N_1 + \left(\frac{h\mu}{\gamma}+a\mu\right)\left(\mathbb{E}\left[B_2(N_1,0,\ell)\right] + \frac{\gamma}{\mu}\mathbb{E}\left[Q(N_1,\ell)\right]\right) \\
&= c_1 N_1 + \left(\frac{h\mu}{\gamma}+a\mu\right)\left(0 + \frac{\gamma}{\mu}\mathbb{E}\left[Q(N_1,\ell)\right]\right) \\
&= \mathcal{C}_{\pi'}(\ell),
\end{aligned}$$

where the last inequality follows by observing from (26) that

$$\begin{aligned}
&\mathbb{E}\left[B_1(N_1,N_2(N_1,\ell),\ell)\right] + \mathbb{E}\left[B_2(N_1,N_2(N_1,\ell),\ell),\ell)\right] + \frac{\gamma}{\mu}\mathbb{E}\left[Q(N_1+N_2(N_1,\ell),\ell)\right] \\
&= \mathbb{E}\left[B_1(N_1,0,\ell)\right] + \mathbb{E}\left[B_2(N_1,0,\ell)\right] + \frac{\gamma}{\mu}\mathbb{E}\left[Q(N_1,\ell)\right] \\
&= \frac{\ell}{\mu},
\end{aligned}$$

and that

$$\mathbb{E}\left[B_1(N_1,N_2(N_1,\ell),\ell)\right] \leq \mathbb{E}\left[B_1(N_1,0,\ell)\right].$$

Thus, it is optimal to set $N_2^*(N_1,\Lambda) = 0$. Q.E.D.

Appendix C: Two-Stage Uncertainty Hedging Rule

For most of the theoretical development starting from this section, we consider the asymptotic behavior of the system as the mean arrival rate λ grows without bound. Thus, throughout Appendices C–E, we add superscript λ to all the quantities that scale with λ. For example, we add the superscript λ in N_1^λ and $N_2^\lambda(N_1^\lambda,\Lambda^\lambda)$ to denote the dependence of the staffing levels on the mean arrival rate. We use U to denote the set of all sequences of admissible staffing polices. The set U contains policies in form of $u = \{\pi^\lambda : \pi^\lambda \in \Pi^\lambda\}$, where u is a sequence of policies that specifies a two-stage staffing decision for each system along the sequence. Whenever needed, we add the subscript u to the costs (e.g., $\mathcal{C}_u^\lambda$) to mark the dependence of the cost on the staffing policy explicitly.

To facilitate the asymptotic analysis, we re-center and scale the total cost by defining

$$\hat{\mathcal{C}}_u^\lambda(\Lambda) := \frac{\mathcal{C}_u^\lambda(\Lambda) - c_1\lambda/\mu}{(\lambda/\mu)^{\max\{\alpha,1/2\}}}, \quad \text{and} \quad \hat{\mathcal{C}}_u^\lambda := \mathbb{E}\left[\hat{\mathcal{C}}_u^\lambda(\Lambda)\right]. \tag{27}$$

To simplify notation, we denote the sum of the surge-stage staffing and queueing-related cost by

$$\mathcal{R}^\lambda(N_1^\lambda, N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda) := c_2 N_2^\lambda(N_1^\lambda,\ell^\lambda) + (h+a\gamma)\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right]. \tag{28}$$

Replacing the realized arrival rate ℓ^λ with Λ^λ in (28), we define

$$\mathcal{R}^\lambda(N_1^\lambda, N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda) := c_2 N_2^\lambda(N_1^\lambda,\Lambda^\lambda) + (h+a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)|\Lambda^\lambda\right],$$

where the expectation operator on the right-hand side is with respect to the queue process. Note that $\mathcal{R}^\lambda(N_1^\lambda, N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)$ is a constant while $\mathcal{R}^\lambda(N_1^\lambda, N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)$ is a random variable.

The proofs of the main theorems require analyzing near-optimal staffing polices. In this section, we propose the *two-stage uncertainty hedging rules* and denote it by $u_{2,UH}$. We characterize the system performance under $u_{2,UH}$ as the mean arrival rate λ increases to infinity. We also show that the two-stage newsvendor solution is a special case of the two-stage uncertainty hedging rule. The proof of Theorem 2 follows.

Consider the following staffing policy, which we denote as $u_2(\beta_1,\beta_2(\beta_1,X))$. At the base stage, the base staffing level is set as

$$N_1^\lambda = \lambda/\mu + \beta_1\,(\lambda/\mu)^{\max\{\alpha,1/2\}} + o((\lambda/\mu)^{\max\{\alpha,1/2\}}),$$

for $\beta_1\in\mathbb{R}$. Note that the base staffing level is set to meet the mean demand, together with a hedging that is of the same order as the arrival-rate uncertainty or system stochasticity, whichever is larger. At the surge stage, after the random arrival rate realizes, the surge staffing level is set to

$$N_2^\lambda(N_1^\lambda,\Lambda^\lambda) = \beta_2(\beta_1,X)\,(\lambda/\mu)^{\max\{\alpha,1/2\}} + o_{UI}((\lambda/\mu)^{\max\{\alpha,1/2\}}),$$

where the coefficient $\beta_2(\beta_1,X)\in\mathbb{R}_+$ depends on both the base staffing level and the realized arrival rate. Note that the surge staffing level serves as another hedging against the larger part of arrival-rate uncertainty and system stochasticity. Importantly, the parameter $(\beta_1,\beta_2(\beta_1,X))$ does not scale with λ.

We also denote

$$D_1^\lambda := N_1^\lambda - \lambda/\mu - \beta_1\,(\lambda/\mu)^{\max\{\alpha,1/2\}} = o((\lambda/\mu)^{\max\{\alpha,1/2\}})$$

and

$$D_2^\lambda(N_1^\lambda,\Lambda^\lambda) := N_2^\lambda(N_1^\lambda,\Lambda^\lambda) - \beta_2(\beta_1,X)\,(\lambda/\mu)^{\max\{\alpha,1/2\}} = o_{UI}((\lambda/\mu)^{\max\{\alpha,1/2\}}).$$

Note that D_1^λ is a constant. On the other hand, $D_2^\lambda(N_1^\lambda,\Lambda^\lambda)$ may depend on the realization of Λ^λ and is thus a random variable. Recall from Section 1.3 that by $D_2^\lambda(N_1^\lambda,\Lambda^\lambda) = o_{UI}((\lambda/\mu)^{\max\{\alpha,1/2\}})$, we mean that $D_2^\lambda(N_1^\lambda,\Lambda^\lambda)/(\lambda/\mu)^{\max\{\alpha,1/2\}}\to 0$ as $\lambda\to\infty$ with probability 1, and there exists some random variable Y with $\mathbb{E}[Y]<\infty$ such that

$$|D_2^\lambda(N_1^\lambda,\Lambda^\lambda)|/(\lambda/\mu)^{\max\{\alpha,1/2\}} < Y \quad \text{for all } \lambda>0. \tag{29}$$

We remark that (29) is not restrictive and allows for a wide range of capacity prescriptions. Examples for $D_2^\lambda(N_1^\lambda,\Lambda^\lambda)$ include $(\lambda/\mu)^\tau$ and $(\lambda/\mu)^\tau X$ for $\tau\in(0,\max\{\alpha,1/2\})$, etc.

The two-stage uncertainty hedging rule is defined by properly optimizing the staffing parameter $(\beta_1,\beta_2(\beta_1,X))$ in $u_2(\beta_1,\beta_2(\beta_1,X))$. In particular, we first derive a proper limit for the scaled total cost under $u_2(\beta_1,\beta_2(\beta_1,X))$. Then, $(\beta_1^*,\beta_2^*(\beta_1^*,X))$ is defined as the optimal solution to the limiting cost function.

C.1. Two-Stage Uncertainty Hedging Rule for $\alpha > 1/2$

For any realized arrival rate $\ell^\lambda = \lambda + x\lambda^\alpha\mu^{1-\alpha}$, under $u_2(\beta_1, \beta_2(\beta_1, x))$ with parameters β_1 and $\beta_2(\beta_1, x)$, the total staffing level can be written as

$$\begin{aligned} N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda) &= \lambda/\mu + (\beta_1 + \beta_2(\beta_1, x))\,(\lambda/\mu)^\alpha + o((\lambda/\mu)^\alpha) \\ &= \frac{\lambda + x\lambda^\alpha\mu^{1-\alpha}}{\mu} + \frac{(\lambda/\mu)^\alpha\,(\beta_1 + \beta_2(\beta_1, x) - x)}{(\lambda/\mu + (\lambda/\mu)^\alpha\, x)^\alpha}\left(\frac{\lambda + \lambda^\alpha\mu^{1-\alpha}x}{\mu}\right)^\alpha + o((\lambda/\mu)^\alpha) \qquad (30) \\ &= \ell^\lambda/\mu + (\beta_1 + \beta_2(\beta_1, x) - x)\left(\ell^\lambda/\mu\right)^\alpha + o((\ell^\lambda/\mu)^\alpha). \end{aligned}$$

Let $\tilde{\beta} := \beta_1 + \beta_2(\beta_1, x) - x$. We first prove an auxiliary lemma on the asymptotic behavior of the steady-state probability of waiting and steady-state probability of abandonment, which facilitates our subsequent analysis on the asymptotic behavior of $\mathcal{R}^\lambda$. The lemma is adapted from Theorem 4.1 and Theorem 4.2 in Maman (2009).

Lemma 2 *Assume that $\alpha > 1/2$. For any sequence of realized arrival rate $\ell^\lambda = \lambda + x\lambda^\alpha\mu^{1-\alpha}$, under $u_2(\beta_1, \beta_2(\beta_1, x))$ with parameters β_1 and $\beta_2(\beta_1, x)$, the multi-server queue with abandonment satisfies:*

(i) If $\beta_1 + \beta_2(\beta_1, x) > x$, then the delay probability converges to zero exponentially fast as $\lambda \to \infty$. Specifically, for λ large enough,

$$\begin{aligned} &\mathbb{P}\left(W(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda) > 0\right) \\ &< \frac{1}{\tilde{\beta}\sqrt{2\pi}}\frac{1}{(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda))^{\alpha - 1/2}}\exp\left\{-\frac{(\ell^\lambda/\mu - (N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda)) + 1)^2}{2((N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda)) - 1)}\right\}. \end{aligned}$$

The probability to abandon of delayed patients decreases at rate $1/(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda))^\alpha$, i.e.,

$$\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda | V(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda) > 0\right) \sim \frac{1}{(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda))^\alpha}\frac{\gamma}{\mu\tilde{\beta}}.$$

(ii) If $\beta_1 + \beta_2(\beta_1, x) < x$, then the delay probability converges to 1 exponentially fast as $\lambda \to \infty$. Specifically, for λ large enough,

$$\mathbb{P}\left(W(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda) = 0\right) < \frac{1}{|\tilde{\beta}|\mu^{1-\alpha}(\ell^\lambda)^\alpha}\exp\left\{-\frac{\tilde{\beta}^2}{8\gamma}\mu^{2-2\alpha}(\ell^\lambda)^{2\alpha-1}\right\}.$$

The probability to abandon of delayed patients decreases at rate $1/(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda))^{1-\alpha}$, i.e.,

$$\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda | V(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda) > 0\right) \sim \frac{|\tilde{\beta}|}{(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda))^{1-\alpha}}.$$

PROOF: Following (30), for total staffing level of the form

$$\ell^\lambda/\mu + (\beta_1 + \beta_2(\beta_1, x) - x)\left(\ell^\lambda/\mu\right)^\alpha + f(\ell),$$

where $f(\ell^\lambda) = o(\sqrt{\ell^\lambda})$, the statement of Lemma 2 follows directly from Theorem 4.1 and Theorem 4.2 from Maman (2009). The work left is to generalize the result to staffing level of the form in (30), where $f(\ell^\lambda) = o((\ell^\lambda)^\alpha)$.

To this end, we show that the proofs of Theorem 4.1 and Theorem 4.2 in Maman (2009) can be generalized to the case where $f(\ell^\lambda) = o((\ell^\lambda)^\alpha)$. Indeed, exactly the same lines of derivation go through when $f(\ell^\lambda) = o((\ell^\lambda)^\alpha)$ (as opposed to $f(\ell^\lambda) = o(\sqrt{\ell^\lambda})$). Just as in Maman (2009), the results follow from Lemmas 4.2 and

4.3 which need to be adapted to this more generalized setting. We next illustrate the generalization of Lemma 4.2 to the general case where $f(\ell^\lambda) = o((\ell^\lambda)^\alpha)$. The other proofs are generalized similarly.

In the proof of Lemma 4.2 in Maman (2009), four places utilize the fact that $f(\ell^\lambda) = o(\sqrt{\ell^\lambda})$. We discuss them one by one. For the rest of this proof, we assume that $\tilde{\beta} > 0$, as in the proof of Lemma 4.2. All numbering of the equations refers to those in Section 4 of Maman (2009).

First, let $\bar{G}(u) := e^{-\gamma u}$ denote the ccdf of the patience time distribution. Following (4.44) and using the definition of δ in (4.40), take

$$\tilde{\gamma} := \frac{1 - \bar{G}(\delta/2)}{2} > 0.$$

Since $\bar{G}(u) < 1$ for all $u > 0$, and $\bar{G}(u) - 1 < -2\tilde{\gamma}$ for all $u > \delta/2$, we get that for λ large enough,

$$\ell^\lambda(\bar{G}(u) - 1) - \tilde{\beta}(\ell^\lambda)^\alpha \mu^{1-\alpha} - f(\ell^\lambda)\mu \le -\tilde{\beta}(\ell^\lambda)^\alpha \mu^{1-\alpha}, \quad \text{for all } u > 0,$$

and

$$\ell^\lambda(\bar{G}(u) - 1) - \tilde{\beta}(\ell^\lambda)^\alpha \mu^{1-\alpha} - f(\ell^\lambda)\mu \le -\tilde{\gamma}\ell^\lambda, \quad \text{for all } u > \delta/2.$$

Therefore, (4.45) and (4.46) hold for the case where $f(\ell^\lambda) = o((\ell^\lambda)^\alpha)$.

Second, in (4.51), define the function

$$r(\ell^\lambda) := \frac{-\tilde{\beta}(\ell^\lambda)^\alpha \mu^{1-\alpha} x - f(\ell^\lambda)\mu x}{\tilde{\beta}\mu^{1-\alpha} x}.$$

Note that for $f(\ell^\lambda) = o((\ell^\lambda)^\alpha)$, we still have $r(\ell^\lambda) \sim (\ell^\lambda)^\alpha$. Therefore, (4.51) still holds by applying Lemma 2.1 in Maman (2009) with $m = 0, k_1 = \alpha, l_1 = 1, k_2 = 1, l_2 = 2$.

Third, utilizing the same fact that $r(\ell^\lambda) \sim (\ell^\lambda)^\alpha$, (4.55) goes through by applying Lemma 2.1 in Maman (2009) with $m = 1, k_1 = \alpha, l_1 = 1, k_2 = 1, l_2 = 2$.

Lastly, for

$$n := N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda) = \ell^\lambda/\mu + \tilde{\beta}\left(\ell^\lambda/\mu\right)^\alpha + o((\ell^\lambda/\mu)^\alpha),$$

it holds that

$$\frac{(\ell^\lambda/\mu - n + 1)^2}{2(n-1)} \sim \frac{\tilde{\beta}^2}{\mu^{2\alpha-1}}(\ell^\lambda)^{2\alpha-1},$$

so the last line in the proof of Lemma 4.2 goes through. Q.E.D.

Lemma 3 *Assume that $\alpha > 1/2$. For any sequence of realized arrival rates $\ell^\lambda = \lambda + x\lambda^\alpha\mu^{1-\alpha}$, under $u_2(\beta_1, \beta_2(\beta_1, x))$ with parameters β_1 and $\beta_2(\beta_1, x)$, we have*

$$\frac{1}{(\lambda/\mu)^\alpha}\mathcal{R}^\lambda(N_1^\lambda, N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda) \to \hat{r}(\beta_1, \beta_2(\beta_1, x), x) \quad \text{as } \lambda \to \infty,$$

where the function $\hat{z} : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is defined as

$$\hat{r}(\beta_1, \beta_2(\beta_1, x), x) := \begin{cases} c_2\beta_2(\beta_1, x) & \text{if } \beta_1 + \beta_2(\beta_1, x) \ge x \\ c_2\beta_2(\beta_1, x) + (h\mu/\gamma + a\mu)\left(x - \beta_1 - \beta_2(\beta_1, x)\right) & \text{if } \beta_1 + \beta_2(\beta_1, x) < x. \end{cases} \tag{31}$$

PROOF: It follows from (2.8)–(2.11) in Maman (2009) that when the patience time is exponentially distributed, we have

$$\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda\right) = \mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda | V(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda) > 0\right)$$
$$\mathbb{P}\left(W(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda) > 0\right).$$

By Lemma 2 and the flow balance equation that

$$\ell^\lambda \mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda\right) = \gamma \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right],$$

the following cases hold:

(i) If $\beta_1 + \beta_2(\beta_1, x) > x$, then for λ large enough,

$$\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda\right)$$
$$< \frac{\gamma}{\mu\tilde{\beta}^2\sqrt{2\pi}} \frac{1}{(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda))^{2\alpha-1/2}} \exp\left\{-\frac{(\ell^\lambda/\mu - (N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda)) + 1)^2}{2((N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda)) - 1)}\right\}.$$

Therefore,

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda/\mu}} \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] = 0. \tag{32}$$

(ii) If $\beta_1 + \beta_2(\beta_1, x) < x$, then for λ large enough,

$$\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda\right) \sim \frac{|\tilde{\beta}|}{(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda))^{1-\alpha}}.$$

Therefore,

$$\lim_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^\alpha} \mathbb{E}\left[Q(N_1^\lambda, N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] = \frac{\mu}{\gamma}(x - \beta_1 - \beta_2(\beta_1, x)). \tag{33}$$

Lastly, when $\beta_1 + \beta_2(\beta_1, x) = x$, we get from Lemma 1 that

$$\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] \le \max\{\mu/\gamma, 1\}\left(\left(\ell^\lambda/\mu - N_1^\lambda - N_2^\lambda(N_1^\lambda, \ell^\lambda)\right)^+ + \sqrt{4\pi/\mu}\sqrt{\ell^\lambda} + 1/\log 2\right)$$
$$= o((\lambda/\mu)^\alpha) + \max\{\mu/\gamma, 1\}\sqrt{4\pi/\mu}\sqrt{\ell^\lambda} + \max\{\mu/\gamma, 1\}/\log 2.$$

Then,

$$\lim_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^\alpha} \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] = 0. \tag{34}$$

The statement of the lemma then follows from (32), (33), and (34). Q.E.D.

Based on Lemma 3, let β_1^* and $\beta_2^*(\beta_1, X)$ be the optimal solution to

$$\min_{\beta_1\in\mathbb{R}}\left\{c_1\beta_1 + \mathbb{E}\left[\min_{\beta_2(\beta_1,X)\in\mathbb{R}_+} \hat{r}(\beta_1, \beta_2(\beta_1, X), X)\right]\right\}, \quad \text{for } \hat{z} \text{ defined in (31).}$$

It is straightforward to derive that

$$\beta_1^* = \arg\min_{\beta\in\mathbb{R}}\ c_1\beta + c_2\mathbb{E}\left[(X-\beta)^+\right] = \bar{F}_X^{-1}(c_1/c_2), \quad \text{and} \quad \beta_2^*(\beta_1, X) = (X - \beta_1)^+. \tag{35}$$

Then, the two-stage uncertainty hedging rule is defined as $u_2(\beta_1, \beta_2(\beta_1, X))$ with parameters β_1^* and $\beta_2^*(\beta_1^*, X)$ in (35). Note that $u_{2,UH}$ is exactly the two-stage newsvendor solution in Definition 1.

The next lemma establishes the asymptotic performance of $u_{2,UH}$.

Lemma 4 *Assume that $\alpha > 1/2$. Under the two-stage uncertainty hedging rule defined in* (35) *(equivalently, the two-stage newsvendor solution), we have*

$$\hat{\mathcal{C}}^\lambda \to c_1\beta_1^* + \mathbb{E}\left[\hat{r}(\beta_1^*, \beta_2^*(\beta_1^*, X), X)\right] \quad \text{as } \lambda \to \infty,$$

for $\hat{z}$ defined in (31).

Proof: It follows from Lemma 3 that

$$\hat{\mathcal{C}}^\lambda(\Lambda^\lambda) \to c_1\beta_1^* + \hat{r}\left(\beta_1^*, \beta_2^*(\beta_1^*, X), X\right) \quad w.p.1 \quad \text{as } \lambda \to \infty.$$

Hence, to prove the claim, it is sufficient to show that

$$\lim_{\lambda\to\infty} \mathbb{E}\left[\hat{\mathcal{C}}^\lambda(\Lambda^\lambda)\right] = \mathbb{E}\left[\lim_{\lambda\to\infty} \hat{\mathcal{C}}^\lambda(\Lambda^\lambda)\right] \tag{36}$$

To this end, we utilize the dominated convergence theorem.

Note that

$$\begin{aligned}\hat{\mathcal{C}}^\lambda(\Lambda^\lambda) =& c_1\beta_1^* + c_2\beta_2^*(\beta_1^*, X) + \frac{1}{(\lambda/\mu)^\alpha}\left(D_1^\lambda + D_2^\lambda(N_1^\lambda, \Lambda^\lambda)\right) \\ &+ \frac{1}{(\lambda/\mu)^\alpha}(h + a\gamma)\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)|\Lambda^\lambda\right].\end{aligned} \tag{37}$$

For the first two terms on the right-hand side of (37), it follows from the definition of $\beta_2^*(\beta_1^*, X)$ that

$$|c_1\beta_1^*| + |c_2\beta_2^*(\beta_1^*, X)| \le c_2\left(|\beta_1^*| + |X|\right),$$

where recall that $\mathbb{E}\left[|X|\right] < \infty$.

