ICASSP 2024 - 2024 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP) | 979-8-3503-4485-1/24/$31.00 ©2024 IEEE | DOI: 10.1109/ICASSP48485.2024.10446728

AN ASYMPTOTICALLY ACHIEVABLE RATE BOUND FOR ESTABLISHING HIGH-FIDELITY ENTANGLEMENTS IN QUANTUM NETWORKS

Zhenyu Liu[†]*, Stefano Marano*[⋆]*, and Moe Z. Win*[‡]

[†]Wireless Information and Network Sciences Laboratory, Massachusetts Institute of Technology
[⋆]Department of Information & Electrical Engineering and Applied Mathematics, University of Salerno
[‡]Laboratory for Information and Decision Systems, Massachusetts Institute of Technology

ABSTRACT

Entangled quantum states serve as important resources in quantum communication, quantum computing, and quantum sensing. Creating entangled states between remote nodes is referred to as remote entanglement establishment (REE). REE typically consists of three types of quantum operations: entanglement generation, distillation, and swapping. By carefully designing the sequence describing the order of these operations, this paper investigates REE in a repeater chain under the requirement that the fidelity of the established entanglements be above a desired threshold. Specifically, the paper derives an asymptotically achievable upper bound on the maximum REE rate.

***Index Terms*—** Entanglement distribution, entanglement distillation, entanglement swapping, quantum networks.

1. INTRODUCTION

The principles of quantum mechanics have inspired numerous signal processing methods [1, 2]. A unique phenomenon in quantum mechanics is the entanglement [3], which can be employed for efficient quantum communication [4–7], quantum computing [8–10], and quantum sensing [11–13]. On the other hand, creating entanglement between remote source nodes and destination nodes, namely remote entanglement establishment (REE), is a challenging task.

An important technique for REE is to use repeater nodes. In this technique, entangled qubit pairs are first generated locally and shared over elementary links, and then entanglement swapping operations [14–16] are performed to establish entangled qubit pairs between the source node and the destination node. The quality of these established qubit pairs is characterized by their fidelities. Efficient REE techniques maximize the number of established qubit pairs while ensuring that their fidelities be above a desired threshold. One method for fidelity improvement is to perform entanglement distillation [17–19] before and/or after entanglement swapping. Specifically, entanglement distillation creates high-fidelity qubit pairs consuming low-fidelity qubit pairs.

The fundamental research described in this paper was supported, in part, by the National Science Foundation under Grant No. CCF-1956211. *Corresponding author: Moe Z. Win (e-mail: moewin@mit.edu).*

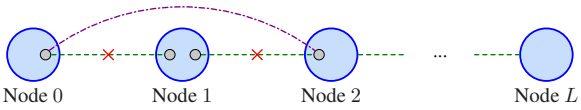

Fig. 1. A repeater chain of length $L+1$. Green dashed lines represent crude qubit pairs (CQPs) over elementary links. Swapping is performed to create qubit pairs (purple dash dotted line) between nodes 0 and 2 by consuming CQPs marked by red crosses.

REE has been studied in the literature [20–23]. Some of the existing works assume ideal quantum channels, whereas some works do not consider entanglement distillation. This paper presents a general model for REE in the presence of noisy quantum channels via entanglement generation, distillation, and swapping. We derive an upper bound on the optimal REE rate that can be achieved asymptotically when the number of qubit pairs generated locally is sufficiently large, while meeting the requirement on the fidelity of the established qubit pairs.

Notations: Random quantities are displayed in sans serif, upright fonts. Vectors are denoted by bold lowercase letters. The expectation of x is denoted by $\mathbb{E}\{\mathsf{x}\}$. The ℓ_1 norm of a vector $\boldsymbol{x}$ is denoted by $|\boldsymbol{x}|$. The relationship that vector $\boldsymbol{x}_1$ is larger than or equal to (resp. smaller than or equal to) vector $\boldsymbol{x}_2$ entry-wise is denoted by $\boldsymbol{x}_1 \succcurlyeq \boldsymbol{x}_2$ (resp. $\boldsymbol{x}_1 \preccurlyeq \boldsymbol{x}_2$). The vector of zeros (resp. ones) is denoted by $\mathbf{0}$ (resp. $\mathbf{1}$).

