SIAM J. Discrete Math.
Vol. 38, No. 3, pp. 2095–2107

PRODUCT STRUCTURE EXTENSION OF THE ALON–SEYMOUR–THOMAS THEOREM*

MARC DISTEL[†], VIDA DUJMOVIĆ[‡], DAVID EPPSTEIN[§], ROBERT HICKINGBOTHAM[†], GWENAËL JORET[¶], PIOTR MICEK[‖], PAT MORIN[#], MICHAŁ T. SEWERYN[¶], AND DAVID R. WOOD[†]

Abstract. Alon, Seymour, and Thomas [*J. Amer. Math. Soc.*, 3 (1990), pp. 801–808] proved that every n-vertex graph excluding K_t as a minor has treewidth less than $t^{3/2}\sqrt{n}$. Illingworth, Scott, and Wood [*Product Structure of Graphs with an Excluded Minor*, preprint, arXiv:2104.06627, 2022] recently refined this result by showing that every such graph is a subgraph of some graph with treewidth $t-2$, where each vertex is blown up by a complete graph of order $\mathcal{O}(\sqrt{tn})$. Solving an open problem of Illingworth, Scott, and Wood [2022], we prove that the treewidth bound can be reduced to 4 while keeping blowups of order $\mathcal{O}_t(\sqrt{n})$. As an extension of the Lipton–Tarjan theorem, in the case of planar graphs, we show that the treewidth can be further reduced to 2, which is best possible. We generalize this result for $K_{3,t}$-minor-free graphs, with blowups of order $\mathcal{O}(t\sqrt{n})$. This setting includes graphs embeddable on any fixed surface.

Key words. graph, minor, treewidth, product, planar graph, separator

MSC codes. 05C83, 05C10

DOI. 10.1137/23M1591773

1. Introduction.

Treewidth is a measure of how similar a given graph is to a tree, and is of fundamental importance in structural and algorithmic graph theory; see [2, 15, 24] for surveys.

In one of the cornerstone results of graph minor theory, Alon, Seymour, and Thomas [1] proved that every n-vertex K_t-minor-free graph G has treewidth $\text{tw}(G) < t^{3/2}n^{1/2}$, which implies that G has a balanced separator of order at most $t^{3/2}n^{1/2}$.

*Received by the editors August 4, 2023; accepted for publication (in revised form) April 4, 2024; published electronically July 9, 2024.
https://doi.org/10.1137/23M1591773

Funding: The research of the first and fourth authors was supported by Australian Government Research Training Program Scholarships. The research of the second author was supported by NSERC and a University of Ottawa Research Chair. The work of the third author was supported in part by NSF grant CCF-2212129. The fifth author was supported by a CDR grant from the Belgian National Fund for Scientific Research (FNRS), by a PDR grant from FNRS, and by the Australian Research Council. The research of the sixth author was supported by the National Science Center of Poland under grant UMO-2018/31/G/ST1/03718 within the BEETHOVEN program. The research of the seventh author was supported by NSERC and the Ontario Ministry of Research and Innovation. The work of the eighth author was supported by a PDR grant from the Belgian National Fund for Scientific Research (FNRS). The research of the ninth author was supported by the Australian Research Council.

[†]School of Mathematics, Monash University, Melbourne, Victoria, Australia (marc.distel@monash.edu, robert.hickingbotham@monash.edu, david.wood@monash.edu).
[‡]School of Computer Science and Electrical Engineering, University of Ottawa, Ottawa, ON, K1N 6N5, Canada (vida.dujmovic@uottawa.ca).
[§]Computer Science Department, University of California, Irvine, Irvine, CA 92697-3425 USA (eppstein@uci.edu).
[¶]Département d'Informatique, Université libre de Bruxelles, Brussels, 1050, Belgium (gwenael.joret@ulb.be, michal.seweryn@ulb.be).
[‖]Department of Theoretical Computer Science, Jagiellonian University, Kraków, Poland (piotr.micek@uj.edu.pl).
[#]School of Computer Science, Carleton University, Ottawa, Ontario, K1S 5B6, Canada (morin@scs.carleton.ca).

For fixed $t \geqslant 5$, this bound is asymptotically tight since the $n^{1/2} \times n^{1/2}$ grid is K_5-minor-free and has treewidth $n^{1/2}$.

Our goal is to prove qualitative strengthenings of the Alon–Seymour–Thomas theorem through the lens of graph product structure theory, which describes graphs in complicated classes as subgraphs of products of simpler graphs. Here we consider products of bounded treewidth graphs and complete graphs. To be precise, for a graph H and $m \in \mathbb{N}$, let $H \boxtimes K_m$ be the *strong product* of H and a complete graph K_m, which is the "complete-blow-up" of H by K_m, that is, the graph obtained by replacing each vertex of H by a copy of K_m and replacing each edge of H by the complete join between the corresponding copies of K_m. Say a graph G is *contained* in a graph X if G is isomorphic to a subgraph of X.

Illingworth, Scott, and Wood [18] showed that for any integer $t \geqslant 4$, every n-vertex K_t-minor-free graph G is contained in $H \boxtimes K_m$ for some graph H with treewidth at most $t-1$, where $m < \sqrt{tn}$. This result implies and strengthens the Alon–Seymour–Thomas theorem since

$$\operatorname{tw}(G) \leqslant \operatorname{tw}(H \boxtimes K_m) \leqslant (\operatorname{tw}(H)+1)m - 1 < t\sqrt{tn}.$$

Importantly, they also showed a similar result with treewidth $t-2$ (and a slightly larger value of m): every n-vertex K_t-minor-free graph G is contained in $H \boxtimes K_m$ for some graph H with treewidth at most $t-2$, where $m < 2\sqrt{tn}$.

The following definition, implicitly introduced by Illingworth et al. [18], naturally arises. For a proper minor-closed graph class $\mathcal{G}$, let $f(\mathcal{G})$ be the minimum integer such that for some c, every n-vertex graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$ for some graph H with treewidth at most $f(\mathcal{G})$, where $m \leqslant c\sqrt{n}$. The above result of Illingworth et al. [18] implies that $f(\mathcal{G})$ is well-defined; in particular, if $\mathcal{G}_t$ is the class of K_t-minor-free graphs, then $f(\mathcal{G}_t) \leqslant t-2$.

Illingworth et al. [18] asked whether $f(\mathcal{G})$ is upper bounded by an absolute constant. This paper answers this question in the affirmative.

Theorem 1.1. *Every n-vertex K_t-minor-free graph G is contained in $H \boxtimes K_m$ for some graph H of treewidth at most* 4*, where $m \in O_t(\sqrt{n})$.*

Theorem 1.1 implies that $f(\mathcal{G}) \leqslant 4$ for every proper minor-closed class $\mathcal{G}$. The proof of Theorem 1.1 actually shows that $\operatorname{tw}(H-v) \leqslant 3$ for some vertex $v \in V(H)$.

We also give improved bounds on $f(\mathcal{G})$ for particular minor-closed classes $\mathcal{G}$. First consider the class $\mathcal{L}$ of planar graphs. The Lipton–Tarjan separator theorem [21] is one of the most important structural results about planar graphs, with numerous algorithmic applications [22]. It is equivalent to saying that every n-vertex planar graph has treewidth $\mathcal{O}(\sqrt{n})$ (see [12]). Since planar graphs are K_5-minor-free, the above result of Illingworth et al. [18] shows that $f(\mathcal{L}) \leqslant 3$. Our next contribution shows that $f(\mathcal{L}) \leqslant 2$, resolving an open problem of Illingworth et al. [18].

