TWO RESULTS ON THE HOMOTHETY CONJECTURE FOR CONVEX BODIES OF FLOTATION ON THE PLANE

M. ANGELES ALFONSECA, FEDOR NAZAROV, DMITRY RYABOGIN, ALINA STANCU, AND VLADYSLAV YASKIN

Abstract. We investigate the homothety conjecture for convex bodies of flotation of planar domains close to the unit disk B. We show that for every density $\mathcal{D} \in (0, \frac{1}{2})$, there exists $\gamma = \gamma(\mathcal{D}) > 0$ such that if $(1-\gamma)B \subset K \subset (1+\gamma)B$ and the convex body of flotation $K^{\mathcal{D}}$ of an origin symmetric body K of density $\mathcal{D}$ is homothetic to K, then K is an ellipse. On the other hand, we also show that if the symmetry assumption is dropped, then there is an infinite set of densities accumulating at $\frac{1}{2}$ for which there is a body K different from an ellipse with the property that $K^{\mathcal{D}}$ is homothetic to K.

1. Introduction

Let K be a body in $\mathbb{R}^2$, i.e., $K \neq \varnothing$, K is compact, the interior of K is connected, and K is the closure of its interior. For every $\theta \in \mathbb{R}$ and the corresponding unit vector $e(\theta) = (\cos\theta, \sin\theta)$ and for every $t \in \mathbb{R}$, define the half-planes

$$W^+(\theta, t) = \{x : \langle x, e(\theta)\rangle \geq t\} \quad \text{and} \quad W^-(\theta, t) = \{x : \langle x, e(\theta)\rangle \leq t\}.$$

If $0 < \mathcal{D} < 1$, then for every $\theta \in \mathbb{R}$, there is a unique $t(\theta)$ such that

$$\mathrm{vol}_2(W^+(\theta, t(\theta)) \cap K) = \mathcal{D}\,\mathrm{vol}_2(K).$$

The corresponding convex body of flotation $K^{\mathcal{D}}$ is defined as

$$K^{\mathcal{D}} = \bigcap_{\theta\in\mathbb{R}} W^-(\theta, t(\theta)).$$

Note that $K^{\mathcal{D}} = \varnothing$ for all $\mathcal{D} \in (\frac{1}{2}, 1)$. The body $K^{\mathcal{D}}$ can be viewed as the set of points that stay above the water level when a solid with shape K of uniform density $\mathcal{D}$ floats in any orientation. For technical reasons, it will be more convenient for us to view $K^{\mathcal{D}}$ as the intersection of half-planes bounded by the lines cutting from K a fixed area $\delta \in (0, \mathrm{vol}_2(K))$ as it is usually done in the literature on convex bodies of flotation (also known as "floating bodies", see [2] and [5]). In this case, we shall use the notation K_δ. We obviously have $K_\delta = K^{\mathcal{D}}$ for $\delta = \mathcal{D}\,\mathrm{vol}_2(K)$.

Key words and phrases. Convex bodies of flotation, homothety conjecture.

The first author is supported in part by the Simons Grant MPS-TSM-00711907. The second and the third authors are supported in part by U.S. National Science Foundation Grants DMS-1900008 and DMS-2247771. The fourth and the fifth authors are supported by NSERC RGPIN 2023-03926 and NSERC RGPIN 2024-05110.

The homothety conjecture in $\mathbb{R}^2$ says that if a convex body is homothetic to one of its convex bodies of flotation, then it is an ellipse. To the best of our knowledge, the question was first raised in 1994 (see [4]). The homothety condition means that $K^{\mathcal{D}} = \lambda K$ for some $\mathcal{D} \in (0, 1)$, $\lambda > 0$. While the full homothety conjecture looks too strong to be true (we will show that it is actually false in $\mathbb{R}^2$), one can also consider various restricted versions of it, imposing additional assumptions on K, $\mathcal{D}$, and/or λ.

In this paper we will prove two theorems. The first one, roughly speaking, says that on the plane the homothety conjecture holds for origin symmetric convex bodies in a small neighborhood of the unit disk. More precisely, we have

Theorem 1.1. *For every compact interval $I \subset (0, \frac{1}{2})$, there is $\gamma > 0$ such that if $K^{\mathcal{D}} = \lambda K$ for some $\mathcal{D} \in I$ and $\lambda > 0$, $K \subset \mathbb{R}^2$ is origin symmetric, and $(1-\gamma)B \subset K \subset (1+\gamma)B$, then K is an ellipse.*

Here $B = \{x \in \mathbb{R}^2 : |x| \le 1\}$ is the unit disk.

We remark that instead of restricting the density $\mathcal{D}$ away from its extreme values, we can just as well restrict the homothety coefficient λ to a compact subinterval $J \subset (0, 1)$ in this theorem. Also, since the problem is affine-invariant, the condition $(1-\gamma)B \subset K \subset (1+\gamma)B$ can be replaced by the condition that the Banach-Mazur distance from K to B is less than γ at no extra cost.

The second theorem shows that in the asymmetric case, the full homothety conjecture fails rather drastically, at least on the plane.

Theorem 1.2. *The equation $K^{\mathcal{D}} = \lambda K$ has infinitely many affinely non-equivalent asymmetric convex solutions $K \subset \mathbb{R}^2$. Moreover, these solutions can be chosen as small perturbations of the unit disk with the corresponding densities $\mathcal{D}$ and homothety coefficients λ accumulating at $\frac{1}{2}$ and 0 respectively.*

2. Quasi-differentiability properties of the mapping $\rho_K \mapsto \rho_{K_\delta}$.

We shall consider the homothety problem in the class of the star-shaped (but not necessarily convex!) bodies $K \subset \mathbb{R}^2$ with continuous radial functions

$$\rho(\theta) = \rho_K(\theta) = \max\{t \ge 0 : te(\theta) \in K\}.$$

For the mapping $\rho_K \mapsto \rho_{K_\delta}$, which we will abbreviate to $\rho \mapsto \rho_\delta$, the homothety condition is equivalent to the equation $\lambda\rho - \rho_\delta = 0$. Note that this equation holds for the unit ball B ($\rho \equiv 1$) with any $\delta \in (0, \frac{\pi}{2})$ and $\lambda = \cos\alpha$, where $\alpha \in (0, \frac{\pi}{2})$ is the angle for which the shaded disk segment $B \cap W^+(\theta, \cos\alpha)$ on Figure 1 spanned by a circular arc of length 2α has area δ.

We shall show that when K is sufficiently close to the unit disk, the mapping $\rho \mapsto \rho_\delta$ is quasi-differentiable with the quasi-differential equal to

$$\Delta\rho \quad \mapsto \quad \frac{1}{2\sin\alpha} \int\limits_{\theta-\alpha}^{\theta+\alpha} \Delta\rho(\tau) d\tau.$$

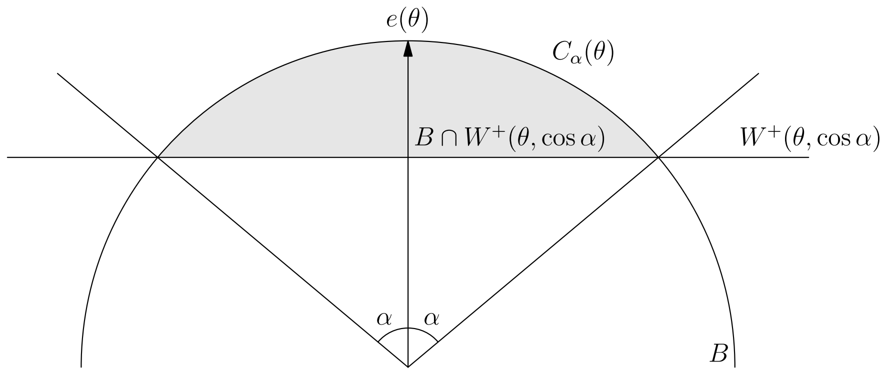

FIGURE 1. Relation between α and δ.