2. SYSTEM MODEL

Consider a repeater chain consisting of $L+1$ nodes, where L represents the length of the repeater chain. These nodes are assigned indices $0, 1, \ldots, L$, as shown in Fig. 1. In particular, the link between nodes $l-1$ and l is referred to as an elementary link for $l = 1, 2, \ldots, L$. In addition, node 0 and node L are referred to as the source node and the destination node, respectively. The aim of REE is to create high-fidelity entangled qubit pairs (EQPs) between nodes 0 and L by performing

the quantum operations described next.

- Entanglement generation: for $1 \leqslant l \leqslant L$, create qubit pairs called CQPs over elementary link $(l-1, l)$. The CQPs are in mixed Bell states, to account for the effect of the noisy quantum channel between nodes $l-1$ and l. The density operator $\boldsymbol{\Xi}^{l-1,l}$ describing the quantum state shared by nodes $l-1$ and l is given by

$$\boldsymbol{\Xi}^{l-1,l} := w^{l-1,l} \left|\boldsymbol{\phi}^{+}\right\rangle\!\left\langle\boldsymbol{\phi}^{+}\right| + (1 - w^{l-1,l}) \left|\boldsymbol{\psi}^{+}\right\rangle\!\left\langle\boldsymbol{\psi}^{+}\right| \tag{1}$$

  where $1/2 < w^{l-1,l} \leqslant 1$ is a scalar representing the fidelity of $\boldsymbol{\Xi}^{l-1,l}$. All fidelities considered in this paper are computed with respect to $|\boldsymbol{\phi}^{+}\rangle\langle\boldsymbol{\phi}^{+}|$. Here and in (1), the kets $|\boldsymbol{\phi}^{+}\rangle$ and $|\boldsymbol{\psi}^{+}\rangle$ describe pure Bell states

$$\left|\boldsymbol{\phi}^{+}\right\rangle := \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$
$$\left|\boldsymbol{\psi}^{+}\right\rangle := \frac{1}{\sqrt{2}}\big(|01\rangle + |10\rangle\big)$$

  where $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ represents the computational basis for two-qubit systems.

- Entanglement distillation: for $0 \leqslant i < j \leqslant L$, consume two qubit pairs between nodes i and j to create one qubit pair called a distilled qubit pair between nodes i and j. This operation is referred to as entanglement distillation between nodes i and j. Entanglement distillation can fail. Specifically, the probability $p_{\mathrm{d}}\big(w^{i,j}\big)$ of successfully creating one distilled qubit pair by consuming two qubit pairs with fidelity $w^{i,j}$ is given by [18]

$$p_{\mathrm{d}}\big(w^{i,j}\big) := \big(w^{i,j}\big)^2 + \big(1 - w^{i,j}\big)^2. \tag{2}$$

  If the distillation operation is successful, the fidelity $g_{\mathrm{d}}\big(w^{i,j}\big)$ of the distilled qubit pair is [18]

$$g_{\mathrm{d}}\big(w^{i,j}\big) := \frac{\big(w^{i,j}\big)^2}{\big(w^{i,j}\big)^2 + \big(1 - w^{i,j}\big)^2}. \tag{3}$$

  Entanglement distillation increases fidelity: it holds that $g_{\mathrm{d}}\big(w^{i,j}\big) \geqslant w^{i,j}$ for any $1/2 < w^{i,j} \leqslant 1$, with equality achieved if and only if $w^{i,j} = 1$.

- Entanglement swapping: for $0 \leqslant i < j < l \leqslant L$, consume one qubit pair between nodes i and j and one qubit pair between nodes j and l to create one qubit pair called a swapped qubit pair between nodes i and l. This operation is referred to as entanglement swapping between nodes i and l at node j. Entanglement swapping can fail. Specifically, the probability of successfully swapping one qubit pair between nodes i and l is $0 < q \leqslant 1$. If the swapping operation is successful, the fidelity $g_{\mathrm{s}}\big(w^{i,j}, w^{j,l}\big)$ of the swapped qubit pair is given by [14]

$$g_{\mathrm{s}}\big(w^{i,j}, w^{j,l}\big) := w^{i,j}\, w^{j,l} + \big(1 - w^{i,j}\big)\big(1 - w^{j,l}\big) \tag{4}$$

  where $w^{i,j}$ represents the fidelity of the qubit pair between nodes i and j, and $w^{j,l}$ represents the fidelity of the qubit pair between nodes j and l. Entanglement swapping reduces fidelity: it holds that $g_{\mathrm{s}}\big(w^{i,j}, w^{j,l}\big) \leqslant \min\big\{w^{i,j}, w^{j,l}\big\}$ for any $1/2 < w^{i,j}, w^{j,l} \leqslant 1$, with equality achieved if and only if $w^{i,j} = 1$ or $w^{j,l} = 1$.

