New bounds for the same-type lemma

Boris Bukh Alexey Vasileuski

Submitted: Sep 26, 2023; Accepted: May 31, 2024; Published: Jun 28, 2024

Abstract

Given finite sets $X_1, \ldots, X_m$ in $\mathbb{R}^d$ (with d fixed), we prove that there are respective subsets $Y_1, \ldots, Y_m$ with $|Y_i| \geqslant \frac{1}{\mathrm{poly}(m)}|X_i|$ such that, for $y_1 \in Y_1, \ldots, y_m \in Y_m$, the orientations of the $(d+1)$-tuples from $y_1, \ldots, y_m$ do not depend on the actual choices of points $y_1, \ldots, y_m$. This generalizes previously known case when all the sets X_i are equal. Furthermore, we give a construction showing that polynomial dependence on m is unavoidable, as well as an algorithm that approximates the best-possible constants in this result.

Mathematics Subject Classifications: 52C10, 52C40

1 Introduction

We say that the sets $Y_1, \ldots, Y_{d+1}$ in $\mathbb{R}^d$ have the *same-type property* if, for every choice of points $y_1 \in Y_1, \ldots, y_{d+1} \in Y_{d+1}$, the orientation of points $y_1, \ldots, y_{d+1}$ is the same. More generally, we say that the sets $Y_1, \ldots, Y_m$ have the same-type property if every $d+1$ of them do. A natural question is the following: given disjoint finite sets $X_1, \ldots, X_m$ in $\mathbb{R}^d$ such that their union is in general position, are there large subsets $Y_i \subseteq X_i$ such that the sets $Y_1, \ldots, Y_m$ have the same-type property? The same-type lemma proved by Bárány and Valtr [2] states that each Y_i may be taken to have a positive fraction of points from the corresponding X-set. How large could this fraction be? Formally, for disjoint sets $X_1, \ldots, X_m$ in $\mathbb{R}^d$, whose union is in general position, denote by $c(X_1, \ldots, X_m)$ the largest constant c for which there exist $Y_1, \ldots, Y_m$ having the same-type property and satisfying $Y_i \subseteq X_i$, $|Y_i| \geqslant c|X_i|$. For fixed number m and dimension d, denote by $c(m, d)$ the infimum of c for all such configurations.

The same-type lemma has been used to prove a number of positive fraction results in discrete geometry, including Radon theorem, Tverberg theorem and Erdős–Szekeres theorem [2]. Notably, a quantitative version of the latter due to Pór and Valtr [14] was a crucial ingredient in Suk's proof [15] of the bound $2^{n+o(n)}$ for the number of points on plane guaranteeing the existence of n points in convex position. Additional results that

Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213, USA (bbukh@math.cmu.edu, avasileu@andrew.cmu.edu)

directly use the same-type lemma are [9, 5]; also, many arguments that are similar to the same-type lemma appear in the literature, e.g., [12, 4, 13, 6, 11].

In their original paper, Bárány and Valtr showed that $c(m, d)$ is at least $(d+1)^{(-2^d-1)\binom{m-1}{d}}$. Fox, Pach and Suk [6] improved this to $c(m, d) \geqslant 2^{-O(d^3 m \log m)}$.

Our first result shows that $c(m, d)$ is polynomial in m, for fixed d.

Theorem 1. *For $d \geqslant 2$ and $m \geqslant d$ the constant $c(m, d)$ satisfies*

$$d^{-50d^3} m^{-d^2} \leqslant c(m, d) \leqslant d^d m^{-d}.$$

Polynomial bounds were previously known only for the special case when the sets $X_1, \dots, X_m$ are all equal (see Lemma 3.2 in [11], with references to [6]). Our upper bound of $d^d m^{-d}$ applies even to this special case; it is a first upper bound both in the special and general cases.

We also show that the constants $c(m, d)$ can be computed with arbitrary precision, at least in principle.

Theorem 2. *There exists an algorithm that computes, for any input $m, d \in \mathbb{N}$ and $\varepsilon > 0$, a constant $c'(m, d)$ satisfying $|c'(m, d) - c(m, d)| < \varepsilon$.*

2 Preliminaries

Sets with the same-type property. For simplicity, we say that a family of sets $X_1, \dots, X_m \subseteq \mathbb{R}^d$ is in *general position* if the sets $X_1, \dots, X_m$ are disjoint and their union is a set of points in general position. We start with a convenient sufficient condition for sets to have the same-type property.

