Inefficient Alliance Formation in Coalitional Blotto Games

Vade Shah, Keith Paarporn, and Jason R. Marden

***Abstract*— When multiple agents are engaged in a network of conflict, some can advance their competitive positions by forming alliances with each other. However, the costs associated with establishing an alliance may outweigh the potential benefits. This study investigates costly alliance formation in the framework of coalitional Blotto games, in which two players compete separately against a common adversary and are able to collude by exchanging resources with one another. Previous work has shown that both players in the alliance can mutually benefit if one player unilaterally donates, or transfers, a portion of their budget to the other. In this letter, we consider a variation where the transfer of resources is inherently inefficient, meaning that the recipient of the transfer only receives a fraction of the donation. Our findings reveal that even in the presence of inefficiencies, mutually beneficial transfers are still possible. More formally, our main result provides necessary and sufficient conditions for the existence of such transfers, offering insights into the robustness of alliance formation in competitive environments with resource constraints.**

I. Introduction

In adversarial settings with multiple competitors, agents can often improve their performance by forming an alliance. Businesses ally to outperform rival products [1], [2], [3], energy providers collaborate to succeed in markets [4], [5], and nations join forces to confront mutual adversaries [6]. A fundamental underlying mechanism for each of these alliances is an exchange of resources, whether they be financial capital, electrical power, or military assets. By exchanging resources, agents can fortify weaknesses, complement strengths, and even intimidate adversaries.

However, when resources are lost in the process of an exchange, deciding whether to form an alliance becomes a more challenging problem. Moreover, inefficient exchanges are ubiquitous: international trades are limited by regulations and tariffs [7], [8], energy transmissions suffer from lossy power lines [9], and sharing defense assets often incurs myriad losses [10], [11]. In the presence of such limitations, a critical question emerges: **At what point do the costs of forming an alliance outweigh its potential benefits?**

Game theory offers a useful set of tools to study the value of alliances in a variety of adversarial contexts [12], [10], [13], [14]. This work focuses on *competitive resource allocation* settings like the ones mentioned above, where agents vie for prizes by strategically distributing their resources. The

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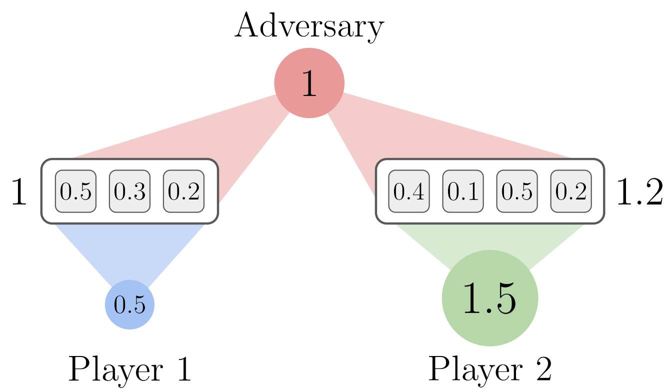

Fig. 1. A cartoon depiction of a coalitional Colonel Blotto game between Players 1 and 2 and a common adversary. Player 1, Player 2, and the adversary are equipped with budgets 0.5, 1.5, and 1, respectively. Player 1 and the adversary compete on the left set of contests with cumulative value 1, and Player 2 and the adversary compete on the right set of contests with cumulative value 1.2.

Colonel Blotto game [15], [16], [17] is a popular model of competitive resource allocation in which two agents compete by allocating their limited resources towards valued contests. An agent wins a contest's valuation if they allocate a greater level of resources to it than their opponent, and each agent plays with the goal of maximizing their accrued valuations.

To analyze opportunities for collaboration, we utilize the model of the *coalitional* Colonel Blotto game, which has been studied extensively in the context of alliance formation [18], [19], [20], [21], [22]. In the coalitional Blotto game (Figure 1), two players compete in separate standard Blotto games against a common adversary. Previous work on these games reveals a surprising opportunity for collaboration between the two players: In certain games, if one player donates, or *transfers*, a portion of their budget to the other, then both players win more contest valuation in their separate competitions than they would have had no transfer occurred [18]. That is, one player becomes 'weaker', the other becomes 'stronger', but they *both* do better because of the transfer. Transferring resources causes the adversary to alter their allocation strategy, so the players can perform a transfer that manipulates the adversary to their advantage.

However, if resources are lost in the process of an exchange, it is unclear whether this strategy remains viable. In fact, it is known that in the extreme case where transfers are lost entirely (i.e., one player simply disposes of some portion of their budget without the other player receiving anything), it is impossible for both players to improve simultaneously [21]. In this work, we probe the granularity of this result to better understand the limiting effects of costly exchange. In particular, we study the case in which one player donates a portion of their budget, but the other player receives only a fraction of the sent amount. We interpret this as a model of *inefficient* alliance formation, where the amount that is lost in the transfer represents the inefficiencies associated with forming the alliance. Within this context, we seek to understand how losses affect the feasibility of budget transfers as a means of *mutually beneficial* alliance formation, meaning that the transfer improves the equilibrium payoff associated with both players.