Theorem 1.2. *Every n-vertex planar graph is contained in $H \boxtimes K_m$, where H is a graph with treewidth* 2 *and $m \in \mathcal{O}(\sqrt{n})$.*

As an aside, since every graph with treewidth 2 is planar, the graph H in Theorem 1.2 is planar (although not necessarily a minor of the original planar graph).

We actually prove a more general result than Theorem 1.2 for graphs that exclude a $K_{3,t}$ minor.

Theorem 1.3. *Every $K_{3,t}$-minor-free n-vertex graph is contained in $H \boxtimes K_m$, where H is a graph with treewidth* 2 *and $m \in \mathcal{O}(t\sqrt{n})$.*

Since $K_{3,3}$ is not planar, Theorem 1.3 with $t=3$ implies Theorem 1.2. More generally, Theorem 1.3 also implies results for graphs embeddable in any fixed surface. The *Euler genus* of a surface with h handles and c cross-caps is $2h+c$. The *Euler genus* of a graph G is the minimum integer $g \geqslant 0$ such that there is an embedding of G in a surface of Euler genus g; see [23] for more about graph embeddings in surfaces. It follows from Euler's formula that $K_{3,2g+3}$ has Euler genus greater than g. Thus Theorem 1.3 implies the following.

COROLLARY 1.4. *Every n-vertex graph with Euler genus g is contained in $H \boxtimes K_m$, where H is a graph with treewidth 2 and $m \in \mathcal{O}((g+1)\sqrt{n})$.*

Note that Gilbert, Hutchinson, and Tarjan [14] and Djidjev [6] proved that n-vertex graphs with Euler genus $g>0$ admit balanced separators of order $\mathcal{O}(\sqrt{gn})$ and thus have treewidth $\mathcal{O}(\sqrt{gn})$. Corollary 1.4 is a qualitative strengthening of these results, with slightly worse dependence on g.

1.1. Related work. We first mention a connection to clustered coloring. A (vertex-)k-coloring of a graph has *clustering* c if every monochromatic component has at most c vertices. This is equivalent to saying that G is contained in $H \boxtimes K_c$ for some graph H with $\chi(H) \leqslant k$. Clustered coloring has been widely studied in recent years; see [28] for a survey. Linial et al. [20] showed that n-vertex planar graphs, and more generally graphs excluding any fixed minor, are 3-colorable with clustering $\mathcal{O}(\sqrt{n})$. Since treewidth 2 graphs are 3-colorable, in the case of planar or $K_{3,t}$-minor-free graphs, Theorems 1.2 and 1.3 are a qualitative improvement over the result of Linial et al. [20].

Clustered colorings also provide lower bounds. Linial et al. [20] constructed a family of planar graphs $\{G_k : k \geqslant 1\}$, where G_k has $2k^3+1$ vertices and every 2-coloring of G_k has a monochromatic component with at least $k^2/2$ vertices. In particular, if G_k is contained in $H \boxtimes K_m$ for some graph H with treewidth 1 (that is, H is a forest), then a proper 2-coloring of H determines a 2-coloring of G_k with clustering m, implying $m \in \Omega(n^{2/3})$ where $n := |V(G_k)|$. Hence $f(\mathcal{L}) > 1$. Therefore the bounds on the treewidth of H in Theorems 1.2 and 1.3 and Corollary 1.4 are best possible. In particular, $f(\mathcal{L}) = 2$, and if $\mathcal{G}_{3,t}$ is the class of $K_{3,t}$-minor-free graphs, then $f(\mathcal{G}_{3,t}) = 2$ for $t \geqslant 3$. These lower bounds lead to the following characterization of minor-closed classes $\mathcal{G}$ with $f(\mathcal{G}) \leqslant 1$.

PROPOSITION 1.5. *For a minor-closed class $\mathcal{G}$, $f(\mathcal{G}) \leqslant 1$ if and only if $\mathcal{G}$ has bounded treewidth.*

Proof. Dvořák and Wood [13, Theorem 8 with $t=1$] proved that every n-vertex graph with treewidth k is contained in $H \boxtimes K_m$, where H is a star and $m \leqslant \sqrt{(k+1)n}$. Since a star has treewidth 1, if $\mathcal{G}$ has bounded treewidth, then $f(\mathcal{G}) \leqslant 1$. For the converse, if $\mathcal{G}$ has unbounded treewidth, then by the grid minor theorem [25], every planar graph is in $\mathcal{G}$, and thus $f(\mathcal{G}) \geqslant f(\mathcal{L}) = 2$, as desired. ◻

We conclude by mentioning the following related definition and results. Campbell et al. [3] defined the *underlying treewidth* of a graph class $\mathcal{G}$ to be the minimum integer k such that for some function g every graph $G \in \mathcal{G}$ is contained in $H \boxtimes K_m$, where $\text{tw}(H) \leqslant k$ and $m \leqslant g(\text{tw}(G))$. Here m is required to depend only on $\text{tw}(G)$, whereas the present paper allows $m \in O(\sqrt{n})$. Among other results, Campbell et al. [3] showed[1] that the underlying treewidth of $\mathcal{G}_t$ equals $t-2$. Thus, in the underlying treewidth setting, no absolute bound on $\text{tw}(H)$ is possible, unlike in the setting of

[1]In the result of Campbell et al. [3], $g(w) \in O_t(w^2 \log w)$, which was improved to $O_t(w)$ by Illingworth et al. [18].

$\mathcal{O}(\sqrt{n})$ blowups, where Theorem 1.1 achieves $\operatorname{tw}(H) \leqslant 4$. There is a similar distinction for planar graphs. Campbell et al. [3] showed that the underlying treewidth of the class of planar graphs equals 3. So in Theorem 1.2 with $\operatorname{tw}(H) \leqslant 2$, the bound of $m \in \mathcal{O}(\sqrt{n})$ cannot be improved to $m \leqslant g(\operatorname{tw}(G))$ for any function g. See [8] for recent results on underlying treewidth.

2. Background.

For $m, n \in \mathbb{Z}$ with $m \leqslant n$, let $[m, n] := \{m, m+1, \ldots, n\}$ and $[n] := [1, n]$.

We consider simple, finite, undirected graphs G with vertex-set $V(G)$ and edge-set $E(G)$. For a graph G and set $S \subseteq V(G)$, let $N_G(S) := \{v \in V(G) \setminus S : \exists vw \in E(G), w \in S\}$ and let $N_G[S] := N_G(S) \cup S$. We drop the subscript G if the graph in question is clear.

A *tree-decomposition* of a graph G is a collection $\mathcal{T} = (B_x : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the vertices of a tree T, such that (a) for every edge $uv \in E(G)$, some bag B_x contains both u and v, and (b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a nonempty (connected) subtree of T. The *width* of $\mathcal{T}$ is $\max\{|B_x| : x \in V(T)\} - 1$. The *treewidth* of a graph G, denoted by $\operatorname{tw}(G)$, is the minimum width of a tree-decomposition of G.

Consider a tree-decomposition $\mathcal{T} = (B_x : x \in V(T))$ of a graph G. The *adhesion* of $\mathcal{T}$ is $\max\{|B_x \cap B_y| : xy \in E(T)\}$. The *torso* of a bag B_x (with respect to $\mathcal{T}$), denoted by $G\langle B_x\rangle$, is the graph obtained from the induced subgraph $G[B_x]$ by adding edges so that $B_x \cap B_y$ is a clique for each edge $xy \in E(T)$. We say $\mathcal{T}$ is *rooted* if T is rooted. Then, for each $x \in V(T)$, a clique C in the torso $G\langle B_x\rangle$ is a *child-adhesion clique* if there is a child y of x such that $C \subseteq B_x \cap B_y$.

A *path-decomposition* is a tree-decomposition in which the underlying tree is a path, simply denoted by the corresponding sequence of bags $(B_1, \ldots, B_n)$.

