

Local theory of stable polynomials and bounded rational functions of several variables

KELLY BICKEL (Lewisburg, PA), GREG KNESE (St. Louis, MO),
JAMES ELDRED PASCOE (Philadelphia, PA) and ALAN SOLA (Stockholm)

Abstract. We provide detailed local descriptions of stable polynomials in terms of their homogeneous decompositions, Puiseux expansions, and transfer function realizations. We use this theory to first prove that bounded rational functions on the polydisk possess nontangential limits at every boundary point. We relate higher nontangential regularity and distinguished boundary behavior of bounded rational functions to geometric properties of the zero sets of stable polynomials via our local descriptions. For a fixed stable polynomial p , we analyze the ideal of numerators q such that q/p is bounded on the bi-upper half-plane. We completely characterize this ideal in several geometrically interesting situations including smooth points, double points, and ordinary multiple points of p . Finally, we analyze integrability properties of bounded rational functions and their derivatives on the bidisk.

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1. Introduction. A multivariate *stable polynomial* $p \in \mathbb{C}[z_1, \dots, z_d]$ is a polynomial that does not vanish on a specified domain $\Omega \subseteq \mathbb{C}^d$. We generally take Ω to be the product \mathbb{H}^d of upper half-planes, where

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\},$$

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or the conformally-equivalent polydisk \mathbb{D}^d with

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}.$$

Historically, the term “stable polynomial” may have arisen in the context of stability of linear time-invariant systems studied by Routh and independently by Hurwitz. Within the past few decades however, multivariate stable polynomials have emerged as critical tools for studying phenomena in fields including complex analysis, probability theory, control theory, electrical engineering, combinatorics, and dynamical systems; see [17, 21, 24, 27, 35, 37, 50]. Some notable highlights include work of Brändén and Borcea [14, 15] classifying operators preserving stable polynomials with applications to statistical physics; Kurasov and Sarnak [36] using stable polynomials to resolve questions on Fourier quasicrystals; and Marcus, Spielman, and Srivastava [38, 39] using the technique of interlacing families of stable polynomials to resolve major problems in graph theory and analysis. For a variety of reasons, which include the existence of simple reflection operations ($z \mapsto \bar{z}$ or $z \mapsto 1/\bar{z}$), the strongest results/applications of stable polynomials are known in the settings of \mathbb{H}^d and \mathbb{D}^d . On the other hand, comparatively little is known about stable polynomials on more general domains in \mathbb{C}^d .

Recently, the present authors established a variety of results about boundary regularity and integrability of rational functions. Along the way a number of *ad hoc* results about the local behavior of stable polynomials were developed. We now standardize and significantly extend this local theory of stable polynomials to create a toolbox powerful enough to address most all of these previous results, at least one conjecture [11, Conjecture 5.2], and several new questions which are posed in Section 1.2. Our local theory gives precise, structural information about the homogeneous expansions, Puiseux factorizations, and realization formulas of stable polynomials—thus illuminating exactly how stable polynomials behave on the boundary of their respective zero-free regions. In addition, we construct stable polynomials with more exotic boundary zero sets.

We investigate the fundamental *numerator criterion question*:

- Given a stable polynomial p on Ω , for which polynomials q is q/p bounded on Ω ?

In the one-variable setting, the zero set of q must include all of the boundary zeros of p by the fundamental theorem of algebra. In several variables, a correct answer requires a detailed analysis of local zero set behavior. We answer the numerator criterion question for stable polynomials with several types of zero set behavior at a distinguished boundary zero, including smooth points, double points, and ordinary multiple points. Below, the *Full Numerator Cri-*

terior (Conjecture 1.3) conjectures a complete characterization of the ideal of admissible numerators.

The local theory of stable polynomials is used to refine and extend a number of results from [10–12, 28] to general bounded rational functions. The properties studied include measures of nontangential polynomial approximation, geometric constraints on boundary regions that witness singularities, and L^p -integrability of rational functions and their derivatives. The next two subsections provide more detailed overviews of these two complementary goals: developing a local theory of stable polynomials and using it to characterize the structure and regularity of general bounded rational functions.

1.1. Local theory of stable polynomials on \mathbb{H}^2 . The local theory of stable polynomials is presented in Section 2, where we collect, refine, and significantly extend results from both [10, 11, 28] and other sources. We note that Agler, McCarthy and Stankus [4, 5] studied the local geometry of zero sets near the boundary of the polydisk. Moreover, in analytic combinatorics and asymptotics in several variables, various authors (see e.g. [45, 48]) have also investigated local aspects of stable polynomials.

Our local theory has three main tools: *homogeneous expansions*, Weierstrass and *Puiseux factorizations*, and *realization formulas*. When possible, we consider d -variable stable polynomials, but some techniques require the restriction to $d = 2$. The $d = 2$ case is special because we have two tools at our disposal: Puiseux series and transfer function type realization formulas [2]. The latter tool is unavailable in more variables due to the failure of Andô's inequality in three or more variables [40, 49].

Let p be a stable polynomial (here taken to mean no zeros in \mathbb{H}^d) with $p(0, \dots, 0) = 0$. Write $p = A + iB$, where A, B are polynomials with real coefficients. Define the *reflection* \bar{p} of p by

$$\bar{p}(z) = \overline{p(\bar{z})} = A(z) - iB(z).$$

The reflection operation creates a natural dichotomy in the class of stable polynomials; each stable p factors as $p = p_1 p_2$ where p_1 is *pure stable* meaning it has no factors in common with \bar{p}_1 and p_2 is *real stable* meaning $p_2 = c\bar{p}_2$ for some constant c with $|c| = 1$. See Section 2.1 for details.

Our primary result on homogeneous expansions is:

THEOREM (Homogeneous expansions). *Assume $p \in \mathbb{C}[z_1, \dots, z_d]$ has no zeros in \mathbb{H}^d . Write $p = A + iB = \sum_j P_j = \sum_j (A_j + iB_j)$, where P_j, A_j, B_j are homogeneous polynomials of degree j and A_j, B_j have real coefficients. Let M be the smallest number with $P_M \neq 0$. Then:*

- (a) *P_M has no zeros on \mathbb{H}^d and there is a unimodular μ such that μP_M has real coefficients.*

- (b) If $\mu = 1$ and p is pure stable, then $P_M = A_M$, $B_{M+1} \not\equiv 0$, and $\frac{A_M}{B_{M+1}}$ maps \mathbb{H}^d to \mathbb{H} .

The condition $\mu = 1$ can be arranged by replacing p with μp . Our formal statements are given as Theorem 2.2 and Propositions 2.5, 2.7. The above theorem is particularly useful for studying the behavior of p in nontangential regions near \mathbb{R}^d , the distinguished boundary of \mathbb{H}^d . When $d = 2$, part (b) implies that the tangents of A and B must interlace in a particular way, see Proposition 2.8. We note that part (a) of our Homogeneous Expansion Theorem was previously obtained in Atiyah–Bott–Gårding [8] and discussed in Pemantle–Wilson [45].

When $d = 2$, more refined local information about stable polynomials can also be extracted via their Puiseux factorizations. Our primary theorem about Puiseux factorizations illustrates the dichotomy between pure and real stable polynomials:

THEOREM (Puiseux factorizations). *Let p have Weierstrass factorization $p = up_1 \cdots p_k$ near $(0, 0)$, where u is a unit and each p_j is an irreducible Weierstrass polynomial in z_2 of degree M_j .*

- (a) *Let p be a pure stable polynomial. Then near $(0, 0)$,*

$$p(z) = u(z) \prod_{j=1}^k \prod_{m=1}^{M_j} (z_2 + q_j(z_1) + z_1^{2L_j} \psi_j(\mu_j^m z_1^{1/M_j})),$$

where each L_j is a positive integer, $\mu_j = e^{2\pi i/M_j}$, $q_j \in \mathbb{R}[z]$ with $q_j(0) = 0$, $q'_j(0) > 0$, $\deg q_j < 2L_j$, and ψ_j is holomorphic near 0 with $\operatorname{Im}(\psi_j(0)) > 0$.

- (b) *Let p be a real stable polynomial without monomial factors. Then each M_j is 1 and near $(0, 0)$,*

$$p(z) = u(z) \prod_{j=1}^k (z_2 + \phi_j(z_1)),$$

where each ϕ_j is holomorphic near 0 with real coefficients, $\phi_j(0) = 0$, and $\phi'_j(0) > 0$.

As complicated as item (a) above looks, a conceptual breakthrough in the present paper is that for most applications the Puiseux series parts, i.e. the ψ_j , can be forgotten and the most important data consists of

- the even integers $2L_1, \dots, 2L_k$, called *local contact orders*,
- the real polynomials q_1, \dots, q_k ,
- the associated multiplicities M_1, \dots, M_k .

The maximum $K := \max \{2L_1, \dots, 2L_k\}$ is called the *contact order* of p at $(0, 0)$ because it measures how the zero set of a pure stable p , denoted \mathcal{Z}_p ,

approaches \mathbb{R}^2 . The papers [10, 11] used contact order to quantify regularity properties of related rational functions.

This data is also reflected in natural perturbations of a pure stable p . Writing $p = A + iB$ as before, it turns out that $A + tB$ is real stable for $t \in \mathbb{R}$ while it is pure stable for $t \in \mathbb{H}$. We are able to deduce two key properties about the family of polynomials $A + tB$. First, for all but finitely many $t \in \mathbb{R}$,

$$(1.1) \quad A(z) + tB(z) = u(z; t) \prod_{j=1}^k \prod_{m=1}^{M_j} (z_2 + q_j(z_1) + z_1^{2L_j} \psi_{j,m}(z_1; t)),$$

using local contact orders and real polynomials from the Puiseux Factorization Theorem. Here $\psi_{j,m}(\cdot; t)$ is analytic at 0 and $u(z; t)$ is a unit. This shows that the initial segments of the branches of the zero sets of $A + tB$ agree with both each other and those of \mathcal{Z}_p , resolving the main problem left open in [11] (Conjecture 5.2). Second, defining $K_{\min} := \min \{2L_1, \dots, 2L_k\}$, the *universal contact order* of p at $(0, 0)$, we find that in the homogeneous expansion of the unit in (1.1):

$$u(z; t) = 1 + \sum_{j \geq 1} u_j(z; t),$$

the polynomials $u_j(z; t)$ are affine in t for $j \leq K_{\min} - 2$. These results appear in Theorems 2.20 and 2.25 and prove critical in our later investigations of rational function regularity.

In the $d = 2$ setting, useful global operator-theoretic formulas related to stable polynomials such as transfer function realizations and determinantal representations have also been established: see Grinshpan, Kaliuzhnyi-Verbovetskyi, Vinnikov, Woerdeman [22, 23], Woerdeman [51], and previous work by the second author [32] which itself was based on recent work of Dritschel [19]. Such formulas often encode local information about p . We provide an overview of some of them in Section 2.4, but we focus now on a formula particularly suited to the study of general bounded rational functions q/p . Here p is pure stable on \mathbb{H}^2 and we normalize so that $|q/p| < 1$ in \mathbb{H}^2 . Let $\gamma : \mathbb{D} \rightarrow \mathbb{H}$ be a Möbius map. Then $f := \gamma \circ (q/p)$ is rational, maps $\mathbb{H}^2 \rightarrow \mathbb{H}$, and is what is called a *rational Pick function*. We show that, up to conformal equivalence, every rational Pick function satisfies a particularly simple formula involving a matrix with positive imaginary part (PIP).

THEOREM (PIP realizations). *Let f be a nonconstant rational Pick function on \mathbb{H}^2 . Then there exist conformal self-maps σ_1, σ_2 of \mathbb{H} with $\sigma_2(0) = 0$ such that for $g := \sigma_1 \circ f \circ (\sigma_2, \sigma_2)$, there exist*

- $c \in \mathbb{C}$, $\alpha, \beta \in \mathbb{C}^n$ for some positive integer n ,
- $n \times n$ matrices P_1, P_2 satisfying $0 \leq P_1, P_2 \leq I$, and $P_1 + P_2 = I$,

- an $n \times n$ matrix S satisfying $\text{Im}(S) := \frac{1}{2i}(S - S^*) \geq 0$

such that

$$(1.2) \quad g(z) = c + \beta^* (S + z_1 P_1 + z_2 P_2)^{-1} \alpha \quad \text{for } z \in \mathbb{H}^2,$$

and $\lim_{t \searrow 0} g(it, it) \neq \infty$.

Note that the limit $\lim_{t \searrow 0} g(it, it)$ exists in $\overline{\mathbb{H}} \cup \{\infty\}$ since g restricted to the diagonal is a one-variable rational Pick function. Formula (1.2) is not valid for all rational Pick functions; Pick functions with certain singular behavior at ∞ must be represented with a more complicated formula as in the Type IV realizations of Agler, Tully-Doyle and Young [7, 43]. In essence, the conformal maps applied to f are designed to avoid these global complications which should be unimportant to a local theory.

The PIP realization theorem gives access to boundary singular behavior of not just p but also to a family of perturbations $q + \lambda p$ where again $|q/p| < 1$. This is in contrast to our previous results which, while very detailed, focus on the less general family of perturbations $A + tB$ which are simply multiples of the perturbations $\bar{p} + \lambda p$ (for proper choice of λ depending on t —see Remark 2.6).

1.2. Structure and regularity of bounded rational functions.

This paper presents a number of applications of the preceding theory to the study of rational functions $f = q/p$ which are bounded and analytic on either \mathbb{H}^d or \mathbb{D}^d . Often we rescale our functions so that $|f| \leq 1$, in which case f is called a *rational Schur function* or *RSF*. While stable polynomials will always be the denominators of RSFs, stable polynomials are directly connected to a special class of RSFs called *rational inner functions* or *RIFs* on \mathbb{D}^d (resp. \mathbb{H}^d). These are rational functions ϕ which are holomorphic on \mathbb{D}^d (resp. \mathbb{H}^d) and whose boundary values satisfy $|\phi(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}^d$ (resp. \mathbb{R}^d). Here, \mathbb{T} is the unit circle and \mathbb{T}^d is the *distinguished boundary* of \mathbb{D}^d .

An example RIF on \mathbb{D}^2 is

$$(1.3) \quad \phi(z_1, z_2) = \frac{2z_1 z_2 - z_1 - z_2}{2 - z_1 - z_2}.$$

RIFs are multivariate analogues of a crucial class of functions called finite Blaschke products (see Garcia, Mashreghi and Ross [20]), but unlike finite Blaschke products, RIFs can have boundary singularities as exhibited by (1.3) at $(1, 1)$. RIFs appear frequently in multivariate function theory investigations for several reasons. Every bounded analytic function on \mathbb{D}^d can be approximated locally uniformly by constant multiples of rational inner functions and so, results about RIFs can sometimes be generalized to all bounded analytic functions; see [26, 46]. In one and two variables, RIFs are the canonical solutions to Nevanlinna–Pick interpolation problems and

serve as essential examples of functions which preserve matrix inequalities, see [2, 6, 42, 44]. RIFs also connect operator theory with systems and control engineering (see e.g. Kummert [34] and Ball, Sadosky, and Vinnikov [9]).

The study of RIFs is made easier by the fact that there is a simple description of all RIFs on \mathbb{D}^d or \mathbb{H}^d . A theorem of Rudin and Stout adapted to \mathbb{H}^d says that every RIF on \mathbb{H}^d is of the form \bar{p}/p for some pure stable p ; see [46, Chapter 5]—a similar description is possible on \mathbb{D}^d . A large amount of prior work has focused on regularity results for RIFs, while the present work aims for general RSFs where no such simple description is known. Indeed, our first question is about exactly this lack of a simple description.

Sections 3–6 are guided by the following questions:

- (Q1) *Numerator characterization.* Given a pure stable polynomial p , for which polynomials q is q/p bounded on \mathbb{H}^2 ? (Section 5)
- (Q2) *Non-tangential regularity.* How much nontangential regularity do bounded rational functions have at singularities on the distinguished boundary? (Section 3)
- (Q3) *Horn regions.* How does the singular behavior of bounded rational functions manifest itself on the distinguished boundary (say, on \mathbb{T}^2 or \mathbb{R}^2)? (Section 4)
- (Q4) *Derivative integrability.* Can we determine the L^p integrability of the partial derivatives of bounded rational functions near boundary singularities? (Section 6)

Our first question (Q1) goes to the heart of the theory and seeks a characterization of RSFs. One might naively conjecture that q/p would be bounded on \mathbb{H}^2 if q vanishes to at least the same order as p at the boundary zeros of p . This condition is necessary but not sufficient. A complete answer requires a detailed local analysis of p at a boundary zero. Let us consider without loss of generality a zero at $(0, 0)$. Since \bar{p}/p is a rational inner function, it is clear that if $q = f_1p + f_2\bar{p}$ for some functions f_1, f_2 analytic at $(0, 0)$, we infer that q/p is bounded on a neighborhood of $(0, 0)$ intersected with \mathbb{H}^2 . In other words, if q belongs to the ideal generated by p and \bar{p} in the ring $R_0 = \mathbb{C}\{z_1, z_2\}$ of convergent power series centered at $(0, 0)$, then q/p is bounded in \mathbb{H}^2 near $(0, 0)$. This ideal, denoted $(p, \bar{p})R_0$, leads to a characterization of numerators in the generic yet simplest case of boundary zeros where p vanishes to order 1. Higher order vanishing makes things more complicated as will be seen. Nonetheless, we have the following local characterization, which appears as Theorem 5.4:

THEOREM 1.1. *Let p be a pure stable polynomial on \mathbb{H}^2 and assume p vanishes to order 1 at $(0, 0)$. For any $q \in \mathbb{C}[z_1, z_2]$, the function q/p is bounded on a neighborhood of $(0, 0)$ intersected with \mathbb{H}^2 if and only if $q \in (p, \bar{p})R_0$.*

The forward and more difficult implication amounts to showing that boundedness of a rational function q/p along certain curves derived from the Puiseux factorization of p forces q to be in the given ideal: To state our results for higher order vanishing, we make reference to the Puiseux Factorization Theorem for pure stable polynomials and its notations. Let us call each $z_2 + q_j(z_1)$ an initial segment with cutoff $2L_j$ and multiplicity M_j . We are able to identify a large subset of numerators q where q/p is bounded near $(0, 0)$ in \mathbb{H}^2 and we also achieve a conceptual reduction of the problem. Define the following product ideal:

$$\mathcal{I} = \prod_{j=1}^k (z_2 + q_j(z_1), z_1^{2L_j})^{M_j} R_0.$$

Here $(z_2 + q_j(z_1), z_1^{2L_j})$ is the ideal generated by $z_2 + q_j(z_1)$ and $z_1^{2L_j}$; $(z_2 + q_j(z_1), z_1^{2L_j})^{M_j}$ is the ideal generated by M_j products of elements of the former ideal; and \mathcal{I} is the product of all such ideals. It is worth noting that \mathcal{I} is generally much larger than $(p, \bar{p})R_0$. Also, define the polynomial

$$[p](z) = \prod_{j=1}^k (z_2 + q_j(z_1) + iz_1^{2L_j})^{M_j}.$$

We say a function is *locally H^∞* if it is analytic and bounded on a neighborhood of $(0, 0)$ intersected with \mathbb{H}^2 . This is to avoid confusion with the concept of “locally bounded”.

THEOREM 1.2. *Let p be a pure stable polynomial on \mathbb{H}^2 with $p(0, 0) = 0$. Let $f \in \mathbb{C}[z_1, z_2]$.*

- *If $f \in \mathcal{I}$, then f/p is locally H^∞ .*
- *f/p is locally H^∞ if and only if $f/[p]$ is locally H^∞ .*
- *Suppose p has either a double point, an ordinary multiple point, or repeated segments (i.e. all $q_j(z_1)$ are the same). If f/p is locally H^∞ then $f \in \mathcal{I}$.*

See Theorems 5.2, 5.3, 5.7. From the terminology of the theory of algebraic curves, a *double point* occurs when p vanishes to order 2 at $(0, 0)$ and an *ordinary multiple point* occurs when p vanishes to order M and has M distinct tangents. The first and last items are the basis for the following general conjecture.

CONJECTURE 1.3 (Full numerator criterion). *Let p be a pure stable polynomial on \mathbb{H}^2 . For any $f \in \mathbb{C}[z_1, z_2]$, f/p is locally H^∞ if and only if $f \in \mathcal{I}$.*

After the posting of this paper in September 2021, Conjecture 1.3 was confirmed by J. Kollar [33]. While this supersedes the third item of Theorem 1.2 (essentially Theorem 5.7), the details of our proof may be of independent interest.

The conjecture suggests that, in many cases, the regularity of RSFs should either mirror (or be better than) that of related RIFs. Our answers to (Q2)–(Q4) align with that intuition. For example, the investigation of (Q2) is motivated by the fact that RIFs ϕ possess nontangential limits, denoted $\phi^*(x)$, at *every* distinguished boundary point x ; see [28]. (Roughly, saying $z \rightarrow x$ nontangentially in \mathbb{H}^d means the quantities $|z_i - x_i|$ and $\text{Im}(z_i)$ are all comparable as $z = (z_1, \dots, z_d) \rightarrow x = (x_1, \dots, x_d)$.) The existence of RSF limits could perhaps *a priori* be more precarious because their modulus functions could encode additional singular behavior. This is illustrated by Figure 1, which displays the modulus of the two-variable bounded rational function on \mathbb{D}^2 :

$$(1.4) \quad f(z_1, z_2) = \frac{(z_1 - 1)(z_2 - 1)}{2 - z_1 - z_2}$$

on \mathbb{T}^2 identified with $[-\pi, \pi]^2$, which is clearly discontinuous at $(1, 1)$.

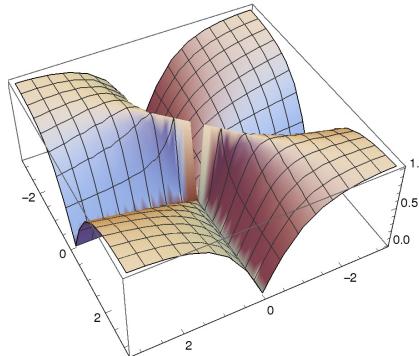


Fig. 1. Modulus of $f(z_1, z_2) = \frac{(z_1 - 1)(z_2 - 1)}{2 - z_1 - z_2}$ on \mathbb{T}^2 .

Despite such apparent obstructions, the following is true:

THEOREM 1.4. *If f is a RSF in \mathbb{H}^d , then $f^*(x)$ exists at every point x in \mathbb{R}^d .*

Theorem 1.4 appears as Theorem 3.2. Though we direct the reader to Section 3 for most details, we note that the local theory of stable polynomials also provides insights into higher order regularity. Our Theorem 3.5 characterizes when a RSF has directional derivatives and Theorem 3.9 characterizes when it has nontangential polynomial approximations to given orders at boundary points. Our analysis leads to an interesting new result for RIFs quantifying the relationships between contact order and nontangential behavior; namely, *if a pure stable polynomial p has universal contact order K at x , then \bar{p}/p has a nontangential polynomial approximation of order $K - 2$ at x* ; see Theorem 3.11. This is a partial converse to a key result [10, Theorem 7.1].

As a complement to nontangential behavior, (Q3) asks about “ultratangential” behavior, namely behavior on the distinguished boundary. For RIFs, the papers [10, 11] addressed this by showing that if ϕ has a singularity at $(0, 0)$, the unimodular level sets $\{x \in \mathbb{R}^2 : \phi(x) = \lambda\}$ for $\lambda \neq \phi^*(0, 0)$ approach $(0, 0)$ inside regions we call horns. A *horn* is a region with at least quadratic pinching (see Figure 2). Section 4 shows that this result follows naturally, and more easily, from the Puiseux Factorization Theorem. One way to interpret this RIF result is that if $(x_n)_n$ is a sequence in \mathbb{R}^2 that manifests the discontinuity of ϕ at $(0, 0)$, namely $x_n \rightarrow (0, 0)$, but $(\phi(x_n))_n$ converges to some value different from $\phi^*(0, 0)$, then the sequence $(x_n)_n$ becomes trapped in a union of horns. Our interpretation generalizes to the class of RSFs.

THEOREM 1.5. *Suppose $f = q/p$ is an RSF with $p(0, 0) = 0$. If a sequence $(x_n) \subset \mathbb{R}^2$ satisfies $x_n \rightarrow (0, 0)$ and $f(x_n) \rightarrow c \neq f^*(0, 0)$, then (x_n) is eventually trapped in a finite union of horns.*

Theorem 1.5 appears as Theorem 4.3 and follows from a delicate local analysis of the formula from the PIP Realization Theorem. A \mathbb{T}^2 -horn associated to the function f in (1.4) is given in Figure 2 below. As before, \mathbb{T}^2 is identified with $(-\pi, \pi]^2$. As $f^*(1, 1) = 0$, our theorem shows that every nonzero level set of this RSF must eventually be caught inside a horn region along the line $y = -x$.

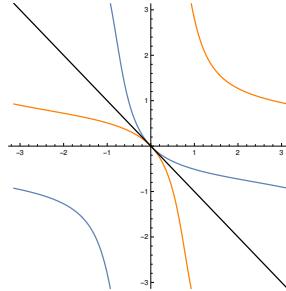


Fig. 2. A horn region for $f(z_1, z_2) = \frac{(z_1-1)(z_2-1)}{2-z_1-z_2}$. The curves come from level sets with $\lambda = 1$ (black), $\lambda = \frac{1}{2}(1+i)$ (blue), and $\lambda = \frac{1}{2}(1-i)$ (orange).

Our last question (Q4) proposes an alternative measure of regularity on the distinguished boundary. Derivative integrability encodes singular behavior because it roughly measures the rate at which a function runs through different values near the singularity. We restrict (Q4) to the bidisk to ensure a bounded domain of integration—allowing us to study global integrability of derivatives without added technical difficulties. For an RIF ϕ , derivative integrability is known to be governed by contact order [10, 11]; indeed, $\frac{\partial \phi}{\partial z_1} \in L^p(\mathbb{T}^2)$ if and only if $p < \frac{K+1}{K}$, where K is the maximum contact order

of ϕ at its singularities on \mathbb{T}^2 . Section 6 shows that the partial derivatives of RSFs possess nice integrability properties:

THEOREM 1.6. *Let p be a stable polynomial on \mathbb{D}^2 with finitely many zeros on \mathbb{T}^2 . Then there is a finite list of numbers $\mathfrak{p}_1, \dots, \mathfrak{p}_M \in (0, \infty]$ such that for any $q \in \mathbb{C}[z_1, z_2]$,*

$$(1.5) \quad \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : \partial_{z_1}(q/p) \in L^{\mathfrak{p}'}(\mathbb{T}^2) \}$$

is equal to one of the \mathfrak{p}_j 's. Moreover, the number M is bounded by an algebraic characteristic of p .

Theorem 1.6 appears as Theorem 6.10. Under additional (generic) assumptions on p , in Theorem 6.11 we obtain an exact characterization of the numbers $\mathfrak{p}_1, \dots, \mathfrak{p}_M$ in terms of contact order. For example, when $p = 2 - z_1 - z_2$, the list of numbers satisfying (1.5) for some q is exactly $\frac{3}{2}, 3, \infty$. Indeed, the function in (1.4) has $\frac{\partial f}{\partial z_1} \in L^{\mathfrak{p}}(\mathbb{T}^2)$ if and only if $\mathfrak{p} < \frac{3}{2}$. Our exact characterization of $\mathfrak{p}_1, \dots, \mathfrak{p}_M$ relies on results in Section 5 (discussed earlier), and hence relies on the Puiseux Factorization Theorem.

1.3. Structure of the paper. Section 2 details our local theory of stable polynomials, including results on homogeneous expansions, Puiseux factorizations, and realization formulas. Section 5 investigates the numerator criterion question while Sections 3, 4, 6 respectively address nontangential regularity, horn regions at singularities, and derivative integrability for RSFs. While this introduction provides an overview of key results, the reader should consult each section for precise definitions, additional results, and a variety of examples which are not mentioned here.

2. A local theory of stable polynomials

2.1. Some global theory of stable polynomials. We now describe a dichotomy for stable polynomials already alluded to in the Puiseux factorization theorem. Very roughly, stable polynomials factor into a polynomial that vanishes very little on the distinguished boundary and a polynomial that vanishes a lot on the distinguished boundary. We also review some basic notions of stable polynomials on \mathbb{D}^d compared to \mathbb{H}^d .

