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CHARACTERIZATIONS OF BALLS AND ELLIPSOIDS BY INFINITESIMAL HOMOTHETIC CONDITIONS

Dedicated to the memory of Professor Delin REN

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Abstract We prove that for a smooth convex body $K \subset \mathbb{R}^d, d \geq 2$, with positive Gauss curvature, its homothety with a certain associated convex body implies that K is either a ball or an ellipsoid, depending on the associated body considered.

Keywords Busemann-Petty problem; convex bodies; dual mixed volumes; floating body; surface of centers

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1 Introduction

Characterizations of balls and ellipsoids among convex bodies in Euclidean space have been used effectively in major results in many areas of mathematics. Currently, a number of conjectured characterizations of balls and ellipsoids are still open and, in this paper, we prove several *infinitesimal versions* of such characterizations for smooth convex bodies with positive Gauss curvature. The meaning of this approach will be made precise shortly, but, for now, let us

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say that the characterizations will depend on the smallness of a parameter $\delta > 0$ representing, in a certain sense, the closeness of the characterization criteria to the boundary of a smooth, strictly convex body. In other words, in this paper, the emphasis is on the effect of the boundary structure on validating the open problems in a restrictive context. The conjectures remain open in larger generality not because the characterization criteria may not be valid farther away from the boundary, but rather because the techniques employed close to the boundary do not apply farther away from it.

One such open problem is whether constant area sections of a convex body tangent to a given ball contained in the body imply that the convex body is a ball. We consider an infinitesimal variant of this problem and, using it, provide a characterization of the ball which is, in some sense, supporting the conjecture. The approach has been prompted by some partial answers to the homothety conjecture [17] for floating bodies, one obtained very recently in [2], for which we will say a few words in Section 3, and another, much older, [18]. As we revisited the older argument in [18], we realized that one can provide some answers to similar questions related to other homothety type conjectures of affine invariant constructions which, to the best of our knowledge, if known, they have never been published. In Sections 3 and 4, we prove two such characterizations of ellipsoids implied by the homothety of some affine invariant constructions. In the first case, a homothety of very small factor between the boundary of a convex body and its surface of centers implies that K is an ellipsoid. In the last section, we propose a version of a classical affine invariant problem, the Busemann-Petty Problem 5.

In this paper, all homotheties between pairs of convex bodies $K_{1,2} \subset \mathbb{R}^d$ are with respect to a fixed point $O \in Int(K_1) \subseteq int(K_2)$, i.e. letting O be the origin of $\mathbb{R}^d$, the convex body K_1 is homothetic to the convex body K_2 if and only if there exists a positive real number λ such that $K_1 = \lambda K_2$. For each of these problems, it makes sense to consider the more general question of characterizing the convex bodies that admit the specific homothetic construction up to translation. This makes sense for convex bodies which do not have a center of symmetry, otherwise the center of homothety is the center of symmetry. This more general question is, however, not considered here.

2 On Convex Bodies with Sections of Constant Volume

A convex body is a compact convex set in $\mathbb{R}^d, d \geq 2$, with non-empty interior. By V_d we mean the usual Lebesgue measure and by V_k, $1 \leq k \leq d-1$, we understand the k-dimensional Hausdorff measure which coincides with the k-dimensional Lebesgue measure when the set lies in a k-dimensional affine subspace of $\mathbb{R}^d$.

In 2001, Barker and Larman proposed the following conjecture [5]: *Suppose that K_1, K_2, L are convex bodies in $\mathbb{R}^d$ with $L \subset int(K_1) \cap int(K_2)$. Assume that whenever $H \subset \mathbb{R}^d$ is a hyperplane supporting L, the $(d-1)$-volumes of $K_1 \cap H$ and $K_2 \cap H$ are equal. Then $K_1 = K_2$.*

One of the natural reference bodies to consider as L is, of course, a Euclidean ball and, so in the same paper, they proved:

Theorem 2.1 [5] Suppose that $K \subset \mathbb{R}^2$ is a convex body containing, in its interior, the Euclidean unit ball $\mathbf{B} \subset \mathbb{R}^2$. If the chords of K cut by the supporting lines of $\mathbf{B}$ have constant length, then K is a larger homothetic copy of $\mathbf{B}$.

