Incomplete and Asymmetric Information in General Lotto Games

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***Abstract*—In this paper, we consider incomplete and asymmetric information formulations of the General Lotto game, which describes two opposing players that strategically allocate limited resources over multiple contests. In particular, we consider scenarios where one of the player's resource budget is common knowledge while the other player's is private. Our main contribution provides complete equilibrium characterizations in the scenario where the private resource budget is drawn from a Bernoulli distribution. We then leverage these characterizations to analyze a resource assignment problem where a commander must decide how to assign resources to sub-colonels that compete against opponents in separate General Lotto games. While optimal deterministic policies have previously been characterized in the literature, we broaden the context by deriving optimal randomized policies, which induce asymmetric information General Lotto games. Leveraging our equilibrium characterizations, we demonstrate that randomization can offer a four-fold improvement in the commander's performance over the optimal deterministic assignments.**

I. Introduction

Obtaining informational advantages is a high priority for competitors engaging in adversarial environments. It is crucial for ensuring the security of many engineered and socio-technical systems, such as cyber-physical systems, cyber networks, and critical infrastructures. More informed adversaries are able to exploit system vulnerabilities, and likewise, a system operator can implement more effective security measures by knowing the adversaries' capabilities. The impact of asymmetric information in strategic interactions is a topic of broad interest, and has been studied extensively in the control theory literature in the context of dynamic games, repeated games, and controlled Markov processes [12], [20], [22], [33]. In these works, effective decision-making policies are derived and computed.

This paper focuses on how information asymmetry impacts a competitor's performance in interactions that concern the strategic allocation of resources. In particular, we consider incomplete and asymmetric information formulations of the General Lotto game [11], [18], a popular variant of the famous Colonel Blotto game. Two opposing players, $\mathcal{A}$ and $\mathcal{B}$, strategically allocate limited resources over a number of valuable contests (or battlefields) [2], [7], [8], [16], [18], [27]. The objective for both players is to accumulate as much value as possible by securing contests of interest. In order to secure an individual contest, one must send a higher amount of resources to that contest than the other player.

In our formulations, we consider a General Lotto game where player $\mathcal{A}$'s resource budget is a Bernoulli random variable, and player $\mathcal{B}$ is uninformed about the true realization. In other words, player $\mathcal{A}$ holds private information about its true strength, while player $\mathcal{B}$'s strength is common knowledge. The primary contribution of this paper is the full derivation of equilibrium payoffs and strategies to the Bayesian game (Theorem 3.1). To do so, we exploit the equilibrium solutions from all-pay auctions with incomplete and one-sided information [30]. To the best of our knowledge, General Lotto games with one-sided budget uncertainty have not been considered in the existing literature.

A practical motivation for this study is the possibility that player $\mathcal{A}$'s randomized resource budget is determined by an exogenous decision-maker. For example, $\mathcal{A}$'s endowed budget comes from a higher-level authority (e.g. a government or budget-deciding entity) that must decide how to assign resources to multiple agents. An important application is ensuring security at an airport. Resources (TSA agents and security equipment) need to be assigned to multiple terminals, and the resources are deployed by local decision-makers. Incorporating randomization in the assignment policy is an essential strategic feature that maintains unpredictability against potential adversaries. This is strongly supported by research on practical implementations for airport security, border security, and the protection of wildlife [26], [31], [34].

Consequently, we also study a hierarchical resource assignment problem in which a high-level commander assigns limited or costly resources to two sub-colonels [15]. The sub-colonels then use their assigned resources to compete in a General Lotto game against their respective opponents. Note that in order to analyze such decision problems, it is necessary to have equilibrium characterizations of the underlying General Lotto games. Indeed, the commander's assignment policy determines the informational environment of the underlying General Lotto games. In particular, a deterministic assignment policy induces complete information games, wherein the opponents are informed of the sub-colonels' assigned resources

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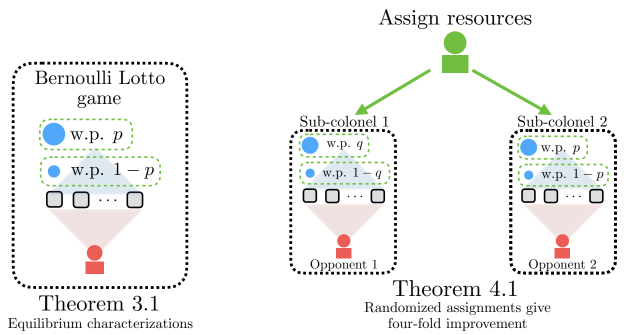

Fig. 1. (Left) In the Bernoulli Lotto game, one player's resource budget is drawn at random from a Bernoulli distribution and becomes private information. The other player's budget is common knowledge. The primary contribution of this paper provides full equilibrium characterizations of this scenario (Theorem 3.1) (Right) A commander assigns resource budgets to two sub-colonels, who subsequently engage in a Lotto game with its respective opponent. Here, the goal of the commander is assign resource so as to maximize the cumulative gains from the two sub-colonels contests. We demonstrate that randomized assignment policies can provide up to a four-fold improvement over the optimal deterministic policies (Theorem 4.1).

(budgets). This is the setting analyzed in originating work by Kovenock [15] as well as several subsequent studies [3], [4], [9], [10].

On the other hand, a randomized assignment policy induces one-sided incomplete and asymmetric information about the sub-colonels' budgets. Thus, our equilibrium characterizations of asymmetric information Lotto games enable us to study the performance of a class of *randomized* assignment policies in the context of this resource assignment problem. The second main contribution of this paper demonstrates that optimal randomized policies can offer a four-fold performance improvement over optimal deterministic policies (Theorem 4.1). A graphical summary of our main contributions in this paper is given in Figure 1.

Related works: The primary literature on Colonel Blotto and General Lotto games, which spans over 100 years, focuses on deriving equilibria under the assumption of complete information. That is, all parameters – player budgets and battlefield valuations – are common knowledge [2], [8], [18], [19], [21], [27]–[29], [32]. Few contributions in the Blotto and Lotto literature have shifted the focus away from complete information settings [1], [6], [13], [24].

The first main result of this manuscript contributes to a growing literature on Blotto and Lotto games with incomplete information. Similar to our budget uncertainty setting, the works [1] and [13] study settings where players have incomplete information about each other's resource endowments. The model studied in [1] considers the players to be equally uninformed (no information asymmetry) about the budget parameters, and hence symmetric Bayes-Nash equilibria are identified. Incomplete information about battlefield valuations has also been featured recently in the literature, with some works considering uncertainty that is symmetric across players [6], [14], and more recently, one-sided uncertainty [23], [24].

The second main result of this manuscript builds upon a well-studied three-stage resource assignment problem [4], [9], [10], [15]. The original formulation is due to Kovenock and Roberson [15], where they derived optimal *deterministic* assignment policies against two opponents. This paper broadens the solutions to randomized assignment policies, where we identify the improvement that randomization offers. Several follow-up studies based on this framework have since appeared, which have demonstrated the benefit of publicly announcing resource transfers between allies [9], [10]. More generally, [5] studies the dynamics of a multi-player network of conflicts.

II. Model: General Lotto Games

We first introduce the classic General Lotto game with complete information. We then formulate our model of one-sided incomplete and asymmetric information on resource budgets, which we term "Bernoulli General Lotto" games.

A. Classic General Lotto games

A (complete information) General Lotto game consists of two players, $\mathcal{A}$ and $\mathcal{B}$. Each player is tasked with allocating their resource budgets $A, B > 0$ across a set of n battlefields. Each battlefield has an associated value $v_j > 0$, $j \in [n] := \{1,2,\ldots,n\}$. Without loss of generality, their sum total is normalized to one, $\sum_{j\in[n]} v_j = 1$. An allocation for $\mathcal{A}$ is any vector $\boldsymbol{x}_\mathcal{A} \in \mathbb{R}^n_{\geq 0}$, and similarly for $\mathcal{B}$. An admissible strategy for $\mathcal{A}$ is a randomization $F_\mathcal{A}$ over allocations such that the expended resources do not exceed the budget A in expectation. Specifically, $F_\mathcal{A}$ is an n-variate (cumulative) distribution that belongs to the family

$$\mathbb{F}(A) \triangleq \left\{ F : \mathbb{E}_{\boldsymbol{x}_\mathcal{A}\sim F}\left[\sum_{j=1}^{n} x_{\mathcal{A},j}\right] \leq A \right\}. \tag{1}$$

and similarly, $F_\mathcal{B} \in \mathbb{F}(B)$. Given a strategy profile $(F_\mathcal{A}, F_\mathcal{B})$, the utility of player $\mathcal{A}$ is

$$u_\mathcal{A}(F_\mathcal{A}, F_\mathcal{B}) \triangleq \mathbb{E}_{\substack{\boldsymbol{x}_\mathcal{A}\sim F_\mathcal{A}\\ \boldsymbol{x}_\mathcal{B}\sim F_\mathcal{B}}}\left[\sum_{j=1}^{n} v_j \cdot \mathbb{1}\{x_{\mathcal{A},j} > x_{\mathcal{B},j}\}\right] \tag{2}$$

where $\mathbb{1}\{\cdot\}$ is 1 if the statement in the bracket is true, and 0 otherwise[1]. It follows that the utility of player $\mathcal{B}$ is

$$u_\mathcal{B}(F_\mathcal{A}, F_\mathcal{B}) \triangleq 1 - u_\mathcal{A}(F_\mathcal{A}, F_\mathcal{B}) \tag{3}$$

The unique equilibrium payoffs in General Lotto games are well-established in the literature [11], [18]:

$$\pi^{\text{CI}}_\mathcal{A}(A,B) \triangleq \begin{cases} \frac{A}{2B}, & \text{if } A < B \\ 1 - \frac{B}{2A}, & \text{if } A \geq B \end{cases} \tag{4}$$

and the equilibrium payoff of player $\mathcal{B}$ is simply $\pi^{\text{CI}}_\mathcal{B}(A,B) = 1 - \pi^{\text{CI}}_\mathcal{A}(A,B)$. Note that the equilibrium payoffs do not depend on the particular battlefield values $v_1, v_2, \ldots, v_n$. They only depend on the total value $\sum_i v_i = 1$, which is a common feature in General Lotto games [11], [18]. We write $\text{GL}(A,B)$ to denote an instance of the complete information General Lotto game.

[1]An arbitrary tie-breaking rule may be selected, without changing our results. This is generally true in General Lotto games [18]. For simplicity, we will assume ties are awarded to player $\mathcal{B}$.

B. Bernoulli General Lotto games

Before play, the budget of player $\mathcal{A}$ is drawn according to a Bernoulli distribution. With probability $p \in [0,1]$, player $\mathcal{A}$ is endowed with a high budget $A^h \geq 0$, and with probability $1-p$, it is endowed with a low budget $A^\ell \geq 0$, where $A^\ell \leq A^h$. We denote this Bernoulli distribution as $\mathbb{P}_\mathcal{A} = (A^h, A^\ell, p)$. The realized budget is private information to player $\mathcal{A}$. An admissible action is thus a pair of strategies $\vec{F}_\mathcal{A} = \{F^h, F^\ell\} \in \mathbb{F}(A^h) \times \mathbb{F}(A^\ell)$, where F^h or F^ℓ is implemented conditional on which budget is realized. Player $\mathcal{B}$ does not observe the true realization, but has knowledge about its distribution p. It thus selects a single strategy $F_\mathcal{B} \in \mathbb{F}(B)$ to implement regardless of which of $\mathcal{A}$'s budget is realized. Given a strategy profile $(\vec{F}_\mathcal{A}, F_\mathcal{B})$, the ex-ante expected utility to $\mathcal{A}$ and $\mathcal{B}$ are:

$$\begin{aligned} U_\mathcal{A}(\vec{F}_\mathcal{A}, F_\mathcal{B}) &\triangleq p \cdot u_\mathcal{A}(F^h_\mathcal{A}, F_\mathcal{B}) + (1-p) \cdot u_\mathcal{A}(F^\ell_\mathcal{A}, F_\mathcal{B}) \\ U_\mathcal{B}(\vec{F}_\mathcal{A}, F_\mathcal{B}) &\triangleq 1 - U_\mathcal{A}(\vec{F}_\mathcal{A}, F_\mathcal{B}) \end{aligned} \tag{5}$$

We will refer to the simultaneous-move game with the above expected utilities as a *Bernoulli General Lotto game*, and denote an instance with $\text{BL}(\mathbb{P}_\mathcal{A}, B)$. This is a Bayesian game extension of the classic Lotto game. In the case that the support of $\mathbb{P}_\mathcal{A}$ is a singleton, it becomes a game of complete information, i.e. reduces to the classic General Lotto game $\text{GL}(A, B)$. A strategy profile $(\vec{F}^*_\mathcal{A}, F^*_\mathcal{B})$ is an *equilibrium* of $\text{BL}(\mathbb{P}, B)$ if

$$U_\mathcal{A}(\vec{F}^*_\mathcal{A}, F^*_\mathcal{B}) \geq U_\mathcal{A}(\vec{F}_\mathcal{A}, F^*_\mathcal{B}) \text{ and } U_\mathcal{B}(\vec{F}^*_\mathcal{A}, F^*_\mathcal{B}) \geq U_\mathcal{B}(\vec{F}^*_\mathcal{A}, F_\mathcal{B}) \tag{6}$$

for any $\vec{F}_\mathcal{A} \in \mathbb{F}(A^h) \times \mathbb{F}(A^\ell)$ and $F_\mathcal{B} \in \mathbb{F}(B)$.

III. Equilibrium payoffs in Bernoulli Lotto games

The following Theorem is the main contribution of the paper, which fully characterizes the players' equilibrium payoffs in the Bernoulli Lotto game. Recall that in the BL game, player $\mathcal{B}$ is the information-disadvantaged player. These payoffs thus quantify the degradation in performance for player $\mathcal{B}$ as a result from not having full knowledge about the strength of player $\mathcal{A}$ (see Figure 2). Analogously, from the perspective of player $\mathcal{A}$, the equilibrium payoff reflects its improvement in performance due to information asymmetry.