Recall that the reflection of a polynomial in the context of \mathbb{H}^d is given by

$$\bar{p}(z) = \overline{p(\bar{z})}$$

for $z \in \mathbb{C}^d$. In the context of \mathbb{D}^d the reflection is degree dependent: if $p \in \mathbb{C}[z_1, \dots, z_d]$ has multidegree $n = (n_1, \dots, n_d)$ (i.e. degree n_j in z_j) then the reflection of p is

$$\tilde{p}(z) = z^n \overline{p(1/\bar{z}_1, \dots, 1/\bar{z}_d)} \quad \text{where} \quad z^n := z_1^{n_1} \cdots z_d^{n_d}.$$

Note that the Cayley transform converts between the two notions of reflection: \tilde{p} in the \mathbb{D}^d setting versus \bar{p} in the \mathbb{H}^d setting. If $p \in \mathbb{C}[z_1, \dots, z_d]$, viewed as a function on \mathbb{D}^d , has multidegree n then we convert to a polynomial in the setting of \mathbb{H}^d via

$$(2.1) \quad \begin{aligned} P(z) &= (\mathbf{1} - iz)^n p\left(\frac{\mathbf{1} + iz}{\mathbf{1} - iz}\right) \\ &= (1 - iz_1)^{n_1} \cdots (1 - iz_d)^{n_d} p\left(\frac{1 + iz_1}{1 - iz_1}, \dots, \frac{1 + iz_d}{1 - iz_d}\right) \end{aligned}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{C}^d$ and we use convenient and temporary component-wise shorthands. Then

$$\bar{P}(z) = (\mathbf{1} + iz)^n \overline{p\left(\frac{\mathbf{1} + iz}{\mathbf{1} - iz}\right)} = (\mathbf{1} - iz)^n \tilde{p}\left(\frac{\mathbf{1} + iz}{\mathbf{1} - iz}\right)$$

shows a direct correspondence between the notions of reflection. It is straightforward to check that if $p \in \mathbb{C}[z_1, \dots, z_d]$ has no zeros in \mathbb{D}^d , then $\phi = \tilde{p}/p$ is a rational inner function on \mathbb{D}^d . Namely, $|\phi| \leq 1$ in \mathbb{D}^d and $|\phi| = 1$ on \mathbb{T}^d outside the zero set of p . (This is obvious if p has no zeros on \mathbb{D}^d by the maximum principle; otherwise one can examine $p(rz)$ as $r \nearrow 1$.) This type of homothety is unavailable in \mathbb{H} so Cayley transform is the easiest way to see that if $p \in \mathbb{C}[z_1, \dots, z_d]$ has no zeros in \mathbb{H}^d then \bar{p}/p is a rational inner function on \mathbb{H}^d .

Let us now review a basic dichotomy of stable polynomials. Any $p \in \mathbb{C}[z_1, \dots, z_d]$ with no zeros in \mathbb{D}^d can be factored into $p = p_1 p_2$ where p_1 has no factors in common with \tilde{p}_1 and p_2 is a constant multiple of \tilde{p}_2 . Indeed, writing $p = p_1 p_2$ and $\tilde{p} = q_1 p_2$ where p_1 and q_1 have no common factors we see that p_2 has no zeros in $\mathbb{D}^d \cup \{z \in \mathbb{C} : |z| > 1\}^d$. The rest follows from the next standard lemma.

LEMMA 2.1. *Any $q \in \mathbb{C}[z_1, \dots, z_d]$ with no zeros in $\mathbb{D}^d \cup \{z \in \mathbb{C} : |z| > 1\}^d$ is a multiple of \tilde{q} .*

Proof. Note that for any $\tau \in \mathbb{T}^d$, the one-variable function $f(z) := q(z\tau)$ only has zeros on \mathbb{T} implying \tilde{f} is a multiple of f and therefore $|\tilde{f}/f| = 1$. But $\tau^n \tilde{f}(z) = \tilde{q}(\tau z)$ so that $\phi = \tilde{q}/q$ attains modulus 1 inside \mathbb{D}^d . Now ϕ is rational inner and therefore, by the maximum principle, the function ϕ must be constant and unimodular. ■

Note that all polynomial factors of q in the lemma satisfy the same hypothesis and conclusion as the lemma.

The following terminology is borrowed from [4]. We shall call polynomials of the type p_1 *atoral stable*. Atoral stable polynomials arise as the denominators of rational inner functions. Polynomials of the type p_2 are called *toral stable*. These arise as defining polynomials for unimodular level sets of ra-

Table 1. Dichotomies of stable polynomials

Nonvanishing region	Reflection	No factors in common with reflection	All factors multiples of their reflection
\mathbb{D}^d	\tilde{p}	atoral stable	toral stable
\mathbb{H}^d	\bar{p}	pure stable	real stable

tional inner functions. Namely, given a nonconstant rational inner function $\phi = z^\alpha \tilde{p}/p$ ($\alpha \in \mathbb{N}^d$) and $\mu \in \mathbb{T}$, the set

$$\{z \in \mathbb{C}^d : \phi(z) = \mu\}$$

can be described by $\{z \in \mathbb{C}^d : z^\alpha \tilde{p}(z) - \mu p(z) = 0\}$ if we ignore zeros of p . Note that $z^\alpha \tilde{p} - \mu p$ is toral stable because it is nonvanishing in \mathbb{D}^d (as a limit of $z^\alpha \tilde{p} - wp$ for $w \rightarrow \mu$ with $|w| \searrow 1$) and is a constant multiple of its reflection.

In the upper half-plane setting we have a dichotomy analogous to atoral versus toral. For $p \in \mathbb{C}[z_1, \dots, z_d]$ with no zeros in \mathbb{H}^d we can factor $p = p_1 p_2$ where p_1 has no factors in common with \bar{p}_1 and p_2 is a constant multiple of \bar{p}_2 . As above, p_2 will have no zeros in $\mathbb{H}^d \cup (-\mathbb{H})^d$ and this property alone implies p_2 is a multiple of \bar{p}_2 . So, every factor of p_2 is a multiple of its reflection. We can then arrange for p_2 and all of its factors to have real coefficients by transferring a constant over to p_1 . We shall call p_1 type polynomials *pure stable* and type p_2 polynomials *real stable*. “Pure stable” is not common parlance. Real stable refers to the fact that in one-variable a real stable polynomial has all of its roots on the real axis. We end this section with remarks about homogeneous polynomials and distinguished boundary zero sets for stable polynomials in two variables.

A homogeneous polynomial $P \in \mathbb{C}[z_1, \dots, z_d]$ with no zeros in \mathbb{H}^d is automatically real stable since $P(-z)$ is a multiple of P . If $d = 2$, then such a homogeneous polynomial factors as

$$P(z_1, z_2) = c \prod_{j=1}^M (a_j z_1 + b_j z_2)$$

with $c \in \mathbb{C}$, $a_j, b_j \in \mathbb{R}$. Note $az_1 + bz_2$ is nonvanishing in \mathbb{H}^2 if and only if a, b have the same sign, so we can further arrange $a_j, b_j \geq 0$ for all j by absorbing sign changes into c . Thus, P is a multiple of a homogeneous polynomial in $\mathbb{R}_+[z_1, z_2]$, the polynomials with nonnegative coefficients.

In two variables, atoral (resp. pure) stable polynomials have finitely many zeros on \mathbb{T}^2 (resp. \mathbb{R}^2) which follows from Bézout’s theorem since zeros on \mathbb{T}^2 are zeros in common with \tilde{p} . In the pure stable case one should keep in mind that there can be common zeros on $(\mathbb{R} \times \{\infty\}) \cup (\{\infty\} \times \mathbb{R}) \cup \{(\infty, \infty)\}$ (e.g. p vanishes at (∞, ∞) if $z_1^{n_1} z_2^{n_2} p(1/z_1, 1/z_2)$ vanishes at $(0, 0)$). Toral

stable and real stable polynomials in two variables have curves of zeros on the distinguished boundaries \mathbb{T}^2 (resp. \mathbb{R}^2) and no isolated zeros. Later on we present a local parametrization theorem for real stable polynomials, Theorem 2.13, which states that locally the zero set of a real stable polynomial on \mathbb{R}^2 is a union of smooth curves. Of course away from finitely many singularities the zero set will locally consist of a single smooth curve.

2.2. Homogeneous expansions. We now discuss the Homogeneous Expansion Theorem from the introduction in Theorem 2.2 and Propositions 2.5 and 2.7.

THEOREM 2.2. *Suppose $p \in \mathbb{C}[z_1, \dots, z_d]$ has no zeros in \mathbb{H}^d and $p(0) = 0$. We may decompose p into homogeneous polynomials*

$$p(z) = \sum_{j=M}^n P_j(z)$$

where $P_j \in \mathbb{C}[z_1, \dots, z_d]$ is homogeneous of degree j , $P_M \not\equiv 0$, and $n = n_1 + \dots + n_d$ is the total degree of p . Then P_M has no zeros in \mathbb{H}^d and by homogeneity P_M is necessarily real stable. In particular, there exists $\mu \in \mathbb{T}$ such that $\mu P_M \in \mathbb{R}[z_1, \dots, z_d]$.

Theorem 2.2 follows from work in Atiyah, Bott and Gårding [8, Lemma 3.42] but as discussed in [45, Proposition 11.1.6] it directly follows from Hurwitz's theorem applied to

$$P_M(z) = \lim_{t \rightarrow 0^+} \frac{1}{t^M} p(tz).$$

EXAMPLE 2.3. The polynomial $p(z_1, z_2) = 4 - 3z_1 - 3z_2 + z_1^2 z_2 + z_1 z_2^2$, taken from [11, Example 7.1], has no zeros on \mathbb{D}^2 . Conformal mapping yields

$$\begin{aligned} P(z_1, z_2) &= (1 - iz_1)^2 (1 - iz_2)^2 p\left(\frac{1 + iz_1}{1 - iz_1}, \frac{1 + iz_2}{1 - iz_2}\right) \\ &= -4(z_2^2 + 4z_1 z_2 + z_1^2 - 2z_1^2 z_2^2 - 4iz_1 z_2(z_1 + z_2)), \end{aligned}$$

which has no zeros in \mathbb{H}^2 . The bottom homogeneous term

$$-4(z_2^2 + 4z_1 z_2 + z_1^2) = -4(z_2 + (2 + \sqrt{3})z_1)(z_2 + (2 - \sqrt{3})z_1)$$

evidently has no zeros in \mathbb{H}^2 and has real coefficients. We return to this example in Examples 2.9 and 2.14. \diamond

When $p \in \mathbb{C}[z_1, \dots, z_d]$ is pure stable its homogeneous expansion has additional structure. The real and imaginary parts (via coefficients) of p are

$$A := \frac{1}{2}(p + \bar{p}) \quad \text{and} \quad B := \frac{1}{2i}(p - \bar{p})$$

so that $A, B \in \mathbb{R}[z_1, \dots, z_d]$ and $p = A + iB$.

REMARK 2.4. Notice that $A + tB$ is real stable for $t \in \mathbb{R}$ since $A + tB = \frac{1}{2}((1-it)p + (1+it)\bar{p}) = 0$ exactly when $\bar{p}/p = -\frac{1-it}{1+it}$ which happens to be a point on \mathbb{T} . Since \bar{p}/p is a nontrivial RIF this can only occur outside of \mathbb{H}^d . Similarly, B is also real stable.

PROPOSITION 2.5. *Suppose $p \in \mathbb{C}[z_1, \dots, z_d]$ is a pure stable polynomial and $p(0) = 0$. Write the homogeneous expansion of $p = \sum_{j \geq M} P_j$. By Theorem 2.2, we may replace p with a unimodular multiple so that $P_M = \bar{P}_M \in \mathbb{R}[z_1, \dots, z_d]$. Then the lowest order homogeneous term of A equals P_M and B vanishes to order exactly $M + 1$.*

Proof. We have arranged for the lowest order homogeneous term of p to have real coefficients so we must have $A_M = P_M$ and $B_M = 0$. Assuming p and \bar{p} have no common factors implies $\phi = \bar{p}/p$ is nonconstant and $|\phi| < 1$ inside \mathbb{H}^d . Set $\tau = (1, \dots, 1)$. This implies $\zeta \mapsto p(\zeta\tau)$ has at least one root in the lower half-plane, as otherwise it would have all zeros real and $\bar{p}(\zeta\tau)/p(\zeta\tau)$ would be constant and unimodular. Now,

$$\begin{aligned} p(\zeta\tau) &= A_M(\tau)\zeta^M + (A_{M+1}(\tau) + iB_{M+1}(\tau))\zeta^{M+1} + \dots \\ &= A_M(\tau)\zeta^M(1 + (a + ib)\zeta + \dots) \end{aligned}$$

where $a + ib = (A_{M+1}(\tau) + iB_{M+1}(\tau))/A_M(\tau)$. On the other hand, if we factor $p(\zeta\tau)$ as

$$\begin{aligned} p(\zeta\tau) &= A_M(\tau)\zeta^M \prod_j (1 + \alpha_j \zeta) \\ &= A_M(\tau)\zeta^M \left(1 + \left(\sum_j \alpha_j\right)\zeta + \dots\right) \end{aligned}$$

where α_j are in the closed lower half-plane we see that $a + ib = \sum_j \alpha_j$. But at least one α_j must be in the open lower half-plane so $b \neq 0$ and hence $B_{M+1}(\tau) \neq 0$. ■

REMARK 2.6. The above proposition has implications for the geometry of unimodular level sets of rational inner functions. The unimodular level sets of $\phi = \bar{p}/p$ are given by $A + tB \equiv 0$ for $t \in \mathbb{R}$ or $B \equiv 0$ (corresponding to $t = \infty$). Indeed, for $\mu \in \mathbb{T}$, the zero set $\bar{p} - \mu p = 0$ is the same as $A(1 - \mu) - iB(1 + \mu) = 0$ or $A + tB = 0$ for $t = i\frac{\mu+1}{\mu-1} \in \mathbb{R}$ when $\mu \neq 1$ and $B = 0$ for $\mu = 1$. Therefore, the polynomials defining the unimodular level sets of ϕ all have initial homogeneous term A_M , with the exception of the level set $B = 0$ which has initial homogeneous term B_{M+1} of one degree higher. In two variables, this has the more direct geometric interpretation that all of the unimodular level curves with the exception of $B = 0$ have the same set of tangents, namely the factors of A_M .

The next proposition refines the relationship between A_M and B_{M+1} . Note that $A/B = i\frac{1+\bar{p}/p}{1-\bar{p}/p}$ maps \mathbb{H}^d into \mathbb{H} .

PROPOSITION 2.7. *With the setup of the previous proposition, let A_M, B_{M+1} be the lowest order homogeneous terms of A, B . Then A_M/B_{M+1} is a Pick function, i.e.*

$$\operatorname{Im}\left(\frac{A_M}{B_{M+1}}\right) \geq 0 \quad \text{in } \mathbb{H}^d.$$

Proof. Since B has no zeros in \mathbb{H}^d , B_{M+1} has no zeros in \mathbb{H}^d . Since A/B is a Pick function we can let $t > 0$ and $z \in \mathbb{H}^d$ and consider

$$0 \leq t \operatorname{Im}\left(\frac{A(tz)}{B(tz)}\right) = \operatorname{Im}\left(\frac{A_M(z) + tA_{M+1}(z) + \dots}{B_{M+1}(z) + tB_{M+2}(z) + \dots}\right).$$

Let $t \rightarrow 0$ to see that A_M/B_{M+1} is a Pick function. ■

In two variables this has the more geometric interpretation that the tangents of A_M interlace the tangents of B_{M+1} . The following proposition encodes that fact and introduces an added level of generality. Namely, it considers pairs of homogeneous polynomials, which we still denote A_M, B_{M+1} , that could possess monomial factors. We will see in the next section that if $p = A + iB$ is pure stable then we can arrange for A_M to have no factors of z_1 or z_2 . Hence, the following proposition handles some pairs A_M, B_{M+1} that do not originate from a pure stable p .

PROPOSITION 2.8. *Write*

$$A_M = az_1^r \prod_{j=1}^{M-r} (z_2 + a_j z_1), \quad B_{M+1} = bz_1^s \prod_{j=1}^{M+1-s} (z_2 + b_j z_1)$$

where $0 \leq a_1 \leq \dots \leq a_{M-r}$, $0 \leq b_1 \leq \dots \leq b_{M+1-s}$. Set $a_j = \infty$ for $j = M-r+1, \dots, M$ and $b_j = \infty$ for $j = M+2-s, \dots, M+1$ if r or s are nonzero. Suppose A_M/B_{M+1} is a Pick function. Then $r = s$ or $r+1 = s$ and

$$b_1 \leq a_1 \leq b_2 \leq \dots \leq a_M \leq b_{M+1}.$$

Proof. Suppose first that A_M, B_{M+1} have no factors of z_1 . Then $A_M(z_1, z_2) = a \prod_{j=1}^M (z_2 + a_j z_1)$ for $a_j \geq 0$ and $B_{M+1}(z_1, z_2) = b \prod_{j=1}^{M+1} (z_2 + b_j z_1)$ for $b_j \geq 0$ because A_M and B_{M+1} and all of their (linear) factors are real stable. Then

$$z_2 \mapsto \frac{A_M(1, z_2)}{B_{M+1}(1, z_2)} = \frac{a}{b} \frac{\prod_{j=1}^M (z_2 + a_j)}{\prod_{j=1}^{M+1} (z_2 + b_j)}$$

is a one-variable real rational Pick function. By [11, Lemma 6.5], the zeros (or rather their negatives) must interlace. Namely, if we write $0 \leq a_1 \leq \dots \leq a_M$,

$0 \leq b_1 \leq \dots \leq b_{M+1}$ then

$$b_1 \leq a_1 \leq b_2 \leq \dots \leq a_M \leq b_{M+1}.$$

Lemma 6.5 of [11] also says $a/b < 0$. If A_M or B_{M+1} has a factor of z_1 , write $A_M = z_1^r A_{M-r}^b$, $B_{M+1} = z_1^s B_{M+1-s}^b$ where A_{M-r}^b, B_{M+1-s}^b have no factors of z_1 . For $t > 0$,

$$t^{s-r} \frac{A_M(tz_1, z_2)}{B_{M+1}(tz_1, z_2)} = z_1^{r-s} \frac{A_{M-r}^b(tz_1, z_2)}{B_{M+1-s}^b(tz_1, z_2)}$$

is still a Pick function and if we send $t \rightarrow 0$ we get the Pick function

$$z_1^{r-s} \frac{A_{M-r}^b(0, z_2)}{B_{M+1-s}^b(0, z_2)} = cz_1^{r-s} z_2^{s-r-1}$$

for some constant c . This is only possible if $s = r + 1$ or $s = r$ and $c < 0$. We view this situation as A_M having r infinite slopes which then implies B_{M+1} has r or $r + 1$ infinite slopes. If $r = s$, A_{M-r}^b and B_{M+1-r}^b have $M - r$ and $M + 1 - r$ slopes that interlace as before and the addition of infinite slopes does not change the interlacing property. If $s = r + 1$, then $A_{M-r}^b(1, z)/B_{M-r}^b(1, z)$ is a one-variable Pick function so the $M - r$ roots of the numerator and denominator interlace. Since the ratio of the leading coefficients is negative, Lemma 6.5 of [11] states that the smallest slope of B_{M+1} is smaller than the smallest slope of A_M . Thus, the $(M - r)$ th slope of B_{M+1} is at most the $(M - r)$ th slope of A_M and the remaining $r + 1$ infinite slopes of B_{M+1} interlace with the remaining r infinite slopes of A_M . ■

If $p = A + iB$ is pure stable with $P_M = A_M$, then the above proposition says that the tangents of A_M and B_{M+1} interlace. Since these are the initial homogeneous terms of A and B , we can say that their tangents interlace as also do those of $A + tB$ and B .

EXAMPLE 2.9. Returning to Example 2.3, we have

$$A = -4(z_2^2 + 4z_1z_2 + z_1^2 - 2z_1^2z_2^2) \quad B = 4z_1z_2(z_1 + z_2).$$

So,

$$\frac{A_2}{B_3} = -\frac{z_2^2 + 4z_1z_2 + z_1^2}{z_1z_2(z_1 + z_2)}$$

is a Pick function.

EXAMPLE 2.10. The Pick function A_M/B_{M+1} need not be especially interesting. Consider the polynomial with no zeros on \mathbb{D}^2 ,

$$p(z_1, z_2) = 4 - 5z_1 - 2z_2 + 2z_1z_2 + 3z_1^2 - z_1^2z_2 - z_1^3z_2$$

(taken from [28, Example 15.3]) converted to the polynomial with no zeros on \mathbb{H}^2 given by

$$P(z_1, z_2) = z_1 + z_2 - 2z_1^3 - 6z_1^2z_2 - i(z_1^2 + z_1z_2 - 4z_1^3z_2).$$

(A rescaling was also involved.) We have

$$\frac{A_1}{B_2} = \frac{(z_1 + z_2)}{-z_1(z_1 + z_2)} = -\frac{1}{z_1}.$$

We return to this example in Examples 2.15, 5.1, and 5.6. \diamond

REMARK 2.11. The above properties of lowest order homogeneous terms transfer to the polydisk via Cayley transform.

Suppose $p \in \mathbb{C}[z_1, \dots, z_d]$, $p(1, \dots, 1) = 0$, and p has multidegree $n = (n_1, \dots, n_d)$. Then

$$P(z) = (1 - iz_1)^{n_1} \cdots (1 - iz_d)^{n_d} p\left(\frac{1 + iz_1}{1 - iz_1}, \dots, \frac{1 + iz_d}{1 - iz_d}\right) \in \mathbb{C}[z_1, \dots, z_d]$$

has zeros in \mathbb{H}^d if and only if p has zeros in \mathbb{D}^d ; also, $P(0) = 0$. In order to compare homogeneous decompositions we write

$$p(1 + z_1, \dots, 1 + z_d) = \sum_{j \geq M} p_j(z) \quad \text{and} \quad P(z) = \sum_{j \geq M} P_j(z).$$

Since $\frac{1+iz}{1-iz} = 1 + \frac{2iz}{1-iz}$ one can directly check by looking at the lowest order terms that

$$P_M(z) = (2i)^M p_M(z).$$

Thus, if p has no zeros in \mathbb{D}^d , then p_M has no zeros in \mathbb{H}^d (or $(c\mathbb{H})^d$ for any $c \neq 0$ by homogeneity). This is how a version of Theorem 2.2 was originally stated in [28]. By this correspondence the theorem from [28] directly implies Theorem 2.2.

The finer properties involving A_M, B_{M+1} would be more technical to state in the context of the polydisk.

2.3. Puiseux expansions. While homogeneous expansions can provide some useful local information about stable polynomials in two and more variables, using Puiseux expansions we can give a nearly complete local description of stable polynomials in two variables. Given $p \in \mathbb{C}[z_1, z_2]$ with no zeros in \mathbb{H}^2 , $p(0, 0) = 0$, we can factor it as

$$p = z_1^{\alpha_1} z_2^{\alpha_2} u p_1 \cdots p_N$$

where α_1, α_2 are nonnegative integers, $u \in \mathbb{C}\{z_1, z_2\}$ is analytic and non-vanishing at 0 and $p_j \in \mathbb{C}\{z_1\}[z_2]$ are monic and irreducible in z_2 with coefficients in the ring of convergent power series $\mathbb{C}\{z_1\}$ that vanish at 0 (i.e. irreducible Weierstrass polynomials). Note that $\mathbb{C}\{z_1, z_2\}, \mathbb{C}\{z_1\}$ denote rings of convergent power series.

By Puiseux's theorem each p_j formally factors into

$$\prod_{k=1}^m (z_2 - g(\mu^k z_1^{1/m}))$$

for $g \in \mathbb{C}\{z_1\}$, $g(0) = 0$, and $\mu = e^{2\pi i/m}$ (see [47, Theorems 3.5.1, 3.5.2]). Alternatively, the zero set of p_j near $(0, 0)$ is parametrized by the map

$$t \mapsto (t^m, g(t))$$

defined for t in a neighborhood of $0 \in \mathbb{C}$. Notice that since p has no zeros in \mathbb{H}^2 , the above map has the property that it is injective and maps into $\mathbb{C}^2 \setminus \mathbb{H}^2$. The following theorem derived from [28] gives a detailed description of the possible $g \in \mathbb{C}\{z_1\}$.

THEOREM 2.12. *Let $g \in \mathbb{C}\{z\}$, $g \not\equiv 0$, $g(0) = 0$, and assume $t \mapsto (t^m, g(t))$ is injective into $\mathbb{C}^2 \setminus \mathbb{H}^2$. Then $\phi(t) := -g(t)$ is of one of the following two forms.*

- *Pure stable type:*

$$\phi(t) = q(t^m) + t^{2mL}\psi(t)$$

where

- ◊ L is a positive integer,
- ◊ $q \in \mathbb{R}[t]$, where $q(0) = 0$, $q'(0) > 0$, $\deg q < 2L$,
- ◊ $\psi \in \mathbb{C}\{t\}$ with $\operatorname{Im} \psi(0) > 0$.

If $m > 1$ then ψ is not of the form $\psi(t) = h(t^m)$ for $h \in \mathbb{C}\{t\}$.

- *Real stable type:* $m = 1$, $\phi \in \mathbb{R}\{t\}$ (analytic with real coefficients) and $\phi(0) = 0, \phi'(0) > 0$.

We shall also say the associated Weierstrass polynomial is of *pure stable type* or *real stable type* depending on the type of the underlying function g in its Puiseux expansion. Using the above description, a Weierstrass polynomial of pure stable type is of the form

$$(2.2) \quad \prod_{n=1}^m (z_2 + q(z_1) + z_1^{2L}\psi(\mu^n z_1^{1/m}))$$

where $\mu = \exp(2\pi i/m)$.

An interesting paper with some antecedents of this theorem is [48] (see Sections 3–4).

Discussion of the proof of Theorem 2.12. This theorem is actually a correction, a refinement, and a rephrasing of [28, Lemma C.3], which states that if ϕ is analytic in a neighborhood of $0 \in \mathbb{C}$ with $\phi(0) = 0$, and $t \mapsto (t^k, \phi(t))$ is injective into $\mathbb{C}^2 \setminus \mathbb{H}^2$, then ϕ has the form

$$\phi(t) = \sum_{j=1}^M a_j t^{jk} + b t^{2kL} + \sum_{j=2kL+1}^{\infty} b_j t^j$$

where $a_1 < 0, a_2, \dots, a_M \in \mathbb{R}$, and $\operatorname{Arg} b \in (\pi, 2\pi)$. A minor correction to

this statement is that there is also the case that ϕ is of the form

$$(2.3) \quad \phi(t) = \sum_{j=1}^{\infty} a_j t^j$$

where $k = 1$, $a_1 < 0$ and $a_j \in \mathbb{R}$ for $j = 2, 3, \dots$. This does not affect any of the results in [28] because the real coefficient case was not relevant there. In any case, the proof given in [28] is essentially correct up to the point where it proves ϕ is necessarily of the form $\phi(t) = at^k + \text{higher order with } a < 0$. However, at this point there are two cases: either ϕ is of the form

$$\phi(t) = \sum_{j=1}^{\infty} a_j t^{jk}$$

with $a_j \in \mathbb{R}$ for all j , or

$$\phi(t) = \sum_{j=1}^M a_j t^{jk} + bt^L + \text{higher order}$$

where either $b \notin \mathbb{R}$ or L is not a multiple of k (we allow for $M = 1$). For this second case the proof proceeds as written in [28]. For the first case, we must have $k = 1$ because we assume $t \mapsto (t^k, \phi(t))$ is injective.

Theorem 2.12 now follows immediately from our corrected Lemma C.3 (with $g(t)$ in the statement of Theorem 2.12 playing the role of ϕ in the statement of [28, Lemma C.3]). ■

The two types in Theorem 2.12 are directly related to our global dichotomy of stable polynomials from Section 2.1. If p is pure stable then locally it can only have irreducible Weierstrass factors of pure stable type—a real stable factor would imply infinitely many zeros on \mathbb{R}^2 . If p is real stable then locally it can only have irreducible Weierstrass factors of real stable type if we ignore monomial factors. Indeed, if we had a pure stable factor parametrized via ϕ as above then for $x > 0$, $p(x, -\phi(x^{1/m})) \equiv 0$. But

$$\text{Im}(\phi(x^{1/m})) = x^{2L}(\text{Im}(\psi(0)) + O(x^{1/m}))$$

is positive for small $x > 0$. So, p has roots in $\mathbb{R} \times (-\mathbb{H})$ which we could perturb to get roots in $(-\mathbb{H})^2$.

The above theorem also shows that real stable polynomials locally factor into smooth branches near a zero on \mathbb{R}^2 since real stable factors are degree 1 Weierstrass polynomials. This was a main result of [11], but it now follows directly from Theorem 2.12. The precise statement follows.

THEOREM 2.13 (Local parametrization of real stable polynomials [11]). *Let $p \in \mathbb{R}[z_1, z_2]$ have no zeros in \mathbb{H}^2 and $p(0, 0) = 0$. Assume that $p(0, \cdot)$, $p(\cdot, 0) \not\equiv 0$. Then there exist $r, R > 0$ and $\phi_1, \dots, \phi_N \in \mathbb{R}\{z_1\}$ convergent on*

$|z_1| \leq r$ with $\phi_j(0) = 0, \phi'_j(0) > 0$ such that $\mathcal{Z}_p \cap \mathbb{R}^2$ is described by

$$\bigcup_{j=1}^N \{(x_1, x_2) : x_2 + \phi_j(x_1) = 0\}$$

for $(x_1, x_2) \in (-r, r) \times (-R, R)$.

The conditions $p(0, \cdot), p(\cdot, 0) \not\equiv 0$ simply rule out monomial factors which can only contribute zero sets $z_1 = 0$ or $z_2 = 0$.