In dimension $d \geq 3$, the conjecture remains open except for the case when K_1, K_2 are convex polytopes and L is a Euclidean ball, see [20]. Several other results were obtained by imposing extra assumptions such as special normalizations, [9], considering data over thick sections of convex bodies [21], and also imposing certain volume assumptions. Some other modifications of the problem were considered in [22]

Here, let $K \subset \mathbb{R}^d$, $d \geq 2$, be a $\mathcal{C}^\infty$ smooth, strictly convex body, containing the origin in its interior, and let $\delta > 0$ be some fixed positive real number assumed to be very small. The assumption on the smoothness of the boundary can be reduced to a lower order of differentiability, $\mathcal{C}^4$, which will be clear from the technique employed in proving the results.

For each $\xi \in \mathbb{S}^{d-1}$, denote by $H_{t(\xi)}(\xi)$ the hyperplane

$$H_{t(\xi)}(\xi) = \{x \in \mathbb{R}^d \mid x \cdot \xi = t(\xi)\},$$

where $t(\xi) > 0$ is the positive number, assuming that it exists, such that the tangent hyperplane to K at the point of outer normal ξ is translated inward until $H_{t(\xi)}$ is so that

$$V_{d-1}(K \cap H_{t(\xi)}(\xi)) = \delta.$$

Let $S_\delta = \cap_{\xi \in \mathbb{S}^{d-1}} H^-_{t(\xi)}(\xi)$ be the corresponding convex body obtained as the intersection of the half-spaces

$$H^-_{t(\xi)}(\xi) = \{x \in \mathbb{R}^d \mid x \cdot \xi \leq t(\xi)\}.$$

We call this convex body the *constant sections body* or the *body of δ-sections*.
The smallness of δ is assumed, for the moment, so that for each direction $\xi \in \mathbb{S}^{d-1}$, there exists a section of $(n-1)$-dimensional volume equal to δ.

Our first result is the following:

Theorem 2.2 Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a $\mathcal{C}^\infty$ smooth, convex body with positive Gauss curvature. If, for some small $\delta > 0$, we have that S_δ is homothetic to K with respect to a point in the interior of S_δ, then K is a Euclidean ball.

Proof As mentioned in the introduction, we take the origin of $\mathbb{R}^d$ to be the center of homothety for S_δ relative to K. The first step of the proof is to derive a relation between the support function of K and the support function of S_δ, both taken with respect to this choice of the origin, along all directions ξ, essentially estimating the distance between the supporting hyperplanes of normal ξ of the two convex bodies.

To do so, choose coordinates $x_1, x_2, \cdots, x_d$ in $\mathbb{R}^d$ such that $\{e_1, \cdots, e_{d-1}, \xi\}$ is an orthonormal basis of $\mathbb{R}^d$ and the supporting point, $\{q\} := H_\xi \cap \partial K$, lies, momentarily, at the origin of this system of coordinates.

Then, ∂K is locally a graph in these coordinates,

$$x_d = -\frac{1}{2}\sum_{i,j=1}^{d-1} b_{ij} x_i x_j + o(|x|^2), \tag{2.1}$$

where b_{ij} is the second fundamental form of ∂K at the point q of outer normal ξ, and $f = o(s)$ means $f/s \to 0$ as $s \to 0$.

Moreover, there exists a special linear transformation $T \in SL(d)$ of the form $T = \begin{pmatrix} T' & 0 \\ 0 & 1 \end{pmatrix}$, with $T' \in SL(d-1)$, that fixes ξ, preserves the $(d-1)$-dimensional volume in any hyperplane of normal ξ, and brings ∂K locally to the form

$$x_d = -\frac{1}{2}\mathcal{K}^{1/(d-1)}(q)\sum_{i=1}^{d-1} x_i^2 + o(|x|^2), \tag{2.2}$$

where $\mathcal{K}(q) = \det[(b_{ij})_{ij}]$ is the Gauss curvature of ∂K at q as the determinant of the coefficients of the second fundamental form at that point. Thereafter, we consider $\mathcal{K}$ as a function of ξ as q is the unique point of the boundary of K of outer normal ξ. With this parametrization, known as the inverse Gauss map parametrization, it is useful to consider the support function h_K of K to represent the boundary of K:

$$h_K : \mathbb{S}^{d-1} \to (0,\infty), \quad h_K(\xi) = \sup_{z\in K} z\cdot\xi = q\cdot\xi.$$

Thus, if $t = t(\xi)$ is defined as above, we have

$$\delta = V_{d-1}(K\cap H_t(\xi)) = \left(\frac{2\mid h_K(\xi)-t\mid}{\sqrt[d-1]{\mathcal{K}(\xi)}}\right)^{\frac{d-1}{2}}\omega_{d-1} + o\left(|h_K(\xi)-t|^{\frac{d-1}{2}}\right), \tag{2.3}$$

where ω_k is the volume of the unit ball in $\mathbb{R}^k$.