In our first contribution, summarized in Theorem 1, we demonstrate that so long as the recipient receives a positive fraction of the transfer, there exists a mutually beneficial budget transfer in a nontrivial subset of games. In particular, **we provide necessary and sufficient conditions for when forming an alliance is mutually beneficial.** Then, we abandon the requirement that transfers must benefit each player individually, and instead study the more general setting where the players seek to maximize their combined performance. In the absence of inefficiencies, it is known that the alliance can almost always improve its joint payoff by performing a transfer [18]. Our second result asserts that this conclusion no longer holds in the presence of inefficiencies. Specifically, Theorem 2 asserts that **inefficiencies eliminate opportunities for the alliance to improve its outcome in a nontrivial subset of games**. Together, these results highlight fundamental limitations and possibilities for alliance formation in competitive settings.

II. Model

A. Colonel Blotto Game

We begin our technical discussion with the classical Colonel Blotto game, where two agents (say, Player 1 and the adversary) simultaneously compete across a set of n contests with valuations $v^1, \dots, v^n \geq 0$. Player 1 and the adversary are endowed with budgets $X_1 \in \mathbb{R}$ and $X_A \in \mathbb{R}$, respectively, which they must simultaneously allocate across the contests. Each agent knows the contest valuations and every agent's budget, but they do not know how their opponent will allocate their budget. We denote a valid allocation decision for Player 1 (and similarly for the adversary) by the tuple $d_1 = (d_1^1, \dots, d_1^n)$, where $d_1^k \geq 0$ and $\sum_{k=1}^n d_1^k \leq X_1$. Player 1's payoff for a given pair (d_1, d_A) is of the form

$$U_1(d_1, d_A) = \sum_{k=1}^{n} v^k \cdot I\{d_1^k \geq d_A^k\}, \tag{1}$$

where $I\{\cdot\}$ is the usual indicator function, and the adversary's payoff is $U_A(d_1, d_A) = \phi - U_1(d_1, d_A)$, where $\phi = \sum_{k=1}^n v^k$ is the cumulative valuation of all of the contests in the game.

Despite the apparent simplicity of the model, characterizing Nash equilibrium allocation decisions in the classical Colonel Blotto game is a historically challenging endeavor that remains an open problem. Thus, in this work, we focus instead on the General Lotto formulation[1], a popular variant of the Colonel Blotto game that admits more tractable solutions by requiring agents' allocation decisions to be less than their actual budgets only *in expectation*. The agents' equilibrium payoffs[2] for General Lotto games have been characterized in [17] and are given by

$$U_1^{\text{NE}}(X_1, X_A, \phi) = \begin{cases} \phi\left(\frac{X_1}{2X_A}\right) & X_1 \leq X_A \\ \phi\left(1 - \frac{X_A}{2X_1}\right) & X_1 > X_A \end{cases} \tag{2}$$

$$\text{and} \quad U_A^{\text{NE}}(X_1, X_A, \phi) = \phi - U_1^{\text{NE}}(X_1, X_A, \phi) \tag{3}$$

for Player 1 and the adversary, respectively.

B. Inefficient Coalitional Colonel Blotto Game

We now examine the coalitional Colonel Blotto game, where two self-interested players compete against a common adversary in disjoint Blotto games (Figure 2). Our goal is to determine whether alliances can improve both players' performance, where an alliance is formed through the unilateral transfer of budgetary resources from one player to another. This analysis specifically focuses on the *inefficiencies* associated with alliance formation, meaning that these transfers are accompanied with certain losses. The coalitional Colonel Blotto game proceeds in stages[3] as follows.

Stage 0: Initial Game: The game is initialized (Figure 2, left). Players 1 and 2 participate in standard Colonel Blotto games 1 and 2, respectively, against a common adversary who competes in both games. Player $i \in \{1,2\}$ uses their budget X_i to compete in Blotto game i for valued contests with total valuation ϕ_i. The adversary is equipped with a normalized budget $X_A = 1$, so that a coalitional Colonel Blotto game G is fully parameterized by $G = (\phi_1, \phi_2, X_1, X_2) \in \mathbf{G} = \mathbb{R}^4_{>0}$. At this stage, every agent has complete knowledge of the tuple G and the subsequent order of play.

Stage 1: Alliance Formation: The players consider the formation of an alliance which allows for the transfer of budget from one player to the other. We denote a transfer by $\tau \in (-X_2, X_1)$, which represents the net amount of budget sent from Player 1 to Player 2. Here, a negative value indicates that the transfer goes in the opposite direction. A transfer effectively impacts the state of the game. More specifically, for a transfer τ, the post-transfer budgets associated with each player are given by

$$\overline{X}_1 \triangleq \begin{cases} X_1 - |\tau| & \tau > 0 \\ X_1 + \beta|\tau| & \tau \leq 0, \end{cases} \qquad \overline{X}_2 \triangleq \begin{cases} X_2 + \beta|\tau| & \tau > 0 \\ X_2 - |\tau| & \tau \leq 0, \end{cases}$$

where $\beta \in (0,1]$ is a parameter known to the agents before the transfer occurs that captures the inefficiencies associated with transferring resources; the case $\beta = 1$ is the fully efficient setting considered in [18]. We say that a transfer τ *induces* a new game $\overline{G} = (\phi_1, \phi_2, \overline{X}_1, \overline{X}_2)$, which every agent has complete knowledge of after the transfer is complete.