Theorem 3.1. *The equilibrium payoff to player $\mathcal{A}$ in the game $\text{BL}(\mathbb{P}_\mathcal{A}, B)$ is*

$$\pi_\mathcal{A}(\mathbb{P}_\mathcal{A}, B) \triangleq \begin{cases} \frac{\bar{A}}{2B}, & (A^h, A^\ell) \in \mathcal{R}_1 \\ 1 - \frac{B}{2\bar{A}}, & (A^h, A^\ell) \in \mathcal{R}_2 \\ p + (1-p)\left(1 - \frac{B}{2A^\ell}\right), & (A^h, A^\ell) \in \mathcal{R}_3 \\ p + (1-p)\frac{A^\ell}{2B}, & (A^h, A^\ell) \in \mathcal{R}_4 \\ p + (1-p)\frac{A^\ell}{A^h} + \frac{\sqrt{\bar{A}(\bar{A}-pA^h)}}{A^h} \\ \qquad - \frac{B\left(\sqrt{(1-p)A^\ell} + \sqrt{\bar{A}}\right)^2}{2(A^h)^2}, & (A^h, A^\ell) \in \mathcal{R}_5 \end{cases} \tag{7}$$

where $\bar{A} \triangleq pA^h + (1-p)A^\ell$, and the $\mathcal{R}_k$ $(k = 1, \dots, 5)$ are disjoint subsets of $\mathcal{R} \triangleq \{(A^h, A^\ell) \in \mathbb{R}^2_{\geq 0} : A^h \geq A^\ell\}$ defined by

$$\begin{aligned} \mathcal{R}_1 &\triangleq \{\bar{A} \leq B\} \setminus \mathcal{R}_5 \\ \mathcal{R}_2 &\triangleq \left\{\bar{A} \geq B \textit{ and } A^\ell \geq \frac{1-p}{2-p}A^h\right\} \\ \mathcal{R}_3 &\triangleq \left\{A^h \geq \left(2 + \frac{p}{1-p}\right)B \textit{ and } A^\ell \in \left[B, \frac{1-p}{2-p}A^h\right)\right\} \\ \mathcal{R}_4 &\triangleq \left\{A^h \geq \left(2 + \frac{p}{1-p}\right)B \textit{ and } \right. \\ &\qquad\qquad \left. A^\ell \in \left[\frac{pB^2}{(1-p)(A^h - 2B)}, B\right)\right\} \\ \mathcal{R}_5 &\triangleq \{A^\ell \leq B \cdot H(A^h/B)\} \end{aligned} \tag{8}$$

with the function $H(a)$ defined as

$$H(a) \triangleq \begin{cases} 0, & \textit{if } a \in [0,1) \\ \frac{p(a-1)^2}{(1-p)(2-a)}, & \textit{if } a \in [1, 2-p) \\ \frac{1-p}{2-p}a, & \textit{if } a \in \left[2-p, 2+\frac{p}{1-p}\right) \\ \frac{p}{(1-p)(a-2)}, & \textit{if } a \geq 2 + \frac{p}{1-p} \end{cases} \tag{9}$$

The equilibrium payoff to player $\mathcal{B}$ is given by $\pi_\mathcal{B}(\mathbb{P}_\mathcal{A}, B) = 1 - \pi_\mathcal{A}(\mathbb{P}_\mathcal{A}, B)$.

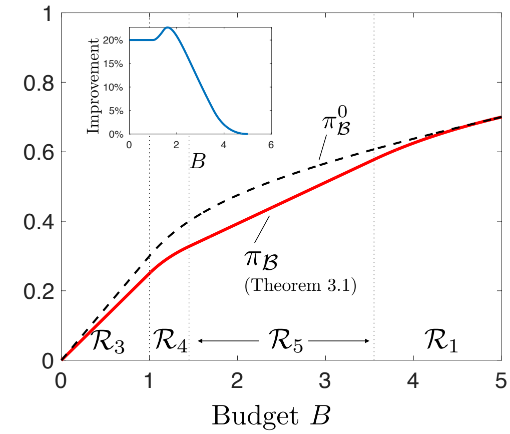

Fig. 2. The equilibrium payoff to player $\boldsymbol{\mathcal{B}}$ in the Bernoulli Lotto game is shown as the solid red line (Theorem 3.1). The dashed black line indicates the equilibrium payoff to player $\boldsymbol{\mathcal{B}}$ when it has full knowledge about player $\boldsymbol{\mathcal{A}}$'s budget type (10). The inset plot shows the percent improvement in payoff that player $\boldsymbol{\mathcal{B}}$ experiences as a result of obtaining full knowledge, i.e. $\mathbf{100} \times (\frac{\pi^0_\mathcal{B}}{\pi_\mathcal{B}} - \mathbf{1})$. In this example, $\boldsymbol{A^h = 5}$, $\boldsymbol{A^\ell = 1}$, $\boldsymbol{p = 0.5}$. In this example, there is up to a $\mathbf{23\%}$ payoff improvement, approximately occurring at $\boldsymbol{B = 2}$. The improvement degrades as player $\boldsymbol{\mathcal{B}}$ becomes stronger.

We devote Section V to the proof of Theorem 3.1, which details the players' equilibrium strategies and contains a diagram of the five regions (8). The equilibrium payoffs depend on the total value of all battlefields and not on any of them individually, which is a common feature in General Lotto games. The result simply generalizes to $\|\boldsymbol{v}\|_1 \cdot \pi_\mathcal{A}$ when the total value of all battlefields is not normalized to 1, where $\|\boldsymbol{v}\|_1$ is the ℓ_1 norm of the vector $\boldsymbol{v}$.

The characterization in Theorem 3.1 allows us to assess the potential performance improvement that player $\mathcal{B}$ can experience as a result from obtaining full information about player $\mathcal{A}$'s budget. Indeed, we highlight the impact of information asymmetry by comparing the equilibrium payoffs in $\mathrm{BL}(\mathbb{P}_\mathcal{A}, B)$ to the scenario where player $\mathcal{B}$ is fully informed about the realization of player $\mathcal{A}$'s budget, i.e. the budget type is public. In the latter scenario, the equilibrium payoff to player $\mathcal{B}$ will simply be:

$$\pi_\mathcal{B}^0 \triangleq p \cdot \pi_\mathcal{B}^{\mathrm{CI}}(A^h, B) + (1-p) \cdot \pi_\mathcal{B}^{\mathrm{CI}}(A^\ell, B) \tag{10}$$

where $\pi_\mathcal{B}^{\mathrm{CI}}$ was defined in (4). It immediately follows that $\pi_\mathcal{B}^0 \geq \pi_\mathcal{B}$, since in this scenario, player $\mathcal{B}$ would have type-dependent strategies (5), and thus can only benefit from the public information.

Figure 2 illustrates player $\mathcal{B}$'s equilibrium payoff in the BL game as compared to its payoff under public budget types (10). The benefit that information provides player $\mathcal{B}$ is shown in the inset figure – the benefit that information provides degrades as $\mathcal{B}$ becomes stronger, and it is highest when $\mathcal{B}$ is weak relative to player $\mathcal{A}$. Indeed, we see that the payoffs $\pi_\mathcal{B}$ (7) and $\pi_\mathcal{B}^0$ (10) coincide when $B = 0$ and when $B \geq A^h$ (occuring in $\mathcal{R}_1$ region). Player $\mathcal{B}$ benefits most from information for a budget value B between these two extremes, as we observe a peak at $B \approx 2$ in this example. A comparison of the form of Figure 2 allows one to weigh the trade-offs between the costs and performance benefits from acquiring information about an opponent's strength.

IV. Commander Assignment Problem: The Value of Randomized Policies

In this section, we apply our equilibrium solutions of BL games to study a resource assignment problem in which a central commander assigns resources to two sub-colonels in order to compete against respective opponents. The primary goal here is to quantify the performance improvement that randomized assignment policies can yield over deterministic assignment policies for the commander.

A. Commander assignment problem

In the following, we define a general model referred to as the "commander assignment problem", which incorporates both per-unit costs for investment as well as a fixed budget limit. Such setups are commonly considered in the literature on Colonel Blotto games [19]. We note, however, that our main analytical results in this section pertain only to two specific formulations of this general setup, wherein the players either have no budget limits (but non-zero costs), or they have zero costs but are budget-limited.

Recall the scenario depicted in Figure 1 (Right) where a commander is responsible for assigning resources to two sub-colonels $\mathcal{A}_1, \mathcal{A}_2$ engaged in separate competitions against respective opponents $\mathcal{B}_1, \mathcal{B}_2$. In this setup, there are limited resource budgets $\bar{A}_C, \bar{B}_1, \bar{B}_2$, as well as per-unit resource costs $c, c_1, c_2 \geq 0$ for the commander and opponents, respectively. The interaction unfolds in the following three-stage extensive-form game, which is also illustrated in Figure 3.

Stage 1: The commander chooses an assignment policy $\mathbb{P}$, which is a distribution on the sub-colonels' resource assignments $(A_1, A_2) \in \mathbb{R}^2_{\geq 0}$. The strategy becomes common

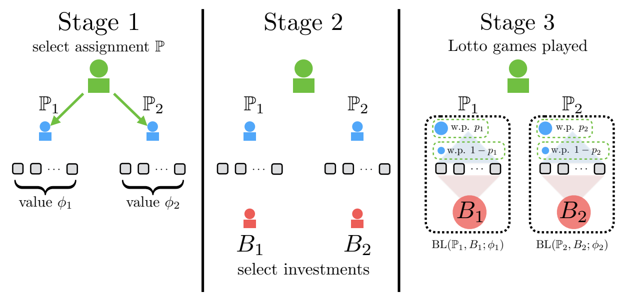

Fig. 3. The three-stage commander assignment problem. In stage 1, the commander decides how to assign resources to two sub-colonels. The assignment policy $\mathbb{P}$ is either randomized or deterministic. Each sub-colonel $i \in \{1, 2\}$ will use its assigned resources to compete over a set of battlefields of value ϕ_i. In Stage 2, the two opponents observe the assignment policy $\mathbb{P}$, but not the actual realizations (if randomized). The opponents invest in resource budgets B_1, B_2, for which they pay a cost $c_i B_i$. In Stage 3, two Bernoulli Lotto games are played between each sub-colonel and opponent using their endowed resources determined in Stages 1 and 2. The commander's payoff is the sum of sub-colonels' equilibrium payoffs in their respective games minus the expected cost to assign the resources from Stage 1 (12).

knowledge to the sub-colonels and the opponents. We say that $\mathbb{P}$ is a *randomized* policy if its support is not a singleton, and that it is *deterministic* otherwise.

We will consider distributions $\mathbb{P}$ such that each marginal distribution $\mathbb{P}_i$ on resource assignments to sub-colonel $\mathcal{A}_i$ has at most two values in its support[2]. The marginal $\mathbb{P}_i$ is thus associated with a Bernoulli distribution $\mathbb{P}_i = (A_i^h, A_i^\ell, p_i)$. Furthermore, we will consider assignment strategies that satisfy the commander's budget *in expectation*:

$$\mathbb{E}_\mathbb{P}[A_1 + A_2] = \sum_{(A_1, A_2) \in \mathrm{supp}(\mathbb{P})} \mathbb{P}(A_1, A_2) \cdot (A_1 + A_2) \leq \bar{A}_C. \tag{11}$$

Let us denote $\mathcal{P}(\bar{A}_C)$ as the set of feasible policies $\mathbb{P}$ where both of its marginals are Bernoulli and it satisfies the above condition.

Stage 2: After observing the assignment policy $\mathbb{P}$ from Stage 1, each opponent individually decides an amount of resources to invest in, $B_1 \leq \bar{B}_1$ and $B_2 \leq \bar{B}_2$.

Stage 3: Two independent Bernoulli Lotto games are played simultaneously: $\mathcal{G}_1 = \mathrm{BL}(\mathbb{P}_1, B_1; \phi_1)$ between sub-colonel $\mathcal{G}_2 = \mathcal{A}_1$ and opponent $\mathcal{B}_1$, and $\mathrm{BL}(\mathbb{P}_2, B_2; \phi_2)$ between sub-colonel $\mathcal{A}_2$ and opponent $\mathcal{B}_2$. Here, $\phi_i > 0$ indicates the total value of the set of battlefields contested in $\mathcal{G}_i$, and $\mathbb{P}_i$ are the marginal distributions of the assignment policy. The final payoff that the commander obtains is given by the sub-colonels' cumulative equilibrium payoffs from $\mathcal{G}_1$ and $\mathcal{G}_2$ minus the costs of resource expenditure:

$$W(\mathbb{P}, B_1, B_2) \triangleq \sum_{i=1,2} \phi_i \cdot \pi_\mathcal{A}(\mathbb{P}_i, B_i) - c\mathbb{E}_\mathbb{P}[A_1 + A_2] \tag{12}$$

[2]We focus on this class of randomized assignment strategies since our results on BL games applies to two-type budget uncertainty. It is of interest to extend these results to more than two randomized budget levels. We note there are analytical challenges associated with these extensions, as outlined in Section V.

The final payoff that opponent $\mathcal{B}_i$ obtains is

$$U_i(\mathbb{P}, B_i, c_i) \triangleq \phi_i \cdot \pi_{\mathcal{B}}(\mathbb{P}_i, B_i) - c_i B_i. \tag{13}$$

The above extensive-form game will be referred to as $\text{CAP}(\bar{A}_C, c, \{\bar{B}_i, c_i, \phi_i\}_{i=1,2})$. In order to elicit the benefit from utilizing randomized policies $\mathcal{P}$, we draw attention to comparisons of the commander's performance in CAP to the scenario in which the commander only has access to deterministic assignment policies. These are the set of all $\mathbb{P} \in \mathcal{P}(\bar{A}_C)$ such that both marginals $\mathbb{P}_i$ are singletons. With this restriction, we refer $\text{CAP}_\text{d}(\bar{A}_C, c, \{\bar{B}_i, c_i, \phi_i\}_{i=1,2})$ as the extensive-form game where the commander is restricted to deterministic policies. In addition, we focus on the following two particular settings on the cost parameters.

- The *fixed budget setting*. Here, there are zero costs associated with using resources, i.e. $c = c_i = 0$, and $\bar{A}_C, \bar{B}_i < \infty$. The commander assignment problem CAP_d under this setting was first featured in [15], where optimal deterministic assignments are provided.
- The *per-unit cost setting*. Here, $c, c_1, c_2 > 0$ and there are no resource budget limits, i.e. $\bar{A}_C, \bar{B}_i = \infty$. Formulations of Colonel Blotto games are commonly analyzed under similar linear cost models [13], [16], [17], [19], but have not been considered in the commander assignment problem.

To the best of our knowledge, the commander assignment problem with randomized policies (in either setting) is novel to the literature. The sub-game perfect equilibrium (SPE) can be derived via backwards induction by leveraging the fact that the final payoffs in Stage 3 are computed directly from Theorem 3.1, and the opponents seek to maximize (13) in Stage 2 given any fixed policy $\mathbb{P} \in \mathcal{P}(\bar{A}_C)$. If there are multiple maximizers of (13), we will assume $\mathcal{B}_i$ chooses the smallest investment level among them.