This leads to a description of the support function with respect to the center of homothety, which we set as origin for this purpose, of the constant sections body in an arbitrary direction $\xi \in \mathbb{S}^{d-1}$ in terms of the support function of the original convex body in the direction ξ:

$$h_{S_\delta}(\xi) = h_K(\xi) - c_d\,\delta^{\frac{2}{d-1}}\,\sqrt[d-1]{\mathcal{K}}(\xi) + o(\delta^{\frac{2}{d-1}}), \tag{2.4}$$

where $c_d = (2\omega_{d-1}^{2/(d-1)})^{-1}$ is a constant depending on the dimension, and all coefficients in the asymptotic expansion are $\mathcal{C}^\infty$ smooth.

Recall now that, by hypothesis, S_δ is homothetic to K, so there exists a $\lambda\in(0,1)$, close to one in this case when δ is close to zero, such that $S_\delta = \lambda K$.

To estimate the homothety factor λ such that $S_\delta = \lambda K$, we note that $S_\delta^\circ = \frac{1}{\lambda}K^\circ$, so we relate λ to the volume ratio of their polar bodies via homogeneity

$$\lambda^{-d} = \frac{V_d(S_\delta^\circ)}{V_d(K^\circ)}.$$

Recall that the polar body of a convex body L containing the origin is the convex body L° defined by

$$L^\circ = \{y\in\mathbb{R}^d \mid x\cdot y \le 1,\ \forall x\in L\}.$$

It follows from the equation (2.4) above that

$$V_d(S_\delta^\circ) = \frac{1}{d}\int_{\mathbb{S}^{d-1}}\frac{1}{h_{S_\delta}^d(\xi)}\,\mathrm{d}\mu(\xi) = V_d(K^\circ) + c_d\delta^{\frac{2}{d-1}}\int_{\mathbb{S}^{d-1}}\frac{\sqrt[d-1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,\mathrm{d}\mu(\xi) + o(\delta^{\frac{2}{d-1}}). \tag{2.5}$$

Thus

$$\lambda^{-d} = 1 + c_d\,\delta^{\frac{2}{d-1}}\frac{1}{V_d(K^\circ)}\int_{\mathbb{S}^{d-1}}\frac{\sqrt[d-1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,\mathrm{d}\mu(\xi) + o(\delta^{\frac{2}{d-1}}), \tag{2.6}$$

and, so,

$$\lambda = 1 - c_d\delta^{\frac{2}{d-1}}\frac{1}{d\,V_d(K^\circ)}\int_{\mathbb{S}^{d-1}}\frac{\sqrt[d-1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,\mathrm{d}\mu(\xi) + o(\delta^{\frac{2}{d-1}}). \tag{2.7}$$

Therefore, using (2.4) and the homothety $h_{S_\delta}(\xi)=\lambda h_K(\xi)$, we have, for each $\xi\in\mathbb{S}^{d-1}$,

$$c_d\,\delta^{\frac{2}{d-1}}\,\frac{h_K(\xi)}{d\,V_d(K^\circ)}\int_{\mathbb{S}^{d-1}}\frac{\sqrt[d-1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,\mathrm{d}\mu(\xi)=c_d\,\delta^{\frac{2}{d-1}}\,\sqrt[d-1]{\mathcal{K}}+o(\delta^{\frac{2}{d-1}}), \tag{2.8}$$

and, consequently,

$$\frac{h_K(\xi)}{V_d(K^\circ)}=\frac{\sqrt[d-1]{\mathcal{K}}(\xi)}{\frac{1}{d}\int_{\mathbb{S}^{d-1}}\frac{\sqrt[d-1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,\mathrm{d}\mu(\xi)}+o(1), \tag{2.9}$$

where $o(1)$ is an error term that goes to zero when δ goes to zero.

