

# RESILIENCE OF CUBE SLICING IN $\ell_p$

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## Abstract

*Ball's celebrated cube slicing theorem asserts that among hyperplane sections of the cube in  $\mathbb{R}^n$ , the central section orthogonal to  $(1, 1, 0, \dots, 0)$  has the greatest volume. We show that the same continues to hold for slicing  $\ell_p$  balls when  $p > 10^{15}$ , as well as that the same hyperplane minimizes the volume of projections of  $\ell_q$  balls for  $1 < q < 1 + 10^{-12}$ . This extends Szarek's optimal Khinchin inequality which corresponds to  $q = 1$ . These results thus address the resilience of the Ball–Szarek hyperplane in the ranges  $2 < p < \infty$  and  $1 < q < 2$ , where analysis of the extremizers has been elusive since the works of Koldobsky, Barthe and Naor, and Oleszkiewicz.*

## 1. Introduction

Fix  $p \in [1, \infty]$  and  $n \in \mathbb{N}$ . The present paper is devoted to the study of geometric parameters of the origin-symmetric convex bodies

$$B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\},$$

which are the closed unit balls of the normed spaces  $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ , where for  $p \in [1, \infty)$ ,

$$\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{1/p}$$

and  $\|x\|_\infty = \max_{i=1, \dots, n} |x_i|$ , when  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . More specifically, we shall address the classical problem of identifying volume extremizing sections and projections of these bodies with respect to hyperplanes passing through the origin. This subject has attracted the interest of mathematicians for decades and a range of tools from probability and Fourier analysis have been employed in its study. We refer to the survey [28] for a detailed account of classical results, recent advances, and further references.

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### 1.1. Sections

Fix  $p \in [1, \infty]$ ,  $n \in \mathbb{N}$ , and consider the following question for sections of  $\mathbf{B}_p^n$ .

#### QUESTION 1

*For which unit vectors  $a$  in  $\mathbb{R}^n$  is the volume of  $\mathbf{B}_p^n \cap a^\perp$  maximal or minimal?*

This problem and its variations have been intensively studied for five decades, since Hadwiger [13] and Hensley [14] showed that sections of the cube  $\mathbf{B}_\infty^n$  with coordinate hyperplanes  $e_i^\perp$  have minimal volume. The reverse question of identifying the volume maximizing sections of the cube was answered in the monumental work of Ball [2], who proved that

$$\text{vol}(\mathbf{B}_\infty^n \cap a^\perp) \leq \text{vol}\left(\mathbf{B}_\infty^n \cap \left(\frac{e_1 + e_2}{\sqrt{2}}\right)^\perp\right). \quad (1)$$

For  $p < \infty$ , the study of Question 1 was initiated by Meyer and Pajor. In [27], they extended the result of Hadwiger and Hensley by proving that sections of  $\mathbf{B}_p^n$  with coordinate hyperplanes  $e_i^\perp$  have minimal volume for any  $p \geq 2$  and maximal volume when  $p \in [1, 2]$ . In the reverse direction, they showed that when  $p = 1$ , the section of the cross-polytope  $\mathbf{B}_1^n$  with the hyperplane orthogonal to  $\frac{e_1 + \dots + e_n}{\sqrt{n}}$  has minimal volume, a result which was later extended to all values of  $p \in [1, 2]$  by Koldobsky [17] (see also [10] for a different probabilistic proof).

In view of the aforementioned results, the only missing case in the study of Question 1 is the identification of volume maximizing sections of  $\mathbf{B}_p^n$  when  $p \in (2, \infty)$ , a problem that has explicitly appeared in the literature multiple times (see, e.g., [4], [9], [18], [19], [21], [25], [28], [30]). In [30], Oleszkiewicz made a crucial remark, showing that for  $p \in (2, 26)$  and  $n$  large enough the section of  $\mathbf{B}_p^n$  with the hyperplane  $(\frac{e_1 + \dots + e_n}{\sqrt{n}})^\perp$  has in fact larger volume than the section with  $(\frac{e_1 + e_2}{\sqrt{2}})^\perp$  and thus one cannot expect a Ball-type extremal for all  $p > 2$ . In the same work, he speculated that Ball-type hyperplanes may maximize the volume of sections for *sufficiently large* values of  $p$ . The first theorem of this work provides a positive answer to Oleszkiewicz's question.

#### THEOREM 1

*There exists  $26 < p_0 < 10^{15}$  such that for every  $n \in \mathbb{N}$ ,  $p \geq p_0$ , and every unit vector  $a$  in  $\mathbb{R}^n$ , we have*

$$\text{vol}(\mathbf{B}_p^n \cap a^\perp) \leq \text{vol}\left(\mathbf{B}_p^n \cap \left(\frac{e_1 + e_2}{\sqrt{2}}\right)^\perp\right). \quad (2)$$

This is the first available result on maximal sections of  $\mathbf{B}_p^n$  for  $p \in (2, \infty)$  and any dimension  $n \geq 3$ . A general conjecture for all choices of  $p$  and  $n$ , predicting

that the extremals undergo a phase transition, was proposed in [32] and [28, Conjecture 2]. Theorem 1 partially confirms it. Let us formulate a more precise version of this conjecture.

## CONJECTURE 2

For every  $n \geq 3$ , there is a unique  $p_0(n)$  such that

$$\max_{a \in \mathbb{S}^{n-1}} \text{vol}(\mathbf{B}_p^n \cap a^\perp) = \begin{cases} \text{vol}(\mathbf{B}_p^n \cap (\frac{e_1 + \dots + e_n}{\sqrt{n}})^\perp) & 2 < p \leq p_0(n), \\ \text{vol}(\mathbf{B}_p^n \cap (\frac{e_1 + e_2}{\sqrt{2}})^\perp) & p \geq p_0(n). \end{cases} \quad (3)$$

Moreover,  $\lim_{n \rightarrow \infty} p_0(n) = 26.265\dots$  is the unique solution to the equation  $2^{2/p} \Gamma(\frac{1}{p})^3 = \pi p^2 \Gamma(\frac{3}{p})$  in the interval  $(1, \infty)$ .

Let us remark that in the above conjecture the critical value  $p = p_0(n)$  is given by the equation

$$\text{vol}\left(\mathbf{B}_p^n \cap \left(\frac{e_1 + \dots + e_n}{\sqrt{n}}\right)^\perp\right) = \text{vol}\left(\mathbf{B}_p^n \cap \left(\frac{e_1 + e_2}{\sqrt{2}}\right)^\perp\right). \quad (4)$$

The limit of the ratio of these two volumes is equal to  $\frac{2^{2/p} \Gamma(\frac{1}{p})^3}{\pi p^2 \Gamma(\frac{3}{p})}$ , as was proved by Oleszkiewicz in [30] using the central limit theorem.

## 1.2. Projections

Fix  $q \in [1, \infty]$ ,  $n \in \mathbb{N}$ , and consider the dual question for projections of  $\mathbf{B}_q^n$ .

## QUESTION 2

For which unit vectors  $a$  in  $\mathbb{R}^n$  is the volume of  $\text{Proj}_{a^\perp} \mathbf{B}_q^n$  maximal or minimal?

The current status of Question 2 is basically identical to that of Question 1. When  $q = \infty$ , Cauchy's projection formula shows that for every unit vector  $a$ , we have

$$\text{vol}(\text{Proj}_{a^\perp} \mathbf{B}_\infty^n) = \|a\|_1 \text{vol}(\mathbf{B}_\infty^{n-1}), \quad (5)$$

which proves that the volume is minimized for  $a = e_i$  and maximized for  $a = \frac{e_1 + \dots + e_n}{\sqrt{n}}$ . In the case of the cross-polytope  $\mathbf{B}_1^n$ , similar reasoning based on Cauchy's formula (see [3]) shows that

$$\text{vol}(\text{Proj}_{a^\perp} \mathbf{B}_1^n) = \frac{2^{n-1}}{(n-1)!} \mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right|, \quad (6)$$

where  $\varepsilon_1, \varepsilon_2, \dots$  is a sequence of independent symmetric  $\pm 1$  random variables. Therefore, Jensen's inequality shows that  $\text{vol}(\text{Proj}_{a^\perp} \mathbf{B}_1^n)$  is maximal when  $a = e_i$ . In view

of (6), identifying the volume minimizing projections of  $\mathbf{B}_1^n$  amounts to finding the sharp constant in the classical  $L_1$ - $L_2$  Khinchin inequality in [16] which was famously discovered by Szarek. In geometric terms, the important result of [31] asserts that  $\text{vol}(\text{Proj}_{a^\perp} \mathbf{B}_1^n)$  is minimized for  $a = \frac{e_1 + e_2}{\sqrt{2}}$ .

The study of Question 2 for  $1 < q < \infty$  was initiated by Barthe and Naor in [4]. In analogy to [27], they showed that projections of  $\mathbf{B}_q^n$  onto coordinate hyperplanes  $e_i^\perp$  have minimal volume for  $q \geq 2$  and maximal volume for  $q \in [1, 2]$ . Moreover, in the spirit of [17] and [27], they proved that when  $q \geq 2$ , the projections of  $\mathbf{B}_q^n$  onto the hyperplane orthogonal to  $\frac{e_1 + \dots + e_n}{\sqrt{n}}$  have maximal volume (see also [20] for a different proof using the Fourier transform).

The volume minimizing hyperplane projections of  $\mathbf{B}_q^n$  remain unknown for  $q \in (1, 2)$ . In analogy with Oleszkiewicz's observation in [30] mentioned earlier, Barthe and Naor noticed that for  $q \in (\frac{4}{3}, 2)$ , the projection of  $\mathbf{B}_q^n$  onto the hyperplane  $(\frac{e_1 + \dots + e_n}{\sqrt{n}})^\perp$  has smaller volume than the projection onto  $(\frac{e_1 + e_2}{\sqrt{2}})^\perp$  and thus one cannot expect a Szarek-type extremal for all  $q \in [1, 2)$ . Our second theorem is the dual to Theorem 1 and addresses Question 2 for  $q$  near 1.

#### THEOREM 3

There exists  $q_0 \in (1 + 10^{-12}, \frac{4}{3})$  such that for every  $n \in \mathbb{N}$ ,  $q \in [1, q_0]$ , and every unit vector  $a$  in  $\mathbb{R}^n$ , we have

$$\text{vol}(\text{Proj}_{a^\perp} \mathbf{B}_q^n) \geq \text{vol}(\text{Proj}_{(\frac{e_1 + e_2}{\sqrt{2}})^\perp} \mathbf{B}_q^n). \quad (7)$$

One can formulate a similar conjecture to the one for sections.

#### CONJECTURE 4

For every  $n \geq 3$ , there is a unique  $q_0(n)$  such that

$$\min_{a \in \mathbb{S}^{n-1}} \text{vol}(\text{Proj}_{a^\perp} \mathbf{B}_q^n) = \begin{cases} \text{vol}(\text{Proj}_{(\frac{e_1 + \dots + e_n}{\sqrt{n}})^\perp} \mathbf{B}_q^n) & q_0(n) < q \leq 2, \\ \text{vol}(\text{Proj}_{(\frac{e_1 + e_2}{\sqrt{2}})^\perp} \mathbf{B}_q^n) & 1 \leq q \leq q_0(n). \end{cases} \quad (8)$$

Moreover,  $\lim_{n \rightarrow \infty} q_0(n) = \frac{4}{3}$ .

#### 1.3. Methods

The delicacy of, say, Theorem 1 lies in the need to find a *universal*  $p_0$ , independent of the unit vector  $a$  and the dimension  $n \in \mathbb{N}$ , such that for every  $p \geq p_0$ ,

$$\text{vol}(\mathbf{B}_p^n \cap a^\perp) \leq \text{vol}\left(\mathbf{B}_p^n \cap \left(\frac{e_1 + e_2}{\sqrt{2}}\right)^\perp\right). \quad (9)$$

On the other hand, finding such a  $p_0(a)$  for a *fixed* unit vector  $a$  in  $\mathbb{R}^n$  is an immediate consequence of the continuity of the section function  $p \mapsto \text{vol}(\mathbf{B}_p^n \cap a^\perp)$ , as the equality cases in Ball's inequality (1) are known to be only the vectors of the form  $\frac{\pm e_i \pm e_j}{\sqrt{2}}$ , where  $i \neq j$ .