A graph H is a *minor* of a graph G if H is isomorphic to a graph that can be obtained from a subgraph of G by contracting edges. A graph G is *H-minor-free* if H is not a minor of G. A graph class $\mathcal{G}$ is *minor-closed* if every minor of every graph in $\mathcal{G}$ is in $\mathcal{G}$. A graph class is *proper* if it is not the class of all graphs. The graph minor structure theorem of Robertson and Seymour [26] shows that every K_t-minor-free graph has a tree-decomposition where each torso can be constructed using three ingredients: graphs on surfaces, vortices, and apex vertices. To describe this formally, we need the following definitions.

Let G_0 be a graph embedded in a surface Σ. A closed disc D in Σ is *G_0-clean* if its only points of intersection with G_0 are vertices of G_0 that lie on the boundary of D. Let $x_1, \ldots, x_b$ be the vertices of G_0 on the boundary of D in the order around D. A *D-vortex* (with respect to G_0) of a graph H is a path-decomposition $(B_1, \ldots, B_b)$ of H such that $x_i \in B_i$ for each $i \in [b]$, and $V(G_0 \cap H) = \{x_1, \ldots, x_b\}$.

For integers $g, p, a \geqslant 0$ and $k \geqslant 1$, a graph G is *(g, p, k, a)-almost-embeddable* if for some set $A \subseteq V(G)$ with $|A| \leqslant a$, there are graphs $G_0, G_1, \ldots, G_p$ such that

- $G - A = G_0 \cup G_1 \cup \cdots \cup G_p$,
- $G_1, \ldots, G_p$ are pairwise vertex-disjoint,
- G_0 is embedded in a surface Σ of Euler genus at most g,
- there are p pairwise disjoint G_0-clean closed discs $D_1, \ldots, D_p$ in Σ, and
- for $i \in [p]$, there is a D_i-vortex $(B_1, \ldots, B_{b_i})$ of G_i of width at most k.

The vertices in A are called *apex* vertices—they can be adjacent to any vertex in G. A graph is *ℓ-almost-embeddable* if it is (g, p, k, a)-almost-embeddable for some $g, p, k, a \leqslant \ell$.

We use the following version of the graph minor structure theorem, which is implied by a result of [4, Theorem 4].

THEOREM 2.1 (see [4]). *For every integer $t \geqslant 1$ there exists an integer $k \geqslant 1$ such that every K_t-minor-free graph G has a rooted tree-decomposition $(B_x : x \in V(T))$ such that for every node $x \in V(T)$, the torso $G\langle B_x \rangle$ is k-almost-embeddable and if A_x is the apex-set of $G\langle B_x\rangle$, then for every child-adhesion clique C of $G\langle B_x\rangle$, either $C \setminus A_x$ is contained in a bag of a vortex of $G\langle B_x\rangle$, or $|C \setminus A_x| \leqslant 3$.*

The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex-set $V(A) \times V(B)$, where distinct vertices $(v,x),(w,y) \in V(A) \times V(B)$ are adjacent if $v = w$ and $xy \in E(B)$, or $x = y$ and $vw \in E(A)$, or $vw \in E(A)$ and $xy \in E(B)$.

Let G be a graph. A *partition* of G is a collection $\mathcal{P}$ of sets of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part.* Empty parts are allowed. The *width* of $\mathcal{P}$ is the maximum number of vertices in a part. The *quotient* of $\mathcal{P}$ (with respect to G) is the graph, denoted by $G/\mathcal{P}$, whose vertices are the nonempty parts in $\mathcal{P}$, where distinct nonempty parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B. For a graph H, an *H-partition* of G is a partition $\mathcal{P} = (\mathcal{P}_x \subseteq V(G) : x \in V(H))$ of G indexed by $V(H)$, such that for each edge $vw \in E(G)$, if $v \in \mathcal{P}_x$ and $w \in \mathcal{P}_y$ then $x = y$ or $xy \in E(H)$. That is, $G/\mathcal{P}$ is contained in H. The following observation connects partitions and products.

Observation 2.2 (see [9]). For all graphs G and H and any integer $p \geqslant 1$, G is contained in $H \boxtimes K_p$ if and only if G has an H-partition with width at most p.

A *layering* of a graph G is a partition $\mathcal{P}$ of G, whose parts are ordered $\mathcal{P} = (V_0, V_1, \dots)$ such that for each edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leqslant 1$. Equivalently, a layering is a P-partition for some path P. Consider a connected graph G. Let $r \in V(G)$ and let $V_i := \{v \in V(G) : \mathrm{dist}_G(v,r) = i\}$ for each $i \geqslant 0$. Then $(V_0, V_1, \dots)$ is a BFS-*layering* of G rooted at r. Let T be a spanning tree of G, where for each non-root vertex $v \in V_i$ there is a unique edge vw in T for some $w \in V_{i-1}$. Then T is called a BFS-*spanning tree* of G. (These trees are a superset of the trees that can be generated by the breadth-first search algorithm.)

If T is a tree rooted at a vertex r, then a nonempty path P in T is *vertical* if the vertex of P closest to r in T is an end-vertex of P.

Many recent results show that certain graphs can be described as subgraphs of the strong product of a graph with bounded treewidth and a path [5, 7, 9, 11, 16, 17, 27]. For example, Distel et al. [5] proved the following result (building on the work of Dujmović et al. [9]).

LEMMA 2.3 (see [5]). *Every connected graph G of Euler genus at most g is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some planar graph H with treewidth* 3*, and for some path P. In particular, for every rooted spanning tree T of G, there is a planar graph H with treewidth at most* 3 *and there is an H-partition $\mathcal{P}$ of G such that each part of $\mathcal{P}$ is a subset of the union of at most* $\max\{2g,3\}$ *vertical paths in T.*

3. Proof of Theorem 1.1.

This section proves Theorem 1.1 for K_t-minor-free graphs, where the graph minor structure theorem is our main tool. We first prove an analogue of Theorem 1.1 for almost-embeddable graphs with several additional properties that will be needed later.

LEMMA 3.1. *For integers $g, p, a \geqslant 0$ and $k, n \geqslant 1$ and $d \geqslant 4$, for every (g,p,k,a)-almost embeddable n-vertex graph G with apex set A, there exists a set $S \subseteq V(G)$*

where $|S| \leqslant \frac{n}{d-3} + a$ *such that* $G - S$ *has an* H*-partition with width at most* $(2g+4p+3)(2\sqrt{(k+1)n}+d+2k+2)$*, where* H *is planar with treewidth at most* 3*. Moreover,* $A \subseteq S$ *and any clique in a vortex of* G *is contained in at most two parts.*

Proof. Let $G_0, G_1, \dots, G_p$ and $D_1, \dots, D_p$ be as in the definition of (g,p,k,a)-almost embeddable. Let G_0' be obtained from G_0 as follows. Initialize $G_0' := G_0$ and add edges to G_0' so that it is connected and is still embedded in the same surface as G_0, and $D_1, \dots, D_p$ are G_0'-clean.

For each $i \in [p]$, modify G_0' as follows. Say the vertices around D_i are $x_1, \dots, x_b$. In G_0', add edges so that $(x_1, \dots, x_b)$ is a path, and add a vertex z_i into the disc D_i adjacent to $x_1, \dots, x_b$. Note that since D_i was initially G_0'-clean for each $i \in [p]$ and $D_1, \dots, D_p$ are pairwise disjoint, this can be done while maintaining an embedding of G_0' in the same surface as G_0.