The latter means that for any continuous 2π-periodic ρ_1, ρ_2 close to 1, we have

$$\rho_{1,\delta}(\theta) - \rho_{2,\delta}(\theta) \approx \frac{1}{2\sin\alpha} \int_{\theta-\alpha}^{\theta+\alpha} (\rho_1(\tau) - \rho_2(\tau))d\tau \tag{1}$$

with an error whose size is substantially smaller than the size of $\rho_1 - \rho_2$.

The idea of the proof is very simple: given $\delta \in (0, \frac{\pi}{2})$ and two star-shaped bodies K_1 and K_2 close to the unit disk, for every $\theta \in \mathbb{R}$, define $t_j(\theta)$ by

$$\mathrm{vol}_2(K_j \cap W^+(\theta, t_j(\theta))) = \delta, \qquad j = 1, 2. \tag{2}$$

Then, up to a small boundary effect, the difference

$$\mathrm{vol}_2(K_1 \cap W^+(\theta, t_2(\theta))) - \mathrm{vol}_2(K_2 \cap W^+(\theta, t_2(\theta)))$$

is determined by the difference of the boundaries of K_1 and K_2 in the angle

$$C_\alpha(\theta) = \{x \in \mathbb{R}^2 : \angle(x, e(\theta)) \leq \alpha\}.$$

So in the first order approximation, this difference is $\int_{\theta-\alpha}^{\theta+\alpha} (\rho_1(\tau) - \rho_2(\tau))d\tau$. To compensate for this difference, we need to move t away from $t_2(\theta)$. Since all cross-sections at the relevant levels are close to those of the unit disk, moving t by Δt units changes the area $\mathrm{vol}_2(K_1 \cap W^+(\theta, t))$ by approximately $-2\sin\alpha\,\Delta t$. Thus, to get $\mathrm{vol}_2(K_1 \cap W^+(\theta, t)) = \delta$, we need to increase $t_2(\theta)$ by the amount $\frac{1}{2\sin\alpha}\int_{\theta-\alpha}^{\theta+\alpha} (\rho_1(\tau) - \rho_2(\tau))d\tau$, i.e.,

$$t_1(\theta) - t_2(\theta) \approx \frac{1}{2\sin\alpha} \int_{\theta-\alpha}^{\theta+\alpha} (\rho_1(\tau) - \rho_2(\tau))d\tau.$$

Note also that the quantity on the right hand side changes very little if we replace θ by a close angle θ': the corresponding domains of integration have a huge common part and only short boundary intervals that are included in one but not the other one. Thus, when switching from K_2 to K_1, all boundary lines of the half-planes $W^-(\theta', t_2(\theta'))$ determining $K_{2,\delta}$ move out by pretty much the same amount

$\frac{1}{2\sin\alpha}\int\limits_{\theta-\alpha}^{\theta+\alpha}(\rho_1(\tau)-\rho_2(\tau))d\tau$ as long as θ' is close to θ. Since the value of the radial function $\rho_{K_\delta}(\theta)$ of the convex body of flotation K_δ of the body K close to the unit disk is determined by $t(\theta')$ with θ' close to θ, this last observation translates into (1) as desired. As usual, the devil is in the details, to which we now turn.

Let $\Gamma > 0$ be small. Consider the disks $(1+\Gamma)B$ and $(1-\Gamma)B$, and let $t_\pm$ be defined by

$$\mathrm{vol}_2((1\pm\Gamma)B\cap W^+(\theta,t_\pm))=\delta.$$

Claim 2.1. *If* $\pi\Gamma\frac{1+\Gamma}{2\sin\alpha}<\cos\alpha$, *then* $t_\pm$ *are well-defined and satisfy* $|t_\pm-\cos\alpha|<\pi\Gamma\frac{1+\Gamma}{2\sin\alpha}$.

Proof. Observe that

$$\mathrm{vol}_2((1+\Gamma)B\cap W^+(\theta,\cos\alpha))\le$$
$$\mathrm{vol}_2(B\cap W^+(\theta,\cos\alpha))+\mathrm{vol}_2(((1+\Gamma)B\setminus B)\cap W^+(\theta,0))\le\delta+\pi(1+\Gamma)\Gamma.$$

Note also that if $t>\cos\alpha$ is so large that the length of the intersection of $(1+\Gamma)B$ with the boundary line of $W^+(\theta,t)$ is less than or equal to $2\sin\alpha$, then we already have

$$\mathrm{vol}_2((1+\Gamma)B\cap W^+(\theta,t))<\mathrm{vol}_2(B\cap W^+(\theta,\cos\alpha))=\delta$$

(the area of the segment of a bigger disk spanned by a not longer chord is smaller).

Thus, to completely compensate for the increase in area of $B\cap W^+(\theta,\cos\alpha)$ when replacing B by $(1+\Gamma)B$, we need to move the initial $t=\cos\alpha$ up within the region where the cross-section of $(1+\Gamma)B$ by the boundary line of $W^+(\theta,t)$ has length at least $2\sin\alpha$. But within this region, the move by Δt units results in the loss of area not less than $2\sin\alpha\,\Delta t$, whence $t_+-\cos\alpha\le\pi\Gamma\frac{1+\Gamma}{2\sin\alpha}$, as claimed.

The bound for t_- is even simpler. Just notice that if we move t by $\pi\Gamma\frac{1}{2\sin\alpha}$ units down from the initial value $t=\cos\alpha$, we will have $\mathrm{vol}_2(B\cap W^+(\theta,t))\ge\delta+\pi\Gamma$. But when replacing B by $(1-\Gamma)B$ here, we can remove the area not exceeding

$$\mathrm{vol}_2((B\setminus(1-\Gamma)B)\cap W^+(\theta,0))<\pi\Gamma,$$

so we went too far and the estimate $t_-\ge\cos\alpha-\pi\Gamma\frac{1}{2\sin\alpha}$ follows. □

Let us now introduce the angles α_- and α_+ by $(1\pm\Gamma)\cos\alpha_\pm=\cos\alpha\mp\pi\Gamma\frac{1+\Gamma}{2\sin\alpha}$. Their geometric meaning can be seen on Figure 2.

Their importance comes from the fact that for every t with $|t-\cos\alpha|\le\pi\Gamma\frac{1+\Gamma}{2\sin\alpha}$, we have

$$((1+\Gamma)B\setminus(1-\Gamma)B)\cap(W^+(\theta,t)\triangle C_\alpha(\theta))\subset C_{\alpha_+}(\theta)\setminus C_{\alpha_-}(\theta).$$

Indeed, the shaded area on the right shows the largest possible piece of $W^+(\theta,t)\setminus C_\alpha(\theta)$ within $(1+\Gamma)B\setminus(1-\Gamma)B$ for such t and the shaded area on the left shows the largest possible piece of $C_\alpha(\theta)\setminus W^+(\theta,t)$.