B. Results: the value of randomization

We present our results concerning the comparison of the commander's performance in CAP and CAP_d. Denote W^* as the SPE payoff to the commander from CAP (randomized assignments) and W_d^* as its SPE payoff from CAP_d (deterministic assignments).

Theorem 4.1. *The following statements hold.*

- *Under the fixed budget setting, i.e. $c = c_i = 0$ and $\bar{A}_C, \bar{B}_i < \infty$, it holds that $W^* = W_d^*$.*
- *Under the per-unit cost setting, i.e. $c, c_i > 0$ and $\bar{A}_C, \bar{B}_i = \infty$, it holds that*

$$0 < W_d^* \leq W^* \leq 4 \cdot W_d^*. \tag{14}$$

The equality $W^* = 4 \cdot W_d^*$ holds if and only if $c > \max\{\frac{1}{2}\sqrt{\frac{c_j}{\phi_j}(c_1\phi_1 + c_2\phi_2)}, c_k(1+\sqrt{3}/2)\}$, where $j = \arg\min\limits_{i=1,2} \frac{\phi_i}{2c_i}$ and $k = \arg\max\limits_{i=1,2} c_i$.

The proof is provided in Appendices A and B. The first statement in Theorem 4.1 asserts that, surprisingly, any randomized resource assignment can do no better than the optimal deterministic assignment under the fixed budget setting. A characterization of the optimal deterministic assignment is available from [15]. Thus, the commander cannot profitably exploit informational asymmetries in this scenario.

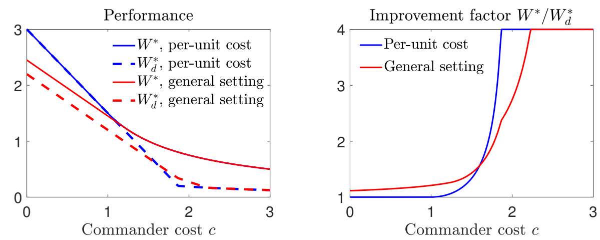

Fig. 4. (Left) Comparison of commander payoffs from randomized and deterministic assignments. The red lines depict performance under a general parameter setting, where equilibrium is computed numerically using the characterizations from Appendices A and B. The blue lines depict performance under the per-unit cost setting, which are analytically characterized. (Right) The improvement factor, $\boldsymbol{W^*/W_\text{d}^*}$. In any setting, the commander can attain up to a four-fold improvement, which is possible when the cost of resources for the commander are sufficiently high. The parameters for the general setting in this example are: $\boldsymbol{\bar{A}_C = 1}$, $\boldsymbol{\bar{B}_i = 0.4}$, $\boldsymbol{c_1 = c_2 = 1}$, $\boldsymbol{\phi_1 = 1}$, $\boldsymbol{\phi_2 = 2}$.

The second statement of Theorem 4.1 asserts that the commander can attain up to a four-fold performance improvement over deterministic assignments when there are non-zero marginal costs involved. The four-fold improvement is attainable in a regime where the commander's per-unit cost is sufficiently high, i.e. when it is expensive to assign resources to the sub-colonels.

In general parameter settings, there can be both positive costs and finite resource budgets, i.e. $c, c_i > 0$ and $A, B_i < \infty$. While we do not provide full solutions in the most generality, we note that their solutions can still coincide with those from the per-unit cost setting or from the fixed budget setting. In particular, the solution will coincide with the per-unit cost setting if the cost c is sufficiently high because the commander will not utilize all of its available resources $\bar{A}_C$. The solution will coincide with the fixed budget setting if all costs are sufficiently low, such that all available resources $\bar{A}_C, \bar{B}_i$ are expended.

Figure 4 shows performance levels in an example setup of this general context. Since the commander in this example has a budget limit of $\bar{A}_C = 1$, it performs worse for lower costs c in comparison to the per-unit cost setting in which it has unlimited budget. We also show a comparison of the improvement factors in the general setting and in the per-unit cost setting. We observe that the performance improvement factor is also upper-bounded by 4 in the general setting. This can only be attained (in both models) for sufficiently high costs c associated with the commander's resources.

V. Proof of Theorem 3.1

This section is devoted to the proof of Theorem 3.1 – the characterization of equilibrium payoffs in the Bernoulli General Lotto game. First it is important to note that the seminal results for (complete information) Blotto and Lotto games first recognized connections to all-pay auctions [18], [27]. It leveraged known equilibria of all-pay auctions to derive the equilibria to Blotto and Lotto games. In our formulation, we leverage equilibrium strategies of incomplete information all-pay auctions [30] to set up first-order conditions for the BL game (subsections V-A, V-B, V-C). This forms a system of non-linear equations associated with the players' expected budget constraints (1). The equations can be completely solved to verify that they constitute equilibria to the BL game (Proposition 5.1), but only for a subset of game parameters (region $\mathcal{R}_5$). We complete the proof by identifying equilibria to the remaining regions $\mathcal{R}_i$, $i = 1, \ldots, 4$. These details are given in subsection V-D. The uniqueness of equilibrium payoffs follows from the BL game being constant-sum in ex-ante utilities.

A. The connection to all-pay auctions

Consider a game instance BL$(\mathbb{P}_\mathcal{A}, B)$. Player $\mathcal{A}$'s ex-interim constrained optimization, given type $t \in \{h, \ell\}$ is realized, can be written as

$$\max_{\{F^t_{\mathcal{A},j}\}_{j\in[n]}} \sum_{j\in[n]} \int_0^\infty \left[v_j F_{B,j}(x_{\mathcal{A},j}) - \lambda^t x_{\mathcal{A},j}\right] dF^t_{\mathcal{A},j} + \lambda^t A^t \tag{15}$$

where λ^t is the multiplier on player $\mathcal{A}$'s expected budget constraint for type $t \in \{h, \ell\}$ (1). Here, $dF^t_{\mathcal{A},j}$ refers to the differential $f^t_{\mathcal{A},j}(x)dx$ where $f^t_{\mathcal{A},j}$ is the corresponding density function. Player $\mathcal{B}$'s constrained optimization is written as

$$\max_{\{F_{B,j}\}_{j\in[n]}} \sum_{j\in[n]} \sum_{t=h,\ell} p^t \int_0^\infty [v_j F^t_{\mathcal{A},j}(x_{B,j}) - \lambda_\mathcal{B} x_{B,j}] dF_{B,j} + \lambda_\mathcal{B} B. \tag{16}$$

where we denote $p^h = p$ and $p^\ell = 1 - p$, and $\lambda_\mathcal{B}$ is the multiplier on player $\mathcal{B}$'s budget constraint. We observe that the above are separable in the elements of the decision variables, e.g. $F^t_{\mathcal{A},j}$ for each $j \in [n]$, and thus may be treated as n independent optimization problems. For $\mathcal{A}$'s problem, a best-response is any $x_{\mathcal{A},j}$ that maximizes $v_j F_{B,j}(x_{\mathcal{A},j}) - \lambda^t x_{\mathcal{A},j}$. In an equilibrium, player $\mathcal{A}$'s strategy $F^t_{\mathcal{A},j}$ places support on all such $x_{\mathcal{A},j}$. Similar conditions must simultaneously hold for player $\mathcal{B}$'s problem and strategy $F_{\mathcal{B},j}$. Consequently, the necessary first-order conditions for equilibrium are

$$\begin{aligned} &\frac{d}{dx_{\mathcal{A},j}}\left[\frac{v_j}{\lambda^t} F_B(x_{\mathcal{A},j}) - x_{\mathcal{A},j}\right] = 0, \quad t = h, \ell \\ &\frac{d}{dx_{B,j}}\left[\sum_{t=h,\ell} p^t \left(\frac{v_j}{\lambda_\mathcal{B}} F^t_{\mathcal{A},j}(x_{B,j}) - x_{B,j}\right)\right] = 0 \end{aligned} \tag{17}$$

for each $j \in [n]$, where we denote $p^h = p$ and $p^\ell = 1 - p$. The conditions (17) are analogous to the conditions stated for complete information Lotto games in [18]. However, the difference in our condition is that player $\mathcal{A}$ has two distinct types, thus requiring optimization for each type t.

Here, we have divided by the associated (positive) multiplier in each condition, since this will not change the optimal choices. The above conditions now coincide with the necessary first-order conditions for equilibrium for n independent two-player first-price all-pay auctions with incomplete and asymmetric information for which player $\mathcal{A}$'s valuation for the item in auction j in type $t \in \{h, \ell\}$ is $\frac{v_j}{\lambda^t}$, and player $\mathcal{B}$'s valuation for the item is $\frac{v_j}{\lambda_\mathcal{B}}$. The equilibrium strategies for each of the n all-pay auctions can be characterized using results from the economics literature [30]. These equilibria are detailed in the next subsection.

B. Equilibrium strategies of all-pay auctions

We summarize the equilibrium strategies to a two-player all-pay auction with incomplete and asymmetric information. These results are derived from [30]. Define

$$\bar{k} \triangleq \begin{cases} 1, & \text{if } p\frac{w_\mathcal{B}}{w^h_\mathcal{A}} \geq 1 \\ 2, & \text{if } p\frac{w_\mathcal{B}}{w^h_\mathcal{A}} + (1-p)\frac{w_\mathcal{B}}{w^\ell_\mathcal{A}} \geq 1 \text{ and } p\frac{w_\mathcal{B}}{w^h_\mathcal{A}} < 1 \\ 3, & \text{if } p\frac{w_\mathcal{B}}{w^h_\mathcal{A}} + (1-p)\frac{w_\mathcal{B}}{w^\ell_\mathcal{A}} < 1 \end{cases} \tag{18}$$

where $w^t_\mathcal{A} \geq 0$ is player $\mathcal{A}$'s valuation of the item given type $t \in \{h, \ell\}$, which satisfies $w^h_\mathcal{A} \geq w^\ell_\mathcal{A}$. Player $\mathcal{B}$'s valuation of the item is $w_\mathcal{B}$. Also, define

$$\begin{aligned} L^h &\triangleq \begin{cases} w^h_\mathcal{A}, & \text{if } \bar{k} = 1 \\ pw_\mathcal{B}, & \text{if } \bar{k} \in \{2, 3\} \end{cases} \\ L^\ell &\triangleq \begin{cases} 0, & \text{if } \bar{k} = 1 \\ w^\ell_\mathcal{A}\left(1 - p\frac{w_\mathcal{B}}{w^h_\mathcal{A}}\right), & \text{if } \bar{k} = 2 \\ (1-p)w_\mathcal{B}, & \text{if } \bar{k} = 3 \end{cases} \\ L &\triangleq L^h + L^\ell. \end{aligned} \tag{19}$$

The L^t are interval lengths where the equilibrium strategies have support. Below, we provide explicit expressions for the equilibrium strategies.

Lemma 5.1 ([30]). *The equilibrium mixed strategies for a two-player first-price all-pay auction with two-type asymmetric information are given as follows*[3]*:*

If $\bar{k} = 1$*:*

$$\begin{aligned} &F^h_\mathcal{A} = \left(1 - \frac{L^h}{pw_\mathcal{B}}\right)\boldsymbol{\delta}_0 + \frac{L^h}{pw_\mathcal{B}}\text{Unif}(0, L^h), \quad F^\ell_\mathcal{A} = \boldsymbol{\delta}_0 \\ &F_\mathcal{B} = \text{Unif}(0, L^h) \end{aligned} \tag{20}$$

If $\bar{k} = 2$*:*

$$\begin{aligned} &F^h_\mathcal{A} = \text{Unif}(L^\ell, L) \\ &F^\ell_\mathcal{A} = \left(1 - \frac{L^\ell}{(1-p)w_\mathcal{B}}\right)\boldsymbol{\delta}_0 + \frac{L^\ell}{(1-p)w_\mathcal{B}}\text{Unif}(0, L^\ell) \\ &F_\mathcal{B} = \frac{L^\ell}{w^\ell_\mathcal{A}}\text{Unif}(0, L^\ell) + \frac{L^h}{w^h_\mathcal{A}}\text{Unif}(L^\ell, L) \end{aligned} \tag{21}$$

If $\bar{k} = 3$*:*

$$\begin{aligned} &F^h_\mathcal{A} = \text{Unif}(L^\ell, L), \quad F^\ell_\mathcal{A} = \text{Unif}(0, L^\ell) \\ &F_\mathcal{B} = \left(1 - \sum_{t=h,\ell}\frac{L^t}{w^t_\mathcal{A}}\right)\boldsymbol{\delta}_0 + \frac{L^\ell}{w^\ell_\mathcal{A}}\text{Unif}(0, L^\ell) + \\ &\qquad + \frac{L^h}{w^h_\mathcal{A}}\text{Unif}(L^\ell, L) \end{aligned} \tag{22}$$

[3]To simplify exposition and notation where convenient, we sometimes explicitly write CDFs as a mixture of uniform and point mass distributions. We write $\text{Unif}(a, b) = \mathbf{1}(x \geq a)\min\{\frac{x}{b-a}, 1\}$ to represent the CDF of the uniform distribution on (a, b) and $\boldsymbol{\delta}_0 := \mathbf{1}(x \geq 0)$ for the CDF of a point mass centered at zero.

In summary, the equilibrium strategies for player $\mathcal{A}$ are a pair of univariate distributions on bids $(F_{\mathcal{A}}^h, F_{\mathcal{A}}^\ell)$, where the shifted support of $F_{\mathcal{A}}^h$ indicates higher effort. Player $\mathcal{B}$'s equilibrium strategy is a single univariate distributions on bids $F_{\mathcal{B}}$, which is piecewise uniform.