Recall now a few elements of the dual mixed volume theory. For any two star bodies $L_{1,2}$ in $\mathbb{R}^d$ containing the origin, and any $i\in\mathbb{R}$, the i-th dual mixed volume of $L_{1,2}$ is defined (see [13]) by

$$\tilde{V}_i(L_1,L_2)=\frac{1}{d}\int_{\mathbb{S}^{d-1}}\rho_{L_1}^{d-i}\rho_{L_2}^{i}(\xi)\,\mathrm{d}\mu(\xi),$$

where $\rho_{L_j}:\mathbb{S}^{d-1}\to(0,\infty)$ denotes the radial function of the star body L_j, $j=1,2$. Of particular interest for us is the case $i=-1$ for which it follows, via Hölder inequality, the mixed volume inequality

$$\tilde{V}_{-1}^{d}(L_1,L_2)\geq V_d^{d+1}(L_1)V_d^{-1}(L_2).$$

Equality is reached above if and only if L_1 and L_2 are dilates of each other and so the radial functions ρ_{L_1} and ρ_{L_2} are multiples of each other. Therefore, given that the reciprocal of the support function of a convex body K containing the origin is the radial function of its polar body, ρ_{K°, equation (2.9) can be re-written in terms of a dual mixed volume and radial functions as follows

$$\frac{\rho_{K^\circ}^{-1}(\xi)}{V_d(K^\circ)}=\frac{\rho_{L^\circ}^{-1}(\xi)}{\tilde{V}_{-1}(K^\circ,L^\circ)}+o(1), \tag{2.10}$$

where L° is the star body whose radial function is $1/\sqrt[d-1]{\mathcal{K}}$.

Due to the dual mixed volume inequality above, if we assume that L° is not a dilation of K°, the inequality is strict and (2.10) implies, after re-arranging terms, that

$$\frac{\rho_{L^\circ}(\xi)}{V_d(L^\circ)^{\frac{1}{d}}}<\frac{\rho_{K^\circ}(\xi)}{V_d(K^\circ)^{\frac{1}{d}}}+o(1). \tag{2.11}$$

If δ is very small, then

$$\frac{\rho_{L^\circ}(\xi)}{V_d(L^\circ)^{\frac{1}{d}}}\leq\frac{\rho_{K^\circ}(\xi)}{V_d(K^\circ)^{\frac{1}{d}}} \tag{2.12}$$

holds and we have two star bodies of volume 1, namely $\frac{1}{V_d^{1/d}(K^\circ)}K^\circ$ and $\frac{1}{V_d^{1/d}(L^\circ)}L^\circ$, such that one is enclosed in the other. Unless the two bodies coincide, this is impossible.

Therefore, equality holds in (2.12) and L° is homothetic to K°. Thus, there exists a $\lambda'>0$ such that $\rho_{K^\circ}(\xi)=\lambda'\rho_{L^\circ}(\xi)$ for all $\xi\in\mathbb{S}^{d-1}$. This means that the convex body K is such that its support and curvature functions satisfy pointwise on the unit sphere the equation

$$\lambda' h_K(\xi)=\sqrt[d-1]{\mathcal{K}}(\xi),\quad\forall\xi\in\mathbb{S}^{d-1}. \tag{2.13}$$

The fact that a convex body K satisfying (2.13) must be a Euclidean ball is precisely the conclusion of the second part of Theorem 1 in [12], concluding also our proposition. □

3 On the Convex Body Bounded by the Surface of Centers Homothetic to K for Small δ

Let $\xi \in \mathbb{S}^{d-1}$ be a unit vector, and let

$$H_t^-(\xi) = \{x \in \mathbb{R}^d \mid x \cdot \xi \le t\}$$

be the half-space of outer normal ξ whose boundary lies at distance t from the origin. Let K be a smooth strictly convex body in $\mathbb{R}^d$ and let $\delta > 0$ be a fixed positive constant.

Recall that the convex floating body K_δ of K (see [4, 17]), if it exists, is defined as the envelope of half-spaces cutting a cap of volume δ from K:

$$K_\delta = \bigcap_{\xi \in \mathbb{S}^{d-1}} H_{t(\xi)}^-(\xi),$$

where $t(\xi)$ is such that

$$V_d(K \cap H_t^+(\xi)) = \delta.$$

The convex floating body is the subject of a homothety conjecture asserting that K homothetic to K_δ, possibly up to a translation (as mentioned in the introduction), implies that K is an ellipsoid. It is expected that the center of homothety is the center of mass of K, and so, the two bodies should be dilations of each other relative to their respective centers of mass. This is obvious in the case of centrally symmetric convex bodies K. However, otherwise, it is still an open problem to find the relation between the centers of mass of the two bodies, see Problem 3 from the list of open problems in [3] and the discussion surrounding it.