For the third term on the right-hand side of (37), note that D_1^λ is a constant that is $o((\lambda/\mu)^\alpha)$. This, together with (29), implies that there exists some random variable $\tilde{Y}$ with $\mathbb{E}[\tilde{Y}] < \infty$ such that

$$\frac{1}{(\lambda/\mu)^\alpha}\left(|D_1^\lambda| + |D_2^\lambda(N_1^\lambda, \Lambda^\lambda)|\right) < \tilde{Y}.$$

For the last term on the right-hand side of (37), we utilize Lemma 1 to get that

$$\begin{aligned}&\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)|\Lambda^\lambda\right] \\ &\le \max\{\mu/\gamma, 1\}\left(\left(\Lambda^\lambda/\mu - N_1^\lambda - N_2^\lambda(N_1^\lambda, \Lambda^\lambda)\right)^+ + \sqrt{4\pi/\mu}\sqrt{\Lambda^\lambda} + 1/\log 2\right) \\ &\le \max\{\mu/\gamma, 1\}\left(\left(\Lambda^\lambda/\mu - N_1^\lambda\right)^+ + \sqrt{4\pi/\mu}\sqrt{\Lambda^\lambda} + 1/\log 2\right) \\ &= \max\{\mu/\gamma, 1\}\left(\left((X - \beta_1^*)(\lambda/\mu)^\alpha - D_1^\lambda\right)^+ + \sqrt{4\pi/\mu}\sqrt{\lambda/\mu + X\lambda^\alpha\mu^{1-\alpha}} + 1/\log 2\right) \\ &\le \max\{\mu/\gamma, 1\}\left(\left(|X| + |\beta_1^*|\right)(\lambda/\mu)^\alpha + |D_1^\lambda| + \sqrt{4\pi/\mu}\sqrt{\lambda/\mu} + \sqrt{4\pi/\mu}\sqrt{|X|\lambda^\alpha\mu^{1-\alpha}} + 1/\log 2\right).\end{aligned} \tag{38}$$

In (38), $D_1^\lambda = o((\lambda/\mu)^\alpha)$ is a constant. In addition, for λ large enough, we have

$$\frac{1}{(\lambda/\mu)^\alpha}\sqrt{4\pi/\mu}\sqrt{|X|\lambda^\alpha\mu^{1-\alpha}} \le \sqrt{4\pi/\mu}\sqrt{|X|}.$$

By Jensen's inequality, $\mathbb{E}\left[\sqrt{|X|}\right] \le \sqrt{\mathbb{E}\left[|X|\right]} < \infty$. Therefore, there exists some random variable Y with $\mathbb{E}\left[Y\right] < \infty$, such that

$$\frac{1}{(\lambda/\mu)^\alpha}(h + a\gamma)\mathbb{E}\left[Q^\lambda(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)|\Lambda^\lambda\right] \le Y.$$

Therefore, $|\hat{\mathcal{C}}^\lambda(\Lambda^\lambda)|$ in (37) is uniformly bounded by an integrable random variable, and (36) is justified. Q.E.D.

C.2. Two-Stage Uncertainty Hedging Rule for $\alpha \le 1/2$

Recall that ϕ and Φ are the pdf and cdf of the standard normal random distribution, respectively. The hazard rate of the standard normal distribution is $H(t) = \phi(t)/\Phi(-t)$, for $t \in \mathbb{R}$.

Lemma 5 *Assume that $\alpha \le 1/2$. For any sequence of realized arrival rate $\ell^\lambda = \lambda + x\lambda^\alpha \mu^{1-\alpha}$, under $u_2(\beta_1, \beta_2(\beta_1, x))$ with parameters β_1 and $\beta_2(\beta_1, x)$, we have*

$$\frac{1}{\sqrt{\lambda/\mu}} \mathcal{R}^\lambda(N_1^\lambda, N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda) \to \hat{r}\left(\beta_1, \beta_2(\beta_1, x), x\right) \quad \text{as } \lambda \to \infty,$$

where the function $\hat{z} : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is defined as

$$\hat{r}\left(\beta_1, \beta_2(\beta_1, x), x\right) := c_2 \beta_2(\beta_1, x) + \left(\frac{h\mu}{\gamma} + a\mu\right) \frac{\sqrt{\frac{\gamma}{\mu}} \left[H\left(\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right) \sqrt{\frac{\mu}{\gamma}} \right) - \left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right) \sqrt{\frac{\mu}{\gamma}} \right]}{1 + \sqrt{\frac{\gamma}{\mu}} \frac{H\left(\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right)\sqrt{\frac{\mu}{\gamma}}\right)}{H\left(-\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right)\right)}}. \tag{39}$$

Proof: For any realized arrival rate $\ell^\lambda = \lambda + \lambda^\alpha \mu^{1-\alpha} x$, the total staffing level satisfies

$$\sqrt{N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda)}(1 - \rho^\lambda) \to \beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}} \quad \text{as } \lambda \to \infty.$$

By Theorem 4.1 in Zeltyn and Mandelbaum (2005), we have

$$\begin{aligned}
&\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda\right) \\
&= \left[1 + \sqrt{\frac{\gamma}{\mu}} \frac{H\left(\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right)\sqrt{\frac{\mu}{\gamma}}\right)}{H\left(-\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right)\right)}\right]^{-1} \frac{1}{\sqrt{N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda)}} \sqrt{\frac{\gamma}{\mu}} \\
&\quad \left[H\left(\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right) \sqrt{\frac{\mu}{\gamma}} \right) - \left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right) \sqrt{\frac{\mu}{\gamma}} \right] + o\left(\frac{1}{\sqrt{N_1^\lambda + N_2^\lambda}}\right) \\
&= \sqrt{\frac{\mu}{\lambda}} \frac{\sqrt{\frac{\gamma}{\mu}} \left[H\left(\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right) \sqrt{\frac{\mu}{\gamma}} \right) - \left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right) \sqrt{\frac{\mu}{\gamma}} \right]}{1 + \sqrt{\frac{\gamma}{\mu}} \frac{H\left(\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right)\sqrt{\frac{\mu}{\gamma}}\right)}{H\left(-\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha = 1/2\}}\right)\right)}} + o\left(\frac{1}{\sqrt{\lambda/\mu}}\right).
\end{aligned}$$

From the steady-state flow balance equation

$$\gamma \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] = \left(\lambda + \lambda^\alpha \mu^{1-\alpha} x\right) \mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda\right),$$

we get that

$$\begin{aligned}
&\frac{1}{\sqrt{\lambda/\mu}} \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] \\
&\to \frac{\mu}{\gamma} \frac{\sqrt{\frac{\gamma}{\mu}} \left[H\left(\left(\beta_1 + \beta_2 - x\mathbb{1}_{\{\alpha = 1/2\}}\right) \sqrt{\frac{\mu}{\gamma}} \right) - \left(\beta_1 + \beta_2 - x\mathbb{1}_{\{\alpha = 1/2\}}\right) \sqrt{\frac{\mu}{\gamma}} \right]}{1 + \sqrt{\frac{\gamma}{\mu}} \frac{H\left(\left(\beta_1 + \beta_2 - x\mathbb{1}_{\{\alpha = 1/2\}}\right)\sqrt{\frac{\mu}{\gamma}}\right)}{H\left(-\left(\beta_1 + \beta_2 - x\mathbb{1}_{\{\alpha = 1/2\}}\right)\right)}}, \quad \text{as } \lambda \to \infty,
\end{aligned}$$

and the statement follows. Q.E.D.

Based on Lemma 5, let β_1^* and $\beta_2^*(\beta_1, X)$ be the optimal solution to

$$\min_{\beta_1 \in \mathbb{R}} \left\{ c_1 \beta_1 + \mathbb{E}\left[\min_{\beta_2(\beta_1, X) \in \mathbb{R}_+} \hat{r}(\beta_1, \beta_2(\beta_1, X), X) \right] \right\}, \quad \text{for } \hat{z} \text{ defined in (39).} \tag{40}$$

Then, the two-stage uncertainty hedging rule, $u_{2,UH}$, is defined as $u_2(\beta_1, \beta_2(\beta_1, X))$ with parameters β_1^* and $\beta_2^*(\beta_1^*, X)$, i.e.,

$$N_1^\lambda = \lambda/\mu + \beta_1^*(\lambda/\mu)^{1/2} + o((\lambda/\mu)^{1/2}), \quad \text{and} \quad N_2^\lambda(N_1^\lambda, \Lambda^\lambda) = \beta_2^*(\beta_1^*, X)(\lambda/\mu)^{1/2} + o_{UI}((\lambda/\mu)^{1/2}).$$

Remark 4 *The existence of β_1^* and $\beta_2^*(\beta_1^*, X)$ follows from the same lines of analysis as those for the conventional single-stage square-root staffing rule considered in the literature (see, e.g., Garnett et al. (2002), Zeltyn and Mandelbaum (2005), Mandelbaum and Zeltyn (2009)). For completeness, we outline the key steps and omit the lengthy algebraic derivation. Given β_1 and $X = x$, it can be seen from* (39) *that $\hat{r}(\beta_1, \beta_2, x)$ is continuous in β_2. In addition, it can be checked that $\hat{r}(\beta_1, \beta_2, x) \to \infty$ as $\beta_2 \to \infty$. Thus, an optimal solution $\beta_2^*(\beta_1, x)$ exists for the inner minimization problem in* (40)*. The existence of β_1^* can be argued similarly. Let $g(\beta_1) := c_1\beta_1 + \mathbb{E}\left[\hat{r}\left(\beta_1, \beta_2^*(\beta_1, X), X\right)\right]$. It can be checked that $g(\beta_1) \to \infty$ as $\beta_1 \to \infty$. In addition, under the condition that $\mu > \gamma$ or $(h + a\gamma)\mu > c_1\gamma$ (this latter condition is implied by Assumption 1), we have $g(\beta_1) \to \infty$ as $\beta_1 \to -\infty$. The existence of an optimal solution β_1^* then follows from the continuity of $g(\beta_1)$ in β_1.*

Before we establish the asymptotic performance of $u_{2,UH}$, we first prove an auxiliary lemma.

Lemma 6 *Assume that $\alpha \le 1/2$. Under the two-stage uncertainty hedging rule defined in* (40)*, there exists a random variable $\tilde{X}$ such that $\beta_2^*(\beta_1, X) \le \tilde{X}$ and $\mathbb{E}[\tilde{X}] < \infty$.*

PROOF: For any realized arrival rate $\ell^\lambda = \lambda + x\lambda^\alpha\mu^{1-\alpha}$, we start by rewriting (39) as

$$
\begin{aligned}
&\hat{r}\left(\beta_1, \beta_2(\beta_1, x), x\right) \\
&:= c_2\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha=1/2\}}\right) - c_2\left(\beta_1 - x\mathbb{1}_{\{\alpha=1/2\}}\right) + \\
&\quad \left(\frac{h\mu}{\gamma} + a\mu\right) \frac{\sqrt{\frac{\gamma}{\mu}}\left[H\left(\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha=1/2\}}\right)\sqrt{\frac{\mu}{\gamma}}\right) - \left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha=1/2\}}\right)\sqrt{\frac{\mu}{\gamma}}\right]}{1 + \sqrt{\frac{\gamma}{\mu}}\frac{H\left(\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha=1/2\}}\right)\sqrt{\frac{\mu}{\gamma}}\right)}{H\left(-\left(\beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha=1/2\}}\right)\right)}}.
\end{aligned}
$$

Let $\tilde{\beta} := \beta_1 + \beta_2(\beta_1, x) - x\mathbb{1}_{\{\alpha=1/2\}}$, and denote

$$
g(\tilde{\beta}) := \left(\frac{h\mu}{\gamma} + a\mu\right) \frac{\sqrt{\frac{\gamma}{\mu}}\left[H\left(\tilde{\beta}\sqrt{\frac{\mu}{\gamma}}\right) - \tilde{\beta}\sqrt{\frac{\mu}{\gamma}}\right]}{1 + \sqrt{\frac{\gamma}{\mu}}\frac{H\left(\tilde{\beta}\sqrt{\frac{\mu}{\gamma}}\right)}{H\left(-\tilde{\beta}\right)}}.
$$

It follows from Section 2.1 in the Online Appendix of Mandelbaum and Zeltyn (2009) that the function g monotonically decreases from infinity to 0.

Define

$$
\tilde{\beta}^* := \underset{\tilde{\beta} \ge \beta_1 - x\mathbb{1}_{\{\alpha=1/2\}}}{\arg\min} \quad c_2\tilde{\beta} + g(\tilde{\beta}). \tag{41}
$$

Note that by construction, we have

$$
\beta_2^*(\beta_1, x) = \tilde{\beta}^* - \beta_1 + x\mathbb{1}_{\{\alpha=1/2\}}.
$$

Corresponding to (41), let

$$
\tilde{\beta}^\dagger := \underset{\tilde{\beta} \in \mathbb{R}}{\arg\min} \quad c_2\tilde{\beta} + g(\tilde{\beta}),
$$

where unlike $\tilde{\beta}^*$, $\tilde{\beta}^\dagger$ is a global minimizer of the objective function over the real line. The existence of $\tilde{\beta}^\dagger$ follows from the same lines of arguments as in Remark 4.

We discuss the following cases:

Case 1: If $\beta_1 - x\mathbb{1}_{\{\alpha=1/2\}} \le \beta^\dagger$, then $\tilde{\beta}^* = \beta^\dagger$, and

$$\beta_2^*(\beta_1, x) = \beta^\dagger - \beta_1 + x\mathbb{1}_{\{\alpha=1/2\}}. \tag{42}$$

Case 2: If $\beta_1 - x\mathbb{1}_{\{\alpha=1/2\}} > \beta^\dagger$, then let $\epsilon > 0$, and let $M \in \mathbb{R}$ be such that (i) $M > \epsilon/c_2$, and (ii) for all $x > M$, we have $0 \le g(x) < \epsilon$. There are two subcases:

Case 2(i): If $\beta_1 - x\mathbb{1}_{\{\alpha=1/2\}} \le M$, then exactly one of the following two scenarios holds:

Case 2(i.a): $\tilde{\beta}^* \le M$, so that

$$\beta_2^*(\beta_1, x) \le M - \beta_1 + x\mathbb{1}_{\{\alpha=1/2\}}. \tag{43}$$

Case 2(i.b): $\tilde{\beta}^* > M$. In this case, (41) can be rewritten as

$$\tilde{\beta}^* = \underset{\tilde{\beta} \ge M}{\arg\min} \quad c_2\tilde{\beta} + g(\tilde{\beta}).$$

Note that for all $y > 2M$, it follows from the definition of M that

$$c_2 M + g(M) < c_2 y + g(y). \tag{44}$$

Therefore, $\tilde{\beta}^* \le 2M$, and

$$\beta_2^*(\beta_1, x) \le 2M - \beta_1 + x\mathbb{1}_{\{\alpha=1/2\}}. \tag{45}$$

Case 2(ii): If $\beta_1 - x\mathbb{1}_{\{\alpha=1/2\}} > M$, then by definition of M, (44) holds for all $y > 2(\beta_1 - x\mathbb{1}_{\{\alpha=1/2\}})$. Hence, $\tilde{\beta}^* \le 2(\beta_1 - x\mathbb{1}_{\{\alpha=1/2\}})$, and

$$\beta_2^*(\beta_1, x) \le 2(\beta_1 - x\mathbb{1}_{\{\alpha=1/2\}}) - \beta_1 + x\mathbb{1}_{\{\alpha=1/2\}} = \beta_1 - x\mathbb{1}_{\{\alpha=1/2\}}. \tag{46}$$

In summary, by (42), (43), (45), and (46), we get that

$$\beta_2^*(\beta_1, x) \le |\beta^\dagger| + 2M + |\beta_1| + |x|. \tag{47}$$

Let $\tilde{X} := |\beta^\dagger| + 2M + |\beta_1| + |X|$. The statement follows from (47) and $\mathbb{E}[|X|] < \infty$. Q.E.D.

The following lemma establishes the asymptotic performance of $u_{2,UL}$.

Lemma 7 *Assume that $\alpha \le 1/2$. Under the two-stage uncertainty hedging rule defined in* (40)*, we have*

$$\hat{\mathcal{C}}^\lambda \to c_1\beta_1^* + \mathbb{E}\left[\hat{r}\left(\beta_1^*, \beta_2^*(\beta_1^*, X), X\right)\right] \quad \text{as } \lambda \to \infty,$$

for $\hat{z}$ defined in (39)*.*

Proof: It follows from Lemma 5 that

$$\hat{\mathcal{C}}^\lambda(\Lambda^\lambda) \to c_1\beta_1^* + \hat{r}\left(\beta_1^*, \beta_2^*(\beta_1^*, X), X\right) \quad w.p.1 \quad \text{as } \lambda \to \infty.$$

Hence, to prove the claim, it is sufficient to show

$$\lim_{\lambda\to\infty} \mathbb{E}\left[\hat{\mathcal{C}}^\lambda(\Lambda^\lambda)\right] = \mathbb{E}\left[\lim_{\lambda\to\infty} \hat{\mathcal{C}}^\lambda(\Lambda^\lambda)\right] \tag{48}$$

To this end, we utilize the dominated convergence theorem.

We start by writing

$$\begin{aligned}\hat{\mathcal{C}}^\lambda(\Lambda^\lambda) =& c_1\beta_1^* + c_2\beta_2^*(\beta_1^*, X) + \frac{1}{\sqrt{\lambda/\mu}}\left(D_1^\lambda + D_2^\lambda(N_1^\lambda, \Lambda^\lambda)\right) \\ &+ \frac{1}{\sqrt{\lambda/\mu}}(h + a\gamma)\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)|\Lambda^\lambda\right] \\ =& c_1\beta_1^* + c_2\beta_2^*(\beta_1^*, X) + \frac{1}{\sqrt{\lambda/\mu}}\left(D_1^\lambda + D_2^\lambda(N_1^\lambda, \Lambda^\lambda)\right) \\ &+ \frac{\Lambda^\lambda}{\sqrt{\lambda/\mu}}\,(h/\gamma + a)\,\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda\right),\end{aligned} \tag{49}$$

where the last equality follows from

$$\gamma\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)|\Lambda^\lambda\right] = \Lambda^\lambda\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda\right).$$

Recall that $\mathbb{P}(BL, m, \lambda)$ is the steady-state blocking probability for an $M/M/m/m$ queue with number of servers equal to m and arrival rate equal to λ. It follows from a simple coupling argument that

$$\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda\right) \le \mathbb{P}\left(BL, N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda\right). \tag{50}$$

Since the Erlang blocking probability is increasing in the offered load and $N_2^\lambda(N_1^\lambda, \Lambda^\lambda) \ge 0$, we further have

$$\mathbb{P}\left(BL, N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda\right) \le \mathbb{P}\left(BL, N_1^\lambda, \lambda + |X|\lambda^\alpha\mu^{1-\alpha}\right). \tag{51}$$

In addition, recall that $L(m, \lambda)$ is the steady-state loss rate in an $M/M/m/m$ queue with number of servers equal to m and arrival rate equal to λ. In particular, $L(m, \lambda)$ satisfies $L(m, \lambda) = \lambda\mathbb{P}(BL, m, \lambda)$, and by Theorem 1 in Smith and Whitt (1981),

$$L(N_1^\lambda, \lambda + |X|\lambda^\alpha\mu^{1-\alpha}) \le L(N_1^\lambda - 1, \lambda) + L(1, |X|\lambda^\alpha\mu^{1-\alpha}). \tag{52}$$

Combining (50)–(52), we have

$$\begin{aligned}\Lambda^\lambda\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda\right) &\le \Lambda^\lambda\mathbb{P}\left(BL, N_1^\lambda, \lambda + |X|\lambda^\alpha\mu^{1-\alpha}\right) \\ &\le \lambda\mathbb{P}\left(BL, N_1^\lambda - 1, \lambda\right) + |X|\lambda^\alpha\mu^{1-\alpha}\mathbb{P}\left(BL, 1, |X|\lambda^\alpha\mu^{1-\alpha}\right) \\ &\le \lambda\mathbb{P}\left(BL, N_1^\lambda - 1, \lambda\right) + |X|\lambda^\alpha\mu^{1-\alpha}.\end{aligned} \tag{53}$$

Dividing both sides of (53) by $\sqrt{\lambda/\mu}$, we get that

$$\frac{\Lambda^\lambda}{\sqrt{\lambda/\mu}}\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda\right) \le \frac{\lambda}{\sqrt{\lambda/\mu}}\mathbb{P}\left(BL, N_1^\lambda - 1, \lambda\right) + |X|\frac{\lambda^\alpha\mu^{1-\alpha}}{\sqrt{\lambda/\mu}}, \tag{54}$$

where the first term on the right-hand side of (54) is a constant. By equation (17) in Whitt (1984),

$$\lim_{\lambda\to\infty}\frac{\lambda}{\sqrt{\lambda/\mu}}\,\mathbb{P}\left(BL, N_1^\lambda - 1, \lambda\right) = \mu\frac{\phi(\beta_1^*)}{\Phi(\beta_1^*)}. \tag{55}$$

Furthermore,

$$\lim_{\lambda\to\infty}|X|\frac{\lambda^\alpha\mu^{1-\alpha}}{\sqrt{\lambda/\mu}} = \begin{cases}\mu|X| & \text{if } \alpha = 1/2 \\ 0 & \text{if } \alpha < 1/2.\end{cases} \tag{56}$$

By (54)–(56), we have for λ large enough,

$$\frac{\Lambda^\lambda}{\sqrt{\lambda/\mu}}\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda\right) \le \mu\frac{\phi(\beta_1^*)}{\Phi(\beta_1^*)} + \mu|X|.$$

This, together with Lemma 6, the assumption that $\mathbb{E}[|X|] < \infty$, and the requirement on D_1^λ and $D_2^\lambda(N_1^\lambda, \Lambda^\lambda)$, implies that $|\hat{\mathcal{C}}^\lambda(\Lambda^\lambda)|$ in (49) is uniformly bounded by an integrable random variable, and the interchange of limit and expectation in (48) is justified. Q.E.D.