C. Equilibria in the $\mathcal{R}_5$ region

We now propose a collection of strategies as a candidate equilibrium for the Bernoulli Lotto game. Recall from (17) that the item valuations in all-pay auction j are given by $w_{\mathcal{A}}^t = v_j/\lambda^t > 0$ for player $\mathcal{A}$ and $w_{\mathcal{B}} = v_j/\lambda_{\mathcal{B}} > 0$ for player $\mathcal{B}$, in type $t \in \{h, \ell\}$. From Lemma 5.1, we have equilibrium marginal distributions for each auction j. We naturally impose the ranking $\lambda^h \leq \lambda^\ell$, which ensures that $w_{\mathcal{A}}^h \geq w_{\mathcal{A}}^\ell$. Since the multipliers $\boldsymbol{\lambda} = (\lambda^\ell, \lambda^h, \lambda_{\mathcal{B}})$ are variables, it is unknown a priori which of the three cases in Lemma 5.1 to use. For now, given a set of multipliers $\boldsymbol{\lambda} = (\lambda^\ell, \lambda^h, \lambda_{\mathcal{B}})$, let us denote the resulting distributions from Lemma 5.1 as $F_{\mathcal{A},j}^{t,\boldsymbol{\lambda}}$ and $F_{\mathcal{B},j}^{\boldsymbol{\lambda}}$ for each $j \in [n]$ and $t \in \{h, \ell\}$. In the Bernoulli Lotto game, the expected budget constraints must be met (1):

$$\begin{aligned} &\sum_{j\in[n]} \mathbb{E}_{x_{\mathcal{A},j}\sim F_{\mathcal{A},j}^{t,\boldsymbol{\lambda}}}[x_{\mathcal{A},j}] = A^t, \quad t = h, \ell \\ &\sum_{j\in[n]} \mathbb{E}_{x_{B,j}\sim F_B^{\boldsymbol{\lambda}}}[x_{B,j}] = B \\ &\text{such that } 0 < \lambda^h \leq \lambda^\ell \end{aligned} \tag{23}$$

This yields a system of three equations that we may use to solve for the multipliers $\boldsymbol{\lambda}$. The value of $\bar{k}$ is not known a priori, as it now depends on $\boldsymbol{\lambda}$. The values it can take, $\bar{k} \in \{1, 2, 3\}$, correspond to transformed multipliers $\boldsymbol{\sigma} = (\sigma^h, \sigma^\ell)$, with $\sigma^t := \frac{\lambda^t}{\lambda_{\mathcal{B}}} > 0$ for $t \in \{h, \ell\}$, lying in three disjoint regions of $\mathbb{R}_+^2$. These regions result directly from (18), and are given below.

$$\begin{aligned} \bar{k} &= 1, \text{ if } \quad p\sigma^h \geq 1 \\ \bar{k} &= 2, \text{ if } \quad p\sigma^h < 1 \text{ and } p\sigma^h + (1-p)\sigma^\ell \geq 1 \\ \bar{k} &= 3, \text{ if } \quad p\sigma^h + (1-p)\sigma^\ell < 1 \end{aligned} \tag{24}$$

The constructed strategies can take one of three different forms described in Lemma 5.1, contingent on the value of $\bar{k}$. Thus, there are three distinct forms that the system of equations (23) can take, which are given in ($\star$) (top of next page).

Observe that the individual battlefield values v_j do not appear in these equations. In fact, one would arrive to the system ($\star$) when considering Lotto games with any number $n \geq 1$ of battlefields whose total value is normalized to one. The individual battlefield values do not play a role in the analysis – only their total value. This is a common feature in the analysis of General Lotto games, given the independence of the strategies' marginal distributions [16], [18]. For simplified exposition, we will henceforth consider players' strategies as allocations to a single battlefield of value one ($F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ with no j dependence).

Below, we detail the complete solutions to ($\star$), and prove their associated strategies (from (20)) constitute equilibria to the BL game only in the $\mathcal{R}_5$ parameter region (Figure 5).

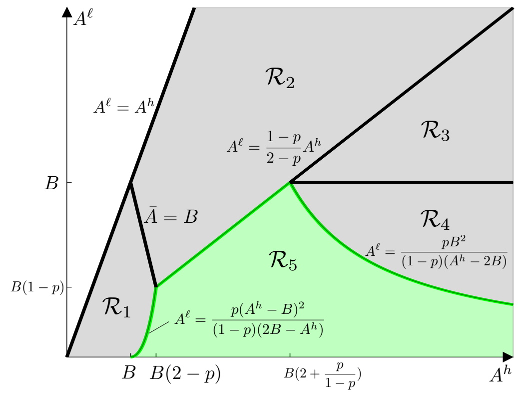

Fig. 5. The five distinct parameter regions (8) that constitute equilibrium characterizations of Bayesian Lotto games BL($\mathbb{P}_{\mathcal{A}}, B$). In this diagram, B and p are fixed.

Proposition 5.1. *Each solution $(\sigma^h, \sigma^\ell, \lambda_{\mathcal{B}})$ to the system ($\star$) corresponds to a particular game instance BL($\mathbb{P}_{\mathcal{A}}, B$), with equilibrium strategies and payoffs given as follows. Define $a^t \triangleq \frac{A^t}{B}$ for $t \in \{h, \ell\}$.*

Case 1: *The solution to ($\star$) is given by $\lambda^h = \frac{1}{2B}$, $\lambda_{\mathcal{B}} = \frac{pA^h}{2B^2}$, and $\lambda^\ell \geq \frac{1}{2B}$ in Case 1. The equilibrium strategies for $a^h \leq 1$ and $a^\ell = 0$ are*

$$\begin{aligned} &F_{\mathcal{A}}^h = (1-a^h)\boldsymbol{\delta}_0 + a^h\text{Unif}(0, 2B), \quad F_{\mathcal{A}}^\ell = \boldsymbol{\delta}_0, \\ &F_{\mathcal{B}} = \text{Unif}(0, 2B) \end{aligned} \tag{25}$$

and the (ex-ante) equilibrium payoffs are

$$\pi_A = \frac{pa^h}{2}, \qquad \pi_B = p(1 - \frac{a^h}{2}) + (1-p) \tag{26}$$

Case 2: *The unique solution to ($\star$) is given by $\sigma^\ell = (1 - \frac{B}{A^h})\sqrt{\frac{A^h/((1-p)A^\ell)}{p+(1-p)A^\ell/A^h}}$, $\sigma^h = \frac{B-(1-p)\sigma^\ell A^\ell}{pA^h}$, and $\lambda_{\mathcal{B}} = \frac{\left(\sqrt{(1-p)A^\ell}+\sqrt{pA^h+(1-p)A^\ell}\right)^2}{2(A^h)^2}$ in Case 2. The equilibrium strategies for $a^\ell \leq H\left(a^h\right)$, where H is defined in (9), are*

$$\begin{aligned} F_{\mathcal{A}}^h &= \text{Unif}\left(L^\ell, L\right), \\ F_{\mathcal{A}}^\ell &= \left(1 - \frac{1-p\sigma^h}{(1-p)\sigma^\ell}\right)\boldsymbol{\delta}_0 + \frac{1-p\sigma^h}{(1-p)\sigma^\ell}\text{Unif}\left(0, L^\ell\right) \\ F_{\mathcal{B}} &= (1-p\sigma^h)\text{Unif}\left(0, L^\ell\right) + p\sigma^h\text{Unif}\left(L^\ell, L\right) \end{aligned} \tag{27}$$

where $L^h = \frac{p}{\lambda_{\mathcal{B}}}$ and $L^\ell = \frac{1-p\sigma^h}{\lambda^\ell}$. The equilibrium payoffs are

$$\begin{aligned} \pi_A &= p(1-p\sigma^h)\left(1 - \frac{\sigma^h}{\sigma^\ell}\right) + \lambda_{\mathcal{B}}B \\ \pi_B &= \lambda_{\mathcal{B}}B - L^\ell + (1-p) \end{aligned} \tag{28}$$

Case 3: *A solution to ($\star$) is of the form $\sigma^h \in \left(\frac{\frac{B}{A^h}\left(2+\frac{p}{1-p}-\frac{A^h}{B}\right)}{p\left(1+\frac{p}{1-p}\right)}, \frac{\frac{B}{A^h}\left(2+\frac{p}{1-p}\right)}{p\left(2+\frac{p}{1-p}+\frac{1-p}{p}\right)}\right)$, $\sigma^\ell = \frac{B-p\sigma^h A^h}{(1-p)A^\ell}$, and $\lambda_{\mathcal{B}} = \frac{2-p}{2A^h}$ in Case 3. The equilibrium strategies for $\frac{A^\ell}{A^h} = \frac{1-p}{2-p}$ and $2-p < \frac{A^h}{B} < 2 + \frac{p}{1-p}$ are*

$$\begin{aligned} &F_{\mathcal{A}}^h = \text{Unif}\left(L^\ell, L\right), \quad F_{\mathcal{A}}^\ell = \text{Unif}\left(0, L^\ell\right) \\ &F_{\mathcal{B}}(x) = (1-p\sigma^h + (1-p)\sigma^\ell)\boldsymbol{\delta}_0 \\ &\quad + (1-p)\sigma^\ell\text{Unif}\left(0, L^\ell\right) + p\sigma^h\text{Unif}\left(L^\ell, L\right) \end{aligned} \tag{29}$$

$$
\begin{array}{llll}
 & \underline{\textbf{Case 1}}\text{: } \bar{k}=1 & \underline{\textbf{Case 2}}\text{: } \bar{k}=2 & \underline{\textbf{Case 3}}\text{: } \bar{k}=3 \\
\text{(i)} & \dfrac{1}{2p(\sigma^h)^2}=\lambda_{\mathcal{B}}A^h & \dfrac{p}{2}+\dfrac{1-p\sigma^h}{\sigma^\ell}=\lambda_{\mathcal{B}}A^h & \dfrac{p}{2}+1-p=\lambda_{\mathcal{B}}A^h \\
\text{(ii)} & 0=A^\ell & \dfrac{\left(1-p\sigma^h\right)^2}{2(1-p)(\sigma^\ell)^2}=\lambda_{\mathcal{B}}A^\ell & \dfrac{1-p}{2}=\lambda_{\mathcal{B}}A^\ell \\
\text{(iii)} & p\sigma^hA^h=B & p\sigma^hA^h+(1-p)\sigma^\ell A^\ell=B & p\sigma^hA^h+(1-p)\sigma^\ell A^\ell=B \\
 & \text{such that} & \text{such that} & \text{such that} \\
\text{(iv)} & p\sigma^h\geq 1 & p\sigma^h<1 \text{ and } p\sigma^h+(1-p)\sigma^\ell\geq 1 & p\sigma^h+(1-p)\sigma^\ell<1 \\
\text{(v)} & \sigma^h\leq\sigma^\ell & \sigma^h\leq\sigma^\ell & \sigma^h\leq\sigma^\ell
\end{array}
\qquad (\star)
$$

where $L^h=\frac{p}{\lambda_{\mathcal{B}}}$ *and* $L^\ell=\frac{1-p}{\lambda_{\mathcal{B}}}=2A^\ell$. *The equilibrium payoffs are given by*

$$\pi_A=1-\lambda_{\mathcal{B}}B, \qquad \pi_B=\lambda_{\mathcal{B}}B. \tag{30}$$

Proof. We divide this proof into two parts. In the first part, we detail the steps used in each Case to calculate the algebraic solution to ($\star$) and the set of game instances for which it is valid. In the second part, we provide a proof that the corresponding strategies recovered from (20) do in fact constitute an equilibrium to the BL game.

Part 1: We will rely on shorthand notations $a_i=A_i/B$ when convenient.

Case 1: The solution to ($\star$) can directly be found to be $\lambda^h=\frac{1}{2B}$, $\lambda_{\mathcal{B}}=\frac{pA^h}{2B^2}$, and any $\lambda^\ell\geq\frac{1}{2B}$ (to satisfy (v)). Such a solution must also satisfy (iv), $p\sigma^h=1/a^h\geq 1$. Combined with (ii), the set of valid game parameters is $a^h\leq 1$ and $a^\ell=0$: player $\mathcal{A}$'s budget in type h is smaller than player $\mathcal{B}$'s budget, and has a budget of zero in type ℓ. Since λ^ℓ does not appear in the algebraic equations of ($\star$) (only in the constraints), this is essentially unique. Plugging these values into (20), we obtain the resulting strategies.

Case 2: To solve for $\lambda_{\mathcal{B}}$, we have $1-p\sigma^h=\sqrt{2(1-p)\lambda_2\sigma_2A^\ell}$ from (ii). Substituting into (i), we obtain a quadratic equation in $\sqrt{\lambda_{\mathcal{B}}}>0$. Its (positive) solution yields the expression for $\lambda_{\mathcal{B}}$.

Multiplying (ii) by $\frac{A^h}{A^\ell}$, the RHS of equations (i) and (ii) become equivalent. From (iv) of ($\star$), we use the substitution $1-p\sigma^h=1-(a^h)^{-1}+(1-p)\sigma^\ell\frac{a^\ell}{a^h}$ to obtain $\sigma^\ell=|1-(a^h)^{-1}|\sqrt{\frac{a^h/((1-p)a^\ell)}{p+(1-p)a^\ell/a^h}}$. The condition (iv) requires $p\sigma^h<1$. Using the substitution $\sigma^h=\frac{1-(1-p)\sigma^\ell a^\ell}{pa^h}$ from (iii), we deduce that $a^h>1$:

$$
\begin{aligned}
&p\sigma^h= \\
&(a^h)^{-1}\left(1-(1-p)a^\ell|1-(a^h)^{-1}|\cdot\sqrt{\frac{a^h/((1-p)a^\ell)}{p+(1-p)a^\ell/a^h}}\right) \\
&=(a^h)^{-1}-|1-(a^h)^{-1}|\cdot\sqrt{\frac{(1-p)a^\ell/a^h}{p+(1-p)a^\ell/a^h}}<1 \\
&\Rightarrow 1-(a^h)^{-1}>-|1-(a^h)^{-1}|\Rightarrow a^h>1
\end{aligned}
\tag{31}
$$

We can also deduce from (iii) and $a^h\geq a^\ell$ that $a^\ell\leq 1$. The condition (iv) also requires $p\sigma^h+(1-p)\sigma^\ell\geq 1$. From this, we obtain

$$a^\ell\leq\frac{1-p}{2-p}a^h. \tag{32}$$

Furthermore, the positivity of σ^ℓ is trivially satisfied. However, positivity of σ^h requires that $1-(1-p)\sigma^\ell a^\ell>0$. Plugging in the expression for σ^ℓ, we obtain $a^\ell(1-p)\left(2-a^h\right)>-p$. Hence, the positivity constraint $\sigma^h>0$ is equivalent to

$$a^h\leq 2,\ \text{or } a^h>2 \text{ and } a^\ell<\frac{p}{(1-p)\left(a^h-2\right)}. \tag{33}$$

Lastly, the constraint (v) requires $\sigma^h\leq\sigma^\ell$. Plugging in the expression for σ^ℓ, we deduce that $a^\ell\left(1-(a^h-1)^2\right)\leq(a^h-1)^2a^h\frac{p}{1-p}$. The term in parentheses on the LHS is positive when $a^h<2$, and negative otherwise. Hence, we obtain

$$a^\ell\begin{cases}\leq\frac{\frac{p}{1-p}(a^h-1)^2}{2-a^h}, & \text{if } 1<a^h\leq 2\\ \geq 0, & \text{if } a^h>2\end{cases} \tag{34}$$

The intersection of conditions (34),(32), and (33) on the budget parameters A^h and A^ℓ, derived directly from (iv) and (v), yields $a^\ell\leq H\left(a^h\right)$, where H was defined in (9). This establishes the set of games for which the system ($\star$) has a solution in Case 2.