Lastly, let us mention two recent partial results on the homothety conjecture for the convex floating body. It was shown in [2] that, in the plane, the homothety conjecture holds if K is centrally-symmetric and close to a Euclidean ball in the Banach-Mazur distance, but that the homothety conjecture is not true for planar convex bodies that are not centrally-symmetric. In its larger generality, the conjecture remains open for centrally-symmetric convex bodies.

The surface of centers, [8], is the locus of centers of mass of the caps cut off from K in defining the floating body

$$x_{C_\delta}(\xi) = \frac{1}{\delta} \int_{K \cap H_t^+(\xi)} x \, \mathrm{d}x.$$

This surface, also called surface of buoyancy, is of uttermost significance in studying the flotation of 3-dimensional objects in a liquid of constant density. In particular, if C_δ is a sphere, then K floats in equilibrium in every direction. The tangent plane to C_δ at $x_{C_\delta}(\xi)$ has normal ξ and is, thus, parallel to the hyperplane bounding the half-space $H_{t(\xi)}^-(\xi)$. There are many other properties of this surface that connect both with the practical aspect of the problem and with many geometric questions, see, for example, [15] for a recent result related to Ulam's 19th problem in the Scottish book and [3] for a survey touching on many related questions.

We will prove that:

Proposition 3.1 Let $K \subset \mathbb{R}^d$, $d \ge 2$, be a $\mathcal{C}^\infty$ smooth, convex body with positive Gauss curvature. If, for some small $\delta > 0$, we have that the convex body of boundary C_δ is homothetic to K with respect to a point in the interior of the domain bounded by C_δ, then K is an ellipsoid.

We want to point out that, using spherical harmonic analysis methods, Reuter proved the equivalent result for $\delta = V_d(K)/2$, and origin-symmetric K close to the Euclidean ball in the Banach-Mazur distance, without *a priori* smoothness or curvature assumptions, see [14]. For the value of $\delta = V_d(K)/2$, we note that the surface of centers is the boundary of the centroid ΓK of K. Without getting into specific details, let us still mention that in [14], Reuter also extended his result to prove that, for any $p > 1$, the p-centroid body $\Gamma_p(K)$ homothetic to K for origin-symmetric K close to the Euclidean ball implies that K is an ellipsoid. Reuter's results, [2], and Ryabogin's [15] suggest that the central-symmetry assumption on K is essential for this type of homothety problems.

Proof The relation between the support functions of K and K_δ for small δ was derived in [18], in a similar manner to the one in the previous section for S_δ. As in the preceding section, in the calculation of the support function in the direction ξ, we may assume a linear transformation that does not affect the directions parallel to ξ in $\mathbb{R}^d$, so that the boundary of K is locally approximated by

$$x_d = -\frac{1}{2}\mathcal{K}^{1/(d-1)}(\xi)\sum_{i=1}^{d-1} x_i^2 + o(|x|^2). \tag{3.1}$$

This representation form assumes that the point on ∂K with support plane of normal ξ is, momentarily, the origin. Then, the difference between the two support functions amounts to estimating the height of the slab $K \cap H_t^+$ of volume δ, and the result is as follows:

$$h_{K_\delta}(\xi) = h_K(\xi) - c_d\,\mathcal{K}^{\frac{1}{d+1}}(\xi)\,\delta^{\frac{2}{d+1}} + o(\delta^{\frac{2}{d+1}}), \tag{3.2}$$

where $c_d := \left(2^{\frac{d-1}{d+1}}\omega_{d-1}^{\frac{2}{d+1}}\right)^{-1}$ is a constant depending on the dimension.

Furthermore, as the surface of centers is known to be the locus of the centers of mass of the slices $K \cap H_t^+(\xi)$ of constant volume, the smallness of δ enables us to derive the relation between the support functions of K and C_δ up to some small error term.