[1]Although the game uses the General Lotto payoffs, we use the terminology 'coalitional Colonel Blotto game' for consistency with the existing literature [18], [19], [20], [22].

[2]Here, we present results purely regarding the equilibrium payoffs and direct the reader to [17] for more details regarding the equilibrium strategies.

[3]The coalitional Colonel Blotto game can be modeled as a sequential (i.e., Stackelberg) game between the players and the adversary, where the players solve a maxmin problem. However, for simplicity and consistency with [18], we adopt the stage-based model described in the text.

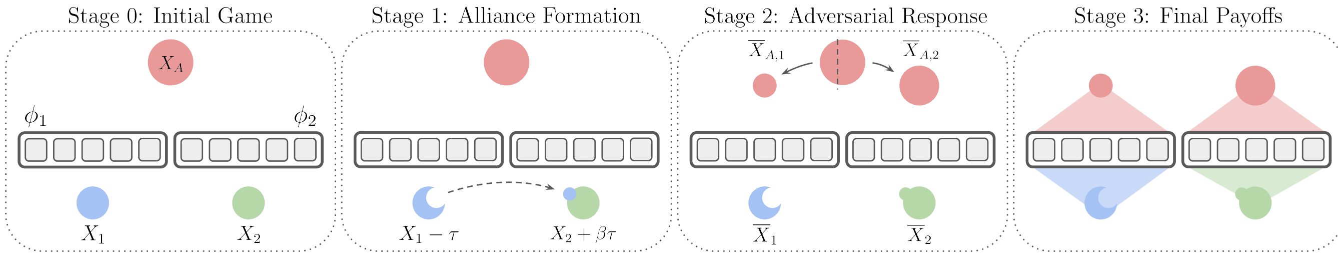

Fig. 2. The stages of the coalitional Blotto game. (Stage 0) The game is initialized; each Player $i \in \{1,2\}$ is endowed with a budget X_i and competes across a set of contests with cumulative valuation ϕ_i. The adversary is endowed with budget $X_A = 1$ and competes across both sets of contests. (Stage 1) One player transfers a portion of their budget, τ, and the other player receives a fraction of the transfer, $\beta\tau$, resulting in a new game $\overline{G} = (\phi_1, \phi_2, \overline{X}_1, \overline{X}_2)$. (Stage 2) The adversary optimally divides their budget between the two games into $\overline{X}_{A,1}$ and $\overline{X}_{A,2}$. (Stage 3) The two disjoint Blotto games are played, and Player 1, Player 2, and the adversary receive their equilibrium payoffs $U_1^{\mathrm{NE}}(\tau;G)$, $U_2^{\mathrm{NE}}(\tau;G)$, and $U_A^{\mathrm{NE}}(\tau;G)$, respectively.

Stage 2: Adversarial Response: After having observed any transfer, the adversary decides how to divide their budget between the two standard Blotto games (Figure 2, center right). Depending on $\overline{G}$, the adversary can optimize their performance by strategically diverting more resources towards one of the Blotto games. In particular, they can solve

$$\underset{\substack{X_{A,1},X_{A,2}\geq 0\\ X_{A,1}+X_{A,2}\leq X_A}}{\arg\max}\; U_A^{\mathrm{NE}}(\overline{X}_1, X_{A,1}, \phi_1) + U_A^{\mathrm{NE}}(\overline{X}_2, X_{A,2}, \phi_2) \qquad (4)$$

to maximize their equilibrium payoff. The optimal solutions to this problem, denoted by $\overline{X}_{A,1}$ and $\overline{X}_{A,2}$, are derived in [18] and summarized in the forthcoming Table I.

Stage 3: Final Payoffs: In the third and final stage, the two separate Colonel Blotto games are played and the agents' equilibrium payoffs are realized (Figure 2, right). In Blotto game i, Player i allocates their post-transfer budget $\overline{X}_i$ and the adversary allocates $\overline{X}_{A,i}$. Each agent then receives their equilibrium payoff as defined in (2) and (3), i.e., the payoffs to Players 1 and 2 are given by $U_1^{\mathrm{NE}}(\overline{X}_1, \overline{X}_{A,1}, \phi_1)$ and $U_2^{\mathrm{NE}}(\overline{X}_2, \overline{X}_{A,2}, \phi_2)$, respectively, while the adversary's payoff is given by $U_A^{\mathrm{NE}}(\overline{X}_1, \overline{X}_{A,1}, \phi_1) + U_A^{\mathrm{NE}}(\overline{X}_2, \overline{X}_{A,2}, \phi_2)$.

C. Mutually Beneficial Alliances

The focus of this work is on identifying whether there are mutually beneficial alliances in inefficient coalitional Colonel Blotto games. Observe that the only decision for the players to make in the above game is that of the transfer τ. Hence, we express the equilibrium payoffs to Player i and the adversary as $U_i^{\mathrm{NE}}(\tau;G)$ and $U_A^{\mathrm{NE}}(\tau;G)$ respectively to specifically highlight this dependence. We say that a game G has a *mutually beneficial transfer* if there exists a transfer τ' such that

$$U_1^{\mathrm{NE}}(\tau';G) > U_1^{\mathrm{NE}}(0;G) \text{ and } U_2^{\mathrm{NE}}(\tau';G) > U_2^{\mathrm{NE}}(0;G),$$

meaning that both players are better off after the transfer. Figure 3 provides an illustrative example demonstrating that in certain games, both the donor and the recipient of a transfer can benefit. In the following sections, we generalize this example and identify all such games where mutually beneficial alliances exist in the presence of inefficiencies.