Entanglement distillation can be performed on multiple qubit pairs. Suppose that there are $n^{i,j}$ qubit pairs between nodes i and j, each with fidelity $w^{i,j}$. These qubit pairs can be consumed to create $\mathsf{n}_{\mathrm{d}}^{i,j}$ distilled qubit pairs between nodes i and j. The described quantum operation that distills all of the qubit pairs is called performing "one round of distillation between nodes i and j" and is denoted by $\mathrm{d}^{i,j}$. The number $\mathsf{n}_{\mathrm{d}}^{i,j}$ of qubit pairs created by $\mathrm{d}^{i,j}$ is a random variable with the following binomial distribution

$$\mathsf{n}_{\mathrm{d}}^{i,j} \sim \mathrm{Bin}\Big(\Big\lfloor \frac{n^{i,j}}{2} \Big\rfloor,\, p_{\mathrm{d}}\big(w^{i,j}\big)\Big) \tag{5}$$

where $p_{\mathrm{d}}(\cdot)$ is given in (2). The fidelity of these $\mathsf{n}_{\mathrm{d}}^{i,j}$ qubit pairs is given in (3).

The qubit pairs created by performing one round of entanglement distillation between nodes i and j can be consumed to perform another round of distillation in order to further improve the fidelity of distilled qubit pairs and so forth. This is called performing multiple rounds of distillation between nodes i and j.

Entanglement swapping can be performed on multiple qubit pairs. Suppose that there are $n^{i,j}$ qubit pairs between nodes i and j, each with fidelity $w^{i,j}$, whereas there are $n^{j,l}$ qubit pairs between nodes j and l, each with fidelity $w^{j,l}$. Then at most $\min\big\{n^{i,j}, n^{j,l}\big\}$ qubit pairs between nodes i and j, together with the same number of qubit pairs between nodes j and l, can be consumed to create $\mathsf{n}_{\mathrm{s}}^{i,l}$ qubit pairs between nodes i and l via entanglement swapping. The quantum operation that swaps the maximum number of qubit pairs is called performing "one round of swapping between nodes i and l at node j" and is denoted by $\mathrm{s}^{i,j,l}$. The number $\mathsf{n}_{\mathrm{s}}^{i,l}$ of qubit pairs created by $\mathrm{s}^{i,j,l}$ is a random variable with the following binomial distribution

$$\mathsf{n}_{\mathrm{s}}^{i,l} \sim \mathrm{Bin}\big(\min\big\{n^{i,j}, n^{j,l}\big\},\, q\big). \tag{6}$$

The fidelity of these $\mathsf{n}_{\mathrm{s}}^{i,l}$ qubit pairs is given in (4).

The REE task is described as follows. The $L+1$ nodes aim to establish EQPs between node 0 and node L by generating CQPs and performing entanglement distillation as well as swapping. In particular, a qubit pair is referred to as an EQP if its fidelity is above a predefined threshold $\underline{w} > 1/2$.

The objective is to maximize the expected number of established EQPs between nodes 0 and L by designing an REE policy, which is a sequence of quantum operations consisting of entanglement generation and rounds of distillation as well as swapping. A constraint on the REE policy is that the total number of CQPs generated between each pair of nodes is within a predefined budget. Specifically, let $b^{l-1,l}$ represent the budget of CQPs between node $l-1$ and l, and let $w^{l-1,l}$ represent the fidelity of the CQPs generated between these two nodes for $l = 1, 2, \ldots, L$. Define CQP budget vector $\boldsymbol{b}$ and CQP fidelity vector $\boldsymbol{w}$ as

$$\boldsymbol{b} := \begin{bmatrix} b^{0,1} & b^{1,2} & \cdots & b^{L-1,L} \end{bmatrix}^{\mathrm{T}} \tag{7a}$$

$$\boldsymbol{w} := \begin{bmatrix} w^{0,1} & w^{1,2} & \cdots & w^{L-1,L} \end{bmatrix}^{\mathrm{T}}. \tag{7b}$$