Let  $a = (a_1, \dots, a_n)$  be a unit vector, and assume without loss of generality that its coordinates are positive and ordered, that is,  $a_1 \geq a_2 \geq \dots \geq a_n \geq 0$ . Choosing  $p_0$  uniformly for (9) to hold requires radically different arguments in the following ranges for  $a$ .

*Case 1. The vector  $a$  is far from the extremizer  $\frac{e_1+e_2}{\sqrt{2}}$ , say  $|a - \frac{e_1+e_2}{\sqrt{2}}| \geq \delta_0$  for some  $\delta_0 > 0$ .*

Here the constant  $\delta_0$  depends on  $p$  and  $|\cdot|$  stands for the standard Euclidean norm. The key ingredient in this range is the dimension-free stability of Ball's inequality (1) with respect to the unit vector  $a$  which has been recently established in [7] and [26] (see also Theorem 9 below for a statement with explicit constants). These works imply that, under the assumption of Case 1, there is a positive deficit in Ball's inequality. Building on the simple-minded argument based on continuity described above, one needs to reason that all functions of the form  $p \mapsto \text{vol}(\mathbf{B}_p^n \cap a^\perp)$  are equicontinuous at  $p = \infty$  with a dimension-independent modulus. This strategy is implemented in Lemma 14 and relies on a combination of Busemann's theorem (see [5]) with a probabilistic formula expressing the volume of sections of  $\mathbf{B}_p^n$  as a negative moment of a sum of independent rotationally invariant random vectors in  $\mathbb{R}^3$ , following [6], [15], and [22].

*Case 2. The vector  $a$  is near the extremizer  $\frac{e_1+e_2}{\sqrt{2}}$ , say  $|a - \frac{e_1+e_2}{\sqrt{2}}| < \delta_0$ .*

This range is evidently the more subtle one, as soft continuity-based arguments are deemed to fail near the equality case. In order to amend this, we introduce a novel inductive strategy. As our starting point, we express again the section function  $\text{vol}(\mathbf{B}_p^n \cap a^\perp)$  as a negative moment of a sum of independent random variables. After a suitable application of Jensen's inequality, we use the inductive hypothesis according to which the desired inequality holds in dimension  $n - 2$  and this reduces the problem to an explicit two-dimensional estimate. Quite stunningly, the resulting estimate does not hold when the unit vector  $a$  is far from the extremizer  $\frac{e_1+e_2}{\sqrt{2}}$  and thus our inductive argument cannot circumvent the stability results which were crucially used in Case 1. Nevertheless, a delicate analysis allows us to deduce the technical estimate under the assumptions of Case 2 for  $\delta_0$  small enough as a function of  $p$  and  $p$  sufficiently large, thus proving Theorem 1.

The proof of Ball's inequality (1) and its stability from [7] crucially use the Fourier transform representation for the volume of sections and properties of a certain special function. However, even in Ball's original proof (see [2]), the Fourier transform method is unable to analyze the case that the largest component  $a_1$  of  $a$  is

greater than  $\frac{1}{\sqrt{2}}$ , which is instead handled by an elegant geometric argument. Unfortunately, a similar geometric argument applied to  $\mathbf{B}_p^n$  for  $p < \infty$  does not yield the optimal bound (9) for  $a_1$  slightly larger than  $\frac{1}{\sqrt{2}}$ , which creates the need for a different method. Surprisingly, our inductive approach outlined above does not use the Fourier transform directly, even though it uses Ball's inequality (1) and its stability as a black box. In a way, this method complements the Fourier-analytic approach with a probabilistic component which permits an analysis near the extremizer. We refer to [11] for extensions of the results of this paper under the additional assumption  $a_1 \leq \frac{1}{\sqrt{2}}$  using the Fourier analytic approach.

The proof of Theorem 3 relies on a very similar strategy apart from purely technical differences. In this case, the probabilistic representation for the volume of projections is due to [4] and the stability of Szarek's inequality was obtained in [8].

## 2. Preliminaries

In this section we present some probabilistic representations for the volume of sections and projections of  $\mathbf{B}_p^n$  (see also [28] and the references therein) along with some crucial technical estimates which will be used in the proofs of Theorems 1 and 3.

### 2.1. Probabilistic representation of the volume of sections

In [15], Kalton and Koldobsky discovered an elegant probabilistic representation of the volume of sections of a convex set  $K$  in  $\mathbb{R}^n$  in terms of negative moments of a random vector  $X$  uniformly distributed on  $K$ . In the case of  $K = \mathbf{B}_p^n$ , this representation takes the following explicit form (see [6] or [28, Lemma 42]).

#### LEMMA 5

Fix  $p \in [1, \infty)$ ,  $n \in \mathbb{N}$ , and let  $Y_1, Y_2, \dots$  be independent and identically distributed (i.i.d.) random variables with density  $e^{-\beta_p^p |x|^p}$ , where  $\beta_p = 2\Gamma(1 + \frac{1}{p})$ . Then for every unit vector  $a$  in  $\mathbb{R}^n$ , we have

$$\frac{\text{vol}(\mathbf{B}_p^n \cap a^\perp)}{\text{vol}(\mathbf{B}_p^{n-1})} = \lim_{s \downarrow -1} \frac{1+s}{2} \mathbb{E} \left| \sum_{j=1}^n a_j Y_j \right|^s. \quad (10)$$

When  $p = \infty$ , the same identity holds with  $Y_1, Y_2, \dots$  being i.i.d. uniform on  $[-1, 1]$ .

Using the representation (10), we derive the following crucial formula for our analysis.

#### PROPOSITION 6

Fix  $p \in [1, \infty)$  and  $n \in \mathbb{N}$ . Let  $R_1, R_2, \dots$  be i.i.d. positive random variables with density  $\alpha_p^{-1} x^p e^{-x^p} \mathbf{1}_{x>0}$ , where  $\alpha_p = \frac{1}{p} \Gamma(1 + \frac{1}{p})$ , and let  $\xi_1, \xi_2, \dots$  be i.i.d. random

vectors uniformly distributed on the unit sphere  $\mathbb{S}^2$  in  $\mathbb{R}^3$ , independent of the random variables  $R_i$ . Then for every unit vector  $a$  in  $\mathbb{R}^n$ , we have

$$\frac{\text{vol}(\mathbf{B}_p^n \cap a^\perp)}{\text{vol}(\mathbf{B}_p^{n-1})} = \Gamma\left(1 + \frac{1}{p}\right) \mathbb{E} \left| \sum_{j=1}^n a_j R_j \xi_j \right|^{-1}, \quad (11)$$

where  $|\cdot|$  denotes the Euclidean norm on the right-hand side. When  $p = \infty$ , the same identity holds with deterministic coefficients  $R_1 = \cdots = R_n = 1$ .

### Proof

We shall assume that  $p < \infty$  and the endpoint case follows (see also [22]). Let  $Y$  have density  $e^{-\beta_p^p |x|^p}$ , let  $R$  have density  $\alpha_p^{-1} x^p e^{-x^p} \mathbf{1}_{x>0}$ , and let  $U$  be uniform on  $[-1, 1]$ , independent of  $R$ . Then  $Y$  has the same distribution as  $\beta_p^{-1} R U$ . More generally, if  $V$  is a random variable with even density  $g$  which is nonincreasing and of class  $C^1$  on  $(0, +\infty)$ , then  $V$  has the same distribution as  $R_0 U$ , where  $R_0$  has density  $-2r g'(r)$  on  $(0, \infty)$ . Indeed, for  $t > 0$  we have

$$\begin{aligned} \mathbb{P}\{R_0 U > t\} &= \mathbb{P}\left\{U > \frac{t}{R_0}\right\} = \int_0^\infty \mathbb{P}\left\{U > \frac{t}{r}\right\} (-2r g'(r)) \, dr \\ &= - \int_t^\infty \left(1 - \frac{t}{r}\right) r g'(r) \, dr = - \int_t^\infty (r - t) g'(r) \, dr \\ &= \int_t^\infty g(r) \, dr = \mathbb{P}\{V > t\}. \end{aligned}$$

Therefore, (10) can be rewritten as

$$\frac{\text{vol}(\mathbf{B}_p^n \cap a^\perp)}{\text{vol}(\mathbf{B}_p^{n-1})} = \lim_{s \downarrow -1} \frac{1+s}{2\beta_p^s} \mathbb{E} \left| \sum_{j=1}^n a_j R_j U_j \right|^s. \quad (12)$$

By a result of König and Kwapien [24, Proposition 4], for every  $x_1, \dots, x_n \in \mathbb{R}$  and  $s > -1$ ,

$$\mathbb{E} \left| \sum_{j=1}^n x_j \xi_j \right|^s = (1+s) \mathbb{E} \left| \sum_{j=1}^n x_j U_j \right|^s. \quad (13)$$

Substituting (13) in (12) conditionally on  $R_j$  and substituting the value of  $\beta_p$  proves (11).  $\square$

### 2.2. Probabilistic representation of the volume of projections

The analogue of Proposition 6 for projections, expressing the normalized volume of projections of  $\mathbf{B}_q^n$  as an  $L_1$ -moment of a sum of independent random variables, has been established in [4, Proposition 2].

PROPOSITION 7 (Barthe and Naor [4])

Fix  $q \in (1, \infty)$  and  $n \in \mathbb{N}$ . Let  $X_1, X_2, \dots$  be i.i.d. random variables with density  $\gamma_q^{-1} |x|^{\frac{2-q}{q-1}} e^{-|x|^{\frac{q}{q-1}}}$ , where  $\gamma_q = 2(q-1)\Gamma(1 + \frac{1}{q})$ . Then for every unit vector  $a$  in  $\mathbb{R}^n$ , we have

$$\frac{\text{vol}(\text{Proj}_{a^\perp} \mathbf{B}_q^n)}{\text{vol}(\mathbf{B}_q^{n-1})} = \Gamma\left(\frac{1}{q}\right) \mathbb{E} \left| \sum_{j=1}^n a_j X_j \right|. \quad (14)$$

When  $q = 1$ , the identity reduces to the consequence (6) of the Cauchy projection formula.

### 2.3. Stability estimates

As explained in the introduction, a crucial step in the proofs of Theorems 1 and 3 is a reduction to sections and projections with respect to hyperplanes near the extremizer  $(\frac{e_1 + e_2}{\sqrt{2}})^\perp$ . This will be a consequence of two recent works [7], [8] establishing the stability of the inequalities of Szarek [31] and Ball [2] with respect to the unit normal vector  $a$ . For the case of projections, we will use the following robust Szarek inequality proved in [8, Theorem 3.2].

THEOREM 8 (De, Diakonikolas, and Servedio [8])

There exists  $\kappa_1 > 0$  such that for every  $n \in \mathbb{N}$  and every unit vector  $a$  in  $\mathbb{R}^n$  with  $a_1 \geq \dots \geq a_n \geq 0$ , we have

$$\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \geq \frac{1}{\sqrt{2}} + \kappa_1 \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|. \quad (15)$$

We can take  $\kappa_1 = 8 \cdot 10^{-5}$  in this inequality.

For the case of sections, we will use the following robust Ball inequality of [7, Equation (5)]. We express it in the equivalent negative moment formulation which follows from Proposition 6.

THEOREM 9 (Chasapis, Nayar, and Tkocz [7])

There exists  $\kappa_\infty > 0$  such that for every  $n \in \mathbb{N}$  and every unit vector  $a$  in  $\mathbb{R}^n$  with  $a_1 \geq \dots \geq a_n \geq 0$ , we have

$$\mathbb{E} \left| \sum_{j=1}^n a_j \xi_j \right|^{-1} \leq \sqrt{2} - \kappa_\infty \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|. \quad (16)$$

We can take  $\kappa_\infty = 6 \cdot 10^{-5}$  in this inequality.