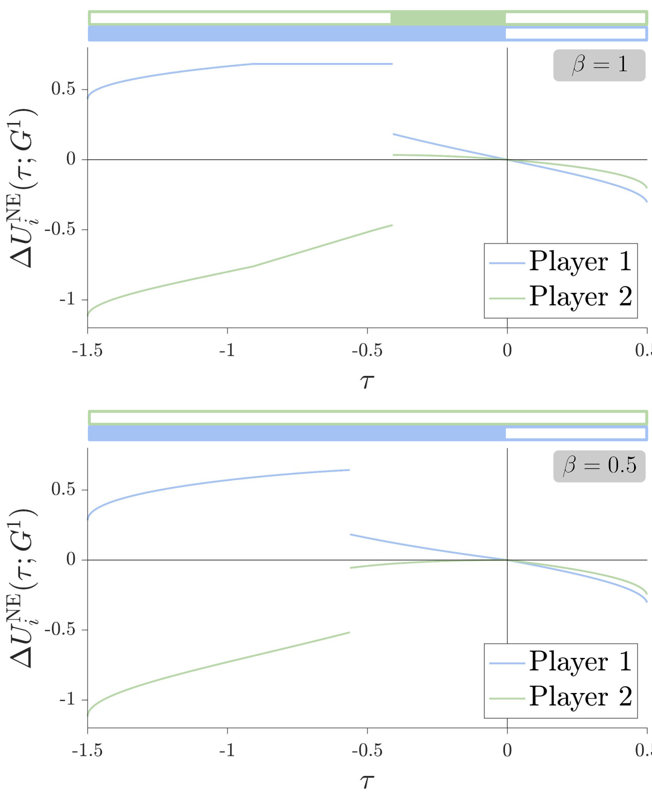

Fig. 3. The change in payoff $\Delta U_i^{\mathrm{NE}}(\tau;G^1) \triangleq U_i^{\mathrm{NE}}(\tau;G^1) - U_i^{\mathrm{NE}}(0;G^1)$ of Player $i \in \{1,2\}$ as a function of τ for the game $G^1 \triangleq (1, 1.2, 0.5, 1.5)$ shown in Figure 1. The blue and green bars above each plot indicate increases in the payoffs of Players 1 and 2, respectively. When $\beta = 1$ (top), there exist transfers for which both players' payoffs increase, but when $\beta = 0.5$ (bottom), there is no transfer for which Player 2's payoff increases.

III. Mutually Beneficial Inefficient Transfers

In this section, we present our first result, which is a complete characterization of the set of games for which mutually beneficial budgetary transfers exist for every inefficiency β.

Theorem 1. *Let $G = (\phi_1, \phi_2, X_1, X_2)$ be a coalitional Blotto game with inefficiency parameter $\beta \in (0,1]$.*

(a) If there exists a mutually beneficial transfer $\tau' < 0$ ($\tau' > 0$), then $\frac{X_1}{\phi_1} \leq \frac{X_2}{\phi_2}$ $\left(\frac{X_1}{\phi_1} \geq \frac{X_2}{\phi_2}\right)$.

(b) Without loss of generality, suppose that $\frac{X_1}{\phi_1} \leq \frac{X_2}{\phi_2}$. There

TABLE I
THE ADVERSARY'S OPTIMAL BUDGET ALLOCATION WHEN $\frac{\overline{X}_1}{\phi_1} \le \frac{\overline{X}_2}{\phi_2}$

Case	Condition	$\overline{X}_{A,1}$
1	$\frac{\overline{X}_1}{\phi_1} \ne \frac{\overline{X}_2}{\phi_2}$ and $\frac{\phi_2}{\phi_1} \le \overline{X}_1\overline{X}_2$	1
2	$0 < 1 - \left(\frac{\phi_1\overline{X}_1\overline{X}_2}{\phi_2}\right)^{\frac{1}{2}} \le \overline{X}_2$	$\left(\frac{\phi_1\overline{X}_1\overline{X}_2}{\phi_2}\right)^{\frac{1}{2}}$
3	$1 - \left(\frac{\phi_1\overline{X}_1\overline{X}_2}{\phi_2}\right)^{\frac{1}{2}} > \overline{X}_2$	$\frac{(\phi_1\overline{X}_1)^{\frac{1}{2}}}{(\phi_1\overline{X}_1)^{\frac{1}{2}}+(\phi_2\overline{X}_2)^{\frac{1}{2}}}$
4	$\frac{\overline{X}_1}{\phi_1} = \frac{\overline{X}_2}{\phi_2}$ and $1 \le \overline{X}_1 + \overline{X}_2$	$\overline{X}_{A,i} \le \overline{X}_i,\ i \in \{1,2\}$

exists a mutually beneficial transfer τ' if and only if

$$\frac{\phi_2}{\phi_1} < X_1X_2 \quad \textit{and} \tag{5}$$

$$\beta > \min\left(\left(\frac{4\phi_2X_1}{\phi_1X_2^3}\right)^{\frac{1}{2}} - \frac{X_1}{X_2}, \left(\frac{4\phi_2X_1}{\phi_1X_2}\right)^{\frac{1}{2}} + \frac{X_1}{X_2}\right). \tag{6}$$