C.3. Optimality Gap of $u_{2,UH}$

In Appendices C.1 and C.2, we propose the two-stage uncertainty hedging rule, which prescribes staffing levels

$$N_1^\lambda = \lambda/\mu + \beta_1^* (\lambda/\mu)^{\max\{\alpha,1/2\}} + o((\lambda/\mu)^{\max\{\alpha,1/2\}})$$
$$N_2^\lambda(N_1^\lambda, \Lambda^\lambda) = \beta_2^*(\beta_1^*, X) (\lambda/\mu)^{\max\{\alpha,1/2\}} + o_{UI}((\lambda/\mu)^{\max\{\alpha,1/2\}}).$$

When $\alpha > 1/2$, β_1^* and $\beta_2^*(\beta_1^*, X)$ are defined in (35), so that the capacity prescription is identical to that under the two-stage newsvendor solution. When $\alpha \le 1/2$, β_1^* and $\beta_2^*(\beta_1^*, X)$ are defined in (40). Let $\mathcal{C}_{2,UH}^\lambda$ be the expected total cost defined under the two-stage uncertainty hedging rules. Recall that $\mathcal{C}_{2,*}^\lambda$ is the optimal total cost for the two-stage optimization problem (2). The next lemma quantifies the optimality gap of the proposed policy to the exact two-stage optimum.

Lemma 8 *For $\alpha \in (0,1)$, we have $\mathcal{C}_{2,UH}^\lambda - \mathcal{C}_{2,*}^\lambda = o(\lambda^{\max\{1/2,\alpha\}})$.*

PROOF: The key of the proof is to show that for any sequence of policies $u \in U$,

$$\liminf_{\lambda\to\infty} \hat{\mathcal{C}}_u^\lambda \ge \lim_{\lambda\to\infty} \hat{\mathcal{C}}_{2,UH}^\lambda. \tag{57}$$

Note that the limit on the right-hand side of (57) is well defined because of Lemma 4 and Lemma 7.

First, it is without loss of generality to consider a sequence of policies $u \in U$ under which

$$\liminf_{\lambda\to\infty} \frac{N_1^\lambda - \lambda/\mu}{(\lambda/\mu)^{\max\{1/2,\alpha\}}} > -\infty. \tag{58}$$

To see this, for any sequence realized arrival rate ℓ^λ, recall from the proof of Proposition 1 that $B_1(N_1^\lambda, N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)$ and $B_2(N_1^\lambda, N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)$ denote the steady-state number of busy servers among the base and surge staff, respectively. It follows that

$$\begin{aligned}
\mathbb{E}\left[\mathcal{R}^\lambda(N_1^\lambda, N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right] &= c_2 N_2^\lambda(N_1^\lambda,\ell^\lambda) + (h+a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right] \\
&\ge \frac{c_2}{\mu}\mu\mathbb{E}\left[B_2(N_1^\lambda, N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right] + \left(\frac{h}{\gamma}+a\right)\gamma\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right] \\
&\ge \min\left\{\frac{c_2}{\mu}, \frac{h}{\gamma}+a\right\}\left(\mu\mathbb{E}\left[B_2(N_1^\lambda, N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right] + \gamma\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right]\right) \\
&= \min\left\{\frac{c_2}{\mu}, \frac{h}{\gamma}+a\right\}\left(\ell^\lambda - \mu\mathbb{E}\left[B_1(N_1^\lambda, N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right]\right) \\
&\ge \min\left\{\frac{c_2}{\mu}, \frac{h}{\gamma}+a\right\}\left(\ell^\lambda - \mu N_1^\lambda\right) \\
&= c_2\left(\frac{\ell^\lambda}{\mu} - N_1^\lambda\right).
\end{aligned}$$

Replacing ℓ^λ with Λ^λ, taking expectation, and recalling that $\mathbb{E}[X] = 0$ give

$$\mathbb{E}\left[\mathcal{R}^\lambda(N_1^\lambda, N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\right] \ge c_2\left(\frac{\lambda}{\mu} - N_1^\lambda\right).$$

Then, the scaled cost $\hat{\mathcal{C}}_u^\lambda$ satisfies

$$\begin{aligned}
\hat{\mathcal{C}}_u^\lambda &= c_1\frac{N_1^\lambda - \lambda/\mu}{(\lambda/\mu)^{\max\{1/2,\alpha\}}} + \frac{\mathbb{E}\left[\mathcal{R}^\lambda(N_1^\lambda, N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\right]}{(\lambda/\mu)^{\max\{1/2,\alpha\}}} \\
&\ge c_1\frac{N_1^\lambda - \lambda/\mu}{(\lambda/\mu)^{\max\{1/2,\alpha\}}} + c_2\frac{\lambda/\mu - N_1^\lambda}{(\lambda/\mu)^{\max\{1/2,\alpha\}}} \\
&= (c_2 - c_1)\frac{\lambda/\mu - N_1^\lambda}{(\lambda/\mu)^{\max\{1/2,\alpha\}}}.
\end{aligned} \tag{59}$$

If (58) does not hold, then it follows from (59) and Assumption 1 that $\liminf_{\lambda\to\infty}\hat{\mathcal{C}}_u^{\lambda}=\infty$. For the purpose of characterizing (near-)optimal staffing rules, we assume without loss of generality that $\liminf_{\lambda\to\infty}\hat{\mathcal{C}}_u^{\lambda}<\infty$.

Now, consider a subsequence of systems indexed by λ_i on which the lim inf in (57) is obtained, namely,

$$\lim_{\lambda_i\to\infty}\hat{\mathcal{C}}_u^{\lambda_i}=\liminf_{\lambda\to\infty}\hat{\mathcal{C}}_u^{\lambda}.$$

Along this subsequence,

$$\hat{\mathcal{C}}_u^{\lambda_i}=\frac{c_1\left(N_1^{\lambda_i}-\lambda_i/\mu\right)}{(\lambda_i/\mu)^{\max\{1/2,\alpha\}}}+\frac{\mathbb{E}\left[\mathcal{R}^{\lambda_i}(N_1^{\lambda_i},N_2^{\lambda_i}(N_1^{\lambda_i},\Lambda^{\lambda_i}),\Lambda^{\lambda_i})\right]}{(\lambda_i/\mu)^{\max\{1/2,\alpha\}}}.$$

Since the second term is non-negative, it must be the case that

$$\limsup_{\lambda_i\to\infty}\frac{c_1\left(N_1^{\lambda_i}-\lambda_i/\mu\right)}{(\lambda_i/\mu)^{\max\{1/2,\alpha\}}}<\infty.$$

Hence,

$$-\infty<\liminf_{\lambda_i\to\infty}\frac{N_1^{\lambda_i}-\lambda_i/\mu}{(\lambda_i/\mu)^{\max\{1/2,\alpha\}}}\le\limsup_{\lambda_i\to\infty}\frac{N_1^{\lambda_i}-\lambda_i/\mu}{(\lambda_i/\mu)^{\max\{1/2,\alpha\}}}<\infty.$$

Then, Bolzano-Weierstrass theorem indicates that any subsequence has a further convergent sub-subsequence indexed by λ_{i_j} along which

$$\frac{N_1^{\lambda_{i_j}}-\lambda_{i_j}/\mu}{\left(\lambda_{i_j}/\mu\right)^{\max\{1/2,\alpha\}}}\to\beta_1\in\mathbb{R}\quad\text{as }\lambda_{i_j}\to\infty. \tag{60}$$

It follows from (60) that

$$\begin{aligned}\lim_{\lambda_{i_j}\to\infty}\hat{\mathcal{C}}_u^{\lambda_{i_j}}&\ge\lim_{\lambda_{i_j}\to\infty}\frac{c_1\left(N_1^{\lambda_{i_j}}-\lambda_{i_j}/\mu\right)}{\left(\lambda_{i_j}/\mu\right)^{\max\{1/2,\alpha\}}}+\liminf_{\lambda_{i_j}\to\infty}\frac{\mathbb{E}\left[\mathcal{R}^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\Lambda^{\lambda_{i_j}}),\Lambda^{\lambda_{i_j}})\right]}{\left(\lambda_{i_j}/\mu\right)^{\max\{1/2,\alpha\}}}\\&=c_1\beta_1+\liminf_{\lambda_{i_j}\to\infty}\frac{\mathbb{E}\left[\mathcal{R}^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\Lambda^{\lambda_{i_j}}),\Lambda^{\lambda_{i_j}})\right]}{\left(\lambda_{i_j}/\mu\right)^{\max\{1/2,\alpha\}}}\\&\ge c_1\beta_1+\mathbb{E}\left[\liminf_{\lambda_{i_j}\to\infty}\frac{\mathcal{R}^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\Lambda^{\lambda_{i_j}}),\Lambda^{\lambda_{i_j}})}{\left(\lambda_{i_j}/\mu\right)^{\max\{1/2,\alpha\}}}\right],\end{aligned} \tag{61}$$

where the last inequality follows from Fatou's lemma.

Next, we are going to establish that for any realized arrival rate $\ell^{\lambda_{i_j}}$,

$$\liminf_{\lambda_{i_j}\to\infty}\frac{\mathcal{R}^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\ell^{\lambda_{i_j}}),\ell^{\lambda_{i_j}})}{\left(\lambda_{i_j}/\mu\right)^{\max\{1/2,\alpha\}}}\ge\hat{r}\left(\beta_1,\beta_2^*(\beta_1,x),x\right). \tag{62}$$

In (62), when $\alpha>1/2$, $\hat{z}$ is defined in (31) and $\beta_2^*(\beta_1,X)$ is defined in (35). In the other case where $\alpha\le1/2$, $\hat{z}$ is defined in (39) and $\beta_2^*(\beta_1,X)$ is defined in (40). To see that (62) holds, define

$$\hat{N}_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\ell^{\lambda_{i_j}}):=N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\ell^{\lambda_{i_j}})/\left(\lambda_{i_j}/\mu\right)^{\max\{1/2,\alpha\}}.$$

Observe that the sequence $\{\hat{N}_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\ell^{\lambda_{i_j}}):\lambda_{i_j}>0\}$ satisfies exactly one of the following three cases:

(i) $\hat{N}_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\ell^{\lambda_{i_j}})\to\beta_2\in\mathbb{R}_+$ as $\lambda_{i_j}\to\infty$.

(ii) $\hat{N}_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\ell^{\lambda_{i_j}})\to\infty$ as $\lambda_{i_j}\to\infty$.

(iii) $\hat{N}_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}},\ell^{\lambda_{i_j}})$ does not converge.

For case (i), (62) follows from Lemma 3, Lemma 5, and the definition of $\beta_2^*(\beta_1, x)$.

For case (ii), we have

$$\begin{aligned}\frac{\mathcal{R}^{\lambda_{i_j}}(N_1^{\lambda_{i_j}}, N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}}, \ell^{\lambda_{i_j}}), \ell^{\lambda_{i_j}})}{(\lambda_{i_j}/\mu)^{\max\{1/2,\alpha\}}} &= \frac{c_2 N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}}, \ell^{\lambda_{i_j}}) + (h+a\gamma)\,\mathbb{E}\left[Q(N_1^{\lambda_{i_j}} + N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}}, \ell^{\lambda_{i_j}}), \ell^{\lambda_{i_j}})\right]}{(\lambda_{i_j}/\mu)^{\max\{1/2,\alpha\}}} \\ &= c_2 \frac{N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}}, \ell^{\lambda_{i_j}})}{(\lambda_{i_j}/\mu)^{\max\{1/2,\alpha\}}} + \frac{(h+a\gamma)\,\mathbb{E}\left[Q(N_1^{\lambda_{i_j}} + N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}}, \ell^{\lambda_{i_j}}), \ell^{\lambda_{i_j}})\right]}{(\lambda_{i_j}/\mu)^{\max\{1/2,\alpha\}}} \\ &\to \infty \quad \text{as } \lambda_{i_j} \to \infty,\end{aligned}$$

and (62) holds.

For case (iii), we can further consider a further subsequence indexed by $\lambda_{i_{j_k}}$ along which $\hat{N}_2^{\lambda_{i_{j_k}}}(N_1^{\lambda_{i_{j_k}}}, \ell^{\lambda_{i_{j_k}}})$ converges. Such subsequence exists because a sequence has no convergent subsequence if and only if it approaches infinity. The same arguments for case (i) can be applied to establish (62).

Now, it follows from (61) and (62) that

$$\begin{aligned}\lim_{\lambda_{i_j}\to\infty} \hat{\mathcal{C}}_u^{\lambda_{i_j}} &\geq c_1\beta_1 + \mathbb{E}\left[\liminf_{\lambda_{i_j}\to\infty} \frac{\mathcal{R}^{\lambda_{i_j}}(N_1^{\lambda_{i_j}}, N_2^{\lambda_{i_j}}(N_1^{\lambda_{i_j}}, \Lambda^{\lambda_{i_j}}), \Lambda^{\lambda_{i_j}})}{(\lambda_{i_j}/\mu)^{\max\{1/2,\alpha\}}}\right] \\ &\geq c_1\beta_1 + \mathbb{E}\left[\hat{r}\left(\beta_1, \beta_2^*(\beta_1, X), X\right)\right].\end{aligned}$$

Furthermore, since β_1^* is constructed such that

$$c_1\beta_1 + \mathbb{E}\left[\hat{r}\left(\beta_1, \beta_2^*(\beta_1, X), X\right)\right] \geq c_1\beta_1^* + \mathbb{E}\left[\hat{r}\left(\beta_1^*, \beta_2^*(\beta_1^*, X), X\right)\right],$$

it follows that

$$\lim_{\lambda_{i_j}\to\infty} \hat{\mathcal{C}}_u^{\lambda_{i_j}} \geq c_1\beta_1^* + \mathbb{E}\left[\hat{r}\left(\beta_1^*, \beta_2^*(\beta_1^*, X), X\right)\right] = \lim_{\lambda_{i_j}\to\infty} \hat{\mathcal{C}}_{2,UH}^{\lambda_{i_j}},$$

where the last equality follows from Lemma 4 and Lemma 7. Since the subsequence indexed by λ_{i_j} is arbitrary, we have established (57).

Next, we apply (57) to the sequence of exact optimal two-stage staffing rules, i.e., $u_{2,*}$, and get that

$$\liminf_{\lambda\to\infty} \hat{\mathcal{C}}_{2,*}^{\lambda} \geq \lim_{\lambda\to\infty} \hat{\mathcal{C}}_{2,UH}^{\lambda}.$$

By the optimality of $u_{2,*}$, we also have

$$\limsup_{\lambda\to\infty} \hat{\mathcal{C}}_{2,*}^{\lambda} \leq \lim_{\lambda\to\infty} \hat{\mathcal{C}}_{2,UH}^{\lambda}.$$

Thus,

$$\lim_{\lambda\to\infty} \hat{\mathcal{C}}_{2,*}^{\lambda} = \lim_{\lambda\to\infty} \hat{\mathcal{C}}_{2,UH}^{\lambda}. \tag{63}$$

The statement follows from (63). Q.E.D.

The following corollary is a direct consequence from the proof of Lemma 8.

Corollary 1 *For $\alpha \in (0,1)$, let β_1^* and $\beta_2^*(\beta_1^*, X)$ be defined in (35) when $\alpha > 1/2$, and defined in (40) when $\alpha \leq 1/2$. Consider a sequence of staffing policies $u = \{\pi^\lambda : \lambda > 0\} = \{N_1^\lambda, N_2^\lambda(N_1^\lambda, \Lambda^\lambda) : \lambda > 0\}$. If there does not exist a subsequence indexed by λ_i along which $\{N_1^{\lambda_i}, N_2^{\lambda_i}(N_1^{\lambda_i}, \Lambda^{\lambda_i}) : \lambda_i > 0\}$ is prescribed as*

$$N_1^{\lambda_i} = \lambda_i/\mu + \beta_1^*\,(\lambda_i/\mu)^{\max\{\alpha,1/2\}} + o((\lambda_i/\mu)^{\max\{\alpha,1/2\}})$$

$$N_2^{\lambda_i}(N_1^{\lambda_i}, \Lambda^{\lambda_i}) = \beta_2^*(\beta_1^*, X)\,(\lambda_i/\mu)^{\max\{\alpha,1/2\}} + o_{UI}((\lambda_i/\mu)^{\max\{\alpha,1/2\}}),$$

then $\mathcal{C}_u^\lambda - \mathcal{C}_{2,UH}^\lambda \geq \Theta(\lambda^{\max\{\alpha,1/2\}})$.

Corollary 1 indicates that it is without loss of optimality to consider the family of two-stage uncertainty hedging rule. To improve upon the $o(\lambda^{\max\{\alpha,1/2\}})$ optimality gap established in Lemma 8, we need to consider refinement which puts further restrictions on the $o((\lambda_i/\mu)^{\max\{\alpha,1/2\}})$ term in N_1^λ and the $o_{UI}((\lambda_i/\mu)^{\max\{\alpha,1/2\}})$ term in $N_2^\lambda(N_1^\lambda,\Lambda^\lambda)$. In the special case when $\alpha>1/2$, it is without loss of optimality to consider the family of two-stage newsvendor solutions. The two-stage QED rule is a refinement of the two-stage newsvendor solution that reduces the optimality gap from $o(\lambda^\alpha)$ to $o(\sqrt{\lambda})$.

C.4. Proof of Theorem 2

PROOF: Note that the two-stage uncertainty hedging rule when $\alpha>1/2$ is equivalent to the two-stage newsvendor solution. The statement follows from Lemma 8. Q.E.D.

Appendix D: Proof of Theorem 1

The proof of Theorem 1 builds on the performance quantification of $u_2(\beta_1,\beta_2(\beta_1,X))$ and $u_{2,UH}$ introduced in Appendix C. For the sequence of systems indexed by λ, recall that $\mathcal{C}_{1,*}^\lambda$ is the optimal total cost for the single-stage optimization problem (4), and $\mathcal{C}_{2,*}^\lambda$ is the optimal total cost for the two-stage optimization problem (2). We establish Theorem 1 for different values of α.

D.1. Benefit of Surge Staffing When $\alpha<1/2$

Lemma 9 *If $\alpha<1/2$, then $\mathcal{C}_{1,*}^\lambda-\mathcal{C}_{2,*}^\lambda=o(\sqrt{\lambda})$.*

PROOF: We start by determining the parameters β_1^* and $\beta_2^*(\beta_1^*,X)$ defined in (40) for the two-stage uncertainty hedging rule when $\alpha<1/2$. In particular, for any realization x of the random variable X, the function $\hat{z}$ in (39) becomes

$$\hat{r}\left(\beta_1,\beta_2(\beta_1,x),x\right)=c_2\beta_2(\beta_1,x)+\left(\frac{h\mu}{\gamma}+a\mu\right)\frac{\sqrt{\frac{\gamma}{\mu}}\left[H\left(\left(\beta_1+\beta_2(\beta_1,x)\right)\sqrt{\frac{\mu}{\gamma}}\right)-\left(\beta_1+\beta_2(\beta_1,x)\right)\sqrt{\frac{\mu}{\gamma}}\right]}{1+\sqrt{\frac{\gamma}{\mu}}\frac{H\left(\left(\beta_1+\beta_2(\beta_1,x)\right)\sqrt{\frac{\mu}{\gamma}}\right)}{H(-(\beta_1+\beta_2(\beta_1,x)))}}.$$

Note that $\hat{r}\left(\beta_1,\beta_2(\beta_1,x),x\right)$ does not depend on the realization x. Hence, given β_1, we have that $\beta_2^*(\beta_1,x)=\arg\min_{\beta_2\in\mathbb{R}_+}\hat{r}(\beta_1,\beta_2(\beta_1,x),x)$ does not depend on x either. Then β_1^* and $\beta_2^*(\beta_1^*,x)$ jointly solve

$$\min_{\beta_1\in\mathbb{R},\,\beta_2(\beta_1,x)\in\mathbb{R}_+}\quad c_1\beta_1+\hat{r}\left(\beta_1,\beta_2(\beta_1,x),x\right).$$

By the assumption that $c_1<c_2$. Thus, it is optimal to set

$$\beta_1^*:=\arg\min_{\beta_1\in\mathbb{R}}\ c_1\beta_1+\left(\frac{h\mu}{\gamma}+a\mu\right)\frac{\sqrt{\frac{\gamma}{\mu}}\left[H\left(\beta_1\sqrt{\frac{\mu}{\gamma}}\right)-\beta_1\sqrt{\frac{\mu}{\gamma}}\right]}{1+\sqrt{\frac{\gamma}{\mu}}\frac{H\left(\beta_1\sqrt{\frac{\mu}{\gamma}}\right)}{H(-\beta_1)}},$$

and $\beta_2^*\left(\beta_1^*,x\right):=0$ for all realizations x of the random variable X.

In this case, the two-stage uncertainty hedging rule is equivalent to the conventional single-stage square-root staffing rule (with staffing cost c_1, holding cost h, and abandonment cost a). Then,

$$\mathcal{C}_{2,UH}^\lambda-\mathcal{C}_{1,*}^\lambda\geq 0\quad\text{for all }\lambda>0. \tag{64}$$

In addition, we establish in Lemma 8 that

$$\mathcal{C}_{2,UH}^\lambda-\mathcal{C}_{2,*}^\lambda=o(\sqrt{\lambda}). \tag{65}$$

The statement follows from (64) and (65). Q.E.D.