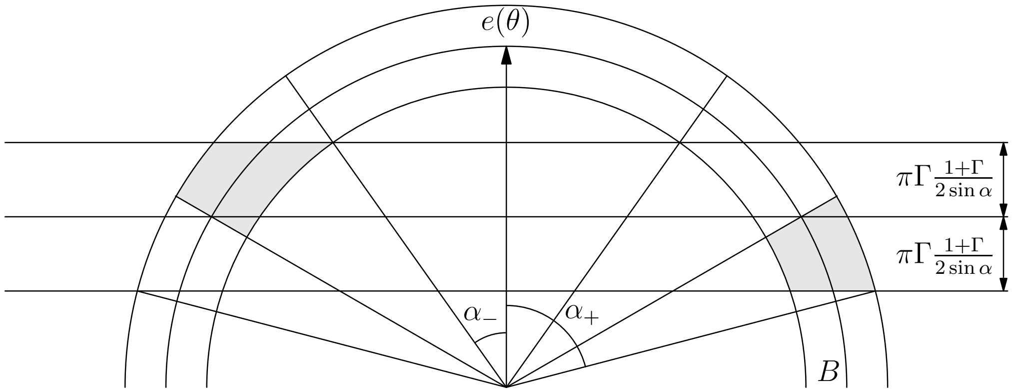

FIGURE 2. Angles α_+ and α_-.

Claim 2.2. *For every compact interval $I \subset (0, \frac{\pi}{2})$, there exist $\gamma = \gamma(I) \in (0,1)$ and $L = L(I) \in (0, +\infty)$*[1] *such that if $\alpha \in I$ and $0 < \Gamma < \gamma$, then $\alpha_\pm$ are well-defined and satisfy $0 < \alpha_- < \alpha < \alpha_+ < \frac{\pi}{2}$, $\alpha_+ - \alpha_- < L\Gamma$.*

Proof. This is tantamount to the claim that the function $\tau \mapsto \arccos\tau$ is well-defined and Lipschitz on $\left[\dfrac{\cos\alpha - \pi\gamma\frac{1+\gamma}{2\sin\alpha}}{1+\gamma}, \dfrac{\cos\alpha + \pi\gamma\frac{1+\gamma}{2\sin\alpha}}{1-\gamma}\right]$ when $\alpha \in (0, \frac{\pi}{2})$ is separated from 0 and $\frac{\pi}{2}$ and $\gamma > 0$ is small enough. $\square$

Now we are ready to prove the main lemma of this section.

Lemma 2.1. *For every compact interval $I \subset (0, \frac{\pi}{2})$, there exist $\gamma = \gamma(I) \in (0,1)$ and $L = L(I) \in (0, +\infty)$ such that if $\alpha \in I$, $0 < \Gamma < \gamma$ and K_1, K_2 are two star-shaped bodies with continuous radial functions ρ_1, ρ_2 respectively satisfying $\|\rho_j - 1\|_C \le \Gamma$, $j = 1, 2$, then the radial functions $\rho_{1,\delta}$, $\rho_{2,\delta}$ of the corresponding convex bodies of flotation $K_{1,\delta}$ and $K_{2,\delta}$ satisfy*

$$\left|\rho_{1,\delta}(\theta) - \rho_{2,\delta}(\theta) - \frac{1}{2\sin\alpha}\int\limits_{\theta-\alpha}^{\theta+\alpha}(\rho_1(\tau) - \rho_2(\tau))d\tau\right| \le L\Big[\Gamma\|\rho_1 - \rho_2\|_{L^1} + Q_{L\sqrt{\Gamma}}|\rho_1 - \rho_2|(\theta)\Big],$$

where, for $\sigma > 0$,

$$Q_\sigma f(\theta) = \int\limits_{\theta-\alpha-\sigma}^{\theta-\alpha+\sigma} f(\tau)d\tau + \int\limits_{\theta+\alpha-\sigma}^{\theta+\alpha+\sigma} f(\tau)d\tau.$$

Proof. Note that for every body K and every $t > 0$, we have

$$\mathrm{vol}_2(K \cap W^+(\theta,t)) =$$
$$\mathrm{vol}_2(K \cap C_\alpha(\theta)) + \mathrm{vol}_2(K \cap (W^+(\theta,t) \setminus C_\alpha(\theta))) - \mathrm{vol}_2(K \cap (C_\alpha(\theta) \setminus W^+(\theta,t))).$$

[1]In this paper we shall denote by L various constants whose values may change from line to line.

Subtracting these identities for K_1 and K_2 and using the "triangle inequality"

$$|\mathrm{vol}_2(K_1\cap E)-\mathrm{vol}_2(K_2\cap E)|\le \mathrm{vol}_2((K_1\triangle K_2)\cap E)$$

valid for every set $E\subset\mathbb{R}^2$, we get

$$\mathrm{vol}_2(K_1\cap W^+(\theta,t))-\mathrm{vol}_2(K_2\cap W^+(\theta,t))\le \\ \mathrm{vol}_2(K_1\cap C_\alpha(\theta))-\mathrm{vol}_2(K_2\cap C_\alpha(\theta))+\mathrm{vol}_2((K_1\triangle K_2)\cap(W^+(\theta,t)\triangle C_\alpha(\theta))).$$

Note now that if $(1-\Gamma)B\subset K_1,K_2\subset(1+\Gamma)B$, then $K_1\triangle K_2\subset(1+\Gamma)B\setminus(1-\Gamma)B$, so for $|t-\cos\alpha|\le\pi\Gamma\frac{1+\Gamma}{2\sin\alpha}$,

$$(K_1\triangle K_2)\cap(W^+(\theta,t)\triangle C_\alpha(\theta))= \\ (K_1\triangle K_2)\cap(W^+(\theta,t)\triangle C_\alpha(\theta))\cap((1+\Gamma)B\setminus(1-\Gamma)B)\subset \\ (K_1\triangle K_2)\cap(C_{\alpha_+}(\theta)\setminus C_{\alpha_-}(\theta)).$$

At last, for $|t-\cos\alpha|\le\pi\Gamma\frac{1+\Gamma}{2\sin\alpha}$, the length of the cross-section of any body K satisfying $(1-\Gamma)B\subset K\subset(1+\Gamma)B$ by the boundary line of $W^+(\theta,t)$ is between $2\sin\alpha-L\Gamma$ and $2\sin\alpha+L\Gamma$ for some $L=L(I)$, provided that the upper bound $\gamma(I)$ for Γ is small enough.

Now we are ready to approximate the difference $t_1(\theta)-t_2(\theta)$ where $t_j(\theta)$, $j=1,2$, are defined by (2). Since $t_{1,2}(\theta)\in[t_-,t_+]$ for all θ, we can use all the above observations for them or any t in between. For $t_2(\theta)$, we have $\mathrm{vol}_2(K_2\cap W^+(\theta,t_2(\theta)))=\delta$, so

$$\mathrm{vol}_2(K_1\cap W^+(\theta,t_2(\theta)))\le \\ \delta+\mathrm{vol}_2(K_1\cap C_\alpha(\theta))-\mathrm{vol}_2(K_2\cap C_\alpha(\theta))+\mathrm{vol}_2((K_1\triangle K_2)\cap(C_{\alpha_+}(\theta)\setminus C_{\alpha_-}(\theta))).$$

The difference

$$\mathrm{vol}_2(K_1\cap C_\alpha(\theta))-\mathrm{vol}_2(K_2\cap C_\alpha(\theta))$$

is

$$\frac12\int\limits_{\theta-\alpha}^{\theta+\alpha}(\rho_1^2(\tau)-\rho_2^2(\tau))d\tau= \\ \frac12\int\limits_{\theta-\alpha}^{\theta+\alpha}(\rho_1(\tau)-\rho_2(\tau))\,2\,d\tau+\frac12\int\limits_{\theta-\alpha}^{\theta+\alpha}(\rho_1(\tau)-\rho_2(\tau))(\rho_1(\tau)+\rho_2(\tau)-2)d\tau\le \\ \int\limits_{\theta-\alpha}^{\theta+\alpha}(\rho_1(\tau)-\rho_2(\tau))d\tau+\Gamma\|\rho_1-\rho_2\|_{L^1}$$

because $|\rho_1+\rho_2-2|\le 2\Gamma$.