The first part of Theorem 1 asserts that mutually beneficial transfers must always go from the relatively stronger to the relatively weaker player, where the measure of fitness of Player i is given by $\frac{X_i}{\phi_i}$. The second part establishes that budget transfers are viable for a nontrivial subset of games so long as the inefficiency parameter β is positive; the case where $\frac{X_1}{\phi_1} \ge \frac{X_2}{\phi_2}$ follows trivially by swapping indices. This is perhaps surprising, as one might expect that beyond a certain point, inefficiencies would unequivocally prohibit alliance formation, but this is not the case as shown in Figure 5. Furthermore, from an implementation perspective, Theorem 1 offers a straightforward tool for players to evaluate opportunities for alliance formation: First, to determine which player may be able to donate their budget, they can check whether $\frac{\phi_2}{\phi_1} \le \frac{X_2}{X_1}$. Then, they can simply verify (5) and (6) to determine whether a transfer is in fact viable.

Proof. The proof proceeds by analyzing the change in each player's equilibrium payoff (2) as a function of the transfer τ. Recall that Player i's payoff depends on the adversary's optimal allocation $\overline{X}_{A,i}$ toward their standard Blotto game, which is derived in [18] and can be comprehensively described by the four cases described in Table I and depicted in Figure 4. Note that when they play optimally, the adversary always uses the entirety of their budget (i.e., $\overline{X}_{A,1} + \overline{X}_{A,2} = 1$).

To simplify our discussion, we define the subsets

$$\mathbf{C}_i \triangleq \left\{ G \in \mathbf{G} \;\middle|\; \frac{X_1}{\phi_1} \le \frac{X_2}{\phi_2} \text{ and } G \text{ belongs to Case } i \right\}$$

for $i \in \{1,2,3,4\}$. When we write that $G = (\phi_1, \phi_2, X_1, X_2) \in \mathbf{C}_i$, the reader should read this as 'the parameters $(\phi_1, \phi_2, X_1, X_2)$ satisfy the conditions for Case i in Table I' with X_1 and X_2 in place of $\overline{X}_1$ and $\overline{X}_2$, respectively.

Case 1: Consider any game $G \in \mathbf{C}_1$. In this case, $\overline{X}_{A,1} = 1$, meaning that the adversary does not allocate any budget to game 2. Thus, Player 2 wins all of their contests to begin with, so there is no transfer that can improve their payoff.

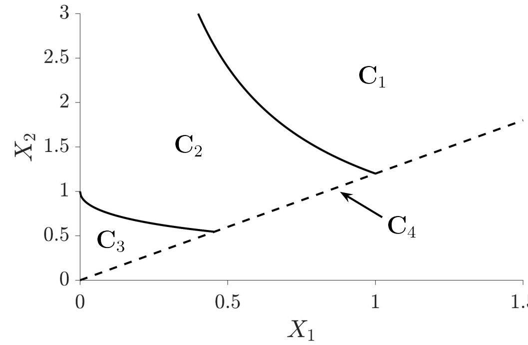

Fig. 4. Graphical illustration of the cases described in Table I for fixed $\phi_1 = 1$ and $\phi_2 = 1.2$ when $\frac{X_1}{\phi_1} \le \frac{X_2}{\phi_2}$. In Case 1, the adversary allocates the entirety of their budget towards Player 1. In Case 2, the adversary reaches a point of diminishing returns in game 1 and begins allocating budget towards game 2. In Case 3, the adversary has a greater budget than both players combined, so they equate their marginal payoffs in each game. In Case 4, the marginal payoff is equal in each game and the adversary is indifferent.

Case 2: First, consider transfers going from Player 1 to Player 2, i.e., positive transfers. If $\tau > 0$, then the induced $\overline{G} = (\phi_1, \phi_2, \overline{X}_1, \overline{X}_2)$ belongs to $\mathbf{C}_2$, so Player 1's payoff is given by $U_1^{\mathrm{NE}}(\tau; G) = \frac{1}{2}\left(\frac{\phi_1\phi_2(X_1-\tau)}{X_2+\beta\tau}\right)^{\frac{1}{2}}$. Since ϕ_1 and ϕ_2 are strictly positive by assumption, and since β and τ are appropriately restricted so that $X_1 - \beta\tau$ and $X_2 + \tau$ are strictly positive, $\frac{d}{d\tau}U_1^{\mathrm{NE}}$ is strictly negative. This implies that for any positive transfer, Player 1's payoff decreases, meaning that Player 1 can never benefit from transferring their budget.