Lemma 3. *Suppose that $Y_1, \dots, Y_{d+1} \subseteq \mathbb{R}^d$ are connected sets and no hyperplane intersects all of them. Then the sets $Y_1, \dots, Y_{d+1}$ have the same-type property.*

Proof. Note that, since the set $Y \stackrel{\text{def}}{=} Y_1 \times \cdots \times Y_{d+1}$ is a product of connected sets, it is itself connected. Suppose that the sets $Y_1, \dots, Y_{d+1}$ lack the same-type property. Then the sets

$$Y_+ \stackrel{\text{def}}{=} \{(y_1, \dots, y_{d+1}) \in Y : \operatorname{orient}(y_1, \dots, y_{d+1}) > 0\},$$
$$Y_- \stackrel{\text{def}}{=} \{(y_1, \dots, y_{d+1}) \in Y : \operatorname{orient}(y_1, \dots, y_{d+1}) < 0\}$$

are both non-empty. Since both Y_+ and Y_- are relatively open in Y, and Y is connected, this implies that $Y_+ \cup Y_- \neq Y$, i.e., there exists $(y_1, \dots, y_{d+1}) \in Y$ such that the points $y_1, \dots, y_{d+1}$ are coplanar. □

The following is a converse to Lemma 3 under slightly different conditions.

Lemma 4. *Suppose that the family of sets $Y_1, \dots, Y_{d+1} \subset \mathbb{R}^d$ is in general position. If some hyperplane intersects each of the sets $\operatorname{conv} Y_i$, then $Y_1, \dots, Y_{d+1}$ do not have the same-type property.*

Proof. Let H be any such hyperplane. We may assume that the sets $Y_1, \dots, Y_{d+1}$ are finite. Indeed, using Carathéodory's theorem, we may replace Y_i by a subset of at most $d+1$ points whose convex hull contains a point of $H \cap \operatorname{conv} Y_i$.

Keeping the condition $H \cap \operatorname{conv} Y_i \neq \emptyset$, perturb H so that it contains points of d sets among $Y_1, \dots, Y_{d+1}$, say the points $y_1 \in Y_1, \dots, y_d \in Y_d$. Since $Y_1 \cup \dots \cup Y_{d+1}$ is in general position, it follows that H contains no point of Y_{d+1}. Because H does intersect $\operatorname{conv} Y_{d+1}$, the set Y_{d+1} contains points y_{d+1} and y'_{d+1} that lie on the opposite sides of H. So, the orientations of the tuples $(y_1, \dots, y_d, y_{d+1})$ and $(y_1, \dots, y_d, y'_{d+1})$ are opposite, which contradicts the same-type property of $Y_1, \dots, Y_{d+1}$. □

Also, in further proofs it will be useful for us to have enough points in each set. The following lemma implies that small constructions can be blown up to arbitrarily large size, with no impact on the same-type constant $c(\cdots)$.

Lemma 5. *Suppose that $X_1, \dots, X_m$ in $\mathbb{R}^d$ is a family of sets in general position. Denote by $X_1^{(n)}$ the set obtained by replacing each point of X_1 by cloud of n points lying close enough to the original and preserving general position. Then,*

$$c(X_1, \dots, X_m) = c(X_1^{(n)}, X_2, \dots, X_m).$$

Proof. First, we prove the inequality $c(X_1, \dots, X_m) \leqslant c(X_1^{(n)}, \dots, X_m)$. Consider arbitrary subsets $Y_1 \subseteq X_1, \dots, Y_m \subseteq X_m$ having the same-type property. Take the subset $Y'_1 \subseteq X_1^{(n)}$, consisting of all points of clouds corresponding to the points of Y_1. Since the cloud are sufficiently small, $Y'_1, Y_2, \dots, Y_m$ also have the same-type property and $|Y_1|/|X_1| = |Y'_1|/|X_1^{(n)}|$.