D.2. Benefit of Surge Staffing When $\alpha = 1/2$

Lemma 10 *If $\alpha = 1/2$, then $\mathcal{C}^{\lambda}_{1,*} - \mathcal{C}^{\lambda}_{2,*} = O(\sqrt{\lambda})$.*

PROOF: Consider $\beta^{\dagger}_2(\beta_1, X) := 0$ for all β_1, and $\beta^{\dagger}_1 := \arg\min_{\beta_1 \in \mathbb{R}} \; c_1\beta_1 + \mathbb{E}\left[\hat{r}\left(\beta_1, \beta^{\dagger}_2(\beta_1, X), X\right)\right]$. Note that $\beta^{\dagger}_1$ and $\beta^{\dagger}_2(\beta_1, X)$ provide a feasible pair of parameters for $u_2(\beta_1, \beta_2(\beta_1, X))$. Let $\mathcal{C}^{\lambda}_{2,\dagger}$ denote the expected total cost under $u_2(\beta^{\dagger}_1, \beta^{\dagger}_2(\beta^{\dagger}_1, X))$. It follows from similar derivation as in the proof of Lemma 7 that

$$\lim_{\lambda \to \infty} \hat{\mathcal{C}}^{\lambda}_{2,\dagger} = c_1\beta^{\dagger}_1 + \mathbb{E}\left[\hat{r}\left(\beta^{\dagger}_1, \beta^{\dagger}_2(\beta^{\dagger}_1, X), X\right)\right].$$

Since $(\beta^{\dagger}_1, \beta^{\dagger}_2(\beta^{\dagger}_1, x))$ is not necessarily optimal for the optimization problem in (40), we have

$$c_1\beta^{\dagger}_1 + \mathbb{E}\left[\hat{r}\left(\beta^{\dagger}_1, \beta^{\dagger}_2(\beta^{\dagger}_1, X), X\right)\right] \geq c_1\beta^{*}_1 + \mathbb{E}\left[\hat{r}\left(\beta^{*}_1, \beta^{*}_2(\beta^{*}_1, X), X\right)\right].$$

It then follows from Lemma 7 that

$$\mathcal{C}^{\lambda}_{2,\dagger} - \mathcal{C}^{\lambda}_{2,UH} = O(\sqrt{\lambda}). \tag{66}$$

Moreover, since $\beta^{\dagger}_2(\beta^{\dagger}_1, X) = 0$, this policy is equivalent to a single-stage staffing rule. By Proposition 3 in Bassamboo et al. (2010), we get that

$$\mathcal{C}^{\lambda}_{2,\dagger} - \mathcal{C}^{\lambda}_{1,*} = O(\sqrt{\lambda}). \tag{67}$$

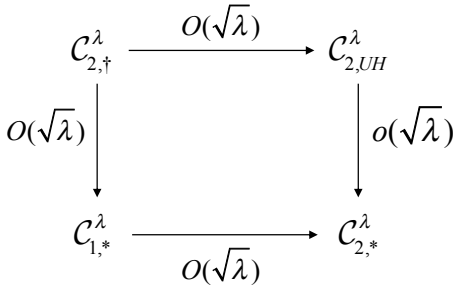
Lastly, by Lemma 8, we have

$$\mathcal{C}^{\lambda}_{2,UH} - \mathcal{C}^{\lambda}_{2,*} = o(\sqrt{\lambda}). \tag{68}$$

The statement follows from (66)–(68). Q.E.D.

Figure 9 below illustrates the performance gap between the employed policies in the proof of Lemma 10.

Figure 9 Cost saving for $\alpha = 1/2$



D.3. Benefit of Surge Staffing When $\alpha > 1/2$

Lemma 11 *If $\alpha > 1/2$, then $\mathcal{C}^{\lambda}_{1,*} - \mathcal{C}^{\lambda}_{2,*} = \Theta(\lambda^{\alpha})$.*

PROOF: Under the two-stage newsvendor solution, the base-stage staffing level is $\lambda/\mu + \beta^{*}_1(\lambda/\mu)^{\alpha} + o((\lambda/\mu)^{\alpha})$, where β^{*}_1 is given by

$$\beta^{*}_1 = \arg\min_{\beta_1 \in \mathbb{R}} \; c_1\beta_1 + c_2\mathbb{E}\left[(X - \beta_1)^{+}\right].$$

Moreover, Lemma 4 establishes that

$$\hat{\mathcal{C}}^{\lambda}_{2,NV} \to c_1\beta^{*}_1 + c_2\mathbb{E}\left[(X - \beta^{*}_1)^{+}\right] \quad \text{as } \lambda \to \infty.$$

In comparison, under the single-stage newsvendor solution, the base-stage staffing level is $\lambda/\mu+\beta_{NV}(\lambda/\mu)^{\alpha}$, where β_{NV} is given by

$$\beta_{NV}=\arg\min_{\beta\in\mathbb{R}} c_1\beta+\left(\frac{h\mu}{\gamma}+a\mu\right)\mathbb{E}\left[(X-\beta)^+\right].$$

Similar lines of arguments as in the proof of Lemma 4 show that

$$\hat{\mathcal{C}}^{\lambda}_{1,NV}\to c_1\beta_{NV}+\left(\frac{h\mu}{\gamma}+a\mu\right)\mathbb{E}\left[(X-\beta_{NV})^+\right] \quad \text{as } \lambda\to\infty.$$

Therefore, if

$$\arg\min_{\beta\in\mathbb{R}} c_1\beta+\left(\frac{h\mu}{\gamma}+a\mu\right)\mathbb{E}\left[(X-\beta)^+\right] > \arg\min_{\beta\in\mathbb{R}} c_1\beta+c_2\mathbb{E}\left[(X-\beta)^+\right], \tag{69}$$

then

$$\lim_{\lambda\to\infty}\hat{\mathcal{C}}^{\lambda}_{1,NV} > \lim_{\lambda\to\infty}\hat{\mathcal{C}}^{\lambda}_{2,NV},$$

so that

$$\mathcal{C}^{\lambda}_{1,NV}-\mathcal{C}^{\lambda}_{2,NV}=\Theta(\lambda^{\alpha}). \tag{70}$$

Note that a sufficient condition for (69) to hold is that X is a continuous random variable, i.e., with a proper density function.

Moreover, by Theorem 1 in Bassamboo et al. (2010), we get that

$$\mathcal{C}^{\lambda}_{1,NV}-\mathcal{C}^{\lambda}_{1,*}=O(\lambda^{1-\alpha})=o(\sqrt{\lambda}). \tag{71}$$

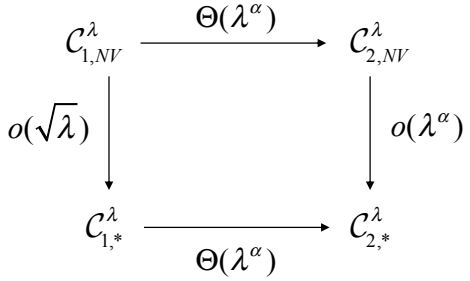
By Lemma 8, we also have

$$\mathcal{C}^{\lambda}_{2,NV}-\mathcal{C}^{\lambda}_{2,*}=o(\lambda^{\alpha}). \tag{72}$$

The statement follows from (70)–(72). Q.E.D.

Figure 10 below illustrates the performance gap between the employed policies in the proof of Lemma 11.

Figure 10 Cost saving for $\alpha>1/2$



Theorem 1 follows from Lemmas 9–11.

Appendix E: Proof of Theorem 3

Before we prove Theorem 3, we first prove an important auxiliary result on the asymptotic equivalence of the family of two-stage newsvendor solutions, and then establish the asymptotic performance of the family of two-stage QED rules. We assume throughout this section that $\alpha > 1/2$.

Recall that the two-stage newsvendor policy takes the form

$$N_1^\lambda = \lambda/\mu + \beta_1^*(\lambda/\mu)^\alpha + D_1^\lambda, \quad N_2^\lambda(N_1^\lambda, \Lambda^\lambda) = \beta_2^*(\beta_1^*, X)(\lambda/\mu)^\alpha + D_2^\lambda(N_1^\lambda, \Lambda^\lambda), \tag{73}$$

for $D_1^\lambda = o((\lambda/\mu)^\alpha)$, and $D_2^\lambda(N_1^\lambda, \Lambda^\lambda) = o_{UI}((\lambda/\mu)^\alpha)$. Let u be a policy of the form (73). Based on u, we can construct another policy $\tilde{u}$, where

$$\tilde{N}_1^\lambda = \lambda/\mu + \beta_1^*(\lambda/\mu)^\alpha + \tilde{D}_1^\lambda, \quad \text{and} \quad \tilde{N}_2^\lambda(\tilde{N}_1^\lambda, \Lambda^\lambda) = \beta_2^*(\beta_1^*, X)(\lambda/\mu)^\alpha + \tilde{D}_2^\lambda(\tilde{N}_1^\lambda, \Lambda^\lambda),$$

for

$$\tilde{D}_1^\lambda := 0, \quad \text{and} \quad \tilde{D}_2^\lambda(\tilde{N}_1^\lambda, \Lambda^\lambda) := \begin{cases} D_2^\lambda(N_1^\lambda, \Lambda^\lambda) & \text{if } X < \beta_1^* \\ D_1^\lambda + D_2^\lambda(N_1^\lambda, \Lambda^\lambda) & \text{if } X \geq \beta_1^*. \end{cases}$$

Let $\mathcal{C}_u^\lambda$ and $\mathcal{C}_{\tilde{u}}^\lambda$ denote the expected total cost under u and $\tilde{u}$, respectively.

Lemma 12 *If $\mathcal{C}_u^\lambda < \mathcal{C}_{\tilde{u}}^\lambda$, then $\mathcal{C}_{\tilde{u}}^\lambda - \mathcal{C}_u^\lambda = o(\sqrt{\lambda})$.*

PROOF: Let $\mathcal{S}_u^\lambda$ and $\mathcal{S}_{\tilde{u}}^\lambda$ denote the expected staffing cost under u and $\tilde{u}$, respectively. By construction, u and $\tilde{u}$ have the same expected staffing cost, namely,

$$\begin{aligned} \mathcal{S}_u^\lambda &= c_1(\lambda/\mu) + c_1\beta_1^*(\lambda/\mu)^\alpha + c_1 D_1^\lambda + \mathbb{E}\left[c_2\beta_2^*(\beta_1^*, X) + c_2 D_2^\lambda(N_1^\lambda, \Lambda^\lambda)\right] \\ &= c_1(\lambda/\mu) + c_1\beta_1^*(\lambda/\mu)^\alpha + c_2\frac{c_1}{c_2} D_1^\lambda + \mathbb{E}\left[c_2\beta_2^*(\beta_1^*, X) + c_2 D_2^\lambda(N_1^\lambda, \Lambda^\lambda)\right] \\ &= c_1(\lambda/\mu) + c_1\beta_1^*(\lambda/\mu)^\alpha + c_2 D_1^\lambda \mathbb{P}(X \geq \beta_1^*) + \mathbb{E}\left[c_2\beta_2^*(\beta_1^*, X) + c_2 D_2^\lambda(N_1^\lambda, \Lambda^\lambda)\right] \\ &= \mathcal{S}_{\tilde{u}}^\lambda, \end{aligned}$$

where the second to last equality follows from $\beta_1^* = \bar{F}_X^{-1}(c_1/c_2)$ and the assumption that X is a continuous random variable.

We next consider queue length. If $D_1^\lambda < 0$, then by construction of $\tilde{u}$, $\tilde{u}$ prescribes a higher staffing level than u when $X < \beta_1^*$, and prescribes the same staffing level as u when $X \geq \beta_1^*$. Thus,

$$\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)\right] \geq \mathbb{E}\left[Q(\tilde{N}_1^\lambda + \tilde{N}_2^\lambda(\tilde{N}_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)\right],$$

and $\mathcal{C}_u^\lambda \geq \mathcal{C}_{\tilde{u}}^\lambda$.

Therefore, it is without loss of generality to assume that $D_1^\lambda \geq 0$ for all $\lambda > 0$. We again divide the discussion into two cases: $X \geq \beta_1^*$ and $X < \beta_1^*$. If the realized random variable satisfies $x \geq \beta_1^*$, then

$$\tilde{D}_1^\lambda + \tilde{D}_2^\lambda(\tilde{N}_1^\lambda, \ell^\lambda) = D_1^\lambda + D_2^\lambda(N_1^\lambda, \ell^\lambda),$$

where $\ell^\lambda = \lambda + x\lambda^\alpha\mu^{1-\alpha}$. This implies that

$$\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)\mathbb{1}_{\{X \geq \beta_1^*\}}\right] = \mathbb{E}\left[Q(\tilde{N}_1^\lambda + \tilde{N}_2^\lambda(\tilde{N}_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)\mathbb{1}_{\{X \geq \beta_1^*\}}\right]. \tag{74}$$

In the other case where $X < \beta_1^*$, it follows from (32) in the proof of Lemma 3 that

$$\begin{aligned}&\lim_{\lambda\to\infty}\frac{1}{(\lambda/\mu)^{1/2}}\mathbb{E}\left[Q(N_1^\lambda+N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X<\beta_1^*\}}|\Lambda^\lambda\right]\\&=\lim_{\lambda\to\infty}\frac{1}{(\lambda/\mu)^{1/2}}\mathbb{E}\left[Q(\tilde{N}_1^\lambda+\tilde{N}_2^\lambda(\tilde{N}_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X<\beta_1^*\}}|\Lambda^\lambda\right]=0.\end{aligned}\tag{75}$$

The above equality and subsequent inequalities involving random variables hold in a path-by-path sense. Furthermore, recall from Lemma 1 that

$$\begin{aligned}\mathbb{E}\left[Q(N_1^\lambda+N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)|\Lambda^\lambda\right]&\le\max\{\mu/\gamma,1\}\left(\left(\Lambda^\lambda/\mu-N_1^\lambda\right)^+\sqrt{4\pi/\mu}\sqrt{\Lambda^\lambda}+1/\log 2\right)\\&\le\max\{\mu/\gamma,1\}\left(\sqrt{4\pi/\mu}\sqrt{\Lambda^\lambda}+1/\log 2\right),\end{aligned}$$

where the second inequality follows because $D_1^\lambda\ge 0$. Thus, there exists a random variable Y with $\mathbb{E}[Y]<\infty$ such that

$$\frac{1}{(\lambda/\mu)^{1/2}}\mathbb{E}\left[Q(N_1^\lambda+N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X<\beta_1^*\}}|\Lambda^\lambda\right]\le Y,\quad\text{for all }\lambda>0.$$

Moreover, the same derivation applies to $\tilde{u}$. Thus, we can apply the dominated convergence theorem to (75) and get that

$$\lim_{\lambda\to\infty}\frac{1}{(\lambda/\mu)^{1/2}}\mathbb{E}\left[Q(N_1^\lambda+N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X<\beta_1^*\}}\right]=\frac{1}{(\lambda/\mu)^{1/2}}\mathbb{E}\left[Q(\tilde{N}_1^\lambda+\tilde{N}_2^\lambda(\tilde{N}_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X<\beta_1^*\}}\right]=0.\tag{76}$$

Now, we write $\mathcal{C}_u^\lambda$ as

$$\begin{aligned}\mathcal{C}_u^\lambda&=\mathcal{S}_u^\lambda+(h+a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda+N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\right]\\&=\mathcal{S}_u^\lambda+(h+a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda+N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X<\beta_1^*\}}\right]+(h+a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda+N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X\ge\beta_1^*\}}\right].\end{aligned}\tag{77}$$

In addition, we write $\mathcal{C}_{\tilde{u}}^\lambda$ as

$$\begin{aligned}\mathcal{C}_{\tilde{u}}^\lambda&=\mathcal{S}_{\tilde{u}}^\lambda+(h+a\gamma)\,\mathbb{E}\left[Q(\tilde{N}_1^\lambda+\tilde{N}_2^\lambda(\tilde{N}_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\right]\\&=\mathcal{S}_{\tilde{u}}^\lambda+(h+a\gamma)\,\mathbb{E}\left[Q(\tilde{N}_1^\lambda+\tilde{N}_2^\lambda(\tilde{N}_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X<\beta_1^*\}}\right]+(h+a\gamma)\,\mathbb{E}\left[Q(\tilde{N}_1^\lambda+\tilde{N}_2^\lambda(\tilde{N}_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X\ge\beta_1^*\}}\right].\end{aligned}\tag{78}$$

Then,

$$\begin{aligned}\mathcal{C}_u^\lambda-\mathcal{C}_{\tilde{u}}^\lambda&=(h+a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda+N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X<\beta_1^*\}}\right]-(h+a\gamma)\,\mathbb{E}\left[Q(\tilde{N}_1^\lambda+\tilde{N}_2^\lambda(\tilde{N}_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)\mathbb{1}_{\{X<\beta_1^*\}}\right]\\&=o(\sqrt{\lambda}),\end{aligned}$$

where the first equality follows from (74), (77) and (78), and the second equality follows from (76). Q.E.D.

Recall from Section 4.2 that $u_{2,QED}$ takes the form

$$N_1^\lambda=\lambda/\mu+\beta_1^*(\lambda/\mu)^\alpha+O(\sqrt{\lambda/\mu}),\quad\text{and}\quad N_2^\lambda(N_1^\lambda,\Lambda^\lambda)=(\Lambda^\lambda/\mu+\eta^*\sqrt{\Lambda^\lambda/\mu}-N_1^\lambda)^+ +o_{UI}(\sqrt{\lambda/\mu}).$$

For a sequence of policies $u\in U$, let

$$\bar{\mathcal{C}}_u^\lambda:=\frac{1}{(\lambda/\mu)^{1/2}}\left(\mathcal{C}_u^\lambda-c_1\frac{\lambda}{\mu}-c_1\beta_1^*\left(\frac{\lambda}{\mu}\right)^\alpha-c_2\mathbb{E}\left[(X-\beta_1^*)^+\right]\left(\frac{\lambda}{\mu}\right)^\alpha\right).\tag{79}$$

In addition, define the mapping $\psi:\mathbb{R}\to\mathbb{R}$ as

$$\psi(x):=\begin{cases}0&\text{if } x<\beta_1^*\\ c_2\eta^*+\left(\frac{h\mu}{\gamma}+a\mu\right)\frac{\sqrt{\frac{\gamma}{\mu}}\left[H\left(\eta^*\sqrt{\frac{\mu}{\gamma}}\right)-\eta^*\sqrt{\frac{\mu}{\gamma}}\right]}{1+\sqrt{\frac{\gamma}{\mu}}\frac{H\left(\eta^*\sqrt{\frac{\mu}{\gamma}}\right)}{H(-\eta^*)}}&\text{if } x\ge\beta_1^*.\end{cases}\tag{80}$$

Lemma 13 *We have*

$$\lim_{\lambda\to\infty} \bar{\mathcal{C}}^{\lambda}_{2,QED} = \mathbb{E}\left[\psi(X)\right].$$

PROOF: Consider an arbitrary two-stage QED policy u of the form

$$N_1^\lambda = \lambda/\mu + \beta_1^*(\lambda/\mu)^\alpha + D_1^\lambda, \quad \text{and} \quad N_2^\lambda(N_1^\lambda,\Lambda^\lambda) = (\Lambda^\lambda/\mu + \eta^*\sqrt{\Lambda^\lambda/\mu} - N_1^\lambda)^+ + J(N_1^\lambda,\Lambda^\lambda),$$

for $D_1^\lambda \in \mathbb{R}$, $D_1^\lambda = O(\sqrt{\lambda/\mu})$, and $J(N_1^\lambda,\Lambda^\lambda) = o_{UI}(\sqrt{\lambda/\mu})$.

For base staffing level, it holds that

$$c_1\left(N_1^\lambda - \lambda/\mu - \beta_1^*(\lambda/\mu)^\alpha - D_1^\lambda\right) = 0.$$

For surge staffing level, we have

$$\lim_{\lambda\to\infty} \frac{1}{\sqrt{\lambda/\mu}} c_2 \left(N_2^\lambda(N_1^\lambda,\Lambda^\lambda) - (X-\beta_1^*)^+ \left(\frac{\lambda}{\mu}\right)^\alpha + D_1^\lambda \mathbb{1}_{\{X>\beta_1^*\}} \right) = \bar{n}(X), \tag{81}$$

where

$$\bar{n}(X) := \begin{cases} 0 & \text{if } X < \beta_1^* \\ c_2\eta^* & \text{if } X > \beta_1^*. \end{cases}$$

We next show that

$$\lim_{\lambda\to\infty} \mathbb{E}\left[\frac{1}{\sqrt{\lambda/\mu}} c_2 \left(N_2^\lambda(N_1^\lambda,\Lambda^\lambda) - (X-\beta_1^*)^+ \left(\frac{\lambda}{\mu}\right)^\alpha + D_1^\lambda \mathbb{1}_{\{X>\beta_1^*\}} \right)\right] = \mathbb{E}\left[\bar{n}(X)\right]. \tag{82}$$

To see (82), note that when $X < \beta_1^*$,

$$\begin{aligned} |N_2^\lambda(N_1^\lambda,\Lambda^\lambda) - (X-\beta_1^*)^+(\lambda/\mu)^\alpha + D_1^\lambda \mathbb{1}_{\{X>\beta_1^*\}}| &= |(\Lambda^\lambda/\mu + \eta^*\sqrt{\Lambda^\lambda/\mu} - N_1^\lambda)^+ + J(N_1^\lambda,\Lambda^\lambda)| \\ &= \left| \left((X-\beta_1^*)(\lambda/\mu)^\alpha + \eta^*\sqrt{\Lambda^\lambda/\mu} - D_1^\lambda \right)^+ + J(N_1^\lambda,\Lambda^\lambda) \right| \\ &\le |\eta^*|\sqrt{\Lambda^\lambda/\mu} + |D_1^\lambda| + |J(N_1^\lambda,\Lambda^\lambda)|. \end{aligned}$$

When $X > \beta_1^*$,

$$\begin{aligned} &|N_2^\lambda(N_1^\lambda,\Lambda^\lambda) - (X-\beta_1^*)^+(\lambda/\mu)^\alpha + D_1^\lambda \mathbb{1}_{\{X>\beta_1^*\}}| \\ &= |(\Lambda^\lambda/\mu + \eta^*\sqrt{\Lambda^\lambda/\mu} - N_1^\lambda)^+ + J(N_1^\lambda,\Lambda^\lambda) - (X-\beta_1^*)^+(\lambda/\mu)^\alpha + D_1^\lambda| \\ &= \left| \left((X-\beta_1^*)(\lambda/\mu)^\alpha + \eta^*\sqrt{\Lambda^\lambda/\mu} - D_1^\lambda \right)^+ + J(N_1^\lambda,\Lambda^\lambda) - (X-\beta_1^*)^+(\lambda/\mu)^\alpha + D_1^\lambda \right| \\ &= \begin{cases} |\eta^*\sqrt{\Lambda^\lambda/\mu} - D_1^\lambda + J(N_1^\lambda,\Lambda^\lambda) + D_1^\lambda| & \text{if } (X-\beta_1^*)(\lambda/\mu)^\alpha \ge -\eta^*\sqrt{\Lambda^\lambda/\mu} + D_1^\lambda \\ |J(N_1^\lambda,\Lambda^\lambda) - (X-\beta_1^*)^+(\lambda/\mu)^\alpha + D_1^\lambda| & \text{if } (X-\beta_1^*)(\lambda/\mu)^\alpha < -\eta^*\sqrt{\Lambda^\lambda/\mu} + D_1^\lambda \end{cases} \\ &\le |\eta^*|\sqrt{\Lambda^\lambda/\mu} + 2|D_1^\lambda| + |J(N_1^\lambda,\Lambda^\lambda)|. \end{aligned}$$

Thus, in both cases, there exists some random variable Y with $\mathbb{E}[Y] < \infty$ such that

$$\left| \frac{1}{\sqrt{\lambda/\mu}} \left(N_2^\lambda(N_1^\lambda,\Lambda^\lambda) - (X-\beta_1^*)^+ \left(\frac{\lambda}{\mu}\right)^\alpha \right) + D_1^\lambda \mathbb{1}_{\{X>\beta_1^*\}} \right| < Y, \quad \text{for all } \lambda > 0.$$

The first equality in (82) can then be justified by (81) and the dominated convergence theorem.