The fact that the center of mass of the cap is also the point of support of C_δ for the hyperplane of outer normal ξ, implies that solely the x_d component, in the sense of the previous approximation, of the center of mass is relevant. Hence,

$$\begin{aligned}
h_{C_\delta}(\xi) &= \frac{1}{\delta}\int_{h_{K_\delta}(\xi)}^{h_K(\xi)} t\,V_{d-1}\left(K\cap H_t(\xi)\right)\,\mathrm{d}t\\
&= \frac{\omega_{d-1}}{\delta}\int_{h_{K_\delta}(\xi)}^{h_K(\xi)}\left[t\left[\frac{2\left(h_K(\xi)-t\right)}{\sqrt[d-1]{\mathcal{K}(\xi)}}\right]^{\frac{d-1}{2}} + o\left(\left(h_K(\xi)-t\right)^{\frac{d+1}{2}}\right)\right]\mathrm{d}t\\
&= h_K(\xi) - \frac{\omega_{d-1}}{\delta}\left[\frac{2^{\frac{d-1}{2}}}{\sqrt{\mathcal{K}(\xi)}}\int_{h_{K_\delta(\xi)}}^{h_K(\xi)}(h_K(\xi)-t)^{\frac{d+1}{2}}\,\mathrm{d}t + o\left(\left(h_K(\xi)-h_{K_\delta}(\xi)\right)^{\frac{d+3}{2}}\right)\right]\\
&= h_K(\xi) - \frac{\omega_{d-1}}{\delta}\left[\frac{2^{\frac{d-1}{2}}}{\sqrt{\mathcal{K}(\xi)}}\left[\left(h_K(\xi)-h_{K_\delta}(\xi)\right)^{\frac{d+3}{2}}\cdot\frac{2}{d+3}\right] + o\left(\left(h_K(\xi)-h_{K_\delta}(\xi)\right)^{\frac{d+3}{2}}\right)\right]\\
&= h_K(\xi) - \frac{1}{\delta}\,\frac{c_d^{-\frac{d+1}{2}}}{\sqrt{\mathcal{K}(\xi)}}\left[\left(c_d\mathcal{K}^{\frac{1}{d+1}}(\xi)\delta^{\frac{2}{d+1}}\right)^{\frac{d+3}{2}}\cdot\frac{2}{d+3}\right] + o\left(\delta^{\frac{2}{d+1}}\right),
\end{aligned}$$

or,

$$h_{C_\delta}(\xi) = h_K(\xi) - \frac{2}{d+3}\,c_d(\mathcal{K}(\xi))^{\frac{1}{d+1}}\delta^{\frac{2}{d+1}} + o\left(\delta^{\frac{2}{d+1}}\right), \tag{3.3}$$

where c_d is the same constant as in the description of the support function of K_δ relative to the support function of K of [18].

For simplicity, in what follows, we will be using C_δ to also denote the convex body whose boundary is the surface of centers. In order to describe the homothety factor λ such that $C_\delta = \lambda K$, we will proceed as in the previous section and use polar bodies

$$\lambda^{-d} = \frac{V_d(C_\delta^\circ)}{V_d(K^\circ)}$$

from which it follows in a similar manner that

$$V_d(C_\delta^\circ) = \frac{1}{d}\int_{\mathbb{S}^{d-1}} \frac{1}{h_{C_\delta}^d(\xi)}\,\mathrm{d}\mu(\xi) = V_d(K^\circ) + \frac{2}{d+3}\, c_d \delta^{\frac{2}{d+1}} \int_{\mathbb{S}^{d-1}} \frac{\sqrt[d+1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,\mathrm{d}\mu(\xi) + o(\delta^{\frac{2}{d+1}}). \quad (3.4)$$

Consequently,

$$\lambda = 1 - c_d \delta^{\frac{2}{d+1}}\, \frac{2}{d+3}\, \frac{1}{d\, V_d(K^\circ)} \int_{\mathbb{S}^{d-1}} \frac{\sqrt[d+1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,\mathrm{d}\mu(\xi) + o(\delta^{\frac{2}{d+1}}). \quad (3.5)$$