Next, we prove the inequality $c(X_1, \dots, X_m) \geqslant c(X_1^{(n)}, \dots, X_m)$. Suppose that the subsets $(Y'_1, Y_2, \dots, Y_m)$ of $(X_1^{(n)}, X_2, \dots, X_m)$ have the same-type property. Define

$$Y_1 \stackrel{\text{def}}{=} \{x \in X_1 : Y'_1 \text{ contains a point of the cloud around } x\}.$$

If each cloud lies sufficiently near the original point, the sets $Y_1, \dots, Y_m$ have the same-type property, and $|Y_1|/|X_1|$ is at least $|Y'_1|/|X_1^{(n)}|$. □

This lemma implies that, in the definition of $c(m, d)$ it is enough to consider only the sets of the same size, which may be assumed to exceed an arbitrarily large constant.

Polynomial partitioning. For the proof of the lower bound we use the polynomial partitioning introduced by Guth and Katz. Since the proof in [7] does not track the dependence on d, we include the relevant calculation. The next lemma is a version of Theorem 4.1 in [7] with a fully explicit bound.

Lemma 6. *If X is a set of n points in $\mathbb{R}^d$ and $J \geqslant 1$ is an integer, then there is a polynomial surface Z of degree $D \leqslant 3d^2 2^{J/d}$ with the following property: each connected component of $\mathbb{R}^d \setminus Z$ contains at most $2^{-J} n$ points of X.*

We rely on the polynomial ham sandwich theorem. We say that the real algebraic hypersurface $\{x \in \mathbb{R}^d : f(x) = 0\}$ *bisects* a point set X if both sets $\{x \in \mathbb{R}^d : f(x) > 0\}$ and $\{x \in \mathbb{R}^d : f(x) < 0\}$ contain at most half of the points of X.

Lemma 7 (Corollary 4.3 in [7])**.** *Let $X_1, \ldots, X_M$ be finite sets of points in $\mathbb{R}^d$ with $M = \binom{D+d}{d} - 1$. Then there is a real algebraic hypersurface of degree at most D that bisects each X_i.*

Proof of the Lemma 6. Given polynomials $p_1, \ldots, p_j$ and a sign vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_j) \in \{-1, +1\}^j$, consider the cell of $\mathbb{R}^d$ on which the first j polynomials have these signs, i.e.,

$$C_\varepsilon \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : \operatorname{sign} p_1(x) = \varepsilon_1, \ldots, \operatorname{sign} p_j(x) = \varepsilon_j\}$$

Write $X_\varepsilon \stackrel{\text{def}}{=} X \cap C_\varepsilon$.

We claim that there are polynomials $p_1, \ldots, p_J$ of degrees $\deg p_{j+1} \leqslant d2^{j/d}$ such that $|X_\varepsilon| \leqslant |X|2^{-j}$ for every $j \leqslant J$ and every $\varepsilon \in \{-1, +1\}^j$.

We find such polynomials one by one. Suppose that the first j polynomials $p_1, \ldots, p_j$ have been defined, and that all 2^j sets X_ε with $\varepsilon \in \{-1, +1\}^j$ satisfy the condition above. Observe the inequality

$$\binom{d2^{j/d} + d}{d} = \prod_{i=0}^{d-1} \frac{d2^{j/d} + d - i}{d - i} > \prod_{i=0}^{d-1} 2^{j/d} = 2^j.$$

Taking this into account, Lemma 7 allows us to find a polynomial p_{j+1} of degree at most $d2^{j/d} + 1$ whose zero set bisects each X_ε.

Let p be the product of $p_1, \ldots, p_J$ and let Z be its zero set, we claim that Z is the desired hypersurface. The degree of p is at most

$$\sum_{j=0}^{J-1} \left(d2^{j/d} + 1\right) = d\left(1 + \frac{2^{J/d} - 1}{2^{1/d} - 1}\right) \leqslant 3d^2 2^{J/d}.$$

Since each connected component of $\mathbb{R}^d \backslash Z$ is a subset of some C_ε, and therefore contains at most $2^{-J}|X|$ points of X, the lemma follows. □

Also, we we will need the following bound due to Warren. One of its consequences is that the number of parts into which the surface from Lemma 6 cuts $\mathbb{R}^d$ is only slightly larger than 2^J.