EXAMPLE 2.14. Let us again return to Example 2.3. Now, $p(0, z_2) = 4z_2^2$ so we can write

$$p(z_1, z_2) = \underbrace{-4(1 - 4iz_1 - 2z_1^2)}_{=:u} \left(z_2^2 + \frac{4z_1(1 - iz_1)}{1 - 4iz_1 - 2z_1^2} z_2 + \frac{z_1^2}{1 - 4iz_1 - 2z_1^2} \right).$$

Near $(0, 0)$ we can factor p/u explicitly into two degree 1 factors of pure stable type $(z_2 + f_1(z))(z_2 + f_2(z))$ where

$$\begin{aligned} f_1(z) &:= z_1 \frac{(2(1 - iz_1) + \sqrt{3 - 4iz_1 - 2z_1^2})}{1 - 4iz_1 - 2z_1^2} \\ &= (2 + \sqrt{3})z_1 + i \left(6 + \frac{10}{\sqrt{3}} \right) z_1^2 + \dots, \\ f_2(z) &:= z_1 \frac{(2(1 - iz_1) - \sqrt{3 - 4iz_1 - 2z_1^2})}{1 - 4iz_1 - 2z_1^2} \\ &= (2 - \sqrt{3})z_1 + i \left(6 - \frac{10}{\sqrt{3}} \right) z_1^2 + \dots. \end{aligned}$$

On the other hand, $A = 4(z_2^2 + 4z_1z_2 + z_1^2 - 2z_1^2z_2^2)$ factors

$$\begin{aligned} A &= 4(1 - 2z_1^2) \left(z_2^2 + \frac{4z_1}{1 - 2z_1^2} z_2 + \frac{z_1^2}{1 - 2z_1^2} \right) \\ &= 4(1 - 2z_1^2) \left(z_2 + z_1 \frac{2 + \sqrt{3 + 2z_1^2}}{1 - 2z_1^2} \right) \left(z_2 + z_1 \frac{2 - \sqrt{3 + 2z_1^2}}{1 - 2z_1^2} \right) \end{aligned}$$

into two degree 1 factors of real stable type. \diamond

EXAMPLE 2.15. Consider again Example 2.10,

$$P(z_1, z_2) = z_1 + z_2 - 2z_1^3 - 6z_1^2z_2 - i(z_1^2 + z_1z_2 - 4z_1^3z_2);$$

although it only vanishes to order 1 at $(0, 0)$, its pure stable Puiseux factorization still has some complexity to it. Observe that

$$\begin{aligned} P(z_1, z_2) &= \underbrace{(1 - iz_1 - 6z_1^2 + 4iz_1^3)}_{=:u(z)} \left(z_2 + z_1 \frac{1 - iz_1 - 2z_1^2}{1 - iz_1 - 6z_1^2 + 4iz_1^3} \right) \\ &= u(z)(z_2 + z_1 + 4z_1^3 + 24z_1^5 + 8iz_1^6 + O(z_1^7)). \end{aligned}$$

Similarly, $A(z_1, z_2) = z_1 + z_2 - 2z_1^3 - 6z_1^2 z_2$ has the (trivial) Puiseux factorization

$$\begin{aligned} A(z_1, z_2) &= (1 - 6z_1^2) \left(z_2 + z_1 \left(1 + \frac{4z_1^2}{1 - 6z_1^2} \right) \right) \\ &= (1 - 6z_1^2)(z_2 + z_1 + 4z_1^3 + 24z_1^5 + O(z_1^7)). \diamond \end{aligned}$$

We have the following local parametrization theorem for pure stable polynomials, a direct result of Theorem 2.12 and Puiseux's theorem.

THEOREM 2.16. *Let $p \in \mathbb{C}[z_1, z_2]$ be pure stable and vanish to order M at $(0, 0)$. Factor $p = up_1 \cdots p_k$ into a unit $u \in \mathbb{C}\{z_1, z_2\}$ and irreducible Weierstrass polynomials of pure stable type. We may write each p_j as*

$$p_j(z) = \prod_{m=1}^{M_j} (z_2 + q_j(z_1) + z_1^{2L_j} \psi_j(\mu_j^m z_1^{1/M_j}))$$

with all of the data associated to a Puiseux expansion of pure stable type from Theorem 2.12.

The following corollary describes the possible lowest order homogeneous terms of p .

COROLLARY 2.17. *Assume the setup and conclusion of the previous theorem. The lowest order homogeneous term of p must be of the form*

$$P_M = c \prod_{j=1}^k (z_2 + a_j z_1)^{M_j}$$

for $a_j > 0$ and $c \in \mathbb{C}$. In particular, if we write $p = A + iB$ and multiply p by a constant to force $P_M = A_M$, then A_M has no factors of z_1 or z_2 and B_{M+1} , the lowest order homogeneous term of B by Proposition 2.5, has at most one factor of z_1 and at most one factor of z_2 .

Proof. Multiplying out p_j in Theorem 2.16 we get

$$p_j(z) = (z_2 + q'_j(0)z_1)^{M_j} + \text{higher order terms},$$

and multiplying p_1, \dots, p_k together we get the desired form for P_M since the unit u only affects this up to a constant multiple.

The observation about A_M is evident while the claim about B_{M+1} follows from Proposition 2.8 (set $r = 0$ in that proposition to see that B_{M+1} has either zero or one factors of z_1). ■

In particular, if $k = 1$ (i.e. we have a single irreducible Weierstrass factor) then the corresponding algebraic curve has a single tangent line through $(0, 0)$, i.e. the lowest order homogeneous term of p is a power of a linear polynomial $(az_1 + bz_2)^M$. This is more generally true. See [1, Chapter 18], where this fact (stated in slightly different language) is called the *Tangent*

Lemma. Stated contra-positively, if p has more than one tangent at $(0, 0)$ then it is locally reducible—i.e. has more than one irreducible Weierstrass factor. If p has all distinct tangents at $(0, 0)$ (i.e. an ordinary multiple point) then it is a product of degree 1 Weierstrass polynomials.

2.3.1. Stable polynomials with nontrivial Puiseux expansions. To our knowledge, the literature contains no examples of stable polynomials with nontrivial Puiseux expansions. It turns out that there is a way to build such examples from our local description. We present an example and a theorem generalizing the example.

EXAMPLE 2.18. A simple example of a nontrivial pure stable branch is

$$h(z) = z + iz^2 + cz^{5/2}$$

where c is a constant. We will see that $|c| \leq \sqrt{2}$ is a convenient assumption. Concretely, for $z = x + iy$ with $0 < |z| < \frac{1}{2}$ and $y \geq 0$

$$\begin{aligned} \operatorname{Im} h(z) &\geq y + x^2 - y^2 - |c| |z|^{5/2} \geq y(1 - |z|) + x^2 - |c| |z|^{5/2} \\ &\geq |z|^2(1 - \sqrt{2} |z|^{1/2}) > 0. \end{aligned}$$

We can then build the polynomial

$$p_1(z_1, z_2) = (z_2 + z_1 + iz_1^2 + cz_1^{5/2})(z_2 + z_1 + iz_1^2 - cz_1^{5/2}) = (z_2 + z_1 + iz_1^2)^2 - c^2 z_1^5$$

which has no zeros when $0 < |z_1| < 1/2$, $\operatorname{Im} z_1 \geq 0$, and $\operatorname{Im} z_2 \geq 0$. In particular, p_1 is nonvanishing on $\{z_1 : |z_1 - i/4| < 1/4\} \times \mathbb{H}$. The map $\sigma(z_1) = \frac{z_1}{1-2iz_1}$ sends \mathbb{H} onto $\{z_1 : |z_1 - i/4| < 1/4\}$ so that

$$\begin{aligned} p_2(z_1, z_2) &= (1 - 2iz_1)^5 p_1(\sigma(z_1), z_2) \\ &= (1 - 2iz_1)((1 - 2iz_1)^2 z_2 + z_1(1 - 2iz_1) + iz_1^2)^2 - c^2 z_1^5 \end{aligned}$$

has no zeros in \mathbb{H}^2 for $|c| \leq \sqrt{2}$. \diamond

The next theorem gives a lower bound on the imaginary part of Puiseux branches of pure stable type which lets us generalize this example.

THEOREM 2.19. *Let*

$$h(z) = q(z) + z^{2L} \psi(z^{1/m})$$

be a branch of pure stable type as in Theorem 2.12. Namely, $q \in \mathbb{R}[z]$ with $\deg q < 2L$, $q(0) = 0$, $a := q'(0) > 0$ and $\psi \in \mathbb{C}\{t\}$ with $b := \operatorname{Im} \psi(0) > 0$. Assume $z^{1/m}$ has a branch cut in the lower half-plane so that h is analytic on a domain containing $\{z : 0 < |z| < \epsilon \text{ and } \operatorname{Im} z \geq 0\}$. Then there exist $r, c > 0$ such that for $0 < |z| < r$ and $\operatorname{Im} z \geq 0$ we have

$$\operatorname{Im} h(z) \geq c|z|^{2L}.$$

In particular, $h(z) \neq 0$ for $0 < |z| < r$ and $\operatorname{Im} z \geq 0$.

Proof. Write $z = x + iy$. Note that $\text{Im}(z^j) = yO(|z|^{j-1})$ for $j \geq 2$ since $z^j - \bar{z}^j = (z - \bar{z}) \sum_{k=0}^{j-1} z^k \bar{z}^{j-1-k}$. This implies

$$\text{Im } q(z) = ay(1 - O(|z|))$$

for $|z|$ sufficiently small since q has real coefficients. Note next that $\text{Re}(z^{2L}) = x^{2L} + yO(|z|^{2L-1})$ by directly expanding $(x + iy)^{2L}$. Then

$$\begin{aligned} \text{Im}(z^{2L} \psi(z^{1/m})) &\geq (\text{Im } z^{2L}) \text{Re}(\psi(0)) + b \text{Re}(z^{2L}) - O(|z|^{2L+1/m}) \\ &\geq bx^{2L} - yO(|z|) - O(|z|^{2L+1/m}). \end{aligned}$$

Combining we get

$$\text{Im } h(z) \geq ay(1 - O(|z|)) + bx^{2L} - O(|z|^{2L+1/m}) \geq c|z|^{2L}$$

for some $c > 0$ when $|z|$ is small enough and $y \geq 0$. ■

If we choose ψ above to be a polynomial then the resulting product

$$p_1(z_1, z_2) = \prod_{j=1}^m (z_2 + q(z_1) + z_1^{2L} \psi(\mu^j z_1^{1/m}))$$

with $\mu = \exp(2\pi i/m)$ will be a polynomial with no zeros in $(\mathbb{D}_{2r} \cap \mathbb{H}) \times \mathbb{H}$ for r sufficiently small. (One can even choose ψ to be rational and clear denominators.) Note that p_1 has no zeros on the domain $\mathbb{D}_r(ir) \times \mathbb{H}$. The map $z \mapsto \sigma(z) = \frac{2rz}{1-iz}$ sends \mathbb{H} onto $\mathbb{D}_r(ir) = \{z : |z - ir| < r\}$. Then

$$p_2(z_1, z_2) = (1 - iz_1)^N p_1(\sigma(z_1), z_2)$$

is a polynomial for N large enough and has no zeros in \mathbb{H}^2 . Note that since $\sigma(0) = 0$, p_2 has a nontrivial Puiseux factorization around $(0, 0)$.

2.3.2. Initial segments and contact order. Given a pure stable polynomial $p \in \mathbb{C}[z_1, z_2]$ with $p(0, 0) = 0$ with real and imaginary coefficient decomposition $p = A + iB$ recall that $A + tB$ is real stable for $t \in \mathbb{R}$ and B is also real stable (Remark 2.4). How do the Puiseux expansions of p and $A + tB$ compare?

Theorem 2.20 below, which is proven in the next subsection, says that generically in t there is an exact correspondence between the smooth branches of $A + tB$ and the Puiseux branches of p . Basically, p factors locally as

$$u(z) \prod_{j=1}^k \prod_{m=1}^{M_j} (z_2 + q_j(z_1) + z_1^{2L_j} \psi_j(z_1^{1/M_j}))$$

and we prove that $A + tB$ generically in t factors *analytically* in the same general way:

$$A(z) + tB(z) = u(z; t) \prod_{j=1}^k \prod_{m=1}^{M_j} (z_2 + q_j(z_1) + z_1^{2L_j} \psi_{j,m}(z_1; t)).$$

Furthermore, we can show that $\psi_{j,m}(0; t)$ must depend on t , so that this generic expansion is the best possible in the sense that we cannot extend one of these branches further in a way that is independent of t .

In order to state this more precisely, let us say that a real polynomial $q \in \mathbb{R}[z_1]$ is an order N *initial segment* of a branch of a two-variable polynomial p if the branch is of the form $z_2 + \phi(z_1)$ where $\phi(z_1) - q(z_1)$ vanishes to order N or higher. We allow for ϕ to be a Puiseux series or a bona fide analytic function.

Now, Theorem 2.16 says p has a complete list of initial segments q_j occurring M_j times with order $2L_j$. Theorem 2.20 below says the same holds generically for $A + tB$ and furthermore none of these generic segments can be extended independently of t . Specifically, we *cannot* write down a “better” factorization

$$A(z) + tB(z) = u(z; t) \prod_{j=1}^k \prod_{m=1}^{M_j} (z_2 + \tilde{q}_j(z_1) + z_1^{2\tilde{L}_j} \tilde{\psi}_{j,m}(z_1; t)).$$

with $\tilde{L}_j \geq L_j$ for all j and $\tilde{L}_j > L_j$ for some j . Here is the formal statement.

THEOREM 2.20. *Assume the setup, conclusion, and notations of Theorem 2.16. Set as usual $A = (p + \bar{p})/2$, $B = (p - \bar{p})/(2i)$. Then for all but finitely many $t \in \mathbb{R}$ we can factor*

$$A(z) + tB(z) = u(z; t) \prod_{j=1}^k \prod_{m=1}^{M_j} (z_2 + q_j(z_1) + z_1^{2L_j} \psi_{j,m}(z_1; t))$$

where $\psi_{j,m}(z_1; t) \in \mathbb{R}\{z_1\}$, $u(z; t) \in \mathbb{C}\{z_1, z_2\}$ for each t and $u(0; t) \neq 0$.

We can do no better in the sense that we cannot find a pair (j, m) , a polynomial \tilde{q}_j of degree less than $2\tilde{L}_j > 2L_j$ and an analytic function $\tilde{\psi}_{j,m}(z_1; t)$ such that the factor above corresponding to (j, m) can generically be replaced with

$$z_2 + \tilde{q}_j(z_1) + z_1^{2\tilde{L}_j} \tilde{\psi}_{j,m}(z_1; t).$$

where $\tilde{q}_j - q_j$ vanishes to order at least $2L_j$.

We prove the above theorem later in this section.

Of particular geometric interest in the above data are the highest order initial segments. Set $K = \max\{2L_1, \dots, 2L_k\}$ and for clarity reorder so that $K = 2L_1 \geq \dots \geq 2L_k$. The quantity K has a couple of interpretations.

First, K measures asymptotically how closely the zero set of p or \bar{p} approaches the distinguished boundary \mathbb{R}^2 . Indeed, for any branch of p , say the branch

$$(2.4) \quad z_2 + q_j(z_1) + z_1^{2L_j} \psi_j(\mu_j^r z_1^{1/M_j}) = 0,$$

if we take $z_1 = x \in \mathbb{R}$ then on this branch

$$(2.5) \quad \operatorname{Im} z_2 = -x^{2L_j} (\operatorname{Im} \psi_j(0) + O(x^{1/M_j}))$$

vanishes to the precise order $2L_j$. This implies that for x close to 0,

$$(2.6) \quad \inf \{|\operatorname{Im} z_2| : p(x, z_2) = 0\} \approx |x|^K.$$

Here “ \approx ” means bounded above and below by constants. The number K satisfying (2.6) is called the *contact order* of p at $(0, 0)$. The notion of contact order is conformally invariant in the sense that if we use a Cayley transform from \mathbb{H}^2 to \mathbb{D}^2 sending $(0, 0)$ to $(1, 1)$ and convert p to an atoral stable polynomial q then for $z_1 \in \mathbb{T}$ close to 1,

$$(2.7) \quad \inf \{1 - |z_2| : \tilde{q}(z_1, z_2) = 0\} \approx |1 - z_1|^K.$$

A main result of [10] connects contact orders to integrability of derivatives of rational inner functions.

A second interpretation of K is that for any $t_1, t_2 \in \mathbb{R}$ outside of some finite set S of exceptional points, $A + t_1 B$ and $A + t_2 B$ have branches, say $z_2 + \phi(z_1; t_1)$ and $z_2 + \phi(z_1; t_2)$, such that $\phi(z_1; t_1) - \phi(z_1; t_2)$ vanishes to order K . Furthermore, K is the largest integer with this property. This interpretation of K was referred to as the *order of contact* of p . One can convert this to a statement about unimodular level sets of rational inner functions on the bidisk via Cayley transform. These two interpretations of K , contact order and order of contact, were how this material was originally approached in [10, 11] and the following fundamental result of [11] can be established from Theorem 2.20 by the above discussion.

THEOREM 2.21 ([11, Theorem 3.1]). *Order of contact equals contact order.*

Theorem 2.20 also constitutes a resolution to [11, Conjecture 5.2] about the concepts of *fine contact order* and *fine order of contact*. Fine contact order refers to the contact order of an individual branch of the zero set of p as in (2.5) where it is shown that an individual branch (2.4) has contact order $2L_j$. The fine contact orders associated to p can be read off from Theorem 2.16: we have fine contact order $2L_j$ occurring M_j times for $j = 1, \dots, k$. Fine order of contact refers to the order of vanishing of the difference of two different (analytic) branches of $A + tB$. Namely, if $z_2 + \phi(z_1, t_1)$ is a branch of $A + t_1 B$ and $z_2 + \phi(z_1, t_2)$ is a branch of $A + t_2 B$ then the fine order of contact of these two branches is the order of vanishing of $\phi(z_1; t_1) - \phi(z_1; t_2)$. Conjecture 5.2 of [11] stated that one could generically (with respect to t) group the branches of $A + t_1 B$ and $A + t_2 B$ so that the fine orders of contact exactly match the fine contact orders of p . Theorem 2.20 does exactly this. The next subsection is occupied with its proof.

2.3.3. Proof of Theorem 2.20. For convenience we will use (x, y) in place of the variables (z_1, z_2) . Assume the setup of Theorems 2.16 and 2.20. Consider

$$(2.8) \quad p(x, x^n y - q(x))$$

where $n \geq 1$, $q \in \mathbb{R}[x]$ and $q(0) = 0$. Let $O(n, q; p)$ be the largest integer N such that x^N divides (2.8). Intuitively, $O(n, q; p)$ helps us measure the presence of $q(x)$ as an initial segment of branches of p . It is difficult to tease out exactly what $O(n, q; p)$ is measuring and the key idea that follows is that it is more fruitful to see how this quantity *changes* with respect to n .

Note that $O(n, q; \bar{p}) = O(n, q; p)$ because q has real coefficients. Since $A + tB$ is a linear combination of p and \bar{p} , we have

$$(2.9) \quad \begin{aligned} O(n, q; A + tB) &\geq O(n, q; p), \\ O(n, q; A + tB) &= O(n, q; p) \quad \text{for all } t \text{ with at most one exception.} \end{aligned}$$

Indeed, if we had $O(n, q; A + tB) > O(n, q; p)$ for $t = s_1, s_2$ distinct then since p is a linear combination of $A + s_1B, A + s_2B$ we would have (2.8) divisible by a higher power of x .

As in the previous section, we say that a branch $y + q_j(x) + x^{2L_j} \psi_j(\mu_j^r x^{1/M_j})$ has initial segment $q \in \mathbb{R}[x]$ of order N if

$$q_j(x) - q(x) + x^{2L_j} \psi_j(\mu_j^r x^{1/M_j})$$

vanishes to order N or higher.

LEMMA 2.22. *Given $q \in \mathbb{R}[x]$ with $q(0) = 0$, the quantity*

$$(2.10) \quad O(n + 1, q; p) - O(n, q; p)$$

counts the number of branches of p such that q is an initial segment of order $n + 1$. Similarly,

$$O(n + 1, q; A + tB) - O(n, q; A + tB)$$

counts the number of branches of $A + tB$ such that q is an initial segment of order $n + 1$.

Proof. Let us look at a single branch $y + q_j(x) + x^{2L_j} \psi_j(\mu_j^r x^{1/M_j})$ of p . Now, if q is an initial segment of order $n + 1$ then

$$(2.11) \quad q_j(x) - q(x) + x^{2L_j} \psi_j(\mu_j^r x^{1/M_j})$$

vanishes to order $n + 1$ or higher. This is only possible if $n + 1 \leq 2L_j$ since $\psi_j(0)$ has nonzero imaginary part. Then

$$(2.12) \quad x^{n+1} y - q(x) + q_j(x) + x^{2L_j} \psi_j(\mu_j^r x^{1/M_j})$$

has largest x -factor x^{n+1} , while

$$(2.13) \quad x^n y - q(x) + q_j(x) + x^{2L_j} \psi_j(\mu_j^r x^{1/M_j})$$

has largest x -factor x^n . So, the contribution to (2.10) is 1.

On the other hand, if (2.11) vanishes to order $s < n + 1$ then both (2.12) and (2.13) are divisible by x^s but no higher power of x , making the contribution to (2.10) equal to 0.

The argument for $A + tB$ is easier because its branches are analytic and have real coefficients. In particular, there is no “ ψ ” term to worry about. ■

As a basic illustration of the lemma we can prove a more precise version of Theorem 2.21. Suppose as before $K = 2L_1 \geq \dots \geq 2L_k$. Then $O(K, q_1; p) > O(K - 1, q_1; p)$ while

$$(2.14) \quad O(n + 1, q; p) = O(n, q; p) \quad \text{for any } q \in \mathbb{R}[x] \text{ and } n \geq K.$$

Now for all but at most one t , say $t \neq t_0$, $O(K - 1, q_1; A + tB) = O(K - 1, q_1; p)$ so that for $t \neq t_0$,

$$O(K, q_1; A + tB) - O(K - 1, q_1; A + tB) > 0,$$

which implies that for $t \neq t_0$, $A + tB$ has a branch with initial segment q_1 of order K . This implies the order of contact is at least K . On the other hand, suppose there exist two values of t , say $t_1 \neq t_2$, and a real polynomial q that is an initial segment of order $\tilde{K} > K$ of $A + t_1B$ and $A + t_2B$. Then by (2.9) we can say without loss of generality that $O(\tilde{K}, q; A + t_1B) = O(\tilde{K}, q; p)$ and then

$O(\tilde{K}, q; p) = O(\tilde{K}, q; A + t_1B) > O(\tilde{K} - 1, q; A + t_1B) \geq O(\tilde{K} - 1, q; p)$ contradicts (2.14).

Conclusion of the proof of Theorem 2.20. With Lemma 2.22 in hand the main issue now is a combinatorial one. We can count occurrences of initial segments in branches of p and these must agree generically with $A + tB$ but the maximal initial segments of p can overlap in a variety of ways so we must perform a type of inclusion-exclusion analysis. Let us write the data from Theorem 2.16 as triples $(q_j(x), L_j, M_j)$. If we ever have $q_i = q_j$ and $L_i = L_j$ let us regroup our triple into $(q_i(x), L_i, M_i + M_j)$. Do this as many times as necessary so that we have a list

$$(Q_1(x), K_1, N_1), \dots, (Q_r(x), K_r, N_r)$$

with the pairs (Q_j, K_j) all distinct, K_j ’s nonincreasing, and $\sum_j N_j = M$, where M is the order of vanishing of p at $(0, 0)$.

With all this preparation,

$$b_j := O(2K_j, Q_j; p) - O(2K_j - 1, Q_j; p)$$

equals the number of branches of p with initial segment Q_j of order $2K_j$. Then for all but two values of t , $A + tB$ has b_j branches with initial segment Q_j of order $2K_j$. Since K_1 is maximal, $b_1 = N_1$. If Q_2 is an initial segment of order $2K_2$ of Q_1 then $b_2 = N_1 + N_2$ otherwise $b_2 = N_2$. In either case

we deduce that p and hence generically $A + tB$ has N_1 branches with initial segment Q_1 and N_2 different branches with initial segment Q_2 . In general,

$$b_j = \sum_{i \in S_j} N_i \quad \text{where}$$

$$S_j = \{i \leq j : Q_j \text{ is an initial segment of } Q_i \text{ of order } 2K_j\}.$$

Since $j \in S_j$, we have

$$b_j = N_j + \sum_{i < j, i \in S_j} N_i,$$

and this shows N_j can be computed in terms of $\{N_1, \dots, N_{j-1}\} \cup \{b_j\}$. By induction then, N_j can be computed entirely in terms of $\{b_1, \dots, b_j\}$ with knowledge of the sets S_j above. Now, N_j equals the number of branches of p that have initial segment Q_j but cannot extend to a longer initial segment (the ψ terms in the Puiseux expansion for p block this). Since we can compute N_j using $\{b_1, \dots, b_j\}$ we see that this statement holds generically for $A + tB$. Thus, generically $A + tB$ has N_j branches with initial segment Q_j that do not extend to longer initial segments. This proves Theorem 2.20. ■

Notice that each use of b_j requires us to potentially avoid two values of t in $A + tB$. Thus, $A + tB$ has the structure in Theorem 2.20 for all but at most $2r$ values of t , where r (see above) is at most k , the number of irreducible Weierstrass factors of p at $(0, 0)$. A simpler yet cruder statement would be to say that $A + tB$ has the desired factorization for all but $2M$ values of t since M , the order of vanishing at $(0, 0)$, can be read off fairly easily from p and equals the total number of branches.

2.3.4. Switching variables. Thus far the two variables z_1, z_2 have played distinct roles. In this section, we prove that when we switch variables in Theorem 2.16, the cutoff data $2L_1, \dots, 2L_k$ are preserved and the initial segments in z_2 can be computed from those in z_1 . In [11, Theorem 4.1] it was already shown that contact order does not depend on whether we examine contact order with respect to z_1 or z_2 .

In order to state our theorem on switching variables we need to introduce some notation. Let $q \in \mathbb{C}[z_1]$ be a polynomial with $q(0) = 0, q'(0) \neq 0$ and let $L \geq 1$. We define $I_{2L}(q) \in \mathbb{C}[z_2]$ to be the polynomial of degree less than $2L$ such that

$$I_{2L}(q)(z_2) - q^{-1}(z_2)$$

vanishes to order at least $2L$. Here q^{-1} is the analytic functional inverse of q guaranteed by the inverse function theorem and $I_{2L}(q)$ is just the power series of q^{-1} cut off past degree $2L - 1$.

THEOREM 2.23. *Assume the setup and conclusion of Theorem 2.16. Then*

$$p(z) = v(z) \prod_{j=1}^k \prod_{m=1}^{M_j} (z_1 - I_{2L_j}(q_j)(-z_2) + z_2^{2L_j} \tilde{\psi}_j(\mu_j^m z_2^{1/M_j}))$$

where $v(z_1, z_2) \in \mathbb{C}\{z_1, z_2\}$ is a unit, $\tilde{\psi}_j \in \mathbb{C}\{z_2\}$, and $\mu_j = \exp(2\pi i/M_j)$.

Thus, the cutoffs are all preserved and the initial segments get transformed to $-I_{2L_j}(q_j)(-z_2)$.

Proof. As in Theorem 2.16, let $p \in \mathbb{C}[z_1, z_2]$ be a pure stable polynomial. To begin, we explain why the number and degrees of the irreducible Weierstrass polynomials of p match when we switch variables. An irreducible Weierstrass polynomial in z_2 of pure stable type is of the form (2.2),

$$g(z) = \prod_{m=1}^M (z_2 + q_1(z_1) + z_1^{2L_1} \psi_1(\mu^m z_1^{1/M})).$$

Since $q_1(0) = 0, q_1'(0) > 0$ we see that $g(z_1, 0)$ vanishes to order M and therefore this can be factored into a unit $v \in \mathbb{C}\{z_1, z_2\}$ times a Weierstrass polynomial in z_1 ,

$$g(z) = v(z)(z_1^M + a_1(z_2)z_1^{M-1} + \cdots + a_M(z_2)).$$

This Weierstrass polynomial in z_1 must be irreducible. If it factored the resulting irreducible Weierstrass polynomial factors of pure stable type would be equal to a unit times Weierstrass polynomials in z_2 by the same reasoning; this would contradict irreducibility of g with respect to z_2 . This proves that the irreducible Weierstrass polynomial factors in z_2 of p are unit multiples of the irreducible Weierstrass polynomial factors in z_1 of p .

Again using g for an irreducible Weierstrass polynomial of pure stable type we also have a pure stable type Puiseux factorization with respect to z_1 ,

$$g(z) = v(z) \prod_{m=1}^M (z_1 + q_2(z_2) + z_2^{2L_2} \psi_2(\mu^m z_2^{1/M})),$$

and we would like to know that $L_1 = L_2$ and to compute q_2 from q_1 . We emphasize that q_2 is not the same as in the statement of Theorem 2.23—since we are isolating a single irreducible Weierstrass polynomial we think it is safe to have q_1 and q_2 play new roles strictly during this proof.

Let

$$\phi_1(t) = -(q_1(t^M) + t^{2L_1 M} \psi_1(t)) \quad \text{and} \quad \phi_2(t) = -(q_2(t^M) + t^{2L_2 M} \psi_2(t))$$

and note

$$\phi_1(t) = t^M(-q_1'(0) + \cdots) \quad \text{and} \quad \phi_2(t) = t^M(-q_2'(0) + \cdots)$$

because $q_1(0) = q_2(0) = 0$ and $q'_1(0), q'_2(0) \neq 0$. By these representations, ϕ_1, ϕ_2 both have analytic M th roots near 0:

$$\begin{aligned}\phi_3(t) &= \phi_1(t)^{1/M} = t((-q'_1(0))^{1/M} + \dots), \\ \phi_4(t) &= \phi_2(t)^{1/M} = t((-q'_2(0))^{1/M} + \dots).\end{aligned}$$

These M th roots are determined by a choice of M th root of $-q'_1(0), -q'_2(0)$. Notice $\phi'_3(0) = (-q'_1(0))^{1/M}, \phi'_4(0) = (-q'_2(0))^{1/M}$ are both nonzero so ϕ_3, ϕ_4 have local analytic functional inverses near 0.