Now, consider the case where $\tau < 0$. A sufficiently large transfer may induce $\overline{G} \in \mathbf{C}_3$, or may result in $\frac{\overline{X}_1}{\phi_1} \ge \frac{\overline{X}_2}{\phi_2}$, but in either scenario, Player 2 would be worse off since their budget would decrease while the adversary's allocation towards their game would increase. Thus, we can limit our attention to transfers that induce $\overline{G} \in \mathbf{C}_2$, for which we have

$$U_1^{\mathrm{NE}}(\tau; G) = \frac{1}{2}\left(\frac{\phi_1\phi_2(X_1 - \beta\tau)}{X_2 + \tau}\right)^{\frac{1}{2}},$$

$$U_2^{\mathrm{NE}}(\tau; G) = \phi_2\left(1 - \frac{1}{2(X_2+\tau)}\right) + \frac{1}{2}\left(\frac{\phi_1\phi_2(X_1 - \beta\tau)}{X_2 + \tau}\right)^{\frac{1}{2}}.$$

Since $\frac{d}{d\tau}U_1^{\mathrm{NE}} < 0$ (i.e., Player 1's payoff increases), Player 1 will always accept a transfer from Player 2. Furthermore, if $\frac{d}{d\tau}U_2^{\mathrm{NE}}\big|_{\tau\to 0^-} < 0$, then Player 2 would benefit from transferring a sufficiently small amount of budget to Player 1. By simple calculus and algebraic manipulation, one can verify that $\frac{d}{d\tau}U_2^{\mathrm{NE}}\big|_{\tau\to 0^-} < 0 \iff \beta > \left(\frac{4\phi_2X_1}{\phi_1X_2^3}\right)^{\frac{1}{2}} - \frac{X_1}{X_2}$.

Case 3: When $\tau > 0$, the induced $\overline{G}$ may belong to $\mathbf{C}_2$ or $\mathbf{C}_3$. If $\overline{G} \in \mathbf{C}_3$, then Player 1's payoff is given by $U_1^{\mathrm{NE}}(\tau; G) = \frac{1}{2}\phi_1(X_1 - \tau) + \frac{1}{2}(\phi_1\phi_2(X_1-\tau)(X_2+\beta\tau))^{\frac{1}{2}}$. It is straightforward to show that $\frac{d}{d\tau}U_1^{\mathrm{NE}}\big|_{\tau\to 0^+} < 0 \iff 2\left(\frac{\phi_1X_1X_2}{\phi_2}\right)^{\frac{1}{2}} < \beta X_1 - X_2$, but this condition is not satisfied by any game where $\frac{X_1}{\phi_1} \le \frac{X_2}{\phi_2}$. Furthermore, given that positive transfers are not mutually beneficial in Case 2, it follows that there is no mutually beneficial transfer such that $\overline{G} \in \mathbf{C}_2$.

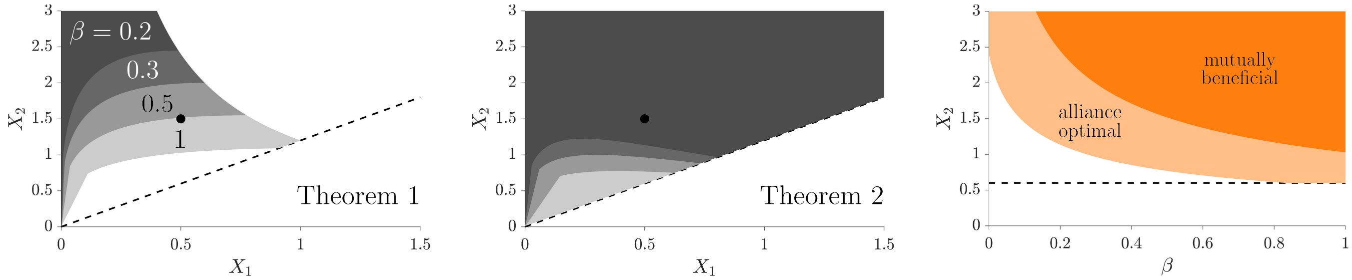

Fig. 5. Subsets of the parameter space where mutually beneficial and nonzero alliance optimal transfers exist for fixed $\phi_1 = 1$ and $\phi_2 = 1.2$. Only games where $\frac{X_1}{\phi_1} \leq \frac{X_2}{\phi_2}$ (dashed black line) are depicted to avoid redundancy. The shading illustrates a hierarchical relationship between the subsets, i.e., darker inner subsets are fully contained in lighter outer subsets. (Left) Subsets of the X_1-X_2 space where mutually beneficial budget transfers exist for various values of β. (Center) Subsets of the X_1-X_2 space where nonzero alliance optimal transfers exist for various values of β. The shades correspond to the same values of β labeled in the left panel. (Right) The subset of the β-X_2 space where mutually beneficial transfers exist (dark orange) and the subset where nonzero alliance optimal transfers exist (light and dark orange) for fixed $X_1 = 0.5$. The former subset is fully contained in the latter. In the light orange region, alliance improvement comes at one player's expense; that is, one player benefits while the other does worse, but their combined payoff increases.