Unfortunately, a direct implementation of the arguments of [7] and [8] does not yield explicit values for the constants  $\kappa_1$  and  $\kappa_\infty$ , which are needed for our estimation of  $p_0$  and  $q_0$  in Theorems 1 and 3. In Section 5, we shall present a new short proof of Theorem 8 which is in the spirit of [7] and gives the numerical constant  $\kappa_1 = 8 \cdot 10^{-5}$ . Moreover, we will explain how to quantify an existential argument used in [7] in order to prove Theorem 9 with  $\kappa_\infty = 6 \cdot 10^{-5}$ .

#### 2.4. A technical lemma

In this section we present the following key lemma, which is crucial for the induction argument sketched in Section 1.3 to work.

LEMMA 10

Let  $c \geq 1$  and  $p > 4\sqrt{2}c$ . If  $0 < a_2 \leq a_1$  satisfy  $\|(a_1, a_2)\|_p \leq 2^{\frac{1}{p}-\frac{1}{2}}$  and  $|a_i - \frac{1}{\sqrt{2}}| \leq \frac{c}{p}$  for  $i = 1, 2$ , then we have

$$|a_1 - a_2| \leq 3.65 \sqrt{\frac{c}{p-2}} \sqrt{1 - a_1^2 - a_2^2}. \quad (17)$$

To prove it, we need an elementary inequality between  $p$ -means with a deficit.

LEMMA 11

Let  $\sigma > 0$ ,  $r \geq \max\{\sigma, 2\}$ , and  $b_1, b_2 \in (0, 1]$  with  $1 - \frac{\sigma}{r} \leq \frac{b_2}{b_1} \leq 1$ . Then we have

$$\left(\frac{b_1^r + b_2^r}{2}\right)^{\frac{1}{r}} \geq \frac{b_1 + b_2}{2} + (r-1) \frac{1 - e^{-\frac{\sigma}{2}}}{4\sigma} |b_1 - b_2|^2. \quad (18)$$

*Proof*

Denote  $c_r \stackrel{\text{def}}{=} (r-1) \frac{1 - e^{-\frac{\sigma}{2}}}{4\sigma}$ . Dividing both sides by  $b_1$ , introducing  $\delta \stackrel{\text{def}}{=} 1 - \frac{b_2}{b_1}$ , raising the inequality to the power  $r$ , and using that  $b_1 \leq 1$ , we see that (18) follows from

$$\frac{1 + (1-\delta)^r}{2} \geq \left(1 - \frac{\delta}{2} + c_r \delta^2\right)^r, \quad \delta \in \left[0, \frac{\sigma}{r}\right].$$

We have equality for  $\delta = 0$  and thus it is enough to show that on  $[0, \frac{\sigma}{r}]$  the derivatives compare,

$$-\frac{r}{2}(1-\delta)^{r-1} \geq r \left(1 - \frac{\delta}{2} + c_r \delta^2\right)^{r-1} \left(-\frac{1}{2} + 2c_r \delta\right).$$

Multiplying both sides by  $\frac{2}{r}$  and rearranging gives an equivalent form

$$1 - 4c_r \delta \geq \left(\frac{1 - \delta}{1 - \frac{\delta}{2} + c_r \delta^2}\right)^{r-1},$$

since  $1 - \frac{\delta}{2} + c_r \delta^2 > 0$  on  $[0, \frac{\sigma}{r}]$ . To prove the last inequality, observe that

$$\left( \frac{1 - \delta}{1 - \frac{\delta}{2} + c_r \delta^2} \right)^{r-1} \leq \left( \frac{1 - \delta}{1 - \frac{\delta}{2}} \right)^{r-1} \leq \left( 1 - \frac{\delta}{2} \right)^{r-1}.$$

It is enough to check the inequality  $(1 - \frac{\delta}{2})^{r-1} \leq 1 - 4c_r \delta$  only for  $\delta \in \{0, \frac{\sigma}{r}\}$ , since the left-hand side is convex in  $\delta$ . For  $\delta = \frac{\sigma}{r}$ , we have  $(1 - \frac{\sigma}{2r})^{r-1} \leq e^{-\frac{\sigma}{2} \cdot \frac{r-1}{r}}$ , so we would like to prove that

$$e^{-\frac{\sigma}{2} \cdot \frac{r-1}{r}} \leq 1 - \frac{r-1}{r} (1 - e^{-\frac{\sigma}{2}}).$$

Since  $u = \frac{r-1}{r} \in [0, 1]$ , we want to verify that  $e^{-\frac{\sigma}{2}u} \leq 1 - u(1 - e^{-\frac{\sigma}{2}})$ , which follows by observing that the left-hand side is a convex function of  $u$  and we have equality for  $u \in \{0, 1\}$ .  $\square$

#### *Proof of Lemma 10*

Since  $p > \sqrt{2}c$ , we have

$$\frac{a_2}{a_1} \geq \frac{\frac{1}{\sqrt{2}} - \frac{c}{p}}{\frac{1}{\sqrt{2}} + \frac{c}{p}} = \frac{1 - \frac{\sqrt{2}c}{p}}{1 + \frac{\sqrt{2}c}{p}} \geq \left( 1 - \frac{\sqrt{2}c}{p} \right)^2 \geq 1 - 2\sqrt{2}\frac{c}{p},$$

so  $\frac{a_2^2}{a_1^2} \geq 1 - 4\sqrt{2}\frac{c}{p} = 1 - \frac{2\sqrt{2}c}{p/2}$ . We can apply Lemma 11 with  $r = \frac{p}{2}$ ,  $b_i = a_i^2$ , and  $\sigma = 2\sqrt{2}c$  to get

$$\frac{1}{2} \geq \left( \frac{a_1^p + a_2^p}{2} \right)^{\frac{2}{p}} \geq \frac{a_1^2 + a_2^2}{2} + \left( \frac{p}{2} - 1 \right) \frac{1 - e^{-\sqrt{2}c}}{8\sqrt{2}c} |a_1^2 - a_2^2|^2,$$

where the leftmost inequality is equivalent to  $\|(a_1, a_2)\|_p \leq 2^{\frac{1}{p}-\frac{1}{2}}$ . By the assumptions, we also have  $a_1 + a_2 \geq \sqrt{2} - \frac{2c}{p} \geq \sqrt{2} - \frac{1}{2\sqrt{2}}$  and  $e^{-c\sqrt{2}} < e^{-\sqrt{2}}$ . Therefore, rearranging gives

$$1 - a_1^2 - a_2^2 \geq \frac{c_0}{c} (p-2) |a_1 - a_2|^2, \quad c_0 = \frac{(\sqrt{2} - \frac{1}{2\sqrt{2}})^2}{8\sqrt{2}} (1 - e^{-\sqrt{2}}).$$

Thus, we conclude that

$$|a_1 - a_2| \leq \frac{\sqrt{c}}{\sqrt{c_0(p-2)}} \sqrt{1 - a_1^2 - a_2^2}, \quad \frac{1}{\sqrt{c_0}} < 3.65,$$

which completes the proof.  $\square$

### 3. Sections

#### 3.1. Ancillary results

We begin with a simple  $L_2$ -bound quantifying that the distribution of the random magnitudes  $R_j$  from (11) is close to the point mass at 1 as  $p$  gets large. Explicit computations using the density show that for every  $s > -p - 1$ , the  $s$ th moment of  $R_1$  is

$$\mathbb{E}R_1^s = \frac{\Gamma(1 + \frac{s+1}{p})}{\Gamma(1 + \frac{1}{p})}. \quad (19)$$

LEMMA 12

For  $p > 5$ , we have

$$\mathbb{E}|R_1 - 1|^2 \leq \frac{2}{\Gamma(1 + 1/p)} p^{-2}. \quad (20)$$

*Proof*

By (19), we can write

$$\mathbb{E}|R_1 - 1|^2 = \mathbb{E}R_1^2 - 2\mathbb{E}R_1 + 1 = \frac{\Gamma(1 + 3/p) - 2\Gamma(1 + 2/p) + \Gamma(1 + 1/p)}{\Gamma(1 + 1/p)}.$$

The function

$$h(x) \stackrel{\text{def}}{=} \Gamma(1 + 3x) - 2\Gamma(1 + 2x) + \Gamma(1 + x)$$

satisfies  $h(0) = h'(0) = 0$ , so for every  $0 < x < \frac{1}{5}$ , by Taylor's expansion with Lagrange's remainder, there exists  $0 < \theta < x$  such that

$$h(x) = \frac{1}{2}x^2h''(\theta) = \frac{1}{2}x^2(9\Gamma''(1 + 3\theta) - 8\Gamma''(1 + 2\theta) + \Gamma''(1 + \theta)). \quad (21)$$

LEMMA 13

The function  $\Gamma''$  is decreasing on  $(0, \frac{8}{5})$ .

Taking this for granted,  $\Gamma''(1 + 3\theta) < \Gamma''(1 + 2\theta)$  and  $\Gamma''(s) < \Gamma''(1) = \gamma^2 + \frac{\pi^2}{6} < 2$  for  $s \in (1, \frac{8}{5})$  (as usual  $\gamma = 0.577..$  is the Euler–Mascheroni constant, and this calculation of  $\Gamma''(1)$  can be done with the aid of the polygamma function; see, for instance, 6.4.2 in the standard reference [1]). Equation (21) thus gives  $h(x) \leq 2x^2$ . This applied to  $x = \frac{1}{p}$  leads to (20).  $\square$

*Proof of Lemma 13*

Note that clearly  $\Gamma^{(4)} > 0$ , so  $\Gamma^{(3)}$  increases, so for  $s \in (0, \frac{8}{5})$ , we have  $\Gamma^{(3)}(s) <$

$\Gamma^{(3)}(\frac{8}{5}) = -0.33..$  (to obtain such numerical values, we refer again to [1]). Therefore,  $\Gamma''$  is decreasing on  $(0, \frac{8}{5})$ .  $\square$

To deal with hyperplanes far from the extremizer, we will crucially rely on the equicontinuity of the section functions at  $p = \infty$  which we will now verify. For  $p \in [1, \infty]$  we introduce the normalized section function,

$$A_{n,p}(a) \stackrel{\text{def}}{=} \frac{\text{vol}(\mathbf{B}_p^n \cap a^\perp)}{\text{vol}(\mathbf{B}_p^{n-1})}, \quad (22)$$

where  $a$  is a unit vector in  $\mathbb{R}^n$ . Additionally, observe that

$$A_{n,\infty}(a) = \frac{\text{vol}(\mathbf{B}_\infty^n \cap a^\perp)}{\text{vol}(\mathbf{B}_\infty^{n-1})} = \text{vol}(\mathbf{Q}_n \cap a^\perp),$$

where  $\mathbf{Q}_n = [-\frac{1}{2}, \frac{1}{2}]^n$  is the unit-volume cube in  $\mathbb{R}^n$ . Recall that from Proposition 6,

$$A_{n,p}(a) = \Gamma\left(1 + \frac{1}{p}\right) \mathbb{E} \left| \sum_{j=1}^n a_j R_j \xi_j \right|^{-1}.$$