Given $\boldsymbol{b}$ and $\boldsymbol{w}$, the number of EQPs established between nodes 0 and L via a policy π is a random variable denoted by $\mathsf{z}_\pi(\boldsymbol{b}, \boldsymbol{w})$. Define the REE rate $r_\pi(\boldsymbol{b}, \boldsymbol{w})$ of policy π as the ratio between the expectation $\mathbb{E}\{\mathsf{z}_\pi(\boldsymbol{b}, \boldsymbol{w})\}$ and the average budget $|\boldsymbol{b}|/L$, i.e.,

$$r_\pi(\boldsymbol{b}, \boldsymbol{w}) := \frac{L}{|\boldsymbol{b}|}\mathbb{E}\{\mathsf{z}_\pi(\boldsymbol{b}, \boldsymbol{w})\}.$$

The optimal REE rate $r^*(\boldsymbol{b}, \boldsymbol{w})$ is defined as the maximum of the REE rates over all policies, i.e.,

$$r^*(\boldsymbol{b}, \boldsymbol{w}) := \max_{\pi} r_\pi(\boldsymbol{b}, \boldsymbol{w}). \tag{8}$$

This paper derives an upper bound on $r^*(\boldsymbol{b}, \boldsymbol{w})$ and shows that such a bound is asymptotically achievable when each entry of $\boldsymbol{b}$ is sufficiently large.

3. UPPER BOUND ON THE REE RATE

This section first introduces some notions including REE procedures, number of established EQPs under certainty equivalence (CE), and allocation vector for an REE procedure. Then the upper bound on REE rates is presented.

3.1. REE Procedures

Definition 1 (REE procedure)**.** An REE procedure for a repeater chain is a sequence of entanglement swapping and entanglement distillation rounds for establishing qubit pairs between the source node and the destination node. □

To clarify this definition, consider as an example the following REE procedure $\boldsymbol{u}_0$ for a repeater chain of length $L = 3$:

$$\boldsymbol{u}_0 := \left(\mathrm{d}^{0,1}, \mathrm{s}^{0,1,2}, \mathrm{s}^{0,2,3}\right). \tag{9}$$

This REE procedure consists of performing one round of distillation between nodes 0 and 1, then one round of swapping between nodes 0 and 2 at node 1, and finally one round of swapping between nodes 0 and 3 at node 2. Note that the sequence $\left(\mathrm{s}^{0,1,2}, \mathrm{d}^{0,1}, \mathrm{s}^{0,2,3}\right)$ is not an REE procedure on this repeater chain. Specifically, after performing $\mathrm{s}^{0,1,2}$, all the qubit pairs between nodes 0 and 1 have been consumed, and thus the round of distillation $\mathrm{d}^{0,1}$ cannot be performed between these two nodes.

Using the notion of REE procedures, an REE policy π can be decomposed into multiple iterations, where each iteration consists of: 1) generating CQPs over each elementary link using a proportion of the budget, and 2) establishing EQPs by performing an REE procedure consuming the generated CQPs. The REE rate of policy π is the sum of expected numbers of established EQPs in all iterations divided by the average budget. Specifically, consider a policy π consisting of K iterations. Let $n_k^{l-1,l}$ represent the number of CQPs generated between nodes $l-1$ and l in the kth iteration of π, and denote the REE procedure performed in the kth iteration by $\boldsymbol{u}_k$. Moreover, define the generation vector $\boldsymbol{n}_k := \begin{bmatrix} n_k^{0,1} & n_k^{1,2} & \cdots & n_k^{L-1,L} \end{bmatrix}^{\mathrm{T}}$ and denote the number of established EQPs in the kth iteration by $\check{\mathsf{z}}_{\boldsymbol{u}_k}(\boldsymbol{n}_k, \boldsymbol{w})$, which is a function of the generation vector $\boldsymbol{n}_k$ and the CQP fidelity vector $\boldsymbol{w}$. Then the REE rate $r_\pi(\boldsymbol{b}, \boldsymbol{w})$ is given by

$$r_\pi(\boldsymbol{b}, \boldsymbol{w}) = \frac{L}{|\boldsymbol{b}|}\sum_{k=1}^{K}\mathbb{E}\{\check{\mathsf{z}}_{\boldsymbol{u}_k}(\boldsymbol{n}_k, \boldsymbol{w})\}. \tag{10}$$