When $\tau < 0$, the induced $\overline{G}$ may belong to either $\mathbf{C}_3$ or $\mathbf{C}_2$. In the former case, we have

$$U_1^{\text{NE}}(\tau; G) = \frac{1}{2}\phi_1(X_1 - \beta\tau) + \frac{1}{2}(\phi_1\phi_2(X_1 - \beta\tau)(X_2 + \tau))^{\frac{1}{2}},$$
$$U_2^{\text{NE}}(\tau; G) = \frac{1}{2}\phi_2(X_2 + \tau) + \frac{1}{2}(\phi_1\phi_2(X_1 - \beta\tau)(X_2 + \tau))^{\frac{1}{2}}.$$

It is straightforward to show that if $\frac{d}{d\tau}U_2^{\text{NE}}\big|_{\tau\to 0^-} \geq 0$, then it will remain nonnegative for all $\tau \leq 0$. Furthermore, since $\frac{d}{d\tau}U_2^{\text{NE}} \geq \frac{d}{d\tau}U_1^{\text{NE}}$, both players will strictly benefit from a net positive transfer when $\frac{d}{d\tau}U_2^{\text{NE}}\big|_{\tau\to 0^-} < 0 \iff \beta > \left(\frac{4\phi_2 X_1}{\phi_1 X_2}\right)^{\frac{1}{2}} + \frac{X_1}{X_2}$. This is also a necessary condition for the existence of a mutually beneficial transfer such that $\overline{G} \in \mathbf{C}_2$, since the set of induced games that satisfy the Case 2 existence condition can be reached through a transfer only from nominal games satisfying this condition.

Case 4: For any game $G \in \mathbf{C}_4$, we show in Theorem 2 that the sum of the players' payoffs is maximized, so there is no mutually beneficial transfer.

□

IV. Alliance Optimal Inefficient Transfers

The previous section demonstrates that in spite of inefficiencies, players can still form mutually beneficial alliances in many cases. In this section, we study how inefficiencies impact opportunities to form alliances that are not necessarily mutually beneficial, but rather jointly optimal. Mathematically speaking, we study transfers that solve

$$\arg\max_{\tau} \; U_{1,2}^{\text{NE}}(\tau; G),$$

where $U_{1,2}^{\text{NE}}(\tau; G) \triangleq U_1^{\text{NE}}(\tau; G) + U_2^{\text{NE}}(\tau; G)$ is the *alliance payoff*. We call a transfer τ^* that maximizes this sum *alliance optimal*. Note that if a transfer is mutually beneficial, then it must also improve the alliance payoff, but the converse need not be true. This is formalized in the following Theorem.

Theorem 2. *Let $G = (\phi_1, \phi_2, X_1, X_2)$ be a coalitional Blotto game with inefficiency parameter $\beta \in (0, 1]$.*

(a) If there exists a nonzero alliance optimal transfer $\tau^* < 0$ ($\tau^* > 0$), then $\frac{X_1}{\phi_1} \leq \frac{X_2}{\phi_2}$ $\left(\frac{X_1}{\phi_1} \geq \frac{X_2}{\phi_2}\right)$.

(b) Without loss of generality, suppose that $\frac{X_1}{\phi_1} \leq \frac{X_2}{\phi_2}$. There exists a strictly negative alliance optimal transfer τ^* if and only if $G \in \mathbf{G} \setminus \mathbf{G}^*(\beta)$, where

$$\mathbf{G}^*(\beta) \triangleq \left\{ G \in \mathbf{C}_2 \;\middle|\; X_1 + \beta X_2 \leq \left(\frac{\phi_2 X_1}{\phi_1 X_2}\right)^{\frac{1}{2}} \right\} \cup \mathbf{C}_4 \; \cup$$
$$\left\{ G \in \mathbf{C}_3 \;\middle|\; \beta\phi_1 - \phi_2 \leq \left(\frac{\phi_1\phi_2}{X_1 X_2}\right)^{\frac{1}{2}} (X_1 - \beta X_2) \right\}.$$

Theorem 2 effectively characterizes the gap between mutually beneficial and alliance optimal transfers as illustrated in Figure 5. However, there is an interesting technical distinction regarding these regions and their dependence on the inefficiency parameter β: Although the players cannot always mutually improve even in efficient environments, the alliance can almost always improve only in efficient environments. That is, the set of games in which mutually beneficial transfers do not exist has positive measure for all values of β, but the set of games in which alliance optimal transfers are zero has positive measure only when $\beta < 1$; when $\beta = 1$, one can easily verify that $\mathbf{G}^*(1)$ is the measure-zero subset of G where $\frac{\phi_2}{\phi_1} = \frac{X_2}{X_1}$. In this measure-theoretic sense, inefficiencies have a more pronounced impact on the outcome of the alliance than they do on the individual.

Proof. The proof proceeds by analyzing the alliance payoff in each of the four cases and computing its derivative.

Case 1: In this case, since the adversary allocates all of their budget towards Player 1, the players can improve the sum of their payoffs by transferring $\tau < 0$ from Player 2 to Player 1 until the induced $\overline{G}$ satisfies either Case 2 (in which case they proceed according to Case 2 below) or Case 4.