LEMMA 14

Let  $p > 5$ . For every unit vector  $a$  in  $\mathbb{R}^n$ , we have

$$|A_{n,p}(a) - A_{n,\infty}(a)| \leq \frac{5}{p}. \quad (23)$$

*Proof*

First recall that for an arbitrary nonzero vector  $x$  in  $\mathbb{R}^n$ ,

$$\mathbf{N}(x) \stackrel{\text{def}}{=} \frac{|x|}{\text{vol}(\mathbf{Q}_n \cap x^\perp)} = \left( \mathbb{E} \left| \sum_{j=1}^n x_j \xi_j \right|^{-1} \right)^{-1}$$

is a norm by Busemann's theorem (see [5]). In particular, using  $1 \leq \text{vol}(\mathbf{Q}_n \cap x^\perp) \leq \sqrt{2}$ , we get

$$\begin{aligned} |\mathbf{N}(y)^{-1} - \mathbf{N}(x)^{-1}| &= \frac{|\mathbf{N}(x) - \mathbf{N}(y)|}{\mathbf{N}(x)\mathbf{N}(y)} \leq \frac{\mathbf{N}(x-y)}{\mathbf{N}(x)\mathbf{N}(y)} \\ &= \frac{|x-y|}{|x||y|} \frac{\text{vol}(\mathbf{Q}_n \cap x^\perp) \text{vol}(\mathbf{Q}_n \cap y^\perp)}{\text{vol}(\mathbf{Q}_n \cap (x-y)^\perp)} \leq 2 \frac{|x-y|}{|x||y|}, \end{aligned}$$

where  $x, y \in \mathbb{R}^n \setminus \{0\}$ . Evoking (11), we can write

$$\frac{A_{n,p}(a)}{\Gamma(1 + 1/p)} = \mathbb{E}_R \mathbb{E}_\xi \left| \sum_{j=1}^n a_j R_j \xi_j \right|^{-1} = \mathbb{E}_R \mathbf{N}(aR)^{-1},$$

where we use the ad hoc notation  $aR$  for the vector  $(a_1 R_1, \dots, a_n R_n)$  in  $\mathbb{R}^n$ . From the previous bound on  $1/N$ , we thus obtain

$$\begin{aligned} \left| \frac{A_{n,p}(a)}{\Gamma(1+1/p)} - A_{n,\infty}(a) \right| &= \left| \mathbb{E}N(aR)^{-1} - N(a)^{-1} \right| \\ &\leq 2\mathbb{E} \frac{|a - aR|}{|a| \cdot |aR|} = 2\mathbb{E} \frac{|a - aR|}{|aR|}. \end{aligned}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned} \mathbb{E} \frac{|a - aR|}{|aR|} &\leq \sqrt{\mathbb{E}|a - aR|^2} \sqrt{\mathbb{E}|aR|^{-2}} \\ &= \sqrt{\mathbb{E} \sum_{j=1}^n a_j^2 (R_j - 1)^2} \sqrt{\mathbb{E} \left( \sum_{j=1}^n a_j^2 R_j^2 \right)^{-1}}. \end{aligned}$$

The first factor on the right-hand side is equal to  $\|R_1 - 1\|_2$ . By the convexity of the function  $s \mapsto \frac{1}{s}$ ,

$$\mathbb{E} \left( \sum_{j=1}^n a_j^2 R_j^2 \right)^{-1} \leq \sum_{j=1}^n a_j^2 \mathbb{E} R_j^{-2} \stackrel{(19)}{=} \frac{\Gamma(1 - \frac{1}{p})}{\Gamma(1 + \frac{1}{p})}.$$

Combining all the above yields

$$\left| A_{n,p}(a) - \Gamma(1+1/p) A_{n,\infty}(a) \right| \leq 2\|R_1 - 1\|_2 \sqrt{\Gamma(1-1/p)\Gamma(1+1/p)}.$$

Using Lemma 12, the right-hand side gets upper-bounded by

$$2\sqrt{\frac{2}{\Gamma(1+1/p)}} p^{-2} \sqrt{\Gamma(1-1/p)\Gamma(1+1/p)} < \frac{2\sqrt{2\Gamma(1/2)}}{p} = \frac{2\sqrt{2}\sqrt[4]{\pi}}{p}$$

using  $p > 2$ . Consequently,

$$\begin{aligned} \left| A_{n,p}(a) - A_{n,\infty}(a) \right| &\leq \frac{2\sqrt{2}\sqrt[4]{\pi}}{p} + (1 - \Gamma(1+1/p)) A_{n,\infty}(a) \\ &\leq \frac{2\sqrt{2}\sqrt[4]{\pi}}{p} + \frac{\sqrt{2}\gamma}{p} < \frac{5}{p}, \end{aligned}$$

because  $1 - \Gamma(1+x) < -\Gamma'(1)x = \gamma x$  for  $0 < x < 1$ , by convexity of  $\Gamma$  on  $(0, \infty)$ .  $\square$

### 3.2. Proof of Theorem 1

Following notation (22), our goal is to prove that for every  $p \geq p_0$  and every unit vector  $a$  in  $\mathbb{R}^n$ , we have

$$A_{n,p}(a) \leq A_{n,p}\left(\frac{e_1 + e_2}{\sqrt{2}}\right), \quad (24)$$

where the right-hand side is explicitly given by

$$\begin{aligned} A_{n,p}\left(\frac{e_1 + e_2}{\sqrt{2}}\right) &= \Gamma\left(1 + \frac{1}{p}\right) \mathbb{E} \left| \frac{R_1 \xi_1 + R_2 \xi_2}{\sqrt{2}} \right|^{-1} \\ &= A_{2,p}\left(\frac{e_1 + e_2}{\sqrt{2}}\right) = \frac{1}{\|(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\|_p} = 2^{\frac{1}{2} - \frac{1}{p}}. \end{aligned}$$

Our proof will proceed by induction on  $n$ . It is directly checked that the theorem holds when  $n = 2$ , as  $A_{2,p}(a) = \|a\|_p^{-1}$  for every unit vector  $a$  in  $\mathbb{R}^2$ . We therefore assume that  $n \geq 3$  and  $a_1 \geq \dots \geq a_n > 0$ . Our analysis will differ depending on the distance of  $a$  to the extremizer. Let

$$\delta(a) \stackrel{\text{def}}{=} \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|^2 = 2 - \sqrt{2}(a_1 + a_2). \quad (25)$$

#### 3.2.1. The vector $a$ is far from the extremizer

Suppose that  $\sqrt{\delta(a)} \geq \frac{c}{p}$  with  $c = 10^5$ . Then, by the equicontinuity proved in Lemma 14 and the stability of Ball's inequality from Theorem 9 with constant  $\kappa_\infty = 6 \cdot 10^{-5}$ , we obtain

$$A_{n,p}(a) \leq \frac{5}{p} + A_{n,\infty}(a) \leq \frac{5}{p} + \sqrt{2} - \kappa_\infty \sqrt{\delta(a)} \leq \sqrt{2} - \frac{\kappa_\infty c - 5}{p}.$$

Since

$$c \geq \frac{\sqrt{2} \log 2 + 5}{\kappa_\infty} = \frac{\sqrt{2} \log 2 + 5}{6} 10^5 = 0.996.. \cdot 10^5,$$

we have

$$\sqrt{2} - \frac{\kappa_\infty c - 5}{p} \leq \sqrt{2} \left(1 - \frac{\log 2}{p}\right) \leq \sqrt{2} e^{-\frac{\log 2}{p}} = 2^{\frac{1}{2} - \frac{1}{p}},$$

which finishes the proof in this case (without using the inductive hypothesis).

#### 3.2.2. The vector $a$ is close to the extremizer

Now, suppose that

$$\sqrt{\delta(a)} < \frac{c}{p},$$

where  $c = 10^5$ . This in particular implies that (as we already assume that  $a_2 \leq a_1$ )

$$\frac{1}{\sqrt{2}} - \frac{c}{p} \leq a_2 \leq a_1 \leq \frac{1}{\sqrt{2}} + \frac{c}{p}.$$

Let us also notice for further use that  $a_2 \leq \frac{1}{\sqrt{2}}$  since  $2a_2^2 \leq a_1^2 + a_2^2 \leq 1$ . We shall consider  $p > Lc + 2$  for a large numerical constant  $L$ . With hindsight, we put  $L = 7.9 \cdot 10^9$ . Observe that our goal (24) is equivalent to the inequality

$$\mathbb{E} \left| \sum_{j=1}^n a_j R_j \xi_j \right|^{-1} \leq C_p \quad (26)$$

with

$$C_p = \mathbb{E} \left| \frac{R_1 \xi_1 + R_2 \xi_2}{\sqrt{2}} \right|^{-1} = \frac{2^{\frac{1}{2} - \frac{1}{p}}}{\Gamma(1 + 1/p)}, \quad (27)$$

which we will now prove by induction on  $n$ . We record for future estimates that when  $p > Lc + 2$ , we have

$$1.41 < C_p < 1.42, \quad (28)$$

since  $2^{10^{-6}} > 2^{1/p} \Gamma(1 + 1/p) > \Gamma(1 + 10^{-6})$ .

Consider the random vectors  $X = a_1 R_1 \xi_1 + a_2 R_2 \xi_2$  and  $Y = \sum_{j>2} a_j R_j \xi_j$  in  $\mathbb{R}^3$ . Since  $X$  and  $Y$  are independent and rotationally invariant, the representation

$$\mathbb{E} \left| \sum_{j=1}^n a_j R_j \xi_j \right|^{-1} = \mathbb{E} \min\{|X|^{-1}, |Y|^{-1}\}$$

holds (see, e.g., [7, Lemma 6.6]). By the inductive hypothesis,

$$\mathbb{E}|Y|^{-1} = \frac{1}{\sqrt{1 - a_1^2 - a_2^2}} \mathbb{E} \left| \frac{\sum_{j>2} a_j R_j \xi_j}{\sqrt{1 - a_1^2 - a_2^2}} \right|^{-1} \leq \frac{C_p}{\sqrt{1 - a_1^2 - a_2^2}},$$

and hence (by the concavity and monotonicity of the function  $t \mapsto \min\{|X|^{-1}, t\}$ ), we get

$$\mathbb{E} \left| \sum_{j=1}^n a_j R_j \xi_j \right|^{-1} = \mathbb{E} \min\{|X|^{-1}, |Y|^{-1}\} \leq \mathbb{E} \min\{|X|^{-1}, \alpha^{-1}\}, \quad (29)$$

where we set

$$\alpha \stackrel{\text{def}}{=} \frac{1}{C_p} \sqrt{1 - a_1^2 - a_2^2}. \quad (30)$$

Observe that

$$\mathbb{E} \min\{|X|^{-1}, \alpha^{-1}\} = \mathbb{E}|X|^{-1} - \mathbb{E}(|X|^{-1} - \alpha^{-1})_+ \quad (31)$$

and

$$\begin{aligned} \mathbb{E}|X|^{-1} &= \frac{1}{\sqrt{a_1^2 + a_2^2}} \mathbb{E} \left| \frac{a_1 R_1 \xi_1 + a_2 R_2 \xi_2}{\sqrt{a_1^2 + a_2^2}} \right|^{-1} \\ &\stackrel{(11)}{=} \frac{1}{\sqrt{a_1^2 + a_2^2}} \frac{1}{\|(a_1, a_2)\|_p \Gamma(1 + 1/p)} = \frac{1}{\|(a_1, a_2)\|_p \Gamma(1 + 1/p)}. \end{aligned} \quad (32)$$

In view of the inductive step (29) and the identities (27), (31), and (32), the desired inequality (26) is a consequence of the following proposition.