For queue length, it follows from (32) in the proof of Lemma 3 (for the case where $X < \beta_1^*$), and the same analysis as in the proof of Lemma 6 (for the case where $X > \beta_1^*$) that

$$\lim_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}} (h + a\gamma)\, \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda) | \Lambda^\lambda \right] = \bar{q}(X), \tag{83}$$

where

$$\bar{q}(X) := \begin{cases} 0 & \text{if } X < \beta_1^* \\ \left(\frac{h\mu}{\gamma} + a\mu\right) \frac{\sqrt{\frac{\gamma}{\mu}}\left[H\left(\eta^*\sqrt{\frac{\mu}{\gamma}}\right) - \eta^*\sqrt{\frac{\mu}{\gamma}}\right]}{1+\sqrt{\frac{\gamma}{\mu}}\frac{H\left(\eta^*\sqrt{\frac{\mu}{\gamma}}\right)}{H(-\eta^*)}} & \text{if } X > \beta_1^*. \end{cases}$$

We next show that

$$\lim_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}} (h + a\gamma)\, \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)\right] = \mathbb{E}\left[\bar{q}(X)\right]. \tag{84}$$

To see (84), it follows from Lemma 1 that

$$\begin{aligned} \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda) | \Lambda^\lambda\right] &\le \max\{\mu/\gamma, 1\} \left(\left(\Lambda^\lambda/\mu - N_1^\lambda - N_2^\lambda(N_1^\lambda, \Lambda^\lambda)\right)^+ + \sqrt{4\pi/\mu}\sqrt{\Lambda^\lambda} + 1/\log 2\right) \\ &\le \begin{cases} \max\{\mu/\gamma, 1\} \left(|D_1^\lambda| + \sqrt{4\pi/\mu}\sqrt{\Lambda^\lambda} + 1/\log 2\right) & \text{if } X < \beta_1^* \\ \max\{\mu/\gamma, 1\} \left(|J(N_1^\lambda, \Lambda^\lambda)| + \sqrt{4\pi/\mu}\sqrt{\Lambda^\lambda} + 1/\log 2\right) & \text{if } X > \beta_1^*. \end{cases} \end{aligned}$$

Thus, there exists some random variable Y with $\mathbb{E}[Y] < \infty$ such that

$$\frac{1}{(\lambda/\mu)^{1/2}} (h + a\gamma)\, \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)\right] < Y, \quad \text{for all } \lambda > 0.$$

The first equality in (84) is justified by (83) and the dominated convergence theorem.

Then, for $\bar{\mathcal{C}}_u^\lambda$ defined in (79) and ψ defined in (80),

$$\begin{aligned} \bar{\mathcal{C}}_u^\lambda &= \frac{1}{(\lambda/\mu)^{1/2}} \Bigg(c_1 N_1^\lambda + c_2 \mathbb{E}\left[N_2^\lambda(N_1^\lambda, \Lambda^\lambda)\right] + (h + a\gamma)\, \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)\right] \\ &\quad - c_1 \frac{\lambda}{\mu} - c_1 \beta_1^* \left(\frac{\lambda}{\mu}\right)^\alpha - c_2 \mathbb{E}\left[(X - \beta_1^*)^+\right] \left(\frac{\lambda}{\mu}\right)^\alpha \Bigg) \\ &= \frac{1}{(\lambda/\mu)^{1/2}} \Bigg(c_1 \left(N_1^\lambda - \frac{\lambda}{\mu} - \beta_1^* \left(\frac{\lambda}{\mu}\right)^\alpha - D_1^\lambda\right) + c_2 \mathbb{E}\left[N_2^\lambda(N_1^\lambda, \Lambda^\lambda) - (X - \beta_1^*)^+ \left(\frac{\lambda}{\mu}\right)^\alpha + D_1^\lambda \mathbb{1}_{\{X > \beta_1^*\}}\right] \\ &\quad + (h + a\gamma)\, \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)\right] \Bigg) \\ &= \mathbb{E}\left[\psi(X)\right], \end{aligned}$$

from which the statement follows. Q.E.D.

We now present the proof of Theorem 3.

Proof: [Proof of Theorem 3] It follows from (57) in the proof of Lemma 8 that for all $u \in U$,

$$\liminf_{\lambda\to\infty} \hat{\mathcal{C}}_u^\lambda \ge \lim_{\lambda\to\infty} \hat{\mathcal{C}}_{2,NV}^\lambda = c_1\beta_1^* + c_2 \mathbb{E}\left[(X - \beta_1^*)^+\right],$$

where $\beta_1^* = \bar{F}_X^{-1}(c_1/c_2)$. Thus, for a sequence of policies $u \in U$, we consider $\bar{\mathcal{C}}_u^\lambda$ defined in (79). We next show that for all $u \in U$,

$$\liminf_{\lambda\to\infty} \bar{\mathcal{C}}_u^\lambda \ge \lim_{\lambda\to\infty} \bar{\mathcal{C}}_{2,QED}^\lambda, \tag{85}$$

where the limit on the right-hand side of (85) is rigorously established in Lemma 13. Similar to the proof of Lemma 8, for the purpose of characterizing (near-)optimal staffing rules, we assume without loss of generality that $\limsup_{\lambda\to\infty} \bar{\mathcal{C}}_u^\lambda < \infty$.

First, by Corollary 1, it is without loss of optimality to consider a sequence of policies u of the form

$$N_1^\lambda = \lambda/\mu + \beta_1^*(\lambda/\mu)^\alpha + D_1^\lambda, \quad N_2^\lambda = (X - \beta_1^*)^+(\lambda/\mu)^\alpha + D_2^\lambda(N_1^\lambda, \Lambda^\lambda),$$

for $D_1^\lambda = o\left((\lambda/\mu)^\alpha\right)$ and $D_2^\lambda(N_1^\lambda, \Lambda^\lambda) = o_{UI}\left((\lambda/\mu)^\alpha\right)$, i.e., the two-stage newsvendor solutions.

In addition, Lemma 12 implies that it is without loss of generality to consider a sequence of policies where $D_1^\lambda = 0$ for all $\lambda > 0$. Thus, we can write

$$\begin{aligned}\bar{\mathcal{C}}_u^\lambda &= \frac{1}{(\lambda/\mu)^{1/2}}\mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, \Lambda^\lambda) + (h + a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)|\Lambda^\lambda\right] - c_2(X - \beta_1^*)^+\left(\frac{\lambda}{\mu}\right)^\alpha\right] \\ &= \frac{1}{(\lambda/\mu)^{1/2}}\mathbb{E}\left[c_2 D_2^\lambda(N_1^\lambda, \Lambda^\lambda) + (h + a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)|\Lambda^\lambda\right]\right].\end{aligned}$$

By Fatou's lemma,

$$\liminf_{\lambda\to\infty} \bar{\mathcal{C}}_u^\lambda \geq \mathbb{E}\left[\liminf_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}}\left(c_2 D_2^\lambda(N_1^\lambda, \Lambda^\lambda) + (h + a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \Lambda^\lambda), \Lambda^\lambda)|\Lambda^\lambda\right]\right)\right]. \tag{86}$$

We are going to establish that for any realized arrival rate $\ell^\lambda = \lambda + x\lambda^\alpha\mu^{1-\alpha}$,

$$\liminf_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}}\left(c_2 D_2^\lambda(N_1^\lambda, \ell^\lambda) + (h + a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right]\right) \geq \psi(x), \tag{87}$$

where ψ is defined in (80). To this end, define

$$\bar{D}_2^\lambda(N_1^\lambda, \ell^\lambda) := \frac{1}{(\lambda/\mu)^{1/2}} D_2^\lambda(N_1^\lambda, \ell^\lambda).$$

Observe that the sequence $\left\{\bar{D}_2^\lambda(N_1^\lambda, \ell^\lambda) : \lambda > 0\right\}$ satisfies exactly one of the following four cases:

(i) $\bar{D}_2^\lambda(N_1^\lambda, \ell^\lambda) \to \infty$ as $\lambda \to \infty$.

(ii) $\bar{D}_2^\lambda(N_1^\lambda, \ell^\lambda) \to -\infty$ as $\lambda \to \infty$.

(iii) $\bar{D}_2^\lambda(N_1^\lambda, \ell^\lambda) \to \eta \in \mathbb{R}$ as $\lambda \to \infty$.

(iv) $\bar{D}_2^\lambda(N_1^\lambda, \ell^\lambda)$ does not converge.

For case (i), since $\mathbb{E}\left[Q^\lambda(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] \geq 0$,

$$\liminf_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}}\left(c_2 D_2^\lambda(N_1^\lambda, \ell^\lambda) + (h + a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right]\right) \geq \liminf_{\lambda\to\infty} c_2 \bar{D}_2^\lambda(N_1^\lambda, \ell^\lambda) = \infty.$$

For case (ii), this case is only possible when $x > \beta_1^*$. This is because otherwise, $\beta_2^* = 0$, so that $D_2^\lambda \geq 0$ for all $\lambda > 0$. Now since $x > \beta_1^*$, we have

$$\begin{aligned}&(h + a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] \\ =& \left(\frac{h}{\gamma} + a\right)\gamma\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] \\ =& \left(\frac{h}{\gamma} + a\right)\left(\ell^\lambda - \mu\mathbb{E}\left[B_2(N_1^\lambda, N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right] - \mu\mathbb{E}\left[B_1(N_1^\lambda, N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right]\right) \\ \geq& \left(\frac{h}{\gamma} + a\right)\left(\ell^\lambda - \mu N_2^\lambda(N_1^\lambda, \ell^\lambda) - \mu N_1^\lambda\right) \\ =& \left(\frac{h\mu}{\gamma} + a\mu\right)\left(-D_2^\lambda(N_1^\lambda, \ell^\lambda)\right),\end{aligned}$$

where recall from the proof of Proposition 1 that $B_1(N_1^\lambda, N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)$ is the steady-state number of busy servers among those that are staffed at the base stage, and $B_2(N_1^\lambda, N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)$ is the steady-state number of busy servers among those that are staffed at the surge stage. Therefore,

$$\begin{aligned}&\liminf_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}}\left(c_2 D_2^\lambda(N_1^\lambda, \ell^\lambda) + (h + a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, \ell^\lambda), \ell^\lambda)\right]\right) \\ \geq& \liminf_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}}\left(c_2 D_2^\lambda(N_1^\lambda, \ell^\lambda) + \left(\frac{h\mu}{\gamma} + a\mu\right)\left(-D_2^\lambda(N_1^\lambda, \ell^\lambda)\right)\right) \\ =& \liminf_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}}\left(c_2 - \frac{h\mu}{\gamma} - a\mu\right) D_2^\lambda(N_1^\lambda, \ell^\lambda) \\ =& \infty.\end{aligned}$$

For case (iii), it follows from (32) in the proof of Lemma 3 (for the case where $x < \beta_1^*$), and the same analysis as in the proof of Lemma 6 (for the case where $x > \beta_1^*$) that

$$\begin{aligned}
&\lim_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}} \left(c_2 D_2^\lambda(N_1^\lambda,\ell^\lambda) + (h+a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right]\right)\\
=&c_2\eta + \lim_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}} (h+a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda,\ell^\lambda),\ell^\lambda)\right]\\
=&\begin{cases} c_2\eta & \text{if } x < \beta_1^* \\ c_2\eta + \left(\frac{h\mu}{\gamma} + a\mu\right) \frac{\sqrt{\frac{\gamma}{\mu}}\left[H\left(\eta\sqrt{\frac{\mu}{\gamma}}\right) - \eta\sqrt{\frac{\mu}{\gamma}}\right]}{1+\sqrt{\frac{\gamma}{\mu}}\frac{H\left(\eta\sqrt{\frac{\mu}{\gamma}}\right)}{H(-\eta)}} & \text{if } x > \beta_1^*. \end{cases}
\end{aligned}$$

Moreover, in the scenario where $x < \beta_1^*$, we have $\beta_2^*(\beta_1^*, x) = 0$, so it must be that $D_2^\lambda \geq 0$ and $\eta \geq 0$. Therefore, (87) follows from the definition of η^* in (11).

For case (iv), we can further consider a subsequence indexed by λ_i along which $\bar{D}_2^{\lambda_i}(N_1^{\lambda_i}, \ell^{\lambda_i})$ converges. Such subsequence exists because a sequence has no convergent subsequence if and only if it approaches infinity. The same arguments for case (iii) can be applied to establish (87).

So far we have established (87). This, together with (86) and Lemma 13, gives that

$$\begin{aligned}
\liminf_{\lambda\to\infty} \bar{\mathcal{C}}_u^\lambda &\geq \mathbb{E}\left[\liminf_{\lambda\to\infty} \frac{1}{(\lambda/\mu)^{1/2}} \left(c_2 D_2^\lambda(N_1^\lambda,\Lambda^\lambda) + (h+a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda,\Lambda^\lambda),\Lambda^\lambda)|\Lambda^\lambda\right]\right)\right]\\
&\geq \mathbb{E}\left[\psi(X)\right]\\
&= \lim_{\lambda\to\infty} \bar{\mathcal{C}}_{2,QED}^\lambda,
\end{aligned}$$

which establishes (85).

In this last step, note that by (85), we have

$$\liminf_{\lambda\to\infty} \bar{\mathcal{C}}_{2,*}^\lambda \geq \lim_{\lambda\to\infty} \bar{\mathcal{C}}_{2,QED}^\lambda.$$

Moreover, by the optimality of $u_{2,*}$, it holds that

$$\limsup_{\lambda\to\infty} \bar{\mathcal{C}}_{2,*}^\lambda \leq \lim_{\lambda\to\infty} \bar{\mathcal{C}}_{2,QED}^\lambda.$$

Therefore,

$$\lim_{\lambda\to\infty} \bar{\mathcal{C}}_{2,*}^\lambda = \lim_{\lambda\to\infty} \bar{\mathcal{C}}_{2,QED}^\lambda,$$

which implies that $\mathcal{C}_{2,QED}^\lambda - \mathcal{C}_{2,*}^\lambda = o(\sqrt{\lambda})$. Q.E.D.

Appendix F: Model with Surge-Stage Prediction Error

Recall that we use F_Y (alternatively, f_Y) and F_Z (alternatively, f_Z) to denote the cdf (alternatively, probability density function) of Y and Z, respectively.

F.1. Small Prediction Error: Proof of Proposition 2

PROOF: Statement (I) follows exactly the same lines of analysis as the proof of Theorem 1 for $\alpha > 1/2$. Statement (II) follows exactly the same lines of analysis as the proof of Theorem 3. Lastly, following the same lines of analysis as the proof of Theorem 3, we can show that $\mathcal{C}_{2,ERR}^{e,\lambda} - \mathcal{C}_{2,*}^{o,\lambda} = o(\sqrt{\lambda})$. This, together with statement (II), implies statement (III). To elaborate on the generalization, we explain why the proof of Proposition 2 follows directly from the analysis of the case with perfect surge-stage prediction. In particular,

when $\nu < 1/2$, the two-stage error policy takes the same form as the two-stage QED rule, with random variable X (alternatively, its realization x) replaced by random variable Y (alternatively, its realization y). For $\ell^\lambda = \lambda + y\lambda^\alpha \mu^{1-\alpha} + z\lambda^\nu \mu^{1-\nu}$, it still holds that if $y < F_Y^{-1}(c_1/c_2)$, then

$$N_1^\lambda + N_2^\lambda(N_1^\lambda, y) = \ell^\lambda/\mu + F_Y^{-1}(c_1/c_2)\left(\ell^\lambda/\mu\right)^\alpha + O(\sqrt{\ell^\lambda/\mu}).$$

In the other case where $y \geq F_Y^{-1}(c_1/c_2)$, we have

$$N_1^\lambda + N_2^\lambda(N_1^\lambda, y) = \ell^\lambda/\mu + \eta^*\left(\ell^\lambda/\mu\right)^\alpha + o(\sqrt{\ell^\lambda/\mu}),$$

for η^* defined in (11). The rest of the analysis is generalized similarly. Q.E.D.

F.2. Moderate to Large Prediction Error: Proof of Proposition 3

Proof: We first show that there exists an optimal solution to (18). In particular, consider the inner-problem in (18):

$$\min_{N_2^\lambda(N_1^\lambda, Y)} \left\{ c_2 N_2^\lambda(N_1^\lambda, Y) + (h\mu/\gamma + a\mu)\mathbb{E}\left[\left(\Lambda^\lambda/\mu - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ | Y\right]\right\}. \tag{88}$$

Note that (88) is a newsvendor problem with unit capacity cost c_2, unit sales price $h\mu/\gamma + a\mu$, random demand $\Lambda^\lambda/\mu - N_1^\lambda | Y$ (where the randomness lies in random variable Z), and capacity decision $N_2^\lambda(N_1^\lambda, Y)$. The optimal solution is given by

$$\bar{N}_2^\lambda(N_1^\lambda, Y) = \left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu + \frac{\lambda}{\mu} + Y\left(\frac{\lambda}{\mu}\right)^\alpha - N_1^\lambda\right)^+.$$

Given $\bar{N}_2^\lambda(N_1^\lambda, Y)$, the outer-problem is given by $\min_{N_1^\lambda} h(N_1^\lambda)$, where

$$h(N_1^\lambda) := c_1 N_1^\lambda + \mathbb{E}\left[c_2 \bar{N}_2^\lambda(N_1^\lambda, Y) + (h\mu/\gamma + a\mu)\left(\Lambda^\lambda/\mu - N_1^\lambda - \bar{N}_2^\lambda(N_1^\lambda, Y)\right)^+\right].$$

Differentiating $h(N_1^\lambda)$ with respect to N_1^λ gives

$$\begin{aligned}\frac{\partial}{\partial N_1^\lambda} h(N_1^\lambda) = c_1 - c_2 \mathbb{P}\Bigg(&\left(\frac{\lambda}{\mu}\right)^\alpha Y > \left(N_1^\lambda - \frac{\lambda}{\mu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu\right)\Bigg) \\ &- \left(\frac{h\mu}{\gamma} + a\mu\right)\mathbb{P}\Bigg(\left(\frac{\lambda}{\mu}\right)^\alpha Y \leq \left(N_1^\lambda - \frac{\lambda}{\mu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu\right), \\ &\left(\frac{\lambda}{\mu}\right)^\alpha Y + \left(\frac{\lambda}{\mu}\right)^\nu Z > N_1^\lambda - \frac{\lambda}{\mu}\Bigg).\end{aligned}$$

By observation, $\frac{\partial}{\partial N_1^\lambda} h(N_1^\lambda)$ is continuous in N_1^λ, and there exist $N_1^{\lambda,L}$ and $N_1^{\lambda,U}$ such that $\frac{\partial}{\partial N_1^\lambda} h(N_1^{\lambda,L}) < 0$ and $\frac{\partial}{\partial N_1^\lambda} h(N_1^{\lambda,H}) > 0$. Thus, the intermediate value theorem implies that there exists critical point $\bar{N}_1^\lambda$ such that $\frac{\partial}{\partial N_1^\lambda} h(\bar{N}_1^\lambda) = 0$. In addition, $h(N_1^\lambda)$ is convex in N_1^λ, because

$$\begin{aligned}&\frac{\partial^2}{\partial (N_1^\lambda)^2} h(N_1^\lambda) \\ &= \left(\frac{h\mu}{\gamma} + a\mu\right)\left(\frac{\lambda}{\mu}\right)^{-\nu} \int_{-\infty}^{\left(\frac{\lambda}{\mu}\right)^{-\alpha}\left(N_1^\lambda - \frac{\lambda}{\mu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu\right)} f_Y(y) f_Z\left(\left(\frac{\lambda}{\mu}\right)^{-\nu}\left(N_1^\lambda - \frac{\lambda}{\mu} - y\left(\frac{\lambda}{\mu}\right)^\alpha\right)\right) dy \geq 0.\end{aligned}$$

Hence, $\bar{N}_1^\lambda$ is a global minimum of $h(N_1^\lambda)$, and $(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y))$ is optimal to (18).

Proof of (I). We discuss the following two cases: $\nu < \alpha$ and $\nu = \alpha$.

Case 1: $\nu < \alpha$. When $\nu < \alpha$, similar lines of analysis as the proof of Theorem 1 for $\alpha < 1/2$ go through. Due to the similarity in the steps, we shall present the key structure of the proof and omit the details.