Locally g 's zero set can be parametrized via the injective maps

$$t \mapsto (t^M, \phi_3(t)^M) \quad \text{or} \quad t \mapsto (\phi_4(t)^M, t^M).$$

Then $(\phi_3(\phi_4(t)))^M = t^M$, which implies that $\phi_3(\phi_4(t))$ is an M th root of unity times t . Namely, $\phi_3(t)$ and $\phi_4(t)$ are functional inverses up to a multiple of an M th root of unity. Reverting back to ϕ_1 we have

$$t^M = \phi_1(\phi_4(t))$$

and using our formula for ϕ_1 we have

$$\begin{aligned}-t^M &= q_1(\phi_4(t)^M) + \phi_4(t)^{2L_1M} \psi_1(\phi_4(t)) \\ &= q_1(\phi_2(t)) + \phi_2(t)^{2L_1} \psi_1(\phi_4(t)) \\ &= q_1(\phi_2(t)) + (-q'_2(0))^{2L_1} \psi_1(0) t^{2L_1M} + O(t^{2L_1M+1})\end{aligned}$$

where the last term is a stand-in for some analytic function vanishing to order at least $2L_1M + 1$. The last equality follows from

$$\begin{aligned}\phi_2(t)^{2L_1} \psi_1(\phi_4(t)) &= t^{2L_1M} (-q'_2(0) + O(t))^{2L_1} (\psi_1(0) + O(t)) \\ &= t^{2L_1M} (-q'_2(0))^{2L_1} \psi_1(0) + O(t^{2L_1M+1})\end{aligned}$$

using initial expansions of ϕ_2, ψ_1 , and ϕ_4 .

Expanding the composition with q_1 we have

$$q_1(-q_2(t^M) - t^{2L_2M} \psi_2(t)) = q_1(-q_2(t^M)) - t^{2L_2M} q'_1(0) \psi_2(0) + O(t^{2L_2M+1}),$$

which follows from examining q_1 term by term via

$$(-q_2(t^M) - t^{2L_2M} \psi_2(t))^j = (-q_2(t^M))^j + O(t^{2L_2M+1}),$$

which holds since q_2 vanishes at 0. We single out the linear term

$$-q_2(t^M) - t^{2L_2M} \psi_2(t) = -q_2(t^M) - t^{2L_2M} \psi_2(0) + O(t^{2L_2M+1})$$

to keep track of the contribution of the first nonreal complex term. Altogether

$$\begin{aligned}-t^M &= q_1(-q_2(t^M)) - t^{2L_2M} q'_1(0) \psi_2(0) + O(t^{2L_2M+1}) \\ &\quad + t^{2L_1M} (-q'_2(0))^{2L_1} \psi_1(0) + O(t^{2L_1M+1}).\end{aligned}$$

Recall that $\psi_1(0), \psi_2(0) \in \mathbb{H}$ while $q'_1(0), q'_2(0) \neq 0$, $q_1, q_2 \in \mathbb{R}[t]$. In order for the imaginary coefficients on the right hand side to vanish we must have

$L_1 = L_2$. Then more simply put

$$-t^M = q_1(-q_2(t^M)) + O(t^{2L_1 M}).$$

We can replace $s = -t^M$ to see that

$$s = q_1(-q_2(-s)) + O(s^{2L_1}).$$

In words, $-q_2(-s)$ is a truncation of the power series for the functional inverse of q_1 (truncated below order $2L_1$); namely, $-q_2(-s) = I_{2L_1}(q_1)(s)$ —exactly what we wanted to show. ■

2.3.5. Universal contact order. Assume the setup and conclusion of Theorem 2.16. In [10, 11] contact order, which is the maximum of the cutoffs in Theorem 2.16,

$$\max \{2L_1, \dots, 2L_k\},$$

appeared naturally in the study of integrability of derivatives of rational inner functions. A quantity which is easier to analyze in the context of boundary regularity of rational inner functions is *universal contact order*. The *universal contact order* of p is the minimum

$$K_{\min} = \min \{2L_1, \dots, 2L_k\}.$$

Its geometric content is that all branches of p have this order of contact or higher with the distinguished boundary. In other terms, generically we can match up branches of $A + t_1 B$ and $A + t_2 B$ so that they all have order of contact K_{\min} or higher.

REMARK 2.24. These interpretations of universal contact order are conformally invariant and give a sensible geometric way of defining this concept for atoral stable polynomials (i.e. the bidisk setting). The explanation parallels that of contact order in Section 2.3.2.

Recall the factorization from Theorem 2.20,

$$(2.15) \quad A(z) + tB(z) = u(z; t) \prod_{j=1}^k \prod_{m=1}^{M_j} (z_2 + q_j(z_1) + z_1^{2L_j} \psi_{j,m}(z_1; t)),$$

which holds for all but finitely many $t \in \mathbb{R}$. Here $u(z; t) \in \mathbb{R}\{z_1, z_2\}$ is a uniquely determined unit for each t but it is not clear how u depends on t . Universal contact order tells us something about this dependence.

THEOREM 2.25. *Assume $p \in \mathbb{C}[z_1, z_2]$ is pure stable and has universal contact order $K_{\min} \geq 2$ (an even integer). Write $p = A + iB$ into real and imaginary (coefficient) polynomials and if necessary, multiply by a constant so that the lowest homogeneous term P_M satisfies $P_M = A_M$ and the coefficient of z_2^M in A_M is 1. Then, for generic values of t , the local factorization (2.15) of $A + tB$ has the property that in the homogeneous expansion of the*

unit,

$$u(z; t) = 1 + \sum_{j \geq 1} u_j(z; t),$$

the polynomials $u_j(z; t)$ are affine in t for $j \leq K_{\min} - 2$.

This theorem is used later when we study boundary regularity of rational inner functions. Note that the theorem is vacuous for $K_{\min} = 2$, as it should be.

Proof of Theorem 2.25. Given universal contact order $K = K_{\min}$ we can rewrite (2.15) in a more convenient form where we “forget” some currently irrelevant information (i.e. portions of initial segments beyond order K):

$$A(z) + tB(z) = u(z; t)W(z; t) = u(z; t) \prod_{j=1}^M (z_2 + h_j(z_1) + z_1^K \psi_j(z_1; t)).$$

Here $h_j \in \mathbb{R}[z_1]$ are polynomials with $h_j(0) = 0$, $h'_j(0) > 0$, $\deg h_j < K$ and $\psi_j(z_1; t) \in \mathbb{R}\{z_1\}$. Expanding we have

$$W(z; t) := A_M(z) + R_{M+1}(z) + \cdots + R_{M+K-2}(z) + F_{M+K-1}(z; t)$$

where R_j is a degree j homogeneous polynomial and F_{M+K-1} is analytic in z and vanishes to order at least $M + K - 1$. The R_j have no t dependence but F_{M+K-1} may. Next, we consider the effect on the unit $u(z; t)$ in the factorization (2.15). Write $u(z; t) = 1 + \sum_{j \geq 1} u_j(z; t)$ where u_j is homogeneous of degree j in z . Then for fixed z ,

$$\frac{A(\lambda z) + tB(\lambda z)}{W(\lambda z; t)} = 1 + \sum_{j \geq 1} \lambda^j u_j(z; t)$$

can be viewed as an analytic function of λ and because the numerator and denominator vanish to order M the result is analytic and nonzero at 0 whenever $A_M(z) \neq 0$. We can perform division of power series (via solving a triangular system) to conclude that $u_j(z; t)$ is affine with respect to t . Indeed,

$$A_{M+1}(z) + tB_{M+1}(z) = A_M(z)u_1(z; t) + R_{M+1}(z)$$

implies

$$u_1(\cdot; t) = \frac{A_{M+1} - R_{M+1}}{A_M} + t \frac{B_{M+1}}{A_M}$$

and then recursively

$$u_n(\cdot; t) = A_M^{-1}(A_{M+n} + tB_{M+n} - (R_{M+1}u_{n-1} + \cdots + R_{M+n}))$$

is affine with respect to t for $n \leq K - 2$. ■

This concludes our local theory related to Puiseux factorizations.

2.4. Realization formulas. Our third and final method for analyzing local behavior of stable polynomials is via transfer function realization formulas. This technique is decidedly “global” and restricted to two dimensions but can be effectively utilized for local questions. It is difficult to disentangle this topic from our applications but it has been so important for theorem discovery and proof that it deserves some discussion here.

Let $p \in \mathbb{C}[z_1, z_2]$ be atoral stable with multidegree $n = (n_1, n_2)$ and corresponding rational inner function $\phi = \tilde{p}/p$. Then there exists a $(1 + |n|) \times (1 + |n|)$ unitary $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ such that

$$\phi(z) = A + Bz_P(I - Dz_P)^{-1}C = A + B(I - z_P D)^{-1}z_P C$$

where $z_P = z_1 P + z_2(I - P)$ for some $|n| \times |n|$ projection P onto an n_1 -dimensional space. Conversely, every such formula produces a rational inner function. This is easiest to see from the formula

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 \\ z_P(I - Dz_P)^{-1}C \end{bmatrix} = \begin{bmatrix} \phi(z) \\ (I - Dz_P)^{-1}C \end{bmatrix}.$$

By Cramer’s rule and a proper accounting of degrees, one can show that

$$p(z) = p(0) \det(I - Dz_P).$$

Thus, every atoral stable polynomial has a contractive determinantal representation of the above form. When $|\alpha| = 1$, $p(z) - \alpha\tilde{p}(z)$ is toral stable and has a unitary determinantal representation

$$p(z) - \alpha\tilde{p}(z) = (p(0) - \alpha\tilde{p}(0)) \det(I - V_\alpha z_P)$$

where

$$V_\alpha = D + \frac{\alpha}{1 - \alpha A} CB$$

is a unitary for $|\alpha| = 1$. It is also contractive for $|\alpha| \leq 1$ and the above formulas hold for $\alpha \neq 1/A = p(0)/\tilde{p}(0)$. Since V_α is unitary for $\alpha \in \mathbb{T}$, we also see that D is a rank 1 perturbation of a unitary. This is presented in [30, Section 9].

One can use a Cayley transform to get determinantal representations of polynomials with no zeros on \mathbb{H}^2 . If $p \in \mathbb{C}[z_1, z_2]$ has no zeros on \mathbb{H}^2 and total degree n then there exist a constant $c \in \mathbb{C}$ and $n \times n$ matrices A_0, A_1, A_2 satisfying $\text{Im}(A_0), A_1, A_2 \geq 0$, $A_1 + A_2 = I$ such that

$$p(z) = c \det(A_0 + A_1 z_1 + A_2 z_2).$$

If p is real stable one can refine the representation so that $\text{Im}(A_0) = 0$. For details and additional discussion in the generic stable case, we refer the reader to [31, Theorem 3.2]. For details about the real stable refinement, we recommend consulting [16, Theorem 6.6] and [29, Corollary 1], with the caveat that their discussions occur in the language of hyperbolic polynomials. (An even deeper result of Helton–Vinnikov [25] implies we can take A_0, A_1, A_2

to be real symmetric.) In terms of local theory, the structure of the (possible) kernel of A_0 in relation to the operators A_1, A_2 can reveal properties of the zero set of p near $(0, 0)$. However, the material in the previous sections on homogeneous and Puiseux expansions seems more appropriate for understanding atoral/pure stable polynomials and their associated toral/real stable perturbations (namely, $p - \alpha\tilde{p}$ in the polydisk setting and $A + tB$ in the upper half-plane setting). More general perturbations, described next, are closely related to rational noninner Schur functions, and more general realization formulas are an effective tool for their study.

Consider a nonconstant rational Schur function on \mathbb{D}^2 , namely $f = q/p$, where $p, q \in \mathbb{C}[z_1, z_2]$ have no common factors, p has no zeros in \mathbb{D}^2 and $|f(z)| \leq 1$ on \mathbb{D}^2 . By Theorem 3.1 below, $\mathcal{Z}_p \cap \mathbb{T}^2 \subset \mathcal{Z}_q \cap \mathbb{T}^2$ and therefore any potential toral factors of p would be factors of q and can be divided out. So, p is necessarily atoral stable. Now, since $p(z) + wq(z) \neq 0$ for $z \in \mathbb{D}^2$ and $w \in \overline{\mathbb{D}}$ we get a more general family of perturbations of stable polynomials than simply $p + w\tilde{p}$ by considering $p + wq$. Further interest in perturbations of stable polynomials comes from a problem studied in [30] of characterizing the extreme points of real rational Pick functions.

Rational Schur functions on \mathbb{D}^2 possess contractive transfer function realizations. Indeed, by [32] Theorem 1.3], there is a finite-dimensional Hilbert space \mathcal{H} , a contraction $U = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ on $\mathbb{C} \oplus \mathcal{H}$, and a projection P on \mathcal{H} such that

$$(2.16) \quad f(z) = A + B(I - z_P D)^{-1} z_P C \quad \text{for } z \in \mathbb{D}^2,$$

where $z_P = z_1 P + z_2(I - P)$. Note that not only is U now merely a contraction, but we also do not have clear control on the (finite) dimension of \mathcal{H} . As we have seen, local behavior is made more apparent in the upper half-plane setting so we perform a change of variables to $f : \mathbb{D}^2 \rightarrow \mathbb{D}$ to obtain a rational Pick function $g : \mathbb{H}^2 \rightarrow \mathbb{H}$. Note that we alter both the domain and range of f to obtain more natural formulas. Specifically, define a conformal map $\gamma : \mathbb{D} \rightarrow \mathbb{H}$ by

$$(2.17) \quad \gamma(z) = i \frac{1+z}{1-z} \quad \text{so that} \quad \gamma^{-1}(w) = \frac{w-i}{w+i}$$

and set

$$(2.18) \quad g(w) = \gamma \circ f(\gamma^{-1}(w_1), \gamma^{-1}(w_2)).$$

PROPOSITION 2.26. *Taking g, \mathcal{H}, U defined as above, if $I - U$ is invertible, then g satisfies*

$$(2.19) \quad g(w) = c - \langle (w_P + S)^{-1} \alpha, \beta \rangle_{\mathcal{H}}$$

for $w \in \mathbb{H}^2$, where

$$(2.20) \quad T := i(I+U)(I-U)^{-1} := \begin{bmatrix} c & \beta^* \\ \alpha & S \end{bmatrix} \text{ for some } c \in \mathbb{C}, \alpha, \beta \in \mathcal{H}, \text{ and } S \in \mathcal{L}(\mathcal{H}).$$

Since U is a contraction, $\text{Im}(T) = \frac{1}{2i}(T - T^*) \geq 0$, $c \in \mathbb{H}$, and $\text{Im}(S) \geq 0$.

This formula follows from [13, Theorems 4.1, 4.2]. Since both sides are rational functions, the formula holds in \mathbb{H}^2 as well as at any points where neither side has a pole. We shall refer to g 's formula as a PIP (positive imaginary part) realization. The formula for g can also be used to construct rational Pick functions since a function g as in (2.19) satisfies

$$\begin{aligned} \text{Im } g(w) &= \left\langle \text{Im}(T) \begin{pmatrix} 1 \\ -(w_P + S)^{-1}\alpha \end{pmatrix}, \begin{pmatrix} 1 \\ -(w_P + S)^{-1}\alpha \end{pmatrix} \right\rangle \\ &\quad + \langle \text{Im}(w_P)(w_P + S)^{-1}\alpha, (w_P + S)^{-1}\alpha \rangle, \end{aligned}$$

which is nonnegative whenever $w \in \mathbb{H}^2$. The representation (2.19) is not valid for all rational Pick functions. In particular, if g satisfies (2.19) then $\lim_{t \rightarrow \infty} g(it, it) = c \neq \infty$, which is not true for all rational Pick functions. Producing a unified representation for all rational Pick functions in two variables turns out to be somewhat technical, and the paper [7] produces what are called type IV Nevanlinna representations to cover all cases. These representations are intricate and necessarily so. This seems to be more of an issue of the behavior of g at ∞ , which is not what we are interested in here. To account for this we find a conformal self-map of \mathbb{H} that fixes 0 and perturbs ∞ so that an arbitrary rational Pick function is conformally equivalent to one of the form (2.19).

THEOREM 2.27. *Let $h : \mathbb{H}^2 \rightarrow \mathbb{H}$ be a nonconstant rational Pick function. Then there exist automorphisms $\sigma_1, \sigma_2 : \mathbb{H} \rightarrow \mathbb{H}$ where $\sigma_2(0) = 0$ such that $g(w) = \sigma_1(h(\sigma_2(w_1), \sigma_2(w_2)))$ has a realization as in (2.19) and (2.20) with $g^*(0, 0) = \lim_{t \searrow 0} g(it, it) \neq \infty$.*

Proof. Let m_1 be a Möbius transformation sending \mathbb{D} to \mathbb{H} and 1 to 0. Then $f = m_1^{-1} \circ h \circ (m_1, m_1) : \mathbb{D}^2 \rightarrow \mathbb{D}$ is an RSF. As discussed above, f has a contractive transfer function realization as in (2.16). If necessary we replace f with a unimodular multiple in order to guarantee $f^*(1, 1) := \lim_{r \nearrow 1} f(r, r) \neq 1$. This limit exists because $\zeta \mapsto f(\zeta, \zeta)$ is a one-variable RSF. Since $\dim \mathcal{H} < \infty$ and f is a nonconstant rational Schur function, there is a $\lambda \in \mathbb{T}$ with $\lambda \neq 1$ such that $1 - \lambda D$ is invertible, f is continuous at (λ, λ) , and $f(\lambda, \lambda) \neq 1$. Then (2.16) extends to (λ, λ) . Define \tilde{f} by $\tilde{f}(z) = f(\lambda z)$

and define a contraction \tilde{U} on $\mathbb{C} \oplus \mathcal{H}$ by

$$\tilde{U} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} A & B \\ \lambda C & \lambda D \end{bmatrix}.$$

Then for $z \in \mathbb{D}^2$, we have

$$\tilde{f}(z) = \tilde{A} + \tilde{B}(I - z_P \tilde{D})^{-1} z_P \tilde{C}.$$

By our choice of λ , $I - \tilde{D}$ is invertible and

$$1 - \tilde{A} - \tilde{B}(I - \tilde{D})^{-1} \tilde{C} = 1 - f(\lambda, \lambda) \neq 0.$$

These two facts paired with the inverse formula for block 2×2 matrices imply that $I - \tilde{U}$ is invertible. Using γ as in (2.17), set $\tilde{g} := \gamma \circ \tilde{f} \circ \gamma^{-1}$ and

$$\tilde{T} := i(I + \tilde{U})(I - \tilde{U})^{-1} := \begin{bmatrix} c & \beta^* \\ \alpha & \tilde{S} \end{bmatrix} \quad \text{for } c \in \mathbb{C}, \alpha, \beta \in \mathcal{H}, \text{ and } \tilde{S} \in \mathcal{L}(\mathcal{H}).$$

By Proposition 2.26, \tilde{g} is a rational Pick function on \mathbb{H}^2 and for $w \in \mathbb{H}^2$,

$$\tilde{g}(w) = c - \langle (w_P + \tilde{S})^{-1} \alpha, \beta \rangle_{\mathcal{H}}.$$

We define our proposed g by $g(w) = \tilde{g}(w + \gamma(\bar{\lambda}))$. Then setting $S = \tilde{S} + \gamma(\bar{\lambda})I$, it is easy to see that

$$T := \begin{bmatrix} c & \beta^* \\ \alpha & S \end{bmatrix}$$

still has positive imaginary part and

$$g(w) = c - \langle (w_P + S)^{-1} \alpha, \beta \rangle_{\mathcal{H}}.$$

In the course of the proof, g was obtained by applying Möbius maps to h and pre-composing h with $\sigma_2(w_1) := m_1(\lambda \gamma^{-1}(w_1 + \gamma(\bar{\lambda})))$ in each component. Evidently, $\sigma_2(0) = 0$ and g satisfies $g^*(0, 0) = \gamma(f^*(1, 1)) \in \mathbb{C}$, since $f^*(1, 1) \neq 1$. ■

With this in hand, we can dig into the kernel structure of S in order to understand the behavior of g near $(0, 0)$ via realizations.

THEOREM 2.28. *Suppose $g : \mathbb{H}^2 \rightarrow \mathbb{H}$ is rational, nonconstant, and satisfies $g^*(0, 0) := \lim_{t \searrow 0} g(it, it) \neq \infty$. Further, suppose g possesses a PIP realization as in Proposition 2.26. Let $\widehat{S} = S|_{\text{Range}(S)} : \text{Range}(S) \rightarrow \text{Range}(S)$ be the compression of S to $\text{Range}(S)$. Then*

- α, β belong to $\text{Range}(S)$,
- \widehat{S} is invertible with positive imaginary part,
- we have

$$\begin{aligned} g(w) &= c - \langle (\widehat{S} + w_{22} - w_{21}w_{11}^{-1}w_{12})^{-1} \alpha, \beta \rangle \\ &= c - \langle \widehat{S}^{-1} (I + (w_{22} - w_{21}w_{11}^{-1}w_{12}) \widehat{S}^{-1})^{-1} \alpha, \beta \rangle \end{aligned}$$

where $w_P = w_1 P + w_2(I - P) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ is the block decomposition of w_P according to $(\text{Range}(S))^\perp \oplus \text{Range}(S)$.

The value of this new decomposition is that singular behavior at $(0, 0)$ is encapsulated within the term $w_{22} - w_{21}w_{11}^{-1}w_{12}$.

LEMMA 2.29. *Let \mathcal{H} be a finite-dimensional Hilbert space and assume that an operator T on $\mathbb{C} \oplus \mathcal{H}$ with positive imaginary part is given by*

$$T = \begin{bmatrix} c & \beta^* \\ \alpha & S \end{bmatrix} \quad \text{for } c \in \mathbb{C}, \alpha, \beta \in \mathcal{H}, \text{ and } S \in \mathcal{L}(\mathcal{H}).$$

Then $\alpha - \beta \in \text{Range}(S)$. Moreover, $\text{Ker}(S) = \text{Ker}(S^)$, so $\mathcal{H} = \text{Ker}(S) \oplus \text{Range}(S)$ and with respect to this decomposition,*

$$S = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{S} \end{bmatrix} \quad \text{for } \widehat{S} = P_{\text{Range}(S)}S|_{\text{Range}(S)}.$$

Proof. We first prove the assertion about $\text{Ker}(S)$. Since T has positive imaginary part, so does S . Thus, if $x \in \text{Ker}(S)$, then $\langle \text{Im}(S)x, x \rangle = 0$, which implies $\|\text{Im}(S)^{1/2}x\| = 0$, which implies $\text{Im}(S)x = 0$, which shows that $S^*x = 0$. A symmetric argument gives the reverse containment so $\text{Ker}(S) = \text{Ker}(S^*)$. From this, $\text{Ker}(S) = (\text{Range}(S))^\perp$ and so $\mathcal{H} = \text{Ker}(S) \oplus \text{Range}(S)$. Writing S with respect to this decomposition immediately gives $S = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{S} \end{bmatrix}$.

Now write $\alpha - \beta = \gamma_1 + \gamma_2$, where $\gamma_1 \in \text{Range}(S)$ and $\gamma_2 \in \text{Ker}(S)$. Then apply $\text{Im}(T)$ to vectors $z \oplus m\gamma_2 \in \mathbb{C} \oplus \mathcal{H}$ where $m \in \mathbb{C}$. If $\gamma_2 \neq 0$, appropriate choices of z and m will give contradictions to $\text{Im}(T) \geq 0$. Thus, $\gamma_2 = 0$ and $\alpha - \beta \in \text{Range}(S)$. ■

Proof of Theorem 2.28. By Lemma 2.29 we know that \widehat{S} is invertible with positive imaginary part. To see that $\alpha, \beta \in \text{Range}(S)$, we write operators and $\beta = \beta_1 + \beta_2$ using the decomposition $\text{Ker}(S) \oplus \text{Range}(S)$ and compute

$$\begin{aligned} g^*(0, 0) &= c - \lim_{t \searrow 0} \langle (itI + S)^{-1}\alpha, \beta \rangle \\ &= c - \lim_{t \searrow 0} \langle (itI + S)^{-1}(\alpha - \beta), \beta \rangle - \lim_{t \searrow 0} \langle (itI + S)^{-1}\beta, \beta \rangle \\ &= c - \lim_{t \searrow 0} \left\langle \begin{bmatrix} it & 0 \\ 0 & it + \widehat{S} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \alpha - \beta \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right\rangle \\ &\quad - \lim_{t \searrow 0} \left\langle \begin{bmatrix} it & 0 \\ 0 & it + \widehat{S} \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right\rangle \end{aligned}$$

$$\begin{aligned}
&= c - \lim_{t \searrow 0} \langle (it + \widehat{S})^{-1}(\alpha - \beta), \beta_2 \rangle \\
&\quad - \lim_{t \searrow 0} \left\langle \begin{bmatrix} 1/(it) & 0 \\ 0 & (it + \widehat{S})^{-1} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right\rangle \\
&= c - \langle \widehat{S}^{-1}(\alpha - \beta), \beta_2 \rangle - \lim_{t \searrow 0} \frac{1}{it} \|\beta_1\|^2 - \langle \widehat{S}^{-1}\beta_2, \beta_2 \rangle.
\end{aligned}$$

Since this limit exists, $\beta_1 = 0$ and we have $\beta = \beta_2 \in \text{Range}(S)$. As $\beta, \alpha - \beta$ are in $\text{Range}(S)$, so is α .

Now we can rewrite

$$g(w) = c - \left\langle \left(\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \widehat{S} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \alpha \end{bmatrix}, \begin{bmatrix} 0 \\ \beta \end{bmatrix} \right\rangle,$$

where w_p has been written using the decomposition $\mathcal{H} = \text{Ker}(S) \oplus \text{Range}(S)$. Since $\text{Im}(w_P) = \text{Im}(w_1)P + \text{Im}(w_2)(I - P)$, it is a strictly positive operator for $w \in \mathbb{H}^2$. Then $\text{Im}(S + w_P)$ is strictly positive as well and so $(S + w_P)^{-1}$ exists. Let $Y = P_{\text{Ker}(S)}PP_{\text{Ker}(S)}$. Then $0 \leq Y \leq I$ and

$$w_{11} = w_1 Y + w_2(I - Y)$$

has strictly positive imaginary part for $w \in \mathbb{H}^2$. This implies that w_{11} is invertible. Then omitting some calculations, the inverse formula for block 2×2 operators implies that $\widehat{S} + w_{22} - w_{21}w_{11}^{-1}w_{12}$ is invertible and one can show

$$\begin{aligned}
g(w) &= c - \langle (\widehat{S} + w_{22} - w_{21}w_{11}^{-1}w_{12})^{-1}\alpha, \beta \rangle \\
&= c - \langle \widehat{S}^{-1}(I + (w_{22} - w_{21}w_{11}^{-1}w_{12})\widehat{S}^{-1})^{-1}\alpha, \beta \rangle. \blacksquare
\end{aligned}$$

3. Nontangential boundary regularity. We now discuss nontangential limits and more general nontangential regularity of rational inner functions and rational Schur functions. The first main result here is that RSFs have nontangential limits at *every* boundary point. We proceed to give a necessary and sufficient condition for higher order boundary regularity in terms of homogeneous expansions. Some of the essence of these ideas is in [28] but many ideas have been simplified and written for the upper half-plane. The final portion of this section goes deeper into two-variable RIFs. One of the main goals is to give a partial converse to a theorem in [10, 11] which says that the nontangential boundary regularity of an RIF implies a certain amount of contact order of the associated stable polynomial. We show that universal contact order (see Section 2.3.5) of a stable polynomial implies nontangential boundary regularity of the associated RIF.

3.1. Non-tangential limits of RSFs. In [28], several approaches and basic results for studying boundary behaviors of rational functions, both on \mathbb{D}^2 and more generally on \mathbb{D}^d , were established. Extending these techniques,

we give a proof of existence of nontangential limits for bounded rational functions $f = q/p$ at every $\zeta \in \mathbb{T}^d$. In order to prove our theorem, we only need to study singular points $\tau \in \mathbb{T}^d$ where $p(\tau) = 0$. As this is a local question, we analyze it at 0 in the upper half-plane setting \mathbb{H}^d .

Broadly speaking we define nontangential approach regions to a boundary point to be regions where the distance to the boundary point in question is comparable to the distance to the boundary. This notion is invariant under conformal maps between \mathbb{D} and \mathbb{H} , so our results have straightforward conversions between \mathbb{D}^d and \mathbb{H}^d . To precisely define nontangential approach regions to 0 via \mathbb{H}^d , we define

$$(3.1) \quad D_z = \{|z_1|, \dots, |z_d|, \operatorname{Im} z_1, \dots, \operatorname{Im} z_d\}$$

for any $z \in \mathbb{H}^d$. Then a *nontangential approach region to 0 via \mathbb{H}^d* is a set

$$AR_c = \{z \in \mathbb{H}^d : c \geq x/y \geq 1/c \text{ for any } x, y \in D_z\}$$

for $c \geq 1$. Letting $z \rightarrow 0$ nontangentially is equivalent to letting $r := |z_1| \rightarrow 0$ while restricting $z \in AR_c$.