Lemma 8 (Lemma 6.2 in [8])**.** *Let f be a real polynomial of degree D in d variables. Then the number of connected components of $\mathbb{R}^d \setminus Z(f)$ is at most $6(2D)^d$.*

3 Proof of Theorem 1: the lower bound

Consider any family $X_1, \dots, X_m \subseteq \mathbb{R}^d$ in general position. Thanks to Lemma 5, we may assume that all of them have the same size n, which is sufficiently large. Also, since slight perturbations of points do not change the orientation, by perturbing the sets generically we may additionally assume that, for any D, no polynomial surface of degree D intersects more than $\binom{D+d}{d} - 1$ points.

Fix $r \overset{\text{def}}{=} m^{d^2} d^{30d^3}$. For each i between 1 and m apply Lemma 6 with $J = \lceil \log_2 r \rceil$ to obtain polynomial surface Z_i of degree at most $D_0 \overset{\text{def}}{=} 6d^2 r^{1/d}$ such that each connected component of $\mathbb{R}^d \setminus Z_i$ contains at most n/r points of X_i. By Lemma 8, the total number of such components is at most $k \overset{\text{def}}{=} 6 \cdot 12^d d^{2d} r \leqslant m^{d^2} d^{50d^3}/4$. Denote by $\mathcal{C}_i$ the set of these components. Having taken n large enough, we observe that each Z_i contains at most $\binom{D_0+d}{d} - 1 \leqslant n/2$ points of X_i. Some components have at most $n/4k$ points of X_i; these account for at most $n/4$ points of X_i in all. Let $\mathcal{C}'_i$ be the set of components containing more than $n/4k$ points. Then, $|\mathcal{C}'_i| \geqslant (n - n/4 - n/2)/(n/r) = (n/4)/(n/r) = r/4$.

Next, we define an auxiliary $(d+1)$-uniform m-partite hypergraph H with parts $\mathcal{C}'_1, \mathcal{C}'_2, \dots, \mathcal{C}'_m$. For any distinct $i_1, i_2, \dots, i_{d+1}$ and any $C_{i_1} \in \mathcal{C}'_{i_1}, C'_{i_2} \in \mathcal{C}'_{i_2}, \dots, C_{i_{d+1}} \in \mathcal{C}'_{i_{d+1}}$ put an edge between vertices $C_{i_1}, C_{i_2}, \dots, C_{i_{d+1}}$ if and only if these components can be pierced by a single hyperplane. The key observation is that the hypergraph H is sparse, thanks to the polynomial partitioning.

Lemma 9. *Any $d+1$ parts of H span at most $d^{20d^2} r^{d+1-1/d}$ edges.*

Proof. Without loss of generality, consider parts indexed by $1, \dots, d+1$. Denote the polynomials defining Z_i by f_i.

Consider the following $d+1$ linear maps from $\mathbb{R}^{d^2+d-1} = (\mathbb{R}^d)^d \times (\mathbb{R}^1)^{d-1}$ to $\mathbb{R}^d$.

$$
\begin{aligned}
l_1(x_1, \dots, x_d, \alpha_1, \dots, \alpha_{d-1}) &\overset{\text{def}}{=} x_1, \\
l_2(x_1, \dots, x_d, \alpha_1, \dots, \alpha_{d-1}) &\overset{\text{def}}{=} x_2, \\
&\vdots \\
l_d(x_1, \dots, x_d, \alpha_1, \dots, \alpha_{d-1}) &\overset{\text{def}}{=} x_d, \\
l_{d+1}(x_1, \dots, x_d, \alpha_1, \dots, \alpha_{d-1}) &\overset{\text{def}}{=} \sum_{i=1}^{d-1} (\alpha_i x_i) + \Big(1 - \sum_{i=1}^{d-1} \alpha_i\Big) x_d.
\end{aligned}
$$

Denote by Z the algebraic hypersurface in $\mathbb{R}^{d^2+d-1} = (\mathbb{R}^d)^d \times (\mathbb{R}^1)^{d-1}$ defined by the polynomial

$$
h(x_1, \dots, x_d, \alpha_1, \dots, \alpha_{d-1}) \overset{\text{def}}{=} \prod_{i=1}^{d+1} f_i\Big(l_i(x_1, \dots, x_d, \alpha_1, \dots, \alpha_{d-1})\Big),
$$

and denote by $\mathcal{C}$ the set of corresponding connected components.