Consider the two-stage staffing rule denoted by u, where the staffing levels are given by

$$N_1^\lambda := \lambda/\mu + \bar{F}_Y^{-1}(c_1/c_2)(\lambda/\mu)^\alpha, \quad \text{and} \quad N_2^\lambda(N_1^\lambda, Y) := \left(Y - \bar{F}_Y^{-1}(c_1/c_2)\right)^+ (\lambda/\mu)^\alpha.$$

Following the definition of $\hat{\mathcal{C}}_u^\lambda$ in (27), we define

$$\hat{\mathcal{C}}_u^{e,\lambda} := \frac{\mathcal{C}_u^{e,\lambda} - c_1\lambda/\mu}{(\lambda/\mu)^{\max\{\alpha,1/2\}}}.$$

Similar lines of arguments as in the proof of Lemma 4 establish that

$$\hat{\mathcal{C}}_u^{e,\lambda} \to c_1\bar{F}_Y^{-1}(c_1/c_2) + c_2\mathbb{E}\left[(Y - \bar{F}_Y^{-1}(c_1/c_2))^+\right] \quad \text{as } \lambda \to \infty.$$

In comparison, consider the single-stage staffing rule denoted by $\tilde{u}$, where the base-stage staffing level is

$$N_1^\lambda := \frac{\lambda}{\mu} + \bar{F}_Y^{-1}\left(\frac{c_1}{h\mu/\gamma + a\mu}\right)(\lambda/\mu)^\alpha.$$

Similar lines of arguments as in the proof of Lemma 4 show that

$$\hat{\mathcal{C}}_{\tilde{u}}^{e,\lambda} \to c_1\bar{F}_Y^{-1}\left(\frac{c_1}{h\mu/\gamma + a\mu}\right) + \left(\frac{h\mu}{\gamma} + a\mu\right)\mathbb{E}\left[\left(Y - \bar{F}_Y^{-1}\left(\frac{c_1}{h\mu/\gamma + a\mu}\right)\right)^+\right] \quad \text{as } \lambda \to \infty,$$

where $\hat{\mathcal{C}}_{\tilde{u}}^{e,\lambda}$ is defined the same way as $\hat{\mathcal{C}}_u^{e,\lambda}$ but for policy $\tilde{u}$ instead.

By Assumption 1 and the continuity of Y, it can be verified that $\lim_{\lambda\to\infty}\hat{\mathcal{C}}_{\tilde{u}}^{e,\lambda} > \lim_{\lambda\to\infty}\hat{\mathcal{C}}_u^{e,\lambda}$. Thus,

$$\mathcal{C}_{\tilde{u}}^{e,\lambda} - \mathcal{C}_u^{e,\lambda} = \Theta(\lambda^\alpha).$$

Moreover, similar derivation as in the proof of Lemma 8 gives that

$$\mathcal{C}_{\tilde{u}}^{e,\lambda} - \mathcal{C}_{1,*}^{e,\lambda} = o(\lambda^\alpha) \quad \text{and} \quad \mathcal{C}_u^{e,\lambda} - \mathcal{C}_{2,*}^{e,\lambda} = o(\lambda^\alpha).$$

The statement follows.

Case 2: $\boldsymbol{\nu = \alpha}$. Consider the two-stage staffing rule denoted by u, where the staffing levels are given by

$$N_1^\lambda := \lambda/\mu + \beta_1^*(\lambda/\mu)^\alpha, \quad \text{and} \quad N_2^\lambda(N_1^\lambda, Y) := \beta_2^*(\beta_1^*, Y)(\lambda/\mu)^\alpha,$$

where β_1^* and $\beta_2^*(\beta_1^*, Y)$ jointly solve

$$\min_{\beta_1}\left\{c_1\beta_1 + \mathbb{E}\left[\min_{\beta_2(\beta_1,Y)\in\mathbb{R}_+}\left\{c_2\beta_2(\beta_1,Y) + (h\mu/\gamma + a\mu)\,\mathbb{E}\left[(Y + Z - \beta_1 - \beta_2(\beta_1,Y))^+\,|Y\right]\right\}\right]\right\}. \tag{89}$$

We first show that an optimal solution to (89) exists. Consider the inner-problem in (89):

$$\min_{\beta_2(\beta_1,Y)\in\mathbb{R}_+} c_2\beta_2(\beta_1,Y) + (h\mu/\gamma + a\mu)\,\mathbb{E}\left[(Y + Z - \beta_1 - \beta_2(\beta_1,Y))^+\,|Y\right]. \tag{90}$$

Note that (90) is a newsvendor problem with unit capacity cost c_2, unit sales price $h\mu/\gamma + a\mu$, random demand $Y + Z - \beta_1|Y$ (where the randomness lies in random variable Z), and capacity decision $\beta_2(\beta_1, Y)$. The optimal solution is given by

$$\beta_2^*(\beta_1, Y) = \left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right) + Y - \beta_1\right)^+. \tag{91}$$

Given $\beta_2^*(\beta_1, Y)$, the outer-problem is given by $\min_{\beta_1\in\mathbb{R}} h(\beta_1)$, where

$$h(\beta_1) := \left\{c_1\beta_1 + \mathbb{E}\left[c_2\beta_2^*(\beta_1,Y) + (h\mu/\gamma + a\mu)\,(Y + Z - \beta_1 - \beta_2^*(\beta_1,Y))^+\right]\right\}.$$

Differentiating $h(\beta_1)$ with respect to β_1 gives

$$\begin{aligned}\frac{\partial}{\partial\beta_1}h(\beta_1) = c_1 - c_2\mathbb{P}\left(Y > \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right) + \beta_1\right) \\ - \left(\frac{h\mu}{\gamma} + a\mu\right)\mathbb{P}\left(Y \le \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right) + \beta_1, Y + Z > \beta_1\right).\end{aligned} \tag{92}$$

By observation, $\frac{\partial}{\partial\beta_1}h(\beta_1)$ is continuous in β_1, and there exist β_1^L and β_1^U such that $\frac{\partial}{\partial\beta_1}h(\beta_1^L) < 0$ and $\frac{\partial}{\partial\beta_1}h(\beta_1^H) > 0$. Thus, the intermediate value theorem implies that there exists critical point β_1^* such that $\frac{\partial}{\partial\beta_1}h(\beta_1^*) = 0$. In addition, $h(\beta_1)$ is convex in β_1, because

$$\frac{\partial^2}{\partial\beta_1^2}h(\beta_1) = \left(\frac{h\mu}{\gamma} + a\mu\right)\int_{-\infty}^{\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma+a\mu}\right)+\beta_1} f_Y(y)f_Z(-y+\beta_1)dy \ge 0.$$

Hence, β_1^* is a global minimum of $h(\beta_1)$.

Following similar lines of arguments as in the proof of Lemma 4 and Lemma 8, we get that

$$\lim_{\lambda\to\infty}\hat{\mathcal{C}}_u^{e,\lambda} = c_1\beta_1^* + \mathbb{E}\left[c_2\beta_2^*(\beta_1^*, Y) + (h\mu/\gamma + a\mu)\left(Y + Z - \beta_1^* - \beta_2^*(\beta_1^*, Y)\right)^+\right],$$

and

$$\mathcal{C}_u^{e,\lambda} - \mathcal{C}_{2,*}^{e,\lambda} = o(\lambda^\alpha). \tag{93}$$

Next, consider the single-stage policy denoted by $\tilde{u}$, where the base-stage staffing level is given by $N_1^\lambda := \lambda/\mu + \tilde{\beta}(\lambda/\mu)^\alpha$, for

$$\tilde{\beta} := \arg\min_{\beta\in\mathbb{R}} c_1\beta + \left(\frac{h\mu}{\gamma} + a\mu\right)\mathbb{E}\left[(Y + Z - \beta)^+\right] = \bar{F}_{Y+Z}^{-1}\left(\frac{c_1}{h\mu/\gamma + a\mu}\right). \tag{94}$$

Similar derivation as in the proof of Lemma 4 gives that

$$\lim_{\lambda\to\infty}\hat{\mathcal{C}}_{\tilde{u}}^{e,\lambda} = c_1\tilde{\beta} + (h\mu/\gamma + a\mu)\,\mathbb{E}\left[\left(Y + Z - \tilde{\beta}\right)^+\right].$$

Theorem 1 in Bassamboo et al. (2010) establishes that

$$\mathcal{C}_{\tilde{u}}^{e,\lambda} - \mathcal{C}_{1,*}^{e,\lambda} = O(\lambda^{1-\alpha}). \tag{95}$$

If Assumption 2 holds, then

$$\beta_2^*(\beta_1^*, Y) = \left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right) + Y - \beta_1^*\right)^+ > 0 \quad \text{with probability } p > 0. \tag{96}$$

To see (96), suppose for the sake of contradiction that $\beta_2^*(\beta_1^*, Y) = 0$ with probability 1. It follows by solving $\frac{\partial}{\partial\beta_1}h(\beta_1^*) = 0$ in (92) that $\beta_1^* = \tilde{\beta}$, for $\tilde{\beta}$ defined in (94). However, plugging in the value of $\tilde{\beta}$ in (91) gives that

$$\beta_2^*(\beta_1^*, Y) = \beta_2^*(\tilde{\beta}_1, Y) = \left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right) + Y - \bar{F}_{Y+Z}^{-1}\left(\frac{c_1}{h\mu/\gamma + a\mu}\right)\right)^+.$$

This, together with Assumption 2, implies that $\beta_2^*(\beta_1^*, Y) > 0$ with probability $p > 0$, a contradiction. Thus, (96) holds. It follows from (96) that $\lim_{\lambda\to\infty}\hat{\mathcal{C}}_{\tilde{u}}^{e,\lambda} > \lim_{\lambda\to\infty}\hat{\mathcal{C}}_u^{e,\lambda}$, so that

$$\mathcal{C}_{\tilde{u}}^{e,\lambda} - \mathcal{C}_u^{e,\lambda} = \Theta(\lambda^\alpha). \tag{97}$$

In the other case where Assumption 2 does not hold, similar derivation shows that $\beta_1^* = \tilde{\beta}$ and $\beta_2^*(\beta_1^*, Y) = \beta_2^*(\tilde{\beta}, Y) = 0$ is optimal to (89), and

$$\mathcal{C}_{\tilde{u}}^{e,\lambda} - \mathcal{C}_u^{e,\lambda} = o(\lambda^\alpha). \tag{98}$$

The statement follows from (93), (95), (97), and (98).

Proof of (II). We discuss the following three cases: $\mu = \gamma$, $\mu > \gamma$, and $\mu < \gamma$.

Case 1: $\mu = \gamma$. It follows from Lemma 3 in Bassamboo et al. (2010) that for any staffing prescriptions N_1^λ and $N_2^\lambda(N_1^\lambda, Y)$, we have

$$\begin{aligned}
&\left(\frac{\Lambda^\lambda}{\mu} - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ \\
\leq &\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), \Lambda^\lambda)|Y, Z\right] \\
\leq &\left(\frac{\Lambda^\lambda}{\mu} - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ + \sqrt{\frac{4\pi}{\mu}}\sqrt{\Lambda^\lambda}\exp\left(-\frac{\mu}{4\Lambda^\lambda}\left(\frac{\Lambda^\lambda}{\mu} - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^2\right) + \frac{1}{\log 2}.
\end{aligned} \tag{99}$$

Taking expectation of (99) conditional on Y gives

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\Lambda^\lambda}{\mu} - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ \middle| Y\right] &\leq \mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), \Lambda^\lambda)|Y\right] \\
&\leq \mathbb{E}\left[\left(\frac{\Lambda^\lambda}{\mu} - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ \middle| Y\right] + \mathbb{E}\left[\sqrt{\frac{4\pi}{\mu}}\sqrt{\Lambda^\lambda}\middle| Y\right] + \frac{1}{\log 2}.
\end{aligned} \tag{100}$$

It follows from (100) that

$$\begin{aligned}
&c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h + a\gamma)\mathbb{E}\left[\left(\Lambda^\lambda/\mu - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ \middle| Y\right]\right] \\
\leq &c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h + a\gamma)\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), \Lambda^\lambda)|Y\right]\right] \\
\leq &c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h + a\gamma)\mathbb{E}\left[\left(\Lambda^\lambda/\mu - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ \middle| Y\right]\right] + \mathbb{E}\left[\sqrt{4\pi/\mu}\sqrt{\Lambda^\lambda}\right] + 1/\log 2 \\
\leq &c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h + a\gamma)\mathbb{E}\left[\left(\Lambda^\lambda/\mu - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ \middle| Y\right]\right] + \sqrt{4\pi/\mu}\sqrt{\lambda} \\
&+ \sqrt{4\pi/\mu}\sqrt{\lambda^\alpha \mu^{1-\alpha}}\mathbb{E}\left[|Y|\right] + \sqrt{4\pi/\mu}\sqrt{\lambda^\nu \mu^{1-\nu}}\mathbb{E}\left[|Z|\right] + 1/\log 2,
\end{aligned} \tag{101}$$

where the last inequality follows from the reverse Jensen's inequality, and the fact that Y and Z are independent.

Let $(N_1^{\lambda,*}, N_2^{\lambda,*}(N_1^{\lambda,*}, Y))$ denotes the optimal solution to problem (16). We have

$$\begin{aligned}
\mathcal{C}_{2,Err}^{e,\lambda} &= c_1\bar{N}_1^\lambda + \mathbb{E}\left[c_2\bar{N}_2^\lambda(\bar{N}_1^\lambda, Y) + (h + a\gamma)\mathbb{E}\left[Q(\bar{N}_1^\lambda + \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y), \Lambda^\lambda)|Y\right]\right] \\
&\overset{(a)}{\leq} c_1\bar{N}_1^\lambda + \mathbb{E}\left[c_2\bar{N}_2^\lambda(\bar{N}_1^\lambda, Y) + (h + a\gamma)\mathbb{E}\left[\left(\Lambda^\lambda - \mu\left(\bar{N}_1^\lambda + \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)\right)\right)^+|Y\right]/\gamma\right] + O(\sqrt{\lambda}) \\
&\overset{(b)}{\leq} c_1 N_1^{\lambda,*} + \mathbb{E}\left[c_2 N_2^{\lambda,*}(N_1^{\lambda,*}, Y) + (h + a\gamma)\mathbb{E}\left[\left(\Lambda^\lambda - \mu\left(N_1^{\lambda,*} + N_2^{\lambda,*}(N_1^{\lambda,*}, Y)\right)\right)^+|Y\right]/\gamma\right] + O(\sqrt{\lambda}) \\
&\overset{(c)}{\leq} c_1 N_1^{\lambda,*} + \mathbb{E}\left[c_2 N_2^{\lambda,*}(N_1^{\lambda,*}, Y) + (h + a\gamma)\mathbb{E}\left[Q(N_1^{\lambda,*} + N_2^{\lambda,*}(N_1^{\lambda,*}, Y), \Lambda^\lambda)|Y\right]\right] + O(\sqrt{\lambda}) \\
&= \mathcal{C}_{2,*}^{e,\lambda} + O(\sqrt{\lambda}),
\end{aligned}$$

where (a) follows from (101), (b) follows from the optimality of $(\bar{N}_1, \bar{N}_2(\bar{N}_1, Y))$ to problem (18), and (c) follows from (101) again.

Case 2: $\mu > \gamma$. To simply notation, define

$$\begin{aligned}\mathcal{C}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) &:= c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h + a\gamma)\,\mathbb{E}\left[Q(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), \Lambda^\lambda)|Y\right]\right] \\ &= c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h/\gamma + a)\,\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), Y\right)\right],\end{aligned} \tag{102}$$

where $\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), Y\right)$ denotes the steady-state abandonment probability conditional on Y, i.e., $\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), Y\right) := \mathbb{E}\left[\mathbb{1}_{(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), \Lambda^\lambda)}|Y\right]$. In addition, define

$$\bar{\mathcal{C}}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) := c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h + a\gamma)\mathbb{E}\left[\left(\Lambda^\lambda - \mu\left(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y)\right)\right)^+|Y\right]/\gamma\right]. \tag{103}$$

Note that $(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) = \arg\min_{N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)} \bar{\mathcal{C}}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y))$.

Consider an auxiliary sequence of systems with the same parameters as the original sequence of systems except that its abandonment rate is equal to μ; that is, systems in this sequence have a higher abandonment rate compared to the original sequence. We refer to this sequence as Sequence II and add the superscript II to all quantities associated with it, e.g., $\mu^{II} = \mu, \gamma^{II} = \mu$. Quantities associated with the original sequence of system are denoted without superscripts. For systems in Sequence II, we choose the cost parameters to be the following: $c_1^{II} = c_1, c_2^{II} = c_2, a^{II} = a$, and $h^{II} = h\mu/\gamma$. The analogues of (102) and (103) for Sequence II are

$$\begin{aligned}\mathcal{C}^{e,\lambda,II}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) &:= c_1^{II} N_1^\lambda + \mathbb{E}\left[c_2^{II} N_2^\lambda(N_1^\lambda, Y) + \left(h^{II}/\gamma^{II} + a^{II}\right)\mathbb{P}\left(AB^{II}, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), Y\right)\right] \\ &= c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h/\gamma + a)\,\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), Y\right)\right] \\ &= \mathcal{C}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)),\end{aligned}$$

and

$$\begin{aligned}\bar{\mathcal{C}}^{e,\lambda,II}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) &:= c_1^{II} N_1^\lambda + \mathbb{E}\left[c_2^{II} N_2^\lambda(N_1^\lambda, Y) + (h^{II} + a^{II}\gamma^{II})\mathbb{E}\left[\left(\Lambda^\lambda - \mu^{II}\left(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y)\right)\right)^+|Y\right]/\gamma^{II}\right] \\ &= c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h + a\gamma)\mathbb{E}\left[\left(\Lambda^\lambda - \mu\left(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y)\right)\right)^+|Y\right]/\gamma\right] \\ &= \bar{\mathcal{C}}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)).\end{aligned} \tag{104}$$

From the proof of Theorem 3 in Bassamboo et al. (2010), we have

$$\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), Y\right) \le \mathbb{P}\left(AB^{II}, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), Y\right),$$

which implies that

$$\mathcal{C}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) \le \mathcal{C}^{e,\lambda,II}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)). \tag{105}$$

Applying (101) to Sequence II, we get that

$$\begin{aligned}&\mathcal{C}^{e,\lambda,II}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) \\ =&c_1^{II} N_1^\lambda + \mathbb{E}\left[c_2^{II} N_2^\lambda(N_1^\lambda, Y) + (h^{II} + a^{II}\gamma^{II})\mathbb{E}\left[Q^{II}(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), \Lambda^\lambda)|Y\right]\right] \\ \le&c_1^{II} N_1^\lambda + \mathbb{E}\left[c_2^{II} N_2^\lambda(N_1^\lambda, Y) + (h^{II} + a^{II}\gamma^{II})\mathbb{E}\left[\left(\Lambda^\lambda/\mu^{II} - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ \middle| Y\right]\right] + O(\sqrt{\lambda}) \\ =&\bar{\mathcal{C}}^{e,\lambda,II}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) + O(\sqrt{\lambda})\end{aligned} \tag{106}$$

Next, consider another auxiliary sequence of systems with the same parameters as the original sequence of systems except that its service rate is equal to γ; that is, systems in this sequence have a lower service rate compared to the original sequence. We refer to this sequence as Sequence III and add the superscript III to

all quantities associated with Sequence III, e.g., $\mu^{III}=\gamma, \gamma^{III}=\gamma$. For systems in Sequence III, we choose the cost parameters to be the following: $c_1^{III}=c_1\gamma/\mu, c_2^{III}=c_2\gamma/\mu, a^{III}=a$, and $h^{III}=h$. The analogues of (102) and (103) for Sequence III are

$$\begin{aligned}\mathcal{C}^{e,\lambda,III}(N_1^\lambda, N_2^\lambda(N_1^\lambda,Y)) :=& c_1^{III}N_1^\lambda + \mathbb{E}\left[c_2^{III}N_2^\lambda(N_1^\lambda,Y) + \left(h^{III}/\gamma^{III}+a^{III}\right)\mathbb{P}\left(AB^{III}, N_1^\lambda+N_2^\lambda(N_1^\lambda,Y),Y\right)\right]\\ =& c_1\gamma/\mu N_1^\lambda + \mathbb{E}\left[c_2\gamma/\mu N_2^\lambda(N_1^\lambda,Y) + (h/\gamma+a)\,\mathbb{P}\left(AB^{III}, N_1^\lambda+N_2^\lambda(N_1^\lambda,Y),Y\right)\right],\end{aligned}$$

and

$$\begin{aligned}&\bar{\mathcal{C}}^{e,\lambda,III}(N_1^\lambda, N_2^\lambda(N_1^\lambda,Y))\\ :=& c_1^{III}N_1^\lambda + \mathbb{E}\left[c_2^{III}N_2^\lambda(N_1^\lambda,Y) + (h^{III}+a^{III}\gamma^{III})\mathbb{E}\left[\left(\Lambda^\lambda - \mu^{III}\left(N_1^\lambda+N_2^\lambda(N_1^\lambda,Y)\right)\right)^+|Y\right]/\gamma^{III}\right]\\ =& c_1\gamma/\mu N_1^\lambda + \mathbb{E}\left[c_2\gamma/\mu N_2^\lambda(N_1^\lambda,Y) + (h+a\gamma)\mathbb{E}\left[\left(\Lambda^\lambda - \gamma\left(N_1^\lambda+N_2^\lambda(N_1^\lambda,Y)\right)\right)^+|Y\right]/\gamma\right]\\ =& \bar{\mathcal{C}}^{e,\lambda}(\gamma/\mu N_1^\lambda, \gamma/\mu N_2^\lambda(N_1^\lambda,Y)).\end{aligned} \tag{107}$$

From the proof of Theorem 3 in Bassamboo et al. (2010), we have

$$\mathbb{P}\left(AB, N_1^\lambda+N_2^\lambda(N_1^\lambda,Y),Y\right) \geq \mathbb{P}\left(AB^{III}, \mu/\gamma\left(N_1^\lambda+N_2^\lambda(N_1^\lambda,Y)\right),Y\right),$$

which implies that

$$\mathcal{C}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda,Y)) \geq \mathcal{C}^{e,\lambda,III}(\mu/\gamma N_1^\lambda, \mu/\gamma N_2^\lambda(N_1^\lambda,Y)). \tag{108}$$

Applying (101) to Sequence III, we get that

$$\begin{aligned}&\mathcal{C}^{e,\lambda,III}(N_1^\lambda, N_2^\lambda(N_1^\lambda,Y))\\ =& c_1^{III}N_1^\lambda + \mathbb{E}\left[c_2^{III}N_2^\lambda(N_1^\lambda,Y) + (h^{III}+a^{III}\gamma^{III})\mathbb{E}\left[Q^{III}(N_1^\lambda+N_2^\lambda(N_1^\lambda,Y),\Lambda^\lambda)|Y\right]\right]\\ \geq& c_1^{III}N_1^\lambda + \mathbb{E}\left[c_2^{III}N_2^\lambda(N_1^\lambda,Y) + (h^{III}+a^{III}\gamma^{III})\mathbb{E}\left[\left(\Lambda^\lambda/\mu^{III} - N_1^\lambda - N_2^\lambda(N_1^\lambda,Y)\right)^+\middle|Y\right]\right]\\ =& \bar{\mathcal{C}}^{e,\lambda,III}(N_1^\lambda, N_2^\lambda(N_1^\lambda,Y)),\end{aligned} \tag{109}$$

which implies that

$$\begin{aligned}\mathcal{C}^{e,\lambda}(N_1^{\lambda,*}, N_2^{\lambda,*}(N_1^{\lambda,*},Y)) =& \min_{N_1^\lambda, N_2^\lambda(N_1^\lambda,Y)} \mathcal{C}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda,Y))\\ \overset{(d)}{\geq}& \min_{N_1^\lambda, N_2^\lambda(N_1^\lambda,Y)} \mathcal{C}^{e,\lambda,III}(\mu/\gamma N_1^\lambda, \mu/\gamma N_2^\lambda(N_1^\lambda,Y))\\ \overset{(e)}{\geq}& \min_{N_1^\lambda, N_2^\lambda(N_1^\lambda,Y)} \bar{\mathcal{C}}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda,Y))\\ =& \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)),\end{aligned} \tag{110}$$

where (d) follows from (108), and (e) follows from (107) and (109).