Then a rational function $f = q/p$ on \mathbb{H}^d is *nontangentially bounded* at 0 if it is bounded on $AR_c \cap \{z \in \mathbb{H}^d : |z_1| < r\}$ for $c \geq 1$ and $r > 0$ sufficiently small. Similarly, f has a nontangential limit ω at 0 if the limit

$$f^*(0) = \lim_{z \rightarrow 0, z \in AR_c} f(z)$$

exists and equals ω .

Luckily, it is not necessary to dwell on these definitions as they have direct connections to homogeneous expansions. Such connections were established in [28] in the setting of the polydisk. The following theorem records the upper half-plane analogues of those key results from [28].

THEOREM 3.1. *Let $p \in \mathbb{C}[z_1, \dots, z_d]$ have no zeros in \mathbb{H}^d and assume p vanishes to order M at 0. Let $q \in \mathbb{C}[z_1, \dots, z_d]$ and $f = q/p$. Then*

- (1) *f is nontangentially bounded at 0 in \mathbb{H}^d if and only if q vanishes to order at least M at 0,*
- (2) *f has a nontangential limit at 0 via \mathbb{H}^d if and only if q vanishes to order at least M and the M th order homogeneous term in q , say Q_M , is a constant multiple of the M th order homogeneous term in p , namely P_M , i.e. $Q_M = bP_M$; in this case*

$$f^*(0) = \lim_{z \rightarrow 0, z \in AR_c} f(z) = b.$$

The main theorem proved in this section is the following:

THEOREM 3.2. *Assume $p, q \in \mathbb{C}[z_1, \dots, z_d]$, p has no zeros in \mathbb{H}^d , and $f = q/p$ is bounded and analytic in \mathbb{H}^d . Then f has a nontangential limit at 0 via \mathbb{H}^d .*

In light of Theorem 3.1, the only thing to prove is that $Q_M = bP_M$ where p vanishes to order $M > 0$ with lowest order homogeneous term P_M , and q has M th order homogeneous term Q_M (which could be zero). This follows immediately from Lemmas 3.3 and 3.4 given below. Note that we are assuming f is nonconstant and $M > 0$ since otherwise the result is trivial.

LEMMA 3.3. *Assuming the above setup, we have $|f| < c$ in \mathbb{H}^d if and only if the $d + 1$ -variable polynomial*

$$c(w + i)p(z) - (w - i)q(z) \in \mathbb{C}[z_1, \dots, z_d, w]$$

has no zeros in \mathbb{H}^{d+1} .

Proof. We have $c > |q/p|$ in \mathbb{H}^d if and only if

$$c - \zeta \frac{q(z)}{p(z)}$$

is nonvanishing for $\zeta \in \overline{\mathbb{D}}$ and $z \in \mathbb{H}^d$, which happens if and only if

$$c - \zeta \frac{q(z)}{p(z)}$$

is nonvanishing for $\zeta \in \mathbb{D}$ and $z \in \mathbb{H}^d$ by the maximum principle (since f is assumed nonconstant). This is equivalent to

$$c - \frac{w - i}{w + i} \frac{q(z)}{p(z)}$$

being nonvanishing for $w \in \mathbb{H}$ and $z \in \mathbb{H}^d$, which in turn holds if and only if

$$c(w + i)p(z) - (w - i)q(z)$$

is nonvanishing on \mathbb{H}^{d+1} . ■

LEMMA 3.4. *Assuming the above setup, if f is bounded on \mathbb{H}^d then Q_M is a constant multiple of P_M .*

Proof. Recall that Theorem 2.2 says that for $p \in \mathbb{C}[z_1, \dots, z_d]$ with no zeros in \mathbb{H}^d and $p(0) = 0$, the lowest order term P_M in the homogeneous expansion of p has no zeros in \mathbb{H}^d and is a multiple of a polynomial with real coefficients. We we may assume without loss of generality that $P_M \in \mathbb{R}[z_1, \dots, z_d]$.

Choose $c > 0$ so that $|q/p| < c$ in \mathbb{H}^d . Then $|e^{i\theta}q/p|$ is also bounded by c for every $\theta \in \mathbb{R}$. By Lemma 3.3 and Theorem 2.2, the lowest order homogeneous term of

$$c(w + i)p(z) - (w - i)e^{i\theta}q(z)$$

has real coefficients up to a unimodular multiple. The bottom homogeneous term is

$$i(cP_M + e^{i\theta}Q_M).$$

Thus, for every $\theta \in \mathbb{R}$ there exists $\psi \in \mathbb{R}$ such that

$$e^{i\psi}(cP_M + e^{i\theta}Q_M)$$

has real coefficients.

Write the coefficients of P_M, Q_M as p_α, q_α . Consider two distinct indices α, β where $p_\alpha \neq 0$. Then

$$e^{i\psi}(cp_\alpha + e^{i\theta}q_\alpha) \quad \text{and} \quad e^{i\psi}(cp_\beta + e^{i\theta}q_\beta)$$

are both real-valued. So,

$$(cp_\alpha + e^{i\theta}q_\alpha)(cp_\beta + e^{-i\theta}\overline{q_\beta})$$

is real for all θ . Viewing it as a trigonometric polynomial we see that the coefficient of $e^{i\theta}$ and the coefficient of $e^{-i\theta}$ must be conjugate, so

$$p_\beta q_\alpha = p_\alpha q_\beta$$

and therefore

$$q_\beta = \frac{q_\alpha}{p_\alpha} p_\beta.$$

This holds for an arbitrary index β , so this implies $Q_M = \frac{q_\alpha}{p_\alpha} P_M$. ■

Similar to [28], we can also study the existence of boundary directional derivatives. For $v \in \mathbb{H}^d$ the directional derivative of f at 0 in direction v is given by

$$D_v f(0) = \lim_{r \rightarrow 0^+} \frac{f(rv) - f^*(0)}{r}$$

where $f^*(0)$ is the nontangential limit of f at 0.

THEOREM 3.5. *Let $p, q \in \mathbb{C}[z_1, \dots, z_d]$ and assume p has no zeros in \mathbb{H}^d . If $f = q/p$ is bounded in \mathbb{H}^d , then $D_v f(0)$ exists for every $v \in \mathbb{H}^d$.*

Proof. As above, we assume p vanishes to order $M > 0$ else f is smooth at 0. Let us write out homogeneous expansions, $p = \sum_{j=M}^n P_j$, $q = \sum_{j=M}^n Q_j$ where n is the maximum of the total degrees of p and q . By Theorems [3.1] and [3.2], we have $Q_M = bP_M$ and $f^*(0) = b$. Then

$$\begin{aligned} \frac{f(rv) - b}{r} &= \frac{1}{r} \left(\frac{q(rv)}{p(rv)} - b \right) \\ &= \frac{1}{r} \frac{Q_{M+1}(rv) - bP_{M+1}(rv) + \sum_{j>M+1} (Q_j(rv) - bP_j(rv))}{P_M(rv) + \sum_{j>M} P_j(rv)} \\ &= \frac{Q_{M+1}(v) - bP_{M+1}(v) + \sum_{j>M+1} r^{j-M-1} (Q_j(v) - bP_j(v))}{P_M(v) + \sum_{j>M} r^{j-M} P_j(v)}. \end{aligned}$$

Letting $r \rightarrow 0^+$ we get

$$D_v f(0) = \frac{Q_{M+1}(v) - bP_{M+1}(v)}{P_M(v)},$$

which exists because P_M is nonvanishing on \mathbb{H}^d . ■

This translates easily to the polydisk \mathbb{D}^d . In particular, for each direction $-\delta$ pointing into \mathbb{D}^d at τ let $D_{-\delta}f(\tau)$ denote the associated directional derivative

$$D_{-\delta}f(\tau) = \lim_{r \rightarrow 0^+} \frac{f(\tau - \delta r) - f^*(\tau)}{r},$$

where $f^*(\tau)$ is the nontangential limit of f at τ .

THEOREM 3.6. *Let $p, q \in \mathbb{C}[z_1, \dots, z_d]$ and assume p has no zeros in \mathbb{D}^d . If $f = q/p$ is bounded in \mathbb{D}^d , then for every $\tau \in \mathbb{T}^d$, f has a directional derivative $D_{-\delta}f(\tau)$ for every direction $-\delta$ pointing into \mathbb{D}^d at τ . In particular, the directional derivative at $\tau = (1, \dots, 1)$ is given by*

$$D_{-\delta}f(\tau) = \frac{Q_{M+1}(\delta) - f^*(\tau)P_{M+1}(\delta)}{P_M(\delta)}$$

where P_M, P_{M+1}, Q_{M+1} are associated homogeneous terms of p, q .

REMARK 3.7. Since Theorem 3.6 gives formulas for the directional derivatives, one can easily test whether a given f has a nontangential gradient at τ (i.e. whether the directional derivative formula is linear in δ). If q vanishes to order N greater than $M + 1$, the formula $D_{-\delta}f(\tau) \equiv 0$ holds and f trivially has a nontangential gradient. If $N = M + 1$ or $N = M$, then f has a nontangential gradient at τ if and only if P_M is a factor of $Q_{M+1} - f^*(\tau)P_{M+1}$.

EXAMPLE 3.8. Let $p(z) = 2 - z_1 - z_2$, so that the associated RIF is $\phi(z) = \frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2}$. Then

$$f(z) := (1 - z_1) \frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2} + 1$$

is in $H^\infty(\mathbb{D}^2)$, the space of bounded analytic functions on \mathbb{D}^2 . Writing $f = q/p$ and computing the homogeneous expansions at $\tau = (1, 1)$ gives

$$p(1+z_1, 1+z_2) = -z_1 - z_2 \quad \text{and} \quad q(1+z_1, 1+z_2) = -z_1 - z_2 - z_1^2 - z_1z_2 - 2z_1^2z_2,$$

so $N = M = 1$ and

$$P_M(z) = -(z_1 + z_2), \quad P_{M+1}(z) = 0,$$

$$Q_M(z) = -(z_1 + z_2), \quad Q_{M+1}(z) = -z_1(z_1 + z_2).$$

Then Remark 3.7 implies that f has a nontangential gradient at $(1, 1)$. Moreover, f is bounded on $\overline{\mathbb{D}^2} \setminus \{(1, 1)\}$. Thus, if we define $f(1, 1) := f^*(1, 1) = 1$, then f is continuous on $\overline{\mathbb{D}^2}$. In contrast, one can show that ϕ is not continuous on $\overline{\mathbb{D}^2}$ and does not have a nontangential gradient at $(1, 1)$. ◇

3.2. Higher nontangential boundary regularity of rational functions. In this section, we study when an analytic function $f : \mathbb{H}^d \rightarrow \mathbb{C}$ has a nontangential polynomial approximation of order k at 0. We specifically look at when there exists a polynomial $F \in \mathbb{C}[z_1, \dots, z_d]$ of degree at most k such that as $z \rightarrow 0$ nontangentially within \mathbb{H}^d we have

$$(3.2) \quad f(z) = F(z) + o(r^k)$$

where $r = |z_1|$. One can take any equivalent quantity in D_z in place of $r = |z_1|$ (recall (3.1)). It turns out that for certain rational functions, if this nontangential “little-o” condition holds, then a nontangential “big-O” condition automatically holds. Following [3, 28], we will say f is *nontangentially C^k* at 0 if (3.2) holds. However, we caution the reader that this does not actually imply (even nontangential) continuity of the k th derivative of f near 0.

It is possible to characterize in simple algebraic terms when a rational function is nontangentially C^k . First, write $f = q/p$ with $p, q \in \mathbb{C}[z_1, \dots, z_d]$ and p having no zeros in \mathbb{H}^d . We assume at the very least that f has a nontangential limit at 0. As in Theorem 3.1, we can write

$$\begin{aligned} p &= P_M + P_{M+1} + \text{higher order terms,} \\ q &= bP_M + Q_{M+1} + \text{higher order terms.} \end{aligned}$$

For fixed $z \in \mathbb{C}^d$, the one-variable function

$$\lambda \mapsto f(\lambda z) = \frac{bP_M(z) + \lambda Q_{M+1}(z) + \dots}{P_M(z) + \lambda P_{M+1}(z) + \dots}$$

is analytic for λ near 0 when $P_M(z) \neq 0$. We can expand it into a power series

$$(3.3) \quad f(\lambda z) = \sum_{j \geq 0} F_j(z) \lambda^j$$

using power series division. Notice that the F_j are well-defined functions on $\{z : P_M(z) \neq 0\} \supset \mathbb{H}^d$. While F_j is homogeneous of order j , it need not be a polynomial. Doing the power series division one can recursively show that the F_j are rational with denominator P_M^j . For instance,

$$F_0 = b, \quad F_1 = \frac{Q_{M+1} - bP_{M+1}}{P_M}, \quad F_2 = \frac{Q_{M+2} - bP_{M+2} - P_{M+1}F_1}{P_M}.$$

THEOREM 3.9. *Let $f : \mathbb{H}^d \rightarrow \mathbb{C}$ be analytic and rational $f = q/p$. Consider the expansion (3.3). Then f is nontangentially C^k at 0 via \mathbb{H}^d if and only if the sum*

$$F(z) := \sum_{j=0}^k F_j(z) \text{ belongs to } \mathbb{C}[z_1, \dots, z_d],$$

in which case we have nontangentially

$$f(z) = F(z) + O(r^{k+1})$$

where r equals $|z_1|$ or any other comparable quantity in D_z .

Proof. If f is nontangentially C^k , then there exists $G \in \mathbb{C}[z_1, \dots, z_d]$ of degree at most k such that $f(z) - G(z) = o(r^k)$ nontangentially at 0. In particular, for fixed $z \in \mathbb{H}^d$,

$$F(\lambda z) - G(\lambda z) = (f(\lambda z) - G(\lambda z)) - (f(\lambda z) - F(\lambda z)) = o(|\lambda|^k),$$

which is only possible if $F \equiv G$.

Conversely, if $F \in \mathbb{C}[z_1, \dots, z_d]$ then by construction, $q - Fp$ vanishes to order at least $M + k + 1$ while $|P_M(z)| > cr^M$ in a nontangential approach region. This implies that

$$f(z) - F(z) = \frac{q(z) - F(z)p(z)}{P_M(z)(1 + \sum_{j \geq 1} \frac{P_{M+j}(z)}{P_M(z)})} = \frac{O(r^{k+1})}{1 + O(r)} = O(r^{k+1}). \blacksquare$$

EXAMPLE 3.10. Consider Example 2.10 (also studied in Example 2.15),

$$P(z) = A(z) + iB(z) = (z_1 + z_2 - 2z_1^3 - 6z_1^2z_2) - i(z_1^2 + z_1z_2 - 4z_1^3z_2),$$

and the associated rational Pick function

$$f = -B/A = \frac{z_1^2 + z_1z_2 - 4z_1^3z_2}{z_1 + z_2 - 2z_1^3 - 6z_1^2z_2}.$$

To examine its regularity we look at

$$f(\lambda z) = \lambda \frac{z_1^2 + z_1z_2 - 4z_1^3z_2\lambda^2}{z_1 + z_2 - (2z_1^3 + 6z_1^2z_2)\lambda^2} = \sum_{j \geq 1} \lambda^j F_j(z).$$

Performing the power series division we get

$$F_1(z) = z_1, \quad F_2 = 0, \quad F_3(z) = 2z_1^3, \quad F_4 = 0, \quad F_5(z) = \frac{4z_1^5(z_1 + 3z_2)}{z_1 + z_2},$$

which shows f is nontangentially C^4 but not C^5 . We will be able to read this off directly from the Puiseux expansion in Example 2.15 using Theorem 3.11 in the next section. It says that f is nontangentially C^4 at $(0, 0)$ since P has universal contact order 6. \diamond

3.3. Universal contact order implies boundary regularity. Let $p \in \mathbb{C}[z_1, z_2]$ be pure stable and $p(0, 0) = 0$. Define the rational inner function on \mathbb{H}^2 , $\phi = \bar{p}/p$. In this section we show that a universal contact order condition on p implies nontangential regularity of ϕ at $(0, 0)$. It is more revealing to study the associated Pick function $f = -B/A$, where $p = A + iB$ is the decomposition of p into real and imaginary (coefficient) polynomials.

It will not be difficult to then convert back and forth between f and ϕ since

$$\frac{1+if}{1-if} = \phi \quad \text{and} \quad f = i \frac{1-\phi}{1+\phi} = i(1-\phi)(1-\phi+\phi^2+\cdots).$$

We shall normalize p so that the lowest order homogeneous term of p , say P_M , has nonnegative real coefficients and the coefficient of z_2^M in P_M equals 1. As a note, Corollary 2.17 both describes how to normalize P_M to have nonnegative real coefficients and gives a formula for P_M that shows z_2^M must have a nonzero coefficient. Then by Proposition 2.5, B vanishes to order $M+1$ and we therefore have $f^*(0,0) = 0$.

THEOREM 3.11. *If p has universal contact order at $(0,0)$ given by the even integer $K_{\min} \geq 2$ then f is nontangentially $C^{K_{\min}-2}$ at 0.*

Proof. The key ingredients are Theorems 2.25 and 3.9. For simplicity, in the proof we write $K_{\min} = K$. As in the proof of Theorem 2.25, let

$$(3.4) \quad A(z) + tB(z) = u(z; t)W(z; t)$$

be our Weierstrass preparation theorem factorization of $A+tB$. For generic t ,

$$W(z; t) = A_M(z) + R_{M+1}(z) + \cdots + R_{M+K-2}(z) + F_{M+K-1}(z; t)$$

where the R_{M+j} are homogeneous polynomials of the indicated order and do not depend on t while F_{M+K-1} is analytic in z , vanishes to order at least $M+K-1$, and may depend on t . By Theorem 2.25, the homogeneous decomposition

$$u(z; t) = 1 + \sum_{j \geq 1} u_j(z; t)$$

satisfies $u_j(z; t) = G_j(z) + tH_j(z)$ for $j \leq K-2$ and generic $t \in \mathbb{R}$, where G_j, H_j are homogeneous polynomials of order j .

Let $(z)^{M-K-1}$ denote the ideal in $\mathbb{C}\{z_1, z_2\}$ generated by homogeneous polynomials of degree $M-K-1$. This is often just a convenient notation for disregarding higher order terms. If we examine (3.4) modulo $(z)^{M+K-1}$ then

$$\begin{aligned} \sum_{j=0}^{K-2} A_{M+j}(z) &= \left(1 + \sum_{j=1}^{K-2} G_j(z)\right) \\ &\quad \times (A_M(z) + R_{M+1}(z) + \cdots + R_{M+K-2}(z)) \bmod (z)^{M+K-1}, \\ \sum_{j=1}^{K-2} B_{M+j}(z) &= \left(\sum_{j=1}^{K-2} H_j(z)\right) \\ &\quad \times (A_M(z) + R_{M+1}(z) + \cdots + R_{M+K-2}(z)) \bmod (z)^{M+K-1}. \end{aligned}$$

As a result,

$$\begin{aligned}
 (3.5) \quad & \frac{\sum_{j=1}^{K-2} B_{M+j}(z) + (z)^{M+K-1}}{\sum_{j=0}^{K-2} A_{M+j}(z) + (z)^{M+K-1}} \\
 &= \frac{(\sum_{j=1}^{K-2} H_j(z))(A_M(z) + R_{M+1}(z) + \cdots + R_{M+K-2}(z))}{(1 + \sum_{j=1}^{K-2} G_j(z))(A_M(z) + R_{M+1}(z) + \cdots + R_{M+K-2}(z))} \\
 &= \frac{\sum_{j=1}^{K-2} H_j(z)}{1 + \sum_{j=1}^{K-2} G_j(z)}.
 \end{aligned}$$

According to Theorem 3.9, the regularity of f is governed by whether the initial terms of

$$f(\lambda z) = \sum_{j \geq 1} \lambda^j F_j(z)$$

are polynomials. Note

$$f(\lambda z) = -\frac{\sum_{j=1}^{K-2} \lambda^j B_{M+j}(z) + \text{higher order terms}}{\sum_{j=0}^{K-2} \lambda^j A_{M+j}(z) + \text{higher order terms}},$$

so the terms F_j up to $j = K - 2$ match those of (3.5). But the last function in (3.5) is analytic at 0 so its homogeneous terms are polynomials. Therefore, each F_j for $j \leq K - 2$ must be a polynomial. Furthermore, since B and A have real coefficients, the F_j have real coefficients. ■

3.4. Intermediate Loewner class and B^J points. In this section we connect the previous result to some work in [10, 11]. Since various past results are stated in upper half-plane or polydisk settings, a certain degree of flexibility is required from the reader.

Past work used the concept of a B^J point to formulate nontangential regularity. In turn, the definition of a B^J point is based on a class of Pick functions (analytic maps from \mathbb{H}^2 to \mathbb{H}) called the intermediate Löwner class which is denoted \mathcal{L}^{J-} . The class \mathcal{L}^{J-} was originally defined in [41] using behavior at (∞, ∞) ; we present a modified version of the class here, as was done in [10], using behavior at $(0, 0)$ instead.

DEFINITION 3.12. For a positive integer J , a two-variable Pick function $g : \mathbb{H}^2 \rightarrow \mathbb{H}$ is in the *intermediate Löwner class at $(0, 0)$* , denoted \mathcal{L}^{J-} , if $\lim_{s \searrow 0} |g(is, is)| = 0$ and if for $1 \leq j \leq 2J - 2$, there exist homogeneous polynomials G_j of degree j with real coefficients such that

$$(3.6) \quad g(w) = \sum_{j=1}^{2J-2} G_j(w) + O(|w|^{2J-1}) \quad \text{nontangentially.}$$

Define the conformal maps

$$\alpha: \mathbb{D} \rightarrow \mathbb{H}, \quad \alpha(z) := i \frac{1-z}{1+z} \quad \text{and} \quad \alpha^{-1}: \mathbb{H} \rightarrow \mathbb{D}, \quad \alpha^{-1}(w) := \frac{1+iw}{1-iw}.$$

Using those, we can translate \mathcal{L}^{J-} to \mathbb{D}^2 and define B^J points:

DEFINITION 3.13. Let ϕ be a RIF on \mathbb{D}^2 with a singularity at $\tau = (1, 1) \in \mathbb{T}^2$ with nontangential value $1 \in \mathbb{T}$. Define

$$(3.7) \quad g_\tau(w) := \alpha(\phi(\alpha^{-1}(w_1), \alpha^{-1}(w_2))) = i \frac{1 - \phi(\alpha^{-1}(w_1), \alpha^{-1}(w_2))}{1 + \phi(\alpha^{-1}(w_1), \alpha^{-1}(w_2))}.$$

Then τ is a B^J point of ϕ if g_τ is in the intermediate Löwner class \mathcal{L}^{J-} at $(0, 0)$.

Combining [10, Theorem 7.1] with [11, Theorem 4.1] yields the following connection between B^J points and contact order. We still let $\tau = (1, 1)$ here.

THEOREM 3.14. Let p be an atoral stable polynomial on \mathbb{D}^2 with $p(\tau) = 0$. Let $\phi = \tilde{p}/p$ and assume $\phi^*(\tau) = 1$. If τ is a B^J point of ϕ , then the contact order K of p at τ satisfies $K/2 \geq J$.

The previous section gives a partial converse to this. Specifically, assume that p has universal contact order K_{\min} (see Remark 2.24) at τ and define a pure stable $P = A + iB$ on \mathbb{H}^2 via (2.1). Then P has universal contact order K_{\min} at $(0, 0)$. Setting $\psi = \bar{P}/P$, we have $\psi^*(0, 0) = 1$, so Theorem 3.1 implies that $A_M = P_M$. Setting $g = -B/A$, Theorems 3.9 and 3.11 show that there exist homogeneous polynomials G_j of degree j such that

$$g(w) = \sum_{j=1}^{K_{\min}-2} G_j(w) + O(|w|^{K_{\min}-1}) \quad \text{nontangentially.}$$

The last sentence of the proof of Theorem 3.11 also shows that the G_j have real coefficients. Tracking through the definitions gives

$$g(w) = \alpha(\phi(\alpha^{-1}(w_1), \alpha^{-1}(w_2))),$$

which implies ϕ has a $B^{K_{\min}/2}$ point at τ . We can summarize this as follows:

COROLLARY 3.15. Let p be an atoral stable polynomial on \mathbb{D}^2 with $p(\tau) = 0$. Let $\phi = \tilde{p}/p$ and assume $\phi^*(\tau) = 1$. If K_{\min} is the universal contact order of p at τ , then τ is a $B^{K_{\min}/2}$ point of ϕ .

4. Horn regions and more general regularity. What can be said about non-nontangential boundary behavior of RIFs or RSFs? One way to examine this behavior is to look ultra-tangentially, specifically on the distinguished boundary. Because RIFs only take unimodular values on \mathbb{R}^2 , we can analyze their ultra-tangential behavior by studying their unimodular level sets restricted to \mathbb{R}^2 . This was addressed in [10, 11] (in the bidisk setting)

where it was basically shown that for RIFs ϕ and unimodular $e^{i\theta_0} \neq \phi^*(0, 0)$, the unimodular level sets

$$\{(x_1, x_2) \in \mathbb{R}^2 : \phi(x_1, x_2) = e^{i\theta_0}\}$$

are constrained to horn shaped regions near $(0, 0)$. This is illustrated in Figure 3, which shows that the unimodular level sets pass through the singularity $(0, 0)$ within a region that has at least quadratic pinching. Such detailed understanding of level sets of a RIF was in turn used to establish results concerning integrability of their partial derivatives [10, Section 5].

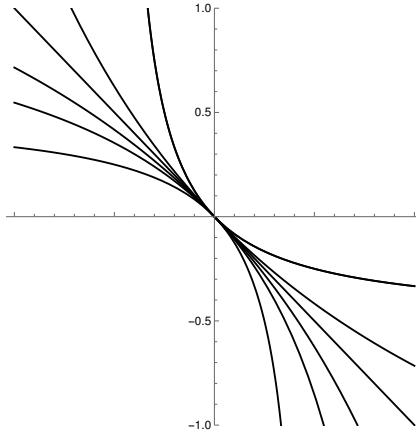


Fig. 3. Several unimodular level sets of $\phi = -\frac{x_1+x_2+2ix_1x_2}{x_1+x_2-2ix_1x_2}$ in \mathbb{R}^2 , indicating the presence of a horn region at $(0, 0)$.

The local theory from Section 2 also reveals this level set behavior and we pursue this line of thought in the next subsection. Specifically, in prior work, the authors used realization theory to establish this. Below, we show how constrained level set behavior actually follows from Puiseux series analysis.

After that we delve into the more difficult question of understanding distinguished boundary behavior of RSFs. Our goal is to obtain an analogue of the level set behavior described above for RIFs. In this setting, some functions, such as the one given in Figure 2, possess some level sets with curves that lie in \mathbb{R}^2 . As shown in that figure, these level sets appear to approach the singularity within a region with quadratic pinching.

However, for general RSFs, it is no longer the case that level sets will necessarily include entire curves that lie within \mathbb{R}^2 . Instead, in what follows, we show that if a level set (not associated to $\phi^*(0, 0)$) contains a sequence of points on \mathbb{R}^2 converging to $(0, 0)$, then those points have to get caught in a horn region that is independent of the particular sequence. This analysis is complicated and requires the use of our local realization formula from Section 2.4.

4.1. Horn regions for RIFs via Puiseux expansions. Let p be pure stable and let $\phi = \bar{p}/p$ be the associated RIF on \mathbb{H}^2 . Write $p = A + iB$ where $A = \frac{1}{2}(p + \bar{p})$, $B = \frac{1}{2i}(p - \bar{p})$. If p vanishes to order $M > 0$ at $(0, 0)$, then we multiply p by a constant so that $P_M = A_M$, and B vanishes to order $M + 1$ by the Homogeneous Expansion Theorem from the introduction. The unimodular level sets of ϕ coincide with the zero sets on \mathbb{R}^2 of $A - tB$ for $t \in \mathbb{R}$ or B (which corresponds to $t = \infty$) if we omit the points of \mathbb{R}^2 such as $(0, 0)$ where $p = 0$ and ϕ is not defined. These unimodular level sets also coincide with the real level sets of the Pick function $f = A/B$ where again $f = \infty$ corresponds to $B = 0$. In what follows, since we are mostly discussing behavior on \mathbb{R}^2 it is convenient to use the variable $x = (x_1, x_2)$ instead of $z = (z_1, z_2)$. As described in Section 2.3, $A - tB$ factors into M analytic and real branches with negative slope at $(0, 0)$. Namely, there exist $\psi(x_1; t, j) \in \mathbb{R}\{x_1\}$ where $t \in \mathbb{R}$, $j = 1, \dots, M$ with $\psi(0; t, j) = 0$, $\psi'(0; t, j) > 0$ where

$$A(x) - tB(x) = u(x; t) \prod_{j=1}^M (x_2 + \psi(x_1; t, j))$$

and we order the functions so that $\psi(x_1; t, j)$ is increasing with respect to j for fixed $x_1 > 0$ and $t \in \mathbb{R}$. We can describe the “level region” described by $s_1 \leq A/B \leq s_2$ in a punctured neighborhood of $(0, 0)$ in \mathbb{R}^2 as a union of regions trapped between graphs of our analytic branches. We will show this for positive x_1 since a similar result holds for negative x_1 . Note that the ordering of the branches $\{\psi(x_1; s, j)\}_j$ can change going from $x_1 > 0$ to $x_1 < 0$ and this is why we focus on $x_1 > 0$.