On the other hand, the area of the intersection $(K_1 \triangle K_2) \cap (C_{\alpha_+}(\theta) \setminus C_{\alpha_-}(\theta))$ is

$$\frac{1}{2}\Big(\int\limits_{\theta-\alpha_+}^{\theta-\alpha_-} + \int\limits_{\theta+\alpha_-}^{\theta+\alpha_+}\Big)|\rho_1^2(\tau) - \rho_2^2(\tau)|d\tau \le 2\Big(\int\limits_{\theta-\alpha_+}^{\theta-\alpha_-} + \int\limits_{\theta+\alpha_-}^{\theta+\alpha_+}\Big)|\rho_1(\tau) - \rho_2(\tau)|d\tau$$

because $|\rho_1^2 - \rho_2^2| = |\rho_1 - \rho_2|(\rho_1 + \rho_2) \le 4|\rho_1 - \rho_2|$, so we obtain

$$\mathrm{vol}_2(K_1 \cap W^+(\theta, t_2(\theta))) \le$$
$$\delta + \int\limits_{\theta-\alpha}^{\theta+\alpha} (\rho_1(\tau) - \rho_2(\tau))d\tau + \Gamma\|\rho_1 - \rho_2\|_{L^1} + 2\Big(\int\limits_{\theta-\alpha_+}^{\theta-\alpha_-} + \int\limits_{\theta+\alpha_-}^{\theta+\alpha_+}\Big)|\rho_1(\tau) - \rho_2(\tau)|d\tau.$$

Moving t from $t_2(\theta)$ to $t_1(\theta)$ diminishes the left hand side by some quantity between $2\sin\alpha\,(t_1(\theta) - t_2(\theta)) - L\Gamma|t_1(\theta) - t_2(\theta)|$ and $2\sin\alpha\,(t_1(\theta) - t_2(\theta)) + L\Gamma|t_1(\theta) - t_2(\theta)|$. Since $\mathrm{vol}_2(K_1 \cap W^+(\theta, t_1(\theta))) = \delta$, we must have the inequality

$$2\sin\alpha\,(t_1(\theta) - t_2(\theta)) - L\Gamma|t_1(\theta) - t_2(\theta)| \le$$
$$\int\limits_{\theta-\alpha}^{\theta+\alpha} (\rho_1(\tau) - \rho_2(\tau))d\tau + \Gamma\|\rho_1 - \rho_2\|_{L^1} + 2\Big(\int\limits_{\theta-\alpha_+}^{\theta-\alpha_-} + \int\limits_{\theta+\alpha_-}^{\theta+\alpha_+}\Big)|\rho_1(\tau) - \rho_2(\tau)|d\tau,$$

whence

$$t_1(\theta) - t_2(\theta) \le \frac{1}{2\sin\alpha}\int\limits_{\theta-\alpha}^{\theta+\alpha} (\rho_1(\tau) - \rho_2(\tau))d\tau +$$
$$L\Big(\Gamma\|\rho_1 - \rho_2\|_{L^1} + 2\Big(\int\limits_{\theta-\alpha_+}^{\theta-\alpha_-} + \int\limits_{\theta+\alpha_-}^{\theta+\alpha_+}\Big)|\rho_1(\tau) - \rho_2(\tau)|d\tau\Big)$$

with some slightly bigger $L = L(I) \in (0, +\infty)$, provided that $\gamma(I)$ is chosen small enough. In the last implication we used the elementary fact that if $ax - b|x| \le y + z$ with $0 < b < \frac{a}{2}$, $z > 0$, then $x \le \frac{y}{a} + 2\frac{b}{a^2}|y| + \frac{2}{a}z$. Indeed,

$$ax - y \le b|x| + z \le \frac{b}{a}|ax - y| + \frac{b}{a}|y| + z \le \frac{1}{2}|ax - y| + \frac{b}{a}|y| + z,$$

so $ax - y \le 2(\frac{b}{a}|y| + z)$, which is equivalent to the inequality claimed.

To switch from $t_1(\theta)$, $t_2(\theta)$ to $\rho_{1,\delta}(\theta)$, $\rho_{2,\delta}(\theta)$, we observe that

$$\rho_{1,\delta}(\theta) = \min_{\theta' :\, \cos(\theta'-\theta)>0} \frac{t_1(\theta')}{\cos(\theta' - \theta)}, \qquad \rho_{2,\delta}(\theta) = \min_{\theta' :\, \cos(\theta'-\theta)>0} \frac{t_2(\theta')}{\cos(\theta' - \theta)}.$$

Note now that $t_- \le t_j(\theta') \le t_+$ for all θ', so when $0 < \cos(\theta - \theta') \le \frac{t_-}{t_+}$, we certainly have

$$\frac{t_j(\theta')}{\cos(\theta - \theta')} \ge \frac{t_-}{t_-/t_+} = t_+ \ge \frac{t_j(\theta)}{\cos(\theta - \theta)}, \qquad j = 1, 2.$$

Thus, in the minimization problem we may restrict ourselves to the angles θ' with $\cos(\theta-\theta') \geq \frac{t_-}{t_+} \geq 1 - L\Gamma$, i.e., $|\theta-\theta'| \leq L\sqrt{\Gamma}$.

Let now θ' be an angle for which $\rho_{2,\delta}(\theta) = \frac{t_2(\theta')}{\cos(\theta-\theta')}$. Then

$$\rho_{1,\delta}(\theta) - \rho_{2,\delta}(\theta) \leq \frac{t_1(\theta') - t_2(\theta')}{\cos(\theta-\theta')}.$$

As we have shown above,

$$t_1(\theta') - t_2(\theta') \leq \frac{1}{2\sin\alpha} \int_{\theta'-\alpha}^{\theta'+\alpha} (\rho_1(\tau) - \rho_2(\tau))d\tau +$$

$$L\Big(\Gamma\|\rho_1-\rho_2\|_{L^1} + 2\Big(\int_{\theta'-\alpha_+}^{\theta'-\alpha_-} + \int_{\theta'+\alpha_-}^{\theta'+\alpha_+}\Big)|\rho_1(\tau)-\rho_2(\tau)|d\tau\Big).$$

Note that

$$\int_{\theta'-\alpha}^{\theta'+\alpha} (\rho_1(\tau)-\rho_2(\tau))d\tau \leq \int_{\theta-\alpha}^{\theta+\alpha} (\rho_1(\tau)-\rho_2(\tau))d\tau + \Big(\int_{\theta-\alpha-|\theta-\theta'|}^{\theta-\alpha+|\theta-\theta'|} + \int_{\theta+\alpha-|\theta-\theta'|}^{\theta+\alpha+|\theta-\theta'|}\Big)|\rho_1(\tau)-\rho_2(\tau)|d\tau,$$

so, taking into account that $1 \leq \frac{1}{\cos(\theta-\theta')} \leq 1 + L\Gamma$, we finally obtain

$$\rho_{1,\delta}(\theta) - \rho_{2,\delta}(\theta) \leq \frac{1}{2\sin\alpha} \int_{\theta-\alpha}^{\theta+\alpha} (\rho_1(\tau) - \rho_2(\tau))d\tau +$$

$$L\Big(\Gamma\|\rho_1-\rho_2\|_{L^1} + \Big(\int_{\theta'-\alpha_+}^{\theta'-\alpha_-} + \int_{\theta-\alpha-|\theta-\theta'|}^{\theta-\alpha+|\theta-\theta'|} + \int_{\theta'+\alpha_-}^{\theta'+\alpha_+} + \int_{\theta+\alpha-|\theta-\theta'|}^{\theta+\alpha+|\theta-\theta'|}\Big)|\rho_1(\tau)-\rho_2(\tau)|d\tau\Big).$$