Case 3: We can directly obtain $\lambda_{\mathcal{B}}=\frac{2-p}{2A^h}$. Note that $\lambda_{\mathcal{B}}=\frac{1-p}{2A^\ell}$ as well, from which we obtain $A^\ell=\frac{1-p}{2-p}A^h$. From (iii), we have $\sigma^\ell=\frac{B-pA^ha^h}{(1-p)A^\ell}$. Substituting this in the condition (iv), $p\sigma^h+(1-p)\sigma^\ell<1$, we obtain $\sigma^h>\frac{(a^h)^{-1}\left(2+\frac{p}{1-p}-a^h\right)}{p\left(1+\frac{p}{1-p}\right)}$. Similarly, constraint (v), $\sigma^h\leq\sigma^\ell$, yields $\sigma^h\leq\frac{(a^h)^{-1}\left(2+\frac{p}{1-p}\right)}{p\left(2+\frac{p}{1-p}+\frac{1-p}{p}\right)}$. A feasible σ^h exists within these constraints if and only if $a^h>1+\frac{1+\frac{1-p}{p}}{2+\frac{p}{1-p}+\frac{1-p}{p}}=2-p$ (upper bound must be larger than lower bound), and $a^h<2+\frac{p}{1-p}$ (lower bound must be positive). Subsequently, (20) recovers the strategies (29). The union of characterized parameter sets in all three cases constitutes the $\mathcal{R}_5$ region in Theorem 3.1.

Part 2: We now verify the profile $(\vec{F}_{\mathcal{A}},F_{\mathcal{B}})$ recovered from (20) is an equilibrium. We can immediately deduce the strategies in Case 1 are equilibria to the BL game by observing that player $\mathcal{A}$ has zero budget in type ℓ, and $(F_{\mathcal{A}}^h,F_{\mathcal{B}})$ forms the unique equilibrium to the complete information General Lotto game [11] with a single battlefield of value p. We will focus here on the strategies produced from Case 2, as the proof for Case 3 follows analogous arguments.

We first calculate the (ex-interim) payoffs from the strategies

(27).

$$
\begin{aligned}
U_A(F_{\mathcal{A}}^h, F_{\mathcal{B}}) &= \int_0^\infty F_{\mathcal{B}}(x)\, dF_{\mathcal{A}}^h \\
&= \int_{L^\ell}^{L} \left[L^\ell \lambda^\ell + \lambda^h (x - L^\ell)\right] \frac{\lambda_{\mathcal{B}}}{p} dx \\
&= (1 - p\sigma^h) \left(1 - \frac{\sigma^h}{\sigma^\ell}\right) + \lambda^h A^h
\end{aligned} \tag{35}
$$

$$
U_A(F_{\mathcal{A}}^\ell, F_{\mathcal{B}}) = \int_0^\infty F_{\mathcal{B}}(x)\, dF_{\mathcal{A}}^\ell = \int_0^{L^\ell} x \frac{\lambda^\ell \lambda_{\mathcal{B}}}{1-p} dx = \lambda^\ell A^\ell
$$

where recall $L = L^\ell + L^h$. The expected payoff (first equation of (5)) to player $\mathcal{A}$ is then $\pi_A = p(1 - p\sigma^h)\left(1 - \frac{\sigma^h}{\sigma^\ell}\right) + \lambda_{\mathcal{B}} B$ (using (iii)). The payoff to player $\mathcal{B}$ is $\pi_B = 1 - \pi_A$. We need to show $F_{\mathcal{A}}$ is a best-response to $F_{\mathcal{B}}$, and vice versa.

For any $F_{\mathcal{A}}'^h \in \mathcal{L}(A^h)$, the payoff in type h is

$$
\begin{aligned}
U_{\mathcal{A}}(F_{\mathcal{A}}'^h, F_{\mathcal{B}}) &= \int_0^{L^\ell} \lambda^\ell x\, dF_{\mathcal{A}}'^h + \\
&+ \int_{L^\ell}^{L} \left[L^\ell \lambda^\ell + \lambda^h (x - L^\ell)\right] dF_{\mathcal{A}}'^h + \int_L^\infty dF_{\mathcal{A}}'^h
\end{aligned} \tag{36}
$$

Using the identities $\int_{L^\ell}^L x\, dF_{\mathcal{A}}'^h = A^h - \int_0^{L^\ell} x\, dF_{\mathcal{A}}'^h - \int_L^\infty x\, dF_{\mathcal{A}}'^h$ and $\int_{L^\ell}^L dF_{\mathcal{A}}'^h = 1 - \int_0^{L^\ell} dF_{\mathcal{A}}'^h - \int_L^\infty dF_{\mathcal{A}}'^h$, we obtain

$$
\begin{aligned}
&= (\lambda^\ell - \lambda^h) \left(\int_0^{L^\ell} x\, dF_{\mathcal{A}}'^h - L^\ell \int_0^{L^\ell} dF_{\mathcal{A}}'^h \right) \\
&\quad + \lambda^h \left(L \int_{L^\ell}^L dF_{\mathcal{A}}'^h - \int_{L^\ell}^L x\, dF_{\mathcal{A}}'^h \right) \\
&\quad + \lambda^h A^h + L^\ell (\lambda^\ell - \lambda^h) \\
&\leq \lambda^h A^h + L^\ell(\lambda^\ell - \lambda^h) = \lambda^h A^h + (1 - \frac{\lambda^h}{\lambda^\ell})(1 - p\sigma^h).
\end{aligned} \tag{37}
$$

The inequality follows from two applications of Markov's inequality: $\int_0^{L^\ell} x\, dF_{\mathcal{A}}'^h \leq L^\ell \int_0^{L^\ell} dF_{\mathcal{A}}'^h$, and $-\int_{L^\ell}^L x\, dF_{\mathcal{A}}'^h \leq -L \int_{L^\ell}^L dF_{\mathcal{A}}'^h$. Hence, the payoff in type h is upper-bounded by $\lambda^h A^h + (1 - \frac{\lambda^h}{\lambda^\ell})(1 - p\sigma^h)$, which can be attained whenever $\text{supp}(F_{\mathcal{A}}'^h) \subseteq [L^\ell, L]$ (for which the Markov inequalities hold with equality).

Analogous calculations for any $F_{\mathcal{A}}'^\ell \in \mathcal{L}(A^\ell)$ yields $U_{\mathcal{A}}(F_{\mathcal{A}}'^\ell, F_{\mathcal{B}}) \leq \lambda^\ell A^\ell$. This upper bound can be attained whenever $\text{supp}(F_{\mathcal{A}}'^\ell) \subseteq [0, L^\ell]$. The strategy $\vec{F}_{\mathcal{A}} = (F_{\mathcal{A}}^h, F_{\mathcal{A}}^\ell)$ satisfies these properties, and hence is a best-response to $F_{\mathcal{B}}$.

For any $F_{\mathcal{B}}' \in \mathcal{L}(B)$, player $\mathcal{B}$'s expected payoff (5) is

$$
\begin{aligned}
U_{\mathcal{B}}(F_{\mathcal{B}}', \vec{F}_{\mathcal{A}}) &= p \left[\int_{L^\ell}^L \frac{\lambda_{\mathcal{B}}}{p} (x - L^\ell)\, dF_{\mathcal{B}}' + \int_L^\infty dF_{\mathcal{B}}' \right] \\
&\quad + (1-p) \left[\int_0^{L^\ell} \left(1 - \frac{\lambda_{\mathcal{B}} L^\ell}{1-p} + \frac{\lambda_{\mathcal{B}}}{1-p} x \right) dF_{\mathcal{B}}' + \int_{L^\ell}^\infty dF_{\mathcal{B}}' \right] \\
&= \lambda_{\mathcal{B}} B - \lambda_{\mathcal{B}} L^\ell + (1-p) + \lambda_{\mathcal{B}} \left(L \int_L^\infty dF_{\mathcal{B}}' - \int_L^\infty x\, dF_{\mathcal{B}}' \right) \\
&\leq \lambda_{\mathcal{B}} B - \lambda_{\mathcal{B}} L^\ell + (1-p)
\end{aligned} \tag{38}
$$

Player $\mathcal{B}$'s upper bound on expected payoff can be attained for any strategy with $\text{supp}(F_{\mathcal{B}}') \subseteq [0, L]$. Because $F_{\mathcal{B}}$ is one such strategy, it is a best-response to $\vec{F}_{\mathcal{A}}$. ■

D. Equilibria in regions $\mathcal{R}_1$ - $\mathcal{R}_4$

Here, we prove Theorem 3.1 for the remaining parameter regions $\mathcal{R}_i$, $i = 1, \ldots, 4$, which are also depicted in Figure 5. We begin with the $\mathcal{R}_3$ region.

Lemma 5.2 (**Region** $\mathcal{R}_3$)**.** *Suppose the game instance $BL(\mathbb{P}_{\mathcal{A}}, B)$ belongs to the region*

$$
\mathcal{R}_3 = \left\{ a^h \geq 2 + \frac{p}{1-p} \text{ and } 1 \leq a^\ell \leq \frac{1-p}{2-p} a^h \right\}. \tag{39}
$$

Then the following profile is an equilibrium

$$
\begin{aligned}
&F_{\mathcal{A}}^h = \mathit{Unif}\left(2A^\ell, 2(A^h - A^\ell)\right), \quad F_{\mathcal{A}}^\ell = \mathit{Unif}\left(0, 2A^\ell\right) \\
&F_{\mathcal{B}} = (1 - (a^\ell)^{-1})\boldsymbol{\delta}_0 + (a^\ell)^{-1} \mathit{Unif}\left(0, 2A^\ell\right)
\end{aligned} \tag{40}
$$

The equilibrium payoff is given by $\pi_{\mathcal{A}} = p + (1-p)\left(1 - \frac{1}{2a^\ell}\right)$.

In the $\mathcal{R}_3$ region, the high budget A^h is disproportionately higher than the low budget A^ℓ. In the equilibrium given above, player $\mathcal{B}$ does not compete with the high budget at all, thus giving a payoff of p to $\mathcal{A}$ outright.

Proof. First, we show $\vec{F}_{\mathcal{A}}$ is a best-response to $F_{\mathcal{B}}$. For any $\{F_{\mathcal{A}}'^t \in \mathcal{L}(A^t)\}_{t=h,\ell}$, player $\mathcal{A}$'s expected payoff is

$$
\begin{aligned}
&p \left[\int_0^{2A^\ell} \left(1 - (a^\ell)^{-1} + \frac{(a^\ell)^{-1}}{2A^\ell} x \right) dF_{\mathcal{A}}'^h + \int_{2A^\ell}^\infty dF_{\mathcal{A}}'^h \right] \\
&+ (1-p) \left[\int_0^{2A^\ell} \left(1 - (a^\ell)^{-1} + \frac{(a^\ell)^{-1}}{2A^\ell} x \right) dF_{\mathcal{A}}'^\ell + \int_{2A^\ell}^\infty dF_{\mathcal{A}}'^\ell \right] \\
&\leq p + (1-p) \left(1 - \frac{(a^\ell)^{-1}}{2} \right)
\end{aligned} \tag{41}
$$

The inequality follows by selecting any $F_{\mathcal{A}}'^h$ such that $\text{supp}(F_{\mathcal{A}}'^h) \subset [2A^\ell, \infty)$, which awards player $\mathcal{A}$ the payoff p from state 1 outright. This is possible because $a^h \geq 2 + \frac{p}{1-p} > 2$, from the assumption. It holds with equality if and only if $\text{supp}(F_{\mathcal{A}}'^\ell) \subseteq [0, 2A^\ell]$. We have thus established an upper bound on $\mathcal{A}$'s payoff to $F_{\mathcal{B}}$ that is achieved by $\vec{F}_{\mathcal{A}}$.

Now we show $F_{\mathcal{B}}$ is a best-response to $F_{\mathcal{A}}$. Let $K \triangleq (1-p) - \frac{pa^\ell}{a^h - 2a^\ell} \geq 0$, which is non-negative due to the assumption $a^\ell \leq \frac{1-p}{2-p} a^h$. For any $F_{\mathcal{B}}' \in \mathcal{L}(B)$, player $\mathcal{B}$'s payoff is

$$
\begin{aligned}
&p \left[\int_{2A^\ell}^{2(A^h - A^\ell)} \frac{x - 2A^\ell}{2(A^h - 2A^\ell)} dF_{\mathcal{B}}' + \int_{2(A^h - A^\ell)}^\infty dF_{\mathcal{B}}' \right] \\
&\quad + (1-p) \left[\int_0^{2A^\ell} \frac{x}{2A^\ell} dF_{\mathcal{B}}' + \int_{2A^\ell}^\infty dF_{\mathcal{B}}' \right] \\
&= \frac{1-p}{2A^\ell} \int_0^{2A^\ell} x\, dF_{\mathcal{B}}' + \frac{p}{2(A^h - 2A^\ell)} \int_{2A^\ell}^{2(A^h - A^\ell)} x\, dF_{\mathcal{B}}' \\
&\quad - \left(\frac{pA^\ell}{A^h - 2A^\ell} - (1-p) \right) \int_{2A^\ell}^{2(A^h - A^\ell)} dF_{\mathcal{B}}' + \int_{2(A^h - A^\ell)}^\infty dF_{\mathcal{B}}'
\end{aligned} \tag{42}
$$

Applying the identity $\int_0^{2A^\ell} x\, dF_{\mathcal{B}}' = B - \int_{2A^\ell}^{2(A^h - A^\ell)} x\, dF_{\mathcal{B}}' -$

$\int_{2(A^h-A^\ell)}^{\infty} x\, dF'_{\mathcal{B}}$, we then obtain

$$
\begin{aligned}
&= K\left(-\frac{1}{2A^\ell}\int_{2A^\ell}^{2(A^h-A^\ell)} x\, dF'_{\mathcal{B}} + \int_{2A^\ell}^{2(A^h-A^\ell)} dF'_{\mathcal{B}}\right)\\
&\quad + (1-p)\frac{(a^\ell)^{-1}}{2} + \int_{2(A^h-A^\ell)}^{\infty} dF'_{\mathcal{B}} - \frac{1-p}{2A^\ell}\int_{2(A^h-A^\ell)}^{\infty} x\, dF'_{\mathcal{B}}\\
&\leq (1-p)\frac{(a^\ell)^{-1}}{2} + \left(p + (1-p)\left(2 - \frac{a^h}{a^\ell}\right)\right)\int_{2(A^h-A^\ell)}^{\infty} dF'_{\mathcal{B}}\\
&\leq (1-p)\frac{(a^\ell)^{-1}}{2}
\end{aligned} \tag{43}
$$

The first inequality results from applying Markov's inequality to $\int_{2A^\ell}^{2(A^h-A^\ell)} x\, dF'_{\mathcal{B}}$ and $\int_{2(A^h-A^\ell)}^{\infty} x\, dF'_{\mathcal{B}}$. The second inequality follows from non-positivity of term in parentheses (from assumption of the Lemma). This inequality holds with equality if and only if $\text{supp}(F'_{\mathcal{B}}) \subseteq [0, 2A^\ell]$. We have thus established an upper bound on player $\mathcal{B}$'s payoff to $F_{\mathcal{A}}$ that is achieved by $F_{\mathcal{B}}$. ■

Lemma 5.3 (**Region** $\mathcal{R}_4$)**.** *Suppose the game instance BL$(\mathbb{P}_{\mathcal{A}}, B)$ belongs to the region*

$$\mathcal{R}_4 := \left\{a^h \geq 2 + \frac{p}{1-p} \text{ and } \frac{p}{(1-p)(a^h-2)} \leq a^\ell \leq 1\right\}. \tag{44}$$

Then the following profile is an equilibrium:

$$
\begin{aligned}
F_{\mathcal{A}}^h &= \mathit{Unif}\big(2B, 2(A^h - B)\big)\\
F_{\mathcal{A}}^\ell &= (1-a^\ell)\boldsymbol{\delta}_0 + a^\ell \mathit{Unif}(0, 2B)\\
F_{\mathcal{B}} &= \mathit{Unif}(0, 2B)
\end{aligned} \tag{45}
$$

The equilibrium payoff is given by $\pi_{\mathcal{A}} = p + (1-p)\frac{a^\ell}{2}$.