Vectors $\boldsymbol{n}_k$ with $k = 1, 2, \ldots, K$ satisfy the constraint that the total number of CQPs generated in all the iterations over each elementary link does not exceed the budget assigned to that link. In other words, $\sum_k \boldsymbol{n}_k \preccurlyeq \boldsymbol{b}$, where $\boldsymbol{b}$ is defined in (7a).

Deriving $r^*(\boldsymbol{b}, \boldsymbol{w})$ is challenging. First, there are multiple REE procedures that can be used for creating EQPs. Second, the expected number of created EQPs is difficult to compute for long repeater chains. To overcome these challenges, we use the technique of CE as described in the next subsection.

3.2. Number of Established EQPs under CE

CE is an approximation method that replaces random quantities by their expectations. This method has been used in control and learning theories. To explain this method, consider a repeater chain of length $L = 3$, and let $\boldsymbol{n}$ and $\boldsymbol{w}$ represent the CQP number vector and the CQP fidelity vector, respectively, for this chain. In the following, we show the approximation via CE $\zeta_{\boldsymbol{u}_0}(\boldsymbol{n}, \boldsymbol{w})$ for the expected number $\mathbb{E}\{\check{\mathsf{z}}_{\boldsymbol{u}_0}(\boldsymbol{n}, \boldsymbol{w})\}$ of established EQPs if the REE procedure $\boldsymbol{u}_0$ given by (9) is performed. The number of qubit pairs between nodes 0 and 1 after performing $\mathrm{d}^{0,1}$ is a random variable whose distribution is given by (5). With the floor function in (5) omitted, the expectation of this random variable is $n^{0,1}p_{\mathrm{d}}(w^{0,1})/2$. We approximate the number of qubit pairs between nodes 0 and 1 after performing $\mathrm{d}^{0,1}$ by this expected value, and consider the expected number of qubit pairs between nodes 0 and 2 after

performing $\mathrm{s}^{0,1,2}$. According to (6), the expected number of qubit pairs between nodes 0 and 2 can be approximated by $q\min\{n^{0,1}p_{\mathrm{d}}(w^{0,1})/2, n^{1,2}\}$. Using this approach leads to

$$\zeta_{\boldsymbol{u}_0}(\boldsymbol{n},\boldsymbol{w}) = q\min\Big\{q\min\{n^{0,1}p_{\mathrm{d}}(w^{0,1})/2,\, n^{1,2}\},\, n^{2,3}\Big\}.$$

The definition of the number $\zeta_{\boldsymbol{u}}(\boldsymbol{n},\boldsymbol{w})$ of established EQPs for a general REE procedure $\boldsymbol{u}$ under CE is detailed in [24].

The next proposition (see [24] for its proof) shows that $\mathbb{E}\{\check{\mathrm{z}}_{\boldsymbol{u}}(\boldsymbol{n},\boldsymbol{w})\}$ is no greater than $\zeta_{\boldsymbol{u}}(\boldsymbol{n},\boldsymbol{w})$.

Proposition 1. For any REE procedure $\boldsymbol{u}$, CQP number vector $\boldsymbol{n}$, and CQP fidelity vector $\boldsymbol{w}$, it holds that

$$\mathbb{E}\{\check{\mathrm{z}}_{\boldsymbol{u}}(\boldsymbol{n},\boldsymbol{w})\} \leqslant \zeta_{\boldsymbol{u}}(\boldsymbol{n},\boldsymbol{w}). \tag{11}$$

□

Next, we define the notion of feasible REE procedures. Let $\boldsymbol{w}$ represent a CQP fidelity vector on a repeater chain and let $\underline{w}$ represent an EQP fidelity threshold. An REE procedure $\boldsymbol{u}$ is said to be feasible for $\boldsymbol{w}$ and $\underline{w}$ if the fidelity of qubit pairs created by performing $\boldsymbol{u}$ exceeds $\underline{w}$. We also introduce the notion of efficient REE procedures in the following.