The homothety $h_{C_\delta}(\xi) = \lambda h_K(\xi)$ and (3.3) imply that, for each $\xi \in \mathbb{S}^{d-1}$,

$$c_d\, \delta^{\frac{2}{d+1}}\, \frac{h_K(\xi)}{d\, V_d(K^\circ)} \int_{\mathbb{S}^{d-1}} \frac{\sqrt[d+1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,\mathrm{d}\mu(\xi) = c_d\, \delta^{\frac{2}{d+1}}\, \sqrt[d+1]{\mathcal{K}} + o(\delta^{\frac{2}{d+1}}), \quad (3.6)$$

and, thus,

$$\frac{h_K(\xi)}{V_d(K^\circ)} = \frac{\sqrt[d+1]{\mathcal{K}}(\xi)}{\frac{1}{d}\int_{\mathbb{S}^{d-1}} \frac{\sqrt[d+1]{\mathcal{K}}(\xi)}{h_K^{d+1}(\xi)}\,\mathrm{d}\mu(\xi)} + o(1), \quad (3.7)$$

from which the conclusion follows exploiting the dual Minkowski theory in the same way as before. □

Note the similarity between the expressions of the support functions of K_δ and, respectively, C_δ in (3.3) which prompts us to state the following proposition regarding the affine surface area of K:

$$\Omega(K) = \int_{\mathbb{S}^{d-1}} \mathcal{K}^{\frac{1}{d+1}}(\xi)\,\mathrm{d}S_K(\xi) = \int_{\mathbb{S}^{d-1}} \mathcal{K}^{-\frac{d}{d+1}}(\xi)\,\mathrm{d}\mu(\xi), \quad (3.8)$$

where $\mathrm{d}S_K$ is the surface area measure of K viewed as a Borel measure defined on the unit sphere. It is likely that the next result is known, but we could not find it stated in the literature.

Lemma 3.2 Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a $\mathcal{C}^\infty$ smooth, convex body with positive Gauss curvature and let C_δ denote the surface of centers of K for δ in some small interval $(0, \delta_0)$. Then, the affine surface area of K, $\Omega(K)$, satisfies

$$\Omega(K) = \frac{d+3}{2c_d} \lim_{\delta \searrow 0} \frac{V_d(K) - V_d(C_\delta)}{\delta^{\frac{2}{d+1}}}. \quad (3.9)$$

Proof Let f be a continuous function on the unit sphere $\mathbb{S}^{d-1}$. Suppose that f defines a perturbation of a convex body L containing the origin in its interior to be the convex body, denoted L_t, of support function $h_t(u) = h_L(u) + tf(u), \forall u \in \mathbb{S}^{d-1}$, where $t \in (-\delta, \delta)$ is taken to be very small so that h_t remains positive. Aleksandrov, [1], showed that the following variational formula holds

$$\frac{\mathrm{d}V_d(L_t)}{\mathrm{d}t} = \lim_{t \to 0} \frac{V_d(L_t) - V_d(L)}{t} = \int_{\mathbb{S}^{d-1}} f(u)\,\mathrm{d}S_L(u). \quad (3.10)$$

Without getting into details, we mention that this variational formula can be made sense of even if h_t is not a support function. To do so, we define the convex body L_t via its support function $h_{L_t} = \sup\{h : \mathbb{S}^{d-1} \to \mathbb{R} \mid h \text{ is support function}, h \leq h_t \text{ pointwise}\}$, and use that $h_{L_t} = h_t$ a.e. with respect to the surface area measure of L_t, [1].

Additionally, it is a known fact that if $h_t \to h$ with respect to the sup-norm on $C(S^{d-1})$, then $K_t \to K$ in the Hausdorff metric and $\mathrm{d}S(K_t)$ converges weakly to $\mathrm{d}S_K$, [11, 16].

Note that our set-up is $h_{K_t} = h_K + tf + o(t)$ where $t = \delta^{\frac{2}{d-1}} > 0$ and f is a power of Gauss curvature of K viewed as a function on the unit sphere $\mathbb{S}^{d-1}$. It was noticed, in particular by Leichtweiss, that Aleksandrov variational formula holds in this case too, [11, 12], so (3.9) is a direct consequence of (3.3).

We mentioned Leichtweiss work in particular as he was also aware early on that such deformations can be used for other functionals. Nowadays, this idea is widespread, note its applications to the log-Miknkowski problem [6], the chord problem [19], and other variational problems, [10], to mention just a few. □

Remark 3.3 This proposition also suggests that, up to a constant depending on dimension, the affine surface area $\Omega(K)$ of a convex body K, not necessarily smooth, can be defined as the limit above via its surfaces of centers.