Case 2: When $\tau > 0$, the alliance payoff is given by

$$U_{1,2}^{\text{NE}}(\tau; G) = \phi_2\left(1 - \frac{1}{2(X_2 + \beta\tau)}\right) + \frac{1}{2}\left(\frac{\phi_1\phi_2(X_1 - \tau)}{X_2 + \beta\tau}\right)^{\frac{1}{2}},$$

and its derivative $\frac{d}{d\tau}U_{1,2}^{\text{NE}}$ is positive when

$$X_1 + \frac{1}{\beta}X_2 < \left(\frac{\phi_2(X_1 - \tau)}{\phi_1(X_2 + \beta\tau)}\right)^{\frac{1}{2}}.$$

However, it is relatively straightforward to show that this condition is not satisfied for any game $\mathbf{C}_2$, so it follows that any positive transfer cannot cause an increase in $U_{1,2}^{\mathrm{NE}}(\tau;G)$. When $\tau < 0$, the alliance payoff is given by

$$U_{1,2}^{\mathrm{NE}}(\tau;G) = \phi_2\left(1 - \frac{1}{2(X_2+\tau)}\right) + \frac{1}{2}\left(\frac{\phi_1\phi_2(X_1-\beta\tau)}{X_2+\tau}\right)^{\frac{1}{2}}.$$

The derivative $\frac{d}{d\tau}U_{1,2}^{\mathrm{NE}}$ remains negative so long as

$$\overline{X}_1 + \beta\overline{X}_2 > \left(\frac{\phi_2\overline{X}_1}{\phi_1\overline{X}_2}\right)^{\frac{1}{2}}.$$

Thus, in any game where $\frac{d}{d\tau}U_{1,2}^{\mathrm{NE}}\big|_{\tau\to 0^-} < 0$, Player 2 should transfer budget to Player 1 until either $\overline{G} \in \mathbf{C}_1$ (in which case they proceed according to Case 1 above) or $\overline{G} \in \mathbf{C}_4$, or until $\overline{X}_1 + \beta\overline{X}_2 = \left(\frac{\phi_2\overline{X}_1}{\phi_1\overline{X}_2}\right)^{\frac{1}{2}}$. In the latter case, the alliance payoff is maximized, and τ^* is the unique transfer that satisfies this condition. Otherwise, if $\frac{d}{d\tau}U_{1,2}^{\mathrm{NE}}\big|_{\tau\to 0^-} > 0$, then the payoff to the alliance is already maximized and $\tau^* = 0$.

Case 3: Following a similar procedure as above, it is straightforward to show that when $\tau > 0$, $\frac{d}{d\tau}U_{1,2}^{\mathrm{NE}} < 0$ for all games in $\mathbf{C}_3$; thus, any positive transfer cannot improve the alliance payoff. When $\tau < 0$, $U_{1,2}^{\mathrm{NE}}(\tau;G)$ is given by

$$\begin{aligned} U_{1,2}^{\mathrm{NE}}(\tau;G) =& \frac{1}{2}\phi_1(X_1-\beta\tau) + \frac{1}{2}\phi_2(X_2+\tau) \\ &+ (\phi_1\phi_2(X_1-\beta\tau)(X_2+\tau))^{\frac{1}{2}}. \end{aligned}$$

The derivative $\frac{d}{d\tau}U_{1,2}^{\mathrm{NE}}$ remains negative so long as

$$\beta\phi_1 - \phi_2 > \left(\frac{\phi_1\phi_2}{\overline{X}_1\overline{X}_2}\right)^{\frac{1}{2}}(\overline{X}_1 - \beta\overline{X}_2).$$

Thus, in any game where $\frac{d}{d\tau}U_{1,2}^{\mathrm{NE}}\big|_{\tau\to 0^-} < 0$, Player 2 should transfer budget to Player 1 until $\overline{G} \in \mathbf{C}_2$ (in which case they proceed according to Case 2 above), or until $\beta\phi_1 - \phi_2 = \left(\frac{\phi_1\phi_2}{\overline{X}_1\overline{X}_2}\right)^{\frac{1}{2}}(\overline{X}_1 - \beta\overline{X}_2)$, at which point the alliance payoff is maximized. Otherwise, if $\frac{d}{d\tau}U_{1,2}^{\mathrm{NE}}\big|_{\tau\to 0^-} > 0$, then the payoff to the alliance is already maximized and $\tau^* = 0$.

Case 4: In Cases 1 and 2, the alliance payoff increases when Player 2 transfers budget to Player 1 until reaching a maximum, or until reaching Case 4. The symmetrical statement is true for all games where $\frac{\phi_2}{\phi_1} \geq \frac{X_2}{X_1}$, meaning that every transfer that improves the alliance payoff moves towards the curve $\frac{\overline{X}_2}{\phi_1} = \frac{\overline{X}_2}{\phi_2}$. Since $U_{1,2}^{\mathrm{NE}}$ is continuous, we can conclude that if $G \in \mathbf{C}_4$, then $U_{1,2}^{\mathrm{NE}}$ is already at a maximum, and $\tau^* = 0$; similarly, if $\overline{G} \in \mathbf{C}_4$, then $U_{1,2}^{\mathrm{NE}}$ is also at a maximum, and τ^* is the transfer that induces $\overline{G}$.

□

V. Conclusion

In this work, we examine a multi-stage coalitional Colonel Blotto game in which two players compete against a common adversary by allocating their limited budgets towards valued contests. We first show that under certain conditions, players can form mutually beneficial alliances by transferring their budgets even in the presence of inefficiencies. Then, we study the alliance's optimal performance, and we demonstrate that inefficiencies limit opportunities for mutual improvement in a nontrivial subset of games. These results lend novel insight into the effects of inefficiencies on alliance formation and prompt further investigation using practical examples.

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