A similar intuition to the $\mathcal{R}_3$ region holds for the $\mathcal{R}_4$ region. In the equilibrium given above, player $\mathcal{B}$ does not compete with the high budget at all, thus giving a payoff of p to player $\mathcal{A}$ outright. The proof follows similar calculations to Lemma 5.2, and hence is omitted.

Regions $\mathcal{R}_1$ and $\mathcal{R}_2$:

Consider the set of $(a^h, a^\ell) \in \mathcal{R}_1 \cup \mathcal{R}_2$ that have a fixed average budget $\bar{a}$. Define the pair of budgets

$$
\boldsymbol{a}^{\text{bd}} \triangleq \begin{cases} (\bar{a}/p, 0) \in \mathcal{R}, & \text{if } \bar{a} \leq p\\ (2 - p/\bar{a}, H(2-p/\bar{a})) \in \mathcal{R}, & \text{if } p < \bar{a} \leq 1\\ ((2-p)\bar{a}, (1-p)\bar{a}) \in \mathcal{R}, & \text{if } 1 < \bar{a} \end{cases} \tag{46}
$$

where H is defined in (9) and $\mathcal{R} = \{(a^h, a^\ell) : a^h \geq a^\ell\}$. The points $\boldsymbol{a}^{\text{bd}}$ specified above for $\bar{a} \leq 1$ are on the border of $\mathcal{R}_5$, whose equilibria are given in Proposition 5.1. The points for $1 < \bar{a}$ are on the upper border of $\mathcal{R}_3$, where an equilibrium is given in Lemma 5.2. Define

$$
\vec{F}_{\mathcal{A}}^{\text{bd}} \triangleq \begin{cases} \text{given by (25) at } \boldsymbol{a}^{\text{bd}}, & \text{if } \bar{a} \leq p\\ \text{given by (27) at } \boldsymbol{a}^{\text{bd}}, & \text{if } p < \bar{a} \leq 1\\ \text{given by (29) at } \boldsymbol{a}^{\text{bd}}, & \text{if } 1 < \bar{a} \end{cases} \tag{47}
$$

Here, $\vec{F}_{\mathcal{A}}^{\text{bd}}$ is an equilibrium strategy for player $\mathcal{A}$ at the boundary point $\boldsymbol{a}^{\text{bd}}$. Let us also define $(\bar{F}_A, \bar{F}_B)$ as the equilibrium at $(\bar{a}, \bar{a}) \in \mathcal{R}$, which is simply the equilibrium in the corresponding complete information game. That is,

$$
\begin{aligned}
\bar{F}_A &\triangleq \begin{cases} (1-\bar{a})\delta_0 + \bar{a}\text{Unif}([0, 2B]), & \text{if } \bar{a} \leq 1\\ \text{Unif}([0, 2\bar{A}]), & \text{if } \bar{a} > 1 \end{cases}\\
\bar{F}_B &\triangleq \begin{cases} \text{Unif}([0, 2B]), & \text{if } \bar{a} \leq 1\\ (1-\bar{a}^{-1})\delta_0 + \bar{a}^{-1}\text{Unif}([0, 2\bar{A}]), & \text{if } \bar{a} > 1 \end{cases}
\end{aligned} \tag{48}
$$

Lemma 5.4. *Suppose the game instance BL$(\mathbb{P}_{\mathcal{A}}, B)$ belongs to $\mathcal{R}_1 \cup \mathcal{R}_2$. Let $\alpha \in [0,1]$ be the unique scaling that gives $\alpha \boldsymbol{a}^{bd} + (1-\alpha)\cdot(\bar{a}, \bar{a}) = (a^h, a^\ell)$. Then the profile*

$$(\alpha \vec{F}_{\mathcal{A}}^{bd} + (1-\alpha)\bar{F}_A, \bar{F}_B) \tag{49}$$

is an equilibrium profile, with equilibrium payoff $\pi_{\mathcal{A}}^{CI}(\bar{A}, B)$.

Player $\mathcal{A}$'s equilibrium strategy in (49) is a convex combination between an equilibrium strategy on the border[4] of $\mathcal{R}_5$ and equilibrium in its corresponding benchmark complete information game GL$(\bar{A}, B)$. As a result, the equilibrium payoff coincides with the equilibrium payoff of the corresponding complete information game.

Proof. Since we know that $(\bar{F}_A, \bar{F}_B)$ is an equilibrium, it will suffice to show that $(\vec{F}_{\mathcal{A}}^{\text{bd}}, \bar{F}_B)$ is also an equilibrium for all $\bar{a}$. One can verify that the payoffs from $(F_{\mathcal{A}}^{\text{bd}}, \bar{F}_B)$ indeed coincide with the payoffs from $(\bar{F}_A, \bar{F}_B)$.

For $\bar{a} \leq p$, the equilibrium at $\boldsymbol{a}^{\text{bd}}$ is given by Case 1 (25), where player $\mathcal{B}$'s strategy is precisely $\bar{F}_B$. For $p < \bar{a} \leq 1$, the equilibrium at $\boldsymbol{a}^{\text{bd}}$ is given by Case 2 (27), where it is also true that player $\mathcal{B}$'s strategy is $\bar{F}_B$ (on these border points, $\sigma^h = \sigma^\ell$). For $1 < \bar{a} \leq \frac{1}{1-p}$, the equilibrium at $\boldsymbol{a}^{\text{bd}}$ is given by Case 3 (29). We note that although player $\mathcal{B}$'s equilibrium strategy here is not unique, $\bar{F}_B$ is one such strategy.

Lastly, for $\frac{1}{1-p} < \bar{a}$, $(\vec{F}_{\mathcal{A}}^{\text{bd}}, \bar{F}_B)$ is an equilibrium at $\boldsymbol{a}^{\text{bd}}$, where $\vec{F}_{\mathcal{A}}^{\text{bd}}$ is player $\mathcal{A}$'s equilibrium strategy at the border of $\mathcal{R}_3$ (40). This strategy is also identical to the monotonic equilibrium strategy from Case 3 (29). Hence, the proof that $(F_{\mathcal{A}}^{\text{bd}}, \bar{F}_B)$ is an equilibrium follows from the analysis in Proposition 5.1. Note that player $\mathcal{B}$'s $\mathcal{R}_3$ equilibrium strategy written in (40) is not $\bar{F}_B$. Indeed, $\bar{F}_B$ in general is not an equilibrium strategy in the interior of $\mathcal{R}_3$. At the border however, we know of at least two equilibria (giving the same payoffs), one of them being $(\vec{F}_{\mathcal{A}}^{\text{bd}}, \bar{F}_B)$. ■

We have established equilibrium strategies and payoffs for each of the five regions, which completes the proof of Theorem 3.1.

VI. Conclusion

This paper considers a class of asymmetric information General Lotto games, where one of the player's resource budget is assigned randomly according to a Bernoulli distribution, while the opponent's endowment is common knowledge. We fully characterize equilibrium strategies and payoffs in

[4]Since equilibria on the border are not necessarily unique, i.e. Case 3 parameters of Proposition 5.1, the equilibria in the regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are not unique. However, all equilibria in one game instance yield identical payoffs, since it is a constant sum game (in ex-ante payoffs).

this class of Bayesian games. Furthermore, these equilibrium characterizations allow us to determine how a high-level commander could benefit from randomly assigning resources to two sub-colonels that engage with two respective opponents in separate General Lotto games. Interestingly, randomized assignments can improve the commander's payoff four-fold in comparison to the optimal deterministic assignment, in settings with a per-unit cost for deployment. In settings with fixed resource budgets, randomized assignments do not offer any strict improvement over deterministic ones.

There are several interesting directions for future research. One can consider extensions in which players are making resource allocation decisions in an online fashion without knowing the opponent's budget, which would necessitate using tools from reinforcement learning and dynamic programming. It would also be of interest to extend the assignment problems to account for more than two sub-colonels, and scenarios where the players enter and leave the contests in a stochastic fashion.

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Appendix

A. Optimal randomized assignments – per-unit cost setting

We proceed to derive the solution of CAP under the per-unit cost setting by first solving the optimal opponent investment decision in Stage 2. Recall that by the beginning of Stage 2, the commander has chosen a feasible assignment policy $\mathbb{P} \in \mathcal{P}$. The marginal distribution $\mathbb{P}_i$ is thus represented by a Bernoulli distribution (A_i^h, A_i^ℓ, p_i). The Stage 2 decision problem for opponent $\mathcal{B}_i$ is therefore

$$\max_{B_i \geq 0} \{\phi_i \cdot \pi_{\mathcal{B}}(\mathbb{P}_i, B_i) - c_i B_i\} \tag{50}$$

where $\pi_{\mathcal{B}}$ is given in Theorem 3.1.

Lemma A.1. *Consider the Stage 2 decision problem for opponent $\mathcal{B}_i$ in CAP (50). If $\frac{A_i^\ell}{A_i^h} \geq \frac{1-p_i}{2-p_i}$, then the optimal investment for $\mathcal{B}_i$ is given by*

$$B_i^*(\mathbb{P}_i) = \begin{cases} \sqrt{\frac{\phi_i \bar{A}_i}{2c_i}}, & \text{if } c_i < \frac{\phi_i}{2\bar{A}_i} \\ 0, & \text{otherwise} \end{cases} \tag{51}$$

where $\bar{A}_i := p_i A_i^h + (1-p_i)A_i^\ell$*. If* $\frac{A_i^\ell}{A_i^h} < \frac{1-p_i}{2-p_i}$*, then*

$$B_i^*(\mathbb{P}_i) = \begin{cases} \sqrt{\frac{\bar{A}_i\phi_i}{2c_i}}, & \text{if } c_i \in [0, \lambda_i) \\ \sqrt{\frac{(1-p_i)A_i^\ell\phi_i}{2c_i}}, & \text{if } c_i \in [\lambda_i, \frac{(1-p_i)\phi_i}{2A_i^\ell}) \\ 0, & \text{if } c_i \geq \frac{(1-p_i)\phi_i}{2A_i^\ell} \end{cases} \tag{52}$$

where we have defined $\lambda_i \triangleq \frac{\phi_i}{2(A_i^h)^2}(\sqrt{(1-p_i)A_i^\ell} + \sqrt{\bar{A}_i})^2$.

Proof. The optimal investment is the maximizer of the optimization problem (50). For simpler exposition, we will drop the subscript i on all variables in this proof since identical analysis applies to both opponents.

First, suppose $\frac{A^\ell}{A^h} \geq \frac{1-p}{2-p}$. From (7), we have $\pi_{\mathcal{B}}(\mathbb{P}_{\mathcal{A}}, B) = \pi_{\mathcal{B}}^{\text{CI}}(\bar{A}, B)$ for all $B' \geq 0$, and therefore B^* is given by (51).