THEOREM 4.1. *Given the above setup, for $s_1 < s_2$ in \mathbb{R} , there exist $r, R > 0$ such that for $x \in (0, r) \times (-R, R) \setminus \{(0, 0)\}$ the level region*

$$\{x : s_1 \leq A(x)/B(x) \leq s_2\}$$

is given by

$$(4.1) \quad \bigcup_{j=1}^M \{(x_1, x_2) : x_2 \in [-\psi(x_1; s_1, j), -\psi(x_1; s_2, j)]\}.$$

Proof. Observe that

$$A(0, z_2) - tB(0, z_2) = z_2^M (A_M(0, 1) + z_2(A_{M+1}(0, 1) - tB_{M+1}(0, 1)) + \dots)$$

and so for t in any fixed compact interval $I = [s_1, s_2] \subset \mathbb{R}$ we can find R such that the above expression is nonzero for $0 < |z_2| \leq R$ and $t \in I$. Then there exists $r > 0$ such that $A(z) - tB(z) \neq 0$ for $|z_1| \leq r$, $|z_2| = R$, $t \in I$. By the argument principle, $z_2 \mapsto A(z_1, z_2) - tB(z_1, z_2)$ has M zeros for $|z_1| \leq r$, $|z_2| < R$, $t \in I$. Since $A - tB$ is real stable, for each fixed $x_1 \in \mathbb{R}$ we see that the univariate polynomial $z_2 \mapsto A(x_1, z_2) - tB(x_1, z_2)$ is

either real stable or identically zero (as follows from Hurwitz's theorem by taking $z_1 \in \mathbb{H} \rightarrow x_1 \in \mathbb{R}$). A real stable univariate polynomial has only real zeros. Therefore, for fixed $x_1 \in [0, r]$ and $t \in I$, the function of $x_2 \in (-R, R)$ given by $x_2 \mapsto A(x_1, x_2) - tB(x_1, x_2)$ has M (real) zeros. We can shrink r to force the zeros to be distinct for $x_1 \in (0, r]$. We can further shrink r if necessary to force the branches $\psi(x_1; s_1, j), \psi(x_1, s_2, j)$ for $j = 1, \dots, M$ to be analytic for $|x_1| \leq r$ and bounded by R . Then the M real zeros of $x_2 \mapsto A(x) - s_1 B(x)$ are exactly given by $x_2 = -\psi(x_1; s_1, j)$ for $j = 1, \dots, M$ and similarly for s_2 .

Define $g(z_2) = A(x_1, z_2)/B(x_1, z_2)$ for fixed $x_1 \in (0, r)$. Now, g is a one-variable nonconstant Pick function (i.e. maps \mathbb{H} to \mathbb{H}), which implies that g is strictly increasing on \mathbb{R} except at poles where it jumps from ∞ to $-\infty$. On the interval $(-R, R)$ we have already established that g attains every value in the interval I exactly M times. The points where g attains the values s_1 and s_2 must interlace since g increases. Among these points we claim that g attains the value s_1 first. Suppose g attains the value s_2 first, say at $y_0 \in (-R, R)$. Then g alternates attaining s_1 and s_2 , say at points $y_1 < y_2 < \dots < y_{2M-1}$ where $g(y_1) = s_1, g(y_2) = s_2, \dots, g(y_{2M-1}) = s_1$. So, g maps the intervals $[y_1, y_2], [y_3, y_4], \dots, [y_{2M-3}, y_{2M-2}]$ onto I , which accounts for $M-1$ times that g attains the values in I . On the other hand, g maps $(-R, y_0]$ onto $(g(-R), s_2]$ and $[y_{2M-1}, R)$ onto $[s_1, g(R))$. Note $g(-R) \geq s_1$ and $g(R) \leq s_2$ since otherwise g would attain s_1 or s_2 more than M times. If $g(R) > g(-R)$ then g attains some values in I more than M times since then the two intervals $(g(-R), s_2]$ and $[s_1, g(R))$ overlap. If $g(R) \leq g(-R)$, then the value $g(R)$ is only attained $M-1$ times since in this case the two intervals $(g(-R), s_2]$ and $[s_1, g(R))$ miss $g(R)$. Thus, we conclude that g must attain the value s_1 first.

Therefore, g alternates attaining the values s_1, s_2 starting with s_1 exactly M times ending with s_2 . These values are attained at points given by consecutive branches and thus g maps $[-\psi(x_1; s_1, j), -\psi(x_1; s_2, j)]$ onto I for $j = 1, \dots, M$. ■

Since $\psi'(0; t, j)$ is constant with respect to t , these regions all have at least quadratic pinching at $(0, 0)$: there exists $k \geq 2$ such that

$$cx_1^k \leq |\psi(x_1; s_1, j) - \psi(x_1; s_2, j)| \leq Cx_1^k.$$

These regions are called *horn regions* and they are defined more formally in the next subsection. If p has contact order K (necessarily even and at least 2) we can always choose j and a pair $s_1 < s_2$ such that

$$cx_1^K \leq |\psi(x_1; s_1, j) - \psi(x_1; s_2, j)| \leq Cx_1^K.$$

One basic conclusion of this is that if $x^n \rightarrow (0, 0)$ in \mathbb{R}^2 and $f(x^n) \rightarrow s \in \mathbb{R}$ then eventually the points x^n are trapped in the region (4.1) for

$s_1 = s - \delta, s_2 = s + \delta$ for $\delta > 0$. Another conclusion is that the closure of $f(\mathbb{D}_r^2 \cap \mathbb{H}^2)$ contains \mathbb{R} for all $r > 0$ since all of the level sets of f pass through $(0, 0)$ and points on these level sets can be perturbed to points of \mathbb{H}^2 .

4.2. Horn regions for RSFs via realization formulas. In this section, we return to rational Schur functions in two variables and prove that they exhibit additional regularity properties possessed by rational inner functions. Let us formally define horn regions in \mathbb{R}^2 .

DEFINITION 4.2. A *nontrivial horn* \mathcal{H} at $(0, 0)$ in \mathbb{R}^2 with slope $a \neq 0$ is a region of points $(x_1, x_2) \in \mathbb{R}^2$ sufficiently close to $(0, 0)$ such that

$$(4.2) \quad |x_2 - ax_1| \leq Bx_1^2 \quad \text{for some fixed } B > 0.$$

A *trivial horn* \mathcal{H} at $(0, 0)$ in \mathbb{R}^2 is either

- oriented along the x_2 -axis and consists of $(x_1, x_2) \in \mathbb{R}^2$ satisfying $|x_1| \leq Bx_2^2$ for some fixed $B > 0$, or
- oriented along the x_1 -axis and consists of $(x_1, x_2) \in \mathbb{R}^2$ satisfying $|x_2| \leq Bx_1^2$ for some fixed $B > 0$.

Definition 4.2 generalizes the notion of a nontrivial horn from [10]. There, the authors consider regions in \mathbb{R}^2 near $(0, 0)$ with boundaries

$$x_2 = \frac{x_1}{m \pm bx_1} \quad \text{for } m < 0 \text{ and } b > 0.$$

A simple power series computation shows that such regions satisfy an inequality of the form (4.2) and so are encompassed by Definition 4.2.

The main theorem of this section is as follows.

THEOREM 4.3. *Let f be a nonconstant rational Schur function on \mathbb{H}^2 with a singularity at $(0, 0)$ and nontangential value $f^*(0, 0)$. If a sequence $(x^n) \subset \mathbb{R}^2$ satisfies*

$$x^n \rightarrow (0, 0) \quad \text{and} \quad f(x^n) \rightarrow \zeta_0 \neq f^*(0, 0),$$

there are a finite number of horns $\mathcal{H}_1, \dots, \mathcal{H}_L$ in \mathbb{R}^2 at $(0, 0)$ such that for n sufficiently large, each x^n is in $\bigcup_{\ell=1}^L \mathcal{H}_\ell$. Moreover, the number L and the slopes of the horns do not depend on the sequence (x^n) .

The first reduction we make is to compose f with a Möbius map $m : \mathbb{D} \rightarrow \mathbb{H}$ and study a rational Pick function $h = m \circ f$. By Theorem 2.27 we can further apply conformal maps to h that fix $(0, 0)$ to obtain a rational Pick function $g : \mathbb{H}^2 \rightarrow \mathbb{H}$ with a local PIP realization formula as in Theorem 2.28. The image of a horn region under a pair of Möbius maps $\mathbb{H} \rightarrow \mathbb{H}$ that fix 0 is still a horn region so we have not lost anything in our reduction. We relabel our sequence x^n accordingly and assume $x^n \rightarrow (0, 0)$ and $g(x^n) \rightarrow \eta_0 \neq g^*(0, 0)$. We can assume $\eta_0 \neq \infty$; for instance if we replace our original

f with $\frac{1}{2}f$ then g will take values in a compact subset of \mathbb{H} . Recall that the formula for g is

$$g(w) = c - \langle \widehat{S}^{-1}(I + (w_{22} - w_{21}w_{11}^{-1}w_{12})\widehat{S}^{-1})^{-1}\alpha, \beta \rangle_{\mathcal{H}}$$

where we reiterate that \mathcal{H} is a finite-dimensional Hilbert space,

- $T = \begin{bmatrix} c & \beta^* \\ \alpha & S \end{bmatrix} \in \mathcal{L}(\mathbb{C} \oplus \mathcal{H})$ has positive imaginary part,
- \widehat{S} is the compression of S to $\text{Range}(S)$, which is reducing for S , and $\alpha, \beta \in \text{Range}(S)$,
- $w_P = w_1P + w_2(I - P) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}$ is the block decomposition of w_P according to $(\text{Ker}(S)) \oplus \text{Range}(S)$, where P is a projection on \mathcal{H} .

We also define the compression $Y = P_{\text{Ker}(S)}P|_{\text{Ker}(S)}$ so that $w_{11} = w_1Y + w_2(I - Y)$. Notice that

$$(4.3) \quad g^*(0, 0) = \lim_{t \searrow 0} g(it, it) = c - \langle \widehat{S}^{-1}\alpha, \beta \rangle$$

since for $w = (it, it)$ we have $w_{22} - w_{21}w_{11}^{-1}w_{12} = itI$.

LEMMA 4.4. *Assume the setup above and fix a constant $C > 0$. Then there exist finitely many Horn regions $\mathcal{H}_1, \dots, \mathcal{H}_L$ with slopes determined by the eigenvalues of Y such that for all $x \in \mathbb{R}^2$ sufficiently close to $(0, 0)$ satisfying $\|x\|^2\|x_{11}^{-1}\| > C$ we have*

$$x \in \bigcup_{j=1}^L \mathcal{H}_j.$$

Proof. Let $0 \leq t_1, \dots, t_L \leq 1$ denote the eigenvalues of the positive matrix Y . Then x_{11} is diagonalizable and has eigenvalues $t_\ell x_1 + (1 - t_\ell)x_2$. This implies

$$\|x_{11}^{-1}\| = \max_{1 \leq \ell \leq L} \frac{1}{|x_1 t_\ell + x_2(1 - t_\ell)|}.$$

Then $\|x\|^2\|x_{11}^{-1}\| > C$ implies that

$$x_1^2 + x_2^2 > C|x_1 t_\ell + x_2(1 - t_\ell)|$$

for some ℓ . The set of such x is the exterior of a union of two circles that are tangent to the line $x_1 t_\ell + x_2(1 - t_\ell) = 0$ at $(0, 0)$. It is a simple computation to show that in an ϵ -neighborhood of $(0, 0)$, if $t_\ell \neq 0, 1$, any such x must satisfy the nontrivial horn inequality

$$\left| x_2 + \frac{t_\ell}{1 - t_\ell} x_1 \right| \leq B x_1^2$$

for a fixed $B > 0$. Similarly, if $t_\ell = 0$, then x satisfies the trivial horn inequality $|x_2| \leq B x_1^2$, and if $t_\ell = 1$, then x satisfies the trivial horn inequality, $|x_1| \leq B x_2^2$. Here, each B depends on C and t_ℓ , but not x . For each ℓ , let

\mathcal{H}_ℓ denote the horn associated to the t_ℓ, C horn inequality. Then if x is sufficiently close to $(0, 0)$ and $\|x\|^2\|x_{11}^{-1}\| > C$, then $x \in \bigcup_{\ell=1}^L \mathcal{H}_\ell$. ■

Proof of Theorem 4.3. For $x \in \mathbb{R}^2$ we write as above $x_P = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ and set $\hat{x} = x_{22} - x_{21}x_{11}^{-1}x_{12}$.

We claim there is a constant $C > 0$ such that for $g(x)$ sufficiently close to η_0 we have

$$\|\hat{x}\| > C.$$

Then since

$$\|\hat{x}\| \leq \|x\| + \|x\|^2\|x_{11}^{-1}\|,$$

we will have $\|x\|^2\|x_{11}^{-1}\| > \tilde{C} > 0$ for $\|x\|$ sufficiently small and $g(x)$ close to η_0 . The theorem will then follow from Lemma 4.4.

First by (4.3), $g(x) - g^*(0, 0) = -\langle \hat{S}^{-1}\hat{x}\hat{S}^{-1}(I + \hat{x}\hat{S}^{-1})^{-1}\alpha, \beta \rangle$. There is no loss in assuming

$$\|\hat{x}\| \leq \frac{1}{2\|\hat{S}^{-1}\|}.$$

Then $\|(I + \hat{x}\hat{S}^{-1})^{-1}\| \leq (1 - \|\hat{x}\hat{S}^{-1}\|)^{-1} \leq 2$ so

$$|g(x) - g^*(0, 0)| \leq 2\|\alpha\| \|\beta\| \|\hat{S}^{-1}\|^2 \|\hat{x}\|.$$

Assuming $|g(x) - \eta_0| \leq \frac{1}{2}|\eta_0 - g^*(0, 0)|$ we then have $|g(x) - g^*(0, 0)| \geq \frac{1}{2}|\eta_0 - g^*(0, 0)|$ and therefore

$$\frac{|\eta_0 - g^*(0, 0)|}{4\|\alpha\| \|\beta\| \|\hat{S}^{-1}\|^2} \leq \|\hat{x}\|,$$

which proves the claim and completes the proof. ■

Using conformal maps, Theorem 4.3 can be easily translated to the bidisk, where it says: Let f be a rational Schur function on \mathbb{D}^2 with a singularity at $(1, 1)$ and nontangential value $f^*(1, 1)$. If a sequence $(\tau^n) \subset \mathbb{T}^2$ satisfies

$$\tau^n \rightarrow (1, 1) \quad \text{and} \quad f(\tau^n) \rightarrow \lambda \neq f^*(1, 1),$$

then there are a finite number of horns $\mathcal{H}_1, \dots, \mathcal{H}_L$ in \mathbb{T}^2 at $(1, 1)$ such that (τ^n) gets stuck inside the union of the horns. Here horns on \mathbb{T}^2 at $(1, 1)$ correspond to sets of points $(e^{i\theta_1}, e^{i\theta_2})$ where $\theta_1, \theta_2 \in [-\pi, \pi]$ satisfy the same inequalities as in Definition 4.2. This conclusion clearly holds for all bounded rational functions on \mathbb{D}^2 as well.

The bidisk setting allows us to easily visualize this horn behavior.

EXAMPLE 4.5. Let $p(z_1, z_2) = 2 - z_1 - z_2$ and let $q = \frac{1}{2}(p + \tilde{p})$. Then

$$(4.4) \quad f(z_1, z_2) := \frac{q(z_1, z_2)}{p(z_1, z_2)} = \frac{(z_1 - 1)(z_2 - 1)}{2 - z_1 - z_2}$$

is a bounded rational function on \mathbb{D}^2 with a singularity at $(1, 1)$ and $f^*(1, 1) = 0$. Consider the following four curves approaching $(1, 1)$ in \mathbb{T}^2 :

$$\begin{aligned}\gamma_1(s) &= (e^{is}, e^{is}), & \gamma_2(s) &= (e^{is}, e^{-is/2}), \\ \gamma_3(s) &= (e^{is}, e^{-is}), & \gamma_4(s) &= (e^{is}, e^{-i \sin(s)}),\end{aligned}$$

which are graphed in Figure 4(a) below, using their arguments on $[-\pi, \pi]^2$. One can check

$$\lim_{s \rightarrow 0} f(\gamma_1(s)) = \lim_{s \rightarrow 0} f(\gamma_2(s)) = 0 = f^*(1, 1),$$

while

$$\lim_{s \rightarrow 0} f(\gamma_3(s)) = \lim_{s \rightarrow 0} f(\gamma_4(s)) = 1 \neq f^*(1, 1).$$

Theorem 4.3 indicates that γ_3 and γ_4 should get stuck in a horn region (or union of horn regions) near $(1, 1)$, or equivalently near $(0, 0)$ when considering their arguments. One can see this in Figure 4(a). The more general situation can be extracted from Figure 4(b), which graphs $|f(e^{i\theta_1}, e^{i\theta_2})|$ for $(\theta_1, \theta_2) \in [-\pi, \pi]^2$. The picture shows that if a sequence $(\tau^n) \subseteq \mathbb{T}^2$ converges to $(1, 1)$ and is not contained in a horn region (mapped onto a narrow ridge in the modulus plot) near the red curves γ_3, γ_4 , then

$$|f(\tau^n)| \rightarrow 0, \quad \text{so} \quad f(\tau^n) \rightarrow f^*(1, 1).$$

Equivalently, if $|f(\tau^n)| \rightarrow c \neq 0$, then (τ^n) must get stuck in a narrow horn region near γ_3, γ_4 .

Some rational Schur functions appear to possess even narrower horn regions.

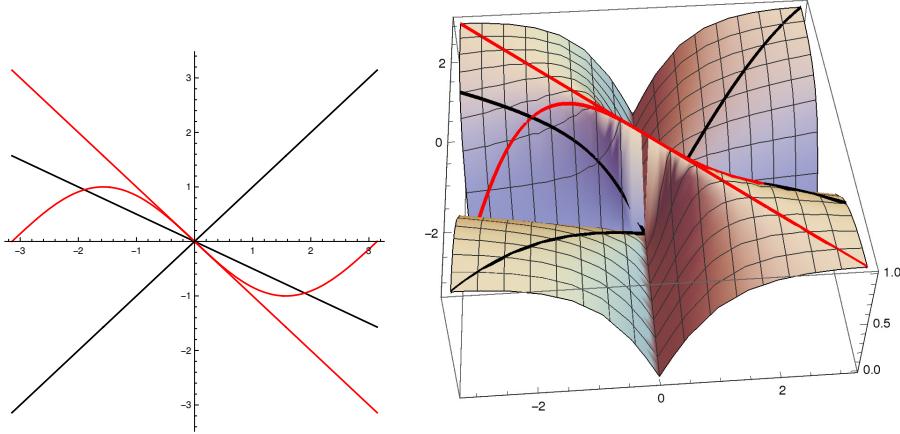
EXAMPLE 4.6. Set $p(z_1, z_2) = 4 - z_2 - 3z_1 - z_1 z_2 + z_1^2$ and $q = \frac{1}{2}(\tilde{p} - p)$. As in the previous example,

$$(4.5) \quad f(z_1, z_2) := \frac{q(z_1, z_2)}{p(z_1, z_2)} = \frac{2z_1^2 z_2 - z_1^2 - z_1 z_2 + z_1 + z_2 - 2}{4 - z_2 - 3z_1 - z_1 z_2 + z_1^2}$$

is bounded on \mathbb{D}^2 with a singularity at $(1, 1)$ and $f^*(1, 1) = -1$. Define the following four curves approaching $(1, 1)$ in \mathbb{T}^2 :

$$\begin{aligned}\gamma_1(s) &= (e^{is}, e^{is}), & \gamma_2(s) &= (e^{is}, e^{-is/2}), \\ \gamma_3(s) &= (e^{is}, h(e^{is})), & \gamma_4(s) &= (e^{is}, h(e^{is}) e^{i(1-\cos(s^4/20))}),\end{aligned}$$

where $h(z) = \frac{2-z+z^2}{1-z+2z^2}$. The function h is actually chosen so that the curve γ_3 parametrizes $\mathcal{Z}_q \cap \mathbb{T}^2$. Then $f(\gamma_3(s)) \equiv 0$ where it is defined, and so must manifestly be different from -1 at $s = 0$. This suggests a way of finding horn regions for some functions: First solve $q = 0$ on \mathbb{T}^2 . If that yields a curve on \mathbb{T}^2 , let that be one curve and then perturb the solution (up to some order, depending on p) to obtain another curve. Together these curves allow one to



(a) The curves γ_1, γ_2 (in black) and γ_3, γ_4 (in red) graphed on $[-\pi, \pi]^2$ using their arguments. The curves γ_3, γ_4 approximate a “horn region” at $(1, 1)$.

(b) The modulus $|f(e^{i\theta_1}, e^{i\theta_2})|$ and the curves γ_1, γ_2 (in black) and γ_3, γ_4 (in red) graphed on $[-\pi, \pi]^2$.

Fig. 4. A horn region of f from (4.4).

visualize a horn region. These curves are graphed in Figure 5(a). Similar to the previous example,

$$\lim_{s \rightarrow 0} f(\gamma_1(s)) = \lim_{s \rightarrow 0} f(\gamma_2(s)) = -1 = f^*(1, 1),$$

while

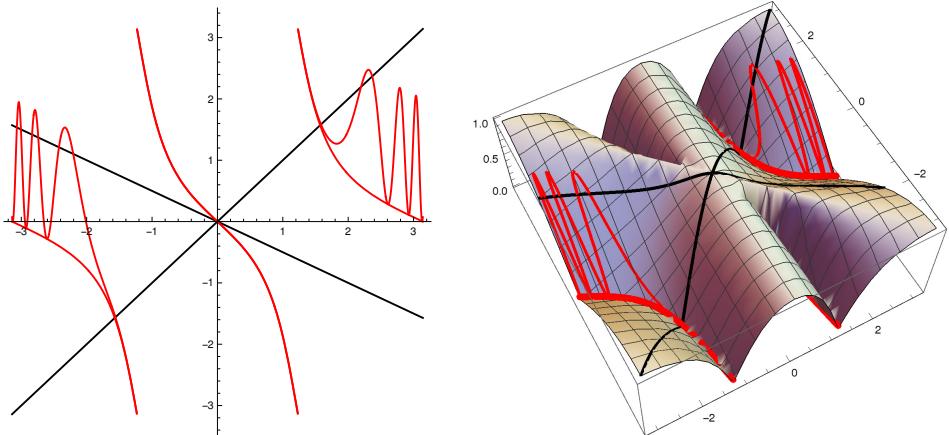
$$\lim_{s \rightarrow 0} f(\gamma_3(s)) = \lim_{s \rightarrow 0} f(\gamma_4(s)) = 0 \neq f^*(1, 1).$$

Thus, γ_3, γ_4 must become trapped in a horn region and as shown in Figure 5(a), the horn region appears to be so narrow that γ_3, γ_4 are indistinguishable near $(1, 1)$. To see the global situation, consider the picture in Figure 5(b). It is clear that if (τ^n) converges to $(1, 1)$ and $|f(\tau^n)| \rightarrow 1$, then (τ^n) must get caught in a very narrow horn region (which is mapped to a steep valley in the modulus plot) containing the (basically) identical parts of γ_3, γ_4 .

5. The ideal of admissible numerators. In this section, we address the question of characterizing the ideal

$$\mathcal{I}_p^\infty = \{q \in \mathbb{C}[z_1, z_2] : q/p \in H^\infty(\mathbb{D}^2)\}$$

for a given stable denominator $p \in \mathbb{C}[z_1, z_2]$. Recall that $H^\infty(\mathbb{D}^2)$ is the set of bounded analytic functions on \mathbb{D}^2 . Recall from Theorems 3.1 and 3.2 (or rather their polydisk counterparts) that if $f = q/p$ is bounded on \mathbb{D}^2 and



(a) The curves γ_1, γ_2 (in black) and γ_3, γ_4 (in red) graphed on $[-\pi, \pi]^2$ via their arguments. The curves γ_3, γ_4 are basically indistinguishable near $(0, 0)$.

(b) The modulus $|f(e^{i\theta_1}, e^{i\theta_2})|$ and the curves γ_1, γ_2 (in black) and γ_3, γ_4 (in red) graphed on $[-\pi, \pi]^2$. There is a very narrow horn region with negative slope along the overlapping red curves where $|f(e^{i\theta_1}, e^{i\theta_2})|$ is near 0.

Fig. 5. A horn region of f from (4.5).

$p(1, 1) = 0$, then assuming $p(1 + z_1, 1 + z_2)$ has lowest order homogeneous term p_M at $(0, 0)$, the M th order homogeneous term of q is a multiple of p_M . We bring this up for two reasons. First, we necessarily have $\mathcal{Z}_p \cap \mathbb{T}^2 \subset \mathcal{Z}_q \cap \mathbb{T}^2$, so if p has any toral factors then they divide every element of \mathcal{I}_p^∞ . We may therefore assume p has no toral factors and hence has only finitely many zeros on \mathbb{T}^2 (see Section 2.1). Second, this condition provides a necessary condition for our problem. Unfortunately, this necessary condition is far from sufficient and is actually somewhat misleading.

EXAMPLE 5.1. Example 2.10 is the polynomial with no zeros on \mathbb{D}^2 ,

$$p(z_1, z_2) = 4 - 5z_1 - 2z_2 + 2z_1z_2 + 3z_1^2 - z_1^2z_2 - z_1^3z_2.$$

In [28] it is shown that for $q \in \mathbb{C}[z_1, z_2]$, $q/p \in L^2(\mathbb{T}^2)$ if and only if $0 = q(1, 1) = \frac{\partial q}{\partial z_1}(1, 1) - \frac{\partial q}{\partial z_2}(1, 1)$ and

$$(5.1) \quad \frac{\partial^2 q}{\partial z_1^2}(1, 1) - 2 \frac{\partial^2 q}{\partial z_1 \partial z_2}(1, 1) + \frac{\partial^2 q}{\partial z_2^2}(1, 1) + 2 \frac{\partial q}{\partial z_1}(1, 1) = 0.$$

The condition $\frac{\partial q}{\partial z_1}(1, 1) - \frac{\partial q}{\partial z_2}(1, 1) = 0$ amounts to the requirement that the first order homogeneous term of q is a multiple of the first order homogeneous term of p (which again implies existence of nontangential limits). The last condition (5.1) is more complicated, thus making the condition of Theorem 3.2 not sufficient to even guarantee $q/p \in L^2(\mathbb{T}^2)$. Moreover, this

example also shows that for certain polynomials p every q with $q/p \in L^2(\mathbb{T}^2)$ has nontangential limits at every point of \mathbb{T}^2 . We continue this example in Example 5.6. \diamond

On the other hand, a simple sufficient condition for $f = q/p$ to be bounded on \mathbb{D}^2 is that q belongs to the polynomial ideal generated by p and \tilde{p} . This sufficient condition is not necessary and we can get broader sufficient conditions by looking at the local factorization of p into Puiseux series, Theorem 2.16, in the upper half-plane setting. Notice that we have reduced to the case where p has finitely many zeros on the distinguished boundary, so we can focus on a single isolated zero of p .

So, let us now work with $p \in \mathbb{C}[z_1, z_2]$ which is pure stable (no zeros on \mathbb{H}^2 and no factors in common with \tilde{p}) such that $p(0, 0) = 0$. We wish to describe all $q \in \mathbb{C}[z_1, z_2]$ such that q/p is bounded on a neighborhood of $(0, 0)$ in \mathbb{C}^2 intersected with \mathbb{H}^2 . In general, if a function is analytic and bounded on \mathbb{H}^2 intersected with a neighborhood of $(0, 0)$ in \mathbb{C}^2 we shall say it is *locally H^∞* at $(0, 0)$. Let us define

$$(5.2) \quad \mathcal{I}^\infty(p, 0) = \{q \in \mathbb{C}[z_1, z_2] : q/p \text{ is locally } H^\infty \text{ at } (0, 0)\}.$$

Let $R_0 = \mathbb{C}\{z_1, z_2\}$ be the ring of convergent power series centered at $(0, 0)$. In order to state our broadest sufficient conditions for inclusion in $\mathcal{I}^\infty(p, 0)$ we recall Theorem 2.16 on the Puiseux expansion for pure stable polynomials. We can factor $p = up_1 \cdots p_k$ where $u \in R_0$ is a unit and each p_j is an irreducible Weierstrass factor in z_2 of pure stable type

$$p_j(z_1, z_2) = \prod_{m=1}^{M_j} (z_2 + q_j(z_1) + z_1^{2L_j} \psi_j(\mu_j^m z_1^{1/M_j})).$$

Here L_j is a positive integer, $q_j(z_1) \in \mathbb{R}[z_1]$, $q_j(0) = 0$, $q'_j(0) > 0$, $\deg q_j < 2L_j$, $\psi_j \in \mathbb{C}\{z_1\}$, $\psi_j(0) \in \mathbb{H}$, $\mu_j = \exp(2\pi i/M_j)$. Let us call each $z_2 + q_j(z_1)$ an *initial segment* with *cutoff* $2L_j$ and *multiplicity* M_j for $j = 1, \dots, k$. Theorem 2.19 shows that

$$H(z_1, z_2) := \frac{z_1^{2L_j}}{z_2 + q_j(z_1) + z_1^{2L_j} \psi_j(\mu_j^m z_1^{1/M_j})}$$

is locally H^∞ since $\text{Im}(q_j(z_1) + z_1^{2L_j} \psi_j(\mu_j^m z_1^{1/M_j})) \geq c|z_1|^{2L_j}$. Note we take z_1^{1/M_j} to have a branch cut in the lower half-plane. Evidently, the function $z_1^{2L_j} \psi_j(\mu_j^m z_1^{1/M_j}) H(z_1, z_2)$ is still locally H^∞ , so we see that

$$1 - z_1^{2L_j} \psi_j(\mu_j^m z_1^{1/M_j}) H(z_1, z_2) = \frac{z_2 + q_j(z_1)}{z_2 + q_j(z_1) + z_1^{2L_j} \psi_j(\mu_j^m z_1^{1/M_j})}$$

is also locally H^∞ . This implies that if the polynomial q belongs to the

product ideal $(z_2 + q_j(z_1), z_1^{2L_j})^{M_j} R_0$ then q/p_j is locally H^∞ . (Recall this means q is in the ideal in R_0 generated by products $f_1 \cdots f_{M_j}$ where each f_i is in $\{z_2 + q_j(z_1), z_1^{2L_j}\}$.) Applied over all irreducible Weierstrass factors, this implies the following theorem.