It remains to notice that all intervals of integration in the last term on the right hand side are contained in the union of intervals centered at $\theta \pm \alpha$ of length $L\sqrt{\Gamma}$, so the corresponding integrals can be bounded by $Q_{L\sqrt{\Gamma}}|\rho_1-\rho_2|(\theta)$ yielding a one-sided bound in the desired inequality. Exchanging the roles of ρ_1 and ρ_2, we get the bound from the other side. □

3. Homothety conjecture for origin-symmetric bodies near the unit disk

Once the quasi-differentiability property of the mapping $\rho \mapsto \rho_\delta$ has been established, we can apply our usual routine, see [1], to obtain a positive result for origin-symmetric bodies near the unit disk. The argument will go along the following lines. At the first step we shall put the body K into the isotropic position (its definition and properties will be discussed below in detail) and normalize its radial function by

$\frac{1}{2\pi}\int\limits_0^{2\pi}\rho(\theta)d\theta = 1$. Both these operations will keep the body that originally was close to the unit disk close to the unit disk. We write $\rho = 1+\varphi$ and decompose φ into its Fourier series $\varphi = \varphi_2 + \varphi_4 + ...$, where $\varphi_k \in \operatorname{span}(\cos(k\theta), \sin(k\theta))$. Due to the isotropic position assumption, $\|\varphi_2\|_{L^2}$ is much less than $\|\varphi\|_{L^2}$.

Now assume that $K_\delta = \lambda K$. Then $\lambda \approx \cos\alpha$ and, applying Lemma 2.1 with $K_1 = K$, $K_2 = B$, we get

$$\left|\rho_\delta(\theta) - \cos\alpha - \frac{1}{2\sin\alpha}\int\limits_{\theta-\alpha}^{\theta+\alpha}\varphi(\tau)d\tau\right| \le L(\Gamma\|\varphi\|_{L^1} + Q_{L\sqrt{\Gamma}}|\varphi|(\theta))$$

with $\Gamma = \|\varphi\|_C < \gamma$. Projecting to non-zero frequencies, we obtain

$$\left\|\lambda\varphi - \frac{1}{2\sin\alpha}\int\limits_{\theta-\alpha}^{\theta+\alpha}\varphi(\tau)d\tau\right\|_{L^2} \le L\sqrt{\Gamma}\|\varphi\|_{L^2}.$$

However, the left hand side squared is at least

$$\sum_{\substack{k\ge2\\k\text{ even}}}\left(\lambda - \frac{\sin(k\alpha)}{k\sin\alpha}\right)^2\|\varphi_k\|^2_{L^2} \approx \sum_{\substack{k\ge2\\k\text{ even}}}\left(\cos\alpha - \frac{\sin(k\alpha)}{k\sin\alpha}\right)^2\|\varphi_k\|^2_{L^2} \ge c(\alpha)\sum_{\substack{k\ge4\\k\text{ even}}}\|\varphi_k\|^2_{L^2} \ge \frac{1}{2}c(\alpha)\|\varphi\|^2_{L^2}, \tag{3}$$

where

$$c(\alpha) = \min_{\substack{k\ge4\\k\text{ even}}}\left(\cos\alpha - \frac{\sin(k\alpha)}{k\sin\alpha}\right)^2 > 0.$$

If Γ is small enough, this will imply $\varphi \equiv 0$, so K is a disk.

Now the details. First of all, we will remind the reader the isotropic position trick. Let K be an arbitrary star-shaped body. Consider the integral $I_K(x) = \int\limits_K \langle x,y\rangle^2 dy$. Opening the parentheses, we see that $I_K(x) = \langle Ax, x\rangle$ for some positive definite self-adjoint matrix A in $\mathbb{R}^2$. Now if we replace K by SK where S is a linear transformation in $\mathbb{R}^2$, then we will get

$$I_{SK}(x) = \int\limits_{SK}\langle x,y\rangle^2 dy = \int\limits_K \langle x, Sz\rangle^2|\det S|dz = |\det S|\int\limits_K\langle S^*x, z\rangle^2 dz = |\det S|I_K(S^*x) = |\det S|\langle SAS^*x, x\rangle.$$

Choosing $S = |\det A|^{\frac14}A^{-\frac12}$, we get $I_{SK}(x) = |\det A|^{\frac12}|x|^2$.

There are two important points here. The first one is that, since the quadratic form $I_{SK}(x)$ is proportional to $|x|^2$, equating the coefficients and switching to polar coordinates, we obtain

$$\frac{1}{4}\int_0^{2\pi} \rho_{SK}(\theta)^4(\cos^2\theta - \sin^2\theta)d\theta = \int_{SK}(y_1^2 - y_2^2)dy = 0$$

and

$$\frac{1}{4}\int_0^{2\pi} \rho_{SK}(\theta)^4 \cos\theta\sin\theta d\theta = \int_{SK} y_1 y_2 dy = 0,$$

i.e., ρ_{SK}^4 is orthogonal to $\operatorname{span}(\cos(2\theta), \sin(2\theta))$. The second point is that if K was close to the unit disk, then so is SK. More precisely, if $(1-\Gamma)B \subset K \subset (1+\Gamma)B$, then

$$(1-\Gamma)\Big(\frac{1-\Gamma}{1+\Gamma}\Big)^2 B \subset SK \subset (1+\Gamma)\Big(\frac{1+\Gamma}{1-\Gamma}\Big)^2 B.$$

Indeed, since

$$I_{(1-\Gamma)B}(x) \le I_K(x) \le I_{(1+\Gamma)B}(x) \quad \text{and} \quad I_B(x) = c|x|^2,$$

we have

$$c(1-\Gamma)^4\,\mathrm{Id} \prec A \prec c(1+\Gamma)^4\,\mathrm{Id},$$

so $\|A\| \le c(1+\Gamma)^4$, $\|A^{-1}\| \le c^{-1}(1-\Gamma)^{-4}$, and $c^2(1-\Gamma)^8 \le \det A \le c^2(1+\Gamma)^8$. Hence, $\|S\|, \|S^{-1}\| \le \left(\frac{1+\Gamma}{1-\Gamma}\right)^2$, and the claimed inclusions follow.

So, from now on, we will assume that our body K is in the isotropic position. We can also normalize K by the condition $\frac{1}{2\pi}\int_0^{2\pi}\rho(\theta)d\theta = 1$, which can be achieved by a pure dilation and also increases Γ at most 3 times if it is small enough.

Now write $\rho = 1+\varphi$ and decompose φ into its Fourier series $\varphi = \varphi_2 + \varphi_4 + \dots$. Since ρ^4 has no second order Fourier component and

$$|\rho^4 - (1+4\varphi)| = |6\varphi^2 + 4\varphi^3 + \varphi^4| \le 11\Gamma|\varphi|,$$

we conclude that the second order Fourier component $-4\varphi_2$ of $\rho^4 - (1+4\varphi)$ has the L^2-norm at most $11\Gamma\|\varphi\|_{L^2}$, i.e., $\|\varphi_2\|_{L^2} \le 3\Gamma\|\varphi\|_{L^2}$.