Now, suppose $\frac{A^\ell}{A^h} < \frac{1-p}{2-p}$. Leveraging the characteriztion from Theorem 3.1, we can write $\pi_{\mathcal{B}}(\mathbb{P}_{\mathcal{A}}, B)$ as

$$\phi \cdot \begin{cases} (1-p)\frac{B}{2A^\ell}, & \text{if } B \leq A^\ell \\ (1-p)\left(1 - \frac{A^\ell}{2B}\right), & \text{if } A^\ell < B \leq Y^\ell \\ (1-p)(1-\frac{A^\ell}{A^h}) - \frac{\sqrt{\bar{A}(\bar{A}-pA^h)}}{A^h} + \lambda B, & \text{if } Y^\ell < B \leq Y^h \\ p\left(1 - \frac{A^h}{2B}\right) + (1-p)\left(1 - \frac{A^\ell}{2B}\right), & \text{if } Y^h < B \end{cases} \tag{53}$$

where we have written $Y^\ell \triangleq \frac{A^h\sqrt{(1-p)A^\ell}}{\sqrt{(1-p)A^\ell}+\sqrt{\bar{A}}}$, and $Y^h \triangleq \frac{A^h\sqrt{\bar{A}}}{\sqrt{(1-p)A^\ell}+\sqrt{\bar{A}}}$. The entries in the expression above correspond to payoffs in regions $\mathcal{R}_3$, $\mathcal{R}_4$, $\mathcal{R}_5$, and $\mathcal{R}_1$, respectively (Theorem 3.1). The values for B^* result from solving $\frac{\partial}{\partial B}(\pi_{\mathcal{B}}(\mathbb{P}_{\mathcal{A}}, B) - cB) = 0$. Note that the first and third entries provide linear returns on investment B. Therefore, there are multiple maximizers when the per-unit cost coincides with these slopes. In particular, when $c = \lambda$ or $c = \frac{1-p}{2A^\ell}$, the payoff $\pi_{\mathcal{B}}(\mathbb{P}_{\mathcal{A}}, B) - c \cdot B$ is constant for all $B \in [Y^\ell, Y^h]$ or $B \in [0, A^\ell]$, respectively. Since player $\mathcal{B}$ prefers the lowest investment level, we thus obtain (52). ■

Now that we have established the optimal decision in Stage 2, we can now consider the optimal commander's assignment decision problem in Stage 1, which can be stated as:

$$\begin{aligned} W^* \triangleq \max_{\mathbb{P}\in\mathcal{P}} \{&\phi_1 \cdot \pi_{\mathcal{A}}(\mathbb{P}_1, B_1^*(\mathbb{P}_1)) + \phi_2 \cdot \pi_{\mathcal{A}}(\mathbb{P}_2, B_2^*(\mathbb{P}_2)) \\ &-c \cdot \mathbb{E}_{\mathbb{P}}[A_1 + A_2]\} \end{aligned} \tag{54}$$

Before addressing (54) fully, we first consider an intermediate step where the commander has a fixed budget $A > 0$ and zero cost $c = 0$. The intermediate decision problem is:

$$W_A^* \triangleq \max_{\mathbb{P}\in\mathcal{P}(A)} \{\phi_1 \cdot \pi_{\mathcal{A}}(\mathbb{P}_1, B_1^*(\mathbb{P}_1)) + \phi_2 \cdot \pi_{\mathcal{A}}(\mathbb{P}_2, B_2^*(\mathbb{P}_2))\} \tag{55}$$

where B_i^* is given in Lemma A.1, and $\mathcal{P}(A)$ denotes the set of all randomized assignment policies with Bernoulli marginals that meets the expenditure budget A in expectation. Note that any feasible $\mathbb{P} \in \mathcal{P}(A)$ induces expected endowments $\bar{A}_i = p_i A_i^h + (1-p_i)A_i^\ell$ for each sub-colonel, where $\bar{A}_1 + \bar{A}_2 = A$ is satisfied. The optimization (55) can thus be re-written as:

$$\max_{\substack{(\bar{A}_1,\bar{A}_2): \\ \bar{A}_1+\bar{A}_2=A}} \left\{ \sum_{i=1,2} \max_{\substack{\mathbb{P}_i=(A_i^h,A_i^\ell,p_i): \\ p_iA_i^h+(1-p_i)A_i^\ell=\bar{A}_i}} \phi_i \cdot \pi_{\mathcal{A}}(\mathbb{P}_i, B_i^*(\mathbb{P}_i)) \right\} \tag{56}$$

The following Lemma provides the optimal Bernoulli distribution for sub-colonel i given a fixed expected endowment $\bar{A}_i$, i.e. the solution of each inner maximization above.

Lemma A.2. *Given a fixed expected endowment* $\bar{A}_i > 0$ *for sub-colonel* i*,*

$$\begin{aligned} \Pi_i^*(\bar{A}_i; \phi_i) &\triangleq \max_{\substack{\mathbb{P}_i=(A_i^h,A_i^\ell,p_i): \\ p_iA_i^h+(1-p_i)A_i^\ell=\bar{A}_i}} \phi_i \cdot \pi_{\mathcal{A}}(\mathbb{P}_i, B_i^*(\mathbb{P}_i)) \\ &= \begin{cases} \sqrt{2c_i\phi_i\bar{A}_i} & \text{if } c_i \in (0, \frac{\phi_i}{2\bar{A}_i}) \\ \phi_i, & \text{if } c_i \geq \frac{\phi_i}{2\bar{A}_i} \end{cases} \end{aligned} \tag{57}$$

The optimal Bernoulli distribution that achieves the above payoff is

$$(A_i^{h*}, A_i^{\ell*}, p_i^*) = \begin{cases} \left(\sqrt{\frac{\bar{A}_i\phi_i}{2c_i}}, 0, \sqrt{\frac{2c_i\bar{A}_i}{\phi_i}}\right) & \text{if } c \in (0, \frac{\phi_i}{2\bar{A}_i}) \\ (\bar{A}_i, \bar{A}_i, \times), & \text{if } c_i \geq \frac{\phi_i}{2\bar{A}_i} \end{cases} \tag{58}$$

From (57), we see that the optimal Bernoulli assignment can double sub-colonel i's payoff compared to the deterministic assignment $\bar{A}_i$.

Proof. We will again drop the subscript $i \in \{1, 2\}$ on all variables since identical analysis applies to both contests. Let us define $\Pi(\mathbb{P}) \triangleq \phi \cdot \pi_{\mathcal{A}}(\mathbb{P}, B^*(\mathbb{P}))$ as the objective of (57) and use $f = \frac{A^\ell}{A^h} \in [0, 1]$ as a change of variable. Note that a choice of p and f determines A^ℓ and A^h through the constraint $\bar{A} = pA^h + (1-p)A^\ell$. Using Lemma A.1, if $f < \frac{1-p}{2-p}$, we can write

$$\Pi(\mathbb{P}) = \begin{cases} \sqrt{\frac{c\phi\bar{A}}{2}}, & \text{if } c \in [0, \lambda) \\ p\phi + \sqrt{\frac{c\phi(1-p)A^\ell}{2}}, & \text{if } c \in [\lambda, \frac{(1-p)\phi}{2A^\ell}) \\ \phi, & \text{if } c \in [\frac{(1-p)\phi}{2A^\ell}, \infty) \end{cases} \tag{59}$$

where $\lambda = \phi\frac{p+(1-p)f}{2\bar{A}}(\sqrt{(1-p)f} + \sqrt{p+(1-p)f})^2$. If $f \geq \frac{1-p}{2-p}$, then

$$\Pi(\mathbb{P}) = \begin{cases} \sqrt{\frac{c\phi\bar{A}}{2}}, & \text{if } c < \frac{\phi}{2\bar{A}} \\ \phi, & \text{else} \end{cases} \tag{60}$$

We first characterize the solution of finding the optimal $f \in [0, 1]$ given a fixed $p \in [0, 1]$ and expected endowment $\bar{A}$:

$$\Pi^*(p) \triangleq \max_{f\in[0,1]} \Pi(\mathbb{P}) \tag{61}$$

under the parameterization $A^h = \frac{\bar{A}}{f(1-p)+p}$, and then optimize $\Pi^*(p)$ over p. Such an approach yields the optimal value of (57) because as we will show, each of the values $\Pi^*(p)$ are well-defined and are attained for some $f^* \in [0, 1]$, and the maximum of $\Pi^*(p)$ is attained for some $p^* \in [0, 1]$.

When $p = 0$ or $p = 1$, the setting becomes a complete information game where the budget is deterministic, so that $A^h = A^\ell = \bar{A}$, or equivalently, $f = 1$. Hence, the optimal payoff $\Pi^*(p)$ is given by (60).

When $p \in (0,1)$, we observe that $\lambda(f)$ is strictly increasing in f, taking the value $\frac{p^2\phi}{2\bar{A}}$ for $f = 0$. From (59), we then have $\Pi(\mathbb{P}) = \sqrt{\frac{c\phi\bar{A}}{2}}$ for all $c < \frac{p^2\phi}{2\bar{A}}$ regardless of f. Therefore, $\Pi^*(p) = \sqrt{\frac{c\phi\bar{A}}{2}}$ for $c < \frac{p^2\phi}{2\bar{A}}$. When $c \geq \frac{\phi}{2\bar{A}}$, we observe f can be set to 1, making (60) active, and ensuring the maximum payoff of ϕ.

For $c \in [\frac{p^2\phi}{2\bar{A}}, \frac{\phi}{2\bar{A}})$, we claim the value f^* that satisfies $\lambda(f^*) = c$ characterizes the solution $A^{h*}, A^{\ell*}$ to (61). Such a value must exist and is unique, since $\lambda(f)$ is strictly increasing in f with $\lambda(1) > \frac{\phi}{2\bar{A}}$. One can solve this equation for f^* as follows: make the substitution $y = p + (1-p)f$ to obtain $y(\sqrt{y-p} + \sqrt{y})^2 = 2c\bar{A}/\phi$, which has the solution $y = \frac{2c\bar{A}/\phi}{2\sqrt{2c\bar{A}/\phi} - p}$. We then get $f^* = \frac{2c\bar{A}/\phi}{(1-p)(2\sqrt{2c\bar{A}/\phi}-p)} - \frac{p}{1-p}$ and subsequent endowments $A^{h*} = \phi\frac{2\sqrt{2c\bar{A}/\phi}-p}{2c}$ and $A^{\ell*} = \frac{1}{1-p}\left(\bar{A} - \frac{p\phi(2\sqrt{2c\bar{A}/\phi}-p)}{2c}\right)$. Denoting $\mathbb{P}^* = (A^{h*}, A^{\ell*}, p)$, from the second entry of (59) (since $c = \lambda$) we have $\Pi(\mathbb{P}^*) = p\phi + \sqrt{\frac{c\phi(1-p)A^{\ell*}}{2}} = \frac{p\phi}{2} + \sqrt{\frac{c\phi\bar{A}}{2}}$. The second equality follows due to $c \geq \frac{p^2\phi}{2\bar{A}}$.

We verify that $\Pi(\mathbb{P}^*) > \Pi(\mathbb{P})$ for any other $\mathbb{P} = (A^h, A^\ell, p)$ satisfying the expected endowment constraint. Let $f = A^\ell/A^h$. For any $f > f^*$, $\lambda(f) > \lambda(f^*)$. Since c is fixed, we thus obtain

$$\Pi(\mathbb{P}) = \sqrt{\frac{c\phi\bar{A}}{2}} < \Pi(\mathbb{P}^*) \tag{62}$$

For any $f < f^*$, we must have $A^\ell < A^{\ell*}$ (and $A^h > A^{h*}$). We then have $\Pi(\mathbb{P}) = p\phi + \sqrt{\frac{c\phi(1-p)A^\ell}{2}} < p\phi + \sqrt{\frac{c\phi(1-p)A^{\ell*}}{2}} = \Pi(\mathbb{P}^*)$. We thus obtain

$$\Pi^*(p) = \begin{cases} \sqrt{\frac{c\phi\bar{A}}{2}}, & \text{if } c < \frac{p^2\phi}{2\bar{A}} \\ \frac{p\phi}{2} + \sqrt{\frac{c\phi\bar{A}}{2}}, & \text{if } \frac{p^2\phi}{2\bar{A}} \leq c < \frac{\phi}{2\bar{A}} \\ \phi, & \text{if } \frac{\phi}{2\bar{A}} \leq c \end{cases} \tag{63}$$

and the optimal randomization is

$$(A^{h*}, A^{\ell*}) = \left(\phi\frac{2\sqrt{2c\bar{A}/\phi}-p}{2c}, \frac{1}{1-p}\left(\bar{A} - \frac{p\phi(2\sqrt{2c\bar{A}/\phi}-p)}{2c}\right)\right) \tag{64}$$

We can now readily obtain the optimal value of (57) by finding the maximum value of $\Pi^*(p)$ (63). We observe sub-colonel i obtains the maximum payoff ϕ for high costs $c \geq \frac{\phi}{2\bar{A}}$, irrespective of p. So, suppose $c < \frac{\phi}{2\bar{A}}$ is fixed. From (63), we may write

$$\Pi^*(p) = \begin{cases} \frac{p\phi}{2} + \sqrt{\frac{c\phi\bar{A}}{2}}, & \text{if } p \in (0, \sqrt{2c\bar{A}/\phi}] \\ \sqrt{\frac{c\phi\bar{A}}{2}}, & \text{if } p \in (\sqrt{2c\bar{A}/\phi}, 1] \end{cases}$$

The expression $\frac{p\phi}{2} + \sqrt{\frac{c\phi\bar{A}}{2}}$ is strictly increasing on $p \in (0, \sqrt{2c\bar{A}/\phi}]$. Hence Π^* is maximized at $p^* = \sqrt{2c\bar{A}/\phi}$. This gives the payoff $\sqrt{2c\phi\bar{A}} = 2 \cdot \pi_{\mathcal{A}}^{\text{CI}}$. ■

With the optimal marginal distributions established with the above Lemma, we are now ready to derive the solution of (55): the commander's optimal assignment given a fixed budget $A > 0$ and zero cost $c = 0$. Below, we provide the optimal assignment $\mathbb{P} \in \mathcal{P}(A)$ and the resulting payoff W_A^*.

Lemma A.3. *Suppose $c = 0$, $A < \infty$. The optimal assignment $\mathbb{P}^*$ that solves (55) is given in the following cases below.*
Case 1: *Suppose $A < \min_{i=1,2} \frac{c_1\phi_1 + c_2\phi_2}{2c_i^2}$. Let $A_1^* = \frac{c_1\phi_1 A}{c_1\phi_1 + c_2\phi_2}$, $A_2^* = \frac{c_2\phi_2 A}{c_1\phi_1 + c_2\phi_2}$, $p_1^* = \sqrt{2c_1A_1^*/\phi_1}$, and $p_2^* = \sqrt{2c_2A_2^*/\phi_2}$. Then $\mathbb{P}^*(0,0) = (1-p_1^*)(1-p_2^*)$, $\mathbb{P}^*\left(0, \sqrt{\frac{A_2^*\phi_2}{2c_2}}\right) = (1-p_1^*)p_2^*$, $\mathbb{P}^*\left(\sqrt{\frac{A_1^*\phi_1}{2c_1}}, 0\right) = p_1^*(1-p_2^*)$, $\mathbb{P}^*\left(\sqrt{\frac{A_1^*\phi_1}{2c_1}}, \sqrt{\frac{A_2^*\phi_2}{2c_2}}\right) = p_1^*p_2^*$. The resulting performance is $W_A^* = \sqrt{2A(c_1\phi_1 + c_2\phi_2)}$.*
Case 2: *Suppose $\min_{i=1,2} \frac{c_1\phi_1 + c_2\phi_2}{2c_i^2} \leq A < \frac{\phi_1}{2c_1} + \frac{\phi_2}{2c_2}$. Let $k = \arg\max_{i=1,2} c_i$, $A_k^* = \frac{\phi_k}{2c_k}$, $A_{-k}^* = A - \frac{\phi_k}{2c_k}$, and $p_{-k}^* = \sqrt{2c_{-k}A_{-k}^*/\phi_{-k}}$. Then*

$$\mathbb{P}_k^*(A_k^*) = 1 \quad \textit{and} \quad \mathbb{P}_{-k}^* = \left(\sqrt{\frac{A_{-k}^*\phi_{-k}}{2c_{-k}}}, 0, p_{-k}^*\right). \tag{65}$$

The resulting performance is $W_A^* = \phi_k + \sqrt{2c_{-k}\phi_{-k}\left(A - \frac{\phi_k}{2c_k}\right)}$.
Case 3: *Suppose $A \geq \frac{\phi_1}{2c_1} + \frac{\phi_2}{2c_2}$. Then any $\mathbb{P}^*$ that satisfies $\mathbb{P}^*(A_1^*, A_2^*) = 1$ for some $A_1^* \in [\frac{\phi_1}{2c_1}, A - \frac{\phi_2}{2c_2}]$ and $A_2^* = A - A_1^*$ is an optimal assignment. The resulting performance is $W_A^* = \phi_1 + \phi_2$.*

The proof was reported in [25]. The result below gives the solution to the optimal Stage 1 assignment problem (54) and corresponding final payoff W^* for the commander in the per-unit costs setting.