THEOREM 5.2. *Assume the setting and subsequent conclusion of Theorem 2.16. Then*

$$\mathbb{C}[z_1, z_2] \cap \prod_{j=1}^k (z_2 + q_j(z_1), z_1^{2L_j})^{M_j} R_0 \subset \mathcal{I}^\infty(p, 0).$$

It is not hard to see that $p_j \in (z_2 + q_j(z_1), z_1^{2L_j})^{M_j} R_0$ since symmetric functions of $\psi(\mu_j^m z_1^{1/M_j})$ with $m = 1, \dots, M_j$ are analytic at 0. Therefore, since $p = up_1 \cdots p_k$ we have

$$p \in \prod_{j=1}^k (z_2 + q_j(z_1), z_1^{2L_j})^{M_j} R_0.$$

Since every $q_j(z_1)$ has real coefficients, we also have \bar{p} in this ideal. Setting

$$(5.3) \quad [p](z_1, z_2) = \prod_{j=1}^k (z_2 + q_j(z_1) + iz_1^{2L_j})^{M_j}$$

we see that

$$\frac{p(z_1, z_2)}{[p](z_1, z_2)} \quad \text{and} \quad \frac{[p](z_1, z_2)}{p(z_1, z_2)}$$

are locally H^∞ by Theorem 2.19. This allows a big conceptual reduction:

THEOREM 5.3. *Assume the setting and subsequent conclusion of Theorem 2.16. Then*

$$\mathcal{I}^\infty(p, 0) = \mathcal{I}^\infty([p], 0).$$

This shows Puiseux series do not play any role in our problem. Instead, the combinatorics of the different ways the initial segments can overlap becomes the crux of the matter. As a warm-up we can give a satisfying answer when p vanishes to order 1 at $(0, 0)$.

THEOREM 5.4. *Assume $p \in \mathbb{C}[z_1, z_2]$ is pure stable and vanishes to order 1 at $(0, 0)$. Then*

$$\mathcal{I}^\infty(p, 0) = (p, \bar{p}) R_0 \cap \mathbb{C}[z_1, z_2].$$

Proof. In this situation, $p = up_1$ has one degree 1 irreducible Weierstrass factor $p_1(z_1, z_2) = z_2 + q(z_1) + z_1^{2L} \psi(z_1)$ as above. The proof breaks down into showing

$$(p, \bar{p}) R_0 = (z_2 + q(z_1), z_1^{2L}) R_0$$

and

$$(5.4) \quad \mathcal{I}^\infty(p, 0) \subset (z_2 + q(z_1), z_1^{2L})R_0.$$

For the first equality, note

$$\bar{p}(z_1, z_2) = \bar{u}(z_1, z_2)(z_2 + q(z_1) + z_1^{2L}\bar{\psi}(z_1)).$$

The factors u, \bar{u} are units in R_0 and so the ideal $(p, \bar{p})R_0$ is then equal to $(p_1, \bar{p}_1)R_0$. Recall that in any commutative ring the ideal generated by A, B equals the ideal generated by $A, B + CA$ for any other element C . We use this fact below:

$$\begin{aligned} & (z_2 + q(z_1) + z_1^{2L}\psi(z_1), z_2 + q(z_1) + z_1^{2L}\bar{\psi}(z_1))R_0 \\ &= (z_2 + q(z_1) + z_1^{2L}\psi(z_1), z_1^{2L}(\psi(z_1) - \bar{\psi}(z_1))R_0 \\ &= (z_2 + q(z_1) + z_1^{2L}\psi(z_1), z_1^{2L})R_0 \\ &= (z_2 + q(z_1), z_1^{2L})R_0. \end{aligned}$$

The second equality is because $\psi(z_1) - \bar{\psi}(z_1)$ is a unit (from $\text{Im } \psi(0) \neq 0$).

Next we prove (5.4) by writing $f \in R_0$ as

$$f(z_1, z_2) = f_0(z_1) + (z_2 + q(z_1))f_1(z_1, z_2)$$

for $f_0 \in \mathbb{C}\{z_1\}$ and $f_1 \in R_0$. This can be accomplished for instance via the change of variable $w_2 = z_2 + q(z_1)$. If $f \in \mathcal{I}^\infty(p, 0)$ then we wish to show $f \equiv 0$ modulo the ideal $\mathcal{I} := (z_2 + q(z_1), z_1^{2L})R_0$. We can freely reduce f mod \mathcal{I} and assume $f_1 \equiv 0$ and $\deg f_0 < 2L$.

Finally, we examine f/p along the path $z_2 + q(z_1) = 0$ for $z_1 \in \mathbb{R}$. Now, $p(z_1, -q(z_1))$ vanishes to order $2L$ and therefore $f(z_1, -q(z_1)) = f_0(z_1)$ must vanish to at least the same order. Otherwise, f/p is unbounded in \mathbb{R}^2 along a path tending to $(0, 0)$ and we can find arbitrarily large values in \mathbb{H}^2 close to $(0, 0)$, contradicting f being locally H^∞ . Since f_0 has degree at most $2L$ we get $f_0 \equiv 0$. ■

We record the following useful corollary of the proof.

COROLLARY 5.5. *If $p \in \mathbb{C}[z_1, z_2]$ is pure stable and vanishes to order 1 at $(0, 0)$, then $\dim R_0/(p, \bar{p})R_0$ is equal to the contact order of p at $(0, 0)$.*

Proof. In the proof above we have $(p, \bar{p})R_0 = (z_2 + q(z_1), z_1^{2L})R_0$ where we note that $2L$ is the contact order of p at $(0, 0)$. Using the change of variables $w_2 = z_2 + q(z_1)$, it is not hard to prove

$$\dim R_0/(z_2 + q(z_1), z_1^{2L})R_0 = \dim R_0/(z_2, z_1^{2L})R_0 = 2L. \blacksquare$$

EXAMPLE 5.6. Example 5.1 falls under the assumptions of the above theorem. The Cayley transformed version of Example 5.1 (designed to send $(1, 1)$ to $(0, 0)$) is

$$p(z_1, z_2) = -4i(z_1 + z_2 - 2z_1^3 - 6z_1^2z_2 - iz_1(z_1 + z_2 - 4z_1^2z_2)),$$

which vanishes to order 1 at $(0, 0)$. This is the same polynomial as that of Examples 2.10 and 2.15. Example 2.15 shows the pure stable Puiseux factor of p starts out

$$z_2 + \underbrace{z_1 + 4z_1^3 + 24z_1^5}_{q(z_1)} + 8iz_1^6 + \dots$$

In this case $L = 3$ and the ideal $\mathcal{I}^\infty(p, 0)$ is therefore

$$(z_2 + z_1 + 4z_1^3 + 24z_1^5, z_1^6)R_0 \cap \mathbb{C}[z_1, z_2]. \diamond$$

We can address three cases where the combinatorics is simple enough to prove the reverse inclusion of Theorem 5.2. Here are the three cases:

Repeated segments: All of the initial segments of p are the same but we allow different cutoffs.

Double points: p vanishes to order 2 at $(0, 0)$.

Ordinary multiple points: p has distinct tangents at $(0, 0)$.

THEOREM 5.7. *Suppose $p \in \mathbb{C}[z_1, z_2]$ is pure stable with $p(0, 0) = 0$. If any of the above conditions hold, then using the notation of Theorem 5.2,*

$$\mathcal{I}^\infty(p, 0) = \mathbb{C}[z_1, z_2] \cap \prod_{j=1}^k (z_2 + q_j(z_1), z_1^{2L_j})^{M_j} R_0.$$

The proofs of all three cases follow the same strategy:

- Find a generating set of

$$\mathcal{I} = \prod_{j=1}^k (z_2 + q_j(z_1), z_1^{2L_j})^{M_j} R_0$$

with elements of the form $z_1^{K_j}$ times a product of j initial segments where we want K_j minimal.

- Write an arbitrary $f \in R_0$ as a combination of products of initial segments and for $f \in \mathcal{I}^\infty(p, 0)$, reduce f modulo \mathcal{I} .
- Show that f 's coefficients are forced to vanish to an order higher than their degree because of the order of vanishing of $[p]$ along certain initial segments (or curves with high order of contact with initial segments).

Proof of Theorem 5.7 for “Repeated segments”. Suppose p has the single initial segment $z_2 + q(z_1)$ repeated with cutoffs $2L_1, \dots, 2L_M$. The cutoffs need not be distinct. We assume without loss of generality that $2L_1 \leq \dots \leq 2L_M$. Let $S_k = 2 \sum_{j=1}^k L_j$ and $S_0 = 0$. The product ideal

$$\mathcal{I} := \prod_{j=1}^M (z_2 + q(z_1), z_1^{2L_j}) R_0$$

is generated by $z_1^{S_k}(z_2 + q(z_1))^{M-k}$ for $k = 0, \dots, M$ with respect to the ring R_0 . Every $f \in R_0$ can be represented as

$$f(z) = \sum_{j=0}^{M-1} f_j(z_1)(z_2 + q(z_1))^j + f_M(z_1, z_2)(z_2 + q(z_1))^M$$

where $f_j \in \mathbb{C}\{z_1\}$, $j = 0, \dots, M-1$, $f_M \in R_0$. If $f \in \mathcal{I}^\infty(p, 0)$ then we can reduce f modulo \mathcal{I} and assume $\deg f_j < S_{M-j}$ and $f_M \equiv 0$. Our goal is to show $f_0, \dots, f_{M-1} \equiv 0$. To achieve this we examine f/p on certain curves in \mathbb{R}^2 and show that boundedness along these curves implies all of the f_j are zero. At this stage there is no harm in replacing z_2 with $z_2 - q(z_1)$ and assuming $q \equiv 0$. With this reduction we have

$$[p](z) = \prod_{j=1}^M (z_2 + iz_1^{2L_j}).$$

To show $f_0 \equiv 0$ note that

$$\frac{f(z_1, 0)}{[p](z_1, 0)} = \frac{f_0(z_1)}{cz_1^{S_M}}.$$

For this to be bounded yet $\deg f_0 < S_M$, we must have $f_0 \equiv 0$.

The inductive argument might be clearer if we present the next step. Note $[p](z_1, tz_1^{2L_M})$ vanishes to order S_M so that $f(z_1, tz_1^{2L_M})$ must also vanish to order at least S_M for every $t \in \mathbb{R}$. Then

$$\lim_{t \rightarrow 0} \frac{1}{t} f(z_1, tz_1^{2L_M}) = f_1(z_1) z_1^{2L_M}$$

must vanish to order at least S_M and so f_1 vanishes to order at least $S_{M-1} = S_M - 2L_M$, implying $f_1 \equiv 0$ since $\deg f_1 < S_{M-1}$.

Now suppose $f_0, \dots, f_j \equiv 0$. Note that $[p](z_1, tz_1^{2L_{M-j}})$ vanishes to order $S_{M-j} + j2L_{M-j}$ and therefore $f(z_1, tz_1^{2L_{M-j}})$ must vanish to at least this order. But

$$\lim_{t \rightarrow 0} \frac{1}{t^{j+1}} f(z_1, tz_1^{2L_{M-j}}) = f_{j+1}(z_1) z_1^{(j+1)2L_{M-j}}$$

must also vanish to at least this order, implying $f_{j+1}(z_1)$ vanishes to order at least $S_{M-j} + j2L_{M-j} - (j+1)2L_{M-j} = S_{M-j-1}$. Since $\deg f_{j+1} < S_{M-j-1}$ we see that $f_{j+1} \equiv 0$. ■

Proof of Theorem 5.7 for “Double points”. For this case, we assume p has two initial segments $z_2 + q_1(z_1), z_2 + q_2(z_1)$ with cutoffs $2L_1, 2L_2$. We may assume $q_1 \neq q_2$ by the previous case and we assume $L_1 \leq L_2$. Let K be the order of vanishing of $q_1(z_1) - q_2(z_1)$. Set

$$\mathcal{I} = (z_2 + q_1(z_1), z_1^{2L_1})(z_2 + q_2(z_1), z_1^{2L_2})R_0.$$

Then

$$\begin{aligned}\mathcal{I} &= ((z_2 + q_1)(z_2 + q_2), z_1^{2L_1}(z_2 + q_2), z_1^{2L_2}(z_2 + q_1), z_1^{2(L_1+L_2)})R_0 \\ &= ((z_2 + q_1)(z_2 + q_2), z_1^{2L_1}(z_2 + q_2), z_1^{2L_2}(q_1 - q_2), z_1^{2(L_1+L_2)})R_0 \\ &= ((z_2 + q_1)(z_2 + q_2), z_1^{2L_1}(z_2 + q_2), z_1^{2L_2+K}, z_1^{2(L_1+L_2)})R_0 \\ &= ((z_2 + q_1)(z_2 + q_2), z_1^{2L_1}(z_2 + q_2), z_1^{2L_2+N})R_0\end{aligned}$$

where $N = \min \{2L_1, K\}$. Also note

$$[p](z) = (z_2 + q_1(z_1) + iz_1^{2L_1})(z_2 + q_2(z_1) + iz_1^{2L_2}).$$

Any $f \in R_0$ can be written

$$f(z_1, z_2) = f_0(z_1) + f_1(z_1)(z_2 + q_2(z_1)) + f_2(z_1, z_2)(z_2 + q_2(z_1))(z_2 + q_1(z_1))$$

for $f_0, f_1 \in \mathbb{C}\{z_1\}$, $f_2 \in R_0$. If $f \in \mathcal{I}^\infty(p, 0)$ we wish to show $f \in \mathcal{I}$, so we may freely reduce f modulo \mathcal{I} and assume $f_2 \equiv 0$, $\deg f_0 < 2L_2 + N$, $\deg f_1 < 2L_1$.

Now, $[p](z_1, -q_2(z_1))$ vanishes to order $2L_2 + N$, hence $f(z_1, -q_2(z_1)) = f_0(z_1)$ must vanish to at least that same order. This implies $f_0 \equiv 0$ by our degree bound.

Next, $q_2(z_1) - q_1(z_1) + tz_1^{2L_1}$ generically (in t) vanishes to order N and so $[p](z_1, tz_1^{2L_1} - q_1(z_1))$ generically vanishes to order $2L_1 + N$. Therefore, $f(z_1, tz_1^{2L_1} - q_1(z_1)) = f_1(z_1)(q_2(z_1) - q_1(z_1) + tz_1^{2L_1})$ vanishes to at least this order, implying that $f_1(z_1)$ vanishes to order at least $2L_1$. Since $\deg f_1 < 2L_1$, we have $f_1 \equiv 0$. This shows $f \in \mathcal{I}$. ■

Proof of Theorem 5.7 for “Ordinary multiple points”. For this case we assume we have initial segments $z_2 + q_j(z_1)$ with cutoffs $2L_j$, $j = 1, \dots, M$, and each difference $q_j(z_1) - q_k(z_1)$ vanishes to order 1 for $j \neq k$. Assume $2L_1 \leq \dots \leq 2L_M$. Define the ideal

$$\mathcal{I} := \prod_{j=1}^M (z_2 + q_j(z_1), z_1^{2L_j})R_0.$$

Our strategy now is to iteratively simplify the generators of this ideal. Since $H_k := z_1^{2L_k} \prod_{j \neq k} (z_2 + q_j(z_1)) \in \mathcal{I}$, we see that $z_1^{2(L_{k+1}-L_k)} H_k - H_{k+1}$ equals $z_1^{2L_{k+1}+1} \prod_{j \neq k, k+1} (z_2 + q_j(z_1))$ times a unit. Repeating this argument we see

$$z_1^{2L_{k+2}+2} \prod_{j \neq k, k+1, k+2} (z_2 + q_j(z_1)) \in \mathcal{I}$$

and more generally

$$z_1^{2L_{k+n}+n} \prod_{j \neq k, \dots, k+n} (z_2 + q_j(z_1)) \in \mathcal{I}$$

for $k + n \leq M$. In particular,

$$z_1^{2L_n+n-1} \prod_{j=n+1}^M (z_2 + q_j(z_1)) \in \mathcal{I},$$

and it is these generators for $n = 1, \dots, M$ that we will use to reduce elements of $\mathcal{I}^\infty(p, 0)$ modulo \mathcal{I} .

Given $f \in R_0$ we may write

$$\begin{aligned} f(z_1, z_2) &= f_0(z_1) + \sum_{n=1}^{M-1} f_n(z_1) \prod_{j=M-n+1}^M (z_2 + q_j(z_1)) + f_M(z_1, z_2) \prod_{j=1}^M (z_2 + q_j(z_1)) \end{aligned}$$

where $f_n(z_1) \in \mathbb{C}\{z_1\}$ for $n = 0, \dots, M-1$ and $f_M \in R_0$. If $f \in \mathcal{I}^\infty(p, 0)$ we may safely reduce f modulo \mathcal{I} and assume $f_M \equiv 0$, while $\deg f_n < 2L_{M-n} + M - n - 1$.

Finally, we compare the order of vanishing of $[p]$ along different segments to f and show that $f \equiv 0$. Note that $[p](z_1, -q_j(z_1))$ vanishes to order $2L_j + M - 1$. Then $f(z_1, -q_M(z_1)) = f_0(z_1)$ must vanish to order at least $2L_M + M - 1$, implying $f_0 \equiv 0$ by our degree bound. Next, $f(z_1, -q_{M-1}(z_1)) = f_1(z_1)(q_M(z_1) - q_{M-1}(z_1))$ vanishes to order at least $2L_{M-1} + M - 1$ but $q_{M-1} - q_M$ vanishes to order 1 because of distinct tangents. So, f_1 vanishes to order at least $2L_{M-1} + M - 2$, again implying $f_1 \equiv 0$. Continuing in this fashion, if $f_0, \dots, f_{n-1} \equiv 0$, then

$$f(z_1, -q_{M-n}(z_1)) = f_n(z_1) \prod_{j=M-n+1}^M (q_j(z_1) - q_{M-n}(z_1))$$

vanishes to order at least $2L_{M-n} + M - 1$ but the product vanishes to order n so f_n vanishes to order at least $2L_{M-n} + M - n - 1$, implying $f_n \equiv 0$. In the end we arrive at $f \equiv 0$ modulo \mathcal{I} . ■

6. Local and global integrability of derivatives. In [10–12], the integrability of an RIF's partial derivatives on \mathbb{T}^2 was used to measure how badly behaved the function was near its singularities. In this section, we partially extend that analysis to rational Schur functions on \mathbb{T}^2 . The denominators of rational Schur functions are exactly the atoral stable polynomials on \mathbb{D}^2 and so those polynomials will be featured in this section.

6.1. Integrability of functions with an atoral stable denominator. We now count the number of integrability indices associated to a given atoral, stable p on \mathbb{D}^2 .

DEFINITION 6.1. A (possibly infinite) number $\mathfrak{p} > 0$ is an *integrability index of p* if there is a $q \in \mathbb{C}[z_1, z_2]$ such that

$$\mathfrak{p} = \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : q/p \in L^{\mathfrak{p}'}(\mathbb{T}^2) \}.$$

Here, q is said to *witness p attaining its integrability index \mathfrak{p}* (or, in brief, *witness \mathfrak{p}*), and \mathfrak{p} is called the *integrability cutoff of q* .

It is not hard to show that each $q \in \mathbb{C}[z_1, z_2]$ has a well-defined integrability cutoff; the details are given below in Remark 6.3. As p has at most finitely many zeros on \mathbb{T}^2 , the integrability of any q/p on \mathbb{T}^2 will only depend on its integrability near those zeros. To examine this local integrability, let $\tau \in \mathbb{T}^2$ be a zero of p and for $\mathfrak{p} > 0$, define

$$L_{\tau}^{\mathfrak{p}}(\mathbb{T}^2) = \{ f : f \in L^{\mathfrak{p}}(U_{\tau}) \text{ for some open } U_{\tau} \subseteq \mathbb{T}^2 \text{ with } \tau \in U_{\tau} \}.$$

Let R_{τ} denote the local ring of convergent power series in z_1, z_2 centered at τ . Then the integrability indices of p at τ are defined as follows:

DEFINITION 6.2. A (possibly infinite) number $\mathfrak{p} > 0$ is an *integrability index of p at τ* if there is a $q \in R_{\tau}$ such that

$$\mathfrak{p} = \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : q/p \in L_{\tau}^{\mathfrak{p}'}(\mathbb{T}^2) \}.$$

Here, q is said to *witness \mathfrak{p}* and \mathfrak{p} is called the τ *integrability cutoff of q* .

Note that $\mathfrak{p} = \infty$ is always an integrability index of p since it is witnessed by $q = p$. To study other q , let $N_{\tau}(p, \tilde{p})$ denote the intersection multiplicity of p and \tilde{p} at τ . The positive integer $N_{\tau}(p, \tilde{p})$ is an algebraic characteristic of p and can for example be computed via the equation

$$(6.1) \quad N_{\tau}(p, \tilde{p}) = \dim(R_{\tau}/(p, \tilde{p})R_{\tau});$$

see [28, Section 12] and [18, Chapter 4, Proposition 2.11]. For more details about intersection multiplicity, we refer the reader to [28, Section 12]. Here since p and \tilde{p} have no common factors, $N_{\tau}(p, \tilde{p})$ is finite. This allows one to show that each $q \in R_{\tau}$ has a well-defined τ integrability cutoff.

REMARK 6.3. Let $M = N_{\tau}(p, \tilde{p})$ and let $[1], [z_1], \dots, [z_1^M]$ denote the cosets of $1, z_1, \dots, z_1^M$ in $R_{\tau}/(p, \tilde{p})R_{\tau}$. By (6.1), these cosets are linearly dependent, thus there is some nonzero polynomial $r \in \mathbb{C}[z_1]$ such that $r \in (p, \tilde{p})R_{\tau}$. Fixing any $q \in R_{\tau}$, there is a small open set $U \subseteq \mathbb{T}^2$ containing τ such that

$$\begin{aligned} \int_U \left| \frac{q}{p}(z) \right|^{\mathfrak{p}} |dz| &= \int_U \left| \frac{rq}{p}(z) \right|^{\mathfrak{p}} \left| \frac{1}{r}(z_1) \right|^{\mathfrak{p}} |dz| \lesssim \int_U \left| \frac{1}{r(z_1)} \right|^{\mathfrak{p}} |dz| \\ &\leq \int_{\mathbb{T}} \left| \frac{1}{r(z_1)} \right|^{\mathfrak{p}} |dz_1|. \end{aligned}$$

Note $|dz|$ is shorthand for $|dz_1||dz_2|$. By the fundamental theorem of algebra, there is certainly some $\mathfrak{p} > 0$ such that $1/r \in L^{\mathfrak{p}}(\mathbb{T})$, which implies that q has a well-defined τ integrability cutoff. Applying this argument at each zero of p on \mathbb{T}^2 also shows that each $q \in \mathbb{C}[z_1, z_2]$ has a well-defined integrability cutoff.

The result below bounds the number of finite integrability indices of p at τ in terms of $N_{\tau}(p, \tilde{p})$.

PROPOSITION 6.4. *Let p be an atoral stable polynomial on \mathbb{D}^2 and let $\tau \in \mathbb{T}^2$ be a zero of p . Then p has at most $N_{\tau}(p, \tilde{p})$ finite integrability indices at τ .*

Proof. First, define the following sets: $Q_0^{\tau} = R_{\tau}$,

$$Q_{\mathfrak{p}}^{\tau} = \{q \in R_{\tau} : q/p \in L_{\tau}^{\mathfrak{p}}(\mathbb{T}^2)\} \quad \text{for } 0 < \mathfrak{p} < \infty,$$

and $Q_{\infty}^{\tau} = \bigcap_{\mathfrak{p} > 0} Q_{\mathfrak{p}}^{\tau}$. These sets are all ideals in R_{τ} and if $\mathfrak{p} \leq \hat{\mathfrak{p}}$, then $Q_{\hat{\mathfrak{p}}}^{\tau} \subseteq Q_{\mathfrak{p}}^{\tau}$. Now, assume that $\mathfrak{p}_1, \dots, \mathfrak{p}_M$ is an increasing sequence of distinct finite integrability indices of p at τ . We will show that $M \leq N_{\tau}(p, \tilde{p})$. Choose $q_1, \dots, q_M \in R_{\tau}$ such that q_j witnesses \mathfrak{p}_j for $j = 1, \dots, M$. Observe that if $\mathfrak{p}'_1 < \mathfrak{p}_j < \mathfrak{p}'_2$, then $q_j \in Q_{\mathfrak{p}'_1}^{\tau}$, but $q_j \notin Q_{\mathfrak{p}'_2}^{\tau}$. Choose numbers $\kappa_1, \dots, \kappa_M$ such that

$$0 < \mathfrak{p}_1 < \kappa_1 < \mathfrak{p}_2 < \kappa_2 < \dots < \mathfrak{p}_M < \kappa_M < \infty.$$

By the above observations and the fact that $q_1 \in Q_0^{\tau}$ automatically, we have

$$q_1 \in Q_0^{\tau} \setminus Q_{\kappa_1}^{\tau}, \quad q_2 \in Q_{\kappa_1}^{\tau} \setminus Q_{\kappa_2}^{\tau}, \dots, \quad q_M \in Q_{\kappa_{M-1}}^{\tau} \setminus Q_{\kappa_M}^{\tau}.$$

Let $[q_1], \dots, [q_M]$ be the cosets generated by q_1, \dots, q_M in $R_{\tau}/(p, \tilde{p})R_{\tau}$. As $q_1/p, \dots, q_M/p$ have different integrability behaviors near τ , it follows that $[q_1], \dots, [q_M]$ must be linearly independent in $R_{\tau}/(p, \tilde{p})R_{\tau}$. By (6.1), we must have $M \leq N_{\tau}(p, \tilde{p})$, and the result follows. ■

To move from local to global bounds, we will let $N_{\mathbb{T}^2}(p, \tilde{p})$ denote the sum of $N_{\tau}(p, \tilde{p})$ over all of the common zeros τ of p, \tilde{p} on \mathbb{T}^2 . As discussed in [28], Bézout's theorem implies that if $\deg p = (n_1, n_2)$, then $N_{\mathbb{T}^2}(p, \tilde{p}) \leq 2n_1n_2$. Then Proposition 6.4 gives an immediate bound on the number of (global) integrability indices of p and the possible integrability cutoffs for derivatives of rational functions with denominator p .

COROLLARY 6.5. *Let p be an atoral stable polynomial on \mathbb{D}^2 . Then p has at most $N_{\mathbb{T}^2}(p, \tilde{p})$ finite integrability indices.*

COROLLARY 6.6. *Let p be an atoral stable polynomial on \mathbb{D}^2 , and let the real numbers $\mathfrak{p}_1, \dots, \mathfrak{p}_M$ denote the finite integrability indices of p^2 and $\mathfrak{p}_{M+1} = \infty$. Then $M \leq 4N_{\mathbb{T}^2}(p, \tilde{p})$ and for each $q \in \mathbb{C}[z_1, z_2]$, there is a j*

with $1 \leq j \leq M + 1$ such that

$$\mathfrak{p}_j = \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : \partial_{z_1}(q/p) \in L^{p'}(\mathbb{T}^2) \}.$$

Proof. Note that $\partial_{z_1}(q/p)$ has denominator p^2 and $4N_{\mathbb{T}^2}(p, \tilde{p}) = N_{\mathbb{T}^2}(p^2, \tilde{p}^2)$. Then the result follows immediately from Corollary 6.5. ■

6.2. Derivative integrability. We now obtain more refined integrability results for partial derivatives of rational Schur functions.

DEFINITION 6.7. A number $\mathfrak{p} > 0$ is a z_1 -derivative integrability index of p if there is a $q \in \mathbb{C}[z_1, z_2]$ such that q/p is a rational Schur function and

$$(6.2) \quad \mathfrak{p} = \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : \partial_{z_1}(q/p) \in L^{p'}(\mathbb{T}^2) \}.$$

Since multiplying by a nonzero constant does not affect integrability, the z_1 -derivative integrability indices of p are exactly the numbers \mathfrak{p} such that (6.2) holds for some $q \in \mathbb{C}[z_1, z_2]$ with $q/p \in H^\infty(\mathbb{D}^2)$. In both cases, q is said to witness \mathfrak{p} and \mathfrak{p} is called the z_1 -derivative integrability cutoff of q .

We can easily identify some z_1 -derivative integrability indices of p , using the concept of contact order defined in Section 2.3.2 and estimates derived from [10]. To avoid a lengthy digression in the proof, we present some of those estimates in the following remark.