Now apply the following operation for each $i \in [p]$. Let $(B_1, \dots, B_b)$ be a D_i-vortex of G_i with width at most k, where $x_j \in B_j$ for each $j \in [b]$. Greedily find an increasing sequence of integers $a_1, \dots, a_{q+1}$ so that $a_1 = 1$, $a_{q+1} = b+1$, and for each $j \in [q]$, if $Z_i := B_{a_1} \cup B_{a_2} \cup \cdots \cup B_{a_q}$ and $Y_{i,j} := (B_{a_j+1} \cup B_{a_j+2} \cup \cdots \cup B_{a_{j+1}-1}) \setminus Z_i$, then $\lceil \sqrt{(k+1)n} \rceil \leqslant |Y_{i,j}| \leqslant \lceil \sqrt{(k+1)n} \rceil + k$ for each $j \in [q-1]$ and $|Y_{i,q}| \leqslant \lceil \sqrt{(k+1)n} \rceil + k$. Note that $n \geqslant (q-1)\sqrt{(k+1)n}$, so $|Z_i| \leqslant (k+1)q \leqslant (k+1)(n/\sqrt{(k+1)n}+1) = \sqrt{(k+1)n}+k+1$.

Every clique in G_i is contained in $Y_{i,j} \cup Z_i$ for some $j \in [q]$. In G_0' contract the path $(x_{a_j+1}, x_{a_j+2}, \dots, x_{a_{j+1}-1})$ into a vertex $y_{i,j}$, for each $j \in [q]$. In G_0' contract the edge $z_i x_{a_j}$ into z_i for each $j \in [q]$. Call the vertices $y_{i,j}$ and z_i of G_0' *special.*

For each $i \in [p]$, let F_i' be some face of G_0' incident to z_i. If $p = 0$, then add a vertex r to G_0' adjacent to some vertex of G_0. If $p \geqslant 1$, then for each $i \in [p-1]$, add a handle to the surface in which G_0' is embedded between F_i' and F_{i+1}'. The resulting embedding of G_0' has a single face F' incident to each of $z_1, \dots, z_p$. Add a vertex r to G_0' adjacent to $z_1, \dots, z_p$. Embed r and the edges incident to r in F'. Note that (for any value of p) the resulting surface has Euler genus at most $g + 2\max\{0, p-1\} \leqslant g + 2p$.

Let T be a BFS-spanning tree of G_0' rooted at r (which exists since G_0' is connected). Let $(V_0, V_1, \dots)$ be the corresponding BFS-layering of G_0'. So $V_0 = \{r\}$, and if $p \geqslant 1$, then $V_1 = \{z_1, \dots, z_p\}$ and V_2 contains all $y_{i,j}$ vertices (plus possibly others). By Lemma 2.3 there is an H'-partition $\mathcal{P}'$ of G_0' where H' is planar with treewidth at most 3 such that each part of $\mathcal{P}'$ is a subset of the union of at most $\max\{2g+4p, 3\} \leqslant 2g+4p+3$ vertical paths in T. Note that each vertical path in T has at most two special vertices (some z_i and some $y_{i,j}$).

For each $i \in [3, d-1]$, let $\widehat{V}_i := V_i \cup V_{i+d} \cup V_{i+2d} \cup \cdots$. Since $|\widehat{V}_3| + |\widehat{V}_4| + \cdots + |\widehat{V}_{d-1}| \leqslant n$, there exists $\ell \in [3, d-1]$ such that $|\widehat{V}_\ell| \leqslant n/(d-3)$. Let $S := \widehat{V}_\ell \cup A$. Then $|S| \leqslant n/(d-3) + a$.

Let $V_i := \varnothing$ for $i < 0$, and for any integer $j \geqslant 0$, let $\mathcal{P}_j'$ be the H_j'-partition of $G_0'[V_{\ell+(j-1)d+1} \cup \cdots \cup V_{\ell+jd-1}]$ induced by $\mathcal{P}'$, where H_j' is a copy of H' (and $H_0', H_1', \dots$ are pairwise disjoint). Then $\mathcal{P}_j'$ has width at most $(2g+4p+3)d$.

Let H be the disjoint union of $H_0', H_1', \dots$. Then H is planar with treewidth at most 3. Now $\mathcal{P}_0' \cup \mathcal{P}_1' \cup \cdots$ is an H-partition of $G_0' - S$ where each part is a subset of the union of at most $(2g+4p+3)$ vertical paths of length at most $d-1$ in T. Hence, the width of this partition is smaller than $(2g+4p+3)d$.

We now modify this partition of $G_0' - S$ into a partition of $G - S$. By construction (since $\ell \geqslant 3$), $\mathcal{P}_0'$ is a partition of $G_0'[V_0 \cup V_1 \cup V_2 \cup \cdots \cup V_{\ell-1}]$. In particular, each vertex $y_{i,j}$ (which is in V_2) is in some part X of $\mathcal{P}_0'$. Replace $y_{i,j}$ in X by $Y_{i,j}$. Similarly, each vertex z_i (which is in V_1) is in some part X of $\mathcal{P}_0'$. Replace z_i in X by Z_i. Remove r

from the part of $\mathcal{P}'_0$ that contains r. This defines an H-partition $\mathcal{P}$ of $G-S$ where every clique in a vortex of G is contained in at most two parts.

It remains to bound the width of $\mathcal{P}$. Let $X \in \mathcal{P}$. If X comes from $\mathcal{P}'_j$ for some $j \geqslant 1$, then $|X| \leqslant (2g+4p+3)d$. Now suppose X comes from $\mathcal{P}'_0$. Each vertical path in T has at most two special vertices (some z_i and some $y_{i,j}$). The corresponding replacements contribute at most $2\sqrt{(k+1)n}+2k+2$ vertices to X. Since X corresponds to the union of at most $2g+4p+3$ vertical paths (before replacement) in T,

$$\begin{aligned}|X| &\leqslant (2g+4p+3)d + (2g+4p+3)(2\sqrt{(k+1)n}+2k+2)\\ &= (2g+4p+3)(2\sqrt{(k+1)n}+d+2k+2).\end{aligned}$$

So $\mathcal{P}$ has width at most $(2g+4p+3)(2\sqrt{(k+1)n}+d+2k+2)$, as required. ∎

To handle tree-decompositions we need the following standard separator lemma. For a tree T rooted at $r \in V(T)$, the root of a subtree T' of T is the vertex in $V(T')$ that is closest to r. A *weighted tree* is a tree T together with a weighting function $\gamma : V(T) \to \mathbb{R}^+$. The *weight* of a subtree T' of T is $\sum_{v \in V(T')} \gamma(v)$.

LEMMA 3.2. *For every integer $q \geqslant 0$ and $n \in \mathbb{R}^+$, every weighted tree T with weight at most n has a set Z of at most q vertices such that each component of $T-Z$ has weight at most $\frac{n}{q+1}$.*

Proof. We proceed by induction on q. The $q=0$ case holds trivially with $Z=\varnothing$. Now assume that $q \geqslant 1$ and the result holds for $q-1$. Root T at an arbitrary vertex r. For each vertex v, let T_v be the maximal subtree of T rooted at v. Let v be a vertex in T furthest from r such that T_v has weight greater than $\frac{n}{q+1}$. (If no such v exists, then T has weight at most $\frac{n}{q+1}$ and $Z=\varnothing$ satisfies the claim). Let $T' := T - V(T_v)$. So T' has weight at most $\frac{qn}{q+1}$. By induction, T' has a set Z' of at most $q-1$ vertices such that each component of $T'-Z'$ has weight at most $\frac{n}{q+1}$. Let $Z := Z' \cup \{v\}$. By the choice of v, each component of $T_v - v$ has weight at most $\frac{n}{q+1}$. Thus each component of $T-Z$ has weight at most $\frac{n}{q+1}$. ∎

The next lemma handles tree-decompositions.