Lastly, we can write

$$\begin{aligned}&\mathcal{C}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) - \mathcal{C}^{e,\lambda}(N_1^{\lambda,*}, N_2^{\lambda,*}(N_1^{\lambda,*},Y))\\ =& \mathcal{C}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) - \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) + \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) - \mathcal{C}^{e,\lambda}(N_1^{\lambda,*}, N_2^{\lambda,*}(N_1^{\lambda,*},Y))\\ \overset{(f)}{\leq}& \mathcal{C}^{e,\lambda,II}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) - \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) + \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) - \mathcal{C}^{e,\lambda}(N_1^{\lambda,*}, N_2^{\lambda,*}(N_1^{\lambda,*},Y))\\ \overset{(g)}{=}& \mathcal{C}^{e,\lambda,II}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) - \bar{\mathcal{C}}^{e,\lambda,II}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) + \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda,Y)) - \mathcal{C}^{e,\lambda}(N_1^{\lambda,*}, N_2^{\lambda,*}(N_1^{\lambda,*},Y))\\ \overset{(h)}{=}& O(\sqrt{\lambda}),\end{aligned}$$

where (f) follows from (105), (g) follows from (104), and (h) follows from (106) and (110).

Case 3: $\mu < \gamma$. The analysis for Case 3 is similar to that for Case 2. In particular, we again consider Sequence II and Sequence III as constructed in Case 2.

For Sequence II, it follows by construction that

$$\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), Y\right) \geq \mathbb{P}\left(AB^{II}, \left(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y)\right), Y\right),$$

which implies that

$$\mathcal{C}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) \geq \mathcal{C}^{e,\lambda,II}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)).$$

Applying (101) to Sequence II, we get that

$$\begin{aligned}
&\mathcal{C}^{e,\lambda,II}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) \\
=&c_1^{II} N_1^\lambda + \mathbb{E}\left[c_2^{II} N_2^\lambda(N_1^\lambda, Y) + (h^{II} + a^{II}\gamma^{II})\mathbb{E}\left[Q^{II}(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), \Lambda^\lambda)|Y\right]\right] \\
\geq&c_1^{II} N_1^\lambda + \mathbb{E}\left[c_2^{II} N_2^\lambda(N_1^\lambda, Y) + (h^{II} + a^{II}\gamma^{II})\mathbb{E}\left[\left(\Lambda/\mu^{II} - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ \middle| Y\right]\right] \\
=&c_1 N_1^\lambda + \mathbb{E}\left[c_2 N_2^\lambda(N_1^\lambda, Y) + (h/\gamma + a)\mathbb{E}\left[\left(\Lambda^\lambda - \mu\left(N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)\right)^+ |Y\right]\right] \\
=&\bar{\mathcal{C}}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)),
\end{aligned}$$

which implies that

$$\begin{aligned}
\mathcal{C}^{e,\lambda}(N_1^{\lambda,*}, N_2^{\lambda,*}(N_1^{\lambda,*}, Y)) &= \min_{N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)} \mathcal{C}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) \\
&\geq \min_{N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)} \mathcal{C}^{e,\lambda,II}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) \\
&\geq \min_{N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)} \bar{\mathcal{C}}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) \\
&= \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)).
\end{aligned} \tag{111}$$

For Sequence III, it follows by construction that

$$\mathbb{P}\left(AB, N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), Y\right) \leq \mathbb{P}\left(AB^{III}, \mu/\gamma\left(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y)\right), Y\right),$$

which implies that

$$\mathcal{C}^{e,\lambda}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) \leq \mathcal{C}^{e,\lambda,III}(\mu/\gamma N_1^\lambda, \mu/\gamma N_2^\lambda(N_1^\lambda, Y)). \tag{112}$$

Applying (101) to Sequence III, we get that

$$\begin{aligned}
&\mathcal{C}^{e,\lambda,III}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) \\
=&c_1^{III} N_1^\lambda + \mathbb{E}\left[c_2^{III} N_2^\lambda(N_1^\lambda, Y) + (h^{III} + a^{III}\gamma^{III})\mathbb{E}\left[Q^{III}(N_1^\lambda + N_2^\lambda(N_1^\lambda, Y), \Lambda^\lambda)|Y\right]\right] \\
\leq&c_1^{III} N_1^\lambda + \mathbb{E}\left[c_2^{III} N_2^\lambda(N_1^\lambda, Y) + (h^{III} + a^{III}\gamma^{III})\mathbb{E}\left[\left(\Lambda^\lambda/\mu^{III} - N_1^\lambda - N_2^\lambda(N_1^\lambda, Y)\right)^+ \middle| Y\right]\right] + O(\sqrt{\lambda}) \\
=&\bar{\mathcal{C}}^{e,\lambda,III}(N_1^\lambda, N_2^\lambda(N_1^\lambda, Y)) + O(\sqrt{\lambda})
\end{aligned} \tag{113}$$

Lastly, we can write

$$\begin{aligned}
&\mathcal{C}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) - \mathcal{C}^{e,\lambda}(N_1^{\lambda,*}, N_2^{\lambda,*}(N_1^{\lambda,*}, Y)) \\
=&\mathcal{C}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) - \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) + \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) - \mathcal{C}^{e,\lambda}(N_1^*, N_2^*(N_1^*, Y)) \\
\overset{(i)}{\leq}&\mathcal{C}^{e,\lambda,III}(\mu/\gamma\bar{N}_1^\lambda, \mu/\gamma\bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) - \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) + \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) - \mathcal{C}^{e,\lambda}(N_1^*, N_2^*(N_1^*, Y)) \\
\overset{(j)}{=}&\mathcal{C}^{e,\lambda,III}(\mu/\gamma\bar{N}_1^\lambda, \mu/\gamma\bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) - \bar{\mathcal{C}}^{e,\lambda,III}(\mu/\gamma\bar{N}_1^\lambda, \mu/\gamma\bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) + \bar{\mathcal{C}}^{e,\lambda}(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y)) \\
&- \mathcal{C}^{e,\lambda}(N_1^{\lambda,*}, N_2^{\lambda,*}(N_1^{\lambda,*}, Y)) \\
\overset{(k)}{=}&O(\sqrt{\lambda}),
\end{aligned}$$

where (i) follows from (112), (j) follows from (107), and (k) follows from (111) and (113).

Proof of (III). For the oracle problem, we consider the following stochastic-fluid optimization problem

$$\min_{N_1^\lambda}\left\{c_1 N_1^\lambda + \mathbb{E}\left[\min_{N_2^\lambda(N_1^\lambda,\Lambda^\lambda)}\left\{c_2 N_2^\lambda(N_1^\lambda,\Lambda^\lambda) + (h\mu/\gamma + a\mu)\mathbb{E}\left[\left(\Lambda^\lambda/\mu - N_1^\lambda - N_2^\lambda(N_1^\lambda,\Lambda^\lambda)\right)^+ |\Lambda^\lambda\right]\right\}\right]\right\}. \quad (114)$$

whose optimal solution is given by

$$\hat{N}_1^\lambda = \bar{F}^{-1}_{\Lambda^\lambda/\mu}(c_1/c_2)\,(\lambda/\mu)^\alpha, \quad \text{and} \quad \hat{N}_2^\lambda(\hat{N}_1^\lambda,\Lambda^\lambda) = (\Lambda^\lambda/\mu - \hat{N}_1^\lambda)^+.$$

We denote the staffing rule that prescribes $(\hat{N}_1^\lambda, \hat{N}_2^\lambda(\hat{N}_1^\lambda,\Lambda^\lambda))$ as $\hat{u}$. The same lines of analysis used to show statement (II) can be applied to establish that

$$\mathcal{C}^{o,\lambda}_{\hat{u}} - \mathcal{C}^{o,\lambda}_{2,*} = O(\sqrt{\lambda}). \quad (115)$$

Recall from the proof of Proposition 3 that $u_{2,ERR}$ prescribes staffing levels $(\bar{N}_1^\lambda, \bar{N}_2^\lambda(\bar{N}_1^\lambda, Y))$ where

$$\bar{N}_2^\lambda(N_1^\lambda, Y) = \left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu + \frac{\lambda}{\mu} + Y\left(\frac{\lambda}{\mu}\right)^\alpha - N_1^\lambda\right)^+,$$

and $\bar{N}_1^\lambda$ solves

$$\begin{aligned} 0 =& c_1 - c_2\mathbb{P}\left(\left(\frac{\lambda}{\mu}\right)^\alpha Y > \left(\bar{N}_1^\lambda - \frac{\lambda}{\mu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu\right)\right) \\ & - \left(\frac{h\mu}{\gamma} + a\mu\right)\mathbb{P}\left(\left(\frac{\lambda}{\mu}\right)^\alpha Y \le \left(\bar{N}_1^\lambda - \frac{\lambda}{\mu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu\right), \right. \\ & \left. \left(\frac{\lambda}{\mu}\right)^\alpha Y + \left(\frac{\lambda}{\mu}\right)^\nu Z > \bar{N}_1^\lambda - \frac{\lambda}{\mu}\right). \end{aligned} \quad (116)$$

Next, we compare the two inner-optimization problems in (18) and (114). It holds that

$$\begin{aligned} & \mathbb{E}\left[\min_{N_2^\lambda(\bar{N}_1^\lambda,Y)}\left\{c_2 N_2^\lambda(\bar{N}_1^\lambda,Y) + \left(\frac{h\mu}{\gamma} + a\mu\right)\mathbb{E}\left[\left(\frac{\Lambda^\lambda}{\mu} - \bar{N}_1^\lambda - N_2^\lambda(\bar{N}_1^\lambda,Y)\right)^+ \middle| Y\right]\right\}\right] \\ & \quad - \mathbb{E}\left[\min_{N_2^\lambda(\bar{N}_1^\lambda,\Lambda^\lambda)}\left\{c_2 N_2^\lambda(\bar{N}_1^\lambda,\Lambda^\lambda) + \left(\frac{h\mu}{\gamma} + a\mu\right)\left(\frac{\Lambda^\lambda}{\mu} - \bar{N}_1^\lambda - N_2^\lambda(\bar{N}_1^\lambda,\Lambda^\lambda)\right)^+\right\}\right] \\ & = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[c_2\left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu + \frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^\alpha - \bar{N}_1^\lambda\right)^+\right. \\ & \quad + \left(\frac{h\mu}{\gamma} + a\mu\right)\left(\frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^\alpha + z\left(\frac{\lambda}{\mu}\right)^\nu - \bar{N}_1^\lambda - \left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu + \frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^\alpha - \bar{N}_1^\lambda\right)^+\right)^+ \\ & \quad \left. - c_2\left(\frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^\alpha + z\left(\frac{\lambda}{\mu}\right)^\nu - \bar{N}_1^\lambda\right)\right] f_Y(y) f_Z(z)\,dy\,dz. \end{aligned} \quad (117)$$

Denote part of the integrand in (117) as

$$\begin{aligned} g^\lambda(y,z) :=& c_2\left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu + \frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^\alpha - \bar{N}_1^\lambda\right)^+ \\ & + \left(\frac{h\mu}{\gamma} + a\mu\right)\left(\frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^\alpha + z\left(\frac{\lambda}{\mu}\right)^\nu - \bar{N}_1^\lambda - \left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^\nu + \frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^\alpha - \bar{N}_1^\lambda\right)^+\right)^+ \\ & - c_2\left(\frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^\alpha + z\left(\frac{\lambda}{\mu}\right)^\nu - \bar{N}_1^\lambda\right)^+. \end{aligned}$$

By construction of the two optimization problems, it holds that $g^\lambda(y,z) \ge 0$ for all $y, z \in \mathbb{R}$. Moreover, it follows from (116) that at least one of the following two cases holds:

(i) $\mathbb{P}\left(\left(\frac{\lambda}{\mu}\right)^{\alpha} Y > \left(\bar{N}_1^{\lambda} - \frac{\lambda}{\mu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^{\nu}\right)\right) > 0$;

(ii) $\mathbb{P}\left(\left(\frac{\lambda}{\mu}\right)^{\alpha} Y \leq \left(\bar{N}_1^{\lambda} - \frac{\lambda}{\mu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^{\nu}\right), \left(\frac{\lambda}{\mu}\right)^{\alpha} Y + \left(\frac{\lambda}{\mu}\right)^{\nu} Z > \bar{N}_1^{\lambda} - \frac{\lambda}{\mu}\right) > 0.$

Note that if $\left(\frac{\lambda}{\mu}\right)^{\alpha} y > \left(\bar{N}_1^{\lambda} - \frac{\lambda}{\mu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^{\nu}\right)$ and $z \neq \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)$, then

$$\begin{aligned} g^{\lambda}(y,z) = {} & c_2\left(\bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^{\nu} + \frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^{\alpha} - \bar{N}_1^{\lambda}\right) \\ & + \left(\frac{h\mu}{\gamma} + a\mu\right)\left(z\left(\frac{\lambda}{\mu}\right)^{\nu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^{\nu}\right)^{+} \\ & - c_2\left(\frac{\lambda}{\mu} + y\left(\frac{\lambda}{\mu}\right)^{\alpha} + z\left(\frac{\lambda}{\mu}\right)^{\nu} - \bar{N}_1^{\lambda}\right)^{+} \\ = {} & \Theta(\lambda^{\nu}). \end{aligned}$$

In addition, if $\left(\frac{\lambda}{\mu}\right)^{\alpha} y \leq \left(\bar{N}_1^{\lambda} - \frac{\lambda}{\mu} - \bar{F}_Z^{-1}\left(\frac{c_2}{h\mu/\gamma + a\mu}\right)\left(\frac{\lambda}{\mu}\right)^{\nu}\right)$ and $\left(\frac{\lambda}{\mu}\right)^{\alpha} y + \left(\frac{\lambda}{\mu}\right)^{\nu} z > \bar{N}_1^{\lambda} - \frac{\lambda}{\mu}$, then

$$g^{\lambda}(y,z) = (h\mu/\gamma + a\mu - c_2)\left(\lambda/\mu + y\,(\lambda/\mu)^{\alpha} + z\,(\lambda/\mu)^{\nu} - \bar{N}_1^{\lambda}\right)^{+} = \Theta(\lambda^{\nu}).$$

Therefore, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^{\lambda}(y,z) f_Y(y) f_Z(z) dy dz = \Theta(\lambda^{\nu}). \tag{118}$$

It follows from (117), (118), and the construction of stochastic-fluid problems (18) and (114) that

$$\mathcal{C}_{2,ERR}^{e,\lambda} - \mathcal{C}_{\hat{u}}^{o,\lambda} = \Theta(\lambda^{\nu}). \tag{119}$$

The statement follows from (115), (119), and statement (II). Q.E.D.

Appendix G: Details on Model Calibration for the ED

In this section, we discuss several model calibration details for the ED application. Section G.1 provides detailed linear regression results for estimating α and σ. Section G.2 investigates an alternative non-parametric estimation method for α and σ, which leads to the same estimation results. Section G.2 elaborates on the statistical procedures to estimate ν and Z, which is an analog to those used to estimate α and X. In Section G.4 we provide normal probability plots for X and Z. Lastly, in Section G.5, we elaborate on how to estimate patients' mean patience time.

G.1. Linear Regression Results for Estimating α and σ

Table 8 provides the detailed estimation results for α and σ using linear regression. The R^2 is 0.821 for the model using 14 observations (obtained by dividing the shifts based on the day of the week and day vs. night), and 0.541 for the model using 56 observations (obtained by dividing the shifts based on the day of the week, day vs. night, and quarter of the year).

G.2. Non-Parametric Estimation of α and σ

In this section, we provide more details of the non-parametric estimation proposed in Maman (2009) to approximate the relationship between α and σ in the random arrival rate (3). In particular, this method does not impose any distributional assumption on X. However, it requires that $\alpha > 1/2$.

Table 8 Linear regression results for estimating α and σ

	Dependent variable:	
	Regression 1: $\lvert\mathcal{I}\rvert = 14$	Regression 2: $\lvert\mathcal{I}\rvert = 56$
$\log(\bar{L}_i)$	0.768^{***} (0.565, 0.971)	0.746^{***} (0.563, 0.929)
Constant	-1.067^{*} (-2.056, -0.077)	-1.017^{**} (-1.909, -0.124)
Observations	14	56
R^2	0.821	0.541
Adjusted R^2	0.806	0.533
Residual Std. Error	0.126 (df = 12)	0.230 (df = 54)
F Statistic	55.051^{**} (df = 1; 12)	63.680^{**} (df = 1; 54)
Note:		.p<0.1; *p<0.05; **p<0.01

Let L_i be a generic random variable denoting the arrival count during a type-i shift, $i \in \mathcal{I}$. Since $L_i|\Lambda_i \sim$Poisson(Λ_i), we have

$$\mathbb{E}[L_i] = \mathbb{E}[\mathbb{E}[L_i|\Lambda_i]] = \lambda_i$$

$$\mathrm{Var}(L_i) = \mathrm{Var}(\mathbb{E}[L_i|\Lambda_i]) + \mathbb{E}[\mathrm{Var}(L_i|\Lambda_i)] = \lambda_i^{2\alpha}\sigma^2 + \lambda_i, \quad i \in \mathcal{I}.$$

Thus,

$$\frac{\mathrm{Std}(L_i)}{\lambda_i^{\alpha}} = \left(\sigma^2 + \lambda_i^{1-2\alpha}\right)^{1/2}, \quad i \in \mathcal{I}.$$

In addition, since $\alpha > 1/2$,

$$\lim_{\lambda\to\infty} (\log \mathrm{Std}(L_i) - \alpha \log \lambda_i) = \log \sigma, \quad i \in \mathcal{I}.$$

Hence, it holds for large λ_i that

$$\log \mathrm{Std}(L_i) \approx \alpha \log \lambda_i + \log \sigma, \quad i \in \mathcal{I}.$$

Using sample mean $\bar{L}_i$ to approximate λ_i and sample standard deviation Σ_i to approximate $\mathrm{Std}(L_i)$, we get that

$$\log \Sigma_i \approx \hat{\alpha} \log \bar{L}_i + \log \hat{\sigma}, \quad i \in \mathcal{I},$$

which is equivalent to (20) in our parametric estimation setting.

G.3. Estimation of ν and Z

We assume that Z follows a normal distribution with a mean equal to 0 and a standard deviation equal to σ_Z. Let $L_i^{(k)}$ and $R_i^{(k)}$ denote the observed arrival count and residual for the kth shift of type i, $1 \le k \le n_i$. Recall from the random arrival-rate model that the residuals for type-i shifts in the surge-stage prediction model are distributed according to $\lambda_i^{\nu}\mu^{1-\nu}Z$. For shifts of type i, $i \in \mathcal{I}$, we define

$$\bar{L}_i := \frac{1}{n_i}\sum_{k=1}^{n_i} L_i^{(k)}, \quad \bar{R}_i := \frac{1}{n_i}\sum_{k=1}^{n_i} R_i^{(k)}, \quad \chi_i^2 := \frac{1}{n_i}\sum_{k=1}^{n_i} (R_i^{(k)} - \bar{R}_i)^2,$$

where $\bar{L}_i$ is the mean of the observed arrival counts, $\bar{R}_i$ is the mean of the residuals, and χ_i^2 is the variance of the residuals. Based on the method of moments, we have the following system of equations for the estimators

$$\bar{L}_i = \hat{\lambda}_i, \quad \chi_i^2 = \hat{\lambda}_i^{2\hat{\nu}}\mu^{2(1-\hat{\nu})}\hat{\sigma}_Z^2, \quad i \in \mathcal{I}. \tag{120}$$

It follows from (120) that

$$\log \chi_i = \hat{\nu} \log \bar{L}_i + \log(\mu^{1-\hat{\nu}} \hat{\sigma}_Z), \quad i \in \mathcal{I}.$$

Then, we can fit $\hat{\nu}$ and $\hat{\sigma}_Z$ by solving the following least squares problem

$$\min_{\nu \in (0,1),\, \gamma \in \mathbb{R}} \sum_{i=1}^{14} \left(\log \chi_i - \gamma - \nu \log \bar{L}_i \right)^2. \tag{121}$$

In particular, let γ^* and ν^* denote the optimal solution to the least squares problem (121). Then, $\hat{\nu} = \nu^*$ and $\mu^{1-\hat{\nu}} \hat{\sigma}_Z = \exp(\gamma^*)$. Following this method, we get that $\hat{v} = 0.508$ and $Z \sim N(0, 1.067)$. Table 9 below provides the results for estimating ν and σ_Z using linear regression.

Table 9 Linear regression results for estimating ν and σ_Z

	Dependent variable:
$\log(\bar{L}_i)$	0.508^{***} (0.326, 0.691)
Constant	0.142 (-0.747, 1.031)
Observations	14
R^2	0.713
Adjusted R^2	0.689
Residual Std. Error	0.111 (df = 12)
F Statistic	29.758^{**} (df = 1; 12)
Note:	.p<0.1; *p<0.05; **p<0.01

G.4. Assumption of Normal Distributions for X and Z

To validate the assumption that X and Z follow normal distributions, we examine their normal probability plots in Figure 11 below. We see that in each plot, the points fall reasonably close to a line, suggesting that our assumption on the normal distribution is reasonable.