By the definition of h, the image of $\mathbb{R}^{d^2+d-1} \setminus Z$ under any l_i belongs to $\mathbb{R}^d \setminus Z_i$. Moreover, since l_i is continuous and the image of a connected set under a continuous map is connected, the image of any set in $\mathcal{C}$ is contained in exactly one of the sets in $\mathcal{C}_i$. This way $\ell_1 \times \ell_2 \times \cdots \times \ell_{d+1}$ induces a well-defined map $L\colon \mathcal{C} \to \mathcal{C}_1 \times \cdots \times \mathcal{C}_{d+1}$.

We observe that any tuple $(C_1, \ldots, C_{d+1}) \in \mathcal{C}'_1 \times \cdots \times \mathcal{C}'_{d+1} \subseteq \mathcal{C}_1 \times \cdots \times \mathcal{C}_{d+1}$ that forms an edge in H is in the image of L. Indeed, the sets $C_1, C_2, \ldots, C_{d+1}$ are open; so if some hyperplane pierces them, one can find points $x_1 \in C_1, x_2 \in C_2, \ldots, x_{d+1} \in C_{d+1}$ such that x_{d+1} is an affine combination of x_1 through x_d, say $x_{d+1} = \sum_{i=1}^d \alpha_i x_i$ with $\sum_{i=1}^d \alpha_i = 1$. In this case, the image of $(x_1, \ldots, x_d, \alpha_1, \ldots, \alpha_{d-1})$ under $l_1 \times l_2 \times \cdots \times l_{d+1}$ lies in $C_1 \times C_2 \times \cdots \times C_{d+1}$.

The reasoning above implies that the number of edges spanned by sets $\mathcal{C}'_1, \ldots, \mathcal{C}'_{d+1}$ in H does not exceed $|\mathcal{C}|$. Since polynomial h depends on d^2+d-1 variables and has degree at most $12d^3r^{1/d}$, Lemma 8 gives the bound of $6(24d^3)^{d^2+d-1}r^{(d^2+d-1)/d} \leqslant d^{20d^2}r^{d+1-1/d}$. $\square$

From each $\mathcal{C}'_i$ pick an element C_i independently at random. We claim that, with positive probability, the $d+1$ vertices $C_1, \ldots, C_{d+1}$ form an independent set in H. This would imply that the sets $C_1 \cap X_1, C_2 \cap X_2, \ldots, C_{d+1} \cap X_{d+1}$ have the same-type property. Since each of these has at last $n/4k$ elements, that would conclude the proof.

The claim follows from Lovász Local Lemma. Indeed, for the set $I \subset [m]$ of size $d+1$ denote by $\mathcal{B}_I$ the event that vertices $C_i \in \mathcal{C}'_i$ with $i \in I$ form an edge in H. Since $|\mathcal{C}'_i| \geqslant r/4$, Lemma 9 shows that probability of such event is at most $4^{d+1}d^{20d^2}r^{-1/d}$. If $I \cap J = \emptyset$, then the events $\mathcal{B}_I$ and $\mathcal{B}_J$ are defined by disjoint sets of random choices. Hence, the natural dependence graph has degree at most $(d+1)\binom{m}{d} \leqslant (d+1)(me)^d/d^d$. Since with our choice of the constant r we have

$$e\frac{(d+1)(me)^d 4^{d+1} d^{20d^2}}{r^{1/d}d^d} \leqslant \frac{d^{30d^2}m^d}{r^{1/d}} = 1,$$

the condition of the symmetric Local Lemma (see e.g., [1, Corollary 5.1.2]) is satisfied and the event $\bigcap_I \mathcal{B}_I$ holds with positive probability.

4 Proof of Theorem 1: the upper bound

To prove an upper bound on $c(r, m)$, we provide a series of constructions of arbitrarily large sets $X_1, \ldots, X_m$ without large subsets with the same-type property. A set $P \subset \mathbb{R}^d$ of size $\binom{n}{d}$ is a *grid set* if there exist n hyperplanes in $\mathbb{R}^d$ whose set of d-wise intersections is P. Our constructions will be suitable small perturbations of grid sets. The purpose of the perturbation is to ensure general position.