Assume now that $K^{\mathcal{D}} = \lambda K$ for some $\mathcal{D} \in I \subset (0, \frac{1}{2})$ and $\lambda > 0$. Since the area $\mathrm{vol}_2(K)$ is squeezed between $(1-\Gamma)^2\pi$ and $(1+\Gamma)^2\pi$, we see that the corresponding area $\delta = \mathcal{D}\,\mathrm{vol}_2(K)$ is separated from both 0 and $\frac{\pi}{2}$ if Γ is not too large. Let α be the angle associated with δ as above, i.e., $\mathrm{vol}_2(B \cap W^+(\theta, \cos\alpha)) = \delta$. Noting now that in this case $(1-\Gamma)B \subset K \subset (1+\Gamma)B$ and $t_- B \subset K_\delta \subset t_+ B$ with $|t_\pm - \cos\alpha| \le \pi\Gamma\frac{1+\Gamma}{2\sin\alpha}$, we conclude that $|\lambda - \cos\alpha| \le L\Gamma$ with some $L = L(I) \in (0, +\infty)$. This observation justifies the approximate equality in (3) and, thereby, completes the proof of Theorem 1.1.

4. Asymmetric convex bodies homothetic to their convex bodies of flotation

This section is devoted to the construction of asymmetric bodies homothetic to their convex bodies of flotation. We use the basic technique from the bifurcation theory that we borrowed from [3], Theorem 3.2, page 171. We believe that this example could have been found long ago if the bifurcation specialists had paid more attention to convex geometry problems or convex geometers were more familiar with the bifurcation theory.

Again, before diving into technicalities, let us (try to) explain the general idea. We are trying to solve the equation $F(\rho) = \lambda\rho - \rho_\delta = 0$ with some positive λ and δ. We have the trivial solution $\rho \equiv 1$, $\lambda = \cos\alpha$ and $\delta = \delta(\alpha)$ (the unit disk). Note that if ρ_0 is the solution of this equation, then $t\rho_0$ is the solution of the equation $\lambda\rho - \rho_{t^2\delta} = 0$. Also, the equation $F(\rho) = 0$ is invariant under rotations. These two degrees of freedom are not so interesting: if we start with a disk, using them will produce nothing but disks.

However, we know also that we can change the disk to an ellipse. Where does this degree of freedom come from? Note that on the Fourier side the quasi-differential

$$dF_\alpha : \Delta\rho \quad \mapsto \quad \cos\alpha\ \Delta\rho - \frac{1}{2\sin\alpha}\int_{\theta-\alpha}^{\theta+\alpha} \Delta\rho(\tau)\, d\tau$$

of the mapping $F_\alpha(\rho) = \cos\alpha\,\rho - \rho_\delta$ is the multiplier operator acting on the k-th frequency (i.e., on the space $\mathrm{span}(\cos(k\theta), \sin(k\theta))$) as the multiplication by

$$\mu_k(\alpha) = \cos\alpha - \frac{\sin(k\alpha)}{k\sin\alpha}$$

(for $k = 0$, $\mu_0(\alpha) = \cos\alpha - \frac{\alpha}{\sin\alpha}$). This multiplier operator is degenerate on the second frequencies regardless of α ($\mu_2(\alpha) = \cos\alpha - \frac{\sin(2\alpha)}{2\sin\alpha} = 0$).

Locally, it allows one to shift away from the radial function $\rho_0 \equiv 1$ of the unit disk along this kernel and, by adjusting other frequencies appropriately, to obtain a one-parametric family of solutions (ellipses) once the rotations and dilations are factored out.

The idea now is to remove the second frequency out of the game entirely and to make another frequency l play its role. Note that while $\mu_2(\alpha) = 0$ for all α, making the differential degenerate on some other frequency requires choosing very special angles. We cannot make $\mu_l(\alpha) = 0$ with even $l > 2$ for any $\alpha \in (0, \frac{\pi}{2})$, but we have our chance with odd $l = 4k + 1 \geq 5$. In this case, the curves $\alpha \mapsto \cos\alpha$ and $\alpha \mapsto \frac{\sin(l\alpha)}{l\sin\alpha}$ intersect once near $\frac{\pi}{2}$, the corresponding angle being $\alpha_l = \frac{\pi}{2} - \frac{\beta_l}{l}$, where β_l is the unique on $(0, \frac{\pi}{2})$ solution of the equation $\cos\beta_l = \frac{l}{2}\sin\frac{2\beta_l}{l}$, which, as $l \to +\infty$, tends to the unique solution β of $\cos\beta = \beta$.

So the idea is to consider the set of all star-shaped bodies with the symmetries of the regular l-gon, i.e., with continuous $\frac{2\pi}{l}$-periodic even radial functions ρ (the class that is preserved by the mapping $\rho \mapsto \rho_\delta$), take $\alpha = \alpha_l$ and try to move away from the radial function 1 of the unit disk by adding $s\cos(l\theta)$ to it. The result will be that the equation $F_\alpha(1+s\cos(l\theta))=0$ will hold in the first (in s) order, but there will be higher order errors in all frequencies divisible by l (no other frequencies can appear due to the symmetry conditions).

To eliminate the errors in all frequencies except the l-th one, we can try to apply $(dF_\alpha)^{-1}$ to the error without the l-th frequency, which is a well-defined multiplier operator with the multiplier sequence $\nu_k=\mu_k(\alpha_l)^{-1}$, $l|k$, $k\neq l$, and subtract the result from the argument of F_α just as it is done in the classical proofs of the inverse or implicit function theorems. This will reduce the size of the error (except for the l-th frequency) and we can do it again and again until only the l-th frequency term remains in the error. The final outcome will be some radial function

$$\rho=\rho_0+s\cos(l\theta)+\rho_{2l}\cos(2l\theta)+\rho_{3l}\cos(3l\theta)+\dots$$

with $\rho_0=1+o(s)$, ρ_{2l}, ρ_{3l}, $\dots=o(s)$ as $s\to 0$, which solves the equation

$$F_{\alpha_l}(\rho)=\mathcal{E}(s,\alpha_l)\cos(l\theta)$$

with $\mathcal{E}(s,\alpha_l)=o(s)$ as $s\to 0$. This is almost what we want, but not quite. To remove the error entirely, we will choose α close to α_l, but not exactly α_l. Note that the curves $\alpha\mapsto\cos\alpha$ and $\alpha\mapsto\frac{\sin(l\alpha)}{l\sin\alpha}$ cross *transversally* at α_l. This is easy to see in terms of $\beta\in(0,\frac{\pi}{2})$ given by $\frac{\beta}{l}=\frac{\pi}{2}-\alpha$, when the equation for β_l becomes $\cos\beta=\frac{l}{2}\sin\frac{2\beta}{l}$ with the left hand side decreasing and the right hand side increasing in β on $(0,\frac{\pi}{2})$. So, with such choice of α, the initial first order error at the l-th frequency will become

$$\mu_l(\alpha)s\cos(l\theta)=(c(\alpha-\alpha_l)+o(\alpha-\alpha_l))s\cos(l\theta)$$

with some $c=c(l)\neq 0$, and then it will change only by $o(s)$ during the rest of the scheme. Thus, for sufficiently small s, we can make the final $\mathcal{E}(s,\alpha)$ both positive and negative by moving α slightly away from α_l. Since $\mathcal{E}(s,\alpha)$ depends continuously on α, there exists some α close to α_l for which it is exactly 0 and, voila, we have our solution; actually even a countable set (indexed by l) of one-parametric (with parameter s) families of solutions.

Now we turn to the pesky details.

We fix an odd number $l=4k+1\geq 5$ and consider the space C_l of continuous even $\frac{2\pi}{l}$-periodic functions. All functions in C_l are represented by pure cosine Fourier series with frequencies divisible by l. We endow C_l with the usual norm $\|f\|=\max\limits_{[0,2\pi]}|f|$. Let C_l' be the subspace of C_l consisting of all functions orthogonal to $\cos(l\theta)$ (in general, for any function space X, we denote by X_l the subspace of $\frac{2\pi}{l}$-periodic even functions from X and by X_l' the subspace of functions from X_l that are orthogonal to $\cos(l\theta)$).