LEMMA 3.3. *Let $a,b,k,n,w \geqslant 1$ be integers, and let G be an n-vertex graph that has a rooted tree-decomposition $(B_x : x \in V(T))$ of adhesion at most k such that for each $x \in V(T)$ there exists $S_x \subseteq B_x$ such that*

- *$|S_x| \leqslant |B_x|/\sqrt{n}+a$,*
- *$G\langle B_x\rangle - S_x$ has a J_x-partition $\mathcal{P}_x$ of width at most b where $\operatorname{tw}(J_x) \leqslant w$, and*
- *for every child-adhesion clique C of $G\langle B_x\rangle$, the set $C \setminus S_x$ is contained in at most w parts in $\mathcal{P}_x$.*

Then G has an H-partition of width at most $\max\{b, (a+2k+1)\lceil\sqrt{n}\rceil\}$ such that $\operatorname{tw}(H) \leqslant w+1$. Moreover, H contains a vertex α such that $\operatorname{tw}(H-\alpha) \leqslant w$.

Proof. Let $r \in V(T)$ be the root of T. For every node $x \in V(T)$ with parent y, let $X_x := B_x \cap B_y$ (where $X_r = \varnothing$) and let $B'_x := B_x - X_x$. For each node $x \in V(T)$, let $\gamma(x) = |B'_x|$. Observe that $(B'_x : x \in V(T))$ is a partition of $V(G)$, so the total weight equals n. By Lemma 3.2 with $q := \lceil\sqrt{n}\rceil - 1$, there is a set $Z' \subseteq V(T)$ where $|Z'| \leqslant q$ such that each component of $T-Z'$ has total weight at most $\frac{n}{q+1} \leqslant \sqrt{n}$. Let $Z := Z' \cup \{r\}$. For each $z \in Z$, let T_z be the maximal subtree of T rooted at z such that $T_z \cap Z = \{z\}$. Let $Q := \bigcup(X_z : z \in Z)$ and observe that $|Q| \leqslant k(q+1) \leqslant k\sqrt{n}$. For each $z \in Z$, let $G_z := G[\bigcup(B'_x : x \in V(T_z))] - X_z$. If there is an edge $xy \in E(T)$, where y is the parent of x, and $x\in V(T_z)$ and $y\in V(T_{z'})$ for some distinct $z,z'\in Z$, then $x=z$. Thus $G-Q$ is the disjoint union of $(G_z\colon z\in Z)$.

CLAIM 3.4. *For each $z\in Z$, there exists $S_z\subseteq V(G_z)$ where $|S_z|\leqslant |B_z|/\sqrt{n}+a$ such that G_z-S_z has an H_z-partition of width at most $\max\{b,\sqrt{n}\}$ for some graph H_z with treewidth at most w.*

Proof. Let $T'_1,\dots,T'_f$ be the components of T_z-z. For each $j\in[f]$, let C_j be the subgraph of G_z induced by $(B'_x\colon x\in V(T'_j))$. Note that $V(G_z)$ is the disjoint union of $B'_z,V(C_1),\dots,V(C_f)$. By Lemma 3.2, $|V(C_j)|\leqslant|\bigcup(B'_x\colon x\in V(T'_j))|=\gamma(T'_j)\leqslant\sqrt{n}$. By assumption, there is a set $S_z\subseteq B_z$ where $|S_z|\leqslant|B_z|/\sqrt{n}+a$ such that $G\langle B_z\rangle-S_z$ has a J_z-partition $\mathcal{P}'_z$ with width at most b where $\operatorname{tw}(J_z)\leqslant w$, and for every child-adhesion clique C in $G\langle B_z\rangle$, $C\setminus S_z$ is contained in at most w parts in $\mathcal{P}'_z$. Let $(W^{(z)}_x\colon x\in V(T^{(z)}))$ be a tree-decomposition of J_z with width at most w. Add $V(C_1),\dots,V(C_f)$ to the partition $\mathcal{P}'_z$ to obtain a partition $\mathcal{P}_z$ of G_z-S_z with quotient H_z. Then $\mathcal{P}_z$ has width at most $\max\{b,\sqrt{n}\}$. For each $j\in[f]$, let $\alpha_j\in V(H_z)$ be the vertex that indexes $V(C_j)$ and let N_j be the neighborhood of α_j. Since the neighborhood of C_j in G_z is a child-adhesion clique of $G\langle B_z\rangle$, it follows that N_j is a clique in J_z of size at most w. Thus there is a node $x\in V(T^{(z)})$ such that $N_j\subseteq W^{(z)}_x$. Add a leaf node ℓ adjacent to x and let $W^{(z)}_\ell:=N_j\cup\{\alpha_j\}$. Repeat this procedure for all $j\in[f]$ to obtain a tree-decomposition of H_z with width at most w. ◻

Observe that

$$\sum_{z\in Z}|B_z|\leqslant\sum_{z\in Z}(|B'_z|+|X_z|)=\left(\sum_{z\in Z}|B'_z|\right)+\left(\sum_{z\in Z}|X_z|\right)\leqslant n+k|Z|.$$

Since $|Q|\leqslant k|Z|$ and $|Z|\leqslant q+1=\lceil\sqrt{n}\rceil$,

$$\left|Q\cup\left(\bigcup S_z:z\in Z\right)\right|\leqslant|Q|+\sum_{z\in Z}\left(|B_z|/\sqrt{n}+a\right)\leqslant(k+a)|Z|+(n+k|Z|)/\sqrt{n}$$
$$<(2k+a+1)\lceil\sqrt{n}\rceil.$$

Let H be the graph obtained from the disjoint union of $(H_z\colon z\in Z)$ by adding one dominant vertex α. So $\operatorname{tw}(H-\alpha)\leqslant w$ and $\operatorname{tw}(H)\leqslant w+1$. By associating $Q\cup(\bigcup S_z: z\in Z)$ with α, we obtain an H-partition of G with width at most $\max\{b,(a+2k+1)\lceil\sqrt{n}\rceil\}$. ◻

Proof of Theorem 1.1. Let G be an n-vertex K_t-minor-free graph. By Theorem 2.1, G has a rooted tree-decomposition $(B_x:x\in V(T))$, such that for each $x\in V(T)$, the torso $G\langle B_x\rangle$ is k-almost-embeddable (for some $k=k(t)$), and if A_x is the apex-set of $G\langle B_x\rangle$, then for every child-adhesion clique C of $G\langle B_x\rangle$, either $C\setminus A_x$ is contained in a vortex of $G\langle B_x\rangle$, or $|C\setminus A_x|\leqslant 3$. [10, Lemma 21] showed that every clique in a k-almost-embeddable graph has at most $9k$ vertices. So the adhesion of $(B_x:x\in V(T))$ is at most $9k$. We may assume that $n\geqslant k$. By Lemma 3.1 with $d:=\lceil\sqrt{n}\rceil+3$, for each torso $G\langle B_x\rangle$ there exists a set $S_x\subseteq B_x$ such that $|S_x|\leqslant\frac{|B_x|}{\lceil\sqrt{n}\rceil}+k\leqslant\frac{|B_x|}{\sqrt{n}}+k$ and $G\langle B_x\rangle-S_x$ has a J_x-partition $\mathcal{P}_x$, where $\operatorname{tw}(J_x)\leqslant 3$ and the width of $\mathcal{P}_x$ is at most $(2k+4k+3)(2\sqrt{(k+1)n}+\lceil\sqrt{n}\rceil+3+2k+2)\leqslant(6k+3)\cdot 9\sqrt{(k+1)n}$ (because $n\geqslant k\geqslant 1$).