Lemma A.4. *The commander's optimal Stage 1 assignment $\mathbb{P}^*$ and corresponding final payoff W^* in the extensive-form game CAP under the per-unit cost setting is given as follows. Let $k = \arg\max_{i=1,2} c_i$.*

- *If $c \leq c_{-k}$, then $W^* = (1 - \frac{c}{2c_1})\phi_1 + (1 - \frac{c}{2c_2})\phi_2$ and $\mathbb{P}^*(\frac{\phi_1}{2c_1}, \frac{\phi_2}{2c_2}) = 1$. The resource expenditure is $A^* = \frac{\phi_1}{2c_1} + \frac{\phi_2}{2c_2}$.*
- *If $c_{-k} < c < c_k$, then $W^* = \phi_k + \frac{c_{-k}}{c}\phi_{-k}$ and*

$$\mathbb{P}_k^*(\frac{\phi_k}{2c_k}) = 1 \quad \textit{and} \quad \mathbb{P}_{-k}^* = \left(\frac{\phi_{-k}}{2c}, 0, \frac{c_{-k}}{c}\right). \tag{66}$$

The expected resource expenditure is $A^* = \frac{c_{-k}\phi_{-k}}{2c^2} + \frac{\phi_k}{2c_k}$.

- *If $c \geq c_k$, then $W^* = \frac{c_1\phi_1 + c_2\phi_2}{2c}$, and $\mathbb{P}^*(0,0) = (1 - c_1/c)(1-c_2/c)$, $\mathbb{P}^*\left(0, \frac{\phi_2}{2c}\right) = (1-c_1/c)\cdot(c_2/c)$, $\mathbb{P}^*\left(\frac{\phi_1}{2c}, 0\right) = (c_1/c)\cdot(1-c_2/c)$, $\mathbb{P}^*\left(\frac{\phi_1}{2c}, \frac{\phi_2}{2c}\right) = \frac{c_1c_2}{c^2}$. The expected resource expenditure is $A^* = \frac{c_1\phi_1 + c_2\phi_2}{2c^2}$.*

Proof. The solution of (54) under per-unit costs follows from solving the optimization problem

$$\max_{A\geq 0}\{W_A^* - c\cdot A\} \tag{67}$$

where we denote W_A^* as the performance from Lemma A.3. It is a concave objective, and the critical point A^* lies in $(0, \min_{i=1,2}\frac{c_1\phi_1+c_2\phi_2}{2c_i^2})$, $[\min_{i=1,2}\frac{c_1\phi_1+c_2\phi_2}{2c_i^2}, \frac{\phi_1}{2c_1}+\frac{\phi_2}{2c_2})$, or at $A^* = \frac{\phi_1}{2c_1}+\frac{\phi_2}{2c_2}$ depending on whether $c \geq \max_i c_i$, $c \in (\min_i c_i, \max_i c_i)$, or $c \leq \min_i c_i$, respectively. ■

We note in the result above that the optimal assignment is completely deterministic when the commander's cost is low ($c \leq c_{-k}$), and is randomized on both marginals if the cost is sufficiently high ($c \geq c_k$).

B. Optimal deterministic assignment – per-unit cost setting

The solution of CAP$_\mathrm{d}$ under the per-unit cost setting, i.e. $A_C = B_1 = B_2 = \infty$ with $c, c_1, c_2 > 0$ is derived in a similar manner to Lemma A.3. Recall that by the beginning of Stage 2, the commander has chosen a deterministic assignment (A_1, A_2). The Stage 2 decision problem for opponent $\mathcal{B}_i$ is therefore

$$\max_{B_i\geq 0}\left\{\phi_i\cdot\pi^{\mathrm{CI}}(B_i, A_i) - c_iB_i\right\} \tag{68}$$

where π^{CI} is defined in (4). Here, a complete information General Lotto game is played at Stage 3 since the opponents have observed the deterministic assignment (A_1, A_2).

Lemma B.1. *Consider the Stage 2 decision for opponent $\mathcal{B}_i$ in CAP$_d$ (68). The optimal investment for $\mathcal{B}_i$ is given by*

$$B_i^*(A_i) = \begin{cases}\sqrt{\frac{\phi_iA_i}{2c_i}}, & \text{if } c_i < \frac{\phi_i}{2A_i}\\ 0, & \text{otherwise}\end{cases} \tag{69}$$

Recall we are using the assumption that $\mathcal{B}_i$ chooses the smallest investment among multiple maximizers of (68). In Lemma B.1, this only arises if $c_i = \frac{\phi_i}{2A_i}$.

Now that we have established the optimal investment in Stage 2, we can address the commander's assignment decision problem in Stage 1. As an intermediate step, we first consider the optimal assignment problem when the commander has a fixed budget $A > 0$ and zero cost $c = 0$:

$$\max_{A_1\in[0,A]}\left\{\phi_1\cdot\pi_{\mathcal{A}_1}^{\mathrm{CI}}(A_1, B_1^*) + \phi_2\cdot\pi_{\mathcal{A}_2}^{\mathrm{CI}}(A - A_1, B_2^*)\right\} \tag{70}$$

Below, we give the optimal assignment of (70).

Lemma B.2. *The optimal deterministic assignment to (70) is given by the following cases. Define $Q_i := \sqrt{\frac{c_i\phi_i(A-\frac{\phi_{-i}}{2c_{-i}})}{2}}$ for $i = 1, 2$.*

Case 1: $A < \min_{i=1,2}\{\frac{\phi_i}{2c_i}\}$. *Then* $A_1^* = \frac{c_1\phi_1}{c_1\phi_1+c_2\phi_2}A$ *and* $W_d^* = \sqrt{\frac{A(c_1\phi_1+c_2\phi_2)}{2}}$.

Case 2: $\min_{i=1,2}\{\frac{\phi_i}{2c_i}\} \leq A < \max_{i=1,2}\{\frac{\phi_i}{2c_i}\}$. *Let* $j = \underset{i\in\{1,2\}}{\arg\min}\{\frac{\phi_i}{2c_i}\}$. *If* $A \geq \frac{c_1\phi_1+c_2\phi_2}{2c_j^2}$, *then* $A_1^* = \frac{\phi_j}{2c_j}$ *and* $W_d^* = \phi_j + Q_{-j}$. *If* $A < \frac{c_1\phi_1+c_2\phi_2}{2c_j^2}$, *then*

$$A_j^* = \begin{cases}\frac{\phi_j}{2c_j}, & \text{if } \phi_j + Q_{-j} \geq \sqrt{\frac{A(c_1\phi_1+c_2\phi_2)}{2}}\\ \frac{c_j\phi_j}{c_1\phi_1+c_2\phi_2}A, & \text{if } \phi_j + Q_{-j} < \sqrt{\frac{A(c_1\phi_1+c_2\phi_2)}{2}}\end{cases}$$

$$W_d(A_1^*) = \max\left\{\phi_j + Q_{-j}, \sqrt{\frac{A(c_1\phi_1+c_2\phi_2)}{2}}\right\} \tag{71}$$

Case 3: $\max_{i=1,2}\{\frac{\phi_i}{2c_i}\} \leq A < \frac{\phi_1}{2c_1}+\frac{\phi_2}{2c_2}$. *Then*

$$A_j^* = \begin{cases}A - \frac{\phi_{-j}}{2c_{-j}}, & \text{if } \phi_{-j}+Q_j \geq \phi_j + Q_{-j}\\ \frac{\phi_j}{2c_j}, & \text{if } \phi_{-j}+Q_j < \phi_j + Q_{-j}\end{cases} \tag{72}$$

$$\text{and} \quad W_d^* = \max_{i=1,2}\{\phi_i + Q_{-i}\}$$

Case 4: $\frac{\phi_1}{2c_1}+\frac{\phi_2}{2c_2} \leq A$. *Then* $A_1^* \in [\frac{\phi_1}{2c_1}, A - \frac{\phi_2}{2c_2})$ *and* $W_d^* = \phi_1 + \phi_2$.

Proof. We omit a proof since it follows similar techniques to Lemma A.3, and was reported in [25]. ■

The next result completely characterizes the commander's optimal Stage 1 assignment in CAP$_\mathrm{d}$ under the per-unit cost setting.

Lemma B.3. *Consider the deterministic commander assignment problem CAP$_d$. Denote $j = \underset{i\in\{1,2\}}{\arg\min}\,\frac{\phi_i}{2c_i}$ and $k = \underset{i\in 1,2}{\arg\max}\, c_i$. We define the intervals $I_\ell := c_{-k}\cdot[1-\frac{\sqrt{3}}{2}, 1+\frac{\sqrt{3}}{2}]$ and $I_r := c_k\cdot[1-\frac{\sqrt{3}}{2}, 1+\frac{\sqrt{3}}{2}]$. Enumerate the following four statements.*

(i) For $c \notin I_\ell \cup I_r$, we have $W_d^* = \frac{c_1\phi_1+c_2\phi_2}{8c}$, $A_j^* = \frac{c_j\phi_j}{8c^2}$, and the expenditure is $A^* = \frac{c_1\phi_1+c_2\phi_2}{8c^2}$.

(ii) For $c \in I_r\backslash I_\ell$ and $j = k$, or $c \in I_\ell\backslash I_r$ and $j \neq k$, we have $W_d^* = \frac{c_{-j}\phi_{-j}}{8c} + (1-\frac{c}{2c_j})\phi_j$, $A_j^* = \frac{\phi_j}{2c_j}$, and $A^* = \frac{c_{-j}\phi_{-j}}{8c^2} + \frac{\phi_j}{2c_j}$.

(iii) For $c \in I_\ell\backslash I_r$ and $j = k$, or $c \in I_r\backslash I_\ell$ and $j \neq k$, we have $W_d^* = \frac{c_j\phi_j}{8c} + (1-\frac{c}{2c_{-j}})\phi_{-j}$, $A_j^* = \frac{c_j\phi_j}{8c^2}$, and $A^* = \frac{c_j\phi_j}{8c^2} + \frac{\phi_{-j}}{2c_{-j}}$.

(iv) For $c \in I_\ell \cap I_r$, we have $W_d^* = (1-\frac{c}{2c_1})\phi_1 + (1-\frac{c}{2c_2})\phi_2$, $A_j^* = \frac{\phi_j}{2c_j}$, and $A^* = \frac{\phi_1}{2c_1}+\frac{\phi_2}{2c_2}$.

Denote $s_1 = \frac{1}{2}\sqrt{\frac{c_j}{\phi_j}(c_1\phi_1+c_2\phi_2)}$ and $s_2 = \frac{c_{-j}}{2}\sqrt{\frac{c_j\phi_{-j}}{c_j\phi_{-j}-c_{-j}\phi_j}}$. Then the optimal commander assignment is given as follows.

- *Suppose $c < \frac{c_j}{2}$. Then $W_d^* = (1-\frac{c}{2c_1})\phi_1 + (1-\frac{c}{2c_2})\phi_2$, $A_j^* = \frac{\phi_j}{2c_j}$, and $A^* = \frac{\phi_1}{2c_1}+\frac{\phi_2}{2c_2}$.*
- *Suppose $\frac{c_j}{2} \leq c < \min\{s_1, s_2\}$. Then the solution follows from whether the condition of (iii) or (iv) holds.*
- *Suppose $\min\{s_1, s_2\} \leq c < \max\{s_1, s_2\}$. If $s_1 \leq s_2$, then the solution follows from whether the condition of (i), (iii), or (iv) holds. If $s_1 > s_2$, then the solution follows from whether the condition of (i), (ii), or (iv) holds.*
- *Suppose $\max\{s_1, s_2\} \leq c$. Then the solution follows from whether the condition of (i), (ii), (iii), or (iv) holds.*

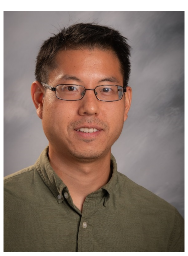
Proof. The commander's optimal assignment in CAP_d under the per-unit cost setting follows from solving the optimization problem

$$\max_{A\geq 0}\{W_\text{d}^*(A)-cA\} \tag{73}$$

where we denote $W_\text{d}^*(A)$ as the commander's optimal payoff given a fixed use-it-or-lose-it budget A, characterized from Lemma B.2. In general, the objective above is not concave and there are at most three points of discontinuity. There may exist up to four critical points in $A\in[0,\infty)$, depending on the value of c. These points are indicated by A^* in the enumerated list (i) - (iv) from the statement. A critical point exists on the interval $A\in(0,\min_{i=1,2}\{\frac{\phi_i}{2c_i}\})$ if $c>\frac{1}{2}\sqrt{\frac{c_j}{\phi_j}(c_1\phi_1+c_2\phi_2)}$, on the interval $A\in[\frac{\phi_j}{2c_j},\frac{\phi_{-j}}{2c_{-j}})$ if $c>\frac{c_{-j}}{2}\sqrt{\frac{c_j\phi_{-j}}{c_j\phi_{-j}-c_{-j}\phi_j}}$, and on the interval $A\in[\frac{\phi_{-j}}{2c_{-j}},\frac{\phi_1}{2c_1}+\frac{\phi_2}{2c_2})$ if $c>\frac{c_j}{2}$. A critical point always exists at $A=\frac{\phi_1}{2c_1}+\frac{\phi_2}{2c_2}$, as 0 is always contained in the sub-differential. The largest critical point is determined by the conditions listed as (i) to (iv) in the statement. ■

We note that the most amount of resources the commander will invest is $\frac{\phi_1}{2c_1}+\frac{\phi_2}{2c_2}$, which occurs for low costs $c<\frac{c_j}{2}$. Indeed, the quantity $\frac{\phi_i}{2c_i}$ is the amount of resources needed to win competition i outright. We can now combine the Lemmas to ascertain the result of Theorem 4.1.

Proof of Theorem 4.1, second part. By comparing the characterizations from Lemmas B.3 and A.4, we identify the four-fold improvement in the regime specified in the statement (first case in Lemma B.3, last case in Lemma A.4). Outside of this regime, the improvement factor is less than four. ■



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