Let $P:C_l\to C_l'$ be the usual orthogonal projection "forgetting" the Fourier coefficient at the frequency l. Note that $\|P\|_{C_l\to C_l}\leq 3$ regardless of l (just because P can be written as the identity operator minus the projection to the l'th frequency, and the norm of the latter is not greater than 2).

Note that the convex body of flotation preserves all symmetries of the original body, so we can view the function $F_\alpha(\rho)=\cos\alpha\,\rho-\rho_\delta$ as a mapping from C_l to itself defined on all functions $\rho \in C_l$ sufficiently close to 1. We shall choose α close to α_l defined above. Let

$$(Tf)(\theta) = \frac{1}{2\sin\alpha} \int\limits_{\theta-\alpha}^{\theta+\alpha} f(\tau)\, d\tau$$

be the quasi-differential of the mapping $\rho \mapsto \rho_\delta$ near the unit disk. The result of Lemma 2.1 implies that

$$\|\rho_{1,\delta} - \rho_{2,\delta} - T(\rho_1 - \rho_2)\|_C \le L\sqrt{\Gamma}\|\rho_1 - \rho_2\|_C$$

with $\Gamma = \max(\|\rho_1 - 1\|_C, \|\rho_2 - 1\|_C)$, say. The quasi-differential of F_α is then $\cos\alpha \operatorname{Id} - T$.

Claim 4.1. *The linear mapping $dF_\alpha = \cos\alpha \operatorname{Id} - T$ is invertible on C'_l and the norm of the inverse (as an operator from C'_l to itself) is uniformly bounded for α sufficiently close to α_l.*

Proof. On the Fourier side, dF_α acts as a multiplier operator with the multiplier sequence $\mu_k(\alpha) = \cos\alpha - \frac{\sin(k\alpha)}{k\sin\alpha}$. We note now that $\mu_0(\alpha) = \cos\alpha - \frac{\alpha}{\sin\alpha}$ is negative and is bounded away from 0 as long as $\alpha \in (0, \frac{\pi}{2})$ is bounded away from 0.

When $k = ml$ $(m = 2, 3, \dots)$, we have

$$\mu_k(\alpha) = \cos\alpha - \frac{\sin(ml\alpha)}{ml\sin\alpha}.$$

Recall that α_l is defined by

$$\cos\alpha_l = \frac{\sin(l\alpha_l)}{l\sin\alpha_l}$$

and that $l\alpha_l \bmod 2\pi$ is neither 0, nor π. Thus, for every $m = 2, 3, ...$, we have $|\sin(ml\alpha_l)| < m|\sin(l\alpha_l)|$, so

$$\left|\frac{\sin(ml\alpha_l)}{ml\sin\alpha_l}\right| < \left|\frac{\sin(l\alpha_l)}{l\sin\alpha_l}\right| = \cos\alpha_l,$$

i.e., $\mu_{ml}(\alpha_l) > 0$, and for every fixed m, this inequality persists in some neighborhood of α_l. On the other hand, for all $\alpha \in (\frac{\alpha_l}{2}, \frac{\pi}{2})$, we have the uniform bound

$$\left|\frac{\sin(ml\alpha)}{ml\sin\alpha}\right| \le \frac{1}{ml\sin\frac{\alpha_l}{2}} < \frac{\cos\alpha_l}{2}$$

when m is large enough. Thus, in a sufficiently small neighborhood of α_l, for all $m = 2, 3, ...$, we have $\mu_{ml}(\alpha) \ge c_l > 0$ with some c_l depending on l only.

Now it is clear that for all α in that neighborhood, $dF_\alpha = \cos\alpha \operatorname{Id} - T$ is invertible in $(L^2_l)'$ and the norm of its inverse S is uniformly bounded there. To get the bound for the norm of S in C'_l, we shall use the resolvent identity

$$(\cos\alpha \operatorname{Id} - T)^{-1} = (\cos\alpha)^{-1}\Big(\operatorname{Id} + T(\cos\alpha \operatorname{Id} - T)^{-1}\Big).$$

Note that the second term in the sum in parentheses can be viewed as the composition of the trivial imbedding $C'_l \hookrightarrow (L^2_l)'$, the resolvent $(\cos\alpha \operatorname{Id} - T)^{-1} : (L^2_l)' \to$

$(L_l^2)'$ of norm uniformly bounded for α sufficiently close to α_l, and the convolution operator $T : (L_l^2)' \to C_l'$, whose norm is bounded for $\alpha \in (0, \frac{\pi}{2})$ separated from 0. $\square$

For small $s > 0$, consider the mapping

$$H_s : \varphi \ \mapsto \ \varphi - SPF_\alpha(1 + s\cos(l\theta) + \varphi)$$

in the closed ball $\|\varphi\|_C \le \Gamma$ in C_l'.

Claim 4.2. *When s is small enough, the mapping H_s sends the closed ball in C_l' of radius comparable to $s^{\frac{3}{2}}$ to itself and is a contraction in that ball.*

Proof. First, let us estimate the norm of $H_s(0)$. By Lemma 2.1, for small enough s, we have

$$F_\alpha(1 + s\cos(l\theta)) = dF_\alpha(s\cos(l\theta)) + \mathcal{E},$$

where

$$\|\mathcal{E}\|_C \le L(\alpha)\sqrt{s}s = L(\alpha)s^{\frac{3}{2}}.$$

Note now that $dF_\alpha(s\cos(l\theta)) = \mu_l(\alpha)s\cos(l\theta)$ and P annihilates it entirely, so

$$\|H_s(0)\|_C = \| - SP\mathcal{E}\|_C \le L(\alpha)s^{\frac{3}{2}}.$$

To show the contraction property, we notice that for $s + \Gamma \le \gamma(\alpha)$ and $\|\varphi_{1,2}\|_C \le \Gamma$, we can write

$$F_\alpha(1 + s\cos(l\theta) + \varphi_1) - F_\alpha(1 + s\cos(l\theta) + \varphi_2) = dF_\alpha(\varphi_1 - \varphi_2) + \mathcal{E}(\varphi_1, \varphi_2),$$

where

$$\|\mathcal{E}(\varphi_1, \varphi_2)\|_C \le L(\alpha)\sqrt{s + \Gamma}\|\varphi_1 - \varphi_2\|_C.$$

Observe now that $SPdF_\alpha$ is the identity operator on C_l', so when $\|\varphi_1\|_C$, $\|\varphi_2\|_C \le \Gamma$, we have

$$\|H_s(\varphi_1) - H_s(\varphi_2)\|_C = \|SP\mathcal{E}(\varphi_1, \varphi_2)\|_C \le L(\alpha, l)\sqrt{s + \Gamma}\|\varphi_1 - \varphi_2\|_C.$$

In order to make H_s a contraction, it suffices to demand that $L(\alpha, l)\sqrt{s + \Gamma} \le \frac{1}{2}$, say. Note that $L(\alpha, l)$ stays bounded and $\gamma(\alpha)$ stays separated from 0 as long as α stays close to α_l. Finally, to ensure that H_s acts from the ball $\|\varphi\|_C \le \Gamma$ to itself, we can write

$$\|H_s(\varphi)\|_C \le \|H_s(0)\|_C + L(\alpha, l)\sqrt{s + \Gamma}\|\varphi\|_C \le L(\alpha, l)(s^{\frac{3}{2}} + \sqrt{s + \Gamma}\,\Gamma)$$

and take $\Gamma = 2L(\alpha, l)s^{\frac{3}{2}}$ and s so small that $L(\alpha, l)\sqrt{s + 2L(\alpha, l)s^{\frac{3}{2}}} \le \frac{1}{2}$. $\square$

Let now φ_s be the fixed point of H_s. Then $\|\varphi_s\|_C = O(s^{\frac{3}{2}})$ as $s \to 0$ and

$$SPF_\alpha(1 + s\cos(l\theta) + \varphi_s) = 0,$$

i.e.,

$$PF_\alpha(1 + s\cos(l\theta) + \varphi_s) = 0,$$

so

$$F_\alpha(1 + s\cos(l\theta) + \varphi_s) = \mathcal{E}(s, \alpha)\cos(l\theta)$$

with some $\mathcal{E}(s, \alpha) \in \mathbb{R}$. It remains to discern the dependence of $\mathcal{E}(s, \alpha)$ on the parameter α of the mapping F_α.