Definition 2. Let $\boldsymbol{u}$ represent a feasible REE procedure for $\boldsymbol{w}$ and $\underline{w} > 1/2$ on a repeater chain. REE procedure $\boldsymbol{u}$ is said to be efficient if there does not exist a different feasible REE procedure $\boldsymbol{u}'$ on this repeater chain such that $\zeta_{\boldsymbol{u}}(\boldsymbol{n},\boldsymbol{w}) \leqslant \zeta_{\boldsymbol{u}'}(\boldsymbol{n},\boldsymbol{w})$ for all $\boldsymbol{n} \succcurlyeq \boldsymbol{0}$. □

Finally, we introduce the notion of allocation vector for an REE procedure. An allocation vector for a repeater chain of length L is an L-dimensional vector. Specifically, the lth entry of this vector represents the number of CQPs that should be generated between nodes $l-1$ and l for establishing one qubit pair between nodes 0 and L under CE. As an example, consider the allocation vector $\boldsymbol{v}_{\boldsymbol{u}_0}(\boldsymbol{w})$ for the REE procedure $\boldsymbol{u}_0$ defined in (9) for a repeater chain of length $L = 3$ with CQP fidelity vector $\boldsymbol{w}$. Note that the last operation in $\boldsymbol{u}_0$ is $\mathrm{s}^{0,2,3}$. In order to establish one EQP between nodes 0 and 3 by performing this operation, $1/q$ qubit pairs are needed on average between nodes 0 and 2 as well as between nodes 2 and 3. The operation preceding $\mathrm{s}^{0,2,3}$ in $\boldsymbol{u}_0$ is $\mathrm{s}^{0,1,2}$. In order to establish $1/q$ qubit pairs between nodes 0 and 2, we need $1/q^2$ qubit pairs on average between nodes 0 and 1 as well as between nodes 1 and 2. Continuing this calculation gives $\boldsymbol{v}_{\boldsymbol{u}_0}(\boldsymbol{w}) = \begin{bmatrix}2/(p_{\mathrm{d}}(w^{0,1})q^2) & 1/q^2 & 1/q\end{bmatrix}^{\mathrm{T}}$. The definition $\boldsymbol{v}_{\boldsymbol{u}}(\boldsymbol{w})$ for a general REE procedure $\boldsymbol{u}$ is presented in [24].

3.3. Asymptotically Achievable Bound on REE Rates

Consider a chain of length L with CQP budget vector $\boldsymbol{b}$, CQP fidelity vector $\boldsymbol{w}$, and EQP fidelity threshold $\underline{w} > 1/2$. Let M represent the number of efficient REE procedures for $\boldsymbol{w}$ and $\underline{w}$, and let $\boldsymbol{u}^{(m)}$ represent the mth efficient REE procedure. The next theorem presents an upper bound on the optimal REE rate.

Theorem 1. The optimal REE rate $r^*(\boldsymbol{b},\boldsymbol{w})$ satisfies

$$r^*(\boldsymbol{b},\boldsymbol{w}) \leqslant \breve{r} \tag{12}$$

where $-\breve{r}$ is the optimal objective value of linear program $\mathscr{P}$ given by

$$\mathscr{P}: \underset{\boldsymbol{x}}{\text{minimize}} \quad -L\big(\mathbf{1}^{\mathrm{T}}\boldsymbol{x}\big) \tag{13a}$$

$$\text{subject to} \quad \boldsymbol{x} \succcurlyeq \boldsymbol{0} \tag{13b}$$

$$\sum_{m=1}^{M} [\boldsymbol{x}]_m\, \boldsymbol{v}_{\boldsymbol{u}^{(m)}}(\boldsymbol{w}) \preccurlyeq \frac{\boldsymbol{b}}{|\boldsymbol{b}|}. \tag{13c}$$

Moreover, there exists a policy π^* such that

$$\lim_{t\to\infty} r_{\pi^*}(t\boldsymbol{b},\boldsymbol{w}) = \lim_{t\to\infty} r^*(t\boldsymbol{b},\boldsymbol{w}) = \breve{r}. \tag{14}$$