Convex sets intersect the grid sets and their perturbations slightly differently. The next lemma says that the difference is small, because the boundary of a convex set meets a grid set in a negligible fraction of points.

Denote by ∂C the boundary of a set $C \subset \mathbb{R}^d$.

Lemma 10. *Suppose that $P \subset \mathbb{R}^d$ is a grid set of size $\binom{n}{d}$. Then for any compact convex set C we have $|\partial C \cap P| \leqslant 2\binom{n}{d-1}$.*

Proof. We prove this by induction on d. If $d = 1$, then C is a segment, which has two boundary points. So, assume that $d > 1$ and that the lemma holds for $d-1$ in place of d. Let $H_1, \ldots, H_n$ be the n hyperplanes generating the grid set P. Put $C_i \stackrel{\text{def}}{=} C \cap H_i$. Write ∂C_i for the relative boundary of C_i inside the hyperplane H_i.

Consider an arbitrary point $x \in \partial C \cap P$. Since $x \in \partial C$, we can find a hyperplane H passing through x such that C lies on one side of H. Then, at least $d-1$ hyperplanes among $H_1, \ldots, H_n$ pass through x but differ from H. For each such H_i, the codimension-2 subspace $H_i \cap H$ contains x and bounds C_i inside H_i, implying that $x \in \partial C_i$.

Since this holds for every $x \in \partial C \cap P$, it follows that

$$|\partial C \cap P|(d-1) \leqslant \sum_{i=1}^{n} |\partial C_i \cap P|. \tag{1}$$

Observe that $C_i \cap P$ is itself a grid set inside the $(d-1)$-dimensional hyperplane H_i. Therefore, bounding $|\partial C_i \cap P|$ by induction, we obtain

$$|\partial C \cap P| \leqslant \frac{n}{d-1} \cdot 2\binom{n-1}{d-2} = 2\binom{n}{d-1}. \qquad \square$$

Fix a grid set X of size $\binom{n}{d}$. Let $X_1, \ldots, X_m$ be small perturbations of X chosen so that the family $X_1, \ldots, X_m$ is in general position. We shall show that these m sets do not contain large subsets with the same-type property.

For $x \in X_i$, write $P(x)$ for its *predecessor*, the point of X that x is a perturbation of. Similarly, write $P(Y)$ for the set of predecessors of a set $Y \subset X_i$. Let $\mathcal{H}$ be the set of n hyperplanes generating the grid set X.

Consider any sets $Y_1, \ldots, Y_m$ with the same-type property such that $Y_i \subseteq X_i$. Writing $\operatorname{int} A$ for the interior of a set A, define, for each $i = 1, 2, \ldots, m$,

$$Z_i \stackrel{\text{def}}{=} X \cap \operatorname{int} \operatorname{conv} Y_i.$$

Breaking the set $P(Y_i)$ into the boundary and the interior parts we obtain

$$\begin{aligned} P(Y_i) &= \big(P(Y_i) \cap \operatorname{int} \operatorname{conv} P(Y_i)\big) \cup \big(P(Y_i) \cap \partial \operatorname{conv} P(Y_i)\big) \\ &\subseteq \big(X \cap \operatorname{int} \operatorname{conv} P(Y_i)\big) \cup \big(X \cap \partial \operatorname{conv} P(Y_i)\big). \end{aligned}$$

If the perturbation defining Y_i is sufficiently small, $X \cap \operatorname{int} \operatorname{conv} P(Y_i) \subseteq Z_i$, and so

$$P(Y_i) \subseteq Z_i \cup \big(X \cap \partial \operatorname{conv} P(Y_i)\big).$$

Since $|Y_i| = |P(Y_i)|$, Lemma 10 tells us that

$$\begin{aligned} |Y_i| &\leqslant |Z_i| + |X \cap \partial \operatorname{conv} P(Y_i)| \\ &\leqslant |Z_i| + o(|X|) \quad \text{as } n \to \infty. \end{aligned} \tag{2}$$

Since $Z_i \subset \operatorname{conv} Y_i$, Lemma 4 implies that no hyperplane in $\mathcal{H}$ intersects more than d sets among $Z_1, \ldots, Z_m$. By the pigeonhole principle, for some i, set Z_i intersects at most

dn/m hyperplanes of $\mathcal{H}$. Hence, this Z_i is contained in the grid set generated by dn/m hyperplanes, implying that

$$|Z_i| \leqslant \binom{dn/m}{d} = (d/m)^d \binom{n}{d}\big(1+o(1)\big) = (d/m)^d |X| \big(1+o(1)\big) \quad \text{as } n \to \infty.$$

From this and (2), it follows that $c(m,d) \leqslant (d/m)^d$.