Moreover, $A_x\subseteq S_x$ and any clique in a vortex of $G\langle B_x\rangle$ is contained in at most two parts in $\mathcal{P}_x$. As such, for every child-adhesion clique C of $G\langle B_x\rangle$, $C\setminus S_x$ is contained in at most three parts of $\mathcal{P}_x$. By Lemma 3.3, G has an H-partition with width at most

$m := \max\{(6k+3)\cdot 9\sqrt{(k+1)n}, (k+18k+1)\lceil\sqrt{n}\rceil\} \leqslant (6k+3)\cdot 9\sqrt{(k+1)n}$, where H contains a vertex α such that $\text{tw}(H-\alpha)\leqslant 3$. It therefore follows from Observation 2.2 that G is contained in $H \boxtimes K_{\lfloor m\rfloor}$ where $\text{tw}(H)\leqslant 4$. ∎

4. Proof of Theorem 1.3. This section proves Theorem 1.3 for $K_{3,t}$-minor-free graphs, where we assume throughout that $t\geqslant 1$. We use the following extremal function for $K^*_{3,t}$-minor-free graphs by Kostochka and Prince [19]. Here $K^*_{3,t}$ is the graph obtained from $K_{3,t}$ by adding an edge between each pair of vertices in the side of the bipartition with three vertices.

LEMMA 4.1 (see [19]). *Every $K^*_{3,t}$-minor-free graph G satisfies $|E(G)|\leqslant \alpha t\,|V(G)|$ for some constant $\alpha\geqslant 1$.*

The following notation will be useful in the proof of Theorem 1.3. For a graph G, an induced subgraph C of G, and sets $X, Y\subseteq V(G)$ such that $X, Y, V(C)$ are pairwise disjoint, let $\kappa_G(X,C,Y)$ be the maximum number of vertex-disjoint paths in C, each with an endpoint in $N_G(X)\cap C$ and an endpoint in $N_G(Y)\cap C$. By Menger's theorem there is a set $S\subseteq V(C)$ of size $\kappa_G(X,C,Y)$ separating $N_G(X)\cap C$ and $N_G(Y)\cap C$ in C. If $X=\{x\}$, then replace X by x in this notation, and similarly for Y.

The following lemma is the key to the proof of Theorem 1.3. Here α is from Lemma 4.1.

LEMMA 4.2. *Let G be a $K^*_{3,t}$-minor-free graph on n vertices. Let X and Y be disjoint nonempty sets of vertices in G such that $G[X]$, $G[Y]$, and $G[X\cup Y]$ are connected. Then there is a set $S\subseteq V(G-X-Y)$ such that*

- $|N_G[S]|\leqslant t\sqrt{3\alpha n}$,
- $\kappa_G(X,C,Y)\leqslant t\sqrt{3\alpha n}$ *for every component C of $G-X-Y-S$, and*
- *$G[X\cup S]$ and $G[Y\cup S]$ are connected.*

Proof. Let $Q_1,\dots,Q_m$ be a maximum-size set of vertex-disjoint paths in $G-X-Y$ between $N_G(X)\setminus Y$ and $N_G(Y)\setminus X$. If $m=0$, then the lemma holds trivially with $S=\varnothing$, and thus we may assume $m\geqslant 1$. Define J to be the auxiliary graph with vertex set $\{q_1,\dots,q_m\}$ where $q_iq_j\in E(J)$ whenever there is a path in $G-X-Y$ joining Q_i and Q_j, and avoiding each Q_ℓ with $\ell\notin\{i,j\}$.

Consider a component J' of J. Let $(V_0,V_1,\dots)$ be a BFS-layering of J'. So $|V_0|=1$. We claim that $|V_i|<t$ for each $i\geqslant 1$. Suppose for the sake of contradiction that $|V_i|\geqslant t$ for some $i\geqslant 1$. Without loss of generality, $q_1,\dots,q_t\in V_i$. Let A be the union of (1) all paths Q_j corresponding to vertices in $V_0\cup\dots\cup V_{i-1}$, (2) all paths in $G-X-Y$ corresponding to edges in $J[V_0\cup\dots\cup V_{i-1}]$, and (3) all paths in $G-X-Y$ corresponding to edges in J between a vertex in V_{i-1} and $q_1,\dots,q_t$, not including the vertex in $Q_1\cup\dots\cup Q_t$. By construction, A is a connected subgraph of $G-X-Y$ disjoint from $Q_1\cup\dots\cup Q_t$ and adjacent to each of $Q_1,\dots,Q_t$. Each of $A,Q_1,\dots,Q_t$ intersect $N_G(X)$ and $N_G(Y)$. Thus $X,Y,A,Q_1,\dots,Q_t$ form a $K^*_{3,t}$-model in G. This contradiction shows that $|V_i|<t$ for each $i\geqslant 0$.

Concatenate the above-mentioned layerings of each component of J to obtain a layering $(V_0,V_1,\dots)$ of J with $|V_i|<t$ for each i. Assign each vertex q_j in J a weight of $|N_G[Q_j]|$. The total weight is at most $|V(G)|+2|E(G)|$, which by Lemma 4.1 is at most $(2\alpha t+1)n\leqslant 3\alpha tn$ since $t\geqslant 1$. Weight each set V_i by the total weight of the vertices in V_i. Let $p:=\lceil\sqrt{3\alpha n}\rceil$. There exists $i\in\{0,\dots,p-1\}$ such that $Z:=\bigcup\{V_j : j\equiv i \pmod p\}$ has weight at most $3\alpha tn/p\leqslant t\sqrt{3\alpha n}$, and each component of $J-Z$ has less than $(p-1)t\leqslant t\sqrt{3\alpha n}$ vertices. Let $S:=\bigcup\{Q_i : q_i\in Z\}$. By construction, $|N_G[S]|$ is at most the weight of Z, which is at most $t\sqrt{3\alpha n}$. Moreover,

since $G[X \cup Q_i]$ and $G[Y \cup Q_i]$ are connected for all $i \in [m]$, it follows that $G[X \cup S]$ and $G[Y \cup S]$ are connected.

Consider a component C of $G - X - Y - S$. Since each component of $J - Z$ has at most $t\sqrt{3\alpha n}$ vertices, the number of paths Q_i that pass through C is at most $t\sqrt{3\alpha n}$. By the choice of $Q_1, \dots, Q_m$, we have $\kappa_G(X, C, Y) \leqslant t\sqrt{3\alpha n}$. ◻

Theorem 1.3 follows from Observation 2.2 and the next lemma.

LEMMA 4.3. *Let G be a $K^*_{3,t}$-minor-free graph on n vertices. Let Q be a clique in G with $|Q| \leqslant 2$ such that if $Q = \{x, y\}$ with $x \neq y$, then $\kappa_G(x, C, y) \leqslant 2t\sqrt{3\alpha n}$ for every component C of $G - x - y$, where α is from Lemma 4.1. Then G has a partition $\mathcal{P}$ with nonempty parts, with width at most $\leqslant 4t\sqrt{3\alpha n}$, with $\operatorname{tw}(G/\mathcal{P}) \leqslant 2$, and with $\{v\} \in \mathcal{P}$ for each $v \in Q$.*

Proof. We proceed by induction on $|V(G) \setminus Q|$. The result is trivial if $V(G) = Q$. Now assume that $V(G) \neq Q$. If $Q = \varnothing$, then the result follows by induction where $Q := \{v\}$ and v is any vertex in G. Now assume that $Q \neq \varnothing$. If G is disconnected, then the result follows by applying induction in each component C of G with the clique $Q \cap V(C)$. Now assume that G is connected. First consider the case in which $Q = \{x\}$. Since G is connected and $V(G) \neq Q$, there is a neighbor v of x. By Lemma 4.2 applied to $(G, \{x\}, \{v\})$, there is a set $S \subseteq V(G - x - v)$ such that

- $|N_G[S]| \leqslant t\sqrt{3\alpha n}$,
- $\kappa_G(x, C, v) \leqslant t\sqrt{3\alpha n}$ for each component C of $G - x - v - S$, and
- $G[S \cup \{v\}]$ is connected.