□

Proof: Detailed proof is presented in [24]. Here, we summarize the outline of the proof for (12). Substituting (10) into (8), and applying Proposition 1, we obtain $r^*(\boldsymbol{b},\boldsymbol{w}) \leqslant r_1$, where r_1 represents the optimal objective value of the following optimization problem $\mathscr{P}_1$

$$\mathscr{P}_1: \underset{\{K,\{\boldsymbol{n}_k,\boldsymbol{u}_k\}_{k=1}^K\}}{\text{maximize}} \quad \frac{L}{|\boldsymbol{b}|}\sum_{k=1}^{K}\zeta_{\boldsymbol{u}_k}(\boldsymbol{n}_k,\boldsymbol{w})$$

$$\text{subject to} \quad \boldsymbol{n}_k \succcurlyeq \boldsymbol{0} \qquad \forall k = 1,2,\ldots,K$$

$$\sum_{k=1}^{K}\boldsymbol{n}_k \preccurlyeq \boldsymbol{b}.$$

In $\mathscr{P}_1$, vectors $\boldsymbol{n}_k$ and $\boldsymbol{u}_k$ represent the CQP number vector and the REE procedure, respectively, for the kth iteration of the REE policy. Note that we only need to consider efficient REE procedures in $\mathscr{P}_1$. Combining the iterations where the same REE procedure is used, we can show that r_1 equals the optimal objective value r_2 of problem $\mathscr{P}_2$ given by

$$\mathscr{P}_2: \underset{\{\breve{\boldsymbol{n}}^{(m)}\}_{m=1}^M}{\text{maximize}} \quad \frac{L}{|\boldsymbol{b}|}\sum_{m=1}^{M}\zeta_{\boldsymbol{u}^{(m)}}\big(\breve{\boldsymbol{n}}^{(m)},\boldsymbol{w}\big) \tag{15a}$$

$$\text{subject to} \quad \breve{\boldsymbol{n}}^{(m)} \succcurlyeq \boldsymbol{0} \qquad \forall m = 1,2,\ldots,M \tag{15b}$$

$$\sum_{m=1}^{M}\breve{\boldsymbol{n}}^{(m)} \preccurlyeq \boldsymbol{b}. \tag{15c}$$

Finally, r_2 can be shown to equal $\breve{r}$. Therefore, the desired result (12) is proved. ⊠

4. CONCLUSION

This paper characterized the optimal REE rate in repeater chains under fidelity requirements. Specifically, an asymptotically achievable upper bound on the optimal REE rate was derived and this bound was shown to be the solution to a linear program. Results in this paper provide guidelines for the design of signal processing techniques in quantum networks.

5. REFERENCES

[1] John Preskill, "Quantum computing in the NISQ era and beyond," *Quantum*, vol. 2, pp. 79, 2018.

[2] Jonathan P. Dowling and Gerard J. Milburn, "Quantum technology: the second quantum revolution," *Phil. Trans. R. Soc. Lond. A*, vol. 361, no. 1809, pp. 1655–1674, June 2003.

[3] Ryszard Horodecki, Paweł Horodecki, Michał Horodecki, and Karol Horodecki, "Quantum entanglement," *Rev. Mod. Phys.*, vol. 81, no. 2, pp. 865–942, June 2009.

[4] Stefano Marano and Moe Z. Win, "Distributing quantum states with finite lifetime," *Phys. Rev. A*, vol. 107, pp. 052413, May 2023.

[5] Marco Chiani, Andrea Conti, and Moe Z. Win, "Piggybacking on quantum streams," *Phys. Rev. A*, vol. 102, pp. 012410, July 2020.

[6] Gianfranco Cariolaro, *Quantum Communications*, Springer International Publishing, Switzerland, 2015.

[7] Stefano Marano and Moe Z. Win, "Distillation and swapping for quantum dragonflies," 2024, in preparation.

[8] Andrew S. Fletcher, Peter W. Shor, and Moe Z. Win, "Channel-adapted quantum error correction for the amplitude damping channel," *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 5705–5718, Dec. 2008.