Remark. The preceding argument also shows that the upper bound of $(d/m)^d$ holds for the non-partite version of the same-type lemma. Precisely, if X is a slight perturbation of a grid set of size $\binom{n}{d}$, and $Y_1, \ldots, Y_m \subset X$ are disjoint sets with the same-type property, then at least one of Y_i is of size at most $\binom{dn/m}{d}\big(1+o(1)\big)$.

5 Arbitrarily good approximations to $c(m,d)$

We now turn to the task of computing arbitrarily good approximations to $c(m,d)$. We use the following well-known result of Vapnik and Červonenkis about the existence of ε-approximants.

Lemma 11 (Section 1.5 of [17])**.** *Let $\mathcal{F} \subseteq 2^X$ be a set family of VC-dimension D. Then, for any $0 < \varepsilon < 1$ there exists a set $A \subseteq X$ of size at most $\frac{32}{\varepsilon^2} D \ln \frac{16D}{\varepsilon^2}$ such that*

$$\left| \frac{|F \cap X|}{|X|} - \frac{|F \cap A|}{|A|} \right| \leqslant \varepsilon \qquad \text{for all } F \in \mathcal{F}.$$

For the purpose of estimating $c(m,n)$, this allows us to limit the search to bounded-size families.

Proposition 12. *For any natural numbers m, d and any $\varepsilon > 0$ there exist sets $A_1, \ldots, A_m \subseteq \mathbb{R}^d$ of size bounded by a computable function of m, d, ε such that $|c(A_1, \ldots, A_m) - c(m,d)| \leqslant \varepsilon$.*

Proof. Pick finite sets $X_1, \ldots, X_m \subset \mathbb{R}^d$ in general position such that $c(X_1, \ldots, X_m) < c(m,d) + \varepsilon/2$. Let $\mathcal{F}$ be the family of open polytopes in $\mathbb{R}^d$ with at most m facets, its VC-dimension is at most $O(dm \log m)$ (see [10, Lemma 10.3.1 and Proposition 10.3.3]). Apply Lemma 11 to each X_i and $\mathcal{F}$ with $\varepsilon/2$ in place of ε to obtain sets A_i of size bounded by some function of m, d, ε.

We claim that $c(m,d) \leqslant c(A_1, \ldots, A_m) \leqslant c(m,d) + \varepsilon$. Since the former inequality follows from the definition of the constant $c(\cdots)$, it remains to show the latter one. To that end, consider arbitrary subsets $Y_1 \subseteq A_1, \ldots, Y_m \subseteq A_m$ with the same-type property. By Lemma 4, for each i the set Y_i is separated from $Y_1 \cup \cdots \cup Y_{i-1} \cup Y_{i+1} \cup \cdots \cup Y_m$ by some hyperplane, which we denote by H_i. The hyperplanes $H_1, \ldots, H_m$ form a hyperplane arrangement; each Y_i is contained within a single cell of this arrangement, which we denote by P_i. Since P_i is an intersection of m halfspaces, each P_i is an open polyhedron with at most m facets. Observe that the sets $P_1, \ldots, P_m$ have the same-type property; this implies that $P_1 \cap X_1, \dots, P_m \cap X_m$ have the same-type property as well, and so, for some i, we have

$$\frac{|P_i \cap X_i|}{|X_i|} \leqslant c(m,d) + \varepsilon/2.$$

Therefore, for this value of i, we have

$$\frac{|Y_i|}{|A_i|} \leqslant \frac{|P_i \cap A_i|}{|A_i|} \leqslant \frac{|P_i \cap X_i|}{|X_i|} + \varepsilon/2 \leqslant c(m,d) + \varepsilon.$$