REMARK 6.8. Let $\phi = \tilde{p}/p$ be the RIF associated to p . If K is the maximum contact order of p at its zeros on \mathbb{T}^2 , then Theorem 4.1 in [10] states that for $1 \leq \mathfrak{p} < \infty$,

$$\frac{\partial \phi}{\partial z_1} \in L^{\mathfrak{p}}(\mathbb{T}^2) \quad \text{if and only if} \quad \mathfrak{p} < \frac{K+1}{K}.$$

The proof of Proposition 6.9 below includes a local version of this for functions of the form $(z_2 - \tau_2)^n \frac{\partial \phi}{\partial z_1}$. We provide some of the necessary estimates here. To that end, fix \mathfrak{p} with $1 \leq \mathfrak{p} < \infty$ and recall that standard properties of finite Blaschke products (as in [10, Lemmas 4.2, 4.3]) show that for every interval $I \subseteq \mathbb{T}$ and almost every $z_2 \in \mathbb{T}$,

$$(6.3) \quad \int_I \left| \frac{\partial \phi}{\partial z_1}(z_1, z_2) \right|^{\mathfrak{p}} |dz_1| \approx \max_j \int_I \left(\frac{1 - |a_j|^2}{|1 - \bar{a}_j z_1|^2} \right)^{\mathfrak{p}} |dz_1|,$$

where the a_j are the zeros of $\tilde{p}(\cdot, z_2)$ in \mathbb{D} . Let $\tau = (\tau_1, \tau_2) \in \mathbb{T}^2$ be a zero of p . Then, as Theorem 2.16 can be translated to atoral stable polynomials on \mathbb{D}^2 (and the roles of z_1, z_2 interchanged), it can be used to obtain a parametrization

$$z_1 = \psi_1(z_2), \dots, z_1 = \psi_L(z_2)$$

of the components of the zero set $\mathcal{Z}_{\tilde{p}}$ that go through τ in some small neighborhood U of τ . Let K_τ^ℓ denote the contact order of each such curve, so that

if z_2 is close to τ_2 , then

$$1 - |\psi_\ell(z_2)| \approx |z_2 - \tau_2|^{K_\tau^\ell}.$$

This is basically (2.7) at the level of branches. Then if we let I_1 and I_2 be sufficiently small intervals in \mathbb{T} around τ_1 and τ_2 respectively, (6.3) implies that for each $z_2 \in I_2$,

$$(6.4) \quad \int_{I_1} |\partial\phi/\partial z_1(z_1, z_2)|^p |dz_1| \approx \max_\ell \int_{I_1} \left(\frac{1 - |\psi_\ell(z_2)|^2}{|1 - \overline{\psi_\ell(z_2)} z_1|^2} \right)^p |dz_1|.$$

The arguments in [10, Lemma 4.3] control the integrals on the right hand side of (6.4) when $I_1 = \mathbb{T}$ and $|\psi_\ell(z_2)| \geq 1/2$. However, the dominating part is the integral over a small interval in \mathbb{T} centered at $e^{i\text{Arg}(\psi_\ell(z_2))}$. Thus, if one shrinks I_2 so that for each $z_2 \in I_2$, $|\psi_\ell(z_2)| > 1/2$ and I_1 contains an interval of fixed length centered at $e^{i\text{Arg}(\psi_\ell(z_2))}$, the estimates in Lemma 4.3 give

$$\int_{I_1} \left(\frac{1 - |\psi_\ell(z_2)|^2}{|1 - \overline{\psi_\ell(z_2)} z_1|^2} \right)^p |dz_1| \approx (1 - |\psi_\ell(z_2)|)^{1-p} \approx |z_2 - \tau_2|^{K_\tau^\ell(1-p)}$$

for all $z_2 \in I_2$. Letting K_τ denote the contact order of ϕ at τ (or equivalently, the largest of the K_τ^ℓ), this gives

$$(6.5) \quad \int_{I_1} \left| \frac{\partial\phi}{\partial z_1}(z_1, z_2) \right|^p |dz_1| \approx |z_2 - \tau_2|^{K_\tau(1-p)}$$

for all $z_2 \in I_2$, which is exactly the estimate we need in the following proof.

PROPOSITION 6.9. *Let p be an atoral stable polynomial on \mathbb{D}^2 with a zero at $\tau \in \mathbb{T}^2$. Let K_τ denote the contact order of $\phi = \tilde{p}/p$ at τ . Then*

$$(6.6) \quad \frac{K_\tau + 1}{K_\tau}, \quad \frac{K_\tau + 1}{K_\tau - 1}, \quad \frac{K_\tau + 1}{K_\tau - 2}, \quad \dots, \quad \frac{K_\tau + 1}{1}, \quad \infty$$

are the z_1 -derivative integrability indices of p .

Proof. We first produce a polynomial that witnesses each index in (6.6) near τ . Write $\tau = (\tau_1, \tau_2)$ and let $q_n = (z_2 - \tau_2)^n \tilde{p}$, for each $0 \leq n \leq K_\tau$. Then $\partial z_1(q_n/p) \in L_\tau^p(\mathbb{T}^2)$ if and only if $(z_2 - \tau_2)^n \frac{\partial\phi}{\partial z_1} \in L_\tau^p(\mathbb{T}^2)$. Now for $1 \leq p < \infty$, recall from Remark 6.8 that we can choose intervals $I_1, I_2 \subseteq \mathbb{T}$ around τ_1, τ_2 respectively such that (6.5) holds for almost every $z_2 \in I_2$. Then

$$\begin{aligned} \int_{I_1 \times I_2} \left| (z_2 - \tau_2)^n \frac{\partial\phi}{\partial z_1}(z) \right|^p |dz| &= \int_{I_2} |z_2 - \tau_2|^{np} \int_{I_1} \left| \frac{\partial\phi}{\partial z_1}(z) \right|^p |dz_1| |dz_2| \\ &\approx \int_{I_2} |z_2 - \tau_2|^{np} |z_2 - \tau_2|^{(1-p)K_\tau} |dz_2|. \end{aligned}$$

It follows immediately that the above L^p integral is finite if and only if $np + (1-p)K_\tau > -1$. Solving for p gives $p < \frac{K_\tau + 1}{K_\tau - n}$ when $n < K_\tau$. If $n = K_\tau$,

then the integral is finite for all \mathfrak{p} . For this last piece, we do not need to worry about whether $\mathfrak{p} \geq 1$. This follows because for each value of n , the $L^{\mathfrak{p}}$ integral is finite for some $\mathfrak{p} > 1$. Then the nested properties of these $L^{\mathfrak{p}}$ spaces imply that the integral is also finite for all $0 < \mathfrak{p} \leq 1$.

To show that each value $\mathfrak{p} = \frac{K_{\tau}+1}{K_{\tau}-n}$ is a global z_1 -derivative integrability index, we must construct a polynomial s_n such that s_n/p is bounded on \mathbb{D}^2 and s_n witnesses \mathfrak{p} . If p has a single zero on \mathbb{T}^2 we are done, so let $\lambda \neq \tau$ be any other zero of p on \mathbb{T}^2 and without loss of generality, assume $\tau_1 \neq \lambda_1$. Then for $N = N_{\lambda}(\tilde{p}, p)$, we know $\dim(R_{\lambda}/(p, \tilde{p})R_{\lambda}) = N$ and so $[1], [z_1], \dots, [z_1^N]$ are linearly dependent in $R_{\lambda}/(p, \tilde{p})R_{\lambda}$. Thus, there exist $a_0, \dots, a_N \in \mathbb{C}$ not all zero such that

$$r_{\lambda}(z) := \sum_{j=0}^N a_j z_1^j \in (p, \tilde{p})R_{\lambda}.$$

Since we can divide out any factors of $z_1 - \tau_1$ without affecting inclusion in $(p, \tilde{p})R_{\lambda}$, we can further assume that $r_{\lambda}(\tau) \neq 0$. Furthermore, by construction,

$$(6.7) \quad \sup_{\mathfrak{p}' > 0} \left\{ \mathfrak{p}' : r_{\lambda} \frac{\partial \phi}{\partial z_1} \in L_{\lambda}^{p'}(\mathbb{T}^2) \right\} = \infty.$$

Let r be the product of these r_{λ} where λ varies over all zeros different from τ on \mathbb{T}^2 and set $s_n = rq_n$. Then s_n/p is a bounded rational function. As $r(\tau) \neq 0$, our previous arguments combined with (6.7) imply that the z_1 -derivative integrability cutoff of s_n is $\mathfrak{p} = \frac{K_{\tau}+1}{K_{\tau}-n}$ if $n < K_{\tau}$ and $\mathfrak{p} = \infty$ if $n = K_{\tau}$. ■

There is also a simple bound for the number of “local” z_1 -derivative integrability indices of p for certain restricted numerators.

THEOREM 6.10. *Let p be an atoral stable polynomial on \mathbb{D}^2 with a zero at $\tau \in \mathbb{T}^2$. Then there is a list of integrability indices $\mathfrak{p}_1, \dots, \mathfrak{p}_M, \mathfrak{p}_{M+1}$ of p^2 at τ with $M \leq N_{\tau}(p, \tilde{p})$ and $\mathfrak{p}_{M+1} = \infty$ such that if $q \in (p, \tilde{p})R_{\tau}$, then*

$$\sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : \partial_{z_1}(q/p) \in L_{\tau}^{p'}(\mathbb{T}^2) \} = \mathfrak{p}_j \quad \text{for some } j \text{ with } 1 \leq j \leq M+1.$$

Proof. We first need to limit the number of finite indices to at most $N_{\tau}(p, \tilde{p})$. To begin, write $\phi = \tilde{p}/p$ and

$$\frac{\partial \phi}{\partial z_1} = \frac{Q}{p^2}$$

for some $Q \in \mathbb{C}[z_1, z_2]$. Mimicking the setup of Proposition 6.4, set $R_0^{\tau} = R_{\tau}$,

$$R_{\mathfrak{p}}^{\tau} = \{ \hat{q} \in R_{\tau} : \hat{q}Q/p^2 \in L_{\tau}^{\mathfrak{p}}(\mathbb{T}^2) \} \quad \text{for } 0 < \mathfrak{p} < \infty,$$

and $R_{\infty}^{\tau} = \bigcap_{\mathfrak{p} > 0} R_{\mathfrak{p}}^{\tau}$. As in Proposition 6.4, these are nested ideals in R_{τ} .

Observe that

$$(6.8) \quad \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : \hat{q} \in R_{\mathfrak{p}'}^\tau \} = \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : \hat{q}Q/p^2 \in L_{\tau}^{\mathfrak{p}'}(\mathbb{T}^2) \}$$

is well-defined for each $\hat{q} \in R_\tau$. Clearly $\mathfrak{p} = \infty$ equals (6.8) when $\hat{q} = p^2$.

Now we will show that there are at most $N_\tau(p, \tilde{p})$ finite values that (6.8) can take. To that end, assume that $\mathfrak{p}_1, \dots, \mathfrak{p}_M$ is an increasing list of distinct finite numbers such that for each j , there is a $\hat{q}_j \in R_\tau$ such that \mathfrak{p}_j equals the quantity in (6.8). Choose numbers $\kappa_1, \dots, \kappa_M$ such that

$$0 < \mathfrak{p}_1 < \kappa_1 < \mathfrak{p}_2 < \kappa_2 < \dots < \mathfrak{p}_M < \kappa_M < \infty.$$

Then $\hat{q}_1, \dots, \hat{q}_M$ satisfy

$$\hat{q}_1 \in R_0^\tau \setminus R_{\kappa_1}^\tau, \quad \hat{q}_2 \in R_{\kappa_1}^\tau \setminus R_{\kappa_2}^\tau, \dots, \quad \hat{q}_M \in R_{\kappa_{M-1}}^\tau \setminus R_{\kappa_M}^\tau.$$

Now let $[\hat{q}_1], \dots, [\hat{q}_M]$ be the cosets generated by $\hat{q}_1, \dots, \hat{q}_M$ in $R_\tau/(p, \tilde{p})R_\tau$. Since $\hat{q}_1Q/p^2, \dots, \hat{q}_MQ/p^2$ have different integrability behaviors near τ on \mathbb{T}^2 , the elements $[\hat{q}_1], \dots, [\hat{q}_M]$ must be linearly independent in $R_\tau/(p, \tilde{p})R_\tau$. Then (6.1) implies that $M \leq N_\tau(p, \tilde{p})$.

This argument shows that (6.8) can take at most $N_\tau(p, \tilde{p})$ finite values as \hat{q} ranges over R_τ . With a slight abuse of notation, we label those values $\mathfrak{p}_1, \dots, \mathfrak{p}_M$ with $M \leq N_\tau(p, \tilde{p})$ and let $\mathfrak{p}_{M+1} = \infty$.

To finish the proof fix $q \in (p, \tilde{p})R_\tau$, so that locally $f := q/p = r_1 + r_2\phi$ for some $r_1, r_2 \in R_\tau$. Then near τ we can write

$$\frac{\partial f}{\partial z_1} = \frac{\partial r_1}{\partial z_1} + r_2 \frac{\partial \phi}{\partial z_1} + \phi \frac{\partial r_2}{\partial z_1}.$$

The only term that is not necessarily bounded near τ is $r_2 \frac{\partial \phi}{\partial z_1}$, so it will determine the integrability of $\frac{\partial f}{\partial z_1}$. Recall that $r_2 \frac{\partial \phi}{\partial z_1} = r_2 \frac{Q}{p^2}$. By our previous argument, one of the \mathfrak{p}_j must satisfy

$$\mathfrak{p}_j = \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : r_2Q/p^2 \in L_{\tau}^{\mathfrak{p}'}(\mathbb{T}^2) \} = \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : \partial_{z_1}(q/p) \in L_{\tau}^{\mathfrak{p}'}(\mathbb{T}^2) \},$$

which completes the proof. ■

In a generic situation Theorem 5.4, coupled with Theorem 6.10 and Proposition 6.9, allows us to precisely identify all of the z_1 -derivative integrability indices of an atoral stable polynomial.

THEOREM 6.11. *Let p be an atoral stable polynomial on \mathbb{D}^2 with zeros $\tau^1, \dots, \tau^m \in \mathbb{T}^2$ and respective contact orders $K_{\tau^1}, \dots, K_{\tau^m}$. Further, assume that p vanishes to order 1 at each τ^j . Then the z_1 -derivative integrability indices of p are exactly*

$$(6.9) \quad \frac{K_{\tau^j} + 1}{K_{\tau^j}}, \quad \frac{K_{\tau^j} + 1}{K_{\tau^j} - 1}, \quad \frac{K_{\tau^j} + 1}{K_{\tau^j} - 2}, \quad \dots, \quad \frac{K_{\tau^j} + 1}{1}$$

for $j = 1, \dots, m$ and ∞ .

Proof. Proposition 6.9 immediately implies that both ∞ and each number in (6.9) are z_1 -derivative integrability indices of p .

Now fix $q \in \mathbb{C}[z_1, z_2]$ so that q/p is a rational Schur function. Since p vanishes to order 1 at each boundary zero, Theorem 5.4 implies $q \in (p, \tilde{p})R_{\tau^j}$ for each j . (Although Theorem 5.4 is stated on the bi-upper half-plane, its conclusions can be easily moved to the bidisk via conformal maps.) Fix j with $1 \leq j \leq m$. By Theorem 6.10,

$$(6.10) \quad \sup_{\mathfrak{p}' > 0} \{ \mathfrak{p}' : \partial_{z_1}(q/p) \in L_{\tau^j}^{p'}(\mathbb{T}^2) \}$$

must equal one of at most $N_{\tau^j}(p, \tilde{p})$ finite numbers or ∞ . By the argument in the proof of Proposition 6.9, K_{τ^j} of those finite numbers are given in (6.9). Because p vanishes to order 1 at τ , $N_{\tau^j}(p, \tilde{p}) = K_{\tau^j}$ by Corollary 5.5 (again by translating to the bidisk). This means (6.9) must contain all finite numbers that could equal (6.10). As the z_1 -derivative integrability cutoff of q is the minimum of (6.10) over $j = 1, \dots, m$, this establishes the claim. ■

EXAMPLE 6.12. Let $p(z_1, z_2) = 2 - z_1 - z_2$. Then p has a single zero on \mathbb{T}^2 at $\tau = (1, 1)$ with contact order $K_\tau = 2$, and p vanishes to order 1 at $(1, 1)$. Theorem 6.11 implies that the z_1 -derivative integrability indices of p are exactly $\frac{3}{2}, 3, \infty$. Furthermore, the proof of Proposition 6.9 implies that $q = \tilde{p}$ witnesses $\mathfrak{p} = \frac{3}{2}$, $q = (z_2 - 1)\tilde{p}$ witnesses $\mathfrak{p} = 3$, and $q = (z_2 - 1)^2\tilde{p}$ witnesses $\mathfrak{p} = \infty$.

Similarly, let $p(z_1, z_2) = 4 - 3z_1 - z_2 - z_1z_2 + z_1^2$. Then p again has a single zero at $\tau = (1, 1)$, this time with contact order $K_\tau = 4$ and order of vanishing 1. In this case, Theorem 6.11 implies that the z_1 -derivative integrability indices of p are exactly $\frac{5}{4}, \frac{5}{3}, \frac{5}{2}, 5, \infty$. As before, these are witnessed by polynomials $q = (z_2 - 1)^n\tilde{p}$ for $n = 0, \dots, 4$.

See [10, p. 298] for both contact order computations. ◇

Our next example has multiple zeros on \mathbb{T}^2 with different associated contact orders.

EXAMPLE 6.13. Consider p defined by

$$p(z_1, z_2) = 4 - z_2 + z_1z_2 - 3z_1^2z_2 - z_1^3z_2.$$

A slight variant of this polynomial is discussed in [11, p. 491]. It is also not hard to check directly that this p does not vanish on \mathbb{D}^2 and has precisely two zeros on \mathbb{T}^2 . These occur at $\tau^1 = (1, 1)$ and $\tau^2 = (-1, 1)$, respectively, with associated contact orders $K_{\tau^1} = 2$ and $K_{\tau^2} = 4$. One can verify these contact orders by first solving $\frac{\tilde{p}}{p}(z_1, z_2) = \lambda$ for z_2 to get $z_2 = \psi(z_1; \lambda)$ and then finding the power series expansions of $\psi(z_1; \lambda)$ around 1 and -1 . The first series has coefficients a_n depending on λ starting with $n = 2$ (giving contact order 2) and the second has coefficients a_n depending on λ starting

with $n = 4$ (giving contact order 4). Since p vanishes to order 1 at both zeros, Theorem 6.11 implies that the z_1 -derivative integrability indices of p are $\frac{3}{2}, 3, \infty$, coming from K_{τ^1} , and $\frac{5}{4}, \frac{5}{3}, \frac{5}{2}, 5, \infty$, from K_{τ^2} . Combining these in increasing order, we obtain the global list of z_1 -integrability indices for p :

$$\frac{5}{4}, \frac{3}{2}, \frac{5}{3}, \frac{5}{2}, 3, 5, \infty.$$

The first part of the proof of Proposition 6.9 implies that the integrability indices $\frac{5}{4}, \frac{5}{3}, \frac{5}{2}, 5, \infty$ are witnessed by $q = (1 - z_2)^n \tilde{p}$ for $n = 0, \dots, 4$. To witness the integrability indices $\frac{3}{2}, 3$, the second part of the proof of Proposition 6.9 suggests that we should find a polynomial r satisfying

$$r \in (p, \tilde{p})R_{\tau^2} \quad \text{and} \quad r(\tau^1) \neq 0.$$

To that end, let $r(z) = (1 + z_1)(1 + z_1 z_2)$. Then $r(\tau^1) \neq 0$ and one can check that

$$r(z) = -\frac{1}{(1 - z_1)^2}(z_1 p(z) + \tilde{p}(z)) \in (p, \tilde{p})R_{\tau^2}.$$

Set $q = (1 - z_2)^n r \tilde{p}$. Then the proof of Proposition 6.9 implies that if $n = 0$, q witnesses $\mathfrak{p} = \frac{3}{2}$ and if $n = 1$, q witnesses $\mathfrak{p} = 3$. \diamond

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References

- [1] S. S. Abhyankar, *Algebraic Geometry for Scientists and Engineers*, Math. Surveys Monogr. 35, Amer. Math. Soc., Providence, RI, 1990.
- [2] J. Agler and J. E. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, Grad. Stud. Math. 44, Amer. Math. Soc., Providence, RI, 2002.
- [3] J. Agler and J. E. McCarthy, *Hankel vector moment sequences and the nontangential regularity at infinity of two variable Pick functions*, Trans. Amer. Math. Soc. 366 (2014), 1379–1411.
- [4] J. Agler, J. E. McCarthy, and M. Stankus, *Toral algebraic sets and function theory on polydisks*, J. Geom. Anal. 16 (2006), 551–562.
- [5] J. Agler, J. E. McCarthy, and M. Stankus, *Local geometry of zero sets of holomorphic functions near the torus*, New York J. Math. 14 (2008), 517–538.
- [6] J. Agler, J. E. McCarthy, and N. J. Young, *Operator monotone functions and Löwner functions of several variables*, Ann. of Math. (2) 176 (2012), 1783–1826.

- [7] J. Agler, R. Tully-Doyle, and N. J. Young, *Nevanlinna representations in several variables*, J. Funct. Anal. 270 (2016), 3000–3046.
- [8] M. F. Atiyah, R. Bott, and L. Gårding, *Lacunas for hyperbolic differential operators with constant coefficients. I*, Acta Math. 124 (1970), 109–189.
- [9] J. Ball, C. Sadosky, and V. Vinnikov, *Scattering systems with several evolutions and multidimensional input/state/output systems*, Integral Equations Operator Theory 52 (2005), 323–393.
- [10] K. Bickel, J. E. Pascoe, and A. Sola, *Derivatives of rational inner functions: geometry of singularities and integrability at the boundary*, Proc. London Math. Soc. (3) 116 (2018), 281–329.
- [11] K. Bickel, J. E. Pascoe, and A. Sola, *Level curve portraits of rational inner functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 21 (2020), 451–494.
- [12] K. Bickel, J. E. Pascoe, and A. Sola, *Singularities of rational inner functions in higher dimensions*, Amer. J. Math. 144 (2022), 1115–1157.
- [13] K. Bickel, J. E. Pascoe, and R. Tully-Doyle, *Analytic continuation of concrete realizations and the McCarthy Champagne conjecture*, Int. Math. Res. Notices 2023, 7845–7882.
- [14] J. Borcea and P. Brändén, *The Lee–Yang and Pólya–Schur programs I. Linear operators preserving stability*, Invent. Math. 177 (2009), 541–569.
- [15] J. Borcea and P. Brändén, *The Lee–Yang and Pólya–Schur programs II. Theory of stable polynomials and applications*, Comm. Pure Appl. Math. 62 (2009), 1595–1631.
- [16] J. Borcea and P. Brändén, *Multivariate Pólya–Schur classification problems in the Weyl algebra*, Proc. London Math. Soc. (3) 101 (2010), 73–104.
- [17] J. Borcea, P. Brändén, and T. M. Liggett, *Negative dependence and the geometry of polynomials*, J. Amer. Math. Soc. 22 (2009), 521–567.
- [18] D. A. Cox, J. Little, and D. O’Shea, *Using Algebraic Geometry*, 2nd ed, Grad. Texts in Math. 185, Springer, New York, 2005.
- [19] M. A. Dritschel, *Factoring non-negative operator valued trigonometric polynomials in two variables*, Math. Ann. (online, 2024).
- [20] S. R. Garcia, J. Mashreghi, and W. T. Ross, *Finite Blaschke Products and Their Connections*, Springer, 2018.
- [21] J. S. Geronimo and H. J. Woerdeman, *Two-variable polynomials: intersecting zeros and stability*, IEEE Trans. Circuits Syst. I Regul. Pap. 53 (2006), 1130–1139.
- [22] A. Grinshpan, D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H. Woerdeman, *Matrix-valued Hermitian Positivstellensatz, lurking contractions, and contractive determinantal representations of stable polynomials*, in: Operator Theory, Function Spaces, and Applications, Oper. Theory Adv. Appl. 255, Birkhäuser/Springer, Cham, 2016, 123–136.
- [23] A. Grinshpan, D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov, and H. Woerdeman, *Stable and real-zero polynomials in two variables*, Multidimens. Systems Signal Process. 27 (2016), 1–26.
- [24] L. Gurvits, *Van der Waerden/Schrijver–Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all* (with a corrigendum), Electron. J. Combin. 15 (2008), no. 1, art. 66, 26 pp.
- [25] J. W. Helton and V. Vinnikov, *Linear matrix inequality representation of sets*, Comm. Pure Appl. Math. 60 (2007), 654–674.
- [26] G. Knese, *Bernstein–Szegő measures on the two dimensional torus*, Indiana Univ. Math. J. 57 (2008), 1353–1376.
- [27] G. Knese, *Polynomials with no zeros on the bidisk*, Anal. PDE 3 (2010), 109–149.
- [28] G. Knese, *Integrability and regularity of rational functions*, Proc. London Math. Soc. 111 (2015), 1261–1306.

- [29] G. Knese, *Determinantal representations of semihyperbolic polynomials*, Michigan Math. J. 65 (2016), 473–487.
- [30] G. Knese, *Extreme points and saturated polynomials*, Illinois J. Math. 63 (2019), 47–74.
- [31] G. Knese, *Global bounds on stable polynomials*, Complex Anal. Oper. Theory 13 (2019), 1895–1915.
- [32] G. Knese, *Kummert's approach to realization on the bidisk*, Indiana Univ. Math. J. 70 (2021), 2369–2403.
- [33] J. Kollár, *Bounded meromorphic functions on the complex 2-disc*, Period. Math. Hungar. 88 (2024), 1–7.
- [34] A. Kummert, *Synthesis of two-dimensional lossless m -ports with prescribed scattering matrix*, Circuits Systems Signal Process. 8 (1989), 97–119.
- [35] A. Kummert, *2-D stable polynomials with parameter-dependent coefficients: generalizations and new results*, IEEE Trans. Circuits Systems I Fund. Theory Appl. 49 (2002), 725–731.
- [36] P. Kurasov and P. Sarnak, *Stable polynomials and crystalline measures*, J. Math. Phys. 61 (2020), art. 083501, 13 pp.
- [37] D. Lind, K. Schmidt, and I. E. Verbitsky, *Homoclinic points, atoral polynomials, and periodic points of algebraic \mathbb{Z}^d actions*, Ergodic Theory Dynam. Systems 33 (2013), 1060–1081.
- [38] A. Marcus, D. Spielman, and N. Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, Ann. of Math. (2) 182 (2015), 307–325.
- [39] A. Marcus, D. Spielman, N. Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem*, Ann. of Math. (2) 182 (2015), 327–350.
- [40] S. Parrott, *Unitary dilations for commuting contractions*, Pacific J. Math. 34 (1970), 481–490.
- [41] J. E. Pascoe, *An inductive Julia–Carathéodory theorem for Pick functions in two variables*, Proc. Edinburgh Math. Soc. (2) 61 (2018), 647–660.
- [42] J. E. Pascoe and R. Tully-Doyle, *Automatic real analyticity and a regal proof of a commutative multivariate Löwner theorem*, Proc. Amer. Math. Soc. 149 (2021), 2019–2024.
- [43] J. E. Pascoe and R. Tully-Doyle, *Free Pick functions: Representations, asymptotic behavior and matrix monotonicity in several noncommuting variables*, J. Funct. Anal. 273 (2017), 283–328.
- [44] J. E. Pascoe and R. Tully-Doyle, *The royal road to automatic noncommutative real analyticity, monotonicity, and convexity*, Adv. Math. 407 (2022), art. 108548, 24 pp.
- [45] R. Pemantle and M. C. Wilson, *Analytic Combinatorics in Several Variables*, Cambridge Stud. Adv. Math. 140, Cambridge Univ. Press, Cambridge, 2013.
- [46] W. Rudin, *Function Theory in Polydiscs*, W. A. Benjamin, New York, 1969.
- [47] B. Simon, *Basic Complex Analysis. A Comprehensive Course in Analysis, Part 2A*, Amer. Math. Soc., Providence, RI, 2015.
- [48] A. K. Tsikh, *Conditions for absolute convergence of series of Taylor coefficients of meromorphic functions of two variables*, Mat. Sb. 182 (1991), 1588–1612 (in Russian); English transl.: Math. USSR-Sb. 74 (1993), 337–360.
- [49] N. Th. Varopoulos, *On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory*, J. Funct. Anal. 16 (1974), 83–100.
- [50] D. G. Wagner, *Multivariate stable polynomials: theory and applications*, Bull. Amer. Math. Soc. 48 (2011), 53–84.
- [51] H. J. Woerdeman, *Determinantal representations of stable polynomials*, in: *Advances in Structured Operator Theory and Related Areas*, Oper. Theory Adv. Appl. 237, Birkhäuser/Springer, Basel, 2013, 241–246.

Kelly Bickel
Department of Mathematics
Bucknell University
Lewisburg, PA 17837, USA
E-mail: kelly.bickel@bucknell.edu

James Eldred Pascoe
Department of Mathematics
Drexel University
Philadelphia, PA 19104, USA
E-mail: jep362@drexel.edu

Greg Knese (corresponding author)
Department of Mathematics
Washington University in St. Louis
St. Louis, MO 63130, USA
E-mail: geknese@wustl.edu

Alan Sola
Department of Mathematics
Stockholm University
106 91 Stockholm, Sweden
E-mail: sola@math.su.se