PROPOSITION 15

Under the assumptions and notation above, for  $p \geq 10^{15}$  we have

$$\mathbb{E}(|X|^{-1} - \alpha^{-1})_+ \geq C_p \left( \frac{2^{\frac{1}{p}-\frac{1}{2}}}{\|(a_1, a_2)\|_p} - 1 \right). \quad (33)$$

*Proof*

If the right-hand side is nonpositive, then we are done. Otherwise,

$$\|(a_1, a_2)\|_p < 2^{\frac{1}{p}-\frac{1}{2}}.$$

Since  $|a_i - \frac{1}{\sqrt{2}}| < \frac{c}{p}$ , Lemma 10 gives

$$\begin{aligned} |a_1 - a_2| &\leq 3.65 \sqrt{\frac{c}{p-2}} \sqrt{1 - a_1^2 - a_2^2} \\ &\stackrel{(30)}{=} 3.65 \sqrt{\frac{c}{p-2}} C_p \alpha \stackrel{(28)}{\leq} \frac{5.25\alpha}{\sqrt{L}}. \end{aligned} \quad (34)$$

To simplify, note that  $\|(a_1, a_2)\|_p \geq 2^{1/p-1/2} \|(a_1, a_2)\|_2$ , so

$$\frac{2^{\frac{1}{p}-\frac{1}{2}}}{\|(a_1, a_2)\|_p} - 1 \leq \frac{1}{\|(a_1, a_2)\|_2} - 1 = \frac{1 - (a_1^2 + a_2^2)}{\sqrt{a_1^2 + a_2^2}(1 + \sqrt{a_1^2 + a_2^2})} < \frac{C_p^2 \alpha^2}{1.95},$$

where we used that

$$a_1^2 + a_2^2 \geq 2a_2^2 \geq 2\left(\frac{1}{\sqrt{2}} - \frac{c}{p}\right)^2 > 1 - \frac{2\sqrt{2}c}{p} > 1 - \frac{2\sqrt{2}}{L} > 0.97 \quad (35)$$



and  $\sqrt{u}(1 + \sqrt{u}) > 1.95$  for  $u > 0.97$ . Since  $\frac{C_p^3}{1.95} < \frac{1.42^3}{1.95} < \frac{3}{2}$ , (33) will follow from

$$\mathbb{E}(|X|^{-1} - \alpha^{-1})_+ \geq \frac{3}{2}\alpha^2. \quad (36)$$

Consider the event

$$\mathcal{E} \stackrel{\text{def}}{=} \left\{ R_1 \leq 1, |R_1 - R_2| < \alpha, |a_1\xi_1 + a_2\xi_2| < \frac{1}{4}\alpha \right\}.$$

On  $\mathcal{E}$ , we have

$$\begin{aligned} |X| &= |a_1 R_1 \xi_1 + a_2 R_2 \xi_2| \leq |a_1 R_1 \xi_1 + a_2 R_1 \xi_2| + |a_2 R_2 \xi_2 - a_2 R_1 \xi_2| \\ &= R_1 |a_1 \xi_1 + a_2 \xi_2| + a_2 |R_2 - R_1| \\ &< \frac{1}{4}\alpha + \frac{1}{\sqrt{2}}\alpha < \frac{24}{25}\alpha, \end{aligned}$$

so

$$\begin{aligned} \mathbb{E}(|X|^{-1} - \alpha^{-1})_+ &\geq \frac{1}{24\alpha} \mathbb{P}(\mathcal{E}) \\ &= \frac{1}{24\alpha} \mathbb{P}\{R_1 \leq 1, |R_1 - R_2| < \alpha\} \mathbb{P}\left\{|a_1 \xi_1 + a_2 \xi_2| < \frac{1}{4}\alpha\right\}. \end{aligned} \quad (37)$$

For the second probability in (37), observe that the random variable  $|a_1 \xi_1 + a_2 \xi_2|^2$  has the same distribution as  $a_1^2 + a_2^2 + 2a_1 a_2 U$ , with  $U$  being uniform on  $[-1, 1]$ . Therefore,

$$\mathbb{P}\left\{|a_1 \xi_1 + a_2 \xi_2| < \frac{1}{4}\alpha\right\} = \mathbb{P}\left\{U < \frac{\alpha^2/16 - a_1^2 - a_2^2}{2a_1 a_2}\right\}.$$

Note that the condition

$$-1 < \frac{\alpha^2/16 - a_1^2 - a_2^2}{2a_1 a_2} < 1 \quad (38)$$

is equivalent to

$$|a_1 - a_2| < \frac{\alpha}{4} < a_1 + a_2.$$

The left inequality holds thanks to (34), provided that  $L > (5.25 \cdot 4)^2 = 441$ , whereas the right inequality holds since  $a_1 + a_2 \geq 2a_2 > \sqrt{2} - \frac{2c}{p} > \sqrt{2} - \frac{2}{L} > 1.2$  which is greater than  $\frac{\alpha}{4}$  since

$$\alpha \leq \frac{1}{C_p} \sqrt{1 - 2a_2^2} \stackrel{(28)}{<} \frac{1}{1.41} \sqrt{1 - 2\left(\frac{1}{\sqrt{2}} - \frac{c}{p}\right)^2} < \frac{1}{1.41} \sqrt{2\sqrt{2}\frac{c}{p}} < \frac{1.2}{\sqrt{L}}. \quad (39)$$

As (38) holds, we have

$$\begin{aligned}\mathbb{P}\left\{|a_1\xi_1 + a_2\xi_2| < \frac{1}{4}\alpha\right\} &= \frac{1}{2}\left(\frac{\alpha^2/16 - a_1^2 - a_2^2}{2a_1a_2} + 1\right) \\ &= \frac{\alpha^2/16 - (a_1 - a_2)^2}{4a_1a_2}.\end{aligned}$$

Using (34) and the estimate  $4a_1a_2 \leq 2(a_1^2 + a_2^2) < 2$ , we get

$$\mathbb{P}\left\{|a_1\xi_1 + a_2\xi_2| < \frac{1}{4}\alpha\right\} > \frac{1 - 441L^{-1}}{32}\alpha^2. \quad (40)$$

For the other probability in (37), it is convenient to place a uniform function of constant mass under the density of  $R_1$ , which is doable due to the following technical lemma.

LEMMA 16

Fix  $p \in (1, \infty)$ , and let  $g_p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be the density of  $R_1$ . Then we have

$$\forall x > 0, \quad g_p(x) \geq \frac{p}{4}\mathbf{1}_{[1-\frac{1}{2p}, 1]}(x). \quad (41)$$

*Proof*

Recall from Proposition 6 that  $g_p(x) = p\Gamma(1 + 1/p)^{-1}x^pe^{-x^p}$  for  $x > 0$ . Since  $g_p$  is log-concave, it suffices to check the inequality at the endpoints  $x = 1 - \frac{1}{2p}$  and  $x = 1$ . For the first endpoint, we have

$$g_p\left(1 - \frac{1}{2p}\right) = \frac{p}{\Gamma(1 + 1/p)}\left(1 - \frac{1}{2p}\right)^pe^{-(1-\frac{1}{2p})^p} > \frac{p}{2}e^{-e^{-1/2}} > \frac{p}{4},$$

using that  $(1 - \frac{1}{2p})^p < e^{-1/2}$  and  $(1 - \frac{1}{2p})^p > \frac{1}{2}$ . Moreover, for the second endpoint,

$$g_p(1) = \frac{p}{e\Gamma(1 + 1/p)} > \frac{p}{4}. \quad \square$$

*Finishing the proof of Proposition 15*

We estimate the first probability in (37) using Lemma 16,

$$\begin{aligned}\mathbb{P}\{R_1 \leq 1, |R_1 - R_2| < \alpha\} \\ &\geq \int \int_{\{x \leq 1, |x-y| < \alpha\}} \left(\frac{p}{4}\right)^2 \mathbf{1}_{[1-\frac{1}{2p}, 1]}(x) \mathbf{1}_{[1-\frac{1}{2p}, 1]}(y) \, dx \, dy \\ &= \begin{cases} \frac{1}{64} & \text{if } \alpha > \frac{1}{2p}, \\ \frac{p^2\alpha}{16}\left(\frac{1}{p} - \alpha\right) & \text{if } \alpha \leq \frac{1}{2p}, \end{cases} \quad (42)\end{aligned}$$

where the equality is an elementary computation. In the case  $\alpha \leq \frac{1}{2p}$ , we further have  $\frac{1}{p} - \alpha \geq \frac{1}{2p}$ , so the probability is further bounded from below by  $\frac{p\alpha}{32}$ , which we will use.

- If  $\alpha > \frac{1}{2p}$ , then inequalities (37), (40), and (42) yield the lower bound

$$\begin{aligned} \mathbb{E}(|X|^{-1} - \alpha^{-1})_+ &\geq \frac{1}{24\alpha} \cdot \frac{1}{64} \cdot \frac{1 - 441L^{-1}}{32} \alpha^2 = \left( \frac{1 - 441L^{-1}}{2^{14} \cdot 3} \cdot \frac{1}{\alpha} \right) \alpha^2 \\ &\stackrel{(39)}{>} \left( \frac{1 - 441L^{-1}}{2^{14} \cdot 3 \cdot 1.2} \sqrt{L} \right) \alpha^2. \end{aligned}$$

Since  $L = 7.9 \cdot 10^9$ , this gives the desired bound (36) by  $\frac{3}{2}\alpha^2$ .

- If  $\alpha \leq \frac{1}{2p}$ , then inequalities (37), (40), and (42) yield the lower bound

$$\begin{aligned} \mathbb{E}(|X|^{-1} - \alpha^{-1})_+ &\geq \frac{1}{24\alpha} \cdot \frac{p\alpha}{32} \cdot \frac{1 - 441L^{-1}}{32} \alpha^2 = \frac{p(1 - 441L^{-1})}{2^{13} \cdot 3} \alpha^2 \\ &> \frac{(L - 441)c}{2^{13} \cdot 3} \alpha^2. \end{aligned}$$

This is at least  $\frac{3}{2}\alpha^2$  for the chosen  $L$ , which completes the proof of (36) for  $p \geq p_0$ , where

$$p_0 = Lc + 2 < 8 \cdot 10^9 \cdot 10^5 < 10^{15}. \quad \square$$

## 4. Projections

The proof here parallels the one from Section 3. For the reader's convenience, we include all the details (which are in fact easier in several places).

### 4.1. Ancillary results

We start by quantifying how close the distribution of the  $X_j$  from (14) is to that of a Rademacher variable (in the Wasserstein-2 distance). Explicit computations using the density show that for every  $s > -\frac{1}{q-1}$ , the  $s$ th moment of  $|X_1|$  is

$$\mathbb{E}|X_1|^s = \frac{\Gamma(1 + \frac{(s-1)(q-1)}{q})}{\Gamma(\frac{1}{q})}. \quad (43)$$

LEMMA 17

For  $1 < q < 2$ , we have

$$\mathbb{E}|X_1 - \text{sgn}(X_1)|^2 \leq 9 \left(1 - \frac{1}{q}\right)^2. \quad (44)$$

*Proof*

Observe that

$$\mathbb{E}|X_1 - \operatorname{sgn}(X_1)|^2 = \mathbb{E}X_1^2 - 2\mathbb{E}|X_1| + 1 \stackrel{(43)}{=} \frac{\Gamma(2 - 1/q) - 2 + \Gamma(1/q)}{\Gamma(1/q)}.$$

Since  $\Gamma$  is decreasing on  $(0, 1)$ ,  $\Gamma(1/q) \geq \Gamma(1) = 1$ . Using Taylor's expansion with Lagrange's remainder, for every  $0 < x < 1$  there exists  $0 < \theta < x$  such that

$$h(x) \stackrel{\text{def}}{=} \Gamma(1 - x) + \Gamma(1 + x) - 2 = \frac{1}{2}x^2(\Gamma''(1 - \theta) + \Gamma''(1 + \theta)).$$

Thus for  $0 < x < 1/2$ , we have

$$h(x) \leq \frac{1}{2}x^2(\|\Gamma''\|_{L_\infty(\frac{1}{2}, 1)} + \|\Gamma''\|_{L_\infty(1, \frac{3}{2})}) = \frac{1}{2}x^2(\Gamma''(1/2) + \Gamma''(1)) < 9x^2$$

since  $\Gamma''$  decreases on  $(\frac{1}{2}, \frac{3}{2})$ , by Lemma 13. Applying this to  $x = 1 - \frac{1}{q}$ , we indeed obtain

$$\mathbb{E}|X_1 - \operatorname{sgn}(X_1)|^2 \leq h\left(1 - \frac{1}{q}\right) \leq 9\left(1 - \frac{1}{q}\right)^2. \quad \square$$

From this estimate, we can easily deduce the equicontinuity of the normalized projection functions, which we state directly in probabilistic terms in view of Proposition 7.