Since the sets $Y_1, \dots, Y_m$ are arbitrary, this completes the proof. □

The approximability of the constants $c(m,d)$ now follows from the famous result of Tarski on the decidability of the theory of real closed fields [16] (see for example [3, Theorem 2.77] for a modern exposition). Indeed, the value of $c(A_1, \dots, A_m)$ depends only on the orientations of $(d+1)$-tuples from $A_1 \cup \dots \cup A_m$. The existence of $A_1, \dots, A_m$ with specified orientations of $(d+1)$-tuples can be expressed as an existential sentence in the language of ordered fields. This sentence is decidable by the aforementioned result of Tarski (though deciding existential sentences can be done more efficiently than general sentences; see, e.g., [3, Algorithm 13.1]). So, the largest value of $c(A_1, \dots, A_m)$, subject to $|A_i| = B_i$, can be computed by iterating over all possible sign patterns and checking if they are realizable by point sets in $\mathbb{R}^d$.

Acknowledgements

We are grateful to Amzi Jeffs for helpful discussions.

The work was supported by U.S. taxpayers through NSF grants DMS-2154063 and CAREER DMS-2042428.

References

[1] Alon, N., and Spencer, J. H. *The probabilistic method*, fourth ed. Wiley Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, 2016.

[2] Bárány, I., and Valtr, P. A positive fraction Erdős–Szekeres theorem. *Discrete Comput. Geom. 19*, 3 (1998), 335–342.

[3] Basu, S., Pollack, R., and Roy, M.-F. *Algorithms in real algebraic geometry*, vol. 10 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2003.

[4] Bukh, B., and Hubard, A. Space crossing numbers. *Combin. Probab. Comput. 21*, 3 (2012), 358–373.

[5] Fabila-Monroy, R., and Huemer, C. Carathéodory's theorem in depth. *Discrete Comput. Geom. 58*, 1 (2017), 51–66.

[6] Fox, J., Pach, J., and Suk, A. A polynomial regularity lemma for semialgebraic hypergraphs and its applications in geometry and property testing. *SIAM Journal on Computing 45*, 6 (2016), 2199–2223. arXiv:1502.01730.

[7] Guth, L., and Katz, N. H. On the Erdős distinct distances problem in the plane. *Ann. of Math. (2) 181*, 1 (2015), 155–190. arXiv:1011.4105.

[8] Kaplan, H., Matoušek, J., and Sharir, M. Simple proofs of classical theorems in discrete geometry via the Guth-Katz polynomial partitioning technique. *Discrete Comput. Geom. 48*, 3 (2012), 499–517. arXiv:1102.5391.

[9] Károlyi, G., and Tóth, G. Erdős-Szekeres theorem for point sets with forbidden subconfigurations. *Discrete Comput. Geom. 48*, 2 (2012), 441–452.

[10] Matoušek, J. *Lectures on discrete geometry*, vol. 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[11] Mirzaei, M., and Suk, A. A positive fraction mutually avoiding sets theorem. *Discrete Math. 343*, 3 (2020), 111730, 6.

[12] Pach, J. A Tverberg-type result on multicolored simplices. *Comput. Geom. 10*, 2 (1998), 71–76. arXiv:math/9603211.

[13] Pach, J., and Solymosi, J. Crossing patterns of segments. *J. Combin. Theory Ser. A 96*, 2 (2001), 316–325.

[14] Pór, A., and Valtr, P. The partitioned version of the Erdős-Szekeres theorem. vol. 28. 2002, pp. 625–637. Discrete and computational geometry and graph drawing (Columbia, SC, 2001).

[15] Suk, A. On the Erdős-Szekeres convex polygon problem. *J. Amer. Math. Soc. 30*, 4 (2017), 1047–1053.

[16] Tarski, A. *A decision method for elementary algebra and geometry*. University of California Press, Berkeley-Los Angeles, Calif., 1951. 2nd ed.

[17] Vapnik, V. N., and Červonenkis, A. J. The uniform convergence of frequencies of the appearance of events to their probabilities. *Teor. Verojatnost. i Primenen. 16* (1971), 264–279. http://mi.mathnet.ru/tvp2146 (English translation is at https://epubs.siam.org/doi/10.1137/1116025).