Let G' be obtained from G by contracting $S \cup \{v\}$ into a single vertex v'. So G' is $K^*_{3,t}$-minor-free and xv' is an edge of G'. For each component C' of $G' - x - v'$, we have $\kappa_{G'}(x, C', v') \leqslant \kappa_G(x, C', v) + |N_G[S]| \leqslant 2t\sqrt{3\alpha n}$. Apply induction to G' and $Q' := \{x, v'\}$ to obtain a partition $\mathcal{P}'$ of G' of width at most $4t\sqrt{3\alpha n}$ such that $\operatorname{tw}(G'/\mathcal{P}') \leqslant 2$ and $\{x\}, \{v'\} \in \mathcal{P}'$. Let $\mathcal{P}$ be the partition of G obtained from $\mathcal{P}'$ by replacing $\{v'\}$ by $S \cup \{v\}$. So $\mathcal{P}$ has width at most $\max\{4t\sqrt{3\alpha n}, |S| + 1\} = 4t\sqrt{3\alpha n}$ and $\{x\} \in \mathcal{P}$. Since $G/\mathcal{P} \cong G'/\mathcal{P}'$ we have $\operatorname{tw}(G/\mathcal{P}) \leqslant 2$.

Now consider the case in which $|Q| = 2$ and $Q = \{x, y\}$.

First, suppose that no component of $G - x - y$ intersects $N_G(x)$ and $N_G(y)$. Let G_x be the subgraph of G induced by $\{x\}$ and the components of $G - x - y$ that intersect $N_G(x)$. Let G_y be the subgraph of G induced by $\{y\}$ and the components of $G - x - y$ that intersect $N_G(y)$. By induction, G_x has a partition $\mathcal{P}_x$ of width at most $4t\sqrt{3\alpha n}$ such that $\operatorname{tw}(G_x/\mathcal{P}_x) \leqslant 2$ and $\{x\} \in \mathcal{P}_x$. Similarly, G_y has a partition $\mathcal{P}_y$ of width at most $4t\sqrt{3\alpha n}$ such that $\operatorname{tw}(G_y/\mathcal{P}_y) \leqslant 2$ and $\{y\} \in \mathcal{P}_y$. Let $\mathcal{P} := \mathcal{P}_x \cup \mathcal{P}_y$. So $\mathcal{P}$ is a partition of G, and $G/\mathcal{P}$ is obtained from the disjoint union of $G_x/\mathcal{P}_x$ and $G_y/\mathcal{P}_y$ by adding the edge $\{x\}\{y\}$. So $\operatorname{tw}(G/\mathcal{P}) \leqslant 2$.

Now assume that some component C of $G - x - y$ intersects both $N_G(x)$ and $N_G(y)$. By assumption, $\kappa_G(x, C, y) \leqslant 2t\sqrt{3\alpha n}$. By Menger's theorem, there exists $S \subseteq V(C)$ such that $|S| \leqslant 2t\sqrt{3\alpha n}$ and S separates $N_G(x) \cap V(C)$ and $N_G(y) \cap V(C)$ in C. Choose S to be minimal. Observe that $S \neq \varnothing$. No component of $C - S$ intersects both $N_G(x)$ and $N_G(y)$. Let D_x be the union of the components of $C - S$ that intersect $N_G(x)$. Let D_y be the union of the components of $C - S$ that intersect $N_G(y)$. Let F be the union of the components of $C - S$ that intersect neither $N_G(x)$ nor $N_G(y)$. Let $G_C := G[(V(C) \cup \{x, y\}) \setminus V(F)]$.

Let $Y := \{y\} \cup V(D_y) \cup S$. By the minimality of S, $G[Y]$ is connected, and $G[\{x\} \cup Y]$ is connected since $xy \in E(G)$. By Lemma 4.2 applied to $(G_C, \{x\}, Y)$, there is a set $S_x \subseteq V(G_C - x - Y) = V(D_x)$ such that

- $|N_{G_C}[S_x]| \leqslant t\sqrt{3\alpha n}$,
- $\kappa_{G_C}(x, C', Y) \leqslant t\sqrt{3\alpha n}$ for every component C' of $D_x - S_x$, and
- $G_C[S_x \cup Y]$ is connected.

Let G_x be the graph obtained from G_C by contracting $S_x \cup Y$ into a single vertex z. Thus G_x is $K^*_{3,t}$-minor-free and xz is an edge of G_x. Consider a component C' of $G_x - x - z$. Then C' is a component of $D_x - S_x$, and

$$\kappa_{G_x}(x, C', z) = \kappa_{G_C}(x, C', S_x \cup Y) \leqslant \kappa_{G_C}(x, C', Y) + |N_{G_C}[S_x]| \leqslant 2t\sqrt{3\alpha n}.$$

By induction, G_x has a partition $\mathcal{P}_x$ of width at most $4t\sqrt{3\alpha n}$ such that $\text{tw}(G_x/\mathcal{P}_x) \leqslant 2$ and $\{x\}, \{z\} \in \mathcal{P}_x$. (Note that $|V(G_x)| < |V(G)|$ since $S \neq \varnothing$, so we may apply induction.)

Let $X := \{x\} \cup V(D_x) \cup S$. By an argument symmetric to the above, there is a set $S_y \subseteq V(D_y)$ such that

- $|N_{G_C}[S_y]| \leqslant t\sqrt{3\alpha n}$,
- $\kappa_{G_C}(y, C', X) \leqslant t\sqrt{3\alpha n}$ for every component C' of $D_y - S_y$, and
- $G_C[S_y \cup X]$ is connected.

Let G_y be the graph obtained from G_C by contracting $S_y \cup X$ into a single vertex z. Thus G_y is $K^*_{3,t}$-minor-free and yz is an edge of G_y. By a symmetric argument, G_y has a partition $\mathcal{P}_y$ of width at most $4t\sqrt{3\alpha n}$ such that $\text{tw}(G_y/\mathcal{P}_y) \leqslant 2$ and $\{y\}, \{z\} \in \mathcal{P}_y$.

Note that $G[X \cup Y]$ is connected. Let G_F be the graph obtained from $G[\{x, y\} \cup V(C)]$ by contracting $X \cup Y$ into a single vertex z. So $V(G_F) = \{z\} \cup V(F)$, and G_F is $K^*_{3,t}$-minor-free. By induction, G_F has a partition $\mathcal{P}_F$ of width at most $4t\sqrt{3\alpha n}$ such that $\text{tw}(G_F/\mathcal{P}_F) \leqslant 2$ and $\{z\} \in \mathcal{P}_F$.

Let $G' := G - V(C)$. So G' is $K^*_{3,t}$-minor-free, and xy is an edge of G'. By induction, G' has a partition $\mathcal{P}'$ of width at most $4t\sqrt{3\alpha n}$ such that $\text{tw}(G'/\mathcal{P}') \leqslant 2$ and $\{x\}, \{y\} \in \mathcal{P}'$.