LEMMA 18

Let  $1 < q < 2$ ,  $X_1, X_2, \dots$  be i.i.d. random variables from (14), and let  $\varepsilon_1, \varepsilon_2, \dots$  be i.i.d. Rademacher random variables. For every unit vector  $a$  in  $\mathbb{R}^n$ , we have

$$\left| \mathbb{E} \left| \sum_{j=1}^n a_j X_j \right| - \mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \right| \leq 3 \left(1 - \frac{1}{q}\right). \quad (45)$$

*Proof*

Since  $\varepsilon_j$  has the same distribution as  $\operatorname{sgn}(X_j)$ , we have

$$\begin{aligned} & \left| \mathbb{E} \left| \sum_{j=1}^n a_j X_j \right| - \mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \right| \\ &= \left| \mathbb{E} \left| \sum_{j=1}^n a_j X_j \right| - \mathbb{E} \left| \sum_{j=1}^n a_j \operatorname{sgn}(X_j) \right| \right| \\ &\leq \mathbb{E} \left| \sum_{j=1}^n a_j (X_j - \operatorname{sgn}(X_j)) \right| \leq \sqrt{\mathbb{E} \left| \sum_{j=1}^n a_j (X_j - \operatorname{sgn}(X_j)) \right|^2} \end{aligned}$$

$$= \sqrt{\mathbb{E}|X_1 - \operatorname{sgn}(X_1)|^2},$$

and Lemma 17 finishes the proof.  $\square$

#### 4.2. Proof of Theorem 3

By virtue of (14), our goal is to show that for every  $1 < q < q_0$  and every unit vector  $a$  in  $\mathbb{R}^n$ , we have

$$\mathbb{E}\left|\sum_{j=1}^n a_j X_j\right| \geq \mathbb{E}\left|\frac{X_1 + X_2}{\sqrt{2}}\right| \stackrel{\text{def}}{=} c_q. \quad (46)$$

For later use, we note that thanks to (14), for every vector  $a$  in  $\mathbb{R}^2$ ,

$$\begin{aligned} \mathbb{E}|a_1 X_1 + a_2 X_2| &= |a| \frac{\operatorname{vol}(\operatorname{Proj}_{a^\perp} \mathbb{B}_q^2)}{2\Gamma(1/q)} \\ &= \frac{|a|}{\Gamma(1/q)} \sup_{x \in \partial \mathbb{B}_q^2} \left| \left\langle x, \frac{1}{|a|}(-a_2, a_1) \right\rangle \right| = \frac{\|a\|_{\frac{q}{q-1}}}{\Gamma(1/q)}. \end{aligned} \quad (47)$$

In particular, we have

$$c_q = \mathbb{E}\left|\frac{X_1 + X_2}{\sqrt{2}}\right| = \frac{2^{\frac{1}{2} - \frac{1}{q}}}{\Gamma(1/q)}. \quad (48)$$

In view of the above explicit expression, inequality (46) clearly holds for  $n = 2$ . We therefore assume that  $n \geq 3$ ,  $a_1 \geq \dots \geq a_n > 0$ , and proceed by induction on  $n$ . Recall the definition of the deficit parameter used earlier,

$$\delta(a) = \left| a - \frac{e_1 + e_2}{\sqrt{2}} \right|^2 = 2 - \sqrt{2}(a_1 + a_2).$$

##### 4.2.1. The vector $a$ is far from the extremizer

Here we consider the case  $\sqrt{\delta(a)} \geq (1 - \frac{1}{q})c$  with numerical constant  $c > 0$ . With hindsight, we set

$$c = \frac{5 - \sqrt{2}}{8} \cdot 10^5.$$

Using the equicontinuity from Lemma 18 and the robust version of Szarek's inequality from Theorem 8, we obtain

$$\begin{aligned} \mathbb{E}\left|\sum_{j=1}^n a_j X_j\right| &\geq \mathbb{E}\left|\sum_{j=1}^n a_j \varepsilon_j\right| - 3\left(1 - \frac{1}{q}\right) \geq \frac{1}{\sqrt{2}} + \kappa_1 \sqrt{\delta(a)} - 3\left(1 - \frac{1}{q}\right) \\ &\geq \frac{1}{\sqrt{2}} + (\kappa_1 c - 3)\left(1 - \frac{1}{q}\right). \end{aligned}$$

Note that by convexity,  $2^x < 1 + 2(\sqrt{2} - 1)x$  for  $0 < x < \frac{1}{2}$ , which with  $x = 1 - \frac{1}{q}$  gives

$$c_q = \frac{2^{\frac{1}{2} - \frac{1}{q}}}{\Gamma(1/q)} \leq 2^{\frac{1}{2} - \frac{1}{q}} = \frac{1}{\sqrt{2}} 2^{1 - \frac{1}{q}} < \frac{1}{\sqrt{2}} + (2 - \sqrt{2}) \left(1 - \frac{1}{q}\right).$$

Since

$$c = \frac{5 - \sqrt{2}}{\kappa_1} = \frac{5 - \sqrt{2}}{8} \cdot 10^5,$$

we get the desired bound (46) (nota bene, without the inductive argument).

#### 4.2.2. The vector $a$ is close to the extremizer

It is left to consider the case when

$$\sqrt{\delta(a)} < \left(1 - \frac{1}{q}\right)c,$$

where  $c = \frac{5 - \sqrt{2}}{8} \cdot 10^5$ , as in the previous case. In particular, we also have that

$$\frac{1}{\sqrt{2}} - c \left(1 - \frac{1}{q}\right) \leq a_2 \leq a_1 \leq \frac{1}{\sqrt{2}} + c \left(1 - \frac{1}{q}\right). \quad (49)$$

Letting  $p = \frac{q}{q-1}$ , we shall assume that  $p$  is large relative to  $c$ , say  $p > Lc + 2$  with a positive numerical constant  $L$ , with hindsight set to be

$$L = 8294400.$$

In particular, when  $\frac{1}{p} = 1 - \frac{1}{q} < 10^{-5}$ ,

$$0.7 < c_q < 0.71, \quad (50)$$

since  $0.7 < \frac{2^{-1/2}}{\Gamma(1-10^{-5})} < \frac{2^{1/2-1/q}}{\Gamma(1/q)} < 2^{-1/2+10^{-5}} < 0.71$ .

To run an inductive argument in order to prove (46), we consider the random variables  $X = a_1 X_1 + a_2 X_2$  and  $Y = \sum_{j>2} a_j X_j$ . By the independence and symmetry of  $X$  and  $Y$ ,

$$\mathbb{E} \left| \sum_{j=1}^n a_j X_j \right| = \mathbb{E}|X + Y| = \mathbb{E} \max\{|X|, |Y|\}.$$

Using the inductive hypothesis,

$$\mathbb{E}|Y| = \sqrt{1 - a_1^2 - a_2^2} \mathbb{E} \left| \frac{\sum_{j>2} a_j X_j}{\sqrt{1 - a_1^2 - a_2^2}} \right| \geq c_q \sqrt{1 - a_1^2 - a_2^2};$$

hence (by the convexity of the nondecreasing function  $t \mapsto \max\{|X|, t\}$ ), we get

$$\mathbb{E} \left| \sum_{j=1}^n a_j X_j \right| = \mathbb{E} \max\{|X|, |Y|\} \geq \mathbb{E} \max\{|X|, \alpha\}, \quad (51)$$

where we set

$$\alpha \stackrel{\text{def}}{=} c_q \sqrt{1 - a_1^2 - a_2^2}. \quad (52)$$

Observe that

$$\mathbb{E} \max\{|X|, \alpha\} = \mathbb{E}|X| + \mathbb{E}(\alpha - |X|)_+ \quad (53)$$

and, by (47),

$$\mathbb{E}|X| = \frac{\|(a_1, a_2)\|_{\frac{q}{q-1}}}{\Gamma(1/q)} \stackrel{(48)}{=} c_q 2^{\frac{1}{q}-\frac{1}{2}} \|(a_1, a_2)\|_{\frac{q}{q-1}}. \quad (54)$$

In view of the inductive step (51) and the identities (53) and (54), the desired inequality (46) is a consequence of the following proposition.

#### PROPOSITION 19

Under the assumptions and notation above, for  $1 \leq q \leq 1 + 10^{-12}$  we have

$$\mathbb{E}(\alpha - |X|)_+ \geq \left(1 - 2^{\frac{1}{q}-\frac{1}{2}} \|(a_1, a_2)\|_{\frac{q}{q-1}}\right) c_q. \quad (55)$$

#### Proof

If the right-hand side is nonpositive, then we are done. Otherwise,

$$\|(a_1, a_2)\|_{\frac{q}{q-1}} < 2^{\frac{1}{2}-\frac{1}{q}}.$$

Letting  $p = \frac{q}{q-1}$  and recalling (49), we see that we can apply Lemma 10 to conclude that

$$|a_1 - a_2| \leq 3.65 \sqrt{\frac{c}{p-2}} \sqrt{1 - a_1^2 - a_2^2} \stackrel{(52)}{=} \frac{3.65}{c_q} \sqrt{\frac{c}{p-2}} \alpha \stackrel{(50)}{<} \frac{5.25\alpha}{\sqrt{L}}. \quad (56)$$

To simplify the right-hand side of (55), we write

$$\begin{aligned} c_q \left(1 - 2^{\frac{1}{q}-\frac{1}{2}} \|(a_1, a_2)\|_{\frac{q}{q-1}}\right) &\leq c_q \left(1 - \|(a_1, a_2)\|_2\right) = c_q \frac{1 - (a_1^2 + a_2^2)}{1 + \sqrt{a_1^2 + a_2^2}} \\ &\stackrel{(52)}{=} \frac{\alpha^2}{c_q (1 + \sqrt{a_1^2 + a_2^2})} \stackrel{(50)}{<} \frac{\alpha^2}{0.7(1 + \sqrt{0.97})} < \frac{3}{4} \alpha^2, \end{aligned}$$

as we have  $a_1^2 + a_2^2 > 0.97$  (see (35)). Therefore, it suffices to show that

$$\mathbb{E}(\alpha - |X|)_+ \geq \frac{3}{4}\alpha^2. \quad (57)$$

Since the  $X_j$  are symmetric random variables, each  $X_j$  has the same distribution as  $\varepsilon_j |X_j|$ , for independent random signs  $\varepsilon_j$ , also independent of all the other random variables. We consider the event

$$\mathcal{E} \stackrel{\text{def}}{=} \left\{ |X_1| \leq 1, ||X_1| - |X_2|| < \alpha, |a_1 \varepsilon_1 + a_2 \varepsilon_2| < \frac{1}{4}\alpha \right\},$$

on which we have

$$\begin{aligned} |a_1 \varepsilon_1 |X_1| + a_2 \varepsilon_2 |X_2|| &\leq |X_1| |a_1 \varepsilon_1 + a_2 \varepsilon_2| + a_2 ||X_2| - |X_1|| \\ &< \frac{1}{4}\alpha + \frac{1}{\sqrt{2}}\alpha < \frac{24}{25}\alpha, \end{aligned}$$

and thus, since  $X$  has the same distribution as  $a_1 \varepsilon_1 |X_1| + a_2 \varepsilon_2 |X_2|$ , we obtain the lower bound

$$\begin{aligned} \mathbb{E}(\alpha - |X|)_+ &\geq \frac{\alpha}{25} \mathbb{P}(\mathcal{E}) \\ &= \frac{\alpha}{25} \mathbb{P}\{|X_1| \leq 1, ||X_1| - |X_2|| < \alpha\} \mathbb{P}\left\{|a_1 \varepsilon_1 + a_2 \varepsilon_2| < \frac{1}{4}\alpha\right\}. \end{aligned} \quad (58)$$

The second probability in (58) is clearly at least  $\frac{1}{2}$  provided that

$$|a_1 - a_2| < \frac{\alpha}{4}.$$

For this to hold, it suffices that  $L > (5.25 \cdot 4)^2 = 441$ , by virtue of (56). For the first probability, analogously to Lemma 16, we will place a constant function under the density  $f_q$  of  $|X_1|$ .