4 Infinitesimal Busemann-Petty 5

In a seminal paper, Busemann and Petty [7] proposed to the larger community working on convex bodies a list of ten problems from which all but one are still open today. One of the them is known as Busemann-Petty 5 (BP5) and pertains to the following construction. Let K be a convex body in $\mathbb{R}^d$ containing the origin in its interior. For each unit vector $\xi \in \mathbb{S}^{d-1}$, consider the section $K(\xi) := K \cap H(\xi)$, where $H(\xi)$ stands for the hyperplane of normal ξ passing through the origin. Let $C(\xi)$ be the cone of base $K(\xi)$ and apex in K that has maximal volume. It is obvious that the apex of the cone is a point of support hyperplane of (outer) normal ξ, where we have chosen the outer for the sake of considering the problem for convex bodies that may not be centrally symmetric and, thus, we follow continuously a fixed orientation.

Problem BP5 asks if ellipsoids are the only convex bodies K characterized by the property that $C(\xi)$ has constant volume, independent of ξ, see [7].

Here, for any smooth, convex body K with positive Gauss curvature, we propose to fix a small constant $\delta > 0$ and look at the cones $C_{t(\xi)}(\xi)$ with apex at the point of support hyperplane of outer normal ξ, and whose base is a non-central section of K denoted $K(t(\xi), \xi) := K \cap H_{t(\xi)}(\xi)$, where

$$H_{t(\xi)}(\xi) = \{x \in \mathbb{R}^d \mid x \cdot \xi = t(\xi)\}$$

is taken so that the value of the d-dimensional volume of the cone, $V_d(C_{t(\xi)}(\xi)) = \delta$, is equal to δ. The envelope of the half-spaces $H^-_{t(\xi)}(\xi)$ forms a convex body denoted by

$$K_\delta^{BP} := \bigcap_{\xi \in \mathbb{S}^{d-1}} H^-_{t(\xi)}(\xi).$$

We prove the following:

Proposition 4.1 Let $K \subset \mathbb{R}^d$, $d \geq 2$, be a $\mathcal{C}^\infty$ smooth, convex body with positive Gauss curvature containing the origin in its interior. If, for some small $\delta > 0$, we have that K_δ^{BP} is homothetic to K with respect to the origin, then K is an ellipsoid.

Proof To describe the height of cones with constant volume δ, fix a direction ξ and letting $t = t(\xi)$ note, assuming again the previous type approximation of the boundary of K near the point of outer normal ξ, that

$$\delta = \frac{1}{d} \mid h_K(\xi) - t \mid \cdot V_{d-1}(K \cap H_t(\xi)) = \frac{1}{d} \mid h_K(\xi) - t \mid \cdot \left(\frac{2 \mid h_K(\xi) - t \mid}{\sqrt[d-1]{\mathcal{K}(\xi)}} \right)^{\frac{d-1}{2}} \omega_{d-1} + o\left(|h_K(\xi) - t|^{\frac{d+1}{2}} \right),$$

and solve for t which is in fact $h_{K_\delta^{BP}}$. This leads to

$$h_{K_\delta^{BP}}(\xi) = h_K(\xi) - d^{\frac{2}{d+1}} c_d \mathcal{K}^{\frac{1}{d+1}}(\xi) \delta^{\frac{2}{d+1}} + o\left(\delta^{\frac{2}{d+1}}\right), \tag{4.1}$$

with the same constant c_d as before. Note the striking similarity with the support function of the floating body K_δ, hence we note on the side the following outcome:

Lemma 4.2 For any K smooth, strictly convex body in $\mathbb{R}^d$ with positive Gauss curvature, we have

$$\Omega(K) = \frac{1}{d^{\frac{2}{d+1}} c_d} \lim_{\delta \searrow 0} \frac{V_d(K) - V_d(K_\delta^{BP})}{\delta^{\frac{2}{d+1}}}, \tag{4.2}$$

where K_δ^{BP} is defined as above for some interval $(0, \delta_0)$.

The proof of Proposition 4.1 follows the same reasoning as the proof of the proposition of the previous section, and same for the proof of the lemma, precisely because of the same type asymptotics where only the constants of the Gauss curvature term differ. First, one gathers information about the factor of homothety using the ratio of volumes of the polars of the two convex bodies, and the fact that Gauss curvature of K raised to power $1/(d+1)$ can be interpreted as a radial function of a star body, concluding the proof in identical manner. □

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Conflict of Interest The authors declare no conflict of interest.

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