Let us now prove that the mapping $(\alpha,\varphi)\mapsto H_s(\varphi)$ is uniformly continuous in α when φ stays in a small closed ball in C_l' and α stays close to α_l. To this end, we can first observe that $\rho_{t^2\delta}=t(t^{-1}\rho)_\delta$, so when $\|\rho-1\|_C$ and $|t-1|$ are less than Γ and $\Gamma>0$ is not too large, Lemma 2.1 implies the estimate

$$\|\rho_{t^2\delta}-\rho_\delta\|_C\le |t-1|\,\|(t^{-1}\rho)_\delta\|_C+\|\rho_\delta-(t^{-1}\rho)_\delta\|_C\le \\ 2|t-1|+L\|\rho-t^{-1}\rho\|_C\le L|t-1|\le L|t^2-1|,$$

or, equivalently, $\|\rho_{\delta'}-\rho_\delta\|_C\le\frac{L}{\delta}|\delta'-\delta|$ for some $L>0$. However, δ is a Lipschitz function of α that is bounded away from 0 in a small neighborhood of α_l, so ρ_δ and, therefore, $F_\alpha(\rho)$ are uniformly Lipschitz in α there when ρ stays close to 1.

Next, we need to show that S depends continuously on α as an operator from C_l' to itself. First, observe that S depends continuously on α as an operator from $(L^2_l)'$ to $(L^2_l)'$. Indeed, if we take two angles α' and α'' close to α_l, the norm of the difference of the corresponding operators S' and S'' is

$$\|S'-S''\|_{(L^2_l)'\to(L^2_l)'}=\sup_{k:\,l|k,k\neq l}|\mu_k(\alpha')^{-1}-\mu_k(\alpha'')^{-1}|\le c_l^{-2}\sup_{k:\,l|k,k\neq l}|\mu_k(\alpha')-\mu_k(\alpha'')|,$$

where $c_l>0$ is the uniform lower bound for $|\mu_k(\alpha)|$ with $l|k$, $k\neq l$, in a small neighborhood of α_l. However, $\mu_k(\alpha)$ is continuous for each k and $\mu_k(\alpha)\to\cos\alpha$ uniformly as $k\to\infty$ in that neighborhood, so the supremum becomes small when the difference $|\alpha'-\alpha''|$ gets small.

Since $S=(\cos\alpha)^{-1}(\mathrm{Id}+TS)$, in order to finish, we just need to show that the convolution operator $\widetilde{T}_\alpha f(\theta)=\int\limits_{\theta-\alpha}^{\theta+\alpha}f(\tau)d\tau$ depends continuously on α as an operator from $(L^2_l)'$ to C_l'. However,

$$\|\widetilde{T}_{\alpha'}f-\widetilde{T}_{\alpha''}f\|_C\le\sqrt{2|\alpha'-\alpha''|}\,\|f\|_{L^2}$$

by Cauchy-Schwarz and we are done. The other components of the mapping H_s do not depend on α at all, so the proof is complete.

Thus, the fixed point φ_s also depends on α continuously. Next, F_α is Lipschitz in its argument as long as the latter remains close to 1, so

$$F_\alpha(1+s\cos(l\theta)+\varphi_s)=F_\alpha(1+s\cos(l\theta))+O(\|\varphi_s\|_C)= \\ dF_\alpha(s\cos(l\theta))+O(s^{\frac32})=\mu_l(\alpha)s\cos(l\theta)+O(s^{\frac32}).$$

The conclusion is that

$$\mathcal{E}(s,\alpha)=\mu_l(\alpha)s+O(s^{\frac32})\quad(s\to0),$$

where the implicit constant in $O()$ stays bounded as long as α stays sufficiently close to α_l (how close exactly depends on l, but not on s).

Since $\mu_l(\alpha)$ changes sign at α_l, we see that $\mathcal{E}(s,\alpha)$ also changes sign in a short interval around α_l if s is small enough. But then, by the intermediate value theorem, for any sufficiently small s, there exists $\alpha=\alpha(s)$ in a small neighborhood of α_l for which $\mathcal{E}(s,\alpha)=0$, i.e.,

$$F_\alpha(1+s\cos(l\theta)+\varphi_s)=0.$$

Then $\rho_s = 1 + s\cos(l\theta) + \varphi_s$ is a continuous radial function of a star-shaped body K_s that is homothetic (with the coefficient $\frac{1}{\cos\alpha}$) to its convex body of flotation and, therefore, convex as well.

At this point, it is already clear that it cannot happen that the K_s corresponding to very different values of s (differing 10 times or more) are affine equivalent. Indeed, since K_s has the symmetries of the regular l-gon, it is centered at the origin and is in the isotropic position, so the only chance to map it to another $K_{s'}$ (which also is centered at the origin and is in the isotropic position) affinely is to use a combination of rotation and dilation (any other affine transformation will destroy the isotropic position or shift K_s off center, or both). However, both rotations and dilations preserve the ratio of the total size of the component of ρ at the l-th frequency to the size of the component at the 0-th frequency, and that ratio for K_s is between $\frac{s}{2}$ and $2s$.

It is reasonable to expect that we have actually obtained a continuous in s continuum size family of pairwise affinely non-equivalent bodies here, but showing it rigorously goes well beyond the scope of this short paper, so we leave it to the interested reader.

References

[1] M. Alfonseca, F. Nazarov, D. Ryabogin, V. Yaskin, Analysis and geometry near the unit ball: proofs, counterexamples and open questions. Harmonic analysis and convexity, *Adv. Anal. Geom.*, De Gruyter, Berlin, **9** (2023), 445–468.

[2] I. Bárány and D. G. Larman, Convex bodies, economic cap coverings, random polytopes, *Mathematika*, **35 (2)** (1988), 274–291.

[3] M. G. Crandall and P. H. Rabinowitz, Bifurcation, Perturbation of Simple Eigenvalues, and Linearized Stability, *Archive for Rational Mechanics and Analysis*, **52** (1973), 161–180.

[4] C. Schütt, E. Werner, Homothetic floating bodies, *Geom. Dedicata,* **49 (3)** (1994), 335–348.

[5] C. Schütt, E. Werner, The convex floating body, *Math. Scand.,* **66** (1990), 275–290.

Department of Mathematics, North Dakota State University, Fargo, ND 58108, USA

Email address: maria.alfonseca@ndsu.edu

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA

Email address: nazarov@math.kent.edu

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA

Email address: ryabogin@math.kent.edu

Department of Mathematics and Statistics, Concorida University, Montreal, Canada

Email address: alina.stancu.concordia@gmail.com

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada

Email address: vladyaskin@math.ualberta.ca