[9] Peter W. Shor, "Fault-tolerant quantum computation," in *Proc. of 37th Annual Symposium on Foundations of Computer Science, IEEEPress*, Los Alamitos, CA, 1996, pp. 56–65.

[10] Andrew S. Fletcher, Peter W. Shor, and Moe Z. Win, "Structured near-optimal channel-adapted quantum error correction," *Phys. Rev. A*, vol. 77, pp. 012320, Jan. 2008.

[11] Stefano Guerrini, Moe Z. Win, Marco Chiani, and Andrea Conti, "Quantum discrimination of noisy photon-added coherent states," *IEEE J. Sel. Areas Inf. Theory*, vol. 1, no. 2, pp. 469–479, Aug. 2020, special issue on *Quantum Information Science*.

[12] Giacomo Mauro D'Ariano, Paoloplacido Lo Presti, and Matteo G. A. Paris, "Using entanglement improves the precision of quantum measurements," *Phys. Rev. Lett.*, vol. 87, no. 27, pp. 270404, Dec. 2001.

[13] Zixin Huang, Chiara Macchiavello, and Lorenzo Maccone, "Usefulness of entanglement-assisted quantum metrology," *Phys. Rev. A*, vol. 94, no. 1, pp. 012101, July 2016.

[14] Brian T. Kirby, Siddhartha Santra, Vladimir S. Malinovsky, and Michael Brodsky, "Entanglement swapping of two arbitrarily degraded entangled states," *Phys. Rev. A*, vol. 94, no. 1, pp. 012336, July 2016.

[15] Jian-Wei Pan, Dik Bouwmeester, Harald Weinfurter, and Anton Zeilinger, "Experimental entanglement swapping: Entangling photons that never interacted," *Phys. Rev. Lett.*, vol. 80, no. 18, pp. 3891–3894, May 1998.

[16] Marek Żukowski, Anton Zeilinger, Mike A. Horne, and Artur K. Ekert, "'Event-ready-detectors' Bell experiment via entanglement swapping," *Phys. Rev. Lett.*, vol. 71, no. 26, pp. 4287–4290, December 1993.

[17] Charles H. Bennett, Gilles Brassard, Sandu Popescu, Benjamin Schumacher, John A. Smolin, and William K. Wootters, "Purification of noisy entanglement and faithful teleportation via noisy channels," *Phys. Rev. Lett.*, vol. 76, no. 5, pp. 722–725, Jan. 1996.

[18] Liangzhong Ruan, Wenhan Dai, and Moe Z. Win, "Adaptive recurrence quantum entanglement distillation for two-Kraus-operator channels," *Phys. Rev. A*, vol. 97, pp. 052332, May 2018.

[19] Liangzhong Ruan, Brian T. Kirby, Michael Brodsky, and Moe Z. Win, "Efficient entanglement distillation for quantum channels with polarization mode dispersion," *Phys. Rev. A*, vol. 103, pp. 032425, Mar. 2021.

[20] Evgeny Shchukin and Peter van Loock, "Optimal entanglement swapping in quantum repeaters," *Phys. Rev. Lett.*, vol. 128, no. 15, pp. 150502, 2022.

[21] Suzanne B. van Dam, Peter C. Humphreys, Filip Rozpędek, Stephanie Wehner, and Ronald Hanson, "Multiplexed entanglement generation over quantum networks using multi-qubit nodes," *Quantum Science and Technology*, vol. 2, no. 3, pp. 034002, Sept. 2017.

[22] Jian Li, Mingjun Wang, Kaiping Xue, Ruidong Li, Nenghai Yu, Qibin Sun, and Jun Lu, "Fidelity-guaranteed entanglement routing in quantum networks," *IEEE Trans. Commun.*, vol. 70, no. 10, pp. 6748–6763, Oct. 2022.

[23] Wolfgang Dür, Hans J. Briegel, J. Ignacio Cirac, and Peter Zoller, "Quantum repeaters based on entanglement purification," *Phys. Rev. A*, vol. 59, no. 1, pp. 169–181, Jan. 1999.

[24] Zhenyu Liu, Stefano Marano, and Moe Z. Win, "Establishing high-fidelity entanglement in quantum repeater chains," *IEEE J. Sel. Areas Commun.*, 2024, to appear.