LEMMA 20

Fix  $q \in (1, \frac{3}{2})$ , and let  $f_q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be the density of  $|X_1|$ . Then we have

$$f_q(x) \geq \frac{1}{4(q-1)} \mathbf{1}_{[1-\frac{q-1}{2}, 1]}(x), \quad x > 0. \quad (59)$$

*Proof*

Recall from Proposition 7 that

$$f_q(x) = \frac{1}{(q-1)\Gamma(1+\frac{1}{q})} x^{\frac{2-q}{q-1}} e^{-x^{\frac{q}{q-1}}}, \quad x > 0.$$



The proof is almost identical to that of Lemma 16. It suffices to check that the inequality holds for  $x = 1 - \frac{q-1}{2}$  and  $x = 1$ . Since  $(1 - \frac{q-1}{2})^{\frac{2-q}{q-1}} > 1 - \frac{2-q}{q-1} \frac{q-1}{2} = \frac{q}{2} > \frac{1}{2}$ , for  $1 < q < \frac{3}{2}$ ,  $(1 - \frac{q-1}{2})^{\frac{q}{q-1}} < e^{-\frac{q}{2}} < e^{-\frac{1}{2}}$ , and  $\Gamma(1 + \frac{1}{q}) < 1$ , we obtain

$$f_q\left(1 - \frac{q-1}{2}\right) > \frac{1}{(q-1)} \frac{1}{2} e^{-e^{-\frac{1}{2}}} > \frac{1}{4(q-1)}.$$

Moreover,

$$f_q(1) = \frac{1}{(q-1)\Gamma(1 + \frac{1}{q})} e^{-1} > \frac{1}{e(q-1)} > \frac{1}{4(q-1)}. \quad \square$$

*Finishing the proof of Proposition 19*

As earlier, Lemma 20 gives

$$\begin{aligned} & \mathbb{P}\{|X_1| \leq 1, ||X_1| - |X_2|| < \alpha\} \\ & \geq \int \int_{\{x \leq 1, |x-y| < \alpha\}} \left(\frac{1}{4(q-1)}\right)^2 \mathbf{1}_{[1-\frac{q-1}{2}, 1]}(x) \mathbf{1}_{[1-\frac{q-1}{2}, 1]}(y) dx dy \\ & \geq \begin{cases} \frac{1}{64} & \alpha > \frac{q-1}{2}, \\ \frac{\alpha}{32(q-1)} & \alpha \leq \frac{q-1}{2}, \end{cases} \end{aligned} \quad (60)$$

where the last inequality follows from (42) with  $p$  replaced by  $\frac{1}{q-1}$ .

- If  $\alpha > \frac{q-1}{2}$ , then inequalities (58) and (60) yield the lower bound

$$\mathbb{E}(\alpha - |X|)_+ \geq \frac{\alpha}{25} \cdot \frac{1}{64} \cdot \frac{1}{2} = \frac{\alpha}{3200}.$$

Since

$$\alpha \stackrel{(52)}{\leq} c_q \sqrt{1 - 2a_2^2} \stackrel{(50)}{<} 0.71 \sqrt{1 - 2\left(\frac{1}{\sqrt{2}} - \frac{c}{p}\right)^2} < 0.71 \sqrt{2\sqrt{2}\frac{c}{p}} < \frac{1.2}{\sqrt{L}},$$

and  $L \geq 8294400$ , we obtain the desired bound (57).

- If  $\alpha \leq \frac{q-1}{2}$ , then inequalities (58) and (60) give

$$\mathbb{E}(\alpha - |X|)_+ \geq \frac{\alpha}{25} \cdot \frac{\alpha}{32(q-1)} \cdot \frac{1}{2} = \frac{\alpha^2}{1600(q-1)}.$$

As we assume that  $1 - \frac{1}{q} \leq \frac{1}{cL+2}$ , this is at least the desired  $\frac{3}{4}\alpha^2$  by a large margin for  $L = 8294400$ . The proof is complete for every  $1 \leq q \leq q_0$ , where

$$q_0 = \frac{Lc + 2}{Lc + 1} > 1 + 10^{-12}. \quad \square$$

### 5. Stability estimates with explicit constants

The proofs of both Theorems 8 and 9 presented here follow the same strategy taken from [7], which we shall now outline. For a unit vector  $a$  in  $\mathbb{R}^n$ , consider again the deficit

$$\delta(a) = \left| a - \frac{e_1 + e_1}{\sqrt{2}} \right|^2 = 2 - \sqrt{2}(a_1 + a_2).$$

Let  $a$  be a unit vector, and without loss of generality assume that  $a_1 \geq \dots \geq a_n \geq 0$ . The approach leading to the stability of the inequalities of Szarek and Ball differs depending on whether the vector  $a$  is *close to* or *far from* the extremizer  $\frac{e_1 + e_2}{\sqrt{2}}$ , as measured by  $\delta(a)$ .

*Case 1.* When  $a$  is close to  $\frac{e_1 + e_2}{\sqrt{2}}$ , we quantitatively sharpen the inequalities of Szarek and Ball by reapplying them only to a *portion* of the vector  $a$ , thus exhibiting their self-improving feature. The probabilistic formulas are crucial for this part.

When  $a$  is far from the extremizer, three things can happen.

*Case 2.* If the largest magnitude of the coordinates of  $a$  is below  $\frac{1}{\sqrt{2}}$ , then the second largest magnitude has to drop *well* below  $\frac{1}{\sqrt{2}}$  on account of  $\delta(a)$  being large and the classical Fourier-analytic approach of Haagerup [12] and Ball [2] allows us to track the deficit.

*Case 3.* If the largest magnitude is *barely* above  $\frac{1}{\sqrt{2}}$ , then a Lipschitz property of the section and projection functions allows us to reduce this case to the one from Case 2.

*Case 4.* If the largest magnitude is bounded below away from  $\frac{1}{\sqrt{2}}$ , then an easy convexity/projection argument gives a strict inequality with a margin.

#### 5.1. Stability of Szarek's inequality

We first deal with the sharp Khinchin inequality of [31].

##### Case 1

We begin with the case that  $a$  is near the extremizer.

##### LEMMA 21

Let  $0 < \delta_0 < \frac{2}{3}$ , and take  $c_0 = \frac{1}{2\sqrt{2}}(\sqrt{\frac{1}{5}(4 - \delta_0)} - \sqrt{\delta_0}) > 0$ . For every unit vector  $a$  in  $\mathbb{R}^n$  with  $a_1 \geq \dots \geq a_n \geq 0$  satisfying  $\delta(a) \leq \delta_0$ , we have

$$\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \geq \frac{1}{\sqrt{2}} + c_0 \sqrt{\delta(a)}. \quad (61)$$

##### Proof

We will assume without loss of generality that  $n \geq 3$  and  $a_1^2 + a_2^2 < 1$  (the remaining

cases can be obtained by taking a limit). Let

$$\theta \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \sqrt{1 - a_1^2 - a_2^2}.$$

Arguing as in the induction of Section 4 and using Jensen's and Szarek's inequalities, we get

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| &= \mathbb{E} \max \left\{ |a_1 \varepsilon_1 + a_2 \varepsilon_2|, \left| \sum_{j=3}^n a_j \varepsilon_j \right| \right\} \\ &\geq \mathbb{E} \max \left\{ |a_1 \varepsilon_1 + a_2 \varepsilon_2|, \mathbb{E} \left| \sum_{j=3}^n a_j \varepsilon_j \right| \right\} \geq \mathbb{E} \max \{ |a_1 \varepsilon_1 + a_2 \varepsilon_2|, \theta \} \\ &= \frac{1}{2} \max \{ a_1 + a_2, \theta \} + \frac{1}{2} \max \{ a_1 - a_2, \theta \} \\ &\geq \frac{a_1 + a_2}{2} + \frac{1}{2} \sqrt{\frac{1}{5} (a_1 - a_2)^2 + \frac{4}{5} \theta^2} \\ &= \frac{2 - \delta(a)}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \sqrt{\frac{1}{5} (4 - \delta(a)) \sqrt{\delta(a)}} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \left( \sqrt{\frac{1}{5} (4 - \delta(a))} - \sqrt{\delta(a)} \right) \sqrt{\delta(a)} \\ &\geq \frac{1}{\sqrt{2}} + c_0 \sqrt{\delta(a)}, \end{aligned}$$

whenever  $\delta(a) \leq \delta_0$ . □

### Case 2

We assume that  $a$  is far from the extremizer and that  $a_1$  is at most  $\frac{1}{\sqrt{2}}$ . A key step in Haagerup's slick Fourier-analytic proof of Szarek's inequality from [12] is the bound

$$\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \geq \sum_{j=1}^n a_j^2 F(a_j^{-2}), \quad (62)$$

for every unit vector  $a$  in  $\mathbb{R}^n$ , where the function  $F : (0, \infty) \rightarrow \mathbb{R}$  is given by

$$F(s) = \frac{2}{\sqrt{\pi s}} \cdot \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s}{2})}, \quad s > 0.$$

Haagerup showed that the function  $F(s)$  is increasing on  $(0, \infty)$ , which will be crucial for us.

## LEMMA 22

Let  $0 < \delta_0 < 2$ . For every unit vector  $a$  in  $\mathbb{R}^n$  with  $a_1 \geq \dots \geq a_n \geq 0$  satisfying  $\delta(a) \geq \delta_0$  and  $a_1 \leq \frac{1}{\sqrt{2}}$ , we have

$$\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \geq \frac{1}{\sqrt{2}} + c_1 \sqrt{\delta(a)}, \quad (63)$$

with  $c_1 = \frac{1}{2\sqrt{2}}(F(\frac{8}{(2-\delta_0)^2}) - F(2))$ .

*Proof*

We have

$$a_2 \leq \frac{a_1 + a_2}{2} = \frac{2 - \delta(a)}{2\sqrt{2}} \leq \frac{2 - \delta_0}{2\sqrt{2}},$$

which shows that  $a_j^{-2} \geq l_0$ , for all  $j \geq 2$ , with  $l_0 = \frac{8}{(2-\delta_0)^2} > 2$ . Employing (62) and using the monotonicity of  $F$ , we therefore have

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| &\geq a_1^2 F(2) + \sum_{j \geq 2} a_j^2 F(l_0) = a_1^2 F(2) + (1 - a_1^2) F(l_0) \\ &= F(l_0) + a_1^2 (F(2) - F(l_0)) \geq F(l_0) + \frac{1}{2} (F(2) - F(l_0)) \\ &= \frac{1}{2} (F(2) + F(l_0)) = \frac{1}{\sqrt{2}} + \frac{1}{2} (F(l_0) - F(2)), \end{aligned}$$

since  $F(2) = \frac{1}{\sqrt{2}}$ . The conclusion follows since  $\delta(a) \leq 2$ .  $\square$

## Case 3

We assume that  $a$  is far from the extremizer, but  $a_1$  is barely larger than  $\frac{1}{\sqrt{2}}$ .

## LEMMA 23

Let  $\gamma_0 \leq 1 - \frac{1}{\sqrt{2}}$  and  $2\sqrt{\gamma_0} < \delta_0 < 2$ . For every unit vector  $a$  in  $\mathbb{R}^n$  with coordinates  $a_1 \geq \dots \geq a_n \geq 0$  satisfying  $\frac{1}{\sqrt{2}} \leq a_1 \leq \frac{1}{\sqrt{2}} + \gamma_0$  and  $\delta(a) \geq \delta_0$ , we have

$$\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \geq \frac{1}{\sqrt{2}} + c_2 \sqrt{\delta(a)}, \quad (64)$$

with

$$\begin{aligned} c_2 &= \frac{1}{2\sqrt{2}} \left( F \left( \frac{8}{(2 + 2\sqrt{\gamma_0} - \delta_0)^2} \right) - F(2) \right) \sqrt{\delta_0 - 2\sqrt{\gamma_0}} \\ &\quad - \sqrt{2\gamma_0 + \gamma_0^2}. \end{aligned} \quad (65)$$

*Proof*

We can assume that  $c_2 \geq 0$  since otherwise it is enough to observe that

$$\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| = \mathbb{E} \left| a_1 + \sum_{j=2}^n a_j \varepsilon_j \right| \geq \left| a_1 + \sum_{j=2}^n a_j \mathbb{E} \varepsilon_j \right| = |a_1| \geq \frac{1}{\sqrt{2}}.$$

Consider the unit vector

$$b \stackrel{\text{def}}{=} \left( \frac{1}{\sqrt{2}}, \sqrt{a_1^2 + a_2^2 - \frac{1}{2}}, a_3, \dots, a_n \right).$$

Then by the triangle inequality, we obtain the following Lipschitz property:

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| &\geq \mathbb{E} \left| \sum_{j=1}^n b_j \varepsilon_j \right| - \mathbb{E} \left| \sum_{j=1}^n (a_j - b_j) \varepsilon_j \right| \\ &\geq \mathbb{E} \left| \sum_{j=1}^n b_j \varepsilon_j \right| - \left( \mathbb{E} \left| \sum_{j=1}^n (a_j - b_j) \varepsilon_j \right|^2 \right)^{1/2} = \left| \sum_{j=1}^n b_j \varepsilon_j \right| - |a - b|. \end{aligned}$$