Let $\mathcal{P}$ be the partition of G obtained from $\mathcal{P}_x \cup \mathcal{P}_y \cup \mathcal{P}_F \cup \mathcal{P}'$ by replacing each of the three instances of $\{z\}$ by $S \cup S_x \cup S_y$. The width of $\mathcal{P}$ is at most $4t\sqrt{3\alpha n}$. Note that $G/\mathcal{P}$ is obtained by pasting the four graphs $G_x/\mathcal{P}_x$, $G_y/\mathcal{P}_y$, $G_F/\mathcal{P}_F$, and $G'/\mathcal{P}'$ on the triangle $\{x\}, \{y\}, S \cup S_x \cup S_y$, where each of the four graphs contains vertices in two of $\{x\}$, $\{y\}$ and $S \cup S_x \cup S_y$. Thus $G/\mathcal{P}$ is obtained from graphs of treewidth at most 2 by pasting on edges. Hence $\text{tw}(G/\mathcal{P}) \leqslant 2$ and $\{x\}, \{y\} \in \mathcal{P}$. ∎

5. Open problems. It is an intriguing open problem to determine $f(\mathcal{G})$ for a given proper minor-class $\mathcal{G}$. It is possible that $f(\mathcal{G}) \leqslant 2$ for every minor-closed class $\mathcal{G}$. This is open even when $\mathcal{G}$ is the class of K_5-minor-free graphs [18]. Let $\mathcal{A}$ be the class of apex graphs,[2] which is minor-closed. It is open whether $f(\mathcal{A}) \leqslant 2$. This is equivalent to the following open problem (which would strengthen Theorem 1.2): for every n-vertex planar graph G, does there exist an apex-forest[3] H such that G is contained in $H \boxtimes K_m$ where $m \in \mathcal{O}(\sqrt{n})$?

It is also open whether treewidth can be replaced by pathwidth in Theorems 1.1 to 1.3. That is, for a proper minor-closed class $\mathcal{G}$, are there integers k, c such that every n-vertex graph in $\mathcal{G}$ is contained in $H \boxtimes K_m$ for some graph H with pathwidth at most k, where $m \leqslant c\sqrt{n}$? Two pieces of evidence suggest a positive answer. First, n-vertex graphs in a proper minor-closed class have pathwidth $\mathcal{O}(\sqrt{n})$; see [2]. Second, if $\mathcal{G}$ has bounded treewidth, then the answer is "yes" with $k = 1$, since Dvořák and Wood [13] showed that n-vertex graphs in $\mathcal{G}$ have H-partitions of width $\mathcal{O}(\sqrt{n})$ where H is a star, which has pathwidth 1. This question is open for planar graphs.

[2] A graph H is *apex* if $H - v$ is planar for some vertex v of H.
[3] A graph H is an *apex forest* if $H - v$ is a forest for some vertex v of H.

Analogous questions are interesting and open for several non-minor-closed classes [13].

Acknowledgment. This work was partially completed at the 10th Annual Workshop on Geometry and Graphs held at Bellairs Research Institute in February 2023.

REFERENCES

[1] N. Alon, P. Seymour, and R. Thomas, *A separator theorem for nonplanar graphs*, J. Amer. Math. Soc., 3 (1990), pp. 801–808.
[2] H. L. Bodlaender, *A partial k-arboretum of graphs with bounded treewidth*, Theoret. Comput. Sci., 209 (1998), pp. 1–45.
[3] R. Campbell, K. Clinch, M. Distel, J. Pascal Gollin, K. Hendrey, R. Hickingbotham, T. Huynh, F. Illingworth, Y. Tamitegama, J. Tan, and D. R. Wood, *Product structure of graph classes with bounded treewidth*, Combin. Probab. Comput., 33 (2024), pp. 351–376.
[4] R. Diestel, K.-i. Kawarabayashi, T. Müller, and P. Wollan, *On the excluded minor structure theorem for graphs of large tree-width*, J. Combin. Theory Ser. B, 102 (2012), pp. 1189–1210.
[5] M. Distel, R. Hickingbotham, T. Huynh, and D. R. Wood, *Improved product structure for graphs on surfaces*, Discrete Math. Theor. Comput. Sci., 24 (2022), 6.
[6] H. N. Djidjev, *A separator theorem for graphs of fixed genus*, Serdica, 11 (1985), pp. 319–329.
[7] V. Dujmović, L. Esperet, P. Morin, B. Walczak, and D. R. Wood, *Clustered* 3*-colouring graphs of bounded degree*, Combin. Probab. Comput., 31 (2022), pp. 123–135.
[8] V. Dujmović, R. Hickingbotham, J. Hodor, G. Joret, H. La, P. Micek, P. Morin, C. Rambaud, and D. R. Wood, *The grid-minor theorem revisited*, in Proceedings 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '24), SIAM, 2024, pp. 1241–1245, https://doi.org/10.1137/1.9781611977912.48, arXiv:2307.02816.
[9] V. Dujmović, G. Joret, P. Micek, P. Morin, T. Ueckerdt, and D. R. Wood, *Planar graphs have bounded queue-number*, J. ACM, 67 (2020), 22.
[10] V. Dujmović, P. Morin, and D. R. Wood, *Layered separators in minor-closed graph classes with applications*, J. Combin. Theory Ser. B, 127 (2017), pp. 111–147.
[11] V. Dujmović, P. Morin, and D. R. Wood, *Graph product structure for non-minor-closed classes*, J. Combin. Theory Ser. B, 162 (2023), pp. 34–67.
[12] Z. Dvořák and S. Norin, *Treewidth of graphs with balanced separations*, J. Combin. Theory Ser. B, 137 (2019), pp. 137–144.
[13] Z. Dvořák and D. R. Wood, *Product Structure of Graph Classes with Strongly Sublinear Separators*, preprint, arXiv:2208.10074, 2022.
[14] J. R. Gilbert, J. P. Hutchinson, and R. E. Tarjan, *A separator theorem for graphs of bounded genus*, J. Algorithms, 5 (1984), pp. 391–407.
[15] D. J. Harvey and D. R. Wood, *Parameters tied to treewidth*, J. Graph Theory, 84 (2017), pp. 364–385.
[16] R. Hickingbotham, P. Jungeblut, L. Merker, and D. R. Wood, *The product structure of squaregraphs*, J. Graph Theory, 105 (2024), pp. 179–191.
[17] R. Hickingbotham and D. R. Wood, *Shallow minors, graph products, and beyond-planar graphs*, SIAM J. Discrete Math., 38 (2024), pp. 1057–1089, https://doi.org/10.1137/22M1540296.
[18] F. Illingworth, A. Scott, and D. R. Wood, *Product Structure of Graphs with an Excluded Minor*, preprint, arXiv:2104.06627, 2022; Trans. Amer. Math. Soc., to appear.
[19] A. V. Kostochka and N. Prince, *Dense graphs have $K_{3,t}$ minors*, Discrete Math., 310 (2010), pp. 2637–2654.
[20] N. Linial, J. Matoušek, O. Sheffet, and G. Tardos, *Graph colouring with no large monochromatic components*, Combin. Probab. Comput., 17 (2008), pp. 577–589.
[21] R. J. Lipton and R. E. Tarjan, *A separator theorem for planar graphs*, SIAM J. Appl. Math., 36 (1979), pp. 177–189, https://doi.org/10.1137/0136016.
[22] R. J. Lipton and R. E. Tarjan, *Applications of a planar separator theorem*, SIAM J. Comput., 9 (1980), pp. 615–627, https://doi.org/10.1137/0209046.
[23] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University Press, 2001.
[24] B. A. Reed, *Algorithmic aspects of tree width*, in Recent Advances in Algorithms and Combinatorics, Vol. 11, Springer, 2003, pp. 85–107.
[25] N. Robertson and P. Seymour, *Graph minors.* V. *Excluding a planar graph*, J. Combin. Theory Ser. B, 41 (1986), pp. 92–114.

[26] N. Robertson and P. Seymour, *Graph minors.* XVI. *Excluding a non-planar graph*, J. Combin. Theory Ser. B, 89 (2003), pp. 43–76.

[27] T. Ueckerdt, D. R. Wood, and W. Yi, *An improved planar graph product structure theorem*, Electron. J. Combin., 29 (2022), P2.51.

[28] D. R. Wood, *Defective and clustered graph colouring*, Electron. J. Combin., DS23 (2018), Version 1.