G.5. Estimation of Mean Patience Time

We use maximum likelihood estimation to derive the mean patience time. In particular, let $\mathcal{M}_1$ and $\mathcal{M}_2$ denote the set of patients who left without being seen and the set of patients who received treatment in the data, respectively. For patient $m \in \mathcal{M}_1$, let w^m be the time between arrival and departure for patient m. For patient $m \in \mathcal{M}_2$, let w^m be the time between arrival and evaluation for patient m. Recall that patients' patience time is assumed to follow an exponential distribution with rate γ. Then the likelihood of observing patient $m \in \mathcal{M}_1$ is $1 - e^{-\gamma w^m}$, and the likelihood of observing patient $m \in \mathcal{M}_2$ is $e^{-\gamma w^m}$. The overall likelihood $L(\gamma)$ is given by

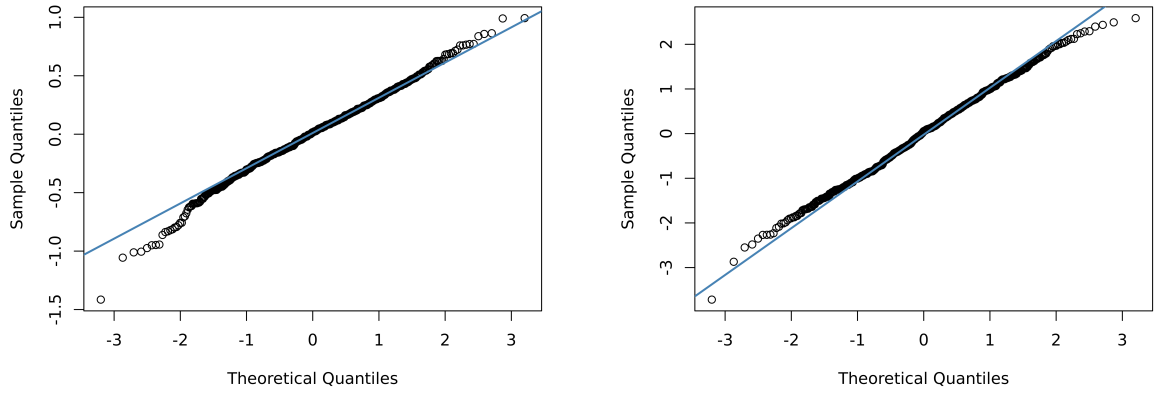
$$L(\gamma) = \prod_{m_1 \in \mathcal{M}_1} \left(1 - e^{-\gamma w^{m_1}}\right) \prod_{m_2 \in \mathcal{M}_2} \left(e^{-\gamma w^{m_2}}\right).$$

Taking the log of the overall likelihood gives

$$\ln(L(\gamma)) = \sum_{m_1 \in \mathcal{M}_1} \ln\left(1 - e^{-\gamma w^{m_1}}\right) - \sum_{m_2 \in \mathcal{M}_2} \gamma w^{m_2}.$$

We let $\hat{\gamma} := \arg\max_{\gamma > 0} \ln(L(\gamma))$, and get $\hat{\gamma} = 27.5$ hours from the data. Hence, in the more complex simulation experiments for the ED, we assume that the mean patience time is 27.5 hours.

Figure 11 Normal probability plots for X and Z



(a) X (b) Z

Appendix H: Model Generalization: Capacity Cap, On-Call Pool, and Nurse No-Show Behavior

In this section, we discuss several generalizations of modeling assumptions. Since it is relatively easy to incorporate different service rates for base and surge nurses (e.g., having μ_1 and μ_2 instead of a single μ), we shall omit its discussion to simplify the exposition. We next elaborate on how to incorporate a capacity cap for surge nurses and random nurse show-up behavior to work.

In particular, we can consider the following generalized model formulation: **At the base stage**, the ED manager has information on the distribution of the random arrival rate Λ, and determines 1) N_1, the number of base nurses with cost rate c_1, and 2) N_2^o, the number of on-call nurses with cost rate c_2^o. These on-call nurses are staffed in advance (i.e., at the base stage) with a small monetary incentive (i.e., c_2^o) and are committed to work as surge staff if they turn out to be needed. **At the surge stage**, the ED manager has information on the realization of the random arrival rate Λ, and determines N_2, the number of surge nurses to call in from the on-call pool with cost rate c_2, subject to $N_2 \leq N_2^o$. We require $N_2 \leq N_2^o$ because the surge staff are exclusively called in from the on-call pool. Moreover, if a nurse stays on call and actually gets called in, then his/her pay rate for this shift is $c_2^o + c_2$. On the other hand, if a nurse stays on call but is not called in to work as a surge staff, then his/her pay rate for this shift is only c_2^o. **At the beginning of the shift**, no-shows are realized among the scheduled base and surge nurses. We assume that the number of base nurses who actually show up to work is $\tilde{N}_1$, which follows a Binomial distribution with parameters N_1 and show-up probability p_1. Similarly, the number of surge nurses who actually show up to work is $\tilde{N}_2$, which follows a Binomial distribution with parameters N_2 and show-up probability p_2. The staffing problem for the generalized model is

$$\min_{N_1, N_2^o} \left\{ c_1 N_1 + c_2^o N_2^o + \mathbb{E}_\Lambda \left[\min_{N_2 \leq N_2^o} \left\{ c_2 N_2 + \mathbb{E}_{Q, \tilde{N}_1, \tilde{N}_2} \left[(h + a\gamma)\, Q\left(\tilde{N}_1, \tilde{N}_2, \Lambda \right) \middle| \Lambda \right] \right\} \right] \right\}. \tag{122}$$

Table 10 Optimal staffing decisions in different cost regimes for the generalized model

Cost parameters	Staffing decisions
$\min\{c_1/p_1, (c_2^o+c_2)/p_2\} \geq h\mu/\gamma + a\mu$	No staffing
$\min\{c_1/p_1, h\mu/\gamma + a\mu\} \geq (c_2^o+c_2)/p_2$	Complete surge staffing
$(c_2^o+c_2)/p_2 \geq h\mu/\gamma + a\mu \geq c_1/p_1$	Complete base staffing
$h\mu/\gamma + a\mu > (c_2^o+c_2)/p_2 > c_1/p_1$	Base + surge staffing

For problem (122), Table 10 summarizes optimal solutions for different parameter regimes, and is an analogue to Table 1 and Proposition 1.

In addition, we can follow similar lines of analysis as those for the two-stage newsvendor solution in the original paper to derive a "generalized" two-stage newsvendor solution.

The high-level structural results as in Theorems 1 and 2 maintain. That is, introducing capped surge-stage staffing levels and nurses' no-show behavior does not change the order of cost savings and optimality gap of the two-stage staffing framework. Intuitively, the staffing level for any realized arrival rate is on the order of $\Theta(\lambda)$. Incorporating nurses' no-show behavior then introduces randomness on the order of $O(\sqrt{\lambda})$ in staffing levels. This is because a Binomial random variable with parameters n (number of trials) and p (success probability) has standard deviation equal to $\sqrt{np(1-p)}$. In comparison to the randomness in staffing levels, the level of uncertainty in the random arrival-rate model is on the order of $\Theta(\lambda^\alpha)$, for $\alpha > 1/2$. Since the uncertainty in random arrival rates dominates the randomness in staffing levels, the generalized two-stage staffing newsvendor solution is still able to achieve a cost saving of $\Theta(\lambda^\alpha)$ and an optimality gap of $o(\lambda^\alpha)$.

Appendix I: Supplementary Numerical Experiments

In this section we conduct additional numerical experiments to support the results in the main paper. Section I.1 investigates effective translation of the two-stage QED staffing rule to finite stochastic systems. Sections I.2–I.4 are devoted to the ED application. Section I.2 provides detailed results for the surge-stage linear regression model. Section I.3 presents sensitivity analysis of the proposed staffing rule with respect to ED-specific patient-flow characteristics, specifically, on the joint impact of lognormal LOS distribution and hourly-varying arrival rates. Section I.4 compares the performance of our proposed heuristic adjustment and the numerically obtained optimal adjustment to account for the transient-shift effects. Lastly, in Section I.5 we develop heuristic policies and conduct numerical experiments regarding non-linear holding costs and multiple patient classes.

I.1. Translation of The Two-Stage QED Staffing Rule

In this appendix we conduct more numerical experiments to examine system performance under the two-stage QED staffing rule with different specifications of k in (12). In what follows, we repeat the experiments in Tables 2 (with $c_2 = 2$) and 3 (with $c_2 = 10$) for other values of surge staffing costs, i.e., $c_2 = 6, 14$. We remark that for the system parameters under consideration, Assumption 1 requires that $c_2 < 18$. The results of these experiments corroborate the efficacy of the particular form of the two-stage QED staffing rule proposed in (13) for small systems.

Table 11 **System performance (optimality gap) under different specifications of the two-stage QED staffing rule with $\beta^* = 0.967, \eta^* = 0.120$ ($\mu = 1, \gamma = 0.1, \alpha = 0.75, h = 1.5, a = 3, c_1 = 1, c_2 = 6$)**

λ \ k	-3	-2	-1	0	1	2	3
25	37.98%	23.30%	9.50%	1.69%	0.58%	5.89%	13.62%
50	29.15%	16.95%	6.79%	1.27%	0.10%	3.42%	9.17%
75	23.57%	13.71%	5.11%	0.85%	0.05%	2.87%	8.01%
100	19.27%	10.04%	3.40%	0.45%	0.23%	3.07%	7.54%

Table 12 **System performance (optimality gap) under different specifications of the two-stage QED staffing rule with $\beta^* = 1.465, \eta^* = -0.380$ ($\mu = 1, \gamma = 0.1, \alpha = 0.75, h = 1.5, a = 3, c_1 = 1, c_2 = 14$)**

λ \ k	-3	-2	-1	0	1	2	3
25	44.04%	24.68%	7.62%	1.15%	5.04%	12.83%	20.36%
50	33.31%	17.89%	5.51%	0.60%	3.94%	10.05%	16.23%
75	27.22%	13.57%	3.25%	0.10%	2.66%	8.07%	13.21%
100	21.75%	10.43%	2.23%	0.07%	2.60%	7.16%	12.21%

I.2. Surge-Stage Linear Regression Model

Table 13 below provides the estimated coefficients in the surge-stage linear regression model.

Table 13: Surge-stage linear regression results

	Dependent variable:
	Observed
Monday day	119.972** (115.275, 124.668)
Tuesday day	97.307** (91.680, 102.934)
Wednesday day	96.277** (91.056, 101.497)
Thursday day	93.560** (88.420, 98.700)
Friday day	83.007** (77.792, 88.222)
Saturday day	57.421** (51.948, 62.894)
Sunday day	53.682** (48.349, 59.014)
Monday night	9.599** (4.116, 15.082)
Tuesday night	6.170. (0.915, 11.426)
Wednesday night	2.755 (−2.481, 7.990)
Thursday night	3.963 (−1.235, 9.161)
Friday night	5.650. (0.213, 11.088)
Saturday night	5.496. (0.161, 10.832)
Winter	3.021 (−0.699, 6.741)
Summer	−1.574 (−4.919, 1.772)
Fall	−2.355 (−5.519, 0.808)
Holiday	−22.392** (−28.168, −16.616)
Holiday − 1 day	−10.137** (−15.761, −4.513)
Holiday + 1 day	16.840** (11.174, 22.507)
Min temperature	0.532** (0.344, 0.719)
Precipitation	−0.160** (−0.249, −0.071)
Snow	−0.169** (−0.224, −0.114)

Wind	0.078^{*} (0.018, 0.139)
Max temperature $\geq 86°$F	-5.761^{**} (-9.292, -2.231)
1-day lag	0.013 (-0.030, 0.055)
7-day lag	0.038 (-0.001, 0.078)
30-day moving average	0.012 (-0.041, 0.065)
Google trend "depression"	-0.098 (-0.231, 0.034)
Google trend "flu"	0.270^{*} (0.087, 0.452)
Average weighted comorbidity score per patient over the last 3 days	14.848. (0.345, 29.352)
Constant	57.365^{**} (22.998, 91.733)
Observations	730
R^2	0.908
Adjusted R^2	0.904
Residual Std. Error	14.316 (df = 699)
F Statistic	231.112^{**} (df = 30; 699)
Note:	.p<0.1; *p<0.05; **p<0.01

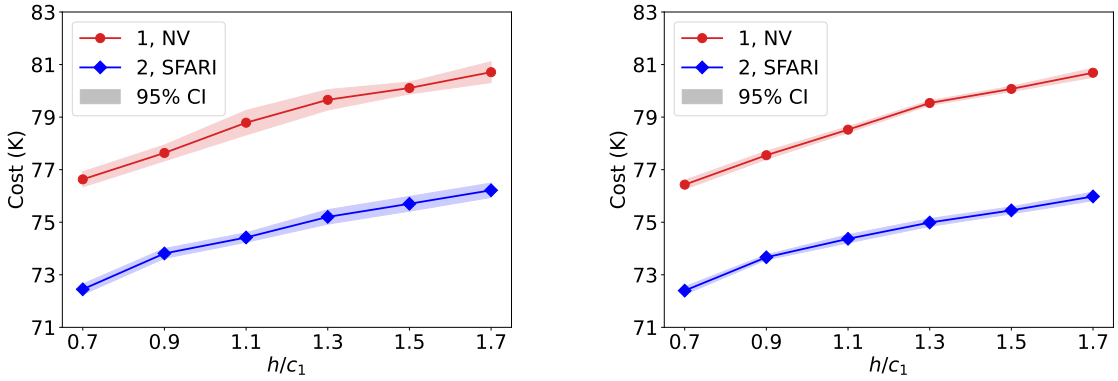
I.3. Robustness of The Proposed Staffing Rule with Respect to ED-Specific Patient-Flow Dynamics

In this section we conduct numerical experiments to check the robustness of the proposed staffing rules with respect to ED-specific patient-flow characteristics. In particular, we consider the parameters associated with Thursday day shifts, and run simulations incorporating different levels of ED-specific features that are not considered in the theoretical model. To prevent prediction error from confounding the results, we assume prefect demand information at the surge stage. In particular, we compare the oracle policy $u_{2,SFARI}$ with the single-stage newsvendor solution $u_{1,NV}$. Figure 12(a) provides a reference to the theoretical setting, where we assume exponential service times, constant arrival rate during the shift (which is equal to the average shift-level arrival rate shown in Table 5), and initialize Thursday day shift at its expected steady-state queue length conditional on the realized arrival rate. The cost curves are generated by increasing the holding cost so that its ratio to the base-stage staffing cost is from 0.7 to 1.7 in increments of 0.2. The 95% confidence intervals are derived by simulating 520 realizations of Thursday day shifts for each holding cost and each policy. With everything else held constant to that in Figure 12(a), Figure 12(b) assumes lognormal (as opposed to exponential) service times and hourly-varying (as opposed to constant) arrival rates. We observe that the cost curves in both figures are very similar. This implies that lognormal service times and hourly-varying arrival rates do not significantly deviate system performance from that in the theoretical setting. (Note that more sensitivity analysis of the proposed staffing rule with respect to lognormal service times is provided in Section 5.3.)

I.4. ED-Catered Staffing Adjustments

In this section we compare the proposed ED-catered staffing adjustment to the optimized one among the same family of adjustment schemes. Recall from Section 7.4.2 that to account for the end-of-shift effects, we propose an adjustment scheme for the two-stage error policy and heuristically set $\xi_1 = 5$ and $\xi_2 = 1$. In what follows, we optimize the adjustment parameters numerically via enumeration. In particular, we simulate the ED over 52 weeks for a wide range of holding costs whose ratio to the base-stage staffing cost range from 0.7

Figure 12 **Impact of LOS distribution and non-stationary arrival rate**

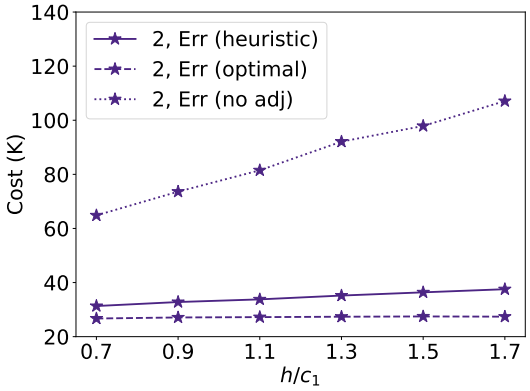


(a) Theoretical setting

(b) Lognormal service time and hourly-varying arrival rate

to 1.7 increment of 0.2. We allow the abandonment cost to grow proportionally to the holding cost by fixing their ratio to be 1.5. For each policy and each holding cost, we enumerate ξ_1 (as well as ξ_2 for the two-stage error policy) from 0 to 10 in increment of 1. Figure 13 demonstrates the expected total cost per shift under $u_{2,ERR}$ using (i) the heuristic adjustment, (ii) the optimized adjustment, and (iii) no adjustment. We note that compared to no adjustment, the heuristic effectively reduces the expected total costs. In addition, the cost curves generated using the heuristic and optimized adjustments are close to each other. These results demonstrate significant value from applying transient-shift adjustment to $u_{2,ERR}$. Given the proximity of the cost curves yielded by the heuristic and optimized adjustments, applying the simple heuristic is effective and circumvents additional computational need.

Figure 13 **Expected total costs per shift under the proposed and optimized adjustment parameters**



I.5. Heuristics for Non-Linear Holding Costs and Multiple Patient Classes

In this section, we elaborate on the heuristic policies to incorporate non-linear holding costs and multiple patient classes. We demonstrate the efficacy of the heuristic policies by comparing the performance of single-stage and two-stage policies.

Non-linear holding costs: In situations where non-linear holding costs are directly concerned, heuristic development of a "generalized two-stage newsvendor solution" is relatively straightforward. Specifically, let

$f : \mathbb{R}_+ \to \mathbb{R}_+$ denote the holding cost (on the queue). The stochastic-fluid approximation of the two-stage staffing problem takes the form

$$\min_{N_1} \left\{ c_1 N_1 + \mathbb{E}\left[\min_{N_2(N_1,\Lambda)} \left\{ c_2 N_2(N_1,\Lambda) + f\left((\Lambda - \mu(N_1 + N_2(N_1,\Lambda)))^+ \right) \right\} \right] \right\}. \tag{123}$$

We refer to the optimal solution to (123) as the "generalized two-stage newsvendor solution". We also propose a "generalized single-stage newsvendor solution", whose base-stage staffing level is the optimal solution to

$$\min_{N_1} \left\{ c_1 N_1 + \mathbb{E}\left[f\left((\Lambda/\mu - N_1)^+ \right) \right] \right\}.$$

We then numerically compare the performance of the single-stage and two-stage newsvendor heuristic policies in a set of simulation experiments. We assume quadratic holding cost, i.e., $f(x) = x^2$, and the rest of the experiments are set up similarly to those in Section 5. Table 14 demonstrates the expected steady-state costs under the two policies. We observe that surge staffing can lead to considerable cost savings, i.e., between 40% and 48%.

Table 14 Performance of the heuristic policies for quadratic holding costs

($\lambda = 20, 40, 60, 80, 100, \mu = 1, \gamma = 1, c_1 = 1, c_2 = 1.5, \alpha = 0.75, \sigma = 1$)

Mean arrival rate	Two-stage heuristic	Single-stage heuristic	Percentage savings by surge staffing
20	27.93	39.13	40.08%
40	54.14	80.38	48.47%
60	79.94	113.47	41.95%
80	105.10	152.34	44.95%
100	129.20	185.75	43.77%

Multiple patient classes: Heuristically, we can incorporate multiple acuity classes by predicting the demand and making staffing decisions for each class individually, and then combining the required nurses for each acuity class. Such a heuristic is applicable to both the single-stage and two-stage staffing policies. We numerically compare the performance of the single-stage and two-stage newsvendor heuristics for a two-class model. We assume Class-1 patients are relatively more urgent, with longer average LOS and higher holding/abandonment costs than those of Class-2 patients. In the simulation experiments, Class-1 patients have priority over Class-2 patients, while patients within the same class are served first come first served. Table 15 lists 1) the expected steady-state costs, 2) the expected queue length for each class, and 3) the LWBS proportion for each class under the two policies. We observe that the two-stage newsvendor heuristic not only achieves significant cost savings, but also considerably reduces the expected queue length and LWBS rates (especially for the less urgent Class 2), compared to the single-stage heuristic (i.e., without surge staffing).

Table 15 Performance of the heuristic policies for the two-class model

(Class 1: $\lambda = 20, 40, 60, 80, 100, \mu = 0.5, \gamma = 1, h = 4, a = 8, c_1 = 1, c_2 = 1.5, \alpha = 0.75, \sigma = 1$;

Class 2: $\lambda = 100, \mu = 1, \gamma = 1, h = 1, a = 2, c_1 = 1, c_2 = 1.5, \alpha = 0.75, \sigma = 1$)

Class-1 mean arrival	Two-stage heuristic					Single-stage heuristic					Percentage savings by surge staffing				
	Cost	Class 1		Class 2		Cost	Class 1		Class 2		Cost	Class 1		Class 2	
		Queue	% LWBS	Queue	% LWBS		Queue	% LWBS	Queue	% LWBS		Queue	% LWBS	Queue	% LWBS
20	174.16	0.04	0.16%	1.60	1.29%	185.68	0.06	0.21%	5.32	3.70%	6.20%	31.98%	23.47%	69.93%	65.07%
40	222.67	0.08	0.18%	1.67	1.39%	237.62	0.11	0.19%	5.44	3.79%	6.29%	28.91%	8.30%	69.24%	63.38%
60	270.88	0.14	0.20%	2.34	1.93%	297.62	0.21	0.26%	8.68	6.04%	8.98%	33.30%	21.79%	73.06%	68.09%
80	316.75	0.18	0.17%	2.29	1.85%	348.60	0.28	0.25%	9.10	6.29%	9.13%	34.98%	30.64%	74.86%	70.53%
100	365.35	0.26	0.21%	3.08	2.49%	401.16	0.39	0.27%	10.49	7.29%	8.93%	34.25%	22.09%	70.63%	65.78%