Note that  $b_1 \geq b_2$  and since  $b_2 \geq a_2$ , also  $b_2 \geq b_3 \geq \dots \geq b_n$ . Moreover,

$$\begin{aligned} \sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2 &= \frac{a_1^2 - \frac{1}{2}}{\sqrt{a_1^2 + a_2^2 - \frac{1}{2}} + a_2} \\ &\leq \sqrt{a_1^2 - \frac{1}{2}} \leq \sqrt{\sqrt{2}\gamma_0 + \gamma_0^2} < \sqrt{2\gamma_0}; \end{aligned} \quad (66)$$

thus

$$|a - b|^2 = \left( a_1 - \frac{1}{\sqrt{2}} \right)^2 + \left( \sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2 \right)^2 < \gamma_0^2 + 2\gamma_0.$$

Observe that, since  $a_1 \geq \frac{1}{\sqrt{2}}$ , we have

$$\begin{aligned} \delta(b) &= 2 - \sqrt{2} \left( \frac{1}{\sqrt{2}} + \sqrt{a_1^2 + a_2^2 - \frac{1}{2}} \right) \\ &= \delta(a) - \sqrt{2} \left( \frac{1}{\sqrt{2}} + \sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_1 - a_2 \right) \\ &\geq \delta_0 - \sqrt{2} \left( \sqrt{a_1^2 + a_2^2 - \frac{1}{2}} - a_2 \right) \stackrel{(66)}{>} \delta_0 - 2\sqrt{\gamma_0}. \end{aligned}$$

Thus, applying Lemma 22 to the vector  $b$  and using the above estimates, we get

$$\begin{aligned}
\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| &\geq \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \left( F \left( \frac{8}{(2 + 2\sqrt{\gamma_0} - \delta_0)^2} \right) - F(2) \right) \sqrt{\delta(b)} \\
&\quad - \sqrt{2\gamma_0 + \gamma_0^2} \\
&\geq \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} \left( F \left( \frac{8}{(2 + 2\sqrt{\gamma_0} - \delta_0)^2} \right) - F(2) \right) \sqrt{\delta_0 - 2\sqrt{\gamma_0}} \\
&\quad - \sqrt{2\gamma_0 + \gamma_0^2}.
\end{aligned}$$

Finally, as  $a_1 \geq \frac{1}{\sqrt{2}}$ , we have  $\delta(a) = 2 - \sqrt{2}(a_1 + a_2) \leq 1 - \sqrt{2}a_2 \leq 1$ , and the proof is complete.  $\square$

#### Case 4

Finally, there is also a simple bound for the case that  $a_1$  is much larger than  $\frac{1}{\sqrt{2}}$ .

#### LEMMA 24

Let  $\gamma_0 > 0$ . For every unit vector  $a$  in  $\mathbb{R}^n$  with  $a_1 \geq \frac{1}{\sqrt{2}} + \gamma_0$ , we have

$$\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \geq \frac{1}{\sqrt{2}} + \gamma_0 \sqrt{\delta(a)}. \quad (67)$$

#### Proof

By Jensen's inequality and the fact that  $\delta(a) \leq 1$ ,

$$\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \geq |a_1| \geq \frac{1}{\sqrt{2}} + \gamma_0 \geq \frac{1}{\sqrt{2}} + \gamma_0 \sqrt{\delta(a)}. \quad \square$$

#### Constants

Combining Lemmas 21, 22, 23, and 24 with  $\delta_0 = 0.66$  (almost the maximal value allowed in Lemma 21) and  $\gamma_0 = 8 \cdot 10^{-5}$ , we conclude that for all unit vectors  $a$  in  $\mathbb{R}^n$ ,

$$\mathbb{E} \left| \sum_{j=1}^n a_j \varepsilon_j \right| \geq \frac{1}{\sqrt{2}} + \kappa_1 \sqrt{\delta(a)}$$

with

$$\begin{aligned}
\kappa_1 &\geq \min\{c_0, c_1, c_2, \gamma_0\} \\
&> \min\{1.7 \cdot 10^{-3}, 1.6 \cdot 10^{-2}, 5.1 \cdot 10^{-4}, 8 \cdot 10^{-5}\} = 8 \cdot 10^{-5}.
\end{aligned}$$

This completes the proof of Theorem 8.

### 5.2. Stability of Ball's inequality

We now turn to the study of Ball's inequality (1). Throughout this section we denote by  $\mathbf{Q}_n = [-\frac{1}{2}, \frac{1}{2}]^n$  the cube of unit volume.

#### Case 1

We begin with the case that  $a$  is near the extremizer.

#### LEMMA 25

For every  $n \geq 2$  and every unit vector  $a$  in  $\mathbb{R}^n$  with  $a_1 \geq \dots \geq a_n \geq 0$  satisfying  $\delta(a) \leq \frac{1}{4}$ , we have

$$\text{vol}(\mathbf{Q}_n \cap a^\perp) \leq \sqrt{2} - c_1 \sqrt{\delta(a)}, \quad (68)$$

where  $c_1 = 0.12$ .

#### Proof

We can assume that  $n \geq 3$  and  $a_1^2 + a_2^2 < 1$  (the missing cases follow by taking a limit). Leveraging a self-improving feature of Ball's inequality, the proof of [7, Lemma 6.7] yields

$$\begin{aligned} \text{vol}(\mathbf{Q}_n \cap a^\perp) &\leq \sqrt{2} \max \left\{ \left( 1 - \delta + \sqrt{\frac{\delta(2-\delta)}{5}} \right)^{-1}, \right. \\ &\quad \left. (1 - \delta)^{-2} \left( 1 - \delta - \frac{\sqrt{\delta(2-\delta)}}{2\sqrt{2}} \right) \right\}, \end{aligned}$$

where  $\delta = \delta(a)$ . Denoting the maximum on the right-hand side by  $M(\delta)$ , we can take

$$c_1 = \inf_{0 < \delta < 1/4} \sqrt{2} \frac{1 - M(\delta)}{\sqrt{\delta}}.$$

Direct numerical calculations show that  $c_1 > 0.12$ . □

#### Cases 2 and 3

We assume that  $a$  is far from the extremizer, but  $a_1$  is not much larger than  $\frac{1}{\sqrt{2}}$ .

#### LEMMA 26

For every  $n \geq 2$  and every unit vector  $a$  in  $\mathbb{R}^n$  with  $a_1 \geq \dots \geq a_n \geq 0$  satisfying  $\delta(a) \geq \frac{1}{4}$  and  $a_1 \leq \frac{1}{\sqrt{2}} + \gamma_0$ , we have

$$\text{vol}(\mathbf{Q}_n \cap a^\perp) \leq \sqrt{2} - c_2, \quad (69)$$

where  $\gamma_0 = 3.2 \cdot 10^{-5}$  and  $c_2 = 0.0002$ .

*Proof*

Here the proof relies on Fourier-analytic arguments. For the special function

$$\Psi(s) = \frac{2}{\pi} \sqrt{s} \int_0^\infty \left| \frac{\sin t}{t} \right|^s dt,$$

Ball showed in [2] that  $\Psi(s) < \Psi(2) = \sqrt{2}$ , for every  $s > 2$ . We need a robust version of this estimate. Using the Nazarov–Podkorytov lemma (see [29]), König and Koldobsky [23] proved that

$$\forall s \geq \frac{9}{4}, \quad \Psi(s) \leq \Psi(\infty) = \sqrt{\frac{6}{\pi}} = \sqrt{2} \left( \frac{3}{\pi} \right)^{1/2} \quad (70)$$

(i.e.,  $\theta_0 = (\frac{3}{\pi})^{1/2}$  in the notation of [7, Lemma 6.8]). The argument now splits into two cases.

- Assume that  $a_1 \leq \frac{1}{\sqrt{2}}$ . Provided that

$$s(a) \stackrel{\text{def}}{=} 2 \left( 1 - \frac{\delta(a)}{2} \right)^{-2} \geq \frac{9}{4},$$

which holds as long as  $\delta(a) \geq 2(1 - \frac{2\sqrt{2}}{3}) = 0.11\dots$ , with the aid of (70), the arguments from [7, Lemma 6.8] give the explicit estimate

$$\text{vol}(\mathbf{Q}_n \cap a^\perp) \leq \left( \frac{3}{\pi} \right)^{1/4} \sqrt{2} = \sqrt{2} - \sqrt{2}(1 - (3/\pi)^{1/4}).$$

Therefore, we can take any

$$c_2 \leq \sqrt{2}(1 - (3/\pi)^{1/4}) = 0.016\dots$$

- Assume that  $\frac{1}{\sqrt{2}} < a_1 \leq \frac{1}{\sqrt{2}} + \gamma_0$ . Using Busemann's theorem (see [5]), this case is reduced in [7, Lemma 6.8] to the previous range, which yields the bound

$$\begin{aligned} \text{vol}(\mathbf{Q}_n \cap a^\perp) &\leq \sqrt{2} - \sqrt{2} \min \left\{ c_1 \sqrt{\frac{1}{8} - \sqrt{\gamma_0}}, 1 - (3/\pi)^{1/4} \right\} \\ &\quad + 2\sqrt{\gamma_0^2 + 2\gamma_0}, \end{aligned}$$

where  $c_1$  is the constant from Lemma 25. With the choice of parameters  $\gamma_0 = 3.2 \cdot 10^{-5}$  and  $c_1 = 0.12$ , this estimate yields  $\text{vol}(\mathbf{Q}_n \cap a^\perp) \leq \sqrt{2} - 0.00021\dots$  and thus completes the proof.  $\square$

*Case 4*

Finally, there is also a simple bound for the case that  $a_1$  is much larger than  $\frac{1}{\sqrt{2}}$ .



## LEMMA 27

For every  $n \geq 2$  and every unit vector  $a$  in  $\mathbb{R}^n$  satisfying  $a_1 \geq \frac{1}{\sqrt{2}} + \gamma_0$ , we have

$$\text{vol}(\mathbf{Q}_n \cap a^\perp) \leq \sqrt{2} - \frac{2\gamma_0}{1 + \gamma_0\sqrt{2}} \sqrt{\delta(a)}, \quad (71)$$

where  $\gamma_0 = 3.2 \cdot 10^{-5}$ .

*Proof*

By Ball's geometric projection argument (see [2]; see also [29]), we have  $\text{vol}(\mathbf{Q}_n \cap a^\perp) \leq \frac{1}{a_1}$ . Since  $a_1 \geq \frac{1}{\sqrt{2}} + \gamma_0$  and hence  $\delta(a) < 1$ , we deduce that

$$\begin{aligned} \text{vol}(\mathbf{Q}_n \cap a^\perp) &\leq \frac{1}{\frac{1}{\sqrt{2}} + \gamma_0} = \sqrt{2} - \sqrt{2} \left( 1 - \frac{1}{1 + \gamma_0\sqrt{2}} \right) \\ &\leq \sqrt{2} - \frac{2\gamma_0}{1 + \gamma_0\sqrt{2}} \sqrt{\delta(a)}. \end{aligned} \quad \square$$

*Constants*

Combining Lemmas 25, 26, and 27, and using that always  $\delta(a) < 2$ , we conclude that for all unit vectors  $a$  in  $\mathbb{R}^n$ , we have the inequality

$$\text{vol}(\mathbf{Q}_n \cap a^\perp) \leq \sqrt{2} - \kappa_\infty \sqrt{\delta(a)}$$

with

$$\kappa_\infty \geq \min \left\{ c_1, \frac{c_2}{\sqrt{2}}, \frac{2\gamma_0}{1 + \gamma_0\sqrt{2}} \right\} > 6 \cdot 10^{-5}.$$

This completes the proof of Theorem 9.

*Remark 28*

We would like to stress that the arguments of this paper have not been optimized to give the best possible constants  $p_0$  and  $q_0$  in Theorems 1 and 3. We instead chose to be fairly generous in various parts of the proof for the sake of clarity